On the Spectrum and Essential Minimum of Heights in Projective Plane

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Abstract

We study the spectrum and essential minimum of the height of projective plane intersecting with the curve $x + y + z = 0 \subset \mathbb{P}^2$. This paper is mainly inspired by the research of Zhang and Zagier. In the first part of the research, we prove that the spectrum of the projective height is dense in $\mathbb{R}$ after a certain value C_0.

In the second part of this research, we prove that the six special points at which Zagier's height gets sharp upper bound can be normalized to one point. Thus, it is plausible to find all the value on the spectrum before C_0 and the essential minimum in our coordinate. We provide a conjecture about the essential minimum and distributions on the spectrum before the essential minimum.

The main ingredients in our analysis are the estimation of capacity of level curve and the theorem of Fekete-Szegö. Thus, we make an intensive use of potential theory.

Our results have been greatly motivated and guided by numerical experiments of capacity and level curves that are described at the end of this paper in details.

Contents

1 Introduction

Height functions play a vital role in the study of arithmetic dynamics over global fields. It measures complexity of the point in an arithmetic sense, and it is another modern view of Lehmer's conjecture.

Let $K/\mathbb{Q}$ be a number field, and let $P \in \mathbb{P}^N(K)$ be a point with homogeneous coordinates

$$P = [x_0 : \cdots : x_N],\ x_0, \cdots, x_N \in K.$$

Then, the *height of P relative to K* is defined by

$$H_K(P) := \prod_{v \in M_K} \max\{|x_0|_v, \cdots, |x_N|_v\}^{n_v}, \text{ where } n_v \text{ is the } \textit{local degree of } v.$$

When working with a point in projective space with algebraic coordinates, it is sometimes easier to work with a height function that does not depend on the number field.

Let $P \in \mathbb{P}^N(\overline{\mathbb{Q}})$ be a point whose coordinates are algebraic numbers. Then the *absolute height of P* is defined by choosing any number field such that $P \in \mathbb{P}^N(K)$ and setting

$$H(P) := H_K(P)^{1/[K:\mathbb{Q}]}.$$

Compared to the standard height defined above, it is always much easier to directly work with logarithmic height.

Keep the notation above, then the *logarithmic height relative K* is defined by

$$h_K : \mathbb{P}^N(K) \longrightarrow \mathbb{R},\ \ h_K(P) := \log H_K(P)$$

and the *absolute logarithmic height* is the function

$$h : \mathbb{P}^N(\overline{\mathbb{Q}}) \longrightarrow \mathbb{R},\ \ h(P) = \log H(P).$$

However, in terms of absolute logarithmic height function, if we use the formula given above to calculate the height of an algebraic number, we will always have a hard time to normalize the absolute value with respect to different number field.

Therefore, for any number field K, and for any algebraic number $\alpha \in K$ with algebraic coordinates, it is alwasy easier to use the following formula of absolute logarithmic height function

$$h(\alpha) = \frac{1}{d} \sum_{\alpha'} \sum_{v \in M_K} \log \max\{|\alpha'|_v, 1\}$$

where d is degree of the minimal polynomial of α over $\mathbb{Q}$, the first sum is summing over all conjugates of α, and the second sum is summing over all places.

In the year of 1992, Zhang discovered a classical result in [Zha92, Theorems 6.3 and 6.4], the simplest version of which is that for all but finitely many pairs of algebraic numbers (α, β) lying on a curve $X \subset \mathbb{P}^1 \times \mathbb{P}^1$, we will always have a universal upper bound $C(X)$, depending on the curve X you choose, such that

$$h(\alpha) + h(\beta) \geq C(X) > 0.$$

Then, in [Zag93, Theorem 1], Zagier applied this result to the curve

$$\{(x,y) : x+y=1, xy \neq 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

He showed that for any algebric numbers $\alpha \in \overline{\mathbb{Q}}$ such that $\alpha \neq 0, 1, (1 \pm \sqrt{-3})/2$, we have

$$h(\alpha) + h(1-\alpha) \geq \frac{1}{2} \log \frac{1+\sqrt{5}}{2}$$

with equality if and only if α or $1-\alpha$ is a primitive 10^{th} root of unity.

Then, in [Zag93, Section 3A], he asked whether there is a whole spectrum of values

$$c_0 = 0 < c_1 = \frac{1+\sqrt{5}}{2} < c_2 < \cdots$$

such that $h(\alpha)+h(1-\alpha) = c_j$ for some finite collection of algebraic numbers α and $h(\alpha)+h(1-\alpha) > \limsup c_j$ for all other $\alpha \in \overline{\mathbb{Q}}$.

With this question, Zagier explored further in [Zag93, Section 3B] with the height function of projective plane intersecting the curve $x+y+z=0 \subset \mathbb{P}^2$, which is arguably more nature and more symmetric, since the only small difference between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ is at the infinity.

In [Zag93, Theorem 1'], he proved for all P lying on $x+y+z=0 \subset \mathbb{P}^2$ with homogenous algebraic coordinates, except for five points that $h_{\text{zz}}(P)$ vanishes, we have

$$h_{\text{zz}}(P) \geq \frac{1}{2} \log \theta$$

where θ is the real root of $\theta^3 - \theta^2 - 1 = 0$, and the bound is sharp for exactly six values, namely at the points $P = (1 : \alpha - 1 : -\alpha)$, where α is a root of the equation $x^6 - 3x^5 + 7x^4 - 9x^3 + 7x^2 - 3x + 1 = 0$.

This paper is mainly motivated by the Zagier's conjecture and [Zag93, Theorem 1'], and we only focus on the spectrum of the height function of projective plane. The main body of this paper is in **Section 3**, which is the first part of our research, in which we prove **Theorem A**.

The second part of this research, which is discussed in **Section 4**, concerns with

$$\mu_{\text{zz}}^{\text{ess}} := \inf\Big\{\theta \in \mathbb{R} : \text{the set } \{\alpha \in \mathbb{Q}^a : h_{\text{zz}}(\alpha) \leq \theta\} \text{ is infinity}\Big\} \tag{1.1}$$

and the set

$$\Big\{h_{\text{zz}}(\alpha) : \alpha \in \mathbb{Q}^a\Big\} \setminus \Big[\, \mu_{\text{zz}}^{\text{ess}}, \infty \Big). \tag{1.2}$$

In **Theorem G**, we prove that, under our transition map, the six special points at which the upper bound of h_{zz} is sharp can be normalized to $\frac{1}{2}$, and another six points at which h_{zz} vanishes can be normalized to 0 and 1. Therefore, it is plausible to find the exact value of $\mu_{\text{zz}}^{\text{ess}}$ and to determine all the values of the set (1.2) in our coordinate, whence solve the problem entirely. In **Conjecture 4.1**, we also provide a conjecture about the value of the essential minimum and the distribution on the spectrum before essential minimum.

Due to the nature and the audience of this paper, we assume basic knowledge of p–adic analysis, affine and projective varieties, and of algebraic number theory. Concepts specific to height functions described above are sufficient for readers of this paper. One could check [Kob84], [Sil07] and [Har77] for p-adic analysis, arithmetic dynamics, and algebraic geometry, respectively.

On the other hand, in the **section 2**, we will introduce some notions in potential theory that are closely related to this paper and to our research, including the notions of potential function, capacity and the theorem of Fekete-Szegö. We will basically follow [Ran95] and [KL95, pg111-127]. One could also read [FS55] for the deep interest in the theorem of Fekete-Szegö.

Further, since [Zag93] is the main inspiration of our research, the **Appendix B** contains the detailed proof of the Theorem1 and Theorem 1' in [Zag93]. Also, since our research is greatly motivated by our numerical experiments of capacity, **Appendix A** described those experiments in details.

2 Preliminaries in Potential Theory

2.1 Harmonic and Subharmonic Functions

We firstly recall some concepts from harmonic analysis. Let U be an open subset of $\mathbb{C}$. A function $h : U \longrightarrow \mathbb{R}$ is called *harmonic* if $h \in C^2(U)$ and $\Delta h = 0$ on U.

Recall that the real part of holomorphic function is harmonic, and the harmonic function satisfies mean-value property, identity principle and maximum principle. [Ran95, Theorem 1.1.6, 1.1.7 and 1.1.8]

Let X be a topological space. We say that a function $u : X \longrightarrow [-\infty, \infty)$ is *upper semicontinuous* if the set $\{x \in X : u(x) < \alpha\}$ is open in X for each $\alpha \in \mathbb{R}$. Also, if $-v$ is upper semicontinuous, then v is *lower semicontinuous*.

Now, let U be an open subset of $\mathbb{C}$. A function $u : U \longrightarrow [-\infty, \infty)$ is called *subharmonic* if it is upper semicontinuous and satisfies the local submean inequality, i.e. given $w \in U$, there exists $\rho > 0$ such that

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{it}) dt \quad (0 \leq r < \rho).$$

Also, if $-v$ is subharmonic, then v is *superharmonic*.

Theorem 1. ***(Criteria for Subharmonicity)*** *Let U be an open subset of $\mathbb{C}$, and let $u \in C^2(U)$. Then u is subharmonic on U if and only if $\Delta u \geq 0$ on U.*

Proof. [Ran95, Theorem 2.4.4] ∎

Theorem 2. ***(Integrability Theorem)*** *Every subharmonic function u on a domain $D \subset \mathbb{C}$, with $u \not\equiv -\infty$ on D is locally integrable on D, i.e. $\int_K |u| dA < \infty$ for each compact subset K of D.*

Proof. [Ran95, Theorem 2.5.1] ∎

2.2 Potential and Measures

Let (X, τ) be a Hausdorff topological space and let $\mathcal{B}$ be the σ-algebra of its Borel sets. A measure $\mu : \mathcal{B} \longrightarrow [0, \infty)$ is called a *regular Borel measure* if it satisfies the following properties:

1. $\mu(K) < \infty$ for every compact set K;
2. If B is a Borel subset of X, then $\mu(B) = \inf\{\mu(O) : O \text{ open and } B \subset O\}$;
3. If O is an open subset of X, then $\mu(O) = \sup\{\mu(K) : K \text{ compact and } K \subset O\}$.

A measure μ on the Borel sets of a topological space that satisfies $\mu(K) < \infty$ for each compact set K is called a *Borel measure (or Borel probability measure)*.

Now, let μ be a finite Borel measure on $\mathbb{C}$ with compact support. Then, its corresponding *potential* is the function $p_\mu : \mathbb{C} \longrightarrow [-\infty, \infty)$ defined by

$$p_\mu(z) := \int \log|z - w| d\mu(w),$$

and its *energy* $I(\mu)$ is given by

$$I(\mu) := \int \int \log|z - w| d\mu(z) d\mu(w) = \int p_\mu(z) d\mu(z).$$

Let K be a compact subset of $\mathbb{C}$, and denote by $\mathcal{P}(K)$ the collection of all Borel probability measures on K. If there exists $v \in \mathcal{P}(K)$ such that

$$I(v) = \sup_{\mu \in \mathcal{P}(K)} I(\mu),$$

then v is called an *equilibrium measure* for K. In particular, every compact set $K \in \mathbb{C}$ has an equilibrium measure.

Remark 2.1. One could also show that every potential function is subharmonic by directly applying the definition and doing a change of variables.

Remark 2.2. A subset E of $\mathbb{C}$ is called *polar* if $I(\mu) = -\infty$ for every finite Borel measure $\mu \neq 0$ with supp μ is a compact subset of E.

2.3 Generalized Laplacian

By **Theorem 1**, a C^2 subharmonic function u satisfies $\Delta u \geq 0$. However, it is important to extend this result to an appropriate generalization for arbitrary subharmonic functions.

Let $D \subset \mathbb{C}$ be a domain. Denote $C_c^\infty(D)$ be the class of all C^∞ functions $\phi : D \longrightarrow \mathbb{R}$ whose support supp ϕ is a compact subset of D. If u is a C^2 subharmonic function on D, then identifying Δu with the positive measure $\Delta u dA$, it then follows from Green's theorem that

$$\int_D \phi \Delta u = \int_D u \Delta \phi dA, \quad \phi \in C_c^\infty(D). \tag{2.1}$$

Now, if u is an arbitrary subharmonic function on D, with $u \not\equiv -\infty$, then by **Theorem 2**, u is locally integrable and thus the right hand side of above identity makes sense. Therefore, we can use it to define the left-hand side.

Thus, let u be a subharmonic function on a domain D in $\mathbb{C}$ with $u \not\equiv -\infty$, then the *generalized Laplacian* of u is Radom measure Δu on D such that (2.1) makes sense. By [Ran95, Theorem 3.7.2], Δu exists and is unique.

Remark 2.3. For this paper, it is sufficient for the readers to know that the generalized Laplacian is a measure, without knowing what exactly the Radom measure is. One could read [Ran95, Section A.3] for more explanations about this concept.

