QUALITATIVE PROPERTIES FOR SOLUTIONS TO A DIFFERENTIAL EQUATION ASSOCIATED WITH THE SKYRME MODEL

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ABSTRACT. The goal of this paper is to discuss a series of qualitative results for the solutions to a differential equation modeling nucleons, as it was introduced by Tony Skyrme in a series of seminal papers [2, 3, 4]. These results are part of the outcomes of an article written by John McLeod and William Troy [1] on the subject.

1. INTRODUCTION

When describing physical behavior in fields, physicists use mathematical concepts to predict behavior. As such, these are typically called *field theories*. An important type of field theory is the *sigma model*, which explains the behavior of an idealized point particle confined to a manifold whose properties and interactions with the particle correspond to physical characteristics of the system. Within field theory manifolds, irregularities known as *topological solitions* correspond to physical features.

This paper focuses on the Skyrme model, which describes such a soliton, known as a *skyrmion*, and for which the relevant equivariant differential equation is given by

$$\begin{aligned}\left(1+\frac{2\alpha^2\sin^2 F}{r^2}\right)(F_{tt}-F_{rr})-\frac{2}{r}F_r \\ +\frac{\sin(2F)}{r^2}\left(1+\alpha^2\left(F_t^2-F_r^2+\frac{\sin^2 F}{r^2}\right)\right)=0,\end{aligned} \tag{1}$$

where α is a constant having the dimension of length and $F = F(t, r)$ is an azimuthal angle. By taking $\alpha = 2$ and considering static (i.e., time independent; $F = F(r)$) configurations one obtains the ordinary differential equation

$$\begin{aligned}\left(\frac{1}{4}r^2+2\sin^2 F\right)F''+\frac{1}{2}rF'+\sin(2F)(F')^2 \\ -\frac{1}{4}\sin(2F)-\frac{\sin^2 F}{r^2}\sin(2F)=0,\end{aligned} \tag{2}$$

which is the main object of study in McLeod-Troy paper. In the context of the Skyrme model, this equation is paired with the boundary conditions

$$F(0) = 0 \qquad \text{and} \qquad F(\infty) = n\pi,$$

where $n \geq 1$ is an integer often called by physicists the *topological charge* of the soliton.

At this moment, it is important to recall that differential equations fall into one of the following four categories: linear, semilinear, quasilinear, and fully nonlinear (in ascending order of intricacy). Both (1) and (2) are quasilinear equations, as the coefficients of the highest order derivatives depend on the unknown function itself. The main point we want to make is that, despite this challenging nature, McLeod and Troy are able to obtain both qualitative and quantitative results for classical solutions[1] to (2) using only relatively primitive, calculus-based math techniques.

In this paper, we cover three of these results which address the size and the end behavior of these solutions. They are as follows:

Theorem 1.1. *Any classical solution to* (2) *is bounded; i.e., there exists* $M > 0$ *such that*

$$|F(r)| \leq M, \qquad (\forall)\, r > 0.$$

Theorem 1.2. *If* $F : (0, \infty) \to \mathbb{R}$ *is a classical solution to* (2) *then there exist* k *and* $l \in \mathbb{Z}$ *such that*

$$\lim_{r \to \infty} F(r) = k\pi \quad \textit{or} \quad \left(k + \frac{1}{2}\right)\pi \tag{3}$$

and

$$\lim_{r \to 0} F(r) = l\pi \quad \textit{or} \quad \left(l + \frac{1}{2}\right)\pi. \tag{4}$$

Theorem 1.3. *If* $F : (0, \infty) \to \mathbb{R}$ *is a classical solution to* (2) *and*

- $\lim_{r\to\infty} F(r) = k\pi$, *then* F *is monotone on an interval of the type* (b, ∞) *and*
$$F - k\pi \sim \frac{1}{r^2} \quad \textit{as} \quad r \to \infty;$$
- $\lim_{r\to\infty} F(r) = (k + 1/2)\pi$, *then* F *oscillates about* $(k + 1/2)\pi$ *and*
$$F - \left(k + \frac{1}{2}\right)\pi = O(r^{-1/2}) \quad \textit{as} \quad r \to \infty;$$
- $\lim_{r\to 0} F(r) = l\pi$, *then* F *is monotone on an interval of the type* $(0, a)$ *and*
$$F - l\pi \sim r \quad \textit{as} \quad r \to 0;$$
- $\lim_{r\to 0} F(r) = (l + 1/2)\pi$, *then* F *oscillates about* $(l + 1/2)\pi$ *and*
$$F - \left(l + \frac{1}{2}\right)\pi = O(r^{1/2}) \quad \textit{as} \quad r \to 0.$$

