

# HINGES IN $\mathbb{Z}_p^d$ AND APPLICATIONS TO PINNED DISTANCE SETS

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## 1. INTRODUCTION

**1.1. Background.** In this paper, we investigate the number of chains in a subset of  $\mathbb{Z}_p^d$ , the  $d$ -dimensional vector space over the finite field with  $p$  elements, for a prime  $p$ .

**Definition 1.1.** A *chain* in  $E \subseteq \mathbb{Z}_p^d$  of length  $n$  with common distance  $t$  is a sequence  $x_1, x_2, \dots, x_{n+1}$  where for  $i = 1, \dots, n$ ,  $\|x_{i+1} - x_i\| = t$ .

Here, and throughout the paper,  $\|x\| = x_1^2 + x_2^2 + \dots + x_d^2$ . Note that chains in  $E$  are equivalent to paths on the distance graph of  $E$ . Theorem 1.1 of [1] gives an estimate for the number of paths of a certain length, which is a very similar result to Theorem 1.4 in this paper; in fact, it is essentially the same but with a better constant on the error term. We give a different proof, and also extend the idea to dot products as well as distances. In other words, we consider sequences with the property  $\|x_{i+1} \cdot x_i\| = t$ .

After obtaining our result estimating the number of chains, we consider an application of the case  $n = 2$  to estimating the size of pinned distance sets.

**Definition 1.2.** The *distance set* of  $E$  pinned at  $x \in E$  is

$$\Delta_x(E) = \{\|x - y\| : y \in E\} \tag{1}$$

In fact, the idea we employ here could be generalized further to relate chains of length  $2n$  to pinned chains of length  $n$ .

## 1.2. Main Results.

**Definition 1.3.** Let  $\Lambda_t^n(E)$  be the number of chains in  $E$  of length  $n$ , with common distance  $t$ , and let  $\Gamma_t^n(E)$  be the number of dot-product chains in  $E$  of length  $n$  with common dot-product  $t$ .

The main focus of this paper is to estimate these quantities. Also, the case  $\Lambda_t^2(E)$  can be used for obtaining a lower bound on the size of pinned distance sets in  $E$ .

**Theorem 1.4.** If  $|E| > p^{\frac{d+1}{2}}$ , then

$$\Lambda_t^n(E) = \frac{|E|^{k+1}}{p^k} + R \quad (2)$$

$$\Gamma_t^n(E) = \frac{|E|^{k+1}}{p^k} + Q \quad (3)$$

Where

$$R \leq 2p^{\frac{d-1}{2}} \frac{|E|^k}{p^{k-1}} \left( \frac{4^{k+1} - 4}{3} \right) \quad (4)$$

## 2. THE FRAMEWORK

We start by proving an estimate of a sum which is essential to our proof for the main theorem. First, let

$$S_t = \{x \in \mathbb{Z}_p^d : \|x\| = t\} \quad (5)$$

For notational convenience, we will identify a set with its indicator function. Also, let  $T_t(x, y)$  be the indicator function for when  $x \cdot y = t$ .

**Theorem 2.1.** For any subset  $E \subseteq \mathbb{Z}_p^d$ , and any nonnegative functions  $f, g : \mathbb{Z}_p^d \rightarrow \mathbb{R}$ , the following estimates hold:

$$\begin{aligned} \sum_{x, y \in \mathbb{Z}_p^d} f(x)g(y)S_t(x - y) &= p^{-1} \|f\|_1 \|g\|_1 + \varepsilon_1 \\ \sum_{x, y \in \mathbb{Z}_p^d} f(x)g(y)T_t(x, y) &= p^{-1} \|f\|_1 \|g\|_1 + \varepsilon_2 \end{aligned} \quad (6)$$

Where the error terms are bounded by

$$\begin{aligned} \varepsilon_1 &\leq 2p^{\frac{d-1}{2}} \|f\|_2 \|g\|_2 \\ \varepsilon_2 &\leq p^{\frac{d-1}{2}} \|f\|_2 \|g\|_2 \end{aligned} \quad (7)$$

Before proving this theorem, we will introduce the following lemma, which we take as a black box.

**Lemma 2.2.** If  $m \neq 0$ , then

$$|\hat{S}_t(m)| \leq 2p^{-\frac{d+1}{2}} \quad (8)$$

Now we prove the theorem.

