APPROACHES FOR THE HINGE AND DISTINCT DISTANCE PROBLEMS

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Abstract. An almost tight lower bound for the Erdős distinct distance problem was found by Guth and Katz[GK15] using a conversion into a set of 3-dimensional point-line incidence problems. A similar open problem is finding a lower bound on the number of equivalence classes of *hinges*, pairs of distances relative to a given point. We review the beginning of the Guth-Katz proof up to an incidence counting theorem, and present similar conversions into weighted incidence problems for the hinge problem, leaving specific bounds to be proven. Unfortunately, several of the controlling quantities for these latter incidence problems would, if known, imply the end-goal bound for the hinge problem. We also estimate the number of distinct distances for a class of regular lattices.

1. Introduction

For a finite set $E \subset \mathbb{R}^2$ with N elements, we define an equivalence relation $\backsim$ over pairs of points in E^2 so that $(x, y) \backsim (x', y')$ when $|x - y| = |x' - y'|$, where $|\cdot|$ denotes the Euclidean norm. Each equivalence class, represented by (x, y), corresponds to the distinct distance $r = |x - y|$, so that the set of equivalence classes is equivalent to the distance set defined by

$$\Delta(E) \coloneqq \{|x - y| : x, y \in E\}$$

The current best general lower bound known for the size of the distance set is from [GK15] and is given by $\Delta(E) \gtrsim N/\log N$.[1] On the other hand, the smallest asymptotic distance set size observed for a class of sets is $\sim N/\sqrt{\log N}$, expected for e.g square or hexagonal grids.[2]

The hinge problem, introduced in [IP18], is the problem of establishing a lower bound on the number of equivalence classes of hinges, namely triplets in E^3 for which $(x, y, z) \backsim (x', y', z')$ iff $|x - y| = |x' - y'|$ and $|x - z| = |x' - z'|$. A trivial upper bound is given by a generic set of N points, in which case all hinges are unique (up to degenerate cases like (x, x, x)), leaving $\sim N^3$ equivalence classes. We call the set of distance pairs produced by hinge equivalence classes $H(E)$.

There is a stronger conjecture which would immediately imply the hinge and distinct distance lower bounds: the Erdős pinned-distance conjecture. In its most relevant form, it claims, given pinned distance sets defined by

$$\Delta_x(E) \coloneqq \{|x - y| : y \in E\},$$

that $\max_{x\in E} |\Delta_x E| \gtrsim N/\sqrt{\log N}$. As the number of hinge equivalence classes represented by hinges with fixed x is $|\Delta_x(E)|^2$, this implies a hinge lower bound

[1] When we say $x \lesssim y$, we mean that there is a universal constant $C > 0$ for which $x \leq Cy$. Furthermore, we define $\sim$ so that $x \sim y$ iff $x \lesssim y$ and $y \lesssim x$.

[2] According to [CSS13], extremely elongated rectangular grids instead yield $\sim N$ distances.

of $|H(E)| \geq |\Delta_x(E)|^2 \gtrsim N^2/\log N$. Similarly, since $\Delta(E) \supseteq \Delta_x(E)$ for all x, it would imply that $|\Delta(E)| \gtrsim N/\sqrt{\log N}$.

We introduce the symbol $\sum\!\!\!\!\!\|$ to concisely express and distinguish dyadic sums,

$$\sum\!\!\!\!\!\|_{X=m}^{N} f(X) := \sum_{i=0}^{\lceil \log_2(N/m) \rceil} f\left(m2^i\right).$$

For later use, we define the POINT-LINE INCIDENCE FUNCTION, applied to a set of points S and set of lines L:

$$I(S,L) := |\{(s,\ell) \in S \times L : s \in \ell\}|.$$

2. Distance set examples

Example 1. The square grid is the best known asymptotic example, within a constant.[3] We pick $S \in \mathbb{N}$, let $N = S^2$, and define $E = \left\{(x,y) \in \mathbb{Z}^2 : 0 \leq x, y < S\right\}$. Note that E has significant translational and reflective symmetry; there are many isometries $\mathscr{I}$ for which $|\mathscr{I}E \cap E| \sim |E|$. The number of distinct distances is $\sim N/\sqrt{\log N}$, as will be calculated later in Section 6. This asymptotic is significantly different for other distance metrics: under ℓ_1 and ℓ_∞ distances, there are $\sim \sqrt{N}$ distinct distances, and a metric produced from a generic convex set produces $\sim N$ distinct distances, since thanks to translational symmetry, $\Delta(E) = \bigcup_{x \in \text{corners}} \Delta_x(E)$ where x is a corner point, e.g., a point $(0,0)$, $(0,S-1)$, $(S-1,0)$, or $(S-1,S-1)$.

Example 2. Another extremal example is optimized to the Euclidean distance metric, rather than to translational symmetry. Here,

$$E = \{(0,0)\} \cup \left\{\left(\cos\frac{2\pi k}{N-1}, \sin\frac{2\pi k}{N-1}\right) : k = 0 \ldots N-2\right\}$$

defines a regular $N-1$ -sided polygon with a central point $O = (0,0)$. The pinned distance set $\Delta_O(E) = \{0,1\}$, while $||\Delta_x(E)| - N/2| \leq 2$ for all other $x \in E, x \neq O$.

We can interpolate between this and the grid by creating $E^\star = \bigcup_{0 \leq x,y \leq k}(E + (x,y))$, which has k^2 points O_i for which $\sim N^\star/k^2 \sim N$ points are a distance of 1 away from O_i.

A "generic" polygon is produced by taking a generic set $F \subset \mathbb{R}/\mathbb{Z}$ for which $|F| = N-1$, and $|F+F| = (N-1)(N-2)/2$; then

$$E = \{(0,0)\} \cup \{(\cos 2\pi f, \sin 2\pi f) : f \in F\}.$$

Example 3. The Penrose tiling has high translational symmetry and does not immediately appear to have grid structure; however, there is a pentagrid construction method[4] which implies that, if E is the subset of a Penrose tiling filling a disk, then the projections of E along five specific directions have size $\sim \sqrt{|E|}$, so that E is a constant-density subset of a lattice. By an argument similar to 7, one can show that $|\Delta(E)| \sim N$.

[3] The best asymptotic class is probably the intersection of a hexagonal/triangular grid with a disk.

[4] This generalizes to seven, nine, etc. sided patterns; see also https://www.mathpages.com/home/kmath621/kmath621.htm, and [dBru81] which introduced the method.

3. Incidence formulations for the distinct distance problem

In this section, we present an minor variation on the method used in [GK15] to convert the lower bound on $\Delta(E)$ into a set of incidence problem upper bounds. Let n_r be the size of the equivalence class of distance r:

$$n_r := \left|\left\{(x,y)\in E^2 : |x-y| = r\right\}\right|$$

Then, counting all *nonzero* equivalence classes, we find $\sum_{r>0} n_r = N^2 - N$, after which by Cauchy-Schwarz we have

$$\begin{aligned} N^4 \sim \left(\sum_{r>0} n_r\right)^2 &\leq \left(\sum_{r>0} 1\right)\left(\sum_{r>0} n_r^2\right) \\ &\leq \left(|\Delta(E)| - 1\right)|Q(E)| \end{aligned}$$

where we define $Q(E)$ to be the set of all quadruples $(x, y, x', y') \in E^4$ for which $|x-y| = |x'-y'| > 0$. Rearranging,

