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Isosceles triangles in fractal subsets of $\mathbb{R}^d$ for $d \geq 4$

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Abstract. We improve on the result in [1] by generalizing the method of recovering equilateral triangles to recovering general isosceles triangles with angle $\theta \in (0, \pi)$. While this result is a specific case of [4], the method we use has hope for a computable dimensional constant s_0 independent of θ.

1 Introduction

Existence of geometric configurations in subsets of Euclidean space is a problem which has been studied in detail. The first problem of this type was studied by Falconer. He asked how large does the hausdorff dimension s of a compact set $E \subset \mathbb{R}^d$ for $d \geq 2$ need to be to ensure the distance set of E, $\Delta(E) = \{|x - y| : x, y \in E\} \subset \mathbb{R}$ has positive lebesgue measure? He intially showed $s > \frac{d}{2}$ is necessary and in [6], he proved a set of haudorff dimension $\geq \frac{d+1}{2}$ contains a distance set of positive lebesgue measure. This exponent was improved in [7] to sets of hausdorff dimension $\geq \frac{d}{2} + \frac{1}{3}$ and it started a slough of questions about how large a subset of d-dimensional euclidean space needs to be for a set of geometric objects formed by the subset to have certain size.

This led to the next question, which was how large did sets need to be to contain geometric configurations? For k-chains with specified gaps, [14] proved that if the dimension is greater than $\frac{d+1}{2}$, such a chaim must exist. This was further generalized to trees in [15]. In the case of triangles, we have [13] which states that in $d = 2$, a set of hausdorff dimension $\geq \frac{8}{5}$ generates a set of triangles which have positive 3-dimensional lebesgue measure.

A more difficult problem is what are the restrictions needed for a set to contain a similar copy of a particular geometric configuration? In the case of distances, this is how large a set needs to be to contain a particular distance, which is much more difficult than asking about the distance set as a whole. For general simplices of k vertices, the problem has been answered for subsets of $\mathbb{R}^d$ with positive lebesgue measure in [5], which gives the following result.

Theorem 1.1. *Let $E \subset \mathbb{R}^d$ be of positive upper Lebesgue density in the sense that*

$$\limsup_{R \to \infty} \frac{L^d(E \cap [-R, R]^d)}{(2R)^d} > 0,$$

where L^d represents d-dimensional Lebesgue measure. Let E_δ denote the δ-neighborhood of E. Let $V = \{\mathbf{0}, v^1, v^2, ..., v^{k-1}\} \subset \mathbb{R}^d$ be where $k \geq 2$ is a positive integer. Then there exists $l_0 > 0$ s.t. for any $l > l_0$ and any $\delta > 0$ there exists $\{x^1, ..., x^k\} \subset E_\delta$ congruent to $lV = \{\mathbf{0}, lv^1, ..., lv^{k-1}\}$.

This result nearly settles the issue of simplexes in sets of positive upper lebesgue density, though it is still open if the δ-neighborhood of E can be adjusted or removed entirely in special cases, such as the case of non-degenerate triangles. Degenerate triangles are a different story as Bourgain shows in [8] the δ-neighborhood is absolutely necessary.

As positive upper lebesgue density is almost completely answered, the next question is what happens in sets of measure 0. More specifically, what happens for compact sets of hausdorff dimension $s_0 < d$, as that is the next natural way of defining size for sets of measure 0. In dimension 1, there is already an example by Keleti in [11] which gives a subset of $[0,1]$ of hausdorff dimension 1 which does not contain any arithmetic progression of length 3 (an equilateral/isosceles triangle in dimension 1). But in [12], they show additional structure assumptions gives a dimensional constant $s_0 < 1$ such that a subset of $[0,1]$ with hausdorff dimension $> s_0$ contains some arithmetic progression of length 3 assuming the additional structural assupmtions.

In dimension 2, a similar problem occurs. Independent examples by [9] and [10] show a subset of $\mathbb{R}^2$ with hausdorff dimension 2 does not need to contain the vertices of an equilateral triangle. But once again, Theorem 1.6 in [2] gives additional structural assumptions which allow for recovery of equilateral triangles in subsets of $\mathbb{R}^d$, $d \geq 2$. More specifically, they require a measure supported on E with decay conditions on its Fourier transform.

The Fourier decay conditions were removed in [1] in the specific case of equilateral triangles. In this paper, we adjust improve the method and seek to extend this to all non-degenerate isosceles triangles with angle $\theta \in (0, \pi)$.

It is worth noting that [4], a more recent paper, managed to generalize this result to k-simplices in $\mathbb{R}^k$ at the cost of losing computability of the dimensional constant. Their result is stated below.

Theorem 1.2. *Let V be a k-simplex in $\mathbb{R}^k$ and $d(V)$ be the diameter of V. Let $r(V)$ be the minimum distance between any vertex and the affine space spanned by the $k-1$ remaning vertices. If $\delta(V) = \frac{r(V)}{d(V)}$, which is positive iff V is non-degenerate, then for $\delta > 0$, there exists $s_0(k, \delta) < k$ s.t. if $E \subset \mathbb{R}^k$ is compact and has hausdorff dimension $\geq s_0$, then E contains the vertices of V' similar to V with $\delta(V') \geq \delta$.*

In the case of the k-simplex containing the vertices of an isosceles triangle, our result is recovered. But the s_0 in the above result is not explicitly computable. Our result has some hope for computability of the dimensional constant.

1.1 Main results

The main result of this paper generalizes that of [1].

Theorem 1.3. *Let E be a compact subset of $\mathbb{R}^d$, $d \geq 4$ and μ a probability Frostman measure on E with $\mu(B(x,r)) \leq c_\mu r^s$ for all $x \in \mathbb{R}^d$, $r > 0$. There exists $s_0 = s_0(c_\mu, d) < d$, s.t. $s > s_0$, then $\forall \theta \in (0, \pi)$, E contains the vertices of an isoceles triangle with angle θ.*

The proof is similar to that of [1], done in 3 similar steps. Throughout the paper, $\approx$ and $\lesssim$ will be used to denote equality or inequality up to a constant depending on d, c_μ, θ.

2 Construction of the measure

First notice it is enough to consider a fixed $\theta \in (0, \pi)$ and through the proof show this choice of θ does not affect the dimensional constant s_0. Consider the surface

$$\Sigma_\theta = \{(x, y) \in \mathbb{R}^{2d} : |x| = |y| = 1, x \cdot y = \cos(\theta)\}.$$

Let σ denote the surface measure of Σ_θ. Let $\phi \in C_0^\infty$ be a bump function supported in the unit ball with $\int \phi = 1$. Denote $\phi_\delta = \delta^{-d}\phi(\frac{\cdot}{\delta})$ which is just dilating the function without changing the integral. Define $\mu_\delta = \mu * \phi_\delta$, which we will use next.

Define a new measure ν on the surface

$$E_\theta = \{(x, y, z) \in E \times E \times E : |z - x| = |z - y| = \frac{|x - y|}{\sqrt{2 - 2\cos(\theta)}}\},$$

by the limit

$$d\nu = \lim_{\delta \to 0} \mu_\delta(z)\mu_\delta(z + tx)\mu_\delta(z + ty)t^{d-1}dzd\sigma(x, y)dt.$$

We define E_θ in this more ugly manner to ensure that it is nonempty. With this definition, we trivially have singletons $(x, x, x) \in E_\theta$ for $x \in E$, but will need to do more work to show nontrivial triangles are also contained in E_θ.