Notation 1.4. *Above, we write* $A \sim B$ *if there exists an absolute constant* $C > 0$ *such that*

$$C^{-1}A \leq B \leq CA.$$

The classical notations $f = O(g)$ *and* $f = o(g)$ *(with* $g \geq 0$*) when* $x \to x_0$ *stand for*

$$\limsup_{x \to x_0} \frac{|f(x)|}{g(x)} < \infty$$

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[1] A classical solution to (2) is a C^2 function $F : (0, \infty) \to \mathbb{R}$ which verifies the equation pointwise.

and

$$\lim_{x\to x_0}\frac{|f(x)|}{g(x)}=0,$$

respectively.

Remark 1.5. *It is important to note, in the context of Theorems 1.2 and 1.3, that once F is is a classical solution to* (2)*, then so is $\pm F+k\pi$, where $k\in\mathbb{Z}$ is arbitrary.*

In the next three sections, we proceed by proving one by one each of the above results.

2. Proof of Theorem 1.1

In arguing for this result, **we show that F is bounded both in a neighborhood of 0 and in a neighborhood of ∞**. This is done with the help of the functional

$$Q(r):=\left(\frac{1}{4}r^2+2\sin^2 F\right)(F')^2-\frac{1}{2}\sin^2 F-\frac{\sin^4 F}{r^2}, \tag{5}$$

for which a direct computation yields

$$\begin{aligned}Q'(r)=2F'\Bigg[\left(\frac{1}{4}r^2+2\sin^2 F\right)F''-\frac{\sin^2 F}{r^2}\sin(2F)-\frac{1}{4}\sin(2F)\\ +\frac{1}{2}rF'+\sin(2F)(F')^2\Bigg]-\frac{1}{2}r(F')^2+\frac{2\sin^4 F}{r^3}.\end{aligned}$$

If F is a classical solution to (2), then

$$Q'=-\frac{1}{2}r(F')^2+\frac{2\sin^4 F}{r^3}\le\frac{2}{r^3}, \tag{6}$$

which can be rearranged to read

$$\left(Q+\frac{1}{r^2}\right)'\le 0.$$

On the other hand, we infer from (5) that

$$Q+\frac{1}{r^2}\ge-\frac{1}{2}\sin^2 F+\frac{1-\sin^4 F}{r^2}\ge-\frac{1}{2}.$$

Thus, it follows that

$$r\mapsto Q+\frac{1}{r^2}$$

is both monotonically decreasing and bounded from below, which implies

$$\lim_{r\to\infty}\left(Q+\frac{1}{r^2}\right)=c\in\mathbb{R}$$

and subsequently

$$\lim_{r\to\infty}Q=c \tag{7}$$

and

$$\lim_{r\to\infty}\left[\left(\frac{1}{4}r^2+2\sin^2 F\right)(F')^2-\frac{1}{2}\sin^2 F\right]=c. \tag{8}$$

Next, **we argue by contradiction that** $c \le 0$. If $c > 0$, then (8) implies that for some $A > 0$ we have

$$\left(\frac{1}{4}r^2 + 2\sin^2 F\right)(F')^2 - \frac{1}{2}\sin^2 F \ge \frac{c}{2}, \qquad (\forall)\, r > A.$$

By eventually imposing $A^2 \ge 8$, we deduce that

$$\frac{1}{2}r^2(F')^2 \ge \left(\frac{1}{4}r^2 + 2\sin^2 F\right)(F')^2 \ge \frac{c}{2}, \qquad (\forall)\, r > A$$

and, hence,

$$(F')^2 \ge \frac{c}{r^2}, \qquad (\forall)\, r > A.$$

By using this estimate in the context of (6), we derive that

$$Q' = -\frac{1}{2}r(F')^2 + \frac{2\sin^4 F}{r^3} \le \frac{-c}{2r} + \frac{2}{r^3}, \qquad (\forall)\, r > A.$$

Readjusting the value of A one more time such that $cA^2 \ge 8$, we infer

$$Q' \le \frac{-c}{4r}, \qquad (\forall)\, r > A,$$

which yields that

$$r \mapsto Q + \frac{c}{4}\ln r$$

is monotonically decreasing on the interval (A, ∞). This implies the existence of

$$\lim_{r\to\infty}\left(Q + \frac{c}{4}\ln r\right) \in \mathbb{R} \cup \{-\infty\}.$$

However, by using (7) and $c > 0$, one obtains

$$\lim_{r\to\infty}\left(Q + \frac{c}{4}\ln r\right) = \infty,$$

thus obtaining the sought out contradiction.