$$|\Delta(E)| \gtrsim N^4 / |Q(E)|. \tag{3.1}$$

To prove that $|\Delta(E)| \gtrsim N/\log N$, we must show that $|Q(E)| \lesssim N^3 \log N$. (This upper bound is the tightest attainable, as shown in Section 7.) First, we split $Q(E)$ into two disjoint sets, $Q_r(E)$ and $Q_t(E)$, namely the *rotated* and *translated* quadruples. The subset $Q_t(E)$ is defined as the set of quadruples $(x, y, x', y') \in E^4$ for which the vector identity $x - y = x' - y'$ holds. Since each such tuple can be identified with $(x, y, x') \in E^3$, as $y' = x' + y - x$, it follows

$$|Q_t(E)| \leq N^3. \tag{3.2}$$

It remains to prove that $|Q_r(E)| \lesssim N^3 \log N$. To do this, let G be the set of positively oriented *non-translational* rigid motions of the plane, and establish a bijection between $Q_r(E)$ and the set $\mathfrak{Q}$, defined as the set of all (x, y, g), where $g \in G$, $x, y \in g^{-1}E \cap E$, and $x \neq y$. By Prop 2.3 of [GK15], for a given tuple $(x, y, x', y') \in E^4$ where $|x-y| = |x'-y'| > 0$, there is a unique g for which $g(x) = x'$ and $g(y) = y'$. Since $g(x) \in E$ and $g(y) \in E$, it follows $x \in g^{-1}E$ and $y \in g^{-1}E$, so that the tuple $(x, y, g) \in \mathfrak{Q}$. On the other hand, given $(x, y, g) \in \mathfrak{Q}$, we define $x' = g(x)$ and $y' = g(y)$; these exist since $x, y \in g^{-1}E$, and since g is an isometry, $|x'-y'| = |x-y|$, which is > 0 since $x \neq y$. Thus $(x, y, x', y') \in Q_r(E)$. Correspondingly, we have

$$\begin{aligned} |Q_r(E)| = |\mathfrak{Q}| &= \sum_{x,y,g\in\mathfrak{Q}} 1 \\ &= \sum_{g\in G} \sum_{x\neq y\in g^{-1}E\cap E} 1 \\ &\leq \sum_{g\in G} \binom{\left|g^{-1}E\cap E\right|}{2} \end{aligned}$$

We partition G into disjoint sets $G_{=k}$ defined so that for all $g \in G_{=k}$, $\left|g^{-1}E \cap E\right| = k$. We also define $G_k = \bigcup_{j \geq k} G_{=k}$, so that by dyadic decomposition

$$|Q_r(E)| \leq \sum_{k=2}^{N} |G_{=k}| \binom{k}{2} \leq \sum_{k=2}^{N} k^2 |G_{=k}| \tag{3.3}$$
$$\leq \sum_{K=2}^{N} \sum_{k=K}^{2K-1} k^2 |G_{=k}|$$
$$\leq 4 \sum_{K=2}^{N} K^2 |G_K \setminus G_{2K}|$$

The set $G_K \setminus G_{2K}$ consists of all g for which $\left|g^{-1}E \cap E\right| \in [K, 2K) \cap \mathbb{N}$; $G_{N+1} = \emptyset$, as $\left|g^{-1}E \cap E\right| \leq |E| = N$.

We now construct an incidence problem which will provide an upper bound on $G_K \setminus G_{2K}$. To a given rigid motion $g_{x,y,\theta} \in G$ which fixes the point (x, y) and rotates by plane by angle θ around that point, we associate the point $\left(x, y, \cot \frac{\theta}{2}\right) \in \mathbb{R}^3$.[5] As g is non-translational, $\theta \not\equiv 0$, so that the point is well defined. Next, we define the set of lines $\mathcal{L} = \{L_{xy} : x, y \in E, x \neq y\}$, where each individual line $L_{pq} \subset \mathbb{R}^3$ contains all $g \in G$ which map p to q. This line, as proven by Prop 2.7 of [GK15], can be parameterized by $t \in \mathbb{R}$ as

$$\frac{p+q}{2} + t\left(\hat{z} \times \frac{p+q}{2} + \hat{z}\right) \tag{3.4}$$

where we consider p and q as points on the $z = 0$ plane in $\mathbb{R}^3$, and $\hat{z}$ the unit $(0, 0, 1)$ vector, and $\times$ the (standard, right-handed) cross product.[6] We claim that $\left|g^{-1}E \cap E\right|$ equals the total number of lines in $\mathcal{L}$ which pass through g (interpreted as a point of $\mathbb{R}^3$). Specifically, for each $x \in g^{-1}E \cap E$, we let $x' = g(x) \in E$, which happens if and only if the line $L_{xx'} \ni g$ (by definition of $L_{xx'}$).

With Proposition 2.8 of [GK15], we can ensure that there are $\leq N$ lines of $\mathcal{L}$ inside a given plane, and $\lesssim N$ such lines contained by any regulus (doubly-ruled surface[7]. Using the bound on the maximum number of lines per regulus, Guth and Katz proved their Theorem 2.10, which implies that there are $\lesssim N^3$ possible intersection points between pairs of lines in $\mathcal{L}$. Splitting off the case $k = 2$, we can write Eq. 3.3 as

$$|Q_r(E)| \lesssim N^3 + \sum_{K=3}^{N} K^2 I(G_K \setminus G_{2K}, \mathcal{L}) \tag{3.5}$$

where the set $G_K \setminus G_{2K}$ is interpreted as a subset of $\mathbb{R}^3$, and I is the point-line incidence counting function. They also find, via a lengthy argument based on polynomial partitioning,

[5] It may be clearest to say that we assign coordinates to the set G; then it is valid to treat g as both a point in $\mathbb{R}^3$ and a non-translational positively oriented rigid motion of $\mathbb{R}^2$.

[6] Evaluating gives $\hat{z} \times (p+q)/2 + \hat{z} = ((q_y - p_y)/2, (p_x - q_x)/2, 1)$.

[7] Specifically, any linear transformation of a hyperbolic paraboloid/parabolic hyperboloid$\{xy = z\}$, or hyperboloid of one sheet $\left\{x^2 + y^2 - z^2 = 1\right\}$; reguli can be defined using three skew lines.

Proposition. *4.22 of [GK15]: Let $k \geq 3$, and $\mathcal{L}$ be a set of L lines in $\mathbb{R}^3$ with $\leq B$ lines in any plane. The set of points in $\mathbb{R}^3$ which meet between k and $2k$ lines of $\mathcal{L}$ is $G_{[k,2k)}$, whose size S can be bounded above by*

$$S \lesssim L^{3/2}k^{-2} + LBk^{-3} + Lk^{-1}$$

Since each intersection in $G_{[k,2k)}$ has line-intersection-richness in $[k, 2k)$, the total number of incidences $I(G_k \setminus G_{2k}, \mathcal{L}) \leq 2kS$.