Intuitively, our measure is taking shrinking δ-neighborhoods around the points $z, z+tx, z+ty$ and using existence of the limit to show existence of the triple $(z, z+tx, z+ty) \in E_\theta$ and conclude existence of an isosceles triangle. One should also note that $t \in [0, 1]$.

But this construction hinges on existence of this limit $d\nu$, which we prove next for sufficiently large hausdorff dimension.

Theorem 2.1. *Let $s > \frac{2}{3}d + 1$, then there exists a sequence $\delta_j \to 0$ s.t. the limit $d\nu$ exists weakly.*

To prove this, it suffices to show

$$\int\int \mu_\delta(z)\mu_\delta(z + tx)\mu_\delta(z + ty)dzd\sigma(x, y) \lesssim C(t).$$

Where $C(t)$ is some function depending on t but independent of δ. If $t^{d-1}C(t)$ is integrable over $t \in [0, 1]$, then by the Banach-Alaoglu theorem we are done.

Now we apply Fourier inversion to get

$$\begin{aligned}
&\int\int \mu_\delta(z)\mu_\delta(z + tx)\mu_\delta(z + ty)dzd\sigma(x, y) \\
&= \int \ldots \int \hat{\mu_\delta}(\xi)\hat{\mu_\delta}(\eta)\hat{\mu_\delta}(\zeta)e^{2\pi i((z,z,z)+(tx,ty,0))\cdot(\xi,\eta,\zeta)}dzd\sigma(x, y)d\xi d\eta d\zeta \\
&= \int\int\int \hat{\mu_\delta}(\xi)\hat{\mu_\delta}(\eta)\hat{\mu_\delta}(\zeta)(e^{2\pi i(z,z,z)\cdot(\xi,\eta,\zeta)}dz)\hat{\sigma}(-t\xi, -t\eta)d\xi d\eta d\zeta
\end{aligned}$$

But as a distribution,

$$\int e^{2\pi i(z,z,z)\cdot(\xi,\eta,\zeta)}dz = \delta(\xi + \eta + \zeta),$$

so we can collapse ζ into $-\xi - \eta$ and get

$$\int\int \hat{\mu_\delta}(\xi)\hat{\mu_\delta}(\eta)\hat{\mu_\delta}(-\xi - \eta)\hat{\sigma}(-t\xi, -t\eta)d\xi d\eta,$$

which after expanding the convolutions gives us

$$\int\int \hat{\mu}(\xi)\hat{\mu}(\eta)\hat{\mu}(-\xi - \eta)\hat{\phi}(\delta\xi)\hat{\phi}(\delta\eta)\hat{\phi}(-\delta\xi - \delta\eta)\hat{\sigma}(-t\xi, -t\eta)d\xi d\eta. \tag{2.1}$$ {?}

Estimating this takes a little more effort. The $\hat{\phi}$ can be estimated out as they are bounded by 1, but the other terms are a little more tricky. After estimating out small values of $|\xi|, |\eta|$, We will decompose the integral based on values for $|\xi|$ and $|\eta|$. Diadically decomposing one variable at a time, an estimate will be given for the range where $\frac{|\xi|}{|\eta|} \in [1, 2]$ and a symmetric estimate gives $\frac{|\xi|}{|\eta|} \in [\frac{1}{2}, 1]$. This estimate will require a few lemmas. We will then reduce the cases where one of $|\xi|, |\eta|$ is disproportionately larger than the other to the cases where they are proportional. This resolves all pairs $(|\xi|, |\eta|)$ and thus estimate the entire integral.

Lemma 2.2. *With the given notation above, we have*

$$|\hat{\sigma}(\xi,\eta)| \le C_{\theta,d}^{-\frac{1}{2}} |\xi + g_\theta \eta|^{-\frac{1}{2}} \cdot |\xi \wedge \eta|^{-\frac{d-2}{2}}.$$

Where $C_{\theta,d} = |\csc(\theta)|^{d-2}$ *and* $|\xi \wedge \eta|^{-\frac{d-2}{2}} = |\xi|^{\frac{d-2}{2}} |\eta|^{\frac{d-2}{2}} \sin(\theta_{\xi,\eta})^{-\frac{d-2}{2}}$, *with* $\theta_{\xi,\eta}$ *being the angle between* ξ *and* η.

This lemma will be proven by a stationary phase argument. The next lemma will be what we need to conclude the bound.

Lemma 2.3.

$$\int_{|\xi|\approx 2^j} |\xi + g_\theta \eta|^{-1} \sin(\theta_{\xi,\eta})^{-(d-2)} \lesssim 2^{j(d-1)}. \quad (2.2)\ \{?\}$$

Now assuming these two lemmas, we will show the integral is bounded. We need to estimate

$$\int\int |\hat{\mu}(\xi)||\hat{\mu}(\eta)||\hat{\mu}(\xi+\eta)||\hat{\sigma}(-t\xi,-t\eta)| d\xi d\eta. \quad (2.3)\ \{?\}$$

Diadically decompose $|\eta|$ and $|\xi|$ into intervals of $[2^j, 2^{j+1})$ and take the two cases $j < 0$ and $j \ge 0$ separately. For $j < 0$, notice that along with $\hat{\phi}$, we can also estimate out $\hat{\mu}$ as well because the fourier transform is uniformly bounded by the L^1 norm and μ being a probability measure means the L^1 norm is just 1. So this reduces to estimating

$$\int\int_{|\xi|,|\eta|<1} |\hat{\sigma}(-t\xi,-t\eta)| d\xi d\eta dt.$$

But we can also notice that $|\hat{\sigma}(-t\xi,-t\eta)|$ is real analytic and thus bounded on $|\xi|, |\eta| < 1$, which finishes bounding the neighborhood of the origin. Noticing that t^{d-1} is integrable over $[0,1]$, we are done with that case.

For the $j \ge 0$ case, more work needs to be done. Instead of estimating out the $|\hat{\mu}|$ terms, we will use the Frostman property of μ to get

$$\int_{|\xi|\in[2^j,2^{j+1})} |\hat{\mu}(\xi)|^2 \lesssim 2^{j(d-s)}.$$

This follows from $\mu(B(x,r)) \le c_\mu r^s$.

With the two lemmas and the above estimate, we can now look at the integral over $|\xi|, |\eta| \ge 1$. We can further split this integral into two cases, when $|\xi|$ and $|\eta|$ are approximately equal and when one is disproportionately larger than the other. For both cases, we can assume that $|\xi| \ge |\eta|$ as the other case can be handled symmetrically.

