ON SPECTRAL PROPERTIES OF SIERPINSKI GASKET

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Abstract. In this paper, we shall explore the p-adic convergence of number of closed prime walks on growing fractal graphs. We will briefly review 3-adic convergence of closed prime walks on the Sierpinski gasket proven by Munch [1] and generalise the result to n-flakes. Furthermore, we also explore the spectral properties of the Sierpinski gasket. In particular, we present an explicit recursive construction of the adjacency eigenvectors of the Sierpinski gasket.

1. Introduction

Fractals are a fascinating subject in mathematics. Fractal and recursive structures serve as a source of research topics in various fields of mathematics. Self-similar strctures such as Cantor sets, Koch snowflakes, and Merger sponge serve as counter examples in topology and analysis. Furthermore, space-filling curves such as Peano curve and Hilbert curve are a special class of fractal curves which not only have interesting properites, but also has practical applications in geolocation systems. In this paper, we shall study a well-known fractal structure called the Sierpinski gasket in context of graph theory.

In 2008, Munch [1] has proven that the number of a special class of walks called closed prime walks on the Sierpinski gasket converges 3-adically. In the first section of this paper, we shall review Munch's proof and generalise the result to a broader class of fractal graphs called n-flakes. In the second section, we shall explore the eigenvalues and eigenvectors of the adjacency matrix of the Sierpinski gasket. We present empricial findings that the multiplicities of the eigenvalues either follows the recursion $x_{n+1} = 3x_n + 3$ or $x_{n+1} = 3x_n + 1$. Furthermore, we provide an explicit recursive construction of the eigenvectors and prove the lower of the mulitplicities which follows the recursion formula.

2. Background

Let us first review the basic definitions on simple graph theory and closed prime walks. An undirected **graph** is a set of vertices and edges (V, E) where $E \subset \{\{u, v\} : u, v \in V\}$. A **walk** is a sequence of vertices $v_0, v_1, \ldots, v_n$ such that $\{v_i, v_{i+1}\} \in E$ for all i. We say that a walk $v_0, v_1 \ldots, v_n$ is **closed** if it starts and ends at the same vertex. i.e. $v_0 = v_n$. Given two walks $P = v_0, v_1, \ldots, v_n$ and $Q = w_0, w_1, \ldots, w_m$, one can define the concatenation of P and Q as $PQ = v_0, v_1, \ldots, v_{n-1}, w_0, w_1, w_2, \ldots, w_m$, assuming that $v_n = w_0$. For closed walk P, we define P^k as the concatenation of k copies of P. A **path** is a walk with no repeated vertices.

A graph is **connected** if there is a walk between any two vertices. In this paper, we will only consider finite, connected, undirected graphs. We will focus on a special class of walks called **closed prime walks** on these graphs. Before we define closed prime walks, we need to define a few more terms.

Definition 2.1. Let $G = (V, E)$ be a graph. A walk $v_0, v_1, \dots, v_n$ has a **backtrack** if there exists consecutive vertices $v_i \neq v_{i+1} \neq v_{i+2}$ such that $v_i = v_{i+2}$.

Definition 2.2. Let $G = (V, E)$ be a graph. A walk $v_0, v_1, \dots, v_n$ has a **tail** if $(v_0, v_1) = (v_n, v_{n-1})$.

For instance, the following are examples of walks with backtracks and tails.

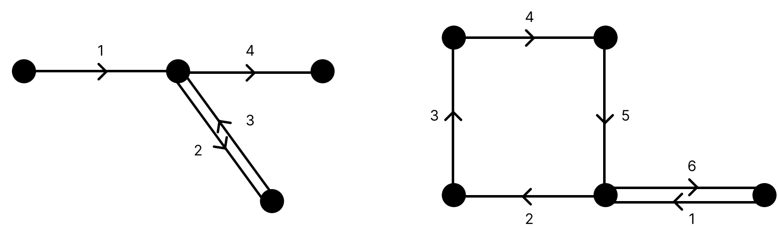


FIGURE 1. Example of walks with backtracks and tails.

We say that a walk is **prime** if it has no backtracks or tails. One can easily define an equivalence relation on the set of closed paths by saying that two closed paths are equivalent if they are cyclic permutations of each other.

Definition 2.3. Let $G = (V, E)$ be a graph. Let $P = (v_0, v_1, \dots, v_n)$ be a closed path on G. Then, the equivalence class of P, denoted by $[P]$ is the set of all cyclic permutations of P. That is to say

$$[P] = \{(v_0, v_1, \dots, v_{n-1}, v_n), (v_1, v_2, \dots, v_n, v_0), \dots (v_n, v_0, \dots, v_{n-2}, v_{n-1})\}$$

A keen reader might have noticed that counting the number of equivalence classes of closed prime cycles under this definition allows for infinite number. For instance, one can define a closed path C on a K_3 which goes around the triangle once. Our definition allows for $[C], [C^2], [C^3], \dots$ to be distinct equivalence classes. Hence, this observation motivates us to redefine the equivalence relation on the set of prime closed walks.

Definition 2.4. Two prime closed walks P and P' are equivalent if

(1) P and P' are cyclic permutations of each other.
(2) there exists a prime closed walk Q such that P and P' are powers of Q.

An example can be worth a thousand words. Let us consider the prime closed walks on K_3.

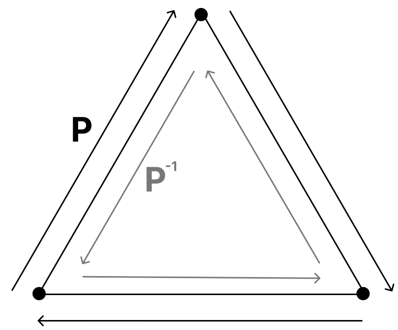


FIGURE 2. Example of prime closed walks on K_3

Given an arbitary starting point on K_3, the walk is forced to go around the triangle in either clockwise or anticlockwise direction. Let us denote the clockwise walk as P and anticlockwise walk as P^{-1}. Combining P and P^{-1} will yield a backtrack and any powers of P and P^{-1} will be equivalent to P and P^{-1}. Hence, there are only two equivalence classes of prime closed walks on K_3.

In fact, there exists a concrete formula for computing the number of prime closed walks on a graph as a generating function. This formulation is called the Ihara zeta function. The formula was first given by Bass [3].

Theorem 2.5. *Let $G = (V, E)$ be a graph. Let A be the adjacency matrix of G.* ***Ihara zeta function*** *of G is defined as the following formal power series in variable u.*

$$\zeta_G(u) = \prod_{[P]} \frac{1}{1 - u^{L([P])}}$$

where $[P]$ is the set of all equivalence classes of prime closed walks on G and $L([P])$ is the length of the shortest closed walk in $[P]$. Then, the coefficients of $\zeta_G(u)$ are given by Bass's formula

$$\zeta_G(u) = \frac{1}{(1-u^2)^{|E|-|V|}\det(I - uA + Qu^2)}$$

where I is the identity matrix, Q is the diagonal matrix with $Q_{ii} = \deg(v_i) - 1$.

Let's use Bass's formula to compute the number of classes of prime closed walks on K_3. If the formula is correct, we should get exactly two equivalence classes of length 3. The matrix A and Q are given by

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The term $\det(I - uA + Qu^2)$ expands to

$$\det(I - uA + Qu^2) = \det \begin{pmatrix} 1+u^2 & -u & -u \\ -u & 1+u^2 & -u \\ -u & -u & 1+u^2 \end{pmatrix}$$
$$= u^6 - 2u^3 + 1$$

Since there are exactly 3 vertices and 3 edges, $(1-u^2)^{|E|-|V|} = (1-u^2)^0 = 1$. Thus, the Bass's formula yields the following result.

$$\begin{aligned}\zeta_G(u) &= \frac{1}{(1-u^2)^{|E|-|V|}\det(I - uA + Qu^2)} \\ &= \frac{1}{u^6 - 2u^3 + 1} \\ &= \frac{1}{(u^3-1)^2} \\ &= \frac{1}{(1-u^3)^2} \\ &= \frac{1}{1-u^{L([P])}} \cdot \frac{1}{1-u^{L([P^{-1}])}}\end{aligned}$$

Therefore, the result from Bass's formula agrees with the number of prime closed walks on K_3 we computed earlier.

While we will be taking a more direct approach to counting closed prime walks, it is worth noting that works such as Terras [2] takes a more complex analytic approach. In particular, due to the reciprocal nature of the Ihara zeta function, it's natural to focus on poles of the Ihara zeta functions on various graphs.

Previously, Munch [1] has proven that the coefficients of Ihara zeta function of the Sierpinski gasket converges 3-adically. In this paper, we shall generalise this result to other fractal graphs and explore the p-adic convergence of closed prime walks on these graphs. Before diving into the details, let us first review the basic definitions of p-adic numbers and review Munch's proof on the Sierpinski gasket.

3. p-adic convergence on the Sierpinski gasket

A **Sierpinski gasket** is a fractal graph which is recursively constructed.

(1) Start with a triangle S_0.
(2) Make three copies of S_i and glue the corner vertices to form S_{i+1}.

The following is an example of first three iterations of Sierpinski gasket.

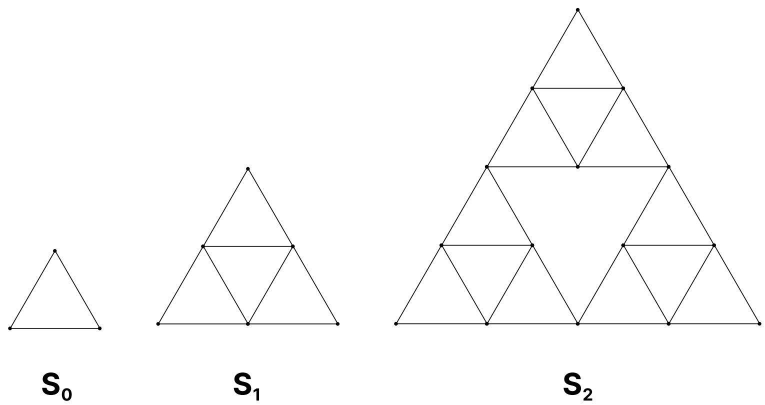


FIGURE 3. Sierpinski gasket

Intuitively, since the Sierpinski gasket roughly grows in size by a factor of 3, it's natural to expect that the number of closed prime walks of a fixed length L grows roughly by a factor of 3. In fact, this guess would be trivially true if Sierpinski gasket did not involve any gluing of vertices. As we will see, the structure and strength of the gluing will be of importance in this paper. In regular notion of convergence, the number of closed prime walks would diverge to infinity as the graph grows. For this reason, we rely on p-adic convergence.

