A More Direct Approach to Lebesgue Integration

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1 Introduction

The study of Lebesgue measurability is typically done in a very specific order, and with specific motivations in mind. The general motivation for our study is the desire to extend the class of functions that can be integrated. To do so, we start by defining the notion of a measure and measurable sets. We then define measurable functions using this already established idea of measurable sets, and finally define the integration of these functions with respect to a given measure accordingly. As an afterthought, we can then consider the space of such measurable functions as a vector space, using Lebesgue integration as a norm on that space.

A drawback of this typical course of study is that the path from our starting point to our ultimate goal is not particularly straightforward or obvious.

One incredibly useful idea that arises in the study of measurability is that of the normed space of Lebesgue measurable functions on a set, called L^1. This space is actually the completion in the L^1-norm of the set of continuous functions, and as such is incredibly useful in the study of functional analysis. But while L^1 is very important in this field, we see the use of this space to functional analysis often plays no role in the motivation of that space.

Professor Peter Lax is a mathematician with significant contributions to the field of functional analysis. As such, the main focus for him is that of the L^1 space itself, rather than the concept of measurability. His present paper defines and motivates L^1 without first utilising the concept of measure. L^1 is constructed as the set of continuous functions in the L^1-norm. From this construction, many of the known properties of L^1 are derived.

2 Setup

We begin with a compact hypercube $K \subset \mathbb{R}^n$ for some $n \in \mathbb{Z}^+$. This means K can be written as the direct product of n closed and bounded intervals $[a_i, b_i] \subset \mathbb{R}$. By defining K in this way, we generalize the notion of a line segment, rectangle, or rectangular prism in 1, 2, or 3 dimensions respectively.

Then for any continuous function $c : K \to \mathbb{R}$, we can define $I(c)$ to be the integral of c over K. By defining I in this way only for continuous functions c, we can use the standard Riemann integral in the evaluation of I. Furthermore, our usage of a compact hypercube K as the domain allows us to evaluate $I(c)$ by computing the 1-dimensional Riemann integral in each component, and ensures this integral is well defined and finite for any such c.

More explicitly, we can write any point $x \in K$ as $x = (x_1, x_2, ..., x_n)$, where $a_i \leq x_i \leq b_i$ for each i. Then for any continuous $c : K \to \mathbb{R}$, we can take

$$I(c) = \int_{a_n}^{b_n} \cdots \int_{a_2}^{b_2} \int_{a_1}^{b_1} c(x_1, x_2, ..., x_n)\, dx_1 dx_2 \cdots dx_n.$$

Because K is compact and c is continuous, we know c must attain a minimum, m, and a maximum, M, on K. Then $m \leq c(x) \leq M$ for all $x \in K$, so we must have

$$m \prod_{i=1}^{n} (b_i - a_i) \leq I(c) \leq M \prod_{i=1}^{n} (b_i - a_i),$$

and so $I(c)$ is bounded for all such c.

We go on to define the volume of an open subset $G \subset K$. For any subset S of K, let χ_S be the characteristic function of S, meaning

$$\chi_S(x) = \begin{cases} 1 & \text{for} \quad x \in S \\ 0 & \text{for} \quad x \notin S. \end{cases}$$

To align with our expectations for the volume of an open set, we would like to have $V(G) = I(\chi_G)$, but I has so far only been defined for continuous functions. This directly motivates the definition used in the paper, which proceeds as follows.

Def: For any $G \subseteq K$, we say $c \in C(K)$ is *admissible* for G iff $c(x) \leq \chi_G(x)$ for all $x \in K$.

From this, we can define $V(G)$ to be the least upper bound of $I(c)$ for all $c \in C(K)$ that are admissible for G:

$$V(G) = \sup\{I(c) : c \in C(K),\ c(x) \leq \chi_G(x) \text{ for all } x \in K\}.$$

From this definition, it can be shown that V is countably subadditive, and additive on pairwise disjoint sets.[1]

[1] For those already familiar with measure theory, we will later show that the measure of any open set G is equal to $V(G)$. However, it is important to note that V can only be defined in this way for open sets. To demonstrate, let $I = [0, 1] \subseteq \mathbb{R}$, and let $S_1 = I \cap \mathbb{Q}$, $S_2 = I \setminus S_1$. If we define volume in the same way as above for all subsets of I, we would have $V(S_1) = V(S_2) = 0$, while $V(S_1 \cup S_2) = V(I) = 1$, contradicting subadditivity.

Using this definition of volume, we then say a subset $B \subseteq K$ is *negligible* iff it can be covered by an open set of arbitrarily small volume. We say a property P is true *almost everywhere* iff there exists a negligible set B such that P is true for all $x \in K \setminus B$.

3 Cauchy Sequences and Complete Spaces

Before we continue, we introduce the notions of Cauchy Sequences and Completeness. Given a normed vector space $(V, ||\cdot||)$, a sequence $\{v_n\}_{n=1}^{\infty} \subset V$ is *Cauchy* iff for any $\epsilon > 0$ there exists some $N \in \mathbb{Z}^+$ such that $m, n \geq N$ implies $||v_m - v_n|| < \epsilon$.

In a general normed vector space, every convergent sequence is Cauchy, but the converse is not necessarily true. As an example, consider the sequence $\{q_n\}_{n=1}$ in the rational numbers $\mathbb{Q}$ defined by

$$q_1 = 3.1, \quad q_2 = 3.14, \quad q_3 = 3.141, \quad q_4 = 3.1415, \quad ...$$

so each q_n takes the first n digits of π after the decimal. We know this sequence is Cauchy, as it converges to π in $\mathbb{R}$, but because its limit π is irrational the sequence can not be convergent in $\mathbb{Q}$.

A normed vector space in which every Cauchy sequence converges is called *complete* or *Banach*. We see from the previous example that $\mathbb{Q}$ is not complete, but as one might expect it can be shown that $\mathbb{R}$ is complete.

Given a normed vector space $(V, ||\cdot||)$ that is not complete, it is possible to define a new space V' with the same norm which is complete, and which contains V as a dense subspace. V' is then called the *completion* of V with respect to that norm.