Assuming $|\eta| \in [2^j, 2^{j+1})$, $1 \le \frac{|\xi|}{|\eta|} \le 2$, then applying Lemma 2.2 gives us

$$\int_{2^j}^{2^{j+1}} \int_{|\xi|\in[|\eta|,2|\eta|]} |\hat{\mu}(\xi)||\hat{\mu}(\eta)||\hat{\mu}(\xi+\eta)||\hat{\sigma}(t\xi,t\eta)| d\xi d\eta$$

$$\lesssim t^{-d+\frac{3}{2}} 2^{-j(d-2)} \int\int_{|\xi|\in[|\eta|,2|\eta|]} |\hat{\mu}(\xi)||\hat{\mu}(\eta)||\hat{\mu}(\xi+\eta)||\xi + g_\theta \eta|^{-\frac{1}{2}} \sin(\theta_{\xi,\eta})^{-\frac{d-2}{2}} d\xi d\eta.$$

Focusing on just the integral in ξ, we get

$$\int_{|\xi|\in[|\eta|,2|\eta|]} |\hat{\mu}(\xi)||\hat{\mu}(\eta)||\hat{\mu}(\xi+\eta)||\xi + g_\theta \eta|^{-\frac{1}{2}} \sin(\theta_{\xi,\eta})^{-\frac{d-2}{2}} d\xi.$$

But we don't want all of the terms since we need cauchy schwartz to apply Lemma 2.3. So remove $|\hat{\mu}(\xi)||\hat{\mu}(\eta)|$ and just consider

$$\int_{|\xi|\in[|\eta|,2|\eta|]} |\hat{\mu}(\xi+\eta)||\xi+g_\theta\eta|^{-\frac{1}{2}}\sin(\theta_{\xi,\eta})^{-\frac{d-2}{2}}d\xi.$$

Applying cauchy schwartz we get

$$\begin{aligned}
&\int_{|\xi|\in[|\eta|,2|\eta|]} |\hat{\mu}(\xi+\eta)||\xi+g_\theta\eta|^{-\frac{1}{2}}\sin(\theta_{\xi,\eta})^{-\frac{d-2}{2}}d\xi \\
&\leq (\int_{|\xi|\in[|\eta|,2|\eta|]} |\hat{\mu}(\xi+\eta)|^2 d\xi)^{\frac{1}{2}}(\int_{|\xi|\in[|\eta|,2|\eta|]} |\xi+g_\theta\eta|^{-1}\sin(\theta_{\xi,\eta})^{-d-2}d\xi)^{\frac{1}{2}} \\
&\lesssim (2^j)^{\frac{d-s+\epsilon}{2}}(\int_{|\xi|\in[|\eta|,2|\eta|]} |\xi+g_\theta\eta|^{-1}\sin(\theta_{\xi,\eta})^{-d-2}d\xi)^{\frac{1}{2}}.
\end{aligned}$$

Where the last line is by the estimate using the frostman property of the measure, with the arbitrary $\epsilon > 0$ coming from the fact that we are dealing with $\hat{\mu}(\xi+\eta)$ instead of $\hat{\mu}(\xi)$. Now applying Lemma 2.3 gives us

$$\begin{aligned}
&(2^j)^{\frac{d-s+\epsilon}{2}}(\int_{|\xi|\in[|\eta|,2|\eta|]} |\xi+g_\theta\eta|^{-1}\sin(\theta_{\xi,\eta})^{-d-2}d\xi)^{\frac{1}{2}} \\
&\lesssim (2^j)^{\frac{d-s+\epsilon}{2}}(2^j)^{\frac{d-1}{2}} = 2^{j(\frac{2d-s-1+\epsilon}{2})}.
\end{aligned}$$

This finishes the estimate for the integral in just ξ. Our range of ξ was $[|\eta|, 2|\eta|]$, but we only really used that the upper and lower bounds for ξ were on the order of 2^j. So integrating $|\xi| \in [2^j, 2^{j+1}]$ would give the same upper bound up to constants. But notice the integrand is approximately symmetric in ξ and η, so we get a symmetric estimate

$$\int_{|\eta|\in[2^j,2^{j+1}]} |\hat{\mu}(\xi+\eta)||\xi+g_\theta\eta|^{-\frac{1}{2}}\sin(\theta_{\xi,\eta})^{-\frac{d-2}{2}}d\eta \lesssim 2^{j(\frac{2d-s-1+\epsilon}{2})}.$$

This lets us apply Schur's test. Let the operator T_j be denoted by

$$T_j f(\eta) = \int_{|\xi|\in[|\eta|,2|\eta|]} |\hat{\mu}(\xi+\eta)||\xi+g_\theta\eta|^{-\frac{1}{2}}\sin(\theta_{\xi,\eta})^{-\frac{d-2}{2}}f(\xi)d\xi.$$

Then T_j is a bounded operator from L^2 to L^2 on each annuli $|\eta| \in [2^j, 2^{j+1}]$ with norm $\lesssim 2^{j(\frac{2d-s-1+\epsilon}{2})}$. In other words, we can write

$$\begin{aligned}
&t^{-d+\frac{3}{2}}2^{-j(d-2)}\int\int_{|\xi|\in[|\eta|,2|\eta|]} |\hat{\mu}(\xi)||\hat{\mu}(\eta)||\hat{\mu}(\xi+\eta)||\xi+g_\theta\eta|^{-\frac{1}{2}}\sin(\theta_{\xi,\eta})^{-\frac{d-2}{2}}d\xi d\eta \\
&= t^{-d+\frac{3}{2}}2^{-j(d-2)}\int T|\hat{\mu}|(\eta)|\hat{\mu}(\eta)|d\eta \leq t^{-d+\frac{3}{2}}2^{-j(d-2)}||\hat{\mu}||_{L^2}||T|\hat{\mu}|||_{L^2} \\
&\lesssim t^{-d+\frac{3}{2}}2^{-j(d-2)+j(\frac{2d-s-1+\epsilon}{2})}||\hat{\mu}||_{L^2}^2.
\end{aligned}$$

Where the second step follows from cauchy schwartz and the L^2 norms are taken over annuli. The last step follows from schur's test. But notice again we have the estimate $||\hat{\mu}||_{L^2}^2 \lesssim 2^{j(d-s)}$, which gives us

$$t^{-d+\frac{3}{2}}2^{-j(d-2)+j(\frac{2d-s-1+\epsilon}{2})+j(d-s)} = t^{-d+\frac{3}{2}}2^{-j\frac{3s-2d-3-\epsilon}{2}}.$$

Now that we have the estimate for each annuli, we can sum them up and get the entire integral is $\lesssim t^{-d+\frac{3}{2}}$ with the condition that $s > \frac{2}{3}d + 1 + \epsilon$ for any $\epsilon > 0$ which just reduces to $s > \frac{2}{3}d + 1$. Adding on the fact that we have a t^{d-1} to account for, the integral in all variables is bounded by

$$\int_0^1 t^{d-1} t^{-d+\frac{3}{2}} dt = \int_0^1 t^{\frac{1}{2}} < \infty.$$

Which gives us the overall integral is bounded.

But this is only the case of the integral when $\frac{|\xi|}{|\eta|} \in [1, 2]$ or $[\frac{1}{2}, 1]$ by symmetry. We still need to cover the cases where one variable is disproportionately larger than the other.