Theorem 3. ***(Riesz Decomposition Theorem)***

Let u be a subharmonic function on a domain D in $\mathbb{C}$ with $u \not\equiv -\infty$. Then, given a relatively compact open subset U of D, we can decompose u as

$$u = p_\mu + h \text{ on } U,$$

where $\mu = (2\pi)^{-1} \Delta u|_U$ and h is harmonic on U.

Proof. [Ran95, Theorem 3.7.9] ∎

Further, it is necessary to recall the **Fundamental Solution of Laplacian** since we made extensive use of this formula, it states that

$$\Delta(\log|\zeta - z_0|) = \delta_{z_0} - \delta_\infty$$

where δ_{z_0} is the direct mass at z_0 and δ_∞ is the direct mass at ∞.

Example 2.4. The **Fundamental Solution of Laplacian** allows us to calculate the laplacian of the log of the absolute value of any polynomial. For instance, say we have $f(z) = (z - w_1) \cdots (z - w_n)$. Then

$$\Delta(\log|f|) = \Delta(\sum_1^n \log|z - w_j|) = \sum_{i=1}^n \delta_{w_i} + \delta_\infty.$$

In fact, by **Fundamental Solution of Laplacian** and by the fact that $\Delta p_\mu = 2\pi\mu$ for any finite Borel measure, we can show that for any holomorphic function f on a domain D with $f \not\equiv 0$, $\Delta(\log|f|)$ consists of (2π)-masses at the zeros of f, counted multiplicity. The proof is in the same fashion as what we did in this example.

Example 2.5. If f is a potential, then

$$f(\zeta) = \int \log|z - \zeta| d(\Delta f)(z).$$

Example 2.6. The function $\log|\zeta + z_0|$ for any $z_0 \in \mathbb{C}$ is a potential function by the above example and the **Fundamental Solution of Laplacian**. Therefore, it is also a subharmonic function.

Example 2.7. $\Delta(\log^+|z|)$ is a Lebesgue measure on the unit circle. This measure is also called uniform measure, and denoted by λ which is defined by

$$\lambda(\text{arc}) := \frac{\text{length of the arc}}{2\pi}$$

Example 2.8. Let ϕ be a holomorphic function, and f is a potential. Then, we have

$$\Delta(f \circ \phi) = \phi^*(\Delta f), \text{ where } \phi^* \text{ is the pull-back of } \phi,$$

therefore we have

$$f \circ \phi(\zeta) = \int \log|z - \zeta| d\phi^*(\Delta f)(Z).$$

This shows that the composition of a potential function with holomorphic function is still a potential function.

2.4 Upper Semicontinuous Regularization

Let X be a topological space, and let $u : X \longrightarrow [-\infty, \infty)$ be a function which is locally bounded above on X. Its *upper semicontinuous regularization* $u^* : X \longrightarrow [-\infty, \infty)$ is defined by

$$u^*(x) := \limsup_{y \to x} u(y) = \inf_N \Big(\sup_{y \in N} u(y) \Big) \ \ (x \in X),$$

and the infimum is taken over all neighborhoods N of x.

Theorem 4. ***(Brelot-Cartan Theorem)***
Let $\mathcal{V}$ be a collection of subharmonic function on an open subset U of $\mathbb{C}$, and suppose that the function $u := \sup_{v \in \mathcal{V}} u$ is locally bounded above on U. Then,

$$(a) \ u^* \text{ is subharmonic on } U;$$

$$(b) \ u^* = u \text{ nearly everywhere on } U.$$

Proof. [Ran95, Theorem 3.4.2] ∎

Example 2.9. The simplest application of **Brelot-Cartan Theorem** is that if g and $\tilde{g}$ are subharmonic, then the function

$$\max\{g, \tilde{g}\}$$

is also subharmonic.

2.5 Capacity and Fekete-Sezgö Theorem

The *logarithmic capacity* of a subset E of $\mathbb{C}$ is given by

$$c(E) := \sup_{\mu} e^{I(\mu)},$$

where the supremum is taken over all Borel probability measures μ on $\mathbb{C}$ whose support is a compact subset of E. In particular, if K is a compact set with equilibrium measure v, then

$$c(K) = e^{I(v)}.$$

Example 2.10. We know that $e^{-\infty} = 0$, so that $c(E) = 0$ when E is a polar set.

Example 2.11. The capacity of a disc with radius r is r.

Example 2.12. The capacity of an ellipse with semi-axes a, b is $\frac{a+b}{2}$.

Theorem 5. *(a) If $E_1 \subset E_2$, then $c(E_1) \leq c(E_2)$;*
(b) If $E \subset \mathbb{C}$, then $c(E) = \sup\{c(K) : K \subset E,\ K$ compact$\}$;
(c) If $E \subset \mathbb{C}$, then $c(\alpha E + \beta) = |\alpha| c(E)$ for all $\alpha, \beta \in \mathbb{C}$.

Proof. [Ran95, Theorem 5.1.2] ■

It is well known that the computation and even only the estimation of capacity are really hard, and there are tons of theorems and methods created for these purposes. Therefore, we did not introduce the concepts related to computation of capacity. In our numerical experiments, we mainly use the calculation results of capacity of circle and ellipse, and the part (a) of the above theorem. One could read [Ran95, Section 5.2 and 5.3] for computation and estimation of capacity.

Theorem 6. ***(Fekete-Szegö Theorem)***
Let $E \subset \mathbb{C}$ be compact and stable under complex conjugation.
If $c(E) < 1$, then there are only finitely many irreducible monic polynomials $p(X) \in \mathbb{Z}[i][X]$ having all their roots in E.
If $c(E) \geq 1$ and if Ω is any open neighborhood of E, then there exists infinitely many irreducible monic polynomials $p(X) \in \mathbb{Z}[i][X]$ having all their roots in Ω.

Proof. [FS55, Theorem A] ■

3 Density of the Spectrum

Before going to the formal proof, let us describe the basic idea. There are six symmetric in the height function of projective plane intersecting with $x+y+z=0$, and the first step of our proof is to mod out those six symmetric. This involves several changes of coordinates within $\mathbb{P}^1$. Then, we want a coordinate such that the height function for the algebraic numbers has the form

$$h(\alpha)=\frac{1}{d}\sum_{\alpha'}\sum_{v,\text{ finite}}\log^+|\alpha|_v+\frac{1}{d}\sum_{\alpha'}g_\infty(\alpha),\text{ where } g_\infty \text{ is some potential function.}$$

We know that $\log^+|\cdot|_v=0$ for any algebraic integers and v finite places, and thus if we only talk about algebraic integers, we can ignore the first part of the height function. Therefore, the height function only has the archimedean part and that part is closely related to potential theory such that we are able to apply **Fekete-Szegö Theorem** on it.

3.1 Basic Setups and Notations

Define $g_v:\mathbb{C}_v\longrightarrow\mathbb{C}_v$ by

$$g_v(\zeta):=\log\Big(\max\Big\{\ \|\zeta+\rho^2\|_v,\ \|\zeta+\rho\|_v,\ \|\zeta+1\|_v\ \Big\}\Big)-\frac{1}{3}\log\|\zeta^3-1\|_v \tag{3.1}$$

where ρ is the primitive cubit root of unity.

Remark 3.1. Note that without $\frac{1}{3}\log\|\zeta^3-1\|_v$, (3.1) is a height function of projective plane composed with the map

$$\iota:\mathbb{P}^1\longrightarrow\mathbb{P}^2\text{ by }\zeta\mapsto[\rho\zeta+1:\zeta+\rho:\rho^2\zeta+\rho^2],$$

and since $\|\rho\|_v=1$, the first term in the max function can be multiplied by $\|\rho\|_v$ and the third term can be divided by $\|\rho^2\|_v$ without changing the function. Finally, by **Product Formula**, $\frac{1}{3}\log\|\zeta^3-1\|_v$ will be zero if we summing over the places.

Hence, g_v can define a valid height function if we summing over the places and over the conjugates.

Now define the transition map:

$$\alpha:\mathbb{P}^1\longrightarrow\mathbb{P}^1\text{ by }\zeta\mapsto\zeta^3 \tag{3.2}$$

Then, if we define $g_v^\dagger:\mathcal{C}_v\longrightarrow\mathcal{C}_v$ by

$$g_v^\dagger(\alpha):=3\log\Big(\max_{\zeta^3=\alpha}\Big\{\|\zeta+1\|_v\Big\}\Big)-\log\|\alpha-1\|_v,$$

we will have

$$g_v=\frac{1}{3}(g_v^\dagger\circ\alpha). \tag{3.3}$$

Now, define another transition map:

$$\mathcal{Z}:\mathbb{P}^1\longrightarrow\mathbb{P}^1\text{ by }\alpha\mapsto\frac{-27}{\alpha+\alpha^{-1}-25}. \tag{3.4}$$

Then, if we define

$$\hat{g}_v:\mathbb{C}_v\longrightarrow\mathbb{C}_v$$

by

$$\hat{g}_v(\mathcal{Z}):=6\log\Big(\max_{\zeta^3=\alpha}\Big\{\|\zeta+1\|_v\Big\}\Big)-\log\|\alpha-1\|_v^2,$$

we will have

$$g_v^\dagger=\frac{1}{2}(\hat{g}_v\circ\mathcal{Z}). \tag{3.5}$$

Finally, define $\tilde{g}_v : \mathbb{C}_v \longrightarrow \mathbb{C}_v$ by

$$\tilde{g}_v(\mathcal{Z}) := \hat{g}_v(\mathcal{Z}) + \log \left\| \mathcal{Z} - \frac{27}{23} \right\|_v. \tag{3.6}$$

This is the final formula we analyze in $\mathcal{Z}$ coordinate, the reason why this is the one we finally need is embodied in the following lemmas and propositions.

With a little bit computation, we can rewrite $\tilde{g}_v$ as

$$\tilde{g}_v(\mathcal{Z}) = 6\log\left(\max_{\zeta^3=\alpha}\left\{\left|\zeta+1\right|_v\right\}\right) - \log\left|\alpha-1\right|_v^2 + \log\left|27-23\mathcal{Z}\right|_v - \log\left|23\right|_v \tag{3.7}$$

The problem with (3.7) is that it is hard to translation ζ into $\mathcal{Z}$, as if we tried to solve ζ for $\mathcal{Z}$, we would be solving a quadratic equation with ζ^6 and ζ^3, which will give different values depending on the $\pm$ you chose in the quadratic formula. Also, there would be a problem with choices of different complex root in the course of solving the formula. (You could see those problems in **Section** 3.5.) Therefore, we want to know the formula of $\tilde{g}_v$ in ζ-coordinate.

Define $\check{g}_v : \mathbb{C}_v \longrightarrow \mathbb{C}_v$ by

$$\check{g}_v(\zeta) := \tilde{g}_v(\mathcal{Z}(\zeta)) \tag{3.8}$$

Then by recalling (3.2) and (3.4) we have

$$\check{g}_v(\zeta) = 6\log\left(\max\left\{|\zeta+1|_v, |\zeta+\rho|_v, |\zeta+\rho^2|_v\right\}\right) - \log\left|\zeta^6 - 25\zeta^3 + 1\right|_v + \log\left|\frac{27}{23}\right|_v \tag{3.9}$$

The whole research is closely related to (3.7) and (3.9), but focuses more on $\tilde{g}_\infty$ and $\check{g}_\infty$.

3.2 Main Theorem and Propositions

With these setups and notations, it is now appropriate to state the main theorem and propositions of our paper.

Theorem A. *Denote by C_0 the value at which*

$$c\left(\{\tilde{g}_\infty = C_0\}\right) = 1.$$

Then, after C_0, the spectrum of the value of heights in projective plane is dense in $\mathbb{R}$.