To do this, we start by letting $\mathcal{V}$ be the set of all Cauchy sequences in V. Then given two Cauchy sequences $X = \{x_n\}, Y = \{y_n\} \in \mathcal{V}$, we define

$$d(X, Y) := \lim_{n\to\infty} ||x_n - y_n||,$$

and we say $X \sim Y$ iff $d(X, Y) = 0$. Finally, we define the *completion* of V to be

$$V' := \mathcal{V}/\sim .$$

This new space does not actually contain V in the most strict sense, as elements of V' are equivalence classes of Cauchy sequences in V, and so are different mathematical objects than the elements of V. However, given any $v \in V$, we know the sequence $\{v\}_{n=1}^{\infty}$ where every element is v, is Cauchy, and so there is exactly one element of V' with this sequence in its equivalence class.

By identifying each $v \in V$ with $[\{v\}]_\sim \in V'$, we see V' can be thought of as containing a copy of V.

Returning to our previous example, this notion of completion actually gives us a way of constructing $\mathbb{R}$ from the set $\mathbb{Q}$. For any $a \in \mathbb{R}$, we again let $\{a_n\}_{n=1}^{\infty}$ be such that a_n takes the first n digits of a after the decimal. We know this sequence is Cauchy in $\mathbb{Q}$ with respect to the absolute value norm, and so for any $a \in \mathbb{R}$ there will be exactly one equivalence class $[\{a_n\}]$ in the completion of $\mathbb{Q}$ corresponding to a.

Using the method described above, we can thus *define* $\mathbb{R}$ to be the completion of $\mathbb{Q}$ in the absolute value norm.

4 Defining L^1

We now have the tools we need to define the L^1 space, which is denoted simply L throughout the rest of this and the original paper. We let $C = C(K)$ be the set of all continuous functions on K. Noting that C is a linear space, we define the L-norm on C by $|c|_L = I(|c|)$.

This is a very natural choice of norm for this space, as it is essentially an extension of the taxicab norm on finite-dimensional vector spaces. This is also the same norm used in the standard definition of the L^1 space.

One important distinction of its usage here, though, is that $I(|c|)$ is truly a norm on C, while it is not a norm on the set of all Lebesgue measurable functions. This is because there exist measurable functions f not identically 0 such that $I(|f|) = 0$, for example the characteristic function of the rationals in K.

We typically deal with this problem by letting L^1 be the *quotient space* of all Lebesgue measurable functions *mod the relation* $\sim$, where $f \sim g$ iff $f(x) = g(x)$ almost everywhere.

By restricting our set of interest to C, we see the L-norm is a true norm on C, as a continuous function is equal to 0 almost everywhere only if it is identically 0. Then as $(C, |\cdot|_L)$ is a normed vector space, we can define L to be the completion of C in the L-norm.

This means the elements of our newly defined space L are not actually the same mathematical objects as the elements of the L^1 space as it is typically defined. Elements of L in this paper are equivalence classes of Cauchy sequences of continuous functions, whereas elements of the standard L^1 space are equivalence classes of Lebesgue measurable functions.

Similarly to $\mathbb{R}$ and the completion of $\mathbb{Q}$, however, we will show throughout the rest of this paper that elements of each of these sets can be clearly identified with one another in a very natural way.

This identification is based on the following idea:
A function $f(x)$ defined almost everywhere on K is said to be a *realization* of f in L iff there is a Cauchy sequence $\{c_n\}$ in the equivalence class f which converges almost everywhere to $f(x)$.

Using this notion of realizations, we can then show that our L is essentially equivalent to the typical L^1 space by demonstrating that:

(i) Every element of $f \in L$ has a realization;

(ii) A function $f(x)$ is measurable iff it is a realization of some $f \in L$; and

(iii) For any $f, g \in L$ with realizations $f(x)$ and $g(x)$ respectively, we have $f(x) = g(x)$ almost everywhere iff $f = g$.

These properties would imply there is a one-to-one correspondence between the elements of our L space, and the equivalence classes of measurable functions that are identical almost everywhere, which is the standard definition of L^1.

5 Important Tools

There are a few important tools and theorems we must introduce before moving on, as they will be vital in proofs and derivations through the rest of this paper.

First, we introduce a version of Chebyshev's Inequality relating volume and integration.

Chebyshev's Inequality: Let $a \in \mathbb{R}^+$, and $d\colon K \to \mathbb{R}_{\geq 0}$ be a nonnegative continuous function. Denote by $G_a \subseteq K$ the set of points $x \in K$ such that $d(x) > a$. Then

$$V(G_a) \leq \frac{I(d)}{a}.$$

Proof: Let c_a be admissible for G_a and $x \in K$. We consider two exhaustive cases, $x \in G_a$ and $x \notin G_a$.

If $x \in G_a$, then we know $d(x) > a$, and so $\frac{d(x)}{a} > 1$. Then by definition of admissible we have

$$c_a(x) \leq \chi_{G_a}(x) = 1,$$

and so we see $c_a(x) < \frac{d(x)}{a}$.

If $x \notin G_a$, we have $c_a(x) \leq \chi_{G_a}(x) = 0$. As we assumed d is nonnegative and $a > 0$, this implies

$$c_a(x) \leq 0 \leq \frac{d(x)}{a}.$$

Thus in either case we see $c_a(x) \leq \frac{d(x)}{a}$ for any $x \in K$, and so integrating both sides over K gives $I(c_a) \leq \frac{I(d)}{a}$. Taking the supremum of both sides over all admissible c_a, we arrive at

$$V(G_a) \leq \frac{I(d)}{a}.$$

This inequality allows us to find upper bounds on the volumes of some sets. We will then make use of these upper bounds in proving that certain subsets of K are in fact negligible, such as by showing a set has volume $\leq \epsilon$ for any $\epsilon > 0$.

Next we introduce the concept of *rapid convergence*. A Cauchy sequence c_n is called *rapidly convergent* iff there exists a constant k for which $|c_n - c_{n+1}|_L < \frac{k}{n^4}$ for all $n \in \mathbb{Z}^+$.

The concept of rapid convergence imposes a restriction on how fast a Cauchy sequence must converge in L. To demonstrate the value of this, we introduce in the following multi-part theorem some useful properties of rapidly convergent sequences of continuous functions, which will be used throughout.

Theorem 1:

(i) Every Cauchy sequence has a rapidly converging subsequence.

Proof: For any Cauchy sequence $\{c_n\}$ and any $j \in \mathbb{Z}^+$, we can let $\epsilon = \frac{1}{j^4}$, and so we know there exists $N_j \in \mathbb{Z}^+$ such that $m, n \geq N_j$ implies $|c_m - c_n| < \frac{1}{j^4}$.