*Proof.* We start with the first estimate.

$$\begin{aligned}
 \sum_{x,y} f(x)g(y)S_t(x-y) &= \sum_{x,y} \sum_m \chi((x-y) \cdot m) \hat{S}_t(m) f(x)g(y) \\
 &= p^{2d} \sum_m \overline{\hat{f}(m)} \hat{g}(m) \hat{S}_t(m) \\
 &= p^{-d} \|f\|_1 \|g\|_1 |S_t| + p^{2d} \sum_{m \neq 0} \overline{\hat{f}(m)} \hat{g}(m) \hat{S}_t(m) \quad (9)
 \end{aligned}$$

The  $m = 0$  term is already the desired main term, considering the fact that  $|S_t| \sim p^{d-1}$ . The error term we bound with Cauchy Schwarz, obtaining

$$\begin{aligned}
 \varepsilon_1 &= p^{2d} \sum_{m \neq 0} \overline{\hat{f}(m)} \hat{g}(m) \hat{S}_t(m) \leq p^{2d} \left(2p^{-\frac{d+1}{2}}\right) \left(\sum_m |\hat{f}(m)|^2\right)^{\frac{1}{2}} \left(\sum_m |\hat{g}(m)|^2\right)^{\frac{1}{2}} \\
 &= 2p^{\frac{d-1}{2}} \|f\|_2 \|g\|_2 \quad (10)
 \end{aligned}$$

Now we move on to the second estimate, in the case of dot products. We start out in similar fashion:

$$\sum_{x,y} f(x)g(y)T_t(x,y) = p^{-1} \|f\|_1 \|g\|_1 + p^{-1} \sum_{m \neq 0} \sum_{x,y} f(x)g(y) \chi(m(x \cdot y - t)) \quad (11)$$

So that

$$\begin{aligned}
 \varepsilon_2^2 &= \left( \sum_{m \neq 0} \sum_{x,y} f(x)g(y) \chi(m(x \cdot y - t)) \right)^2 \\
 &\leq p^{-2} \|f\|_2^2 \sum_x \sum_{\substack{m, m' \neq 0 \\ y, y'}} g(y)g(y') \chi(x \cdot (my - m'y')) \chi(-t(m - m')) \quad (12)
 \end{aligned}$$

Now in the above sum, when  $my - m'y' \neq 0$  the sum over  $x$  vanishes. Thus, we have

$$\varepsilon_2^2 = p^{d-2} \|f\|_2^2 \sum_{my=m'y'} g(y)g(y') \chi(t(m' - m)) \quad (13)$$

Now, when  $m = m'$ , we have  $y = y'$  also, and so we have

$$p^{d-2} \|f\|_2^2 \sum_{\substack{m=m' \neq 0 \\ y}} |g(y)|^2 \leq p^{d-1} \|f\|_2^2 \|g\|_2^2 \quad (14)$$

On the other hand, if  $m \neq m'$ , then we use a substitution  $a = \frac{m}{m'}$ ,  $b = m'$ , to obtain

$$p^{d-2} \|f\|_2^2 \sum_{\substack{y, y' \\ a \neq 0, 1 \\ b \neq 0}} g(y)g(y') \chi(tb(1-a)) \quad (15)$$

But notice that

$$\sum_{b \neq 0} \chi(tb(1-a)) = -1 \quad (16)$$

so the  $m \neq m'$  terms actually give a negative contribution, so we may just ignore them. Therefore, we have the desired bound on the error term.  $\square$

In fact, this theorem can be done in a more general setting; the dot product can be replaced with any non-degenerate bilinear form [2]. The other cases require a more detailed analysis, and this is all we need for our main result.

Here we recursively define two sequences of functions which will encode a lot of information about  $\Lambda_t^n(E)$ ,  $\Gamma_t^n(E)$ .

**Definition 2.3.** Let  $\psi_0 = \varphi_0 = 1$ , and let

$$\begin{aligned}\psi_k(x) &= (E\varphi_{k-1}) * S_t(x) = \sum_y E(y)\varphi_{k-1}(y)S_t(x-y) \\ \varphi_k(x) &= \sum_y E(y)\varphi_{k-1}(y)T_t(x,y)\end{aligned}\tag{17}$$

Note that this definition really only makes sense for  $x \in E$ , The reason these are useful is the following lemma, in particular the case  $k = n$ .