Definition 3.1. Let p be a prime number. We define the **p-adic norm** on $\mathbb{Q}$ as follows. For any nonzero $x \in \mathbb{Q}$, let $x = p^k \frac{a}{b}$ where $p \nmid a$ and $p \nmid b$. Then, we define $|x|_p = p^{-k}$.

Example 3.2. For instance, the 3-adic and 5-adic norm of $\frac{24}{5}$ are

$$\left|\frac{24}{5}\right|_3 = 3^{-1} = \frac{1}{3}$$
$$\left|\frac{24}{5}\right|_5 = 5^{-(-1)} = 5$$

Given this nortion of p-adic norm, we say that a sequence of rational numbers $\{x_n\}$ **converges to** x **p-adically** if $|x_n - x|_p \to 0$ as $n \to \infty$.

Example 3.3. A divergent geometric series $\sum_{n=0}^{\infty} 3^n$ under the usual norm converges 3-adically to $-\frac{1}{2}$.

$$\begin{aligned}
\left|\sum_{n=0}^{\infty} 3^n - (-\frac{1}{2})\right|_3 &= \lim_{N\to\infty} \left|\sum_{n=0}^{N} 3^n + \frac{1}{2}\right|_3 \\
&= \lim_{N\to\infty} \left|\frac{3^{N+1}-1}{2} + \frac{1}{2}\right|_3 \\
&= \lim_{N\to\infty} \left|\frac{3^{N+1}}{2}\right|_3 \\
&= \lim_{N\to\infty} \frac{1}{3^{N+1}} = 0
\end{aligned}$$

Lemma 3.4. *Let S_i be the Sierpinski gasket at i^{th} stage. Then, the length of a shortest walk from one gluing vertex to another gluing vertex is 2^{i-1}.*

Proof. Let us prove this by induction. For $i = 1$, the answer is trivially 1. Suppose that the statement holds for $i = k$. To go from one gluing vertex to another gluing vertex on S_{k+1}, one must walk to the nearest gluing vertex on S_k, and walk to a gluing vertex on S_{k+1}. By induction hypothesis, the length of the shortest walk from one gluing vertex to another gluing vertex on S_k is $2 \cdot 2^{k-1} = 2^k$. □

Lemma 3.5. *Given a fixed length L, let N_n^L be the number of closed prime walks of length L on S_i. Then, there exists C such that if $i > C$, then the following equation holds for some K independent of i.*

$$N_i^L = 3N_{i-1}^L + 3K$$

Proof. Given the Sierpinski gasket at i^{th} stage S_i, the subsequent stage S_{i+1} contains three copies of S_i. There are three types of closed prime walks on S_{i+1}.

(1) Closed prime walks entirely contained in one copy of S_i.
(2) Closed prime walks contained in two copies of S_i.
(3) Closed prime walks contained in three copies of S_i.

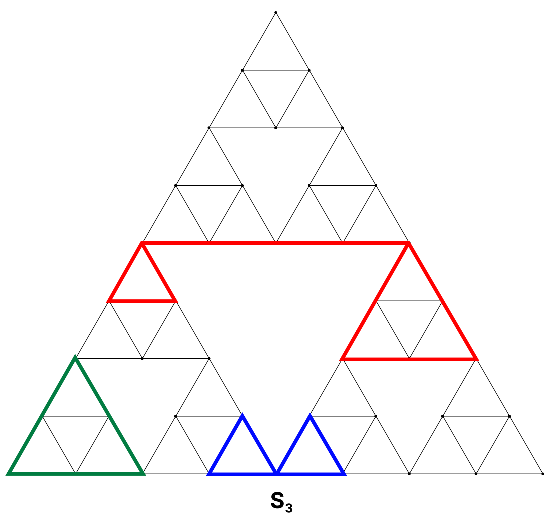


FIGURE 4. Three types of closed prime walks on S_4 going through one (green), two (blue), and three (red) copies of S_3 respectively.

By construction, there are $3N_i^L$ closed prime walks of type 1. We argue that for i large enough, the number of closed prime walks of type 2 is a constant independent of i. Furthemore, the number of closed prime walks of type 3 is 0.

Let us prove the latter claim first. For a closed prime walk to be contained in three copies of S_i, it must satisfy the following conditions.

(1) The walk starts at S_i, walks to nearest gluing vertex.
(2) The walk walks across S_i to the next gluing vertex.
(3) The walk walks into the third copy of S_i.

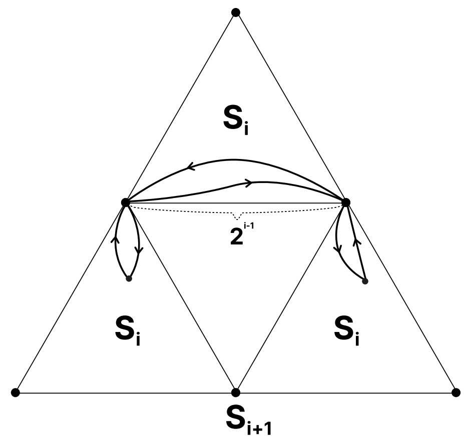


FIGURE 5. A closed prime walk of type 3 on S_{i+1}. The walk starts at S_i, walks to nearest gluing vertex, and walks across S_i which requires 2^{i-1} steps.

By Lemma 3.4, for $i > \log_2(L) + 1$, the number of closed prime walks of type 3 is 0. It remains to show that the number of closed prime walks of type 2 is a constant independent of i.

Let P be a closed prime walk of type 2. By construction, a closed prime walk of type 2 starts at some copy of S_i, walks to a neighboring copy of S_i, and walks back to the original copy of S_i. Therefore, P must go through a gluing vertex and P can be decomposed into a concatenation of closed prime walks $P = W_1 \cdots W_k$. Here, W_i is a closed prime walk entirely within a copy of S_i which starts and ends at a gluing vertex. As $k \geq 2$ and $\sum_i L(W_i) = L$, we have that $L(W_i) < L$ for all i. Hence, for $i > \log_2(L) + 1$, we have that

$$L(W_i) < L < 2^{i-1}$$

Therefore, the number of closed prime walks of type 2 is remains constant for $\log_2(L) + 1 < i, i+1, i+2 \ldots$ (See Figure 6).

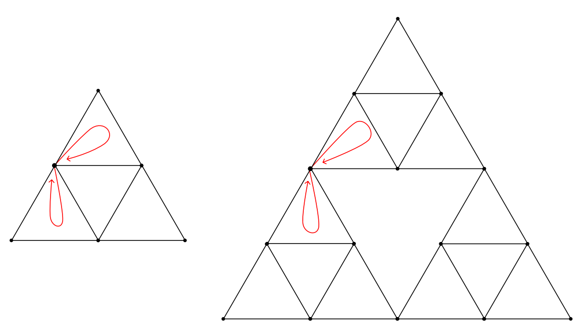


FIGURE 6. A closed prime walk of type 2 on S_{i+1}. The walk can be decomposed into a concatenation of closed prime walks $W_1 \cdots W_k$ (in red). Each W_i has length $< L$ so that the number of closed prime walks of type 2 does not change after $i > \log_2(L) + 1$.

□

Theorem 3.6. *Let N_n^L be the number of closed prime walks of length L on S_n. Then, N_n^L converges 3-adically as $n \to \infty$.*

Proof. Let $C = \lceil \log_2(L) + 1 \rceil$. By the previous lemma, for $n > C$, we have the following result.

$$\begin{aligned}
N_n^L &= 3N_{n-1}^L + 3K \\
&= 3(3N_{n-2}^L + 3K) + 3K = 3^2 N_{n-2}^L + 3^2 K + 3K \\
&= 3^2(3N_{n-3}^L + 3K) + 3^2 K + 3K = 3^3 N_{n-3}^L + 3^3 K + 3^2 K + 3K \\
&\vdots \\
&= 3^{n-C} N_C^L + 3^{n-C} K + 3^{n-C-1} K + \cdots + 3K \\
&= 3^{n-C} N_C^L + 3K \cdot \frac{3^{n-C} - 1}{2} \\
&= 3^{n-C}(\frac{3}{2}K + N_C^L) - \frac{3}{2}K
\end{aligned}$$

Hence, the 3-adic norm between N_n^L and $-\frac{3}{2}K$ is $|3^{n-C}(\frac{3}{2}K + N_C^L)|_3 \leq \frac{1}{3^{n-C}}$. Therefore, N_n^L converges 3-adically to $-\frac{3}{2}K$ as $n \to \infty$. □

Corollary 3.7. *All coefficients of the inverse of Ihara zeta function of the Sierpinski gasket converges 3-adically.*

4. p-adic convergence on n-flakes

The key argument of Munch's result can be distilled into two key observations.

(1) As the graph grows, only finitely many vertices can walk to another copy of the graph with fixed length L.
(2) The number of closed prime walks that contains a vertex in a neighboring copy grows by a factor of $p = 3$.

These two observations allow us to easily extend the result to other fractal graph, namely n-flakes.

Definition 4.1. A **n-flake** is a fractal graph which is recursively constructed.

(1) Start with a regular n-gon S_0.
(2) Make n copies of S_i and place it around the n-gon to form S_{i+1}.

The following is an first three iterations of 5-flake.