Following this, we deduce with the help of (5) that

$$\frac{1}{4}r^2(F')^2 \le Q + \frac{1}{2} + \frac{1}{r^2}$$

and, by also factoring in (7), this implies

$$F' = O(r^{-1}) \quad \text{as} \quad r \to \infty. \tag{9}$$

In fact, **we can prove by contradiction that** $F' = o(r^{-1})$ **as** $r \to \infty$. The alternative is that there exists $\delta > 0$ and a strictly increasing sequence $(r_n)_{n\ge1} \to \infty$ such that

$$|F'(r_n)| \ge \frac{\delta}{r_n}, \qquad (\forall)\, n \ge 1. \tag{10}$$

On the other hand, if we rely on (2), we can infer that

$$\frac{1}{4}r^2|F''| \le \frac{1}{2}r|F'| + (F')^2 + \frac{1}{4} + \frac{1}{r^2},$$

which coupled with (9) yields

$$F'' = O(r^{-2}) \quad \text{as} \quad r \to \infty.$$

Hence, we can assume, without loss of generality, that there exists $K > 0$ such that

$$|F''(r)| \le \frac{K}{r^2}, \qquad (\forall)\, r > r_1.$$

With the help of this estimate, we derive that

$$|F'(r)-F'(r_n)| \leq \int_{r_n}^{r} |F''(s)|\,ds \leq \int_{r_n}^{r} \frac{K}{s^2}\,ds = K\left(\frac{1}{r_n}-\frac{1}{r}\right), \qquad (\forall)\, r \geq r_n,\, n \geq 1.$$

If we factor in now (10) and we also rely on the triangle inequality, then we obtain that

$$|F'(r)| \geq \frac{\delta}{2r_n}, \qquad (\forall)\, r_n \leq r \leq \left(1+\frac{\delta}{2K}\right) r_n,\, n \geq 1.$$

Using this fact in the context of (6), we further deduce that

$$\left(Q+\frac{\delta}{4}\ln r+\frac{1}{r^2}\right)' = Q'+\frac{\delta}{4r}-\frac{2}{r^3} \leq 0, \qquad (\forall)\, r_n \leq r \leq \left(1+\frac{\delta}{2K}\right) r_n,\, n \geq 1,$$

which implies

$$Q(r_n)+\frac{\delta}{4}\ln r_n+\frac{1}{r_n^2} \geq Q\left(\left(1+\frac{\delta}{2K}\right) r_n\right)+\frac{\delta}{4}\ln\left(\left(1+\frac{\delta}{2K}\right) r_n\right) + \frac{1}{\left(1+\frac{\delta}{2K}\right)^2 r_n^2}, \quad (\forall)\, n \geq 1.$$

By rearranging some of the terms, it follows that

$$Q(r_n)+\frac{1}{r_n^2}-Q\left(\left(1+\frac{\delta}{2K}\right) r_n\right)-\frac{1}{\left(1+\frac{\delta}{2K}\right)^2 r_n^2} \geq \frac{\delta}{4}\ln\left(1+\frac{\delta}{2K}\right), \quad (\forall)\, n \geq 1.$$

Finally, if we let $n \to \infty$ and take advantage of (7), we conclude that

$$\frac{\delta}{4}\ln\left(1+\frac{\delta}{2K}\right) \leq 0,$$

which is obviously a contradiction.

Now, we have all the prerequisites to **show that F is bounded in a neighborhood of ∞**. Since $F' = o(r^{-1})$ as $r \to \infty$, we infer based on (5) that

$$\lim_{r\to\infty}\left(Q+\frac{1}{2}\sin^2 F\right) = 0,$$

which jointly with (7) yields

$$\lim_{r\to\infty} \sin^2 F = -\frac{c}{2}.$$

If we rely one more time on $F' = o(r^{-1})$ as $r \to \infty$, we derive that

$$\lim_{r\to\infty} F \in \mathbb{R}$$

and the claim is proven.

Next, we turn our attention to the behavior of F close to 0 and **we start by arguing that**

$$r^2 Q + \frac{r^2}{2} \geq \lim_{r\to 0} r^2 Q = a \in \mathbb{R}. \tag{11}$$

By combining (5) and (6), we easily obtain that

$$Q' + \frac{2Q}{r} = \frac{4\sin^2 F}{r}((F')^2 - \frac{1}{4}) \geq -\frac{1}{r}, \tag{12}$$

which can be rearranged to read as

$$\left(r^2Q+\frac{r^2}{2}\right)'=r^2\left(Q'+\frac{2Q}{r}+\frac{1}{r}\right)\geq 0.$$

Thus, $r\mapsto r^2Q+\frac{r^2}{2}$ is monotonically increasing and this implies

$$r^2Q+\frac{r^2}{2}\geq\lim_{r\to 0}\left(r^2Q+\frac{r^2}{2}\right)=\lim_{r\to 0}r^2Q\in\mathbb{R}\cup\{-\infty\}.$$

On the other hand, we deduce from (5) that

$$Q\geq-\frac{1}{2}-\frac{1}{r^2},$$

which leads to

$$\liminf_{r\to 0}\, r^2Q\geq -1,$$

hence, reaching the desired conclusion.