Then letting $C_j = c_{N_j}$ for every $j \in \mathbb{Z}^+$, it is clear that $\{C_j\}$ is a rapidly converging subsequence of $\{c_n\}$.

(ii) A rapidly convergent sequence of continuous functions $\{c_n(x)\}$ converges almost everywhere.

Proof: Let $d_n(x) := |c_n(x) - c_{n+1}(x)|$ for each $n \in \mathbb{Z}^+$, and define

$$D_n := \{x \in K : |c_n(x) - c_{n+1}(x)| > \frac{1}{n^2}\}.$$

We know each d_n is a nonnegative continuous function, and so by Chebyshev's Inequality we know

$$V(D_n) \leq \frac{I(d_n)}{1/n^2} = \frac{|c_n - c_{n+1}|_L}{1/n^2}.$$

By definition of rapid convergence, we know there exists some k such that $|c_n - c_{n+1}|_L < \frac{k}{n^4}$, which means

$$V(D_n) \leq \frac{|c_n - c_{n+1}|_L}{1/n^2} < \frac{k/n^4}{1/n^2} = \frac{k}{n^2}$$

for each $n \in \mathbb{Z}^+$.

Now let $B \subseteq K$ be the set of all $x \in K$ such that $\{c_n(x)\}$ does not converge. For any $x \in B$, we see there must be some $N \in \mathbb{Z}^+$ such that $x \in D_N \cup D_{N+1} \cup \cdots$, as otherwise we would have $|c_m(x) - c_{m+1}(x)| \leq \frac{1}{m^2}$ for all $m > N$, which would imply convergence of $\{c_n(x)\}$.

Then we can define $G_N := \bigcup_{i=N+1}^{\infty} D_i$, and we see this implies $B \subseteq G_N$ for any $N \in \mathbb{Z}^+$. As volume is countably subadditive, we thus get

$$V(B) \le V(G_N) \le \sum_{N+1}^{\infty} V(D_n) < \sum_{N+1}^{\infty} \frac{k}{n^2} < \frac{k}{N}.$$

We see $\frac{k}{N} \to 0$ as $N \to \infty$, and so we must have $V(B) = 0$. Thus the set of $x \in K$ for which $\{c_n(x)\}$ does not converge must be negligible, meaning $\{c_n(x)\}$ converges almost everywhere.

(iii) For any $\epsilon > 0$, a rapidly convergent sequence converges uniformly except for a set of x contained in an open set of volume less than ϵ.

Proof: Using the definitions from the previous proof, we see $\{c_n\}$ does not necessarily converge uniformly in the compliment of B, but *does* converges uniformly in the complement of $G_N = \bigcup_{i=N+1}^{\infty} D_i$ for every $N \in \mathbb{Z}^+$.

For any $\epsilon > 0$, we can thus choose $N > \frac{k}{\epsilon}$, and we see $V(G_N) < \frac{k}{N} < \epsilon$. Thus for any $\epsilon > 0$, $\{c_n\}$ converges uniformly except for a set of x contained in an open set of volume less than ϵ.

The above property (i) leads us to a useful corollary: Every equivalence class $f \in L$ contains a Rapidly converging sequence. Every $f \in L$ must have some Cauchy sequence in its equivalence class, and that sequence must have a rapidly converging subsequence, which means that rapidly converging subsequence is also in f.

6 Realizations

Using these tools, we now demonstrate some fundamental properties of realizations of functions in L^1

Theorem 2.1: Every f in L has a realization.

Proof: Let $\{c_n\}$ be a rapidly converging sequence of continuous functions in the equivalence class f. Then by property (i) above we know $\{c_n\}$ converges to some function $f(x)$ almost everywhere, and we see this $f(x)$ satisfies the definition of a realization of f.

Theorem 2.2: Two realizations of f are equal almost everywhere.

Proof: Let $f_1(x)$ and $f_2(x)$ be two realizations of f, and let $k \in \mathbb{Z}^+$ be given. By definition of realization, there must exist Cauchy sequences of continuous functions $\{c_n\}$ and $\{d_n\}$ in the equivalence class f converging pointwise to $f_1(x)$ and $f_2(x)$ respectively. As these sequences are Cauchy in L, we know they both converge to f in the L-norm, and so there exists some $N_k \in \mathbb{Z}^+$ such that $m \geq N_k$ implies

$$|c_m - f|_L < \frac{1}{j^4} \quad \text{and} \quad |d_m - f|_L < \frac{1}{j^4}.$$

Setting $b_k = c_{N_k}$ for odd k and $b_k = d_{N_k}$ for even k, we see $\{b_k\}$ is rapidly convergent and contains infinitely many elements of both $\{c_n\}$ and $\{d_n\}$.

This implies by Theorem 1(ii) that $\{b_k(x)\}$ must converge to both $f_1(x)$ and $f_2(x)$ almost everywhere, which means $\{x \in K : f_1(x) \neq f_2(x)\}$ must be negligible. Thus two arbitrarily chosen realizations of some $f \in L$ must be equal almost everywhere.

Theorem 3.1: Given $a \in \mathbb{R}$ and $f, g \in L$ with realizations $f(x)$ and $g(x)$ respectively, $af(x) + g(x)$ is a realization of $af + g \in L$.

Proof: By definition of realization, we know there exist rapidly convergent sequences of continuous functions $\{c_n\} \in f$ and $\{d_n\} \in g$ such that

$$c_n(x) \to f(x) \quad \text{and} \quad d_n(x) \to g(x)$$

almost everywhere.

Then as $\{c_n\}, \{d_n\}$ are rapidly convergent, we also know $\{ac_n + d_n\}$ is also rapidly convergent, so by Theorem 1(ii) we know $\{ac_n(x) + d_n(x)\}$ converges almost everywhere. Thus it must converge to $af(x) + g(x)$ almost everywhere.

Because $\{c_n\} \in f$ and $\{d_n\} \in g$, we know $\{ac_n + d_n\} \in af + g$, and so we see $af(x) + g(x)$ is a realization of $af + g$,

Theorem 3.2: Given a sequence $\{f_n\} \subseteq L$ that converges in norm to $f \in L$, and if $\lim f_n(x)$ exists almost everywhere, then $\lim f_n(x) = f(x)$ almost everywhere.

Proof: As $\{f_n\}$ converges to f, we know it is Cauchy in L, and so has a rapidly convergent subsequence, call this $\{g_n\}$.

Now let $n \in \mathbb{Z}^+$ be chosen arbitrarily, and let $h = g_n \in L$.