To apply Prop 4.22 of [GK15], we need an upper bound on the total number of lines in a given plane. We introduce

Lemma 4. *Let H be the horizontal ($z = 0$) plane. For each non-horizontal plane P in $\mathbb{R}^3$, let the line $\ell = P \cap H$, and let $\hat{\ell}$ be a direction vector chosen along ℓ. Then let $1/s$ be its slope relative to the horizontal plane (when seen as a graph $w \mapsto z$ on the vertical plane which has coordinate vectors $\hat{w} = \hat{\ell} \times \hat{z}$ and $\hat{z}$), in which case we can define an associated glide reflection $\mathscr{R}_{\ell,s}$ which reflects points across ℓ and shifts them by $2s$ in the direction of $\hat{\ell}$. (When $s = 0$ the plane is vertical.) Then the line $L_{pq} \in P$ iff $\mathscr{R}_{\ell,s}(p) = q$.*

Proof. We can establish coordinates in the (x, y)-plane containing E so that ℓ is the line $\{(x, y, z) : x = z = 0\}$, and $\hat{\ell} = \hat{y}$. Then $\hat{w} = \hat{x}$. The plane P is the set $\{(x, y, z) : z = x/s\}$, and points p, q have coordinates $(p_x, p_y, 0)$ and $(q_x, q_y, 0)$. Then

$$L_{pq} = \left(\frac{p_x + q_x}{2} + t\frac{q_y - p_y}{2}, \frac{p_y + q_y}{2} + t\frac{p_x - q_x}{2}, t\right)$$

which fulfills $sz = x$ for all t iff

$$\left(\frac{p_x + p_y}{2} + t\frac{q_y - p_y}{2}\right) = st$$

iff $p_x = -q_x$ and $q_y = 2s + p_y$, which is the case iff $q = \mathscr{R}_{\ell,s}(p)$. □

As a given plane can contain no more than N lines (since points in E have unique image and pre-image under $\mathscr{R}_{\ell,s}$), we can set $B = N$. Since $|\mathcal{L}| = L \leq N^2$, combining Prop 4.22 of [GK15] with Eq. 3.5 produces

$$\begin{aligned}|Q_r(E)| &\lesssim N^3 + \sum_{K=3}^{N} K^2 \left[N^3/K^2 + N^3/K^3 + N^2/K\right] \\ &\lesssim N^3 + \sum_{K=3}^{N} \left[N^3 + N^3/K + N^2K\right] \\ &\lesssim N^3 + N^3 \log N + N^3 + N^3 \lesssim N^3 \log N \qquad (3.6)\end{aligned}$$

Combining Eq. 3.6 with Eq. 3.2, it follows $|Q(E)| \lesssim N^3 \log N$, after which by Eq. 3.1 we have $|\Delta(E)| \gtrsim N/\log N$.

4. Weighted incidence formulations of the hinge problem

It is useful to define the circular weight functions for the set E,

$$\omega_s(x) := |\{y \in E : |x - y| = s > 0\}| \qquad (4.1)$$

These functions have several convenient properties: most importantly, $\omega_s(x) = 0$ when $s \notin \Delta_x(E)$, so that we can be lazy with quantifiers. We have the identity

$\sum_s \omega_s(x) = N-1$, which counts points in $E\backslash\{x\}$ grouped by distance from x. Then $\sum_x \sum_s \omega_s(x) = N^2 - N$. There are several upper bounds: trivially, $\omega_s(x) \leq N-1$, which is sharp by Example 2. By substituting Eq. 4.1, we find that $\sum_x \omega_s(x)$ gives the size of the equivalence class corresponding to distance s; thus by the argument from [GK15],

$$|Q(E)| = \sum_s \left(\sum_x \omega_s(x)\right)^2 \lesssim N^3 \log N. \tag{4.2}$$

This expression is the strongest nontrivial bound of its type which has been proven so far.

Conjecture 5. *Assuming maximum values are obtained for the grid example, it is not unreasonable to conjecture*

$$\sum_s \sum_x \omega_s(x)^2 \lesssim N^2 \log N$$

For sufficiently large exponents, the assumption that the maximum is obtained for spatially uniform $\omega_s(x)$ might break down, depending on how common very "concentrated" points, like the center of a polygon in Example 2, for which $\omega_1(O) = N-1$, may be.

4.1. **Flipping the hinge problem.**

As mentioned in the introduction, $H(E)$ is the set of equivalence classes of hinges. We provide notation to count the number of elements in each equivalence class,

$$n_{r,s} = |\{(x,y,z) : |x-y| = r > 0 \quad ; \quad |x-z| = s > 0\}|$$

and identify each class with a pair $(r,s) \in \mathbb{R}^2$. Then by Cauchy-Schwarz,

$$N^6 \sim \left(\sum_{r,s\in H(E)} n_{r,s}\right)^2 \leq \left(\sum_{r,s\in H(E)} 1\right)\left(\sum_{r,s\in H(E)} n_{r,s}^2\right)$$
$$\leq |H(E)|\,|U(E)|$$
$$|H(E)| \gtrsim N^6 / |U(E)| \tag{4.3}$$

where $U(E)$ is the set of tuples $(x,y,z,x',y',z') \in E^6$ so that $|x-y| = |x'-y'| > 0$ and $|x-z| = |x'-z'| > 0$. Using the circular weight functions, we count the number of (r,s) hinges rooted at x, and sum:

$$n_{r,s} = \sum_{x\in E} \omega_r(x)\,\omega_s(x)$$

from which follows

$$|U(E)| = \sum_{r,s} \left(\sum_x \omega_r(x)\,\omega_s(x)\right)^2 \tag{4.4}$$

If we assume Conjecture 5, then we get by AM-GM

$$\begin{aligned}
|U(E)| &= \sum_{r,s}\left(\sum_{x}\omega_r(x)\,\omega_s(x)\right)^2 \\
&= \sum_{r,s}\sum_{x,x'}\omega_r(x)\,\omega_r(x')\,\omega_s(x)\,\omega_s(x') \\
&\le \frac{1}{4}\sum_{x,x'}\sum_{r,s}\left(\omega_r(x)^2+\omega_r(x')^2\right)\left(\omega_s(x)^2+\omega_s(x')^2\right) \\
&\le \frac{1}{2}N\sum_{x}\sum_{r,s}\omega_r(x)^2\,\omega_s(x)^2+\frac{1}{2}\sum_{x}\sum_{x'}\sum_{r,s}\omega_r(x)^2\,\omega_s(x')^2 \\
&\le \frac{1}{2}N\sum_{x}\left(\sum_{t}\omega_t(x)\right)^2+\frac{1}{2}\left(\sum_{t}\sum_{x}\omega_t(x)^2\right)^2 \\
&\lesssim N^4+\left(N^2\log N\right)^2=N^4(\log N)^2
\end{aligned}$$

which implies that if we must assume Conjecture 5 then our argument is either circular or recursive[8].

4.2. **Conversion to decision point before weighted incidence problems.** Next, we convert $|U(E)|$ into a weighted computation over $Q(E)$. After all,

$$\begin{aligned}
|U(E)| &= \sum_{(x,y,z,x',y',z')\in U(E)} 1 \\
&= \sum_{(x,y,x',y')\in Q(E)}\ \sum_{|x-z|=|x'-z'|} 1 \\
&= \sum_{(x,y,x',y')\in Q(E)}\ \sum_{r\in\Delta}\omega_r(x)\,\omega_r(x') \\
&= \sum_{r\in\Delta}\ \sum_{(x,y,x',y')\in Q(E)}\omega_r(x)\,\omega_r(x')
\end{aligned}$$

As with the distinct distance problem, we first partition $Q(E)$ into disjoint subsets, the translational part $Q_t(E)$ and the rotational part $Q_r(E)$, as defined in Section 3. We split $U(E)$ into $U_t(E)$ and $U_r(E)$ in accordance with the split of $Q(E)$. The former is easy to bound, since the last component of (x,y,x',y') in $Q_t(E)$ is defined by the first three components, after which by Eq. 4.2

$$\begin{aligned}
|U_t(E)| &= \sum_{r\in\Delta}\ \sum_{x,y,x',y'\in Q_t(E)}\omega_r(x)\,\omega_r(x') \\
&\le \sum_{r\in\Delta}\ \sum_{(x,y,x')\in E^3}\omega_r(x)\,\omega_r(x') \\
&\le N\sum_{r\in\Delta}\left(\sum_{x}\omega_r(x)\right)^2 \lesssim N^4\log N \qquad (4.5)
\end{aligned}$$

[8]For example, if $f(N)\le g(N)+\alpha f(N)$ then $f(N)\lesssim g(N)$ if $\alpha<1$; but if $\alpha>1$, the inequality is useless.