Proposition B. *Given a function $h : (-\infty, 0] \longrightarrow \mathbb{R}_{>0}$, define a function ν on $(-\infty, 0]$ by*

$$\nu := h \cdot Leb\left([a,b]\right), \text{ where } Leb\left([a,b]\right) \text{ denotes the Lebesgue measure on } [a,b].$$

Then, we have the following results:

1. $\nu = \int_a^b h(x)dx$, *where* $b \leq 0$.
2. *For any given* $\epsilon > 0$, *we have*

$$h \cdot Leb\left([a,b]\right) \sim h(\zeta)|b-a|, \ \forall \zeta \in [a,b], \text{ whenever } |b-a| < \epsilon;$$

3. *Denote by λ the uniform measure on S^1, and by ξ an arbitrary arc on S^1. Recall that*

$$\lambda(\xi) = \frac{|\xi|}{2\pi}.$$

Then, Define the Möbius transformation $\Phi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ by

$$\Phi(\zeta) := \frac{\zeta+\rho}{\zeta+\rho^2}, \text{ where } \rho \text{ is the primitive cubic root of unity.}$$

Now, Let $\epsilon > 0$, then for all $a, b \in (-\infty, 0]$ with $|b-a| < \epsilon$ and for all $\zeta \in [a,b]$, we have:

$$h(\zeta)|b-a| \sim (hLeb)\left([a,b]\right) = \lambda\left[\Phi\left([a,b]\right)\right] = \frac{\left|(\Phi)'(\zeta)\right| \cdot |b-a|}{2\pi};$$

4. *Finally, h has explicit formulas in each coordinate.*

Proposition C. $\tilde{g}(\mathcal{Z}) \sim \log|\mathcal{Z}|$ *near infinity.*

Proposition D. $\check{g}_\infty$ *is a potential function in* $\mathbb{C} \setminus \{0\}$. *Further, the generalized laplacian of* $\check{g}_\infty$ *has an explicit formula in the whole* $\mathbb{C}$. *In particular,*

$$\Delta \tilde{g}_\infty = \hat{h} \cdot Leb|_{[0,1]},$$

where $\hat{h}$ *is the h defined in* ***Proposition B*** *but in* $\mathcal{Z}$ *coordinate.*

Proposition E. *For* $\mathcal{Z} \in [0,1]$, $\tilde{g}_\infty$ *achieves local maximum in* $[0,1]$ *at* $\mathcal{Z}_{\max} = \dfrac{1}{3\sqrt{3}-2}$ *with the maximum value* $\tilde{g}_\infty(\mathcal{Z}_{\max}) := C_{\max} \approx 0.64631336460006861463197 8759$.

Proposition F. *For all* $C \in \mathbb{R}$ *such that* $C > C_{\max}$, *the level curve* $\tilde{g}_\infty^{-1}(C)$ *is analytic.*

3.3 Lemmas

Lemma 1. *Define the Möbius transformation* $\Phi : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ *by*

$$\Phi(\zeta) := \frac{\zeta + \rho}{\zeta + \rho^2}, \text{ where } \rho \text{ is the primitive cubic root of unity.}$$

Then, we have

$$\Phi\Big((-\infty, 0]\Big) = \Big\{z \in \mathcal{S}^1 : Arg(z) \in [-\frac{2\pi}{3}, 0)\Big\}.$$

Proof. Let $x \in (-\infty, 0]$. Then, observe that $|x+\rho| = |x+\rho^2|$. Thus, we have $\Phi(x) \in \mathcal{S}^1$. On the other hand, a little bit computation yields us

$$\Phi(x) = \frac{(x-\frac{1}{2})^2 - \frac{3}{4}}{(x-\frac{1}{2})^2 + \frac{3}{4}} + i\frac{\sqrt{3}(x-\frac{1}{2})}{(x-\frac{1}{2})^2+\frac{3}{4}}$$

Observe that $u(x) := \mathfrak{Re}\Big(\Phi(x)\Big)$ satisfies

$$u(0) = -\frac{1}{2}, \text{ and } \lim_{x \to -\infty} u(x) = 1.$$

Further, $u(x)$ is clearly continuous and by identifying $u(x_1) = u(x_2)$ one could check that $u(x)$ is also injective on $(-\infty, 0]$. Therefore, it must be monotone. Therefore, we have

$$u(x) \in \Big[-\frac{1}{2}, 1\Big).$$

The argument of $\Phi(x)$ at which $u(x) = -\frac{1}{2}$ can be $-\frac{2\pi}{3}$, and $\frac{2\pi}{3}$, but we can get rid of $\frac{2\pi}{3}$ since $v(0) = -\frac{\sqrt{3}}{2}$. It then follows immediately that $Arg\Big(\Phi(x)\Big) \in \Big[-\frac{2\pi}{3}, 0\Big)$.

∎

Lemma 2. *Define* $S := \Big\{\zeta \in \mathbb{C} \setminus \{0\} : Arg(\zeta) \notin [-\frac{\pi}{3}, \frac{\pi}{3}]\Big\}$. *Then there exists a rational map R of degree 6 such that for all* $\zeta \in S$, *we have*

$$\check{g}_\infty(\zeta) = 6 \log^+ \Big|\Phi(\zeta)\Big| - \log\Big|R(\zeta)\Big| + C \text{ , where } C \text{ is a constant.}$$

Proof. The result follows immediately by noting that for all $\zeta \in S$, we have

$$\check{g}_\infty(\zeta) = 6\log\Big(\max\{|\zeta+\rho|, |\zeta+\rho^2|\}\Big) - \log|\zeta^6 - 25\zeta^3 + 1| + \log(27/23)$$

$$\implies \check{g}(\zeta) = 6\log^+\left|\Phi(\zeta)\right| - \log|\zeta^6 - 25\zeta^3 + 1| + 6\log|\zeta+\rho^2| + \log(27/23)$$

$$\implies \check{g}(\zeta) = 6\log^+\left|\Phi(\zeta)\right| - \log\left|R(\zeta)\right| + C$$

where $R(\zeta) = \dfrac{\zeta^6 - 25\zeta^3 + 1}{(\zeta+\rho^2)^6}$ and $C = \log(27/23)$.

■

Lemma 3. *For $x \in [-1, 0]$, the function*

$$\check{g}_\infty(x) = 6\log\Big(\max\Big\{|x+1|, \sqrt{x^2 - x + 1}\Big\}\Big) - \log\left|x^6 - 25x^3 + 1\right| + \log\Big(\frac{27}{23}\Big)$$

achieves local maximum at

$$x_{\max} = \frac{1}{2}\Big((1 - 3\sqrt{3}) + \sqrt{24 - 6\sqrt{3})}\Big)$$

which is the larger negative root of

$$x^4 - 2x^3 - 24x^2 - 2x + 1.$$

Proof. Firstly note that for all $x \in [-1, 0]$, we have $\sqrt{x^2 - x + 1} > |x + 1|$. Then, computing the derivative and setting it to be zero yields us

$$(x-1)(x+1)(x^4 - 2x^3 - 24x^2 - 2x + 1) = 0.$$

As $x \in [-1, 0]$, we can get rid of $x = 1$ and all other three roots of $x^4 - 2x^3 - 24x^2 - 2x + 1$.
The result then follows immediately by directly comparing

$$\check{g}_\infty(-1) = \check{g}_\infty(0) \approx 0.16$$

$$\text{and } \check{g}_\infty(x_{\max}) := \mathcal{C}_{\max} \approx 0.64631336460006861463197 8759.$$

■

Lemma 4. *Keep the notation, then for finite places v and any algebraic integers $\alpha \in \mathbb{Q}$, we have*

$$\check{g}_v(\alpha) = \frac{1}{d}\sum_{\alpha'}\sum_{v}\log^+|\alpha'|_v,$$

where d = the degree of minimal polynomial of α, and the first sum is summing over all the conjugates of α.

3.4 Proof of Theorem A

Proof. Let C be any real number such that $C \geq C_0$ and set $E := \check{g}_\infty^{-1}(C)$. Then by the first part of **Theorem 5**, we know that $c(E) \geq 1$.

Now, it follows from **Proposition F** that E is closed, and from **Proposition C** that E is bounded. Therefore, E is a compact subset of $\mathbb{C}$.

Further, one could check $\check{g}_\infty(\zeta) = \overline{\check{g}_\infty(\zeta)}$, as $\rho^{-1} = \overline{\rho^2}$. Also, $\mathcal{Z}(\zeta) = \overline{\mathcal{Z}(\zeta)}$. Therefore, E is stable under complex conjugation.

Thus, by **Fekete-Szegö Theorem (Theorem 6)**, we know that for every open neighborhood $\mathcal{U} \supset E$, there exists infinitely many irreducible monic polynomials $P(X) \in \mathbb{Z}[i][X]$ that have all the roots in $\mathcal{U}$.

Let $\mathcal{O}$ be any open set containing E such that

$$\mathcal{O} \subset \check{g}_\infty^{-1}\Big((C - \epsilon, C + \epsilon)\Big), \text{ where } \epsilon > 0 \text{ is arbitrarily fixed.}$$

Let $\alpha \in \mathcal{O}$ be an algebraic integer, then by **Lemma 4**, we know that the height function

$$\tilde{h}(\alpha) = \frac{1}{d} \sum_{\alpha'} \tilde{g}_\infty(\alpha') \in (C - \epsilon, C + \epsilon).$$

As $\epsilon > 0$ is arbitrarily small, we can conclude that for any real number C such that $C \geq C_0$, it can be approximated by a sequence of values of height function in projective plane.

■

3.5 Proof of Proposition B

Proof. The first part of the proposition follows immediately from the fact that

$$(h \cdot \mu)(A) = \int_A h(z) d\mu(z),$$

for any Borel measure μ and set A.

The second part follows immediately from the first part. The third part follows from **Lemma 1**.

The fourth part requires a little bit long computation of the push-forward of ν.

Firstly, by the third part, we immediately have

$$h(\zeta) = \frac{\left|\Phi'(\zeta)\right|}{2\pi} = \frac{\sqrt{3}}{2\pi} \cdot \frac{1}{\zeta^2 - \zeta + 1}. \tag{3.10}$$

Now, define

$$\nu^\dagger := h^\dagger \cdot Leb\Big((-\infty, 0]\Big) = \alpha_* \nu \text{ where } \alpha_* \text{ is the push-forward of } \alpha.$$

Let $\epsilon > 0$ be arbitrarily fixed, then for all $a \in (-\infty, 0]$, we have

$$\nu^\dagger\Big([a, a+\epsilon]\Big) := \alpha_* \nu\Big([a, a+\epsilon]\Big) = \nu\Big(\alpha^{-1}([a, a+\epsilon])\Big) = \nu\Big([a^{1/3}, (a+\epsilon)^{1/3}]\Big)$$

$$\sim h(a^{1/3}) \cdot \left|(a+\epsilon)^{1/3} - a^{1/3}\right| = h(a^{1/3}) \cdot \left|(\alpha^{-1})'(a)\right| \cdot |a + \epsilon - a| = h(a^{1/3}) \cdot \frac{1}{3} a^{-2/3} \cdot \epsilon.$$

On the other hand, we have

$$\nu^\dagger\Big([a, a+\epsilon]\Big) \sim h^\dagger(a)|(a+\epsilon) - a| = h^\dagger(a) \cdot \epsilon.$$

Therefore, for all $a \in (-\infty, 0]$, we have

$$h^\dagger(a) = h(a^{1/3}) \cdot \frac{1}{3} a^{-2/3} = \frac{\sqrt{3}}{6} \cdot \frac{1}{a(a^{1/3} + a^{-1/3} - 1)}. \tag{3.11}$$

Denote by $\mathcal{Z}_*$ the push-forward of $\mathcal{Z}$. Then we define

$$\hat{\nu} := \hat{h} Leb\Big([0, 1]\Big) = \mathcal{Z}_* \nu^\dagger \tag{3.12}$$

Let $\epsilon > 0$ and for all $a \in [0, 1]$, set

$$I := [a, a+\epsilon],\ \mathcal{Z}^{-1}(I) = J_1 \cup J_2, \text{ and } \mathcal{Z}^{-1}(a) = \{a_1, a_2\}$$

where $J_i \subset (-\infty, 0]$ and $a_i \in (-\infty, 0]$ for all $i \in \{1, 2\}$.

Then, we have

$$\hat{\nu}(I) = \mathcal{Z}_* \nu^\dagger(I) = \nu^\dagger\Big(\mathcal{Z}^{-1}(I)\Big) = \nu^\dagger(J_1) + \nu^\dagger(J_2) \sim h^\dagger(a_1)|J_1| + h^\dagger(a_2)|J_2|$$

Further, for all $i \in \{1, 2\}$, we have

$$\epsilon = |I| = \left|\mathcal{Z}'(a_i)\right| \cdot |J_i|$$

which gives us

$$|J_i| = \frac{\epsilon}{|\mathcal{Z}'(a_i)|}$$

On the other hand, we have

$$\hat{\nu}(I) \sim \hat{h}(a) \cdot \epsilon$$

Therefore, for all $a \in [0,1]$,we have

$$\hat{h}(a) = \frac{h^{\dagger}(a_1)}{|\mathcal{Z}'(a_1)|} + \frac{h^{\dagger}(a_2)}{|\mathcal{Z}'(a_2)|}, \text{ where } a_1, a_2 \in \mathcal{Z}^{-1}(a). \tag{3.13}$$

Now, solving the inverse map of (3.4) involves solving a quadratic equation. The solution provides us the following information:

1. $a_1 = a_2^{-1}$;
2. $a_1 + a_1^{-1} - 25 = \dfrac{-27}{a}$ and thus $a_1^2 - 25a_1 + 1 = \dfrac{-27}{a} \cdot a_1$ for $a \in [0,1]$;
3. $a_1 - a_2 = \dfrac{\pm\sqrt{\Delta}}{a}$;
4. $\mathcal{Z}^{-1}(a) = \dfrac{(25a-27) \pm \sqrt{\Delta}}{2a}$, where $\Delta = (27-25a)^2 - 4a^2 = 27(27-23a)(1-a)$.