**Lemma 2.4.** For  $k = 0, \dots, n$ ,

$$\Lambda_n^t(E) = \sum_{x_1, \dots, x_{n+1-k}} \psi_k(x_{n+1-k}) \prod_{i=1}^{n+1-k} E(x_i) \prod_{i=1}^{n-k} S_t(x_{i+1} - x_i),\tag{18}$$

$$\Gamma_n^t(E) = \sum_{x_1, \dots, x_{n+1-k}} \varphi_k(x_{n+1-k}) \prod_{i=1}^{n+1-k} E(x_i) \prod_{i=1}^{n-k} S_t(x_{i+1} \cdot x_i)\tag{19}$$

*Proof.* This can be shown by induction, noting that the case  $k = 0$  is trivial, and that

$$\begin{aligned}\Lambda_t^n(E) &= \sum_{x_1, \dots, x_{n+2-k}} \psi_{k-1}(x_{n+2-k}) \prod_{i=1}^{n+2-k} E(x_i) \prod_{i=1}^{n+1-k} S_t(x_{i+1} - x_i) \\ &= \sum_{x_1, \dots, x_{n+1-k}} \prod_{i=1}^{n+1-k} E(x_i) \prod_{i=1}^{n-k} S_t(x_{i+1} - x_i) \sum_{x_{n+2-k}} E(x_{n+2-k}) \psi_{k-1}(x_{n+2-k}) S_t(x_{n+2-k} - x_{n+1-k}) \\ &= \sum_{x_1, \dots, x_{n+1-k}} \psi_k(x_{n+1-k}) \prod_{i=1}^{n+1-k} E(x_i) \prod_{i=1}^{n-k} S_t(x_{i+1} - x_i)\end{aligned}\tag{20}$$

Since

$$\psi_k(x_{n+1-k}) = \sum_{x_{n+2-k}} E(x_{n+2-k}) \psi_{k-1}(x_{n+2-k}) S_t(x_{n+2-k} - x_{n+1-k})\tag{21}$$

Therefore, the first identity holds. The proof of the second one is identical, replacing  $S_t(x_{i+1} - x_i)$  with  $T_t(x_{i+1}, x_i)$  and  $\psi$  with  $\varphi$  throughout.  $\square$

**Corollary 2.5.**

$$\Lambda_t^n(E) = \sum_x E(x)\psi_n(x) = \|E\psi_n\|_1, \quad (22)$$

$$\Gamma_t^n(E) = \sum_x E(x)\varphi_n(x) = \|E\varphi_n\|_1 \quad (23)$$

*Proof.* This is the special case  $k = n$ .  $\square$

We have now reduced the original problem to estimating  $\|E\psi_n\|_1$  and  $\|E\varphi_n\|_1$ , which we will do using a few recursive inequalities. Until this stage, we have been simultaneously handling the distance case and the dot product case in parallel, since they have been structurally identical. Now the two proofs will diverge - not because they are different in a particularly meaningful way, but because the small differences between the two cases will start to add up, and so it would be less efficient to do them together. Note that the only difference between the two cases lies in the error term from Theorem 2.1, where there is an extra factor of 2 in the distance case, when compared to the dot product case. This seems insignificant enough that it might make sense to continue handling the two cases simultaneously, but since our plan of attack is to use a recursive inequality based on Theorem 2.1, this difference will propagate.

### 3. PROOF OF MAIN THEOREM

**3.1. Distances.** In order to prove Theorem 1.4, we first prove a lemma which allows us to recursively bound  $a_k := \|E\psi_k\|_1$ . We will get an inequality involving  $L^1$  and  $L^2$  norms, and then we will get a similar recursive inequality for the sequence of  $L^2$  norms,  $b_k := \|E\psi_k\|_2$ . We will be able to combine these to achieve an estimate of  $a_k$ .

**Lemma 3.1.**

$$a_k \leq \frac{|E|}{p} a_{k-1} + 2p^{\frac{d-1}{2}} |E|^{\frac{1}{2}} b_{k-1} \quad (24)$$

*Proof.* This is a straightforward application of Theorem 2.1, with  $f(x) = E(x)$ ,  $g(y) = E(y)\psi_{k-1}(y)$ . We have

$$\begin{aligned} a_k &= \sum_x E(x)\psi_k(x) = \sum_{x,y} E(x)E(y)\psi_{k-1}(y)S_t(x-y) \\ &\leq p^{-1}|E| \cdot \|E\psi_{k-1}\|_1 + 2p^{\frac{d-1}{2}} |E|^{\frac{1}{2}} \|E\psi_{k-1}\|_2 \\ &= \frac{|E|}{p} a_{k-1} + 2p^{\frac{d-1}{2}} |E|^{\frac{1}{2}} b_{k-1} \end{aligned} \quad (25)$$