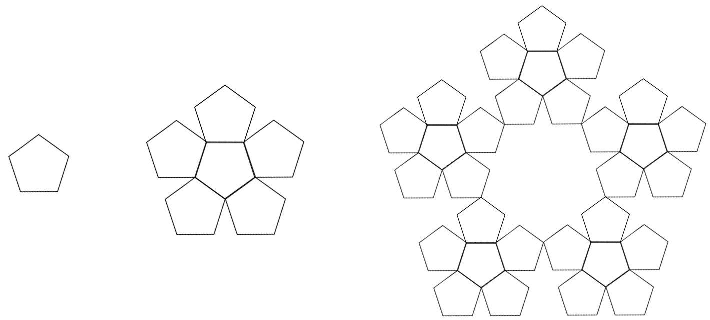
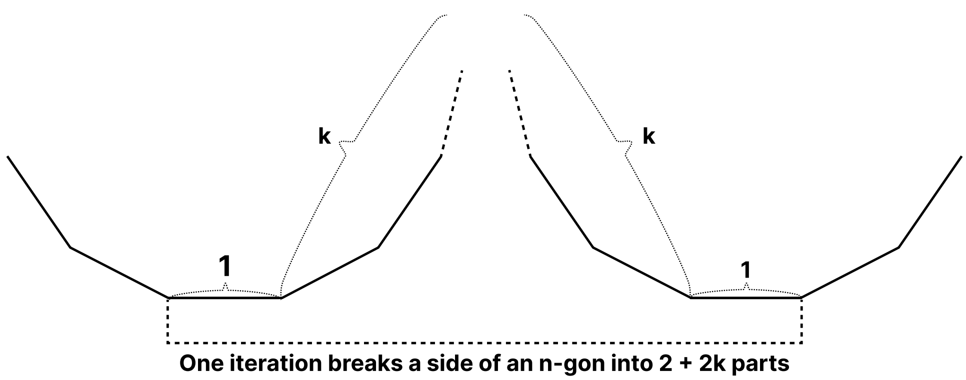


FIGURE 7. The first three iteration of 5-flake. Each iteration is constructed by placing 5 copies of the previous iteration around the pentagon.

The argument for n-flakes natural generalisation of the argument for Sierpinski gasket as you will see below.

Lemma 4.2. *Given an n-flake, at i^{th} iteration, the minimum distance from a vertex from S_0 to another vertex to S_0 is at least 2^{i-1}.*



Proof. Since the $i+1^{th}$ iteration is constructed by placing n copies of S_i around the n-gon, a side of an n-gon is broken into $2+2k$ segments where $k \geq 0$. Hence, for each iteration, the minimum distance from a vertex from S_0 to another vertex to S_0 at least doubles. □

Theorem 4.3. *Given an n-flake, the number of closed prime walks of length L converges p-adically as $i \to \infty$ for $p|n$.*

Proof. The proof is almost identical to the proof for Sierpinski gasket. Let $C = \lceil \log_2(L) + 1 \rceil$. By the same argument in the proof of Lemma 3.5, we have the following result for $i \geq C$.

$$
\begin{aligned}
N_i^L &= nN_{i-1}^L + nK \\
&= n(nN_{i-2}^L + nK) + nK = n^2 N_{i-2}^L + n^2K + nK \\
&= n^2(nN_{i-n}^L + nK) + n^2K + nK = n^3 N_{i-n}^L + n^3K + n^2K + nK \\
&\vdots \\
&= n^{i-C} N_C^L + n^{i-C}K + n^{i-C-1}K + \cdots + nK \\
&= n^{i-C} N_C^L + nK \cdot \frac{n^{i-C}-1}{n-1} \\
&= n^{i-C}\left(\frac{n}{n-1}K + N_C^L\right) - \frac{nK}{(n-1)K}
\end{aligned}
$$

Hence, for $p|n$, the p-adic norm between N_i^L and $-\frac{n}{n-1}K$ is $|n^{i-C}(\frac{n}{n-1}K + N_C^L)|_p \leq \frac{1}{p^{i-C}}$. Therefore, N_i^L converges p-adically to $-\frac{n}{n-1}K$ as $i \to \infty$. $\square$

As demonstrated, this argument is easily generalisable to other fractal graphs.

5. Spectral properties of fractal graphs

In this section, we shall explore both empirical and theoretical findings in the spectral properties of Sierpinski gasket. Analyzing eigenvalues and eigenvectors of the adjacency matrix and Laplacian matrix are of our paticular interest. Before diving into the details, let us first review major definitions and results we will use.

Theorem 5.1. ***Spectral theorem*** *Let A be a Hermitian matrix. Then, there exists an orthonormal basis of eigenvectors of A.*

Since the adjacency matrix of a graph is real and symmetric, the spectral theorem implies the following result.

Corollary 5.2. *An adjacency matrix of a graph has an orthonormal basis of eigenvectors. In other words, the adjacency matrix is diagonalisable and has real eigenvalues.*

The spectral theorem rules out the possibility of complex eigenvalues as well as non-diagonalisable cases which makes the study of spectral properties of graphs much more tractable. Another crucial result in spectral graph theory is the following.

Theorem 5.3. ***Perron-Frobeinus theorem*** *Let A be a real positive matrix. Then, there exists an unique largest eigenvalue λ of A.*

The largest eigenvalue of A is called the **spectral radius** of A and is denoted by $\rho(A)$. One may ask why the spectral radius is of particular interest. Many models that describe the real world (including graphs) can be often described as a linear discrete ODE of the form $x_{t+1} = Ax_t$ where x_t is a vector representing the state of the system at time t. A common question is to ask given an initial state x_0, whether the system will converge to a stable state or diverge. In the context of graphs, a "state" of the system can be thought of as a distribution of some quantity on the vertices of the graph. The linear discrete ODE described by $x_{t+1} = Ax_t$ changes the value on each vertex by summation of the values of its neighbors. That is to say

$$x_{t+1} = Ax_t \implies x_{t+1}(v) = \sum_{u \in N(v)} A_{vu} x_t(u)$$

where $N(v)$ is the set of neighbors of vertex v.

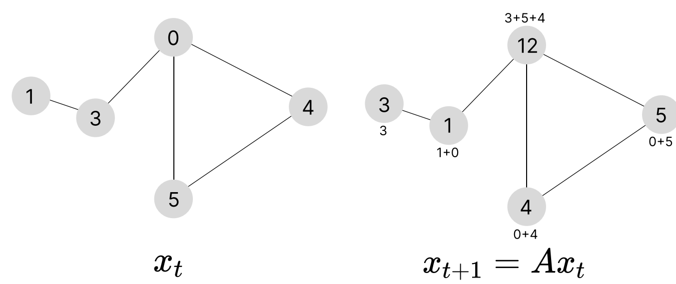


FIGURE 8. An example of a state update rule on a graph. The value on each vertex is updated by the sum of the values of its neighbors.

This view will be crucial in our analysis of how eigenvectors behave on the Sierpinski gasket. Combined with spectral theorem, the Perron-Frobenius theorem allows us to answer this question. In the context of our question, the number of closed walks of length L is precisely given by $\mathrm{tr}(A^L)$. Therefore, the spectral radius provides a bound for how fast the number of closed walks grows as the graph grows. Namely, the following results are particularly useful.

Theorem 5.4. *Let A be the adjacency matrix of a graph. Then, the absolute value of the eigenvalues of A are bounded by the maximum degree of the graph.*

Proof. Let λ be an eigenvalue of A and x be the corresponding eigenvector. Then, we have

$$|\lambda|||x|| = ||Ax|| = \left|\left|\sum_{u \in V} A_{vu} x_u\right|\right| \leq \sum_{u \in V} |A_{vu}||x_u| \leq \deg(v)||x||$$

Therefore, $|\lambda| \leq \deg(v)$. □

In the case of Simplepinski gasket, the degrees of the vertices are either 2 or 4. Therefore, the spectral radius of the adjacency matrix is always bounded by 4. Hence, we obtain the following result.

Corollary 5.5. *The number of closed prime walks of length L on the Sierpinski gasket is bounded by 4^L.*

Proof. Let A be the adjacency matrix of the Sierpinski gasket. Then, the number of closed walks of length L is given by $\mathrm{tr}(A^L)$. By spectral theorem, $A = UDU^T$ where U is the matrix of eigenvectors and D is the diagonal matrix of eigenvalues. Therefore, $\mathrm{tr}(A^L) = \mathrm{tr}(UD^LU^T) = \mathrm{tr}(D^L) \leq 4^L$. By definition, the number of closed prime walks is less than the number of closed walks. We have the desired result. □

Definition 5.6. The **Laplacian matrix** of a graph $G = (V, E)$ is defined as $L = D - A$ where A is the adjacency matrix and D is the degree matrix.

The eigenvalue of the Laplacian matrix reveals many information about the graph such as the number of connected components and the number of spanning trees. It's also useful for our empricial analysis because of the following result.

Lemma 5.7. *Let $G = (V, E)$ be a graph. Let L be the Laplacian matrix of G. Then, the eigenvalues of L are non-negative.*

This result makes numerical analysis easier since the signs of the eigenvalues are guranteed to be positive, making comparisons and applying log transformations much easier. Now, we shall present several empirical findings on the spectral properties of Sierpinski gasket. All computations were done using python package `networkx`. The code used for the computation can be found in the appendix.

5.1. **On spectral distribution of Sierpinski gasket.** A well-known fact in spectral graph theory is that the distribution of adjacency and laplacian eigenvalues of a graph converges to a limiting distribution as the size of the graph grows under certain conditions. More precisely, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a matrix A. We define the **empirical spectral distribution** of A as the probability measure μ_A defined by

$$\mu_A = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}$$

where δ_{λ_i} is the Dirac measure at λ_i. As an example, the following are empirical spectral distributions of the Laplacian matrix of a cycle graph and a grid graph.