Define the inversion map $I : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ by $\zeta \mapsto \zeta^{-1}$, then it is easy to check that

$$g_\infty \circ I(\zeta) = g_\infty(\zeta).$$

Thus, $\Delta g_\infty = \Delta(g_\infty \circ I)$ yielding us $I_*\nu = \nu$.
Thus, for all $a \in (-\infty, 0]$ and $\epsilon > 0$ arbitrarily fixed, we have

$$h(a) \cdot \epsilon \sim \nu\Big([a, a+\epsilon]\Big) = I_*\nu\Big([a, a+\epsilon]\Big) = \nu\Big([\frac{1}{a}, \frac{1}{a+\epsilon}]\Big) \sim h\Big(\frac{1}{a}\Big) \cdot \frac{\epsilon}{a^2}.$$

Therefore, we have

$$h(a) = h\Big(\frac{1}{a}\Big) \cdot \frac{1}{a^2}.$$

Now, note that $\alpha \circ I = I \circ \alpha$, and thus we have

$$I_*\nu^{\dagger} = I_*(\alpha_*\nu) = (\alpha \circ I)_*(\nu) = (I \circ \alpha)_*(\nu) = \alpha_*(I_*\nu) = \alpha_*\nu = \nu^{\dagger}.$$

Thus, we go back to the same case for $I_*\nu = \nu$, and thus we have

$$h^{\dagger}(a) = h^{\dagger}\Big(\frac{1}{a}\Big) \cdot \frac{1}{a^2}.$$

Therefore,

$$h^{\dagger}(a_2) = h^{\dagger}\Big(\frac{1}{a_1}\Big) = a_1^2 h^{\dagger}(a_1) \tag{3.14}$$

Note that by the information of a_1 and a_2, we know that

$$I(a_1) = a_2 \text{ and thus } I(J_1) = J_2$$

Also, observe that

$$\mathcal{Z} \circ I = \mathcal{Z}$$

and thus we have

$$\mathcal{Z}' = (\mathcal{Z} \circ I)' = (\mathcal{Z}' \circ I) \cdot I'$$

which gives us

$$Z'(a_2) = \mathcal{Z}' \circ I(a_2) \cdot I'(a_2) = \mathcal{Z}(a_1) \cdot I'(a_2 = \mathcal{Z}'(a_1) \cdot -\frac{1}{a_2^2} = -a_1^2 \cdot \mathcal{Z}'(a_1)$$

Therefore, we have

$$\mathcal{Z}'(a_2) = -a_1^2 \cdot \mathcal{Z}'(a_1) \tag{3.15}$$

Then, plugging (3.14) and (3.15) into (3.13) yields us the desired result.
Therefore, we have

$$\hat{h}(a) = 2\frac{h^{\dagger}(a_1)}{|\mathcal{Z}'(a_1)|} \tag{3.16}$$

and thus it suffices for us now to simplify only one thing.

Consider the results above, and we will have

$$\begin{aligned}
\hat{h}(a) = 2\frac{h^{\dagger}(a_1)}{|\mathcal{Z}'(a_1)|} &= \frac{\sqrt{3}}{3\pi \cdot 27} \cdot \frac{(a_1^2 - 25a_1 + 1)^2}{a_1\left|(a_1+1)(a_1-1)\right|(a_1^{1/3} + a_1^{-1/3} - 1)} \\
&= \frac{9\sqrt{3}}{\pi} \cdot \frac{a_1}{a^2} \cdot \frac{1}{|a_1^2 - 1| \cdot (a_1^{1/3} + a_1^{-1/3} - 1)} \\
&= -\frac{9\sqrt{3}}{\pi} \cdot \frac{1}{a^2} \cdot \frac{1}{\left|a_1 - a_1^{-1}\right| \cdot (a_1^{1/3} + a_1^{-1/3} - 1)} \\
&= -\frac{9\sqrt{3}}{\pi} \cdot \frac{1}{a^2} \cdot \frac{1}{\left|a_1 - a_2\right|(a_1^{1/3} + a_1^{-1/3} - 1)} \\
&= -\frac{9\sqrt{3}}{\pi} \cdot \frac{1}{a \cdot \sqrt{\Delta}} \cdot \frac{1}{a_1^{1/3} + a_1^{-1/3} - 1} \\
&= -\frac{3}{\pi} \cdot \frac{1}{a} \cdot \frac{1}{(1-a)^{1/2} \cdot (27 - 23a)^{1/2}} \cdot \frac{1}{a_1^{1/3} + a_1^{-1/3} - 1}
\end{aligned} \tag{3.17}$$

Set $\xi = a_1^{1/3}$, then by **point 2** from **the information of** a_1 **and** a_2, we have

$$25 - \frac{27}{a} = a_1 + a_1^{-1} = \xi^3 + \xi^{-3} = (\xi + \xi^{-1})(\xi^2 - 1 + \xi^{-2})$$

Define $\mathcal{X} := (\xi + \xi^{-1}) = a_1^{1/3} + a_1^{-1/3}$, then $(\xi^2 - 1 + \xi^{-2}) = \mathcal{X}^2 - 3$
Thus, we have:

$$25 - \frac{27}{a} = \mathcal{X}(\mathcal{X}^2 - 3) \text{ where } \mathcal{X} = (\xi + \xi^{-1}),\ \xi = a_1^{1/3} \tag{3.18}$$

Now, set $\mathcal{Y} := \mathcal{X} - 1$ and then by **point 2** from **the information of** a_1 **and** a_2, we have

$$25 - \frac{27}{a} = (\mathcal{Y}+1)(\mathcal{Y}^2 + 2\mathcal{Y} + 1 - 3) = \mathcal{Y}^3 + 3\mathcal{Y}^2 - 2$$

Thus, we have

$$\mathcal{Y}^3 + 3\mathcal{Y}^2 = 27(1 - \frac{1}{a}), \text{ where } \mathcal{Y} \in (-\infty, -3] \tag{3.19}$$

Set $\mathcal{Y} = -\frac{3\hat{\mathcal{Y}}}{a^{1/3}}$, then (3.19) gives us

$$1 - a = \hat{\mathcal{Y}}^3 - a^{1/3}\hat{\mathcal{Y}}^2 \tag{3.20}$$

Finally, taking (3.18), (3.19), (3.20) into the consideration of (3.17), we have our final formula of $\hat{h}(a)$ where $a \in [0, 1]$:

$$\hat{h}(a) = \frac{1}{\pi} \cdot \frac{1}{a^{2/3}(1-a)^{1/2}} \cdot \frac{1}{(27 - 23a)^{1/2} \cdot \hat{\mathcal{Y}}} \tag{3.21}$$

where $\hat{\mathcal{Y}}(a)$ is the unique solution of $\hat{\mathcal{Y}}^3 - a^{1/3}\hat{\mathcal{Y}}^2 + (a-1) = 0$ in $[1, \infty)$

■

3.6 Proof of Proposition C

Proof. Consider the three rays that connects the origin to $-1, -\rho, -\rho^2$ respectively.

Denote by μ the measure of those three rays. Then, if we define

$$L : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \text{ by } \zeta \mapsto \rho\zeta$$

we have

$$\mu = \nu + L_* \nu + L_*^2 \nu.$$

Then by observing that

$$\frac{1}{3}\log\|\zeta^3 - 1\|_v = \frac{1}{3}\Big(\log\|\zeta - 1\|_v + \log\|\zeta - \rho\|_v + \log\|\zeta - \rho^2\|_v\Big)$$

and by the Fundamental Formula of Laplacian, we have

$$\Delta g_v = \mu - \frac{1}{3}\Big(\delta_1 + \delta_\rho + \delta_{\rho^2}\Big).$$

Now recall (3.3), and we have

$$\Delta g_v = \frac{1}{3}\alpha^*\Big(\Delta g_v^\dagger\Big) \text{ where } \alpha^* \text{ is the pull-back of } \alpha.$$

Then, we have

$$\begin{aligned} \alpha_*\Big(\Delta g_v\Big) &= \frac{1}{3} \cdot \alpha_* \alpha^* \Big(\Delta g_v^\dagger\Big) \\ &= \frac{1}{3} \cdot 3 \cdot Id\Big(\Delta g_v^\dagger\Big) = \Delta g_v^\dagger. \end{aligned}$$

Thus, we have

$$\begin{aligned} \Delta g_v^\dagger = \alpha_*\Big(\Delta g_v\Big) &= \alpha_*\Big(\nu + L_*\nu + L_*^2\nu - \frac{1}{3}(\delta_1 + \delta_\rho + \delta_{\rho^2})\Big) \\ &= 3\alpha_*\nu - \frac{1}{3}\cdot 3\delta_1 = 3\nu^\dagger - \delta_1. \end{aligned}$$

Therefore, we have

$$\Delta g_v^\dagger = 3\nu^\dagger - \delta_1. \tag{3.22}$$

Similarly, recalling (3.5) yields us

$$\Delta g_v^\dagger = \frac{1}{2}\mathcal{Z}^*\Big(\Delta \hat{g}_v\Big) \text{ where } \mathcal{Z}^* \text{ is the pull-back of } \mathcal{Z}$$

and thus

$$\mathcal{Z}_*\Delta g_v^\dagger = \frac{1}{2}\mathcal{Z}_*\mathcal{Z}^*\Big(\Delta \hat{g}_v\Big) = \frac{1}{2}\cdot 2 \cdot Id\Big(\Delta \hat{g}_v\Big) = \Delta \hat{g}_v.$$

Therefore, we have

$$\Delta \hat{g}_v = \mathcal{Z}_*\Big(\Delta g_v^\dagger\Big) = \mathcal{Z}_*\Big(3\nu^\dagger - \delta_1\Big) = 3\hat{\nu} - \delta_{\frac{27}{23}}.$$

Thus, we have

$$\Delta \hat{g}_v = 3\hat{\nu} - \delta_{\frac{27}{23}}. \tag{3.23}$$

It is immediate from (3.6) and (3.23) that

$$\Delta \tilde{g}_v = 3\hat{\nu} - \delta_{\frac{27}{23}} + \delta_{\frac{27}{23}} - \delta_\infty = 3\hat{\nu} - \delta_\infty, \tag{3.24}$$

which implies that

$$\Delta \tilde{g}_v = \text{probability measure} - \delta_\infty,$$

this happens if and only if

$$\tilde{g}_\infty(\mathcal{Z}) \sim \log\|\mathcal{Z}\| \text{ near infinity.}$$

∎

3.7 Proof of Proposition D

Proof. Firstly, it is clear that $\log|\zeta+1|$, $\log|\zeta+\rho|$ and $\log|\zeta+\rho^2|$ are subharmonic.

Therefore, by **Brelot-Carton Theorem**,

$$6\log\Big(\max\{|\zeta+1|,|\zeta+\rho|,|\zeta+\rho^2|\}\Big) \text{ is subharmonic.}$$

Also, $\zeta^6-25\zeta^3+1$ is a polynomial and thus $\log|\zeta^6-25\zeta^3+1|$ is a potential.

Now, set $h := 6\log^+\left|\Phi(\zeta)\right| - \log\left|R(\zeta)\right| + C$, where the right hand side is defined as in **Lemma 2**.

By **Lemma 2**, $\check{g}_\infty - h = 0$ on S, which implies

$$\Delta\check{g}_\infty|_S = \Delta h|_S.$$

Thus,

$$\Delta(\check{g}_\infty - h) = \Delta(0) = 0.$$

Therefore, $\check{g}_\infty$ is a potential on $\mathbb{C}\setminus\{0\}$.

By **Lemma 2**, we have

$$\Delta\check{g}_\infty\Big|_S = \Big(\Delta\log^+\left|\Phi(\zeta)\right|\Big)_S - \Big(\Delta\log\left|R(\zeta)\right|\Big)_S.$$

As $R(\zeta)$ is a rational function, the log of it can be rewritten as the difference of two polynomials. Then, each polynomial can be factored into a product all of the roots. Then, by the **Fundamental Solution of Laplacian**, we have

$$\log|R(\zeta)| = \int \log|z-\zeta|\,d\mu_0(z), \text{ where } \mu_0 = \sum_{\zeta \text{ zeros}} \delta_\zeta - \sum_{w \text{ poles}} \delta_w$$

On the other hand,

$$\Big(\Delta\log^+\left|\Phi(\zeta)\right|\Big)_S = \Phi^*\Big(\Delta\log^+|\cdot|\Big)_{S^1}, \text{ where } \Phi^* \text{ is the pull-back of } \Phi.$$

Therefore,

$$\log^+|\Phi(\zeta)| = \int \log|z-\zeta|\,d(\Phi^*\lambda)(z), \text{ where } \lambda \text{ is the uniform measure on } S^1.$$

Thus, $\Delta\check{g}_\infty\Big|_S$ is has an explicit formula.