It turns out that these cases can reduce to the cases where $\frac{|\xi}{|\eta|} \in [\frac{1}{2}, 2]$. Assume for now that $\frac{|\xi}{|\eta|} > 2$ and cover the $< \frac{1}{2}$ case by symmetry. But notice this means $\frac{|\xi|}{|\eta+\xi|} \in [\frac{1}{2}, 2]$, which we will exploit.

Now consider the original integrand which we were considering. We wanted to estimate

$$|\hat{\mu}(\xi)||\hat{\mu}(\eta)|\hat{\mu}(\xi + \eta)||\hat{\sigma}(-t\xi, -t\eta)| \lesssim |\hat{\mu}(\xi)||\hat{\mu}(\eta)|\hat{\mu}(\xi + \eta)|C_{\theta,d}^{-\frac{1}{2}} |\xi + g_\theta \eta|^{-\frac{1}{2}} \cdot |\xi \wedge \eta|^{-\frac{d-2}{2}}.$$

But one interesting thing to note is that $|\xi \wedge \eta|^{-\frac{d-2}{2}} = |\xi \wedge (\eta + \xi)|^{-\frac{d-2}{2}}$. So changing η with $\eta + \xi$ does not affect the wedge product part of the estimate. On the other hand, we have $|\xi + g_\theta \eta|^{-\frac{1}{2}}$ and we want to replace it with $|\xi + g_\theta(\eta + \xi)|^{-\frac{1}{2}}$. Notice that $|\xi + g_\theta \eta| \in [\frac{1}{2}|\xi|, \frac{3}{2}|\xi|]$ by $\frac{|\xi|}{|\eta|} > 2$ and rotations preserving norm. On the other hand, $|\xi + g_\theta(\eta + \xi)| \in (0, \frac{5}{2}|\xi|)$ by the same reasoning. So $|\xi + g_\theta \eta|^{-\frac{1}{2}} \leq 5|\xi + g_\theta(\eta + \xi)|^{-\frac{1}{2}}$. So we really can replace η with $\eta + \xi$ and it will not affect the estimates.

More specifically, we have

$$\begin{aligned}
&\int_{2^j}^{2^{j+1}} \int_{2|\eta|<|\xi|} |\hat{\mu}(\xi)||\hat{\mu}(\eta)||\hat{\mu}(\xi + \eta)||\hat{\sigma}(-t\xi, -t\eta)| d\eta d\xi \\
\lesssim &\int \int_{2|\eta|<|\xi|} |\hat{\mu}(\xi)||\hat{\mu}(\eta)||\hat{\mu}(\xi + \eta)||\xi + g_\theta \eta|^{-\frac{1}{2}} \cdot |\xi \wedge \eta|^{-\frac{d-2}{2}} d\eta d\xi \\
\lesssim &\int \int_{2|\eta|<|\xi|} |\hat{\mu}(\xi)||\hat{\mu}(\eta)||\hat{\mu}(\xi + \eta)||\xi + g_\theta(\eta + \xi)|^{-\frac{1}{2}} \cdot |\xi \wedge (\eta + \xi)|^{-\frac{d-2}{2}} d\eta d\xi \\
\lesssim &\int \int_{|\zeta| \in [\frac{1}{2}|\xi|, 2|\xi|]} |\hat{\mu}(\xi)||\hat{\mu}(\zeta - \xi)||\hat{\mu}(\zeta)||\xi + g_\theta \zeta|^{-\frac{1}{2}} \cdot |\xi \wedge \zeta|^{-\frac{d-2}{2}} d\zeta d\xi.
\end{aligned}$$

In the final step we let $\zeta = \eta + \xi$ and note the inequality by the fact that the integrand is nonnegative and we are increasing the region of integration. This gives us an integral which can be split into $\zeta \in [\frac{1}{2}|\xi|, |\xi|)$ and $\zeta \in [|\xi|, 2|\xi|]$ and estimated accordingly by the methods previously used.

So combining the two cases gives the boundedness of the original integral which completes the construction of the measure.

3 Positivity of the integral

After we have shown the measure to be well defined and give what we want, we need to show our initial set E contains an isosceles triangle of angle θ by applying the measure ν. This is done through an integration argument.

Before getting into the argument, we need to set up a bit of machinery.

Definition 3.1. *Fix integers $n \geq 2$, $p \geq 3$, and $m = n\lceil \frac{p+1}{2} \rceil$. If $B_1, ..., B_p$ are $n \times (m-n)$ matrices, we say the collection $\{B_1, ..., B_p\}$ is non-degenerate if*

$$rank \begin{bmatrix} B_{i_2} - B_{i_1} \\ \vdots \\ B_{i_{\frac{m}{n}}} - B_{i_1} \end{bmatrix} = m - n$$

for any choice of indices $i_1, ..., i_{\frac{m}{n}} \in \{1, ..., p\}$.

With this definition, we have Proposition 5.1 from [2] which says

Theorem 3.2. *Let*

$$\Lambda(f) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \prod_{j=1}^{k} f(z + B_j x) dz dx.$$

Where $\{B_1, ..., B_k\}$ are non-degenerate. Then for every $\lambda, M > 0$, there exists a constant $c(\lambda, M) > 0$ s.t. if $f : [0,1]^n \to \mathbb{R}$, $f \in [0, M]$, $\int f \geq \lambda$, then $\Lambda(f) \geq c(\lambda, M)$.

In essence, this theorem is a way to get positivity for an integral of a product of functions which have inputs transformed by non-degenerate matrices, which would applies to what we have.

One notable observation is that

$$\int\int\int \mu_\delta(z)\mu_\delta(z+tx)\mu_\delta(z+ty)t^{d-1} dz d\sigma(x,y) dt = \int\int\int \mu_\delta(z)\mu_\delta(z+x)\mu_\delta(z+g_\theta x) dz dx dg_\theta.$$

Where dg_θ is the measure on the set of θ-rotation (which is a compact set). Also notice that $||\mu_\delta||_\infty \lesssim \delta^{s-d}$ and $\int \mu_\delta = 1$, which are conditions for the above theorem.

Applying the theorem with $n = 2m = 2d$, $k = 3$, $B_1 = 0$, $B_2 = id$, and $B_3 = g_\theta$, we get

$$\int\int \mu_\delta(z)\mu_\delta(z+x)\mu_\delta(z+g_\theta x) dz dx \gtrsim c(\delta^{s-d}).$$

Using compactness of the set of θ-rotations, we can integrate in dg_θ and get the overall integral (denoted by I_δ) is still bounded above by $c(\delta^{s-d})$ mutliplied by a constant depending on the size of the set of θ-rotations.