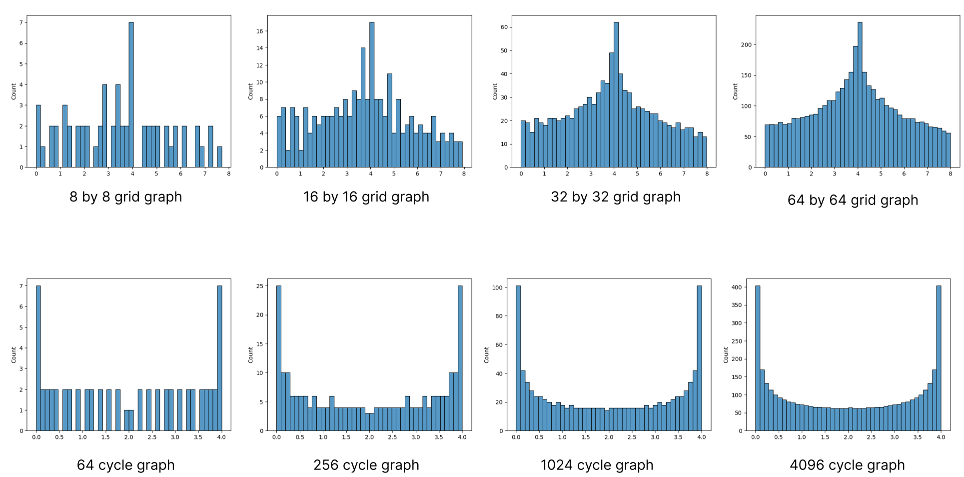


FIGURE 9. Empirical spectral distribution of Laplacian matrix of a cycle graph and a grid graph. The histogram is binned into 42 buckets for visualisation.

As figure 9 demonstrates, the empirical spectral distribution for cycle graph seems to converge to a Laplace distribution and that of cycle graph seems to converge to a beta distribution. A conjecture for the Sierpinski gasket is that the empirical spectral distribution will have either an exponential or a distribution with repeating shapes. However, contrary to our expectation, the computed empirical spectral distribution of the Sierpinski gasket does not seem to have any discernible pattern as shown below.

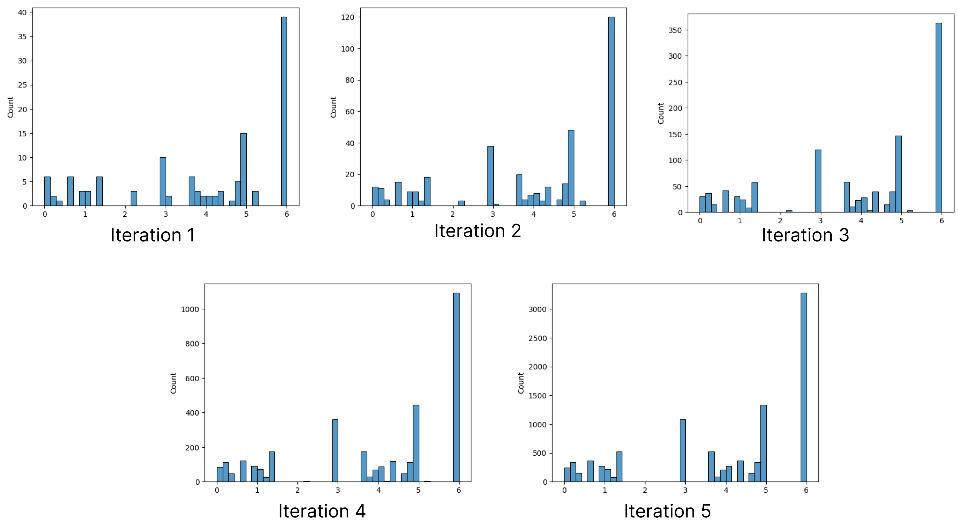


FIGURE 10. Empirical spectral distribution of Laplacian matrix of the Sierpinski gasket.

Similarly, applying log transformation to the eigenvalues does not seem to reveal any pattern.

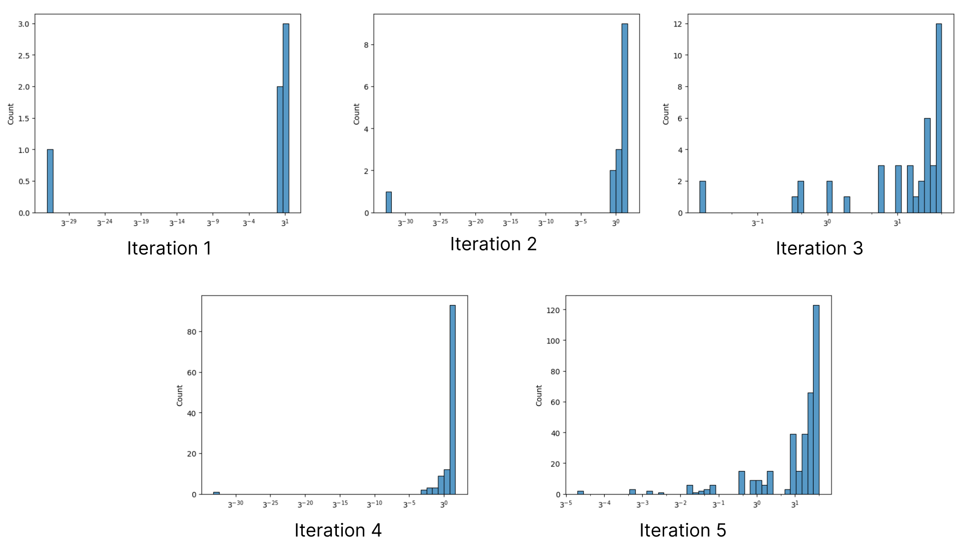


FIGURE 11. Empirical spectral distribution of Laplacian matrix of the Sierpinski gasket after log (base 3) transformation.

The distribution certainly does not display a "smooth" pattern as we found it for the case of cycle and grid graphs. It's clear that the distribution display a sparse, irregular pattern which seems to converge. While a clear pattern is not visible, a more detailed analysis of the mulitplicity of the eigenvalues reveals a more interesting result.

S_3	
1	4
2	4
3	1

S_4	
1	7
2	8
3	1
4	1
12	1

S_5	
1	24
2	31
3	5
4	4
12	2
13	2
39	1
40	1
120	1

S_6	
1	47
2	62
3	10
4	8
12	4
13	4
39	2
40	2
120	1
121	1
363	1

S_7	
1	90
2	121
3	23
4	16
12	8
13	8
39	4
40	4
120	2
121	2
363	1
364	1
1092	1

S_8	
1	180
2	243
3	45
4	32
12	16
13	16
39	8
40	8
120	4
121	4
363	2
364	2
1092	1
1093	1
3279	1

TABLE 1. Multiplicity of eigenvalues of the adjacency matrix of the Sierpinski gasket. The left column is the mulitplicity and the right column is the number of eigenvalues with that multiplicity. For example, the second row for table S_5 means that there were 31 eigenvalues which had mulitplicity 2.

5.2. **On spectral multiplicity of Sierpinski gasket.** Table 1 shows the mulitplicity of the eigenvalues of the adjacency matrix of the Sierpinski gasket. From the table, we observe three clear patterns. Let $m_i^{(n)}$ be the multiplicity of i-th adjacency eigenvalue of S_n (in ascending order). Let $c_k^{(n)}$ be the number of adjacency eigenvalues of S_n with multiplicity k. Then, we observe the following patterns.

(1) $c_i^{(n)} \leq c_i^{(n+1)}$ for all i and n.
(2) $m_i^{(n)} = 3m_{i-1}^{(n)}$ or $m_i^{(n)} = m_{i-1}^{(n)} + 1$ for all i.
(3) For large enough S_n, tail of $c_i^{(n)}$ is $1, 1, 1, 2, 2, 4, 4, 8, 8 \cdots$.

The first pattern is reasonable to expect since the Sierpinski gasket is constructed recursively. The second pattern is also expected since S_i is constructed by gluing three copies of S_{i-1}. What's interesting is that the multiplicities increase in an alternating pattern. For instance, in table S_8, we observe that the progression is $1, 2 \cdots 13, 39, 40, 120, 121, 363, 364, 1092, 1093, 3279$. It either triples or increases by one in an alternating fashion. The last pattern also displays a clear converging pattern as the size of S_n grows, which ties back to the idea of our proof for p-adic convergence of number of prime paths.

5.3. **On properties of leading eigenvector on Sierpinski gasket.** In spectral graph theory, the eigenvalues receive much attention, and the eigenvectors are often overlooked. Motivated by Perron-Frobenius theorem, a reasonable eigenvector to start our analysis is the leading eigenvector of the adjacency matrix.

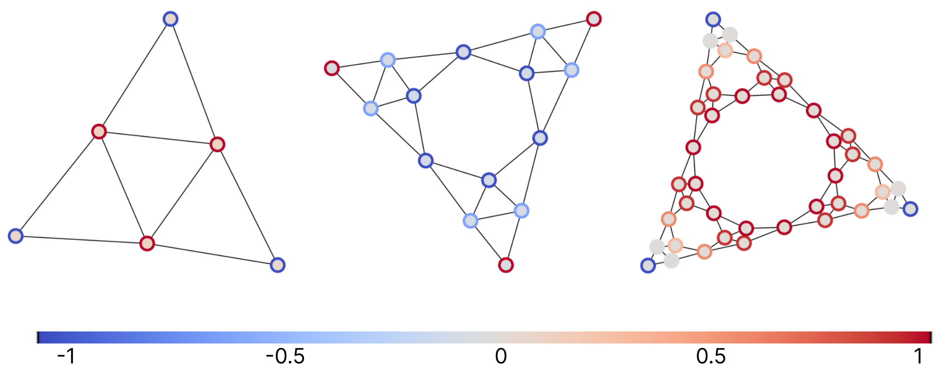


FIGURE 12. Distribution of values of leading eigenvector on first 3 iterations of Sierpinski gasket.

In figure 12, we plot the values of leading eigenvector on the Sierpinski gasket. Since each value of an eigenvector corresponds to a vertex, we can visualise the eigenvector as a distribution of values on the vertices. To do this, we use `numpy`'s `eigh` function to compute orthonormal eigenvectors. After overlaying the eigenvector values on the vertices, in all iterations, we observe a rotational symmetry of values of the leading eigenvector. Furthermore, in odd iterations, the values are maximal gluing vertices and minimal at the corners of the triangle. In even iterations, the pattern manifests in the opposite way. The symmetry of the leading eigenvector has a clear theoretical explanation.