$\square$

**Lemma 3.2.**

$$b_k^2 \leq p^{-1} a_k a_{k-1} + 2p^{\frac{d-1}{2}} b_k b_{k-1} \quad (26)$$

*Proof.* This is the same idea as the previous lemma, but with  $f(x) = E(x)\psi_k(x)$ ,  $g(y) = E(y)\psi_{k-1}$ . We have

$$\begin{aligned} b_k^2 &= \sum_x E(x)\psi_k(x)^2 = \sum_{x,y} E(x)E(y)\psi_k(x)\psi_{k-1}(y)S_t(x-y) \\ &\leq p^{-1}\|E\psi_k\|_1\|E\psi_{k-1}\|_1 + 2p^{\frac{d-1}{2}}\|E\psi_k\|_2\|E\psi_{k-1}\|_2 \\ &= p^{-1}a_k a_{k-1} + 2p^{\frac{d-1}{2}}b_k b_{k-1} \end{aligned} \tag{27}$$

□

Now the idea is to combine these two inequalities to bound  $a_k, b_k$  above by induction. Then we will use that upper bound to prove our main result.

**Lemma 3.3.** *For some constant  $C_k$  depending only on  $k$ , we have*

$$a_k \leq C_k \frac{|E|^{k+1}}{p^k}, \text{ and } b_k \leq C_k \frac{|E|^{k+\frac{1}{2}}}{p^k} \tag{28}$$

*Proof.* It is clearly true in the case  $k = 0$  with the choice of constant  $C_0 = 1$ . Assume it is true for indices less than  $k$ . We have by Lemma 3.1 that

$$\begin{aligned} (p^{-1}a_k a_{k-1}) &\leq p^{-2}|E|a_{k-1}^2 + 2p^{\frac{d-3}{2}}|E|^{\frac{1}{2}}a_{k-1}b_{k-1} \\ &\leq C_{k-1}^2 \frac{|E|^{2k+1}}{p^{2k}} + 2C_{k-1}^2 \frac{|E|^{2k+1}}{p^{2k}} \\ &= 3C_{k-1}^2 \frac{|E|^{2k+1}}{p^{2k}} \end{aligned} \tag{29}$$

Also,

$$2p^{\frac{d-1}{2}}b_{k-1} \leq 2C_{k-1} \frac{|E|^{k+\frac{1}{2}}}{p^k} \tag{30}$$

The following simple algebra is helpful: for  $A, B > 0$ , if  $x \leq \sqrt{A+Bx}$ , then we can solve the corresponding quadratic polynomial to find that

$$x \leq \frac{B + \sqrt{B^2 + 4A}}{2} \leq \max(B, \sqrt{B^2 + 4A}) \leq \sqrt{B^2 + 4A} \tag{31}$$

We use this inequality with

$$\begin{aligned} x &= b_k, \\ A &= 3C_{k-1}^2 \frac{|E|^{2k+1}}{p^{2k}}, \\ B &= 2C_{k-1} \frac{|E|^{k+\frac{1}{2}}}{p^k} \end{aligned} \tag{32}$$

And we obtain

$$\begin{aligned} b_k &\leq \left( 4C_{k-1} \frac{|E|^{2k+1}}{p^{2k}} + 12C_{k-1}^2 \frac{|E|^{2k+1}}{p^{2k}} \right)^{\frac{1}{2}} \\ &\leq 4C_{k-1} \frac{|E|^{k+\frac{1}{2}}}{p^k} \end{aligned} \quad (33)$$

Therefore,  $C_k = 4^k$  works as long as we can show that we obtain the desired inequality for  $a_k$  also.

$$\begin{aligned} a_k &\leq \frac{|E|}{p} a_{k-1} + 2p^{\frac{d-1}{2}} |E|^{\frac{1}{2}} b_{k-1} \\ &\leq C_{k-1} \frac{|E|^{k+1}}{p^k} + 2 \frac{|E|^{\frac{3}{2}}}{p} b_{k-1} \\ &\leq 3C_{k-1} \frac{|E|^{k+1}}{p^k} \\ &< C_k \frac{|E|^{k+1}}{p^k} \end{aligned} \quad (34)$$

□

We are now ready to prove the main theorem.