Another observation is that

$$\begin{aligned} \int d\nu &= \lim_{\delta \to 0} \int\int\int \hat{\mu}(\xi)\hat{\mu}(\eta)\hat{\mu}(-\xi-\eta)\hat{\phi}(\delta\xi)\hat{\phi}(\delta\eta)\hat{\phi}(-\delta\xi-\delta\eta)\hat{\sigma}(-t\xi,-t\eta)t^{d-1} d\xi d\eta dt \\ &= \int\int\int \hat{\mu}(\xi)\hat{\mu}(\eta)\hat{\mu}(-\xi-\eta)\hat{\sigma}(-t\xi,-t\eta)t^{d-1} d\xi d\eta dt. \end{aligned}$$

So we have the estimate

$$\begin{aligned} |I_\delta - \int d\nu| &\lesssim \int\int\int_{|\xi|,|\eta|>\delta^{-1}} |\hat{\mu}(\xi)||\hat{\mu}(\eta)||\hat{\mu}(\xi+\eta)|\hat{\sigma}(-t\xi,-t\eta)|t^{d-1} d\xi d\eta dt \\ &\lesssim \delta^{\frac{3s-2d-3}{2}}. \end{aligned}$$

The last inequality follows from the exact same method of estimating for $\delta = 1$ done earlier in this paper. So to get positivity of $\int d\nu$, we need to have $\delta^{\frac{3s-2d-3}{2}} < c(\delta^{s-d})$. Now we can choose an appropriate value for δ.

Letting $\delta = e^{\frac{1}{s-d}}$, we get $\delta^{\frac{3s-2d-3}{2}} = e^{\frac{3s-2d-3}{2(s-d)}} \to 0$ as $s \to d$ and $c(\delta^{s-d}) = c(e) > 0$ which is independent of s. So for s sufficiently close to d, the result follows. We can find some s_0 to denote this threshold, which will be discussed later.

3.1 Non-degeneracy of the measure ν

While we have shown $\int d\nu > 0$, we need to show that our set Σ_θ contains something nontrivial. This requires a dominated convergence argument and some additional work. This argument is also the same as that presented in [1]

Define the function

$$F_\delta(t) = \int\int \mu_\delta(z)\mu_\delta(z+tx)\mu_\delta(z+ty)t^{d-1}dzd\sigma(x,y)$$
$$= \int\int \hat{\mu}(\xi)\hat{\mu}(\eta)\hat{\mu}(-\xi-\eta)\hat{\phi}(\delta\xi)\hat{\phi}(\delta\eta)\hat{\phi}(-\delta\xi-\delta\eta)\hat{\sigma}(-t\xi,-t\eta)t^{d-1}d\xi d\eta.$$

From the work in defining the measure properly, we have the estimate that

$$|F_\delta(t)| \leq \int\int |\hat{\mu}(\xi)||\hat{\mu}(\eta)||\hat{\mu}(\xi+\eta)||\hat{\sigma}(t\xi,t\eta)|t^{d-1}d\xi d\eta \lesssim t^{\frac{1}{2}}.$$

As we are integrating $t \in [0,1]$ and the upper bound is independent of δ, we can apply dominated convergence theorem to get

$$\lim_{\delta\to 0} F_\delta(t) = \int\int \hat{\mu}(\xi)\hat{\mu}(\eta)\hat{\mu}(-\xi-\eta)\hat{\sigma}(-t\xi,-t\eta)t^{d-1}d\xi d\eta$$

is a function in $L^1([0,1])$ and that

$$\int_0^1 \lim_{\delta\to 0} F_\delta(t)dt = \lim_{\delta\to 0}\int_0^1 F_\delta(t)dt.$$

However, we showed earlier that this integral is positive. Namely that

$$0 < \int_0^1 \lim_{\delta\to 0} F_\delta(t)dt = \lim_{\delta\to 0}\int_0^1 F_\delta(t)dt < \infty.$$

But $\lim_{\delta\to 0} F_\delta(t) \in L^1([0,1])$ implies that there exists $t_0 > 0$ s.t.

$$0 < \int_{t_0}^1 \lim_{\delta\to 0} F_\delta(t)dt = \lim_{\delta\to 0}\int_{t_0}^1 F_\delta(t)dt < \infty.$$

Expanding things out and using our sequence $\delta_j \to 0$, we get that

$$\lim_{\delta_j\to 0}(\int_{t_0}^1 F_\delta(t)dt) = \lim_{\delta_j\to 0}(\int_{t_0}^1\int\int \mu_{\delta_j}(x)\mu_{\delta_j}(x+ty)\mu_{\delta_j}(x+tz)t^{d-1}dxd\sigma(y,z)dt).$$

is a well defined measure for $t \in [t_0, 1]$. The set that this restricted measure is defined on is

$$\{(x,y,z) \in E\times E\times E : |x-y| = |x-z| = \frac{|y-z|}{\sqrt{2-2\cos(\theta)}} > t_0\}.$$

And positivity of this measure means the set it is defined on must be nonempty. So we have E containing some nontrivial isosceles triangle of angle $\theta \in (0,\pi)$ which is arbitrary.

4 Proof of lemmas

4.1 Proof of Lemma 2.2

The proof of Lemma 2.2 requires a stationary phase argument. More specifically, if K represents the determinant of the hessian of the phase function evaulated at the critical point, then $|\hat{\sigma}(\xi,\eta)| \lesssim K^{-\frac{1}{2}}$. So we just need to compute K, which means computing the determinant of the hessian and the critical point.

For notational reasons, let us denote $u = (u_1, u') \in \mathbb{R}^{d-1}$ and I to be the identity matrix. Let $f(u) = \sqrt{1-|u|^2}$ and compute

$$Df(u) = -\frac{u}{f}$$

$$D^2 f(u) = -\frac{\delta_{i,j}}{f} - \frac{u^{\otimes 2}}{f^3}.$$

Now define the following vectors $u^0 = (0,0)$, $v^0 = (\sin(\theta), 0)$, $x^0 = (u^0, f(u^0))$, and $y^0 = (v^0, f(v^0))$. Using a partition of unity and rotation invariance of σ, we can restrict ourselves to just a neighborhood of (x^0, y^0). Computations will be done up to $O(|u,v|^3)$, as indicated by $\approx_2$.

Let (x,y) be an isosceles triangle about (x^0, y^0) which is contained in Σ_θ. This means that $|x| = |y| = 1$, which gives us

$$\begin{aligned} x(u) &\approx_2 (u + u^0, 1 - \frac{1}{2}|u|^2) \\ y(u) &\approx_2 (v + v^0, \cos(\theta) - \tan(\theta)v_1 - \frac{1}{2}\sec(\theta)|v|^2 - \frac{1}{2}\tan^2(\theta)\sec(\theta)v_1^2) \\ &= (v + v^0, \cos(\theta) - \tan(\theta)v_1 - \frac{\sec^3(\theta)}{2}v_1^2 - \frac{\sec^2(\theta)}{2}|v'|^2) \end{aligned}$$

Using the further restriction that $x \cdot y = \cos(\theta)$, we can conclude

$$\frac{\sec^3(\theta)}{2}v_1^2 + (\tan(\theta) - u_1)v_1 = u' \cdot v' + \sin(\theta)u_1 - \frac{\cos(\theta)}{2}|u|^2 - \frac{\sec(\theta)}{2}|v'|^2.$$

Now letting $t = a_1 s + a_2 s^2$, we get $s \approx_2 \frac{t}{a_1} - \frac{a_2}{a_1^3}t^2$. We now use the implicit function theorem in one variable to solve for v_1 in terms of the remaining variables.

$$v_1 \approx_2 \cot(\theta)u' \cdot v' + \cos(\theta)u_1 - \frac{\sin(\theta)}{2}u_1^2 - \frac{\csc(\theta)\cos^2(\theta)}{2}|u'|^2 - \frac{\csc(\theta)}{2}|v'|^2.$$