Now, define $L : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ by $L(\zeta) := \zeta\rho$, which is an isomorphism of order 3. Then, we have

$$\mathbb{C}\setminus\{0\} = S \cup L(S) \cup L^{-1}(S)$$

.

Observe that

$$\check{g}_\infty \circ L = \check{g}_\infty$$

and thus we have

$$\Delta(\check{g}_\infty \circ L) = \Delta\check{g}_\infty$$

which implies

$$L^*(\Delta\check{g}_\infty) = \Delta\check{g}_\infty, \text{ where } L^* \text{ is the pull-back of } L.$$

Notice that as $L : L(S) \longrightarrow L^{-1}(S)$ is an isomorphism, the pull-back of any measure on $L^{-1}(S)$ via L is the measure on $L(S)$. Therefore, we have

$$L^*\Big(\Delta\check{g}_\infty\Big|_{L^{-1}(S)}\Big) = \Big(L^*\Delta\check{g}_\infty\Big)\Big|_{L(S)} = \Delta\check{g}_\infty\Big|_{L(S)}.$$

Similarly, as $L : L^{-1}(S) \longrightarrow S$ is an isomorphism, the pull-back of any measure on S via L is a measure on $L^{-1}(S)$. Thus,

$$L^*\left(\Delta\check{g}_\infty\big|_S\right) = \left(L^*\Delta\check{g}_\infty\right)\big|_{L^{-1}(S)} = \Delta\check{g}_\infty\big|_{L^{-1}(S)}$$

As $\Delta\check{g}_\infty\big|_S$ has an explicit formula, by the above two calculation, $\Delta\check{g}_\infty\big|_{L^{-1}(S)}$ and $\Delta\check{g}_\infty\big|_{L(S)}$ both have explicit formulas.

Finally, $\check{g}_\infty(0) = \log(27/23)$ and thus there is no direct mass, $\Delta\check{g}(0) = 0$.

For the last part of the proposition, by (3.12) and (3.24), we know that $\Delta\tilde{g}_\infty$ is supported on $[0,1]$. Then for any Borel set $A \subset [0,1]$, we have

$$\Delta\tilde{g}(A) = \int_A h(x)dx = \hat{h} \cdot Leb|_A.$$

Therefore, we have

$$\Delta\tilde{g} = \hat{h} \cdot Leb|_{[0,1]}.$$

■

3.8 Proof of Proposition E

Proof. Note that for all $\zeta \in [-1,0]$, we have $\mathcal{Z}(\zeta) \in [0,1]$. Thus, it suffices to analyze parametrized curve

$$\zeta \mapsto \left(\mathcal{Z}(\zeta), g^\dagger_\infty(\mathcal{Z}(\zeta))\right), \text{ for } \zeta \in [-1,0].$$

However, note that by definition we have

$$\check{g}_\infty(\zeta) = g^\dagger\left(\mathcal{Z}(\zeta)\right) \text{ and thus } \check{g}'_\infty(\zeta) = g^{\dagger'}_\infty\left(\mathcal{Z}(\zeta)\right) \cdot \mathcal{Z}'(\zeta).$$

Also, for all $\zeta \in [-1,0]$, we have

$$\mathcal{Z}'(\zeta) = \frac{81\zeta^2(\zeta^6 - 1)}{(\zeta^6 - 25\zeta^3 + 1)^2} \le 0.$$

Therefore, as long as the critical point ζ_0 is not -1 and 0, the vanishing of $\check{g}'_\infty$ must imply the vanishing of $g^{\dagger'}_\infty$.

Further, by **Lemma 3**, we know that $\forall x \in [-1,0] \setminus \{x_{\max}\}$, we have

$$\check{g}_\infty(x_{\max}) > \check{g}_\infty(x).$$

Set $\mathcal{Z}_{\max} = \mathcal{Z}(x_{\max})$. As $\mathcal{Z} : [-1,0] \longrightarrow [0,1]$ is a bijection, then $\forall \mathcal{Z} \in [0,1] \setminus \{\mathcal{Z}_{\max}\}$, we have

$$\check{g}_\infty(x_{\max}) = \tilde{g}_\infty(\mathcal{Z}_{\max}) > \tilde{g}_\infty(\mathcal{Z}) = \check{g}_\infty(x).$$

Hence, it suffices to analyze the behavior of $\check{g}_\infty(\zeta)$ for $\zeta \in [-1,0]$.

Therefore, Proposition E follows immediately from **Lemma 3** by noting that $\mathcal{Z}(x_{\max}) = \dfrac{1}{3\sqrt{3}-2}$.

■

3.9 Proof of Proposition F

Proof. By **Proposition E**, for all $x \in [0,1]$, we have

$$\tilde{g}(x) \le C_{\max}.$$

Therefore,

$$\tilde{g}^{-1}\left((-\infty, C_{\max}]\right) \ge [0,1].$$

Therefore, for all $C \in \mathbb{R}$ such that $C > C_{\max}$, we have

$$\tilde{g}^{-1}(C) \cap \tilde{g}^{-1}\Big((-\infty, C_{\max}]\Big) = \varnothing,$$

which implies

$$\tilde{g}^{-1}(C) \cap [0, 1] = \varnothing.$$

Now, by **Proposition B**, $\Delta\tilde{g}_\infty$ is supported on $[0, 1]$, and thus $\tilde{g}_\infty$ is harmonic and thus analytic on $\mathbb{C}\backslash[0, 1]$. The result then follows immediately from **Implicit Function Theorem**.

■

4 Values on the Spectrum

4.1 Conjecture of the Spectrum

Conjecture 4.1. *In **Theorem A**, we showed that the spectrum of height function in projective plane is dense after C_0. Here, we also conjecture that*

$$\mu_{zz}^{\text{ess}} = C_0.$$

Also, we conjecture that the set (1.2) is finite.

4.2 Theorem G and Discussions

Theorem G. *Under the the map $\mathcal{Z} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ defined by*

$$\mathcal{Z}(\zeta) := \frac{-27}{\zeta^3 + \zeta^{-3} - 25},$$

the six special points at which the projective height intersecting with $x + y + z = 0$ gets sharp upper bound are normalized to one point $\mathcal{Z}^\dagger = \frac{1}{2}$.

Proof. Recall that the six special points mentioned in the lemma has the form $(1 : \alpha - 1 : -\alpha)$ where α is a root of the equation $z^6 - 3z^5 + 7z^4 - 9z^3 + 7z^2 - 3z + 1$. We firstly need to identify those three coordinates with the coordinates we use in $\mathbb{P}^2$, which is of the form $[\rho\zeta + 1 : \zeta + \rho : \rho^2\zeta + \rho^2]$.

As height function is symmetric, we can identity in this way:

$$[\zeta + \rho : \rho^2\zeta + \rho^2 : \rho\zeta + 1] = [1, \alpha - 1, -\alpha]$$

and since the coordinates are homogenous, the above identity can be converted to

$$\left[1 : -\frac{\rho^2\zeta}{\zeta + \rho} : \frac{\rho\zeta + 1}{\zeta + \rho}\right] = [1, \alpha - 1, -\alpha].$$

Therefore, we have

$$\alpha = -\frac{\rho\zeta + 1}{\zeta + \rho}.$$

Since α is a root, we have

$$\alpha^6 - 3\alpha^5 + 7\alpha^4 - 9\alpha^3 + 7\alpha^2 - 3\alpha + 1 = 0$$

and thus

$$(\zeta + \rho)^6\left[(\alpha^6 + 1) - 3(\alpha^5 + \alpha) + 7(\alpha^4 + \alpha^2) - 9\alpha^3\right] = 0.$$

Set $y = \zeta + \zeta^{-1}$, then, through a long computation, we have

$$(3\zeta)^3\left[1 + \left(\frac{y-1}{3}\right)^2 + \left(\frac{y-1}{3}\right)^3\right] = 0,$$

which then gives us

$$\zeta^3 + \frac{1}{\zeta^3} + 29 = 0.$$

Plugging the above equation into $\mathcal{Z}(\zeta)$ yields us the desired result:

$$\mathcal{Z}^\dagger = \frac{1}{2}.$$

∎

Remark 4.2. In the similar fashion, one could also check that another six points at which h_{zz} vanished can be normalized to $\mathcal{Z} = 1$ and $\mathcal{Z} = 0$.

Therefore, under our normalization, those twelve points in Zagier's paper can be now normalized to only three points. This implies that, in our coordinate, the distributions of algebraic numbers and the corresponding values of height function are much simpler than Zagier's coordinate. Hence, it is plausible for us to find the essential minimum and all the values before essential minimum.

Remark 4.3. Our numerical experiment suggests that the level curve $\tilde{g}_\infty(C_{\max})$ has logarithmic capacity less than 1. Thus, by **Proposition F** , the level curve with logarithmic capacity 1 must be smooth, since we need to increase C in order to increase the capacity. Also, the level curve with capacity 1 contains the level curve $\tilde{g}_\infty(C_{\max})$, and thus contains all the three normalized points.

We strongly suspect that, if $C_0 = \mu_{zz}^{\text{ess}}$, then the level curve with capacity 1 would contain all the algebraic numbers that give you the values of height function before the essential minimum.

To prove or disprove the **Conjecture 4.1**, it is necessary to find an algebraic formula of the curve that contains the level curve $\tilde{g}_\infty(C_{\max})$, which could be the level curve with capacity 1. In this way, we could try to understand the spectrum by understanding the algebraic formula since there must be some number theory on it.

For now, we are trying to drive an algebraic equation form

$$\check{g}(\zeta)_\infty = C_0$$

but only considering one of the three components in the max function.

For example, we only consider

$$\frac{|\zeta+1|^6}{|\zeta^6 - 25\zeta^3 + 1|} = \exp[C_0 - \log(27/23)].$$

Then, we want to fit the right hand side into some algebraic equation by identifying the left hand side as a root of some algebraic equation.

The major problem here is that C_0 is directly related to the capacity, which is hard to estimate. The way we estimate C_0 is described in **Section A**, and one could find that it wholly depends on the luck.

For now, we find the right hand side could be a root of the polynomial

$$x^3 - 5x^2 - 3x + 4.$$

Thus, if we set

$$A = |\zeta+1|^6, \text{ and } B = |\zeta^6 - 25\zeta^3 + 1|,$$

we would have

$$\left(\frac{A}{B}\right)^3 - 5\left(\frac{A}{B}\right)^2 - 3\left(\frac{A}{B}\right) + 4 = 0,$$

which gives us

$$A(A^2 - 3B^2) = B(5A^2 - 4B^2).$$

To get rid of the square root, we need to square both side, which yields us

$$A^2(A^2 - 3B^2)^2 = B^2(5A^2 - 4B^2)^2.$$

However, this equation involves 36^{th} power, and there is no way to cancel the left hand side and right hand side in a beautiful way such that the final equation is clean or at least understandable.

Getting the algebraic formula is just the first step, and it is a pretty new and unknown field for us to know what will be going on in the several steps.

Hence, the conjecture is still an open and a totally new problem, and we will be working on this in the next few semesters. At the very least, **Theorem A** has provided a freshly new result of the spectrum of the height function in projective plane.

A Numerical Experiment

A.1 Level Curve in original coordinate

To plot the graph of $\check{g}_\infty(\zeta)$ with $\zeta \in \mathbb{C}$, we will directly use (3.9) but replacing $\zeta = x + iy$ in the software.

We will firstly plot some general graphs within different domains and then we will plot the level curves with specific values.