Theorem 5.8. *The eigenvectors of multiplicity 1 of the adjacency matrix of the Sierpinski gasket are symmetric with respect to Dihedral group of order 6. Simply put, the eigenvectors with multiplicity 1 are invariant under rotation and reflection.*

Proof. Suppose v is an eigenvector of the adjacency matrix of the Sierpinski gasket. Let $G = \{I, R, R^2, F, FR, FR^2\}$ be the dihedral group of order 6 where I is the identity, R is a permutation matrix which rotates the triangle by $\frac{2\pi}{3}$, and F is a permutation matrix which reflects the triangle along the vertical axis. Since v has mulitplicity one, it follows that $R \cdot v = c_R v$ for some constant c_R. Therefore, $R^3 v = v = c_R^3 v$ which implies $c_R = 1$. Therefore, the values of eigenvector on left, right, and top corners of the Sierpinski gasket must be the same. Similarly, $F^2 v = v = c_F^2 v$ which implies $c_F = \pm 1$. Therefore, the values of eigenvector must be symmetric with respect to the reflective axis (upto a sign). □

Remark 5.9. While the proof is only done in context of the Sierpinski gasket and Dihedral group of order 6, one should see that this argument easily extends to other graphs with symmetries. In particular, the core idea of the proof came from the fact that there exists an element in the symmetry group of odd order which allowed us to conclude that the eigenvector is symmetric.

Combined with Perron-Frobenius theorem, we obtain the following corollary which explains the empirical observation in figure 12.

Corollary 5.10. *The leading eigenvector of the adjacency matrix of the Sierpinski gasket is symmetric with respect to D_6.*

5.4. **Alternating recurrence relations on multiplicity of eigenvalues.** In previous sections, we observed a number of interesting patterns in the multiplicity of eigenvalues of the adjacency matrix of the Sierpinski gasket. In this section, we shall offer an conjectures on where these patterns come.

S_4		S_5		S_6		S_7	
-2	39	-2	120	-2	363	-2	1092
-1	13	-1	40	-1	121	-1	364
1	12	1	39	1	120	1	363
2.618	4	2.618	13	2.618	40	2.618	121
0.382	4	0.382	13	0.382	40	0.382	121
3.303	4	3.303	12	3.303	39	3.303	120
-0.302	4	-0.302	12	-0.302	39	-0.302	120

TABLE 2. Multiplicity of eigenvalues of the adjacency matrix, sorted from highest to lowest. Eigenvalues were rounded to 3 decimal places.

The table above shows how multiplicity of the eigenvalues change as the size of the Sierpinski gasket grows. We see that there are two recurrence patterns $R_1 : x_{n+1} = 3(x_n + 1)$ and $R_2 : x_{n+1} = 3x_n + 1$. For example, the multiplicity of -2 follow pattern R_1 with progression $39, 120, 363, 1092$, whereas the multiplicity of -1 follow pattern R_2 with progression $13, 40, 121, 364$. What's interesting is that the recurrence rule alternates between R_1 and R_2 in powers of 2. We see that eigenvalue -2 follows R_1, -1 follows R_2, 1 follows R_1, 2.618 follows R_2, 0.382 follows R_2, 3.303 follows R_1, and -0.302 follows R_1. Table 3 shows the recurrence pattern of eigenvalues for S_6.

Eigenvalue	-2	-1	1	2.618	0.382	3.303	-0.302
Multiplicity	363	121	120	40	40	39	39
Recurrence	R_1	R_2	R_1	R_2	R_2	R_1	R_1

Eigenvalue	3.706	3.122	-0.706	-0.122	3.856	2.895	-0.856	0.104
Multiplicity	13	13	13	13	12	12	12	12
Recurrence	R_2	R_2	R_2	R_2	R_1	R_1	R_1	R_1

TABLE 3. Reccurence pattern of eigenvalues for S_6

Assuming this reccurence persists for all S_i, this observation explains patterns 1,2, and 3 we observed in section 5.2. The question is where are these reccurence patterns coming from.

Theorem 5.11. *Let $u_1, u_2, \cdots, u_k$ be the eigenvectors of the adjacency matrix of S_n with eigenvalue λ, whose eigenvalues vanish on the corner vertices. Then, there are at least $3k$ eigenvectors of the adjacency matrix of S_{n+1} with eigenvalue λ.*

Due to the recursive nature of the Sierpinski gasket, the reader's first intuition might be that one can simply copy the eigenvectors of S_{n-1} and glue them together to form the eigenvectors of S_n. However, for this to work, the eigenvectors of S_{n-1} must vanish on the corners. Otherwise, in the subsequent iteration, the neighboring vertices of the corners will receive nonzero values and the previous eigenvectors will no longer work as eigenvectors of S_n.

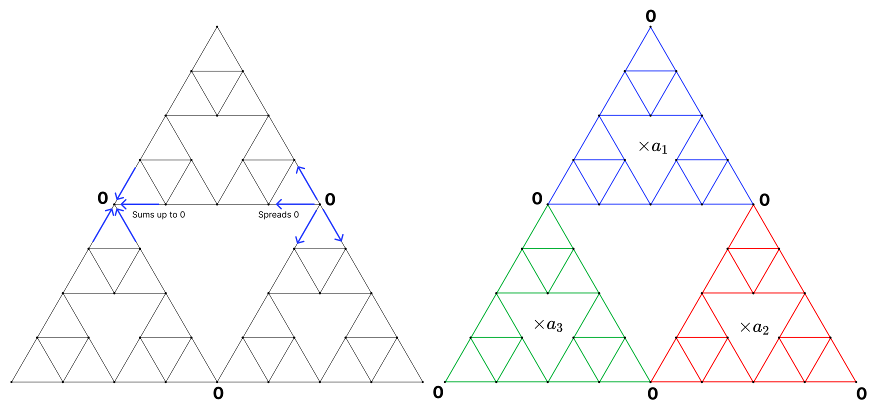


FIGURE 13. The eigenvector values must vanish on the gluing vertices

On the other hand, if the eigenvectors vanish on the corners, then one can simply copy the eigenvectors of S_n three times and glue them together to form the eigenvectors of S_{n+1}. The new eigenvector can be written as a linear combination of the previous eigenvectors (See figure 13). Therefore, a single eigenvector of S_n whose eigenvalues vanish on the corners creates at least three eigenvectors of S_{n+1}. Moreover, since the new eigenvectors in S_{n+1} also vanish on the corners of S_{n+1}, we can inductively apply the same argument to obtain that at least 3^k eigenvectors of the same eigenvalue exist in S_{n+k}. For eigenvalues, -2, -1, 1, and 2.618, one can empirically verify that the eigenvectors indeed vanish on the corners (See figure 14).

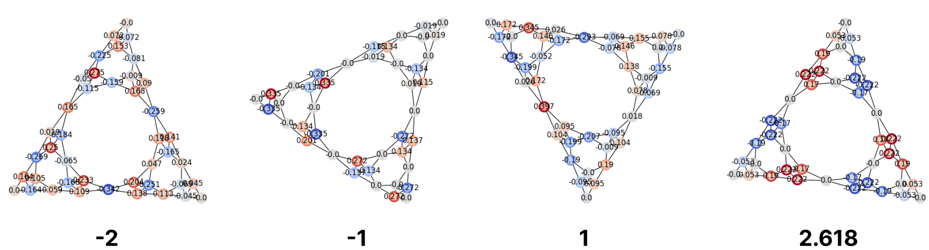


FIGURE 14. Eigenvectors of eigenvalues -2, -1, 1, and 2.618 vanish on the corners. We only display one eigenvector for each eigenvalue, but the same pattern persists for all eigenvectors.

Now, we shall formally prove the theorem 5.11.

Proof. Let $u_1, u_2, \cdots, u_k$ be the eigenvectors of the adjacency matrix of S_n with eigenvalue λ whose eigenvalues vanish on the corners. Let $(u_i)_{\text{top}}, (u_i)_{\text{left}}, (u_i)_{\text{right}}$ be a vector in S_{n+1} by placing u_i on the top, left, and right corners of the Sierpinski gasket and setting the values of the other vertices to 0.

Let $[Au]_v$ be the value of the vector Au at vertex v. As discussed above, the value of $[Au]_v$ only depends on the values of the neighbors of v. Consider the case of $(u_i)_{\text{top}}$. Since the values of $(u_i)_{\text{top}}$ are all zeros except on the top S_n, it suffices to check the values at ther left/right corners and its neighbors (See figure 13). If c is a left/right corner of top triangle, we have $[(u_i)_{\text{top}}]_c = 0$ and the neighboring values of c sum to 0. Therefore, $[A(u_i)_{\text{top}}]_c = \lambda(u_i)_{\text{top}} = 0$. Moreover, since $[(u_i)_{\text{top}}]_c = 0$, the values of neighboring vertices of c after applying A will be same as the previous iteration.

It remains to show that the eigenvectors $(u_i)_{\text{top}}, (u_i)_{\text{left}}, (u_i)_{\text{right}}$ for $i = 1, 2, \cdots, k$ are linearly independent. Consider the following linear combination

$$\sum_{i=1}^{k} \alpha_{i,\text{top}}(u_i)_{\text{top}} + \sum_{i=1}^{k} \alpha_{i,\text{left}}(u_i)_{\text{left}} + \sum_{i=1}^{k} \alpha_{i,\text{right}}(u_i)_{\text{right}} = 0$$

Since the values of $(u_i)_{\text{left}}$ and $u_{i,\text{right}}$ are zeros on the top S_n, we have

$$\sum_{i=1}^{k} \alpha_{i,\text{top}}(u_i)_{\text{top}} = 0$$

By assumption, the eigenvectors $(u_i)_{\text{top}}$ are linearly independent. Therefore, $\alpha_{i,\text{top}} = 0$ for all i. Similarly, by the same argument, we have $\alpha_{i,\text{left}} = 0$ and $\alpha_{i,\text{right}} = 0$ for all i. Therefore, the eigenvectors $(u_i)_{\text{top}}, (u_i)_{\text{left}}, (u_i)_{\text{right}}$ are linearly independent and we have at least $3k$ eigenvectors of the adjacency matrix of S_{n+1} with eigenvalue λ. □

6. Conjectures on the recursion patterns

In previous section, we have shown that "if eigenvectors vanish on the corners", then the number of eigenvectors of the same eigenvalue grows by at least a factor of 3. However, this doesn't explain the recursion patterns $x_{n+1} = 3(x_n + 1)$ and $x_{n+1} = 3x_n + 1$ we observed. One would naturally expect that the number of eigenvectors should grow roughly by a factor of 3, but it seems unintuitive where $+1$ is coming from. In this section, we shall derive a lower bound on the number of eigenvectors of the same eigenvalue which obeys the recursion patterns.