With v_1 written in terms of the remaining variables u_1, u', v', we get the following chart for a neighborhood of (x^0, y^0),

$$\begin{aligned} (x,y) \approx_2 (u, 1 - \frac{1}{2}|u|^2, \sin(\theta) + \cot(\theta)u' \cdot v' + \cos(\theta)u_1 - \frac{\sin(\theta)}{2}u_1^2 - \frac{\csc(\theta)\cos^2(\theta)}{2}|u'|^2 \\ - \frac{\csc(\theta)}{2}|v'|^2, v', \cos(\theta) - u' \cdot v' - \sin(\theta)u_1 - \frac{\cos(\theta)}{2}u_1^2 + \frac{\cos(\theta)}{2}|u'|^2). \end{aligned}$$

Where the parameters $(u_1, u', v') \in \mathbb{R}^{2d-3}$. Under this parametrization, we can define a smooth function ψ near the origin which satisfies

$$d\sigma = \psi(u_1, u', v')du_1 du' dv'.$$

integraing and applying a fourier transform to both sides, we get

$$\hat{\sigma}(\xi, \eta) = \int e^{-2\pi i (x,y)\cdot(\xi,\eta)} \psi du du' dv'.$$

Here, (x, y) is defined by the parametrization done previously. So the phase is $\Phi = x\xi + y\eta$ and we need to compute $\nabla_{u,v'}\Phi$ and $\nabla^2_{u,v'}\Phi$.

For the computation of $\nabla_{u,v'}\Phi$, we get

$$\begin{aligned}
\partial_{u_1} &= \xi_1 + \cos(\theta)\eta_1 - \sin(\theta)\eta_d - (\xi_d + \sin(\theta)\eta_1 + \cos(\theta)\eta_d)u_1 \\
\partial_{u'} &= \xi' + (\cos(\theta)\eta_d - \xi_d - \csc(\theta)\cos^2(\theta)\eta_1)u' + (\cot(\theta)\eta_1 - \eta_d)v' \\
\partial_{v'} &= (\cot(\theta)\eta_1 - \eta_d)u' - \csc(\theta)\eta_1 v' + \eta'.
\end{aligned}$$

Here we use the notation $\xi = (\xi_1, \xi', \xi_d) \in \mathbb{R}^d$. Notice that a critical point must have

$$\begin{gathered}
\xi_1 + \cos(\theta)\eta_1 = \sin(\theta)\eta_d \\
\xi' = \eta' = 0
\end{gathered}$$

These conditions allow us to compute and evaluate the second derivative $\nabla^2_{u,v'}\Phi$. Let I_{d-2} denote the $(d-2)\times(d-2)$ identity matrix. Then $\nabla^2_{u,v'}\Phi$ can be computed as

$$\begin{aligned}
\partial_{u_1u_1} &= -(\xi_d + \sin(\theta)\eta_1 + \cos(\theta)\eta_d) \\
\partial_{u_1u'} &= \partial_{u_1v'} = 0 \\
\partial_{u'u_1} &= 0 \\
\partial_{u'u'} &= (\cos(\theta)\eta_d - \xi_d - \cot(\theta)\cos(\theta)\eta_1)I_{d-2} \\
\partial_{u'v'} &= (\cot(\theta)\eta_1 - \eta_d)I_{d-2} \\
\partial_{v'u_1} &= 0 \\
\partial_{v'u'} &= (\cot(\theta)\eta_1 - \eta_d)I_{d-2} \\
\partial_{v'v'} &= -\csc(\theta)\eta_1 I_{d-2}.
\end{aligned}$$

We can simplify a few terms using restrictions for the critical point

$$\begin{aligned}
\partial_{u'u'} &= (\cot(\theta)\xi_1 - \xi_d)I_{d-2} \\
\partial_{u'v'} &= -\csc(\theta)\xi_1 I_{d-2}.
\end{aligned}$$

Now we make an observation. If a pair of points (a, b) forms an isosceles triangle of angle θ with the origin, then we can write

$$\int f(x, y)d\sigma(x, y) = \int_{O(d)} f(ga, gb)dg.$$

Where $O(d)$ is the orthogonal group and dg is the Haar measure on $O(d)$, independent of choice of a, b. This allows for us to consider $0 \in \mathbb{R}^d$ to be the critical point due to rotation invariance. This is also why all the computations can be done up to $O(|u, v|^3)$ since all higher order terms in the hessian will disappear after evaluating at 0.

Now we compute the determinant of the hessian at the critical point 0.

$$\begin{aligned}
|\det \nabla^2\Phi_{u,v'}(0)| &= |\xi_d + \sin(\theta)\eta_1 + \cos(\theta)\eta_d| \cdot |\csc(\theta)|^{d-2} \cdot |\eta_1\xi_d - \xi_1\eta_d|^{d-2} \\
&= |\xi_d + \sin(\theta)\eta_1 + \cos(\theta)\eta_d| \cdot |\csc(\theta)|^{d-2} \cdot |\xi \wedge \eta|^{d-2}.
\end{aligned}$$

This is almost what we want. One more step has to be done, using the properties of the critical point to show

$$|\xi_d + \sin(\theta)\eta_1 + \cos(\theta)\eta_d| = |\begin{bmatrix} \xi_1 + \cos(\theta)\eta_1 - \sin(\theta)\eta_d \\ \xi_d + \sin(\theta)\eta_1 + \cos(\theta)\eta_d \end{bmatrix}| = |\xi + g_\theta \eta|.$$

This gives us the final conclusion

$$|\det \nabla^2 \Phi_{u,v'}(0)| = |\csc(\theta)|^{d-2} \cdot |\xi + g_\theta \eta| \cdot |\xi \wedge \eta|^{d-2}.$$

Which is exactly what we want.

4.2 Proof of Lemma 2.3

Notice that in the application of the lemma, we are always assuming $\frac{|\xi|}{|\eta|} \in [\frac{1}{2}, 2]$. For $\theta_{\xi,\eta}$ denoting the angle between ξ and η, we only need to estimate the cases where $\theta_{\xi,\eta} \leq \frac{\theta}{2}$ or $|\xi + g_\theta \eta| \not\approx |\xi| \approx 2^j$ since

$$\int_{|\xi| \approx 2^j} |\xi + g_\theta \eta|^{-1} \sin(\theta_{\xi,\eta})^{-(d-2)} \lesssim 2^j$$

is a relatively simply bounded otherwise.