Figure A.1: 3D and Contour Plot of $\check{g}_\infty(\zeta)$ for $x, y \in [-1, 1]$

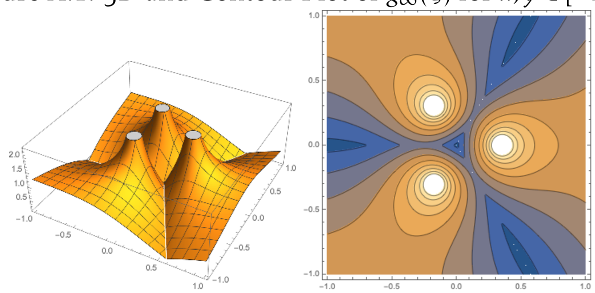


Figure A.2: 3D and Contour Plot of $\check{g}_\infty(\zeta)$ for $x, y \in [-5, 5]$

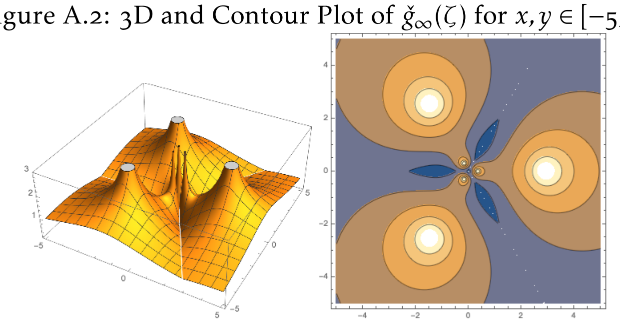


Figure A.3: 3D and Contour Plot of $\check{g}_\infty(\zeta)$ for $x, y \in [-15, 15]$

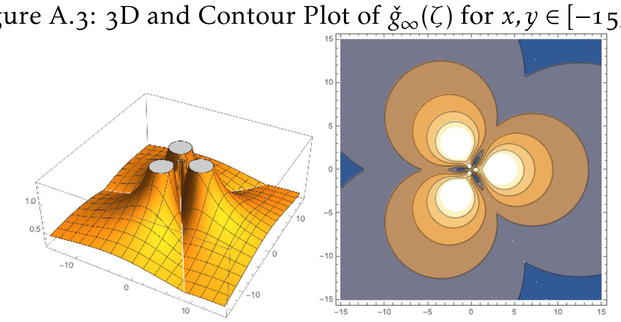


Figure A.4: 3D and Contour Plot of $\check{g}_\infty(\zeta)$ for $x, y \in [-200, 200]$

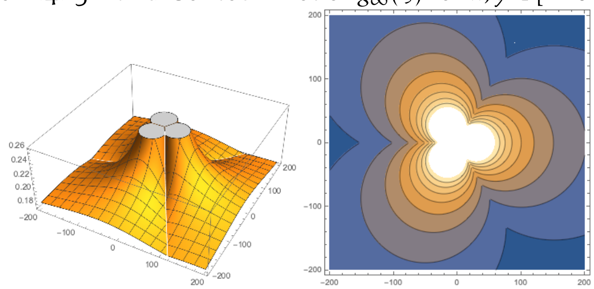


Figure A.5: Left $\check{g}_\infty(\zeta) = 1/2$ for $x, y \in [-3, 3]$, Right $\check{g}_\infty(\zeta) = 1$ for $x, y \in [-10, 10]$

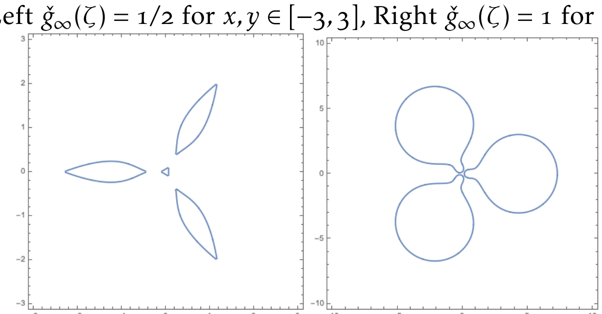
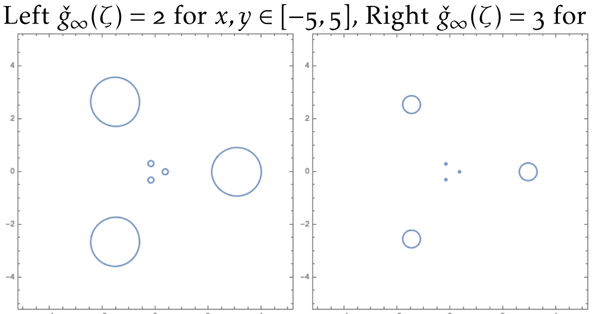


Figure A.6: Left $\check{g}_\infty(\zeta) = 2$ for $x, y \in [-5, 5]$, Right $\check{g}_\infty(\zeta) = 3$ for $x, y \in [-5, 5]$



A.2 Level Curve in Final Coordinate

As we tried many times that if we used the formula (3.7), the plot will not really be nice behaved, especially around the origin. This is because the choice of different complex root and the choice of the ± in terms of solving the quadratic formula. However, since the formula of $\check{g}_\infty(\zeta)$ is really nice, we can then directly let Mathematica to compute the curve of the composition map

$$(\check{g}_\infty \circ \mathcal{Z}^{-1})(\zeta).$$

Figure A7 below indicates that Mathematica does not compute the wrong graph, since in the graph, $\check{g}_\infty(\zeta)$ tends to be circle when ζ is large, which is consistent with **Proposition C**.

Figures A8, A9, A10 give us a closer look of the level curve. The behavior is consistent with what we guessed. The curve becomes not smooth anymore around $[0,1]$. **Figure A11** gives us the look of the level curve with respect to the local maximum value. **Figure A11, A12, A13** provides our insight about the change of the shape when we increase the value. In fact, it changes to a circle at a really fast speed.

Figure A.7: 3D and Contour Plot of $\tilde{g}_\infty(\mathcal{Z})$ for $x, y \in [-500, 500]$

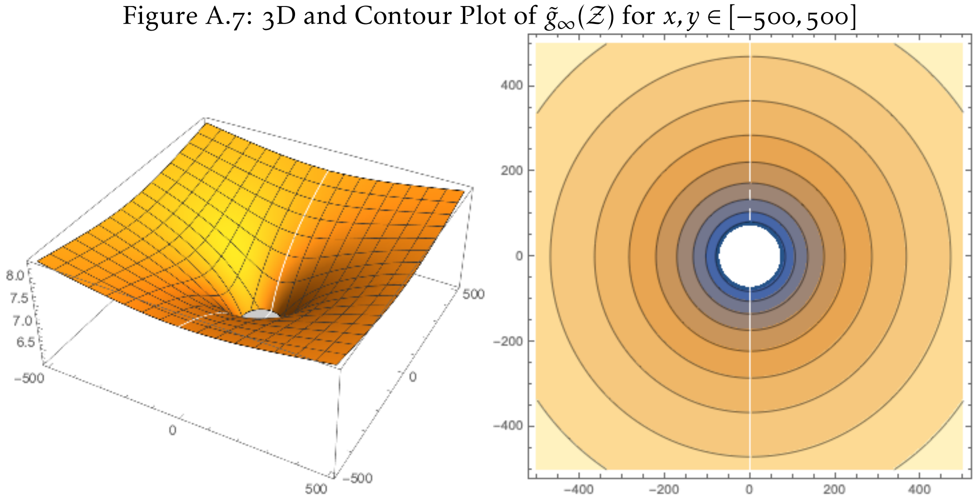


Figure A.8: 3D and Contour Plot of $\tilde{g}_\infty(\mathcal{Z})$ for $x, y \in [-2, 2]$

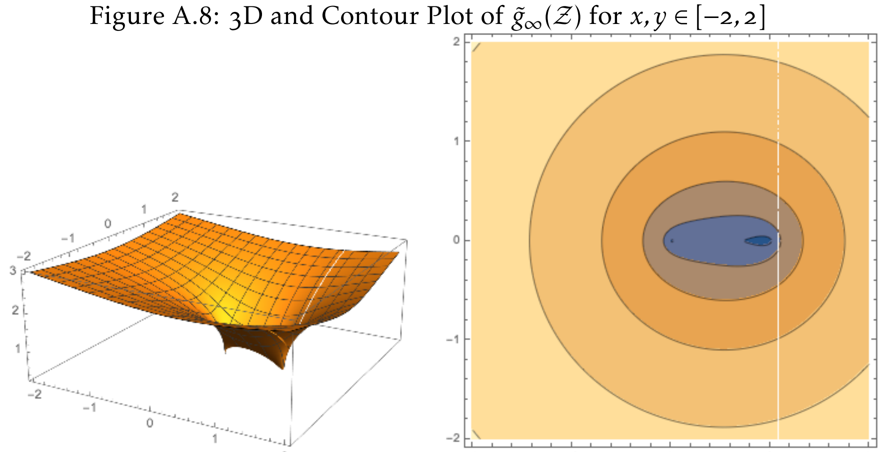


Figure A.9: 3D and Contour Plot of $\tilde{g}_\infty(\mathcal{Z})$ for $x, y \in [-1, 1]$

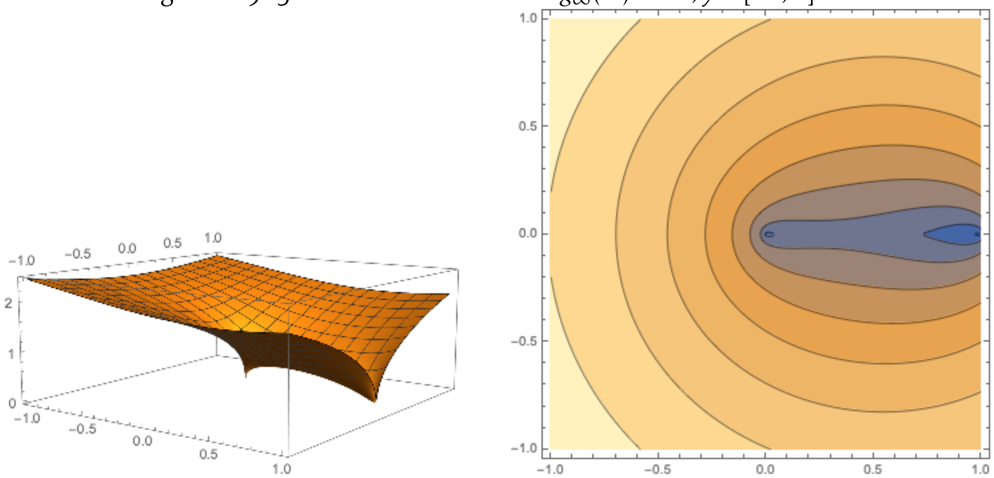


Figure A.10: 3D and Contour Plot of $\tilde{g}_\infty(\mathcal{Z})$ for $\{x, -1, 3\}, \{y, -2, 2\}$

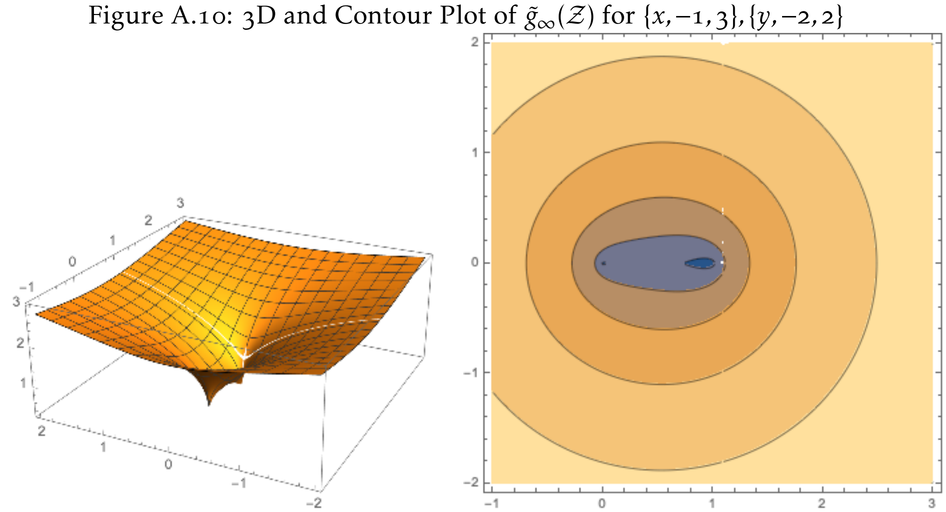


Figure A.11: $\tilde{g}_\infty(\mathcal{Z}) = \tilde{g}_\infty(x_0)$ for $\{x, -1/2, 3/2\}, \{y, -1/2, 3/2\}$