**Proof of Theorem 1.4 (Distances Case).** The key trick in passing from the previous lemma to the desired theorem is the observation that although losing a constant factor on  $a_k$  was necessary in order to bound  $b_k$  in the first place, once we know our bound for  $b_k$  it is not hard to use the exact same recursion to get a more precise estimate for  $a_k$ . We know by Theorem 2.1 along with the previous lemma that

$$\begin{aligned} a_k &= \frac{|E|}{p} a_{k-1} + R_k, \\ R_k &\leq 2p^{\frac{d-1}{2}} C_k \frac{|E|^k}{p^{k-1}} \end{aligned} \quad (35)$$

Since  $a_0 = |E|$ , we can use this recursion to estimate  $a_k$ . We have

$$\begin{aligned} a_k &= \frac{|E|}{p} a_{k-1} + R_k, \\ &= \frac{|E|^2}{p^2} a_{k-2} + \frac{|E|}{p} R_{k-1} + R_k \\ &= \dots \\ &= \frac{|E|^k}{p^k} a_0 + \sum_{i=0}^{k-1} \frac{|E|^i}{p^i} R_{k-i} \\ &= \frac{|E|^{k+1}}{p^k} + R \end{aligned} \quad (36)$$

Where

$$\begin{aligned}
R &:= \sum_{i=0}^{k-1} \frac{|E|^i}{p^i} R_{k-i} \\
&\leq 2p^{\frac{d-1}{2}} \frac{|E|^k}{p^{k-1}} \sum_{i=0}^{k-1} 4^{k-i} \\
&= 2p^{\frac{d-1}{2}} \frac{|E|^k}{p^{k-1}} \left( \frac{4^{k+1} - 4}{3} \right)
\end{aligned} \tag{37}$$

In light of the fact that  $a_k = \Lambda_t^k(E)$ , we are done.  $\square$

**3.2. Dot Products.** The proof in the dot product case goes in essentially the same fashion. As mentioned earlier, the only difference in the results from Section 2 is that the error term is missing a factor of 2 when compared to the error term from the distance case. We will trace through the lemmas building up to the proof of Theorem 1.4, making sure to keep track of that factor of 2 in the error term. To continue with notation similar to before, let  $c_k := \|E\varphi_k\|_1$ ,  $d_k := \|E\varphi_k\|_2$ .

**Lemma 3.4.**

$$c_k \leq \frac{|E|}{p} c_{k-1} + p^{\frac{d-1}{2}} |E|^{\frac{1}{2}} d_{k-1} \tag{38}$$

*Proof.* In the proof of Lemma 3.1, replace  $\psi$  with  $\varphi$  and  $S_t(x - y)$  with  $S_t(x \cdot y)$ , and we lose a factor of 2 in the error term because of the difference in Theorem 2.1.  $\square$

**Lemma 3.5. ??**

$$d_k^2 \leq p^{-1} c_k c_{k-1} + p^{\frac{d-1}{2}} d_k d_{k-1} \tag{39}$$

*Proof.* This is to Lemma 3.2 as the previous lemma is to 3.1.  $\square$

**Lemma 3.6.** For some constant  $D_k$  depending only on  $k$ , we have

$$c_k \leq D_k \frac{|E|^{k+1}}{p^k}, \quad d_k \leq D_k \frac{|E|^{k+1}}{p^k} \tag{40}$$

*Proof.* It is clearly true in the case  $k = 0$  with the choice of constant  $D_0 = 1$ . Assume it is true for indices less than  $k$ . We have by Lemma 3.4 that

$$\begin{aligned}
(p^{-1} c_k c_{k-1}) &\leq p^{-2} |E| c_{k-1}^2 + p^{\frac{d-3}{2}} |E|^{\frac{1}{2}} c_{k-1} d_{k-1} \\
&\leq D_{k-1}^2 \frac{|E|^{2k+1}}{p^{2k}} + D_{k-1}^2 \frac{|E|^{2k+1}}{p^{2k}} \\
&= 2D_{k-1}^2 \frac{|E|^{2k+1}}{p^{2k}}
\end{aligned} \tag{41}$$

Also,

$$p^{\frac{d-1}{2}} d_{k-1} \leq C_{k-1} \frac{|E|^{k+\frac{1}{2}}}{p^k} \tag{42}$$



Similarly to before, we will have  $d_k \leq \sqrt{B^2 + 4A}$ , where

$$\begin{aligned} A &= 2D_{k-1}^2 \frac{|E|^{2k+1}}{p^{2k}}, \\ B &= d_{k-1} \frac{|E|^{k+\frac{1}{2}}}{p^k} \end{aligned} \quad (43)$$