Theorem 6.1. *Let u be an eigenvector of the adjacency matrix of S_n with eigenvalue λ. Suppose u vanishes on the vertices on the reflective axis of the Sierpinski gasket. Futhermore, suppose the values of u are invariant under rotation by $\frac{2\pi}{3}$ and symmetric upto a sign with respect to the reflective axis. Then, multiplicity of λ in S_{n+1} is at least $3 \cdot 1 + 1 = 4$.*

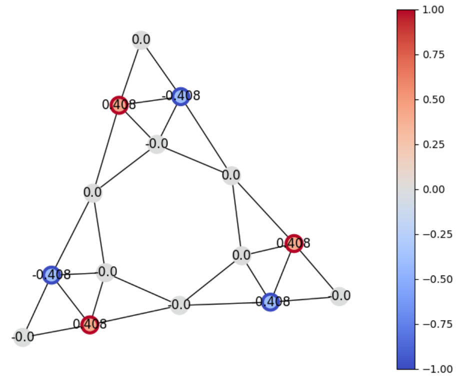


FIGURE 15. Eigenvalue of -1 on the Sierpinski gasket

Before we prove the theorem, we shall motivate the theorem with an empirical observation for $\lambda = -1$ which exhibits recursion pattern $x_{n+1} = 3x_n + 1$. As shown in figure above, the eigenvalues of -1 are zeros on the reflective axis and are symmetric upto a sign with respect to the reflective axis. Hence, apart from the $3x_n$ eigenvectors we obtained from theorem 5.11, we can construct one more eigenvector by cleverly piecing together the eigenvectors of S_n (See figure 16).

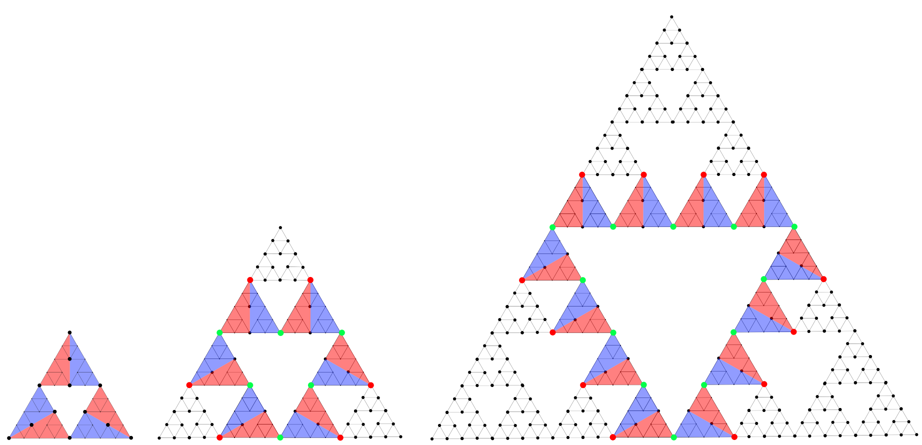


FIGURE 16. Constructing an eigenvector of eigenvalue -1 for S_{n+1} and S_{n+2} from eigenvectors of S_n. The area with the same color ahve the same eigenvector. Blue and red areas are negatives of each other. The dotted vertices indicate vertices with zero values.

For the purpose of our discussion, let us introduce some notations.

Definition 6.2. Let u be an adjacency eigenvector of S_n with eigenvalue λ. We denote $u^{(p)}$ for $p \in \{\text{top}, \text{left}, \text{right}\}$ as the values of u on the top, left, and right corners of the Sierpinski gasket. Furthermore, we denote $(u^{(p)})^{(q)}$ for $p, q \in \{\text{top}, \text{left}, \text{right}\}$ as $u^{(pq)}$ and $u^{(p\cdots p)} = u^{(p^n)}$.

Definition 6.3. Let u be an adjacency eigenvector of S_n with eigenvalue λ. We denote $u_{\text{top}}, u_{\text{left}}, u_{\text{right}}$ as the eigenvectors of S_{n+1} constructed from u by placing u on the top, left, and right corners of the Sierpinski gasket and setting the values of the other vertices to 0.

Proof. By the same argument as theorem 5.11, we can show that the eigenvector constructed in figure 16 is indeed an eigenvector of S_{n+1}. In other words, it suffices to check the values of the vertices on the gluing vertices (red/green on the figure) and its neighbors under action of the adjacency matrix. For red vertices, the argument from the theorem 5.11 applies. For the green vertices, we see that the values of the neighbors are identical to that of the previous iteration.

Therefore, the eigenvector constructed in figure 16 is indeed an eigenvector of S_{n+1}. It remains to show that the $3x_n$ eigenvectors along with the one constructed in figure are linearly independent.

Let v be the eigenvector constructed in figure 16. Consider the following linear combination

$$\alpha_{\text{top}} u_{\text{top}} + \alpha_{\text{left}} u_{\text{left}} + \alpha_{\text{right}} u_{\text{right}} + \alpha_v v = 0$$

Let us focus our attention on the values of the vertices on top S_n. On the top S_n, we must have that

$$\alpha_{\text{top}} u + \alpha_v v^{(\text{top})} = 0$$

We see that there're vertices where v are zeros, but u_{top} are nonzero. (See figure 17). Therefore, we have that $\alpha_{\text{top}} = 0$ which also implies that $\alpha_v = 0$. The same argument applies to α_{left} and α_{right}. Hence, $\alpha_{\text{left}} = \alpha_{\text{right}} = \alpha_{\text{left}} = 0$. ☐

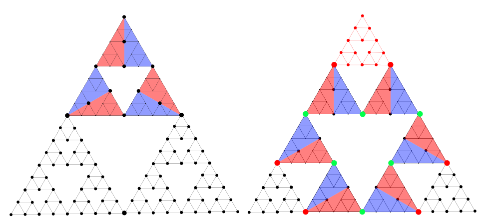


FIGURE 17. On the top S_n, we see that the red vertices have zero values for v but nonzero values for u_{top}.

Remark 6.4. It is no suprise that the eigenvector constructed in figure 16 is invariant under rotation by $\frac{2\pi}{3}$. If it wasn't, then instead of getting just one new eigenvector, we would have gotten at least three types of rotation. While the contruction may seem obvious in hindsight, these insights are crucial in coming up with the potential candiates for the eigenvectors.

Theorem 6.5. *Let u be an eigenvector of the adjacency matrix of S_n with eigenvalue λ. Suppose u vanishes on the corners of the Sierpinski gasket and the values of u "not" invariant under rotation by $\frac{2\pi}{3}$. More precisely, u, Ru, R^2u are linearly independent where R is a permutation matrix which rotates the triangle by $\frac{2\pi}{3}$. Then, multiplicity of λ in S_{n+1} is at least $3 \cdot (3+1) = 12$.*

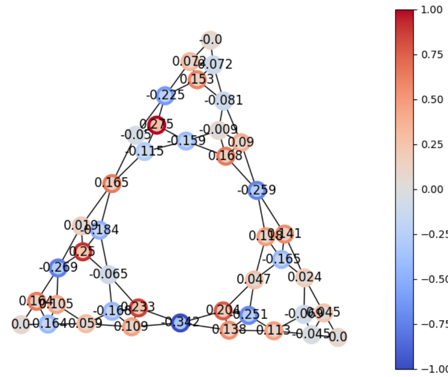


FIGURE 18. Eigenvalue of -2 on the Sierpinski gasket

Again, we motivate the theorem with an empirical observation for $\lambda = -2$ which exhibits recursion pattern $x_{n+1} = 3(x_n + 1)$ and starts with mulitplicity 3 on S_2. As shown in figure above, the eigenvalues of 2 are zeros on the corners and are not invariant under rotation by $\frac{2\pi}{3}$. Hence, for eigenvectors of these form on S_n, we can contruct an eigenvector of S_{n+1} by piecing together the eigenvectors of S_n (See figure 19).