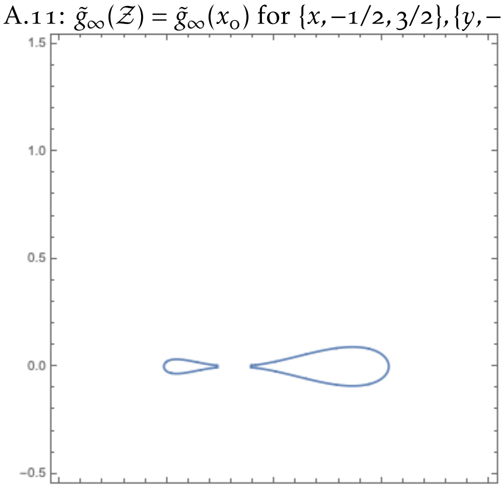


Figure A.12: $\tilde{g}_\infty(\mathcal{Z}) = 1$ for $x, y \in [-1/2, 3/2]$

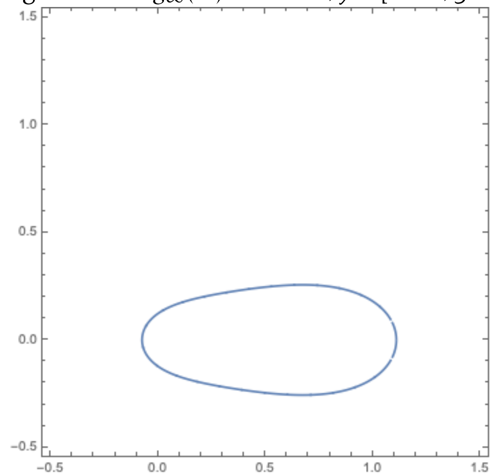


Figure A.13: $\tilde{g}_\infty(\mathcal{Z}) = 1.5$ for $x, y \in [-1.5, 1.5]$

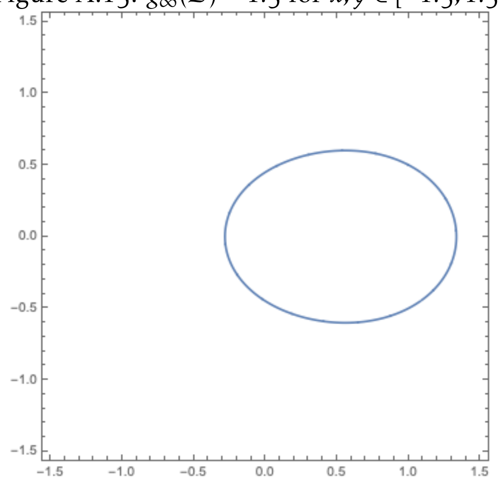
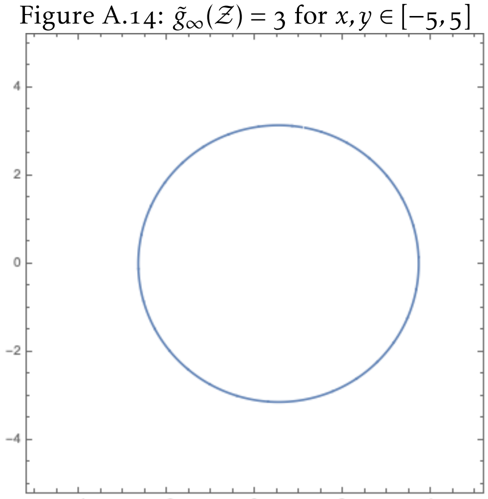


Figure A.14: $\tilde{g}_\infty(\mathcal{Z}) = 3$ for $x, y \in [-5, 5]$



A.3 Computation of Capacity

We aim to compute find a C such that the capacity of the set of level curve $\{\tilde{g}_\infty \leq C\}$, $\text{cap}\Big(\{\tilde{g}_\infty \leq C\}\Big) = 1$.

The hardness of computing capacity is well-known. Some ways to get information of capacity is to estimate it by some numerical method, which is also hard. However, the capacity some certain shapes is known. For instance, the capacity of a circle is its radius, and the capacity of an ellipse is $(a+b)/2$. Since the shape of our level curve is clearly shown in the Figures above, we could interact those level curves with circle and ellipse to estimate C such that the level curve is almost like a circle or ellipse.

Firstly, **Figures A15-A18** show the experiments we did. We could see that for circle with radius less than 1, even we can adjust the center, the level curve cannot be even close to a circle. However, the level curve with respect to the value 3 gives us an insight about trying ellipse, because it is pretty like an ellipse and it pretty much tends to be an "flat" circle. Thus, we get the **Figure A17**. We are now trying to estimate this value as specific as possible for the purpose described in **Section 4**. The value provided in **Figure A18** is the C_0 that gives us the cubic polynomial.

Figure A.15: **Left:**$\check{g}_\infty(x) = \check{g}_\infty(x_0)$ with B(center,radius)=B(1/2,0.53). **Right:**$\check{g}_\infty(x) = 1$ with B(1/2, 0.615).

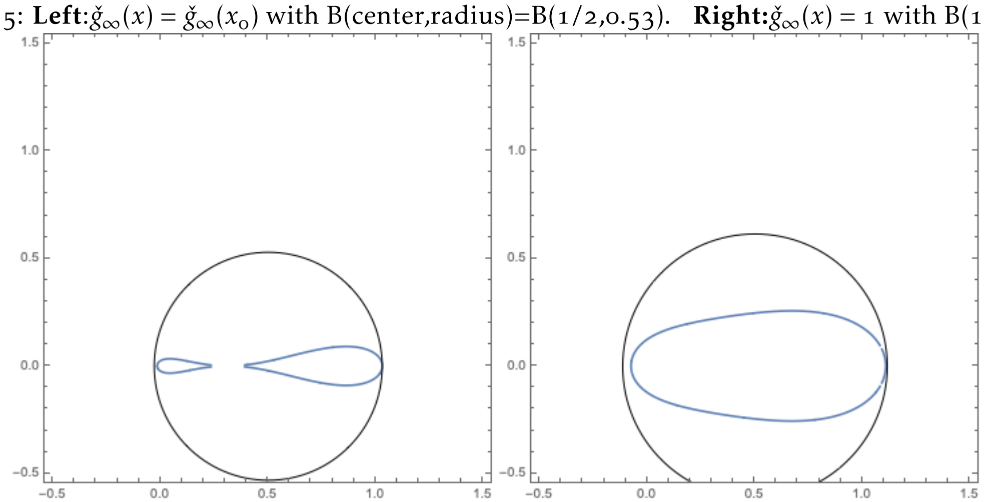


Figure A.16: **Left:**$\check{g}_\infty(x) = 1.7$ with B(1/2,1). **Right:**$\check{g}_\infty(x) = 3$ with B(1/2,3.21).

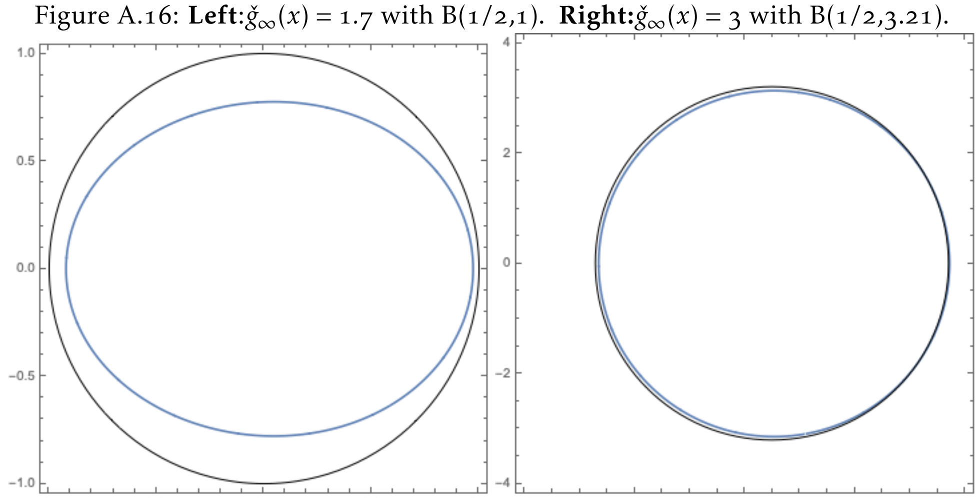


Figure A.17: $\check{g}_\infty(x) = 5$ with Ball(1/2, 23.2).

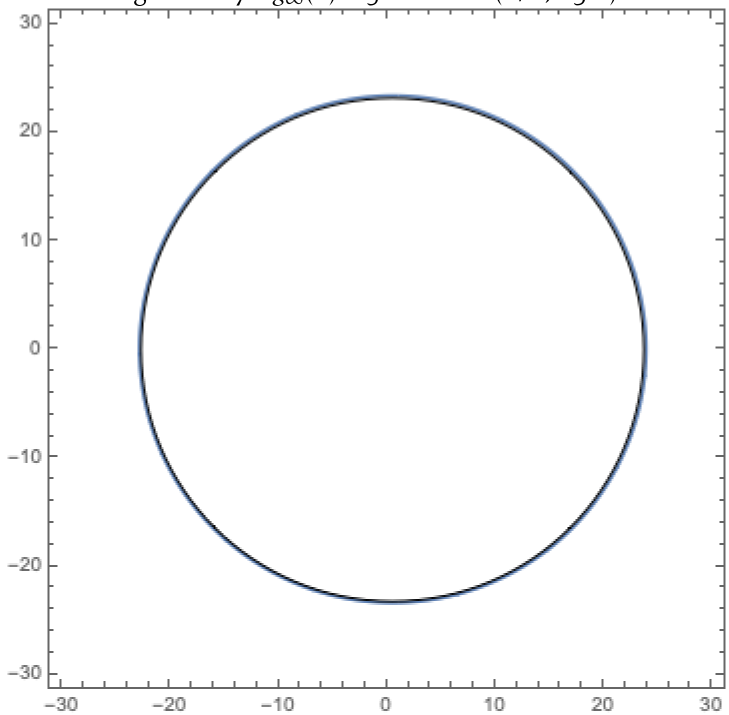
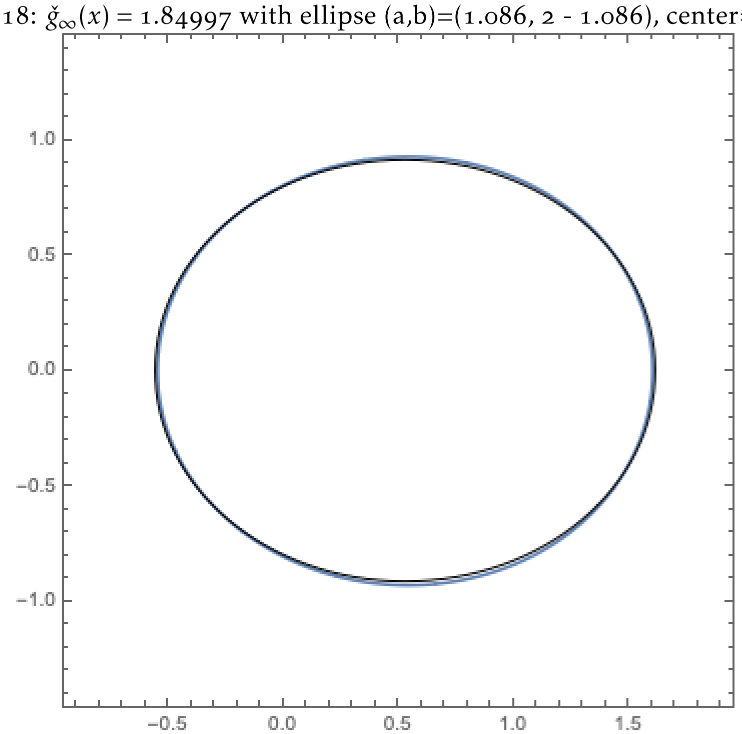


Figure A.18: $\check{g}_\infty(x) = 1.84997$ with ellipse (a,b)=(1.086, 2 - 1.086), center=(0.525,0)



B Proof of Zagier's Paper

B.1 Theorem 1 of Zagier

Theorem 7. ***Zagier's Theorem*** *For all algebraic number $\alpha \neq 0, 1, \frac{(1 \pm \sqrt{-3})}{2}$, we have,*

$$H(\alpha) + H(1-\alpha) \geq \frac{1}{2} \log \frac{1+\sqrt{5}}{2},$$

with equality if and only if α or $1-\alpha$ is a primitive 10^{th} root of unity.

Lemma 5. *Let w and $\overline{w}$ denote the complex roots of $x^2 - x + 1 = 0$. There is a universal constant $A \geq 1$ such that for every complex number $\alpha \notin \{0, 1, w, \overline{w}\}$, we have*

$$\log|\alpha^2 - \alpha + 1|_v + n_v \leq A(|\log|\alpha|_v| + |\log|1-\alpha|_v|), \tag{B.1}$$

for all places v in an algebraic number field K, where $n_v = \begin{cases} 1 & \text{if } v \text{ is real} \\ 2 & \text{if } v \text{ is complex} \\ 0 & \text{if } v \text{ is non-Archimedean.} \end{cases}$

Proof. For v being non-Archimedean, by Theorem 2.4, we have $|\alpha|_v \leq 1$ for all $\alpha \in \mathbb{C}$. Therefore, $|\alpha^2 - \alpha + 1|_v \leq 1$ and hence the left-hand side of (3.1.1) ≤ 0. But the right-hand side of (3.1.1) is always ≥ 0. Therefore, the inequality holds for the non-Archimedean case.

If v is Archimedean, then we need to consider the function,

$$f(z) = \frac{\log|z^2 - z + 1| + 1}{|\log|z|| + |\log|1-z||},$$

for all $z \in \mathbb{C}$ $\{0, 1, w, \overline{w}\}$.