The second component, $Q_r(E)$, is in bijection with $\mathfrak{Q}$, the set of (x, y, g) where $g \in G$, the non-translational subset of the group of positively oriented rigid motions, and $x, y \in g^{-1}E \cap E$. Then we have

$$\begin{aligned} |U_r(E)| &= \sum_{r\in\Delta} \sum_{x,y,g\in\mathfrak{Q}} \omega_r(x)\,\omega_r(g(x)) \\ &\leq \sum_{r\in\Delta} \sum_{g\in G} \sum_{x,y\in g^{-1}E\cap E} \omega_r(x)\,\omega_r(g(x)) \end{aligned}$$

At this point, it is convenient to define the weights

$$m_r(g) \coloneqq \sum_{x\in g^{-1}E\cap E} \omega_r(x)\,\omega_r(g(x)),$$

and $m(g) = \sum_{r\in\Delta} m_r(g)$. As before, dyadic decomposition of G by the richness $k = \left|g^{-1}E\cap E\right|$ of a given transformation yields

$$\begin{aligned} |U_r(E)| &\leq \sum_{r\in\Delta} \sum_{g\in G} \left|g^{-1}E\cap E\right| m_r(g) \\ &\leq \sum_{r\in\Delta} \sum_{K=2}^{N} K \sum_{g\in G_K\setminus G_{2K}} m_r(g) \\ &= \sum_{K=2}^{N} K \sum_{g\in G_K\setminus G_{2K}} m(g) \end{aligned} \tag{4.6}$$

As in Section 3, we can identify each $g \in G$ with a point in $\mathbb{R}^3$, and note that $\left|g^{-1}E\cap E\right|$ is the number of lines in the set $\mathcal{L} = \{L_{pq} : p, q \in E\}$ which intersect with g. Furthermore, since each line $L_{xx'} \ni g$ has the property that $x' = g(x)$, and $x' \in g^{-1}E \cap E$, it follows that if we associate a weight $m_r(L_{xx'}) = \omega_r(x)\,\omega_r(x')$, the sum of the weights over all the lines intersecting g gives $m_r(g)$. As the expression up to now is linear, we could alternatively associate weights $m(L_{xx'}) = \sum_r \omega_r(x)\,\omega_r(x')$, so that the total over all lines intersecting g is $m(g)$.

4.3. **Expected results conditional on the best possible generic lemma.** Assume that there is a lemma similar to Prop 4.22 of [GK15], which is agnostic to the structure of the line weights, except for their average value, and the maximum over all planes in $\mathbb{R}^3$ of the average weight in a given line. In total, we have $L = |\mathcal{L}| = N^2$ lines, and a maximum of $B = N$ lines per plane; and average line weight μ_r, and maximum average line weight υ_r in a plane. Based on Eq. 3.6, and assuming that all line weights are uniform, we would expect linearity in the line weights:[9]

$$\begin{aligned} |U_r(E)| &\lesssim \sum_{r\in\Delta} \left[L^{3/2}\mu_r + L^{3/2}\mu_r \log N + LB\mu_r^{1/2}\nu_r^{1/2} + L\mu_r N\right] \\ &\lesssim L^{3/2}\mu + L^{3/2}\mu\log N + LB\left(\sum_r \mu_r^{1/2}\nu_r^{1/2}\right) + L\mu N \end{aligned} \tag{4.7}$$

[9]The exact $\mu_r^{1/2}\nu_r^{1/2}$ tradeoff may differ, but the term almost certainly depends on ν_r, in which case the analysis of this section analysis is unchanged.

To compute $\mu = \sum_{r\in\Delta} \mu_r$, we apply Eq. 4.2

$$\mu = \frac{1}{N^2} \sum_{L_{xx'}} m\left(L_{xx'}\right) = \frac{1}{N^2} \sum_{r\in\Delta} \sum_{x,x'} \omega_r\left(x\right)\omega_r\left(x\right) \lesssim N \log N$$

More difficult is the computation of $\sum_r \mu_r^{1/2} \nu_r^{1/2}$. Thanks to Lemma 4, we know that

$$\nu_r = \frac{1}{N} \max_{\ell,s} \sum_{x\in\mathscr{R}_{\ell,s}^{-1}E\cap E} \omega_r\left(x\right)\omega_r\left(\mathscr{R}_{\ell,s}\left(x\right)\right)$$

for which $N^{-1}\sum_{x\in E}\omega_r\left(x\right)^2$ is a trivial upper bound. For sets with perfect reflection symmetry across a given line ℓ, this upper bound is tight:

$$\begin{aligned}\sum_{x\in\mathscr{R}_{\ell,0}^{-1}E\cap E} \omega_r\left(x\right)\omega_r\left(\mathscr{R}_{\ell,0}\left(x\right)\right) &= \sum_{x\in E} \omega_r\left(x\right)\omega_r\left(\mathscr{R}_{\ell,0}\left(x\right)\right) \\ &= \sum_{x\in E} \omega_r\left(x\right)^2\end{aligned}$$

since a reflection $\mathscr{R}_{\ell,0}$ is an isometry, and hence preserves the number of points within a given distance of x. (Note that for $s \neq 0$, E cannot be invariant under $\mathscr{R}_{\ell,s}$, since the shift operation increases the y coordinates of all points when we set coordinates so that y is aligned with ℓ. It may be that $\left|\mathscr{R}_{\ell,s}^{-1}E\cap E\right| \sim |E|$, and also $\omega_r\left(x\right) \sim \omega_r\left(\mathscr{R}_{\ell,s}\left(x\right)\right)$, in which case $\mu_r \gtrsim \sum_{x\in E}\omega_r\left(x\right)^2$.)

If we were to take $\nu_r = \frac{1}{N}\sum_{x\in E}\omega_r\left(x\right)^2$, then

$$\sum_r \mu_r^{1/2}\nu_r^{1/2} = \frac{1}{N^{3/2}} \sum_r \left(\sum_x \omega_r\left(x\right)^2\right)^{1/2} \left(\sum_x \omega_r\left(x\right)\right) \tag{4.8}$$

which is slightly more concentrated in r than Eq. 4.2, and slightly less concentrated in r than Conjecture 5.[10] We could alternatively move the $\sum_r$ inside the weight expressions on 4.7; but the end result is equivalent to applying Cauchy-Schwarz on $\sum_r \left(\mu_r\nu_r\right)^{1/2}$.

It may be that such an ideal generic lemma requires constraints on the ℓ_p norms of the weights; however, for $p \geq 2$, a bound in terms of N would imply Conjecture 5 by Hölder's inequality. Note also that Eq. 4.7 uses the lowest reasonable exponents on the line weights – the bound is false if they are less than linear – and it may be easier to prove a result for which $L^{3/2}\mu_r$ is replaced by $L^{3/2}\mu_r^{3/2}N^{-1/2}$; but then it makes a difference to the final result whether or not the incidence problems are computed with the line weights summed before, after, or interpolating between the two. However, with the $L^{3/2}\mu_r^{3/2}N^{-1/2}$ example, if one computes incidences before summation, then the first term bounding $|U_r\left(E\right)|$ becomes $N\sum_r\left(\sum_x \omega_r\left(x\right)\right)^3$ which is stronger than Conjecture 5.

Of course, including distribution information, such as the distribution of total line weights within planes, might provide a more complicated but tractable upper bound.

[10]Applying Muirhead, AM-GM, or Cauchy-Schwarz, we find Conjecture 5 implies Eq. 4.8 which implies Eq. 4.2.

One of the obstacles to finding a "best possible generic lemma" is that the weights $m(g)$ of an intersection are not always strictly greater than than the intersection multiplicity; it may often be that $m_r(\ell) = 0$, or $m(\ell) < N$.