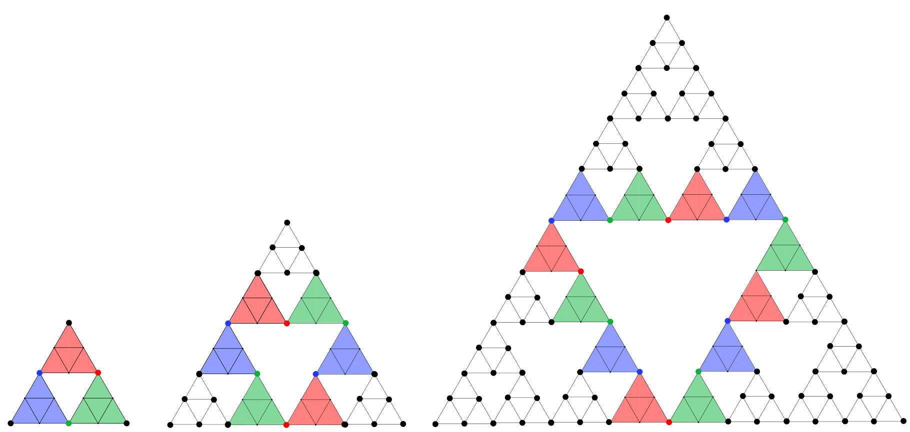
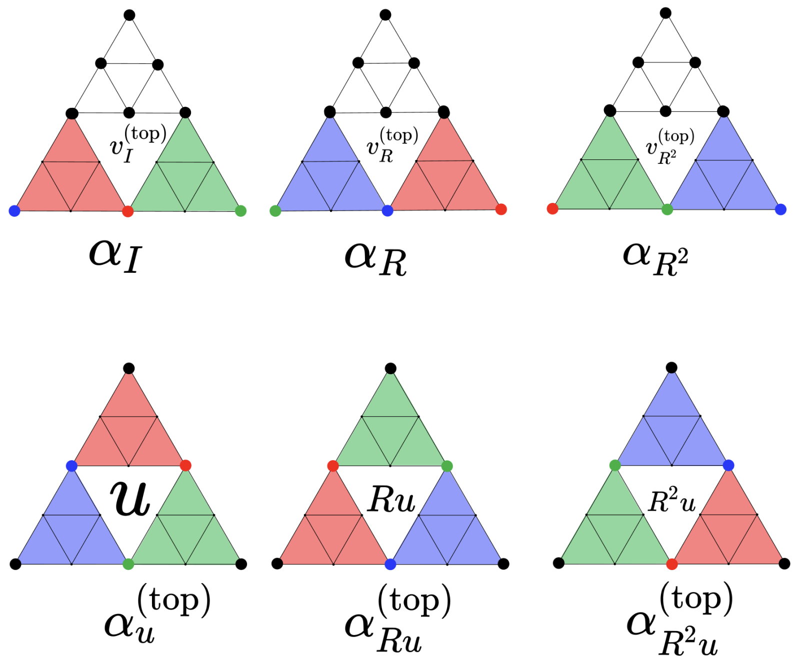


FIGURE 19. Constructing an eigenvector of eigenvalue -2 for S_{n+1} and S_{n+2} from eigenvectors of S_n. The areas of the same color have the same value. The dotted vertices indicate vertices with zero values.

Proof. Let v_I be a vector in S_{n+1} constructed in figure 19. As in the previous theorem, it suffices to check the values of the vertices on the gluing vertices and its neighbors (i.e red/green/blue vertices which glue red/green/blue triangles in the figure). Clearly, the values of the neighbors of the gluing vertices are identical to that of the previous iteration. Since the constructed eigenvector is "not" invariant under rotation by $\frac{2\pi}{3}$, one can create additional two eigenvectors by rotating the constructed eigenvector by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. Let v_R, v_{R^2} be these three eigenvectors. It remains to show that v_I, v_R, v_{R^2} and $(R^i u)_{\text{top}}, (R^i u)_{\text{left}}, (R^i u)_{\text{right}}$ for $i = 0, 1, 2$ are linearly independent. Here we use the convention that $R^0 = I$. Consider the following linear combination

$$\sum_{p \in \{\text{top,left,right}\}} \sum_{i=0}^{2} \alpha^{(\text{p})}_{R^i u} (R^i u)_p + \alpha_I v_I + \alpha_R v_R + \alpha_{R^2} v_{R^2}$$

Similar to last proof, let us focus our attention on top S_n of S_{n+1}. For above equation to hold, the values at the top S_n must vanish.



Then, it follows that the following equation must hold

$$\alpha_I v_I^{(\text{top})} + \alpha_R v_R^{(\text{top})} + \alpha_{R^2} v_{R^2}^{(\text{top})} + \alpha_u^{(\text{top})} u + \alpha_{Ru}^{(\text{top})} Ru + \alpha_{R^2 u}^{(\text{top})} R^2 u = 0$$

We see that $v_I^{(\text{top})}, v_R^{(\text{top})}, v_{R^2}^{(\text{top})}$ vanish on the top S_{n-1}. However, $u, Ru, R^2 u$ are nonzero on top S_{n-1}. Hence, the above equation decouples into two equations

$$\alpha_I v_I^{(\text{top})} + \alpha_R v_R^{(\text{top})} + \alpha_{R^2} v_{R^2}^{(\text{top})} = 0 \tag{6.1}$$

$$\alpha_u^{(\text{top})} u + \alpha_{Ru}^{(\text{top})} Ru + \alpha_{R^2 u}^{(\text{top})} R^2 u = 0 \tag{6.2}$$

By linear independence of $u, Ru, R^2 u$, we have that $\alpha_u^{(\text{top})} = \alpha_{Ru}^{(\text{top})} = \alpha_{R^2 u}^{(\text{top})} = 0$. For contradiction, suppose that there exists nontrivial $\alpha_I, \alpha_R, \alpha_{R^2}$ which satisfy equation (6.1). (i.e there exists nontrivial linear combination of red, green, blue triangles which sum to zero). But, we see that this would imply $\alpha_I u + \alpha_R Ru + \alpha_{R^2} R^2 u = 0$ which would be a contradiction. Repeating this argument for left/right S_n of S_{n+1}, we have that all α's are zeros and we are done. $\square$

The previous two theorems have that if the eigenvectors at S_N obey certain boundary conditions and symmetries, then one can obtain eigenvectors for S_{N+1} which follows the recursion patterns. From figure 19 and figure 16, one can certainly keep constructing eigenvectors for $S_{N+2}, S_{N+3}, \cdots$ which follow the recursion patterns. (See figure 22 in Appendix). However, it's entirely possible that these eigenvectors are not linearly independent. The following corallaries show that this is indeed the case and we obtain a lower bound on the multiplicity of adjacency eigenvectors.

Lemma 6.6. *Let u be an adjacency eigenvector of S_N with eigenvalue λ which satisfies the conditions of theorem 6.1. By applying the process in theorem 6.1, recursively construct eigenvectors for $S_{N+1}, S_{N+2}, \cdots$. Let U_n be the set of eigenvectors constructed for S_n. Let v_n be the eigenvector constructed by tiling u in the manner of figure 16 for S_n. Let $w_n = v_{n+1}^{(top)}$. Then, $w_n \notin$ span(U_n) for all $n \geq N$. (See figure 20)*

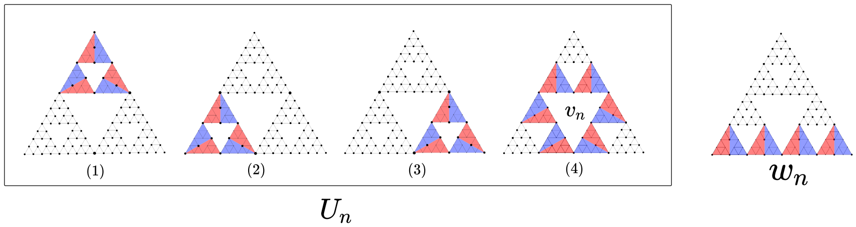


FIGURE 20. Can we construct w_n from U_n?

Proof. Let us proceed by induction. By theorem 6.1, the base case $n = N$ holds. Suppose the lemma holds upto $n = k \geq N$. An eigenvector $u_i \in U_n$ falls into four categories.

(1) $u_i = u_{\text{top}}$ for $u \in U_{k-1}$.
(2) $u_i = u_{\text{left}}$ for $u \in U_{k-1}$.
(3) $u_i = u_{\text{right}}$ for $u \in U_{k-1}$.
(4) $u_i = v_k$.

It's clear that w_k only has nonzero values on left S_{k-1} and right S_{k-1} of S_k. Therefore, (1) cannot be used to construct w_k. We also see that (4) has nonzero values on top S_{k-1} of S_k, where (2) and (3) are zeros. Hence, (4) also cannot be used to construct w_k. Lastly, (2) and (3) orthnogonal complements of each other and have no overlap. By symmetry, the problem now reduces to showing that $(w_k)_{\text{left}}$ cannot be constructed from U_{k-1}. But, $(w_k)_{\text{left}} = w_{k-1}$ and by induction hypothesis, $w_{k-1} \notin \text{span}(U_{k-1})$. $\square$

Corollary 6.7. *Let u be an adjacency eigenvector of S_N with eigenvalue λ which satisfies the conditions of theorem 6.1. Let x_n be the multiplicity of λ in S_n. Then, $x_{n+1} \geq 3x_n + 1$ for all $n \geq N$.*

Proof. Let U_n, v_n and w_n be as above lemma. To check for independence, our goal is to show that $v_n \notin \operatorname{span}(U_n \setminus \{v_n\})$. The set of eigenvectors in $U_n \setminus \{v_n\}$ are either $u_{\text{top}}, u_{\text{left}}, u_{\text{right}}$ for $u \in U_{n-1}$. Consider the linear combination

$$\sum_i \alpha_{i,\text{top}} u_{i,\text{top}} + \sum_i \alpha_{i,\text{left}} u_{i,\text{left}} + \sum_i \alpha_{i,\text{right}} u_{i,\text{right}} = \alpha_{v_n} v_n$$

where $u_i \in U_{n-1}$. Focusing on the top S_{n-1} of S_n, for above equation to hold, we must have that

$$\sum_i \alpha_{i,\text{top}} u_i = \alpha_{v_n} v_n^{(\text{top})}$$

But, $v_n^{(\text{top})} = w_{n-1}$ by definition. Hence, we have that $w_{n-1} \in \operatorname{span}(U_{n-1})$ if above equation holds for nontrivial coefficients. By the lemma, this cannot happen and we are done.

□

Remark 6.8. The above lemma and corollary only relies on where zero and nonzero values are placed on the Sierpinski gasket. A similar argument can be applied to show that the eigenvectors contructed recursively from theorem 6.5 are also linearly independent.

7. Conclusion and future work

In this paper, we have stuided Sieprinski gasket from two different perspectives: spectral graph theory and p-adic analysis on the number of prime paths. For prime path counting, this paper has shown that Munch's argument can be easily extended to similar fractals beyond the Sierpinski gasket. A natural question is to test how far this argument can be extended to other fractals beyond n-flakes. From spectral graph theory perspective, this paper has provided a both empirical and theoretical insights on where the multiplicity of eigenvalues and the structure of eigenvectors. However, there are still two open questions which we have not addressed in this paper.