Notice that as $|z| \to \infty$, $f(z) \to 1$. For z which is near the points w or $\overline{w}$, $f(z)$ is large and negative. Finally the function is continuous everywhere except at z satisfying $|z| = |1-z| = 1$, where again $f(x)$ is large and negative. It follows that the function is bounded above uniformly on all of $\mathbb{C}$. ∎

Corollary B.1. *For all algebraic number $\alpha \neq 0, 1, \frac{(1 \pm \sqrt{-3})}{2}$, we have*

$$H(\alpha) + H(1-\alpha) \geq \frac{1}{2A}.$$

Proof. By **Product Formula** and observation, we have

$$\sum_v \log|\beta|_v = 0, \quad \sum_v n_v = [K : \mathbb{Q}], \quad \sum_v |\log|\beta|_v| = 2H_K(\beta). \tag{B.2}$$

Now, summing (**B.1**) over all places v with considering (**B.2**), we can conclude that

$$H(\alpha) + H(1-\alpha) \geq \frac{1}{2A}.$$

∎

Lemma 6. *For $z \in \mathbb{C}$, we have*

$$\max(0, \log|z|) + \max(0, \log|1-z|) \geq \frac{\sqrt{5}-1}{2\sqrt{5}} \log|z^2 - z| + \frac{1}{2\sqrt{5}} \log|z^2 - z + 1| + \frac{1}{2} \log \frac{1+\sqrt{5}}{2}, \tag{B.3}$$

and the equality holds if and only if z or $1-z$ equals $e^{\frac{\pm \pi i}{5}}$ or $e^{\frac{\pm 3\pi i}{5}}$.

Proof. Denote $\max(0, \log|z|) = \log^+|z|$ for brevity. Define a function f by

$$f(z) = \frac{\sqrt{5}-1}{2\sqrt{5}} \log|z^2 - z| + \frac{1}{2\sqrt{5}} \log|z^2 - z + 1| + \frac{1}{2} \log \frac{1+\sqrt{5}}{2} - \log^+|z| - \log^+|1-z|.$$

Notice that if $|z|$ is large, then $f(z) \sim \frac{\sqrt{5}-1}{2\sqrt{5}} \log|z^2| + \frac{1}{2\sqrt{5}} \log|z^2| - \log|z| - \log|z| = -\log|z|$. In particular, $f(z) \to -\infty$ as $|z| \to \infty$. Similarly, if z is close to one of the points $0, 1, w, \overline{w}$, then $f(z)$ is large and negative. Away from these points, $f(z)$ is continuous and therefore can attain its maximum on some finite point or points. Further, off the circles $|z| = 1$ and $|1-z| = 1$, the function is the real part of a holomorphic function, and thus by the maximal Maximum Modulus Principle for harmonic function, the maxima must be attained on these circles. Finally, observe that $z \mapsto 1-z$ and $z \mapsto \overline{z}$ preserve f, and thus it suffices to only consider the case of $z = e^{i\theta}$ where $0 \leq \theta \leq \pi$. Thus, (3.4.2) can be simplified a little to the following since $|z| = 1$ always holds in this proof.

It now remains only to consider the size between $|1-z|$ and 1.

Firstly, consider the case where $0 \leq \theta \leq \frac{\pi}{3}$, so that $|1-z| \leq 1$. Replace z by $z = e^{i\theta}$ in (3.4.3) and use Euler's formula to expand $e^{i\theta}$. Simplify and we can get

$$f(z) = \frac{\sqrt{5}-1}{2\sqrt{5}} \log(2\sin\frac{\theta}{2}) + \frac{1}{2\sqrt{5}} \log(2\cos\theta - 1) + \frac{1}{2} \log \frac{1+\sqrt{5}}{2}$$

.

Wirte $S = 4\sin^2 \frac{\theta}{2}$ so that $0 \leq S \leq 1$, then we get

$$f(z) = \frac{\sqrt{5}-1}{4\sqrt{5}} \log(S) + \frac{1}{2\sqrt{5}} \log(1-S) + \frac{1}{2} \log \frac{1+\sqrt{5}}{2}.$$

Differentiating the above equation with respect to S, and we can find that the maximum of f for $S \in (0,1)$ is achieved at $S = \frac{3-\sqrt{5}}{2}$, where $f = 0$ and $\theta = \frac{\pi}{5}$.

Now, consider the case where $\frac{\pi}{3} < \theta \leq \pi$ so that $|1-z| > 1$. We use the same S, so that $1 < S \leq 4$ and

$$f(z) = \frac{-\sqrt{5}-1}{4\sqrt{5}} \log(S) + \frac{1}{2\sqrt{5}} \log(S-1) + \frac{1}{2} \log \frac{1+\sqrt{5}}{2}.$$

Differentiating above equation with respect to S, and we can find that the maximum of f for $S \in (1,4)$ is achieved at $S = \frac{3+\sqrt{5}}{2}$, where $f = 0$ and $\theta = \frac{3\pi}{5}$.

Therefore, $f \leq 0$ and achieves equality if and only if z or $1-z$ equals $e^{\frac{\pi i}{5}}$ or $e^{\frac{3\pi i}{5}}$. This conclusion is equivalent to (**B.3**).

■

Proof of Theorem 7 (Zagier's Theorem):

Proof. **Lemma 6** immediately gives us

$$\max(0, \log|z|_v) + \max(0, \log|1-z|_v) \geq \frac{\sqrt{5}-1}{2\sqrt{5}} \log|z^2 - z|_v + \frac{1}{2\sqrt{5}} \log|z^2 - z + 1|_v + \frac{1}{2} \log \frac{1+\sqrt{5}}{2},$$

since for v to be Archimedean, $|\alpha|_v = |\alpha|^{n_v}$, and for v to be non-Archimedean, we can apply the same argument as in the proof of **Lemma 6** by only considering the cases $|\alpha|_v \leq 1$. Now summing over all places v, using (**B.3**), we can conclude the theorem immediately.

■

B.2 Projective Version of Zagier's Theorem

Observe that we have the following inequality:

$$\max(|x|,1)+\max(|y|,1)\geq\max(|x|,|y|,1)\geq\frac{\max(|x|,1)+\max(|y|,1)}{2}.$$

To show this inequality, one can assume without the loss of generality that $|x|\geq 1$, then separate the cases $|x|\leq|y|$ and $|x|\geq|y|$, and to show the second inequality, one may need to additionally separate the cases $|y|\leq 1$ and $|y|\geq 1$.

The combination of this observation with **Theorem 7** implies that

$$H(x)+H(y)\geq H_{\mathbb{P}^2}(x:y:1)\geq\tfrac{1}{2}(H(x)+H(y))$$

and hence the minimum of $H_{\mathbb{P}^2}$ restricted to the curve $x+y+z=0$ is some value between $C/2$ and C, where C is the optimal constant such that

$$H(\alpha)+H(1-\alpha)\geq C>0$$

Theorem 8. ***(Projective Version of Zagier's Theorem)*** *Let $C\subset\mathbb{P}^2$ be the curve $x+y+z=0$, and let $\theta=1.46556\cdots$ be the real root of $\theta^3-\theta^2-1=0$. Then, we have:*

$$H_{\mathbb{P}^2}(P)\geq\frac{1}{2}\log\theta=0.1911255\cdots$$

for all $P\in C(\overline{\mathbb{Q}})$ except for the five points:

$$P=(1:-1:0),\quad(1:0:-1),\quad(0:1:-1),\quad(1:w:w^2),\quad(1:w^2:w),$$

where w = nontrivial cube root of unity for which $H_{\mathbb{P}^2}(P)$ vanishes.

The equality holds if and only if $P=(1:\alpha-1:-\alpha)$, where α is a root of the equation $\alpha^6-3\alpha^5+7\alpha^4-9\alpha^3+7\alpha^2-3\alpha+1=0$

Lemma 7. *For $(x:y:z)\in\mathbb{C}$, we have*

$$\log\max(|x|,|y|,|z|)\geq\frac{1}{2}\log\theta+\frac{1}{6\theta-4}\log|xy+yz+zx|+\frac{\theta-1}{3\theta-2}\log|xyz|,\tag{B.4}$$

with $x+y+z=0$, where $\theta=1.46557\cdots$ is the real root of $\theta^3-\theta^2-1=0$.

Proof. Since the calculation is symmetric in $x+y+z=0$, we can always assume $|z|=|x+y|\geq\max\{|x|,|y|\}$ by interchanging x, y, z if necessary. Apply this change to (**B.4**) and we will get the inequality

$$\log|x+y|\geq\frac{1}{2}\log\theta+\frac{1}{6\theta-4}\log|(x+y)^2-xy|+\frac{\theta-1}{3\theta-2}\log|xy(x+y)|.$$

Define function F to be the right-hand side of the above equation subtracting the left-hand side.

$$F=\frac{1}{2}\log\theta+\frac{1}{6\theta-4}\log|(x+y)^2-xy|+\frac{\theta-1}{3\theta-2}\log|xy|+\frac{1-2\theta}{3\theta-2}\log|x+y|$$

For now, let's assume $|x+y|\geq|x|$ and $|x+y|\geq|y|$. By similar argument in the proof of Lemma 3.4, we know that F is a hormonic function, and thus the maximal value only happens on the boundary. Therefore, we can assume $|x+y|=|x|$ or $|x+y|=|y|$ with $|x+y|,|x|,|y|\neq 0$.

Without loss of generality, assume $|x+y|=|x|$, so that $\frac{|x+y|}{|x|}=|1+\frac{y}{x}|=1$. Let $\zeta=1+\frac{y}{x}$ with noting that $|\zeta|=1$. Replace ζ into F and get

$$F=\frac{1}{2}\log\theta+\frac{1}{6\theta-4}\log|\zeta^2-\zeta+1|+\frac{\theta-1}{3\theta-2}\log|\zeta-1|+\frac{1-2\theta}{3\theta-2}\log|\zeta|.$$

Since $|\zeta|= 1$ and $\zeta \in \mathbb{C}$, we can parametrize $\zeta = e^{i\alpha}$. Note that since $|x+y|= |x|$ and $|x+y|\geq |y|$, we have $|x+y|= |x|\geq |y|= |x||\zeta - 1|$, and hence we have $|\zeta - 1|\leq 1$. Therefore, $\alpha \in [0, \frac{\pi}{3}]$ or $[0, -\frac{\pi}{3}]$. By symmetric, we assume that $\alpha \in [0, \frac{\pi}{3}]$.

Replace $\zeta = e^{i\alpha}$ and get

$$F = \frac{1}{2}\log\theta + \frac{1}{6\theta - 4}\log|e^{i2\alpha} - e^{i\alpha} + 1|+\frac{\theta - 1}{3\theta - 2}\log|e^{i\alpha} - 1|.$$

Use Euler's formula, we have $|e^{i2\alpha} - e^{i\alpha} + 1|^2 = (2\cos\alpha - 1)^2$ and $|e^{i\alpha} - 1|^2 = 1 - (2\cos\alpha - 1)$. Let $S = 2\cos\alpha - 1$ where $\alpha \in [0, \frac{\pi}{3}]$. Let's first consider the case where $\alpha \in (0, \frac{\pi}{3})$, so that $1 < 2\cos\alpha < 2$ and thus $0 < S < 1$. Then, $|e^{i2\alpha} - e^{i\alpha} + 1|= S$ and $|e^{i\alpha} - 1|= (1 - S)^{\frac{1}{2}}$. Replace S into the above equation, we get

$$F = \frac{1}{2}\log\theta + \frac{1}{6\theta - 4}\log(S) + \frac{\theta - 1}{6\theta - 4}\log(1 - S), \text{ for } S \in (0, 1).$$

Differentiating above equation with respect to S, and we get that F achieves maximum at $S = \dfrac{1}{\theta}$, with $F_{\max} = F(\dfrac{1}{\theta}) = 0$.

Lastly, for the cases $\alpha = 0$ and $\alpha = \dfrac{\pi}{3}$, $F \to -\infty$, and thus the maximum cannot occur.

Hence, $F \leq 0$ as desired.

■

Proof of Theorem 8 (Projective Version of Zagier's Theorem):

Proof. Firstly, we have argued in the proof of **Lemma 7** that the maximum for F cannot happen for $\alpha = \dfrac{\pi}{3}, 0, -\dfrac{\pi}{3}$. For $\alpha = \dfrac{\pi}{3}$, we have the corresponding coordinates to be $(1 : w : w^2)$, where w is a primitive cube root of unity. For $\alpha = -\dfrac{\pi}{3}$, the corresponding coordinates are $(1 : w^2 : w)$. For $\alpha = 0$, we have $(1 : 0 : -1)$. We also argued that by symmetry, we can interchange x, y, z, so that we also have another two points at which the maximal cannot occur, $(1 : -1 : 0)$, $(0 : 1 : -1)$.

Similarly as in the proof of **Theorem 7**, **Lemma 7** immediately gives us

$$\log\max(|x|_v, |y|_v, |z|_v) \geq \frac{n_v}{2}\log\theta + \frac{1}{6\theta - 4}\log|xy + yz + zx|_v + \frac{\theta - 1}{3\theta - 2}\log|xyz|_v$$

where n_v is the same as before.

Now, summing all over the places v, using (**B.2**) and dividing by $\dfrac{1}{[K : \mathbb{Q}]}$ will yield the desired result.

■

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