4.4. **Double partitioning fails without further constraints.** There is a second way to compute the expressions at the end of 4.2: we can separately partition over points, by the number of lines they meet, and over lines, by their weight. We introduce notation for the second case: $\mathcal{L}_{[J,2J)}$ is the set of all lines ℓ in $\mathcal{L}$ for which $m(\ell) \in [J, 2J)$.

Specifically, Eq. 4.6 becomes

$$\begin{aligned}
|U_r(E)| &\lesssim \sum_{K=2}^{N} K \sum_{g \in G_K \setminus G_{2K}} \sum_{L_{xx'} \ni g} m(L_{xx'}) \\
&\lesssim \sum_{K=2}^{N} K \sum_{J=1}^{N^2} \sum_{g \in G_K \setminus G_{2K}} \sum_{\substack{L_{xx'} \ni g \\ L_{xx'} \in \mathcal{L}_{[J,2J)}}} m(L_{xx'}) \\
&\lesssim \sum_{K=2}^{N} \sum_{J=1}^{N^2} KJ \cdot I\big(G_{[K,2K)}, \mathcal{L}_{[J,2J)}\big)
\end{aligned}$$

where I is the point-line incidence counting function. The following theorem is particularly useful:

Theorem. *12.1 of [Gut16]. Let $\mathcal{S}$ be a set of S points and $\mathfrak{L}$ a set of L lines in $\mathbb{R}^3$. Suppose that there are at most B lines of $\mathfrak{L}$ in any plane, and that $B \geq L^{1/2}$. Then the number of incidences is bounded as follows:*

$$I(\mathcal{S}, \mathfrak{L}) \lesssim S^{1/2} L^{3/4} + B^{1/3} L^{1/3} S^{2/3} + L + S$$

By this theorem, with the total number of lines $L_J = \left|\mathcal{L}_{[J,2J)}\right|$, the total number of points $S_K \lesssim N^3/K^2$, and maximum number of lines in a plane, $B \leq N$, we get

$$\begin{aligned}
|U_r(E)| &\lesssim \sum_{K=2}^{N} \sum_{J=1}^{N^2} KJ \left[S_K^{1/2} L_J^{3/4} + B^{1/3} L_J^{1/3} S_K^{2/3} + L_J + S_K\right] \\
&\lesssim \sum_{K=2}^{N} \sum_{J=1}^{N^2} KJ \left[N^{3/2} K^{-1} \left|\mathcal{L}_{[J,2J)}\right|^{3/4} + N^{7/3} K^{-4/3} \left|\mathcal{L}_{[J,2J)}\right|^{1/3}\right] \\
&\quad + \sum_{K=2}^{N} \sum_{J=1}^{N^2} KJ \left[\left|\mathcal{L}_{[J,2J)}\right| + N^3/K^2\right] \\
&\lesssim N^{3/2} \log N \left(\sum_{J=1}^{N^2} J \left|\mathcal{L}_{[J,2J)}\right|^{3/4}\right) + N^{7/3} \sum_{J=1}^{N^2} J \left|\mathcal{L}_{[J,2J)}\right|^{1/3} \\
&\quad + N \sum_{J=1}^{N^2} J \left|\mathcal{L}_{[J,2J)}\right| + N^5
\end{aligned}$$

By Chebyshev's inequality, we find $J\left|\mathcal{L}_{[J,2J)}\right| \lesssim \sum_{\ell\in\mathcal{L}} m(\ell)$, which implies $\left|\mathcal{L}_{[J,2J)}\right| \lesssim \left(N^3 \log N\right)/J$; substituting this in yields:

$$\begin{aligned}|U_r(E)| &\lesssim N^{15/4}(\log N)^{7/4}\left(\sum_{J=1}^{N^2} J^{1/4}\right) + N^{10/3}(\log N)^{1/3}\sum_{J=1}^{N^2} J^{2/3} \\ &\quad + N^4(\log N)^2 + N^5 \\ &\lesssim N^4\left(N^{1/4}(\log N)^{7/4}\right) + N^4\left(N^{2/3}(\log N)^{1/3}\right) + N^4(\log N)^2 + N^5\end{aligned}$$

While the first three terms might be reducible with Chebyshev bounds on e.g. $J^{4/3}\left|\mathcal{L}_{[J,2J)}\right|$ and $J^3\left|\mathcal{L}_{[J,2J)}\right|$, the right hand side expressions $\sum_{\ell\in\mathcal{L}} m(\ell)^{4/3}$ and $\sum_{\ell\in\mathcal{L}} m(\ell)^3$ are difficult to control. The trailing N^5 term is an artifact of the specific incidence expression used: the $+S$ term of Theorem 12.1 of [Gut16] is maximal when all S_K points are placed on unique lines in L_J, which is unlikely since there are, in total, more points than lines.

5. Line-based incidence formulations of the hinge problem

In this section, we present another approach to finding an upper bound for $|U(E)|$ (which was defined in Section 4.1). In order to obtain an incidence problem for lines in $\mathbb{R}^3$, rather than lines in $\mathbb{R}P^3$, we partition $U(E)$ into three (not necessarily disjoint) components, U_y, U_z, and $U_\star$. First, we note that for each tuple (x, y, z, x', y', z'), there are unique positively oriented rigid motions g_y and g_z for which $g_y x = g_z x = x'$, $g_y y = y'$, and $g_z z = z'$. First, if g_y is a translation, then the tuple is in U_y; if g_z is a translation, we put it in U_z; and $U_\star = U \setminus (U_y \cup U_z)$. For sets U_z, we fall back to the weighted incidence argument used to express U_z as a sum over Q_t, after which applying Eq. 4.5 gives $|U_z| \lesssim N^4 \log N$. As $|U_z| = |U_y|$ by y-z exchange symmetry, $|U_y| \lesssim N^4 \log N$ as well.

To control $|U_\star|$, we establish an isomorphism between $U_\star$ and the set $\mathfrak{U}$ of triplets of transformation-representing lines $L_{x,x'}$, $L_{y,y'}$, $L_{z,z'}$ in $\mathbb{R}^3$, defined as per Section 3, with the additional constraint $L_{x,x'}$ intersects both $L_{y,y'}$ and $L_{z,z'}$. Note, just as y and z, or y' and z', need not be distinct, neither do $L_{y,y'}$ and $L_{z,z'}$. However, $x \neq y$ and $x \neq z$, so $L_{x,x'} \neq L_{y,y'}$ and $L_{x,x'} \neq L_{z,z'}$. That $U_\star$ maps injectively into $\mathfrak{U}$ is follows from the formula for a line, Eq. 3.4; for the reverse direction, note that $g_y = L_{y,y'} \cap L_{x,x'}$ and $g_z = L_{z,z'} \cap L_{z,z'}$ are precisely the unique positively oriented rigid motions between (x, y) and (x', y'), and (x, z) and (x', z'). Since $L_{z,z'}$ and $L_{y,y'}$ are independent of each other, with $\mathcal{L}$ the set of all lines L_{pq} for $p, q \in E$, $p \neq q$,

$$|U_\star| = \sum_{L_{xx'}\in\mathcal{L}} m_{\mathcal{L}}(L_{xx'})^2 \tag{5.1}$$

where we define the weight $m_{\mathcal{L}}(L_{xx'})$ as the total number of lines $\mathcal{L} \setminus \{L_{xx'}\}$ which intersect $L_{xx'}$. As the uniform bound is not particularly useful, we partition $\mathcal{L}$ into disjoint sets $\mathcal{L}_{=r,\mathcal{L}}$, so that $L_{xx'} \in \mathcal{L}_{=r,\mathcal{L}}$ iff $r = m_{\mathcal{L}}(L_{xx'})$, and then define $\mathcal{L}_{r,\mathcal{L}} \coloneqq \bigcup_{s=r}^{N^2} \mathcal{L}_{=s,\mathcal{L}}$. (A convenient upper bound for the weight $m_{\mathcal{L}}(L_{xx'})$ is N^2, since one can obtain $m_{\mathcal{L}}(L_{xx'}) \sim N^2$ for a generic polygon as described in Example 2, because there are $\sim N^2$ distinct rotation angles corresponding to the maps which fix the polygon center.)