Then by Theorem 2.1, we know h has a realization $h(x)$, meaning there exists a rapidly convergent sequence of continuous functions $\{c_m\}$ in the equivalence class of h such that $c_m(x)$ converges to $h(x)$ almost everywhere. As $\{c_m\}$ is rapidly convergent, we know by Theorem 1(iii) that $c_m(x)$ converges to $h(x)$ uniformly except for a on set of volume less than $\frac{1}{n^2}$ (letting $\epsilon = \frac{1}{n^2}$ in the original theorem).

Then by definition of uniform convergence, we know there exists $M \in \mathbb{Z}^+$ such that $m \geq M$ implies $|c_m(x) - h(x)| < \frac{1}{n}$.

Now consider the sequence $\{c_j\}_{j=M}^{\infty}$. As this converges in the L-norm to $h = g_n$, we know there exists some $J_n \geq M$ such that $|c_{J_n} - h|_L < \frac{1}{n^4}$.

As n was chosen arbitrarily, do this for each $n \in \mathbb{Z}^+$, and define a new sequence $\{C_n\}_{n=1}^{\infty}$ by $C_n := c_{J_n}$.

We have thus shown that for each g_n there exists a continuous function C_n such that

$$|g_n - C_n|_L < \frac{1}{n^4} \quad \text{and} \quad |g_n(x) - C_n(x)| < \frac{1}{n}$$

except for a set of volume $< \frac{1}{n^2}$.

The first inequality together with the rapid convergence of $\{g_n\}$ implies $\{C_n\}$ is a rapidly convergent sequence of continuous functions converging to f in L. Thus by Theorem 1(ii), we know $\{C_n(x)\}$ converges to some function $f(x)$ almost everywhere, and as $\{C_n\}$ is in the equivalence class f we know $f(x)$ is a realization of f.

7 Introducing Lipschitz Continuous Functions

Our next goal is to define a sort of functional calculus on L, which will allow us to manipulate elements of L without first determining their realizations. To do this, we introduce the concept of Lipschitz continuous functions.

Given any normed vector space $(V, ||\cdot||)$, A function $\varphi : V \to \mathbb{R}$ is called *Lipschitz continuous* iff there exists some constant $k \in \mathbb{R}^+$ such that

$$|\varphi(x) - \varphi(y)| \leq k||x - y||$$

for all $x, y \in V$.

We see all Lipschitz continuous functions are uniformly continuous, as for any $\epsilon > 0$ we can take $\delta = \frac{\epsilon}{k}$, and we see $||x - y|| < \delta$ implies

$$|\varphi(x) - \varphi(y)| \leq k||x - y|| < k\delta = \epsilon$$

for any $x, y \in V$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous, so for all $s, t \in \mathbb{R}$ we have

$$|\varphi(s) - \varphi(t)| \leq k|s - t|.$$

We now show there is a natural way to assign to each $f \in L$ a unique $\varphi(f) \in L$, and then demonstrate some nice properties of this mapping.

First, note that for any pair of continuous functions c, d and any $x \in K$ we have

$$|\varphi(c(x)) - \varphi(d(x))| \leq k|c(x) - d(x)|.$$

Because compositions of continuous functions are continuous, we can integrate both sides over all of K to get

$$|\varphi(c) - \varphi(d)|_L \leq k|c - d|_L. \tag{$*$}$$

Then given any Cauchy sequence of continuous functions $\{c_n\}$, the previous equation implies $\{\varphi(c_n)\}$ is also Cauchy in L.

Now for any $f \in L$, we can take some Cauchy sequence $\{c_n\}$ in the equivalence class f. As L is complete we know $\{\varphi(c_n)\}$ will have a limit, so we can define $\varphi(f)$ to be the limit of $\{\varphi(c_n)\}$ in L.

We know this limit is independent of the choice of $\{c_n\}$, as for any two Cauchy sequences $\{c_n\}, \{d_n\}$ in the equivalence class of f, we know

$$|\lim \varphi(c_n) - \lim \varphi(d_n)| = \lim |\varphi(c_n) - \varphi(d_n)|_L \leq k \lim |c_n - d_n|_L = k|f - f| = 0,$$

and so we must have $\lim \varphi(c_n) = \lim \varphi(d_n)$.

Now considering φ as a functional from L to L as defined previously, we demonstrate that it satisfies the following properties.

Theorem 4:

(i) $|\varphi(f) - \varphi(g)|_L \leq k|f - g|_L$ for all $f, g \in L$

(ii) For any $f \in L$, $\varphi(f)(x) = \varphi(f(x))$ almost everywhere.

Proof (i): Let $f, g \in L$ have Cauchy sequences $\{c_n\}$ and $\{d_n\}$ in their respective equivalence classes. Then we see by equation $(*)$ that

$$|\varphi(f) - \varphi(g)|_L = \lim |\varphi(c_n) - \varphi(d_n)|_L \leq k \lim |c_n - g_n|_L = k|f - g|_L.$$

Proof (ii): Let $f \in L$ with realization $f(x)$ be given and let $\{c_n\}$ be a rapidly convergent Cauchy sequence of continuous functions in the equivalence class f converging to $f(x)$ almost everywhere. This means there exists some λ such that

$$|c_n - c_{n+1}| < \frac{\lambda}{n^4}$$

for all $n \in \mathbb{Z}^+$.

Then by equation $(*)$, we see

$$|\varphi(c_n) - \varphi(c_{n+1})|_L \leq k|c_n - c_{n+1}|_L < \frac{k\lambda}{n^4},$$

and so $\{\varphi(c_n)\}$ is also a rapidly convergent sequence of continuous functions and converges to $\varphi(f)$. Then we know $\{\varphi(c_n)(x)\}$ converges almost everywhere to $\varphi(f(x))$, and so we must have $\varphi(f)(x) = \varphi(f(x))$ almost everywhere.

8 Using Lipschitz Continuous Functions

These functions will prove incredibly useful in defining new elements of L given some other element, without dealing with realizations.

As a particularly nice example, consider $\varphi_+, \varphi_- : \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi_+(s) = \begin{cases} s & \text{for} \quad 0 \leq s \\ 0 & \text{for} \quad s < 0 \end{cases}, \qquad \varphi_-(s) = \begin{cases} 0 & \text{for} \quad 0 \leq s \\ s & \text{for} \quad s < 0 \end{cases}.$$

Then for any $f \in L$, we can define

$$f_+ = \varphi_+(f) \text{ and } f_- = \varphi_-(f).$$

to be the *positive* and *negative parts* of f respectively.