(1) We have shown that if conditions of the theorems 6.1 and 6.5 are satisfied, then one can obtain a lower bound on the multiplicity which obeys the empirical recursion patterns. The empirical result suggests that the multiplicity of eigenvalues are fully determined by the construction presented in theorem 5.11, 6.1, and 6.5. For instance, why is it that the eigenvectors with -2 are only obtained from the recursion described in theorem 6.5?
(2) It also seems that it is a necessary condition for the eigenvectors to vanish on the corners in order to obtain the recursion patterns. Is there another way to obtain the recursion patterns without the eigenvectors vanishing on the corners? If not, what is the reason behind this?

As for potential directions in addressing these questions, one potential approach is to first show that the adjacency matrix of the Sierpinski gasket is always full rank and show that the lower bounds of each multiplicies add up to the total number of vertices. Since we have obtained a lower bound by recursively constructing eigenvectors from the previous iteration, it would be natural to try to obtain an upper bound by starting with an eigenvector from iteration above and decompose it to form an eigenvector in preceding iteration. Furthermore, applying the theory of group actions on graphs even at a shallow level has already provided us with a lot of insights. It may be wise to further explore this direction. For instance, one could treat assignment of values to the vertices as colorings and apply Burnside's lemma to count the number of eigenvectors.

Another underdeveloped approach is to study Sierpinski gasket by performing a similar analysis on the Laplacian matrix which allows us to leverage tools like Graph Fourier Transform. Even doing analysis on adjacency matrix shows that orthonormal basis of fractal graphs show alternating patterns and values vanishing on the "boundaries" of the fractal. A spectral analysis on the Laplacian matrix may provide us with more insights on the structure of the eigenvectors for adjacency matrices since a Laplacian eigenvector is also an eigenvector of the adjacency matrix for d-regular graphs.

Lemma 7.1. *Let G be a d-regular graph. (i.e all vertices have degree d) Then, an eigenvector of the Laplacian matrix of G is also an eigenvector of the adjacency matrix of G.*

Proof. Let u be an eigenvector of the Laplacian matrix of G with eigenvalue λ. Then, we have that

$$Lu = \lambda u \iff (D - A)u = \lambda u \iff Au = Du - \lambda u \iff Au = (d - \lambda)u$$

Therefore, u is an eigenvector of the adjacency matrix of G with eigenvalue $d - \lambda$. □

While the Sierpinski gasket is not a regular graph, it "converges" to a 4-regular graph since all vertices except the corners have degree 4. Therefore, one could potentially approximate the adjacency eigenvectors of the Sierpinski gasket with the Laplacian eigenvectors.

8. Acknowledgements

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References

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Appendix

Python code for Sierpinski gasket generation.

```
import networkx as nx

def _sierpinski(last, ends):
    n = len(last.nodes)

    top = nx.relabel_nodes(last, { i:i for i in last.nodes })
    left = nx.relabel_nodes(last, { i:i+n for i in last.nodes })
    right = nx.relabel_nodes(last, { i:i+2*n for i in last.nodes })

    top = nx.relabel_nodes(
        top, { ends["left"] : ends["top"] + n })
    left = nx.relabel_nodes(
        left, { ends["right"] + n : ends["left"] + 2 * n })
    right = nx.relabel_nodes(
        right, { ends["top"] + 2 * n : ends["right"] })

    ends = {
        "top":ends["top"],
        "left":ends["left"] + n,
        "right":ends["right"] + 2 * n
    }

    x = nx.compose_all([top, left, right])
    update = { node : idx for idx, node in enumerate(x.nodes) }
    x = nx.relabel_nodes(x, update)
    ends = {  end: update[node] for end, node in ends.items() }

    return x, ends

def sierpinski(n:int):
    last = nx.complete_graph(3)
    ends = { "top":0, "left":1, "right":2 }
    for _ in range(n):
        last, ends = _sierpinski(last, ends)
    return last, ends
```

Example usage.

```
G, ends = sierpinski(2)
nx.draw(G)
```

Python code for empirical spectral distribution.

```
    import seaborn as sns
    import numpy as onp

    # cycle graph
    G = nx.cycle_graph(64)
```

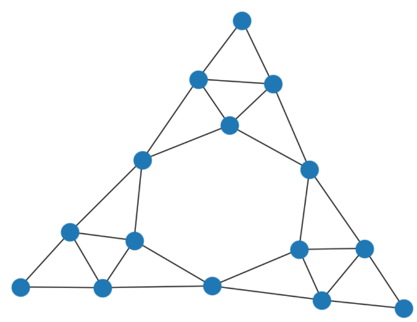

FIGURE 21. Sierpinski gasket of 2nd iteration.

```
A = nx.laplacian_spectrum(G)
A = A.astype(float)

sns.histplot(A, bins=42)

# grid graph
G = nx.grid_graph((64,64))
A = nx.laplacian_spectrum(G)
A = A.astype(float)

sns.histplot(A, bins=42)

# sierpinski gasket
G, ends = sierpinski(5)
A = nx.adjacency_spectrum(G)
A = A.astype(float)

sns.histplot(A, bins=42)
```

Python code for dominant eigenvector analysis.

```
import matplotlib.pyplot as plt
import numpy as np
G, corners = sierpinski(1)
A = nx.to_numpy_array(G)
eigenvalues, eigenvectors = np.linalg.eig(A)


# Sort eigenvalues and eigenvectors based on eigenvalues
sorted_indices = np.argsort(eigenvalues)[::-1]
sorted_eigenvalues = eigenvalues[sorted_indices]
sorted_eigenvectors = eigenvectors[:, sorted_indices]

# Select the eigenvector you want to plot
eigenvector_to_plot = sorted_eigenvectors[:, 0]
```

```
# Create a dictionary mapping nodes to their eigenvector values
eigenvector_values = {
    node: eigenvector_to_plot[i]
    for i, node in enumerate(G.nodes)
}

# Draw the graph with node colors based on eigenvector values
pos = nx.spring_layout(G)  # positions for all nodes
nx.draw(
    G, pos, node_color=list(eigenvector_values.values()),
    cmap=plt.cm.viridis
)
nx.draw_networkx_nodes(
    G, pos, node_color=list(eigenvector_values.values()),
    cmap=plt.cm.viridis, node_size=128
)

# Add a colorbar
sm = plt.cm.ScalarMappable(cmap=plt.cm.viridis)
sm.set_array([])
plt.colorbar(sm)

# Display the plot
plt.show()
```

Python code for multiplicity analysis.

```
import numpy as onp
from collections import Counter

for i in range(10):
    print(f"Sierpinski iteration: {i}")
    G, corners = sierpinski(i)

    spectrums = nx.adjacency_spectrum(G)
    counter = Counter([onp.round(n,8) for n in spectrums])
    multiples = Counter([count for _, count in counter.items()])

    spectrums = sorted(spectrums)
    # print spectral gap as well
    print("the gap", spectrums[-1] - spectrums[-2])

    for factor, count in sorted(multiples.items()):
        print(factor, "^", count)

    print("\n#####\n")

    for spectrum, count in counter.most_common():
        print(spectrum, count)
```

```
    print("\n")
```

Example output.

```
Sierpinski iteration: 0
the gap (2.9999999999999999+0j)
1 ^ 1
2 ^ 1

#####

(-1+0j) 2
(2+0j) 1


Sierpinski iteration: 1
the gap (2.6180339887498945+0j)
1 ^ 2
2 ^ 2

#####

(0.61803399+0j) 2
(-1.61803399+0j) 2
(3.23606798+0j) 1
(-1.23606798+0j) 1


Sierpinski iteration: 2
the gap (0.8096973139654096+0j)
1 ^ 4
2 ^ 4
3 ^ 1

#####

(-2+0j) 3
(2.9687598+0j) 2
(0.84179978+0j) 2
(-0.25767832+0j) 2
(-1.55288127+0j) 2
(3.77845712+0j) 1
(0.71083145+0j) 1
(-1.48928857+0j) 1
(-1+0j) 1
```

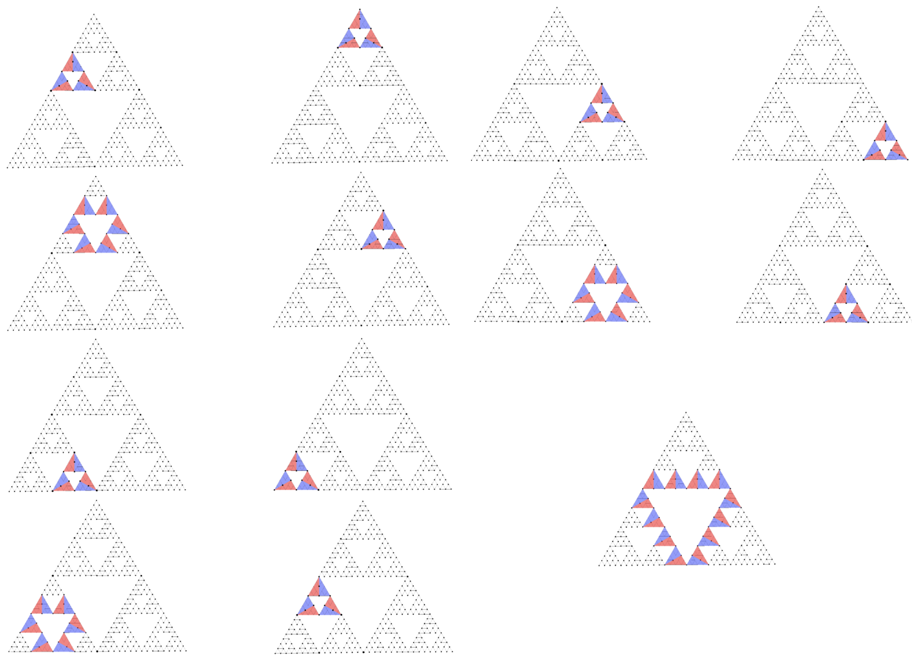

FIGURE 22. Constructing eigenvectors of eigenvalue -2 for S_{N+2} from eigenvectors of S_N. These are all eigenvectors of -2 for S_{N+2}. The question is whether these eigenvectors are linearly independent.