Then straightforward dyadic decomposition gives

$$\begin{aligned}
|U_\star| &= \sum_{k=1}^{N^2} \sum_{L_{xx'} \in \mathcal{L}_{=k,\mathcal{L}}} m_{\mathcal{L}} (L_{xx'})^2 \\
&\le \sum_{K=1}^{N^2} \sum_{L_{xx'} \in \mathcal{L}_{K,\mathcal{L}} \setminus \mathcal{L}_{2K,\mathcal{L}}} m_{\mathcal{L}} (L_{xx'})^2 \\
&\le 4 \sum_{K=1}^{N^2} K^2 |\mathcal{L}_{K,\mathcal{L}} \setminus \mathcal{L}_{2K,\mathcal{L}}|
\end{aligned}$$

in which case an upper bound on the number of lines in $\mathcal{L}$ with line-richness between K and $2K$ would be required. As Example 10 shows, for the square grid it is possible that $|\mathcal{L}_{K,\mathcal{L}} \setminus \mathcal{L}_{2K,\mathcal{L}}| \sim N^2$ for $K \lesssim N \log N$; the difficulty is in establishing the upper bound on $|\mathcal{L}_{K,\mathcal{L}} \setminus \mathcal{L}_{2K,\mathcal{L}}|$ for lines with line-richness $\ge N$. Of course, thanks to Proposition 2.8 of [GK15], there are at $\lesssim N$ lines per regulus, and hence via their Theorem 2.10, $\lesssim N^3$ total intersection points, so we can rule out the case that all N^2 lines each intersect all N^2 other lines.

One might expect that a double partitioning argument, as in Section 4.4, would be helpful; however, defining $m(g) := |\{L_{xy} \in \mathcal{L} : g \in L_{xy}\}|$ as the line-richness of point g, we get

$$|U_\star| = \sum_{k=1}^{N^2} \sum_{L_{xx'} \in \mathcal{L}_{=k,\mathcal{L}}} \left(\sum_{g \in L_{xx'}} m(g) \right)^2 .$$

Applying Cauchy-Schwarz over the squared sum produces an expression amenable to such partitioning; however, for the square grid, for a constant fraction of lines there are $\sim N$ intersections per line, for which $\left(\sum_{g\in\ell} m(g)\right)^2 \sim (N \log N)^2$ but $N \sum_{g\in\ell} m(g)^2 \sim N^3$; the resulting upper bound produced by Cauchy-Schwarz is then at least N^5.

6. Counting distances for sufficiently simple lattices

As noted by [CSS13], the method of finding a lower bound for $|\Delta(E)|$ by computing an upper bound for $|Q(E)|$ produces a lower bound which is off by a factor of $\sqrt{\log N}$ in the square grid case, in large part because the equivalence class sizes n_r are non-uniform.

An alternative approach to proving a tight lower bound on $|\Delta(E)|$ is to prove the bound for a class of sets $\mathcal{E}$, and then show, using symmetry or other constraints, that if $|\Delta(E)|$ is below some threshold then $E \subset \mathcal{E}$. For example, one might hypothesize that if $|\Delta(E)| \lesssim N$, then either E is almost a polygon (i.e., there is a polygon F so that $|(E \setminus F) \cup (F \setminus E)| \ll |E|$), or E is a constant density subset of a finite lattice Ξ. ("constant density subset", just like "almost", is a heuristic; perhaps $|\Xi| / |E| \le \log |E|$.)

Definition 6. A finite lattice Ξ is a set of points describable as $\pi_1^{-1} A \cap \pi_2^{-1} B$ where π_1 and π_2 are distinct projections. Then $|\Xi| = |A| |B| = N$. It is convenient to let $\theta \in (0, \pi)$ be the smaller angle between π_1 and π_2.

As the points in Ξ can be identified with their projections, we can use the law of cosines to compute distances, with $c = 2\cos\theta$.

$$\Delta(\Xi) = \left\{ \left| \sqrt{a^2 + b^2 + cab} : a \in A - A, b \in B - B \right| \right\}$$

The notation $A - A := \{a - a' : a, a' \in A\}$; since $A - A$ is symmetric around 0, $\Delta(E)$ is unchanged if we replace c with $-c$ in the above expression. For lattices Ξ where A and B have high translational symmetry, that is, $|A| \sim |A - A|$, and $|B| \sim |B - B|$, it follows immediately that $|\Delta(\Xi)| \le |A - A| |B - B| \sim |A| |B| = N$.

We now address the specific cases where A and B are both arithmetic progressions. We also define $U_k := \mathbb{Z} \cap [-k, k]$.

Lemma 7. *Let $A = \alpha U_{n_A}$ and $B = \beta U_{n_B}$; let Ξ be the finite lattice determined by the intersection of their inverse projections at cross angle θ. If $(\beta/\alpha)^2 \notin \mathbb{Q}$ or $(\beta/\alpha)\cos\theta \notin \mathbb{Q}$, then*

$$2 \max_{p \in \Xi} |\Delta_p(\Xi)| \ge |\Delta(\Xi)| \ge N/4$$

Proof. Thanks to scale invariance, we can assume $\alpha = 1$. Then $A - A = U_{2n_A}$ and $B - B = \beta U_{2n_B}$. For $x \in A - A$, and $y = \beta z \in B - B$, the distance corresponding to $(x, z) \in U_{2n_A} \times U_{2n_B}$ is $f(x, z) = x^2 + \beta^2 y^2 + c\beta xy$, for $c = 2\cos\theta$. Furthermore, when points $p, q \in \Xi$ have coordinates $(p_x, p_z) = (-n_A, -n_B)$ and $(q_x, q_z) = (n_A, -n_B)$, then $\Delta(\Xi) = \Delta_p(\Xi) \cup \Delta_q(\Xi)$. There are three cases:

Case 1. $\beta^2 \in \mathbb{R} \setminus \mathbb{Q}$, $c\beta \in \mathbb{Q}$.

Proof. Assume that there are x, z, x', z' for which $f(x, z) = f(x', z')$. If $z \ne z'$, then from $x^2 + \beta^2 z^2 + c\beta xz = x'^2 + \beta^2 z'^2 + c\beta x'z'$, we derive $\left(x^2 - x'^2 + c\beta xz - c\beta x'z'\right) / \left(z'^2 - z^2\right) = \beta^2$, so $\beta^2 \in \mathbb{Q}$ all terms in the left hand side are in $\mathbb{Q}$. If $z = z'$, then $x^2 - x'^2 = (x - x')(x + x') = c\beta z(x' - x)$ which has two solutions classes ($x = x'$, $x + x' = c\beta z$), neither of which provides more than one solution per value of $f(x, z)$. Thus $|\Delta| \ge N/2$. □

Case 2. $\beta^2 \in \mathbb{Q}$, $c\beta \in \mathbb{R} \setminus \mathbb{Q}$.