If $\theta_{\xi,\eta} \leq \frac{\theta}{2}$, then notice that $\theta_{\xi, g_\theta \eta} \geq \frac{\theta}{2}$. This means that

$$|\xi + g_\theta \eta| \approx_\theta |\xi| \approx 2^j.$$

Use polar coordinates to get

$$\begin{aligned} &\int_{|\xi| \approx 2^j, \theta_{\xi,\eta} \leq \frac{\theta}{2}} |\xi + g_\theta \eta|^{-1} \sin(\theta_{\xi,\eta})^{-(d-2)} \\ &\approx 2^{-j} \int_{r \approx 2^j} r^{d-1} \int_{S^{d-1}} \sin(\theta_{\xi,\eta})^{-d+2} d\omega_{d-1} \\ &\approx 2^{j(d-1)} \int_{S^{d-1}} \sin(\theta_{\xi,\eta})^{2-d} d\omega_{d-1} \end{aligned}$$

To complete the computation, notice that $d\omega_{d-1} = \sin^{d-2}(\alpha) d\alpha d\omega_{d-2}$ which means we can use orthogonality of sin to only keep $\theta_{\xi,\eta} = \alpha$ in the integrand as all other fixed values will integrate to 0. So we get

$$2^{j(d-1)} \int_{S^{d-1}} \sin(\theta_{\xi,\eta})^{2-d} d\omega_{d-1} = 2^{j(d-1)} \int_0^{2\pi} 1 d\alpha = 2\pi \cdot 2^{j(d-1)}.$$

This is approximately $2^{j(d-1)}$ so we are done in this case.

The second case is where $|\xi + g_\theta \eta| \not\approx |\xi|$. It follows that $\theta_{\xi,\eta} \geq \frac{\theta}{2}$ as $\frac{|\xi|}{|\eta|} \in [\frac{1}{2}, 2]$. So with $\zeta = \xi + g_\theta \eta$, we have

$$\begin{aligned} &\int_{|\xi + g_\theta \eta| \not\approx |\xi| \approx 2^j} |\xi + g_\theta \eta|^{-1} \sin(\theta_{\xi,\eta})^{2-d} d\xi \\ &\lesssim_\theta \int_{|\zeta| \lesssim 2^j} |\zeta|^{-1} d\zeta \\ &\lesssim 2^{j(d-1)}. \end{aligned}$$

This proves the other case and completes the proof of the lemma.

5 Some remarks about the constant s_0.

While this result is less general than [4], perhaps one of the significant differences is the constant can be traced and perhaps computed. Notice that we need to choose s_0 so that

$$e^{\frac{3s-2d-3}{2(s-d)}} \leq c(e).$$

A potential function c is actually computed explicitly in [2], but the bounds on the function require constants from [3] which are less clearly defined and obtained through iteration. We will go into more detail below.

To begin, let us go through some theorems. We obviously have Proposition 5.1 from [2] which introduces this $c(e)$ constant. The $c(e)$ constant actually comes from several sources. It is found by a decomposition done on the function f. This shows up in Proposition 5.2 of [2].

Theorem 5.1. *Let f be as defined in Proposition 5.1. Suppose $f = g + b$ where*

$$||g||_\infty, ||b||_\infty \leq M$$

$$||g||_1, ||b||_1 = \delta.$$

Then

$$\Lambda(f) \geq \Lambda(g) - (2^k - 1)(M\delta)^r ||\hat{b}||_\infty.$$

Where r is the unique positive integer s.t. $n(r-1) < nk - m \leq nr$.

This theorem now reduces finding $c(e)$ to computing estimates for the decompositon.

Now this reduces to estimating $\Lambda(g)$ and $||\hat{b}||_\infty$, one of which is much easier than the other. The easier one is $\Lambda(g)$ and is estimated by the following lemma (Lemma 5.6) from [2].

Lemma 5.2. *Let $K \in \mathbb{N}$, $M > 0$, $\delta \in (0,1)$, and*

$$\sigma \in (0, \frac{\delta^k}{4kM^{k-1}}].$$

Then there exists some constant $\frac{\delta^k c(\epsilon,K)}{4} = c(K,\delta,M) > 0$ such that any non-negative (σ, K)-almost periodic function g bounded by M and obeying $\int g \geq \delta$ has

$$\Lambda(g) \geq c(K, \delta, M).$$

Here, the definition of (σ, K)-almost periodic is not that important other than the fact that the bound for $||\hat{b}||_\infty$ will come from the fact that $f = g+b$ is a decomposition into a (σ, K)-almost periodic g and a remainder term b which will be estimated by the decomposition.

For the purposes of the constant, we need to understand what $\frac{\delta^k c(\epsilon,K)}{4}$ is. [2] take $\epsilon = \frac{\delta^k}{4kM^{k-1}}$ and manage to give an explicit lower bound for $c(\epsilon, K)$ using Lemma B.1 and Corollary B.2 of [2].

Lemma 5.3. *For $\epsilon \in (0,1)$ and $K \geq 1$, there exists $c'(\epsilon, K)$ such that*

$$|\{t \in [0,1] : ||tv_l|| \leq \epsilon foralll \in [1,K]\}| \geq c'(\epsilon, K) = \frac{\epsilon^K}{2^{K+1}}.$$

Corollary 5.4. *Given $\epsilon \in (0,1)$ and integers $k, K, m, n \in \mathbb{N}$, $m > n$. there exists a positive constant depending on all these quantities, for which the set*

$$C_\epsilon = \{y \in \mathbb{R}^{m-n} : ||A_j^t v_l \cdot y|| \leq \epsilon forallj \in [1,k], l \in [1,K]\}$$

defined in Lemma 5.6 obeys the size estimate

$$|C_\epsilon| \geq c = c(\epsilon, K) = c'(\frac{\epsilon}{m-n}, K)^{k(m-n)} = \frac{\epsilon^{Kk(m-n)}}{2^{(K+1)k(m-n)}(m-n)^{Kk(m-n)}}.$$

So at this point, the constants seem to be getting somewhat complicated. But they are still able to be computed. The main problem comes when trying to estimate $||\hat{b}||_\infty$. [2] use a result adapted from [3] (Lemma 5.11 in [2]) to provide the decomposition and estimate for this term.

Lemma 5.5. *Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be an arbitrary function. Let $\delta \in (0, 1]$ and $f \geq 0$ be a function bounded by M with $\int f \geq \delta$. Let σ be as in Lemma 5.6. Then there exists a K with $K \in (0, C(F, \delta)]$ and a decomposition $f = g + b$ where $g \geq 0$ is a bounded (σ, K)-almost periodic function with $\int g \geq \delta$ and b obeys the bounds*

$$||\hat{b}||_\infty \leq F(\delta, K).$$

The bound for $\Lambda(f)$ then follows from the fact that since $F(\delta, K)$ is an arbitrary function, we can choose it to be a smaller multiple of the lower bound for $\Lambda(g)$, using that M is a constant as appropriate. An explicit function which works could be

$$F(\delta, K) = \frac{1}{2(2^k - 1)(M\delta)^r} \frac{\delta^k \epsilon^{Kk(m-n)}}{2^{(K+1)k(m-n)} 4(m-n)^{Kk(m-n)}} = \frac{1}{(2^k - 1)(M\delta)^r} \frac{c(K, \delta, M)}{2}.$$

At this point we have been able to explicitly compute everything. The root of all of our pain and reason why this bound is so hard to actually compute is the upper bound $C(F, \delta)$ for K.