This implies $f = f_- + f_+$ and $|f|_L = I(f_+) - I(f_-)$, which we would expect if these definitions of positive and negative part coincide with the typical definition of the positive and negative parts of a function.

Based on this, we say an element $g \in L$ is *positive* iff $g = g_+$ and *negative* iff $g = g_-$. We also say $f \leq g$ iff $g - f$ is positive.

We now demonstrate more properties of elements of L using these concepts.

Theorem 5:

(i) The sum of two positive elements is positive

(ii) If f is positive, $f(x) \geq 0$ almost everywhere.

(iii) If f is positive, $|f|_L = I(f)$.

Proof (i): Let $f, g \in L$ be positive, and let $\{c_n\}$, $\{d_n\}$ be Cauchy sequences in the equivalence class f and g respectively. Then by definition of a positive element of L, we know

$$f = \varphi_+(f) = \lim_{n\to\infty} \varphi_+(c_n)$$

and

$$g = \varphi_+(g) = \lim_{n\to\infty} \varphi_+(d_n).$$

This means $\{\varphi_+(c_n)\}$ and $\{\varphi_+(d_n)\}$ are also Cauchy sequences of continuous functions in the equivalence classes f and g respectively, which implies $\{\varphi_+(c_n) + \varphi_+(d_n)\}$ is a Cauchy sequence of continuous functions in the equivalence class $f + g$. Noting that $\varphi_+(c_n(x)) + \varphi_+(d_n(x)) \geq 0$ for all $x \in K$ by definition, this implies

$$\varphi_+(f + g) = \lim_{n\to\infty} \varphi_+(\varphi_+(c_n) + \varphi_+(d_n)) = \lim_{n\to\infty} [\varphi_+(c_n) + \varphi_+(d_n)] = f + g,$$

and so $f + g$ is positive.

Proof (ii): Let $f \in L$ be positive with realization $f(x)$. Then we know there exists a rapidly convergent sequence of continuous functions $\{c_n\}$ in the equivalence class f such that

$$\lim c_n(x) = f(x) \text{ almost everywhere.}$$

From this, the continuity of φ_+ as well as Theorem 4(ii) imply

$$\lim \varphi_+(c_n(x)) = \varphi_+(\lim c_n(x)) = \varphi_+(f(x)) = \varphi_+(f)(x) = f(x)$$

almost everywhere.

We know by definition of φ_+ that $\varphi_+(c_n(x)) \geq 0$ for all $x \in K$ and all $n \in \mathbb{Z}^+$, and so the previous equality implies $f(x) \geq 0$ almost everywhere.

Proof (iii): Let $f \in L$ be positive, so $f_+ = f$. We know $f = f_- + f_+$ for any $f \in L$, and so $f = f_+$ implies $f_- = 0 \in L$, and so $I(f_-) = 0$. Thus we have

$$|f|_L = I(f_+) - I(f_-) = I(f_+) = I(f).$$

Using this new partial ordering on L given by $\geq$ defined above, we can show the following version of the Monotone Convergence Theorem.

Theorem 6: Let $\{f_n\}$ be an increasing sequence of elements of L, that is $f_n \leq f_{n+1}$ for all $n \in \mathbb{Z}^+$. Suppose that $I(f_n)$ is bounded; then $\{f_n\}$ converges in the L-norm to a limit $f \in L$, and

$$\lim f_n(x) = f(x) \text{ almost everywhere.}$$

Proof: We see for any $m \leq n$, we have

$$f_n - f_m = (f_n - f_{n-1}) + (f_{n-1} + f_{n-2}) + ... + (f_{m+2} - f_{m+1}) + (f_{m+1} - f_m).$$

By assumption each $f_{i+1} - f_i$ is positive, and by Theorem 5(i) the sum of positive elements is positive, so we know $f_n - f_m$ is positive, and thus $f_m \leq f_n$.

Then Theorem 5(iii) implies

$$|f_n - f_m|_L = I(f_n - f_m) = I(f_n) - I(f_m).$$

This means $n \geq m$ implies $I(f_n) \geq I(f_m)$, and so $\{I(f_n)\}$ is an increasing sequence of real numbers.

By our initial assumption that $I(f_n)$ is bounded, it thus must converge. Convergent sequences are Cauchy, which means $\{f_n\}$ is also Cauchy in the L-norm, and so by the completeness of L must converge to some $f \in L$.

Then as $f_{n+1} - f_n$ is positive for all $n \in \mathbb{Z}^+$, we know by Theorem 5(ii) that for almost all $x \in K$ we have $(f_{n+1} - f_n)(x) \geq 0$, meaning $\{f_n(x)\}$ is an increasing sequence of $\mathbb{R}$ for almost all $x \in K$.

Accepting ∞ as a possible limit, this would imply $\lim f_n(x)$ exists for all such x. Thus by Theorem 3.2 we arrive at $\lim f_n(x) = f(x)$ for almost all $x \in K$.

We can also define the property of *boundedness* for elements of L using Lipschitz continuous functions. For any $a \in \mathbb{R}^+$, define

$$\varphi_a(s) = \begin{cases} -a & \text{for} \quad s \leq -a \\ s & \text{for} \quad -a < s < a \\ a & \text{for} \quad a \leq s \end{cases}$$

Then for any $f \in L$, we define $f_a := \varphi_a(f) \in L$. We say f is *bounded* iff $f = f_a$ for some $a \in \mathbb{R}^+$.

For any bounded $f \in L$, we can show from this definition that there must be some *smallest* $a \in \mathbb{R}$. We then define the *sup norm* $|f|_{\text{sup}}$ to be this smallest a.

We know f is bounded if and only if there is a Cauchy sequence $\{c_n\}$ of uniformly bounded continuous functions in the equivalence class f, specifically with $|c_n(x)| \leq a$ for all $x \in K, n \in \mathbb{Z}^+$.

As K is compact, we also know that for any $f \in L$, we have

$$\lim_{a \to \infty} f_a = f,$$

meaning $f_a \to f$ in the L-norm as $a \to \infty$.

This boundedness property can then be used to define a product for some elements of L with certain nice properties.

First let $f, g \in L$ be bounded. Then there exist Cauchy sequences $\{c_n\}, \{d_n\}$ of uniformly bounded continuous functions converging to f and g respectively.