Proof. Assume there are x, z, x', z' where $xz \ne x'z'$ for which $f(x, z) = f(x', z')$. This would be a contradiction, since then

$$c\beta = \frac{x^2 - x'^2 + \beta^2 z^2 - \beta^2 z'^2}{x'z' - xz}$$

which is in $\mathbb{Q}$ since $\mathbb{Q}$ is closed under field operations. On the other hand, if there are x, z, x', z' where $xz = x'z'$ while $f(x, z) = f(x', z')$, then $x^2 + \beta^2 z^2 = x'^2 + \beta^2 z'^2$. For fixed xz and $x^2 + \beta^2 z^2$, there are at most four solutions in $\mathbb{R}$, hence at most four solutions in $\mathbb{N}$. As, by the earlier argument, there cannot be more than one value of xz per value of $f(x, z)$ (lest $c\beta \in \mathbb{Q}$), $f(x, z) = f(x', z')$ has at most four solutions. Thus $|\Delta| \ge N/4$. □

Case 3. $\beta^2 \in \mathbb{R} \setminus \mathbb{Q}$ and $c\beta \in \mathbb{R} \setminus \mathbb{Q}$.

Proof. If $c = \frac{p}{q} \in \mathbb{Q}$, then we scale $x \mapsto x/\beta$, $z \mapsto z/\beta$, and exchange the roles of x and z, falling back to case 1. Otherwise, the map $(x, z) \mapsto x^2 + \beta^2 z^2 + c\beta xz$ is almost injective, since 1, β^2, and $c\beta$ form an independent basis over $\mathbb{Z}$ for the space of possible solutions; $f(x, z) = f(x', z')$ would then require $x^2 = x'^2$, $z^2 = z'^2$, and (since $c\beta \notin \mathbb{Q} \implies c\beta \ne 0$)

$xz = xz'$; there are precisely two solutions, $(x,z) = (x',z')$ and $(x,z) = (-x',-z')$. Thus $|\Delta| \geq N/2$. □

Combining all three cases gives $|\Delta| \geq N/4$. □

Lemma 8. *Let $A = \alpha U_{n_A}$ and $B = \beta U_{n_B}$, and Ξ as before, with cross angle θ. Assume furthermore that $(\beta/\alpha)^2 \in \mathbb{Q}$ and $(\beta/\alpha)\cos\theta \in \mathbb{Q}$. Then $|\Delta(\Xi)| \gtrsim C_{\alpha,\beta,\theta}\min(n_a,n_b)/\sqrt{\log N}$ for some constant $C_{\alpha,\beta,\theta}$ depending on α, β, and θ.*

Proof. We use a similar setup as for the proof of Lemma 7. Since $(\beta/\alpha)^2 \in \mathbb{Q}$, and $(\beta/\alpha)\cos\theta \in \mathbb{Q}$, we can scale Ξ be an integral factor without affecting Δ, so that we can write $f(x,z)$ as a binary integral quadratic form, $f(x,y) = px^2 + qxy + ry^2$, where $\gcd(p,q,r) = 1$ (i.e, f is primitive). Then $|\Delta| = |f(U_{n_A}, U_{n_B})|$. Since f computes a squared distance, it is positive definite, and its discriminant $D = q^2 - 4pr < 0$. Now, consider $U_{2n_A} \times U_{2n_B}$ as a rectangular lattice, one which the level sets of f are ellipses; with $n_A < n_B$, there is some maximal $K_{inner} \sim n_a^2$ for which for all $k < K_{inner}$, $f^{-1}(k)$ is contained in $U_{n_a} \times U_{n_b}$; and some minimal $K_{outer} \sim n_B^2\sqrt{-D}$ for so that $f(U_{2n_A} \times U_{2n_B}) \leq K_{outer}$. According to Paul Bernay's thesis[Ber12], the total number of integers $\leq m$ which are representable by f is $B_f(m) = C(D)\frac{m}{\sqrt{\log m}} + O\left(\frac{m}{(\log m)^{1/2+\epsilon}}\right)$ for some $\epsilon > 0$. Thus

$$C(D)\frac{n_A^2}{\sqrt{\log n_A}} \sim B_f(K_{inner}) \leq |\Delta| \leq B_f(K_{outer}) \sim C(D)\sqrt{-D}\frac{n_B^2}{\sqrt{\log n_B}}$$

controls the size of the distance set. By [MO06], the hex lattice has minimal $C(D)\cdot\sqrt{-D}$ with $D = -3$, i.e, the number of distances per covolume[11], so if the lattice is of similar aspect ratio as the level set ellipses, then $K_{inner} \sim n_a^2\sqrt{-D}$, and there is a universal constant for which $|\Delta| \gtrsim \min(n_a,n_b)/\sqrt{\log N}$. □

Remark. It is easy to empirically investigate the distinct distance set sizes for lattices whose coordinates are arithmetic progressions. For quadratic binary forms f, there is an $\Theta(n_a n_b)$ time-and-memory algorithm, which, in addition to computing $|f(U_{n_a}, U_{n_b})|$, incidentally produces $f(U_k, U_j)$ for all $k \leq n_a, j \leq n_b$. Consider the partial combined (x and y)-sums of the distance set sizes, so that $|\Delta_{n_a,n_b}| = \sum_{x=0}^{n_a}\sum_{y=0}^{n_b} p_{x,y}$. Then $p_{x,y} = 1 - d_{x,y}$, where to compute $d_{x,y}$ we first compute for all $k \in \mathbb{G}_f$ (the set of f-representable integers) the lists ℓ_k of values for which $f(x,y) = k$, and remove all (x,y) pairs which dominate some other pair (x',y') in the list (so that $x' < x$ and $y' < y$). W initialize $d_{x,y} \leftarrow 0$, and then for each dominating element, increment the corresponding $d_{x,y}$ counter. The remaining elements $(x_{k,i}, y_{k,i})_{i=1}^{s_k}$ are sorted ascending by $x_{k,i}$, after which, for each $i = 2\ldots s_k$, we increment $d_{x_{k,i},y_{k,i-1}}$. This procedure ensures that for a rectangle containing the entire list ℓ_k, the squared distance k is counted exactly once.

With suitable uniformity conditions, it is possible to show that as the lattices Ξ become progressively elongated (and cover progressively smaller fractions of any minimal ball containing them), $\Delta \to N$. A specific case was proven by [CSS13], who showed that for $f = x^2 + y^2$, if $n_A = N^{1/2-\epsilon}$, and $n_B = N^{1/2+\epsilon}$, then $\Delta = \Theta(N)$, but their method does not easily generalize to general binary quadratic forms, and is incapable of handling e.g. $n_A = N^{1/2}\log N$ and $n_B = N^{1/2}\log N$.

[11]The computation of $C(D)$ is covered in more detail by [BMO11].

We define $\nu_\Xi(k)$ to be the number of points (x,y) on $\Gamma = U_{2n_A} \times U_{2n_B}$ for which $f(x,y) = k$. Following [BG06], we define $r_f(k)$ to be the total number of $(x,y) \in \mathbb{Z}^2$ for which $f(x,y) = k$. Clearly $\nu_\Xi(k) \le r_f(k)$. By counting the pairs in $U_{2n_A} \times U_{2n_B}$ and the mean number of values of k that they contribute, we obtain

$$|\Delta| = \sum_{x,y\in\Gamma} \frac{1}{\nu_\Xi(f(x,y))}$$

Note that for $k < K_{inner}$, $\nu_\Xi(f(x,y)) = r_f(f(x,y))$, and the sum over $k \le K_{inner}$ evaluates to $\sim K_{inner}/\sqrt{\log N}$, while when we have $|x| < 2n_a \ll |y| < 2n_b$, $\nu_\Xi(f(x,y)) = 1$ for almost all x, y, and then $\sum_{x,y\in\Gamma: f(x,y)\gg K_{inner}} 1 \sim n_a n_b = N$. The tricky part in proving a lower bound on Δ is showing that for k close to K_{inner}, the sets of points representing a value k are not very spatially clustered. Some work has been done in this area; for instance, Theorem 5.11 of [Dia15] considers the ideals of the ring of integers of $\mathbb{Q}\left(i\sqrt{D}\right)$ (whose representation counting function is similar to that for binary quadratic forms) and finds a maximum discrepancy bound between the total number of representations in a sector of given angle and the number of representations that would be expected if all representations were uniformly distributed over a circle.