And we obtain

$$\begin{aligned} d_k &\leq \left( D_{k-1}^2 \frac{|E|^{2k+1}}{p^{2k}} + 8D_{k-1}^2 \frac{|E|^{2k+1}}{p^{2k}} \right)^{\frac{1}{2}} \\ &= 3D_{k-1} \frac{|E|^{1+\frac{1}{2}}}{p^k} \end{aligned} \quad (44)$$

Therefore,  $D_k = 3^k$  works as long as we can show that we obtain the desired inequality for  $c_k$  also.

$$\begin{aligned} c_k &\leq \frac{|E|}{p} c_{k-1} + p^{\frac{d-1}{2}} |E|^{\frac{1}{2}} d_{k-1} \\ &\leq D_{k-1} \frac{|E|^{k+1}}{p^k} + \frac{|E|^{\frac{3}{2}}}{p} d_{k-1} \\ &\leq 2D_{k-1} \frac{|E|^{k+1}}{p^k} \\ &< D_k \frac{|E|^{k+1}}{p^k} \end{aligned} \quad (45)$$

□

We can now conclude the dot product case of Theorem 1.4.

*Proof of Theorem 1.4 (Dot Product Case).* Similarly to in the proof for the distance case, we obtain

$$\begin{aligned} c_k &= \frac{|E|^{k+1}}{p^k} + Q, \\ Q &:= \sum_{i=0}^{k-1} \frac{|E|^i}{p^i} Q_{k-i}, \\ Q_i &\leq p^{\frac{d-1}{2}} D_i \frac{|E|^i}{p^{i-1}} \end{aligned} \quad (46)$$

Thus,

$$Q \leq p^{\frac{d-1}{2}} \frac{|E|^k}{p^{k-1}} \left( \frac{3^{k+1} - 3}{2} \right) \quad (47)$$

We have finished proving Theorem 1.4 in both cases.

□

## 4. APPLICATION TO PINNED DISTANCE SETS

**Definition 4.1** (Pinned Distance Sets). *For  $x \in E \subseteq \mathbb{Z}_p^d$ , let*

$$\Delta_x(E) = \{ \|x - y\| : y \in E \} \quad (48)$$

*We call this the distance set of  $E$  pinned at  $x$ .*

Our goal is to find a lower bound for  $|\Delta_x(E)|$  via estimating certain sums over  $\mathbb{Z}_p^d$ . Consider the quantity

$$\nu_x(t) := |\{y \in E : \|x - y\| = t\}| = \sum_y E(y) S_t(x - y) \quad (49)$$

Note that for every  $y \in E$ , there is precisely one  $t$  (namely  $t = \|x - y\|$ ) for which  $y \in \{y \in E : \|x - y\| = t\}$ . This trivial observation, along with an application of Cauchy-Schwarz, allows us to write

$$|E|^2 = \left( \sum_t \nu_x(t) \right)^2 \leq |\Delta_x(E)| \sum_t \nu_x(t)^2 \quad (50)$$

Therefore, bounding  $\Delta_x(t)$  below can be done by bounding  $\sum_t \nu_x(t)^2$  above. Rather than doing this for a fixed  $x$ , we will look at what happens when we sum over  $x$ .

$$\begin{aligned} \sum_{x,t} \nu_x(t)^2 &= \sum_t \sum_{x,y,z} E(x)E(y)E(z)S_t(x-y)S_t(x-z) \\ &\leq \sum_t \Lambda_t^2(E) \leq p \left( \frac{|E|^3}{p^2} + 40p^{\frac{d-1}{2}} \frac{|E|^2}{p} \right) \\ &= \frac{|E|^3}{p} + 40p^{\frac{d-1}{2}} |E|^2 \end{aligned} \quad (51)$$

By Chebyshev's inequality, the number of  $x \in E$  such that  $\sum_t \nu_x(t) \geq a$  is bounded above by

$$\frac{1}{a} \left( \frac{|E|^3}{p} + 40p^{\frac{d-1}{2}} |E|^2 \right) \quad (52)$$

If we take

$$a = \frac{|E|^2 \log(p)}{p} \quad (53)$$

then the number of bad  $x \in E$  is  $o(E)$ . Therefore, as a result of the preceding discussion, we obtain the following theorem.

**Theorem 4.2.** *For almost all  $x \in E$ ,*

$$|\Delta_x(E)| \geq \frac{p}{\log(p)} \quad (54)$$

Note that in our choice of  $a$ , the factor of  $\log(p)$  was rather arbitrary, and was chosen mostly for convenience. It could be replaced by any function of  $p$  which goes to infinity.

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