Then we know the sequence $\{c_n d_n\}$ is also Cauchy, as for any $n, m \in \mathbb{Z}^+$ we see

$$c_n d_n - c_m d_m = (c_n - c_m) d_n + c_m (d_n - d_m).$$

We can thus define the product fg by

$$fg := \lim_{n \to \infty} c_n d_n.$$

Now let $f \in L$ be bounded and choose $g \in L$ arbitrarily. Then we know $g_a \to g$ as $a \to \infty$, we must have that fg_a also converges in L, and so we can define the product fg by

$$fg := \lim_{a \to \infty} fg_a.$$

For any bounded $f \in L$, this product can be shown to have the following nice properties:

(i) $f(g+h) = fg + fh$ for all $g, h \in L$

(ii) For any $g \in L$, $(fg)(x) = f(x)g(x)$ almost everywhere

(iii) For any $g \in L$ we have $|fg|_L \leq |f|_{\sup}|g|_L$

9 Final properties of L

For the remaining properties of L, we will need to develop a way to determine properties of an element $f \in L$ from the properties of its realization $f(x)$.

Lemma 7: Suppose that $f \in L$ satisfies $f(x) \leq 0$ almost everywhere. Then $I(f) \leq 0$.

Proof: We know $f_a \to f$ in the L norm and $I(f_a) \to I(f)$ in $\mathbb{R}$ as $a \to \infty$. Thus it suffices to show that $I(f_a) \leq 0$ for all $a > 0$.

Let $a > 0$ be chosen arbitrarily. We know by Theorem 4(ii) that $f(x) \leq 0$ almost everywhere implies $f_a(x) \leq 0$ almost everywhere.

Using Theorem 1(iii), one can then show that for any $\epsilon > 0$ there exists a continuous function c_ϵ also bounded by a such that

$$|f_a - c_\epsilon|_L < \epsilon,$$

and there exists an open set G_ϵ with volume $\leq \epsilon$ such that $c_\epsilon(x) \leq \epsilon$ for all $x \in K \setminus G_\epsilon$.

This second inequality implies

$$\frac{c_\epsilon(x) - \epsilon}{a} \leq \frac{\epsilon - \epsilon}{a} = 0$$

for all $x \in K \setminus G_\epsilon$, and the fact that c_ϵ is bounded by a implies

$$\frac{c_\epsilon(x) - \epsilon}{a} \leq \frac{|c_\epsilon(x)|}{a} \leq 1$$

for all $x \in G_\epsilon$.

We thus see $(c_\epsilon - \epsilon)/a$ is a continuous function such that $\frac{c_\epsilon(x)-\epsilon}{a} \leq \chi_{G_\epsilon}(x)$ for all $x \in K$, which means $(c_\epsilon - \epsilon)/a$ is admissible for G_ϵ.

By our definition of volume, we thus know

$$I\left(\frac{c_\epsilon - \epsilon}{a}\right) = \frac{I(c_\epsilon) - \epsilon I(1)}{a} \leq V(G_\epsilon) < \epsilon,$$

and rearranging thus implies

$$I(c_\epsilon) < \epsilon(a + I(1)).$$

Recall that one of our defining properties of c_ϵ was that $|f_a - c_\epsilon|_L < \epsilon$. This implies

$$\lim_{\epsilon \to 0} I(c_\epsilon) = I(f),$$

and so $I(c_\epsilon) < \epsilon(a + I(1))$ for all $\epsilon > 0$ implies

$$I(f) \leq 0.$$

This result leads directly to the following theorem.

Theorem 8: Let f, g be elements of L with realizations $f(x)$ and $g(x)$ respectively. If $f(x) = g(x)$ for almost all $x \in K$, then $f = g$ as elements of L.

Proof: Let $f \in L$ be chosen arbitrarily, with realization $f(x)$. We first show that $f(x) = 0$ for almost all $x \in K$ implies $f = 0 \in L$.

Assume $f(x) = 0$ almost everywhere. Then we know

$$f_+(x) = (\varphi_+(f))(x) = \varphi_+(f(x)) = 0 \text{ almost everywhere,}$$

and so by the previous lemma we know $I(f_+) \leq 0$. Then by Theorem 5(iii) we know $|f_+|_L = I(f_+) \leq 0$, meaning which as $|\cdot|_L$ is a norm on L implies $f_+ = 0$.

Similarly, $f(x) = 0$ almost everywhere implies

$$-f_-(x) = -(\varphi_-(f))(x) = -\varphi_-(f(x)) = 0 \text{ almost everywhere,}$$

and so by the previous lemma we know $I(-f_-) \leq 0$, meaning $I(f_-) \geq 0$. Then by Theorem 5(iii) we know $|f_-|_L = -I(f_-) \leq 0$, which as $|\cdot|_L$ is a norm on L implies $f_- = 0$. Thus we arrive at

$$f = f_+ + f_- = 0.$$

Now let $f, g \in L$ be chosen arbitrarily with realizations $f(x)$ and $g(x)$ respectively. If $f(x) = g(x)$ for almost all x, we know $f(x) - g(x) = 0$, which by Theorem 3.1 implies $(f - g)(x) = 0$ almost everywhere. Thus by our previous result we know $f - g = 0$, and so $f = g$ as elements of L.

This theorem is what ultimately allows us to determine properties of an element of L based only on the properties of its realization. For instance, this theorem gives us the following properties:

For any $f \in L$ with realization $f(x)$, we have **(i):** $f(x) \geq 0$ almost everywhere implies f is positive. **(ii):** $f(x) \leq 0$ almost everywhere implies f is negative. **(iii):** $|f(x)| \leq a$ almost everywhere implies f is bounded.

These results are intuitively true, but we are only now able to prove them with Theorem 8.

10 Principle of Dominated Convergence

We now move on to another important result in the study of Lebesgue integration, the principle of dominated convergence. This result is very useful in defining functions as limits of a sequence of functions, as it does not require such strict conditions on that sequence as we have needed in previous theorems.

To set up our proof, we introduce a new Lipschitz continuous function, but now on two real variables rather than one. Define $\varphi_{\max} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\varphi_{\max}(a, b) := \max(a, b).$$

Noting

$$|\varphi_{\max}(a, b) - \varphi_{\max}(s, t)| \leq \max(|a - s|, |b - t|),$$

we see $\varphi_{\max}$ is Lipschitz continuous.