The origin of this upper bound comes from [3]. The theorem where this bound comes from is an incrementation result (Proposition 2.11 in [3]),

Theorem 5.6. *Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be an arbitrary function. Let $\delta \in (0, 1]$ and $f \geq 0$ be a function bounded on $\mathbb{Z}/N\mathbb{Z}$ with expectation $\geq \delta$. Let $\sigma = \frac{\delta^3}{100}$. Then there exists a K with $K \in (0, C(F, \delta)]$ and a decomposition $f = g + b$ where $g \geq 0$ is a bounded (σ, K)-almost periodic function with $\int g \geq \delta$ and b obeys the bounds*

$$||\hat{b}||_\infty \leq F(\delta, K).$$

This result is over $\mathbb{Z}/N\mathbb{Z}$, but the domain is not important. The proof is more important which requires iteration of the function F. I will give a sketch of the proof below.

We have a function $E(f, B)$ which takes expectation of f with respect to some σ-algebra B. Decomposing the σ-algebra B into B_{ϵ_i,χ_i} atoms which depend on an $\epsilon_i > 0$ and a linear phase function χ_i, we can find a $K = K(n, \sigma, \epsilon_1, ..., \epsilon_n)$ which gives us $E(f, B)$ is a $(\frac{\sigma}{2}, K)$-almost periodic function. If we have the desired estimate in the theorem, we stop and this is the upper bound for K. Otherwise we add additional atoms to the generators of the σ-algebra to get a new σ-algebra B' which is B with more B_{ϵ_m,χ_m} added. Fortunately, each addition uses the same $\epsilon = \frac{F(\delta,K)}{C}$ and character χ so the growth of K is not too random.

With each addition, we are actually increasing the quantity $E(f, B)$, which we have an upper bound for. So the process must eventually terminate, but only after $O(\sigma^{-2})$ steps. Not only this, but the number of additional atoms added in each step is $\frac{C\sigma^2}{F(\delta,K')^2}$ where K' is the K associated with the last step. So not only do we have a K which is a product of many terms, but the number of terms is increasing as we allow our function $F(\delta, K)$ to be smaller and requires iteration of our function F to calculate the precise amount.

If we are able to get past this step (which could be possible, but highly unlikely), we would also need to find the relation between $K = K(n, \sigma, \epsilon_1, ..., \epsilon_n)$. The result for multiple character pretty easily reduces to a product of $K_i = K(\sigma, \epsilon_i)$'s since a product of (σ_i, K_i)-almost periodic functions gives a $(\sum_i \sigma_i, \prod_i K_i)$-almost periodic function. So if $\sigma_i = \frac{\sigma}{n}$ and K_i is as defined, we just need to compute each K_i and $K = \prod_i K_i$. But we run into another wall since the upper bound for K_i depends on the degree of a weierstrass approximating polynomial for the indicator function, which I do not know how to compute. This discussion is taken from Lemma 2.8 and Corollary 2.9 in [3].

Another constant to estimate for an exact dimensional constant would be the constant C which appears in Lemma 2.10 of [3]. This constant C appears in the proof of Proposition 2.11 and is part of the iteration mechanism, but could potentially be estimated out by a better choice of $F(\delta, K)$.

The iteration mechanism also can not be simplified by taking $F(\delta, K) = h(\delta)$ independent of K. This is because we need

$$c(K, \delta, M) - (2^k - 1)(M\delta)^r F(\delta, K) \geq C' c(K, \delta, M).$$

Where $C' > 0$ is some constant. But notice that if F does not depend on K, we can simplify the iteration mechanism by noticing $\frac{C\sigma^2}{F(\delta,K')^2} = \frac{C\sigma^2}{F(\delta,1)^2}$ and so we can take the number of characters to be approximately $\frac{C}{F(\delta,1)^2}$. This means $C(F, \delta)$ can be approximated by $a^{\frac{C}{F(\delta,1)^2}}$ for some real number $a > 0$. So we have

$$c(C(F, \delta), \delta, M) \approx b^{(a^{\frac{C}{F(\delta,1)^2}})}.$$

Where $b \in (0, 1)$. However, this choice of F now gives us a negative value for the difference if F is small since the negative second term is decaying linearly in F while the positive first term is decaying faster than exponentially in F. So F really needs to decay in K for the argument to work, making computation of the constant that much harder.

References

Iosevich Liu [1] Iosevich, A., Liu, B. "Equilateral triangles in subsets of $\mathbb{R}^d$ of large Hausdorff dimension". *Isr. J. Math.* **231,** 123-137 (2019) https://doi.org/10.1007/s11856-019-1848-8

a Laba Pramanik [2] Chan, V, Łaba, I. & Pramanik, M. "Finite configurations in sparse sets". *JAMA* **128,** 289-335 (2016) https://doi.org/10.1007/s11854-016-0010-3

Tao [3] Tao, T. "Arithmetic progressions and the primes". *Collect. Math.*, (Vol. Extra):37–88, 2006. 9, 23, 25, 30, 31

osevich Magyar [4] Iosevich, A. Magyar, A. "Simplices in thin subsets of Euclidean spaces". https://arxiv.org/abs/2009.04902 (2020).

Ziegler [5] Ziegler, T. "Nilfactors of $\mathbb{R}^d$ actions and configurations in sets of positive upper density in $\mathbb{R}^m$", *J. Anal. Math.* **99**, 249-266 (2006).

Falconer [6] Falconer, K. J. "On the Hausdorff dimensions of distance sets", *Mathematika*, **32** (2): 206–212 (1986), doi:10.1112/S0025579300010998

Erdogan [7] Erdoğan, M. Burak, "A bilinear Fourier extension theorem and applications to the distance set problem", *International Mathematics Research Notices*, **23**: 1411–1425 (2005), doi:10.1155/IMRN.2005.1411

Bourgain [8] Bourgain, J. "A Szemeredi type theorem for sets of positive density", *Israel J. Math.* **54** (1986), no. 3, 307-331.

Falconer2 [9] Falconer, K.J. "Some problems in measure combinatorial geometry associated with Paul Erdős", http://www.renyi.hu/conferences/erdos100/slides/falconer.pdf

Maga [10] Maga, P. "Full dimensional sets without given patterns", *Real Anal. Exchange* **36** (2010/11), no. 1, 79-90.

Keleti [11] Keleti, T. "A 1-dimensional subset of the reals that intersects each of its translates in at most a single point", *Real Anal. Exchange* 24:2 (1998/99), 843-844.

Laba Pramanik [12] Laba, I. and Pramanik, M. "Arithmetic progressions in sets of fractional dimension", *Geom. Funct. Anal.* **19** (2009), no. 2, 429-456.

enleaf Iosevich [13] Greenleaf, A. Iosevich, A. Liu, B. Palsson, E. "A group theoretic viewpoint on Erdős–Falconer type problems and the Mattila integral". *Rev. Mat. Iberoam.* **31** (2015), no. 3, 799–810 https://doi.org/10.4171/rmi/854

ennett Iosevich [14] Michael Bennett. Alexander Iosevich. Krystal Taylor. "Finite chains inside thin subsets of $\mathbb{R}^d$." *Anal. PDE* **9** (3) 597 - 614, (2016). https://doi.org/10.2140/apde.2016.9.597

Iosevich Taylor [15] A. Iosevich and K. Taylor, "Finite trees inside thin subsets of $\mathbb{R}^d$". https://arxiv.org/abs/1903.02662 (2019).