7. Computations for the Grid

Example 9. For a constant fraction subset of points E' (the central quarter), and a constant fraction subset of radii Δ' (those are less than a quarter the diameter of E), it is the case that $\omega_s(x) = \omega_s(x')$ for all $x, x' \in E'$, since the circle of radius s around x intersects precisely the gridpoints which also lie in E (which, for a point x more than s away from the boundary, is all of them.) Then $\omega_s(x)$ equals the representation counting function $r_f\left(s^2\right)$ for the quadratic form $f = x^2 + y^2$. Thanks to [BG06], the various asymptotic moments of $r_f(x)$ are straightforward to compute, and so are *lower* bounds on the upper bounds for expressions based on $\omega_s(x)$. From their Corollary 1, for any binary quadratic form f (such as $x^2 + y^2$), and $\beta \in 1$,

$$\sum_{s^2<N/4} r_f\left(s^2\right)^\beta \sim N(\log N)^{2^{\beta-1}-1}$$

and hence for any $x_\star \in E'$, we have specifically

$$\sum_s \omega_s(x_\star) \sim N \qquad \sum_s \omega_s(x_\star)^3 \sim N(\log N)^3$$
$$\sum_s \omega_s(x_\star)^2 \sim N\log N \qquad \sum_s \omega_s(x_\star)^4 \sim N(\log N)^7$$

This makes it straightforward to compute that, for the square grid, via Eq. 4.2,

$$|Q(E)| = \sum_s \left(\sum_x \omega_s(x)\right)^2 \ge \sum_{s\in\Delta'} \left(\sum_{x\in E'} \omega_s(x)\right)^2$$
$$\gtrsim N^2 \sum_s \omega_s(x_\star)^2 \sim N^3 \log N$$

where we use a single representative $x_\star$ for all the $x \in E'$. Similarly, for the hinge problem, we obtain for Eq. 4.4,

$$\begin{aligned} |U(E)| &= \sum_{r,s}\left(\sum_x \omega_r(x)\,\omega_s(x)\right)^2 \\ &\geq \sum_{r,s\in\Delta'}\left(\sum_{x\in E'} \omega_r(x)\,\omega_s(x)\right)^2 \\ &= \sum_{x\in E'}\sum_{x'\in E'}\sum_{r\in\Delta'}\sum_{s\in\Delta'} \omega_r(x)\,\omega_s(x)\,\omega_r(x')\,\omega_s(x') \\ &= \sum_{x,x'\in E'}\left(\sum_{t\in\Delta'} \omega_t(x)\,\omega_t(x')\right)^2 \\ &\gtrsim N^2\left(\sum_r \omega_r(x_\star)^2\right)^2 \sim N^4(\log N)^2 \end{aligned}$$

Example 10. The appendix of [GK15] shows that for the square grid, the number of k-line-intersection-rich points in the 3d incidence problem is at least $|P_k(\mathcal{L})| \gtrsim N^3/k^2$. We re-purpose their proof to show that, for the square grid, a constant fraction of lines have point-richness $\sim N$ and line-richness $\sim N\log N$; i.e, $|\mathcal{L}_{\sim N}(P)| \sim N^2$, and $|\mathcal{L}_{\sim N\log N}(\mathcal{L})| \sim N^2$.

Let $E = \left(\mathbb{Z}\cap\left(-2N^{1/2}, 2N^{1/2}\right)\right)\times\left(\mathbb{Z}\cap\left(-2N^{1/2}, 2N^{1/2}\right)\right)$ be a grid of points centered on the origin. Then $|E| \sim N$. We let $\mathcal{L} = \{L_{pq} : p, q \in E\}$ be the set of lines in $\mathbb{R}^3$ corresponding to non-translational positively oriented rigid motions mapping e.g. p to q, with parameterization according to Eq. 3.4. By Lemma A.1 of [GK15], there is a subset $\mathcal{L}_0 \subset \mathcal{L}$ for which $|\mathcal{L}_0| \sim |\mathcal{L}|$, where the lines $\mathcal{L}_0$ are those which pass through the points $(a,b,0)$ and $(c,d,1)$ in $\mathbb{R}^3$, where $a,b,c,d \in \mathbb{Z}\cap\left(-N^{1/2}, N^{1/2}\right)$. For every integer $1 \leq q \leq N^{1/2}/2$, and integer $0 \leq p \leq q$, the plane at height p/q intersects all $\sim N^2$ of the lines in $\mathcal{L}_0$. Furthermore, since the lines in $\mathcal{L}_0$ pass through integer lattices at heights $z = 0$ and $z = 1$, at height p/q (with p/q in lowest terms), the lines in $\mathcal{L}_0$ pass through a uniform lattice with spacing $1/q$ and $\sim Nq^2$ points in total.

We define $\mathfrak{L}$ to be the set of lines in $\mathcal{L}$ for which $|a-c|, |b-d| < N^{1/2}$; we have$|\mathfrak{L}| \sim N^2$. Then by translation invariance[12] for motions shorter than $N^{1/2}$, all lines in $\mathcal{L}_0$ pass through the same number of lines in $\mathfrak{L}$ at height p/q, namely, $\sim N^2/\left(Nq^2\right) \sim N/q^2$ of them. As any pair of lines can intersect at most twice, the total number of intersections [13] between a line ℓ in $\mathcal{L}_0$ and the lines of $\mathfrak{L}$, is

$$m_{\mathcal{L}}(\ell) = \sum_{p/q}^{q\lesssim N^{1/2}} \frac{N}{q^2} = N\sum_{q=1}^{N^{1/2}} \frac{\phi(q)}{q^2} \sim N\log N$$

[12]To be precise, this only ensures that the number of lines passing through $(a + a'/q, b + b'/q)$ at height p/q is the same as the number passing through $(c + c'/q, d + d'/q)$. Because shifting an endpoint of the line segment from $(r,s,0)$ to $(u,v,1)$ by one translates the intersection point of the line by p/q (or $1-p/q$), and $\gcd(p,q) = 1$ (so that we can attain any fractional x or y coordinate), we can transform the bundle of all lines passing through e.g. $(a + a'/q, b + b'/q)$ to pass through $(c + c'/q, d + d'/q)$.

[13]$+N$, as we overcounted intersections between ℓ as a member of $\mathcal{L}_0$ and as a member of $\mathfrak{L}$.

where $\phi(q)$ is Euler's totient function and the identity $\sum_{q=1}^{N^{1/2}} \frac{\phi(q)}{q^2} \sim \log N$ is easily derived from $\sum_{q=1}^{n} \phi(q)/q = \frac{6}{\pi^2}n$. (See for instance [PS92], section 3.)

As $m_{\mathcal{L}}(\ell) \sim N \log N$ for $\sim N^2$ different lines $\ell \in \mathcal{L}_0$, it follows $|\mathcal{L}_{\sim N \log N}(\mathcal{L})| \sim N^2$. Furthermore, since each line $\ell \in \mathcal{L}_0$ intersects at least one other line in L for $\sum_{p/q}^{q \lesssim N^{1/2}} 1 = \sum_{q=1}^{\sim N^{1/2}} \phi(q) \sim N$ different heights, $|\mathcal{L}_{\sim N}(P)| \sim N^2$.

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