Through the same method as with one variable, we can also define $\varphi_{\max}(f, g)$ as an element of L for any pair of elements $f, g \in L$. Now using the notations $\max(f, g) := \varphi_{\max}(f, g)$, we know by properties of Lipschitz continuous functions that $\max(f, g)(x) = \max(f(x), g(x))$ almost everywhere. We can then define $\max(f_1, ..., f_n)$ recursively as

$$\max(f_1, ..., f_n) := \max(f_n, \max(f_1, ..., f_{n-1})).$$

Theorem 9: Let $\{g_n\}_{n=1}^{\infty} \subseteq L$ be such that there exists $g \in L$ for which $|g_n(x)| \leq g(x)$ almost everywhere for each $n \in \mathbb{Z}^+$. Then $h = \max\{g_n\}$ can be defined as an element of L so that

$$\sup\{g_n(x) : x \in K\} = h(x) \text{ almost everywhere.}$$

Proof: For each $n \in \mathbb{Z}^+$, define

$$h_n := \max(g_1, ..., g_n).$$

From this definition we see $\{h_n\}$ is an increasing sequence of functions, and as each g_n is bounded by g we know each h_n is as well.

This means $h_n(x) - g(x) \leq 0$ almost everywhere, and so by Lemma 7 we know $I(h_n - g) \leq 0$, which implies $I(h_n) \leq I(g)$.

Thus by the Monotone Convergence Theorem, Theorem 6, we know $\{h_n\}$ converges in L to a limit $h \in L$ with

$$\lim_{n\to\infty} h_n(x) = h(x) \text{ almost everywhere.}$$

By definition of h_n, we see

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \max\{g_m(x) : 1 \leq m \leq n\} = \sup\{g_n(x)\},$$

and so we arrive at $\sup\{g_n(x)\} = h(x)$ almost everywhere.

Theorem 10 (Principle of Dominated Convergence):
If a sequence of elements $\{f_n\}$ in L converge pointwise almost everywhere, and if all f_n are dominated by a single element g in L:

$$|f_n(x)| \leq g(x) \text{ almost everywhere,}$$

then $\{f_n\}$ converges in norm to a limit f in L, and

$$f(x) = \lim f_n(x) \text{ almost everywhere.}$$

Proof: By Theorem 9, we know for each $n \in \mathbb{Z}^+$ we can define

$$f_n^{\max} := \max\{f_j\}_{j=n}^{\infty}; \qquad f_n^{\min} := -\max\{-f_j\}_{j=n}^{\infty}$$

as elements of L.

Then we see $\{f_n^{\max}\}$ is decreasing and bounded below by $-g$, while $\{f_n^{\min}\}$ is increasing and bounded above by g, so by the monotone convergence theorem we know they each converge to a limit in L, call these $f^{\max}$ and $f^{\min}$ respectively, with

$$\lim_{n\to\infty} f_n^{\max}(x) = f^{\max}(x) \quad \text{and} \quad \lim_{n\to\infty} f_n^{\min}(x) = f^{\min}(x)$$

almost everywhere.

As $\{f_n(x)\}$ converges almost everywhere, we thus must have

$$\lim f_n(x) = f^{\min}(x) = f^{\max}(x) \text{ almost everywhere,}$$

and so by Theorem 8 we know $f^{max} = f^{min} = f$ as elements of L.

Now as $f_n^{\min} \leq f_n \leq f_n^{\max}$ for each n, and $\lim f_n^{\max} = \lim f_n^{\min}$, we thus know $\lim f_n = f$ as well.

11 Measure

In this final section, we define the measure of a set $S \subseteq K$ using our already defined concepts, and confirm that it is equivalent with the typical definition of measure.

Definition: A set $S \subseteq K$ is *measurable* iff there exists $f_S \in L$ such that $f_S(x) = \chi_S(x)$ for almost all x.

By Theorem 7 we know there is at most one such f_S, and so we define the *measure* of S by

$$m(S) := I(f_S).$$

We now demonstrate that these definitions of measure and measurability are equivalent to the definitions of these concepts in typical study. To do this, we show that our new definitions satisfy all of the desired properties.

(i) Every open set $G \subset K$ is measurable, and $m(G) = V(G)$.

Proof: For open $G \subset K$, let $p_G(x)$ be the distance from x to $K \setminus G$. We see $p_G(x) = 0$ for all $x \in K \setminus G$. Then because G is open, we know for any $x \in G$ there exists an open ball B centered at x that is fully contained in G. The distance from x to $K \setminus G$ then must be at least the radius of B, and so we know $p_G(x) > 0$.

Now for each $n \in \mathbb{Z}^+$, define

$$\varphi^n(s) := \begin{cases} 0 & \text{for} \quad s \leq 0 \\ ns & \text{for} \quad 0 < s \leq \frac{1}{n} \\ 1 & \text{for} \quad \frac{1}{n} < s \end{cases}$$

and define $b_n(x) := \varphi^n(p_G(x))$ for all $x \in K$.

We see then that $\{b_n\}$ is an increasing sequence of continuous functions bounded above by 1, and so by the monotone convergence theorem we know $\{b_n\}$ converges to some $b_G \in L$ with

$$\lim b_n(x) = b_G(x) \text{ almost everywhere.}$$

From the definition of b_n we see $\lim b_n(x) = \chi_G(x)$ for all $x \in K$, and so we must have $(\lim b_n)(x) = b_G(x)$ is the characteristic function of G.

We know each b_n is admissible for G, and so $\lim b_n = b_G$ must be as well, which means $I(b_G) \leq V(G)$.

We also know by definition that every admissible function c for G satisfies $c(x) \leq b_G(x)$ almost everywhere, which by Lemma 7 implies $I(c) \leq I(b_G)$ for all such c. Taking the supremum over all c admissible for G thus gives

$$V(G) \leq I(b_G).$$

Thus we must have $V(G) = I(b_G) = m(G)$.

(ii) For every measurable set S, $m(S) = \inf\{V(G) : G \text{ open }, S \subset G \subset K\}$.

Proof: Let $S \subseteq K$ be chosen arbitrarily.

First, note that if $S \subseteq G$, then $f_G(x) \geq f_S(x)$ almost everywhere. This implies $I(f_G) \geq I(f_S)$, and so we know $V(G) \geq m(S)$.

Next, by Theorem 1(iii) one can show that for any $\epsilon > 0$ there exists a nonnegative continuous function c such that $|c - f_S|_L < \epsilon$ and there exists an open set D with $V(D) < \epsilon$ such that

$$|c(x) - f_S(x)| < \epsilon \text{ for all } x \notin D.$$

Then define $H := \{x \in K : c(x) > 1 - \epsilon\}$. We see then that $x \in S \setminus D$ implies $x \in H$, and so we know $S \subset H \cup D$.

By Chebyshev's inequality we then know $V(H) \leq I(c)/(1-\epsilon)$, and by $|c - f_S|_L < \epsilon$ we know $I(c) < I(f_S) + \epsilon$. Using both of these inequalities with the subadditivity of volume, we thus know

$$V(H \cup D) \leq V(H) + V(D) < \frac{I(f_S) + \epsilon}{1 - \epsilon} + \epsilon = \frac{m(S) + \epsilon}{1 - \epsilon} + \epsilon.$$

Noting again that $S \subset H \cup D$, We thus know for any $\epsilon > 0$ there exists an open set $G \supset S$ such that $V(G) \leq \frac{m(S)+\epsilon}{1-\epsilon} + \epsilon$.

This implies

$$\inf\{V(G) : G \text{ open}, S \subset G\} \leq \lim_{\epsilon \to 0}(\frac{m(S) + \epsilon}{1 - \epsilon} + \epsilon) = m(S).$$

This inequality together with $V(G) \geq m(S)$ for all open $G \supset S$ then implies $\inf\{V(G) : G \text{ open}, S \subset G\} = m(S)$

(iii) For any $f \in L$, $\{x \in K : f(x) < a\}$ is measurable for any $a \in \mathbb{R}$.

Proof: It suffices to show that $S = \{x \in K : f(x) < 0\}$ is measurable for any $f \in L$, as any element of L minus a constant function is still an element of L.

Define

$$\varphi_n(s) := \begin{cases} 1 & \text{for} \quad s \leq -1/n \\ -ns & \text{for} \quad -1/n < s < 0 \\ 0 & \text{for} \quad 0 \leq s \end{cases}$$

and define $g_n := \varphi_n(f)$ for each $n \in \mathbb{Z}^+$.

We see $\{g_n\}$ is an increasing sequence in L bounded above by 1. Thus by the monotone convergence theorem we know $\{g_n\}$ converges to a limit $g \in L$, with

$$g(x) = \lim g_n(x) \text{ almost everywhere.}$$

We see then that $\lim g_n(x) = g(x) = \chi_S(x)$ almost everywhere, and so $S = \{x \in K : f(x) < 0\}$ is measurable.

(iv) The measurable sets form a σ-algebra.

Proof: Let $\mathcal{S}$ be the collection of all measurable sets. To show $\mathcal{S}$ is a σ-algebra, we show it satisfies the necessary properties.

(a): We first show $B \in \mathcal{S}$ implies $K \setminus B \in \mathcal{S}$.

Given $B \in \mathcal{S}$, we know there exists $f_B \in L$ such that $f_B(x) = \chi_B(x)$ almost everywhere.

This means we must have $1 - f_B \in L$ with

$$(1 - f_B)(x) = 1 - \chi_B(x) = \chi_{K \setminus B}(x) \text{ almost everywhere,}$$

which implies $K \setminus B \in \mathcal{S}$.

(b): Next we show that for any $B_1, B_2 \in \mathcal{S}$, we have $B_1 \cap B_2 \in \mathcal{S}$.

Given $B_1, B_2 \in \mathcal{S}$, we know there exist $f_1, f_2 \in L$ such that

$$f_1(x) = \chi_{B_1}(x) \text{ and } f_2(x) = \chi_{B_2}(x) \text{ almost everywhere.}$$

We know f_1 and f_2 are both bounded by the constant 1, so we can take their product $f = f_1 \cdot f_2 \in L$. We then know

$$f(x) = \chi_{B_1}(x)\chi_{B_2}(x) \text{ almost everywhere.}$$

We see for $x \in B_1 \cap B_2$ we have $\chi_{B_1}(x) = \chi_{B_2}(x) = 1$, and so $\chi_{B_1}(x)\chi_{B_2}(x) = 1$.

Then for $x \notin B_1 \cap B_2$, we have either $x \notin B_1$ or $x \notin B_2$. This means at least one of $\chi_{B_1}(x)$ and $\chi_{B_2}(x)$ is 0, and so $\chi_{B_1}(x)\chi_{B_2}(x) = 0$.

Thus we see

$$\chi_{B_1}(x)\chi_{B_2}(x) = \chi_{B_1 \cap B_2}(x) \text{ for all } x \in K,$$

and so $f(x) = \chi_{B_1 \cap B_2}(x)$ almost everywhere, which implies $B_1 \cap B_2 \in \mathcal{S}$.

(c): Finally, we show for any countable subcollection $\{B_n\}_{n=1}^{\infty} \subset \mathcal{S}$, we also have $\bigcup_{n=1}^{\infty} B_n \in \mathcal{S}$.

First, note that for any two $B_1, B_2 \in \mathcal{S}$, we know

$$((B_1^c) \cap (B_2^c))^c = B_1 \cup B_2 \in \mathcal{S}.$$

Thus by induction we know $\mathcal{S}$ is closed under finite unions. We now use this to show $\mathcal{S}$ is also closed under countably infinite unions.

Let $\{B_n\}_{n=1}^{\infty} \subset \mathcal{S}$. For each $k \in \mathbb{Z}^+$, define $T_k := \bigcup_{n=1}^{k} B_n$, noting that $T_k \subset T_{k+1}$ for each k.

We know for each k there exists some $f_k \in L$ such that $f_k(x) = \chi_{T_k}(x)$ almost everywhere.

Then by $T_k \subset T_{k+1}$, we know $f_k(x) \leq f_{k+1}(x)$ almost everywhere, which implies $f_k \leq f_{k+1}$ as elements of L. We also know $f_k(x) \leq 1$ almost everywhere by definition of the characteristic function.

Thus $\{f_k\}$ is an increasing sequence in L that is bounded above, and so by the monotone convergence theorem we know $\{f_k\}$ converges to some $f \in L$ with

$$\lim f_k(x) = f(x) \text{ almost everywhere.}$$

Now define $T := \bigcup_{k=1}^{\infty} T_k = \bigcup_{n=1}^{\infty} B_n$. We see then that

$$\lim \chi_{T_k}(x) = \chi_T(x) \text{ for all } x \in K,$$

which then implies $f(x) = \chi_T(x)$ almost everywhere, and so $T = \bigcup_{n=1}^{\infty} B_n \in \mathcal{S}$.

■