Following this, let us **derive a couple of estimates related to the size of rF' and its derivative for r close to** 0. First, we infer from (5) and (11) that

$$(rF')^2=\frac{r^2Q+\frac{1}{2}r^2\sin^2F+\sin^4F}{\frac{1}{4}r^2+2\sin^2F}\leq\frac{r^2Q}{2\sin^2F}+\frac{5}{2}\sin^2F\leq\frac{a+1}{2\sin^2F}+\frac{5}{2}\tag{13}$$

holds true if r is sufficiently small and $\sin F\neq 0$. Second, we obtain also with the help of (5) and (11) that

$$(rF')^2\geq\frac{a+\sin^4F-\frac{1}{2}r^2\cos^2F}{\frac{1}{4}r^2+2\sin^2F}\geq\frac{a+\sin^4F-\frac{1}{2}r^2}{2\sin^2F},$$

is valid for all $r>0$ with $\sin F\neq 0$. In particular, these two inequalities imply that if one has

$$0<\gamma\leq\min\{\sin^2F,|a+\sin^4F|\},\tag{14}$$

and $0<r<\gamma^{1/2}$ is sufficiently small, then

$$\frac{1}{4}\leq(rF')^2\leq\frac{a+1}{2\gamma}+\frac{5}{2}.\tag{15}$$

Finally, let us denote $H(r)=rF'(r)$ and argue by relying on (2) that

$$H'=F'+rF''=\frac{\sin^2F\sin(2F)}{r(\frac{1}{4}r^2+2\sin^2F)}+\frac{r\sin(2F)\left(\frac{1}{4}-(F')^2\right)}{\frac{1}{4}r^2+2\sin^2F}+\frac{\left(\frac{-1}{4}r^2+2\sin^2F\right)F'}{\frac{1}{4}r^2+2\sin^2F},$$

which leads to

$$|H'|\leq C\left(\frac{1}{r}+1+(F')^2+|F'|\right)\leq 3C\left(\frac{1}{r}+(F')^2\right)=3C\left(\frac{1}{r}+\frac{H^2}{r^2}\right)$$

if $0<r<1$ and $C>0$ is a suitable chosen absolute constant. Based on more involved integral estimates (e.g., Gronwall-type inequalities), one is able to control the size of

$$|H(r)-H(\tilde{r})|$$

in terms of the size of $|\ln r-\ln\tilde{r}|$. This means that for all $\epsilon>0$ there exists $\omega>0$ such that

$$|\tilde{r}F'(\tilde{r})-rF'(r)|<\epsilon,\qquad(\forall)\,0<\tilde{r},r<1,\ \left|\frac{\tilde{r}}{r}-1\right|<\omega.$$

Now, we have all the prerequisites to **show that F is bounded in a neighborhood of** 0. As in the case for r large, this is done by proving that

$$\lim_{r\to 0} \sin F \in \mathbb{R}.$$

We argue by contradiction and, hence, deduce the existence of $\gamma > 0$ and of a strictly decreasing sequence $(r_n)_{n\geq 1} \to 0$ such that both $0 < r_n < \gamma^{1/2}$ and (14) for $r = r_n$ hold true for all $n \geq 1$. This implies that (15) is valid for all $n \geq 1$ and, jointly with (2), it yields

$$|rF'(r)| \geq \frac{1}{3}, \qquad (\forall)\, 0 < r < 1,\ \left|\frac{r}{r_n} - 1\right| < \omega_0,\ n \geq 1,$$

for some fixed ω_0 independent of n. By using (12), it follows that

$$(r^2Q)' = 4r\sin^2 F\left((F')^2 - \frac{1}{4}\right) \geq 4r\sin^2 F\left(\frac{1}{3r^2} - \frac{1}{4}\right)$$

holds true under the same conditions for r as above. Thus, if we take n sufficiently large, we can guarantee that

$$(r^2Q)' \geq \frac{\tilde{C}}{r_n}, \qquad (\forall)\, 0 < r < 1,\ \left|\frac{r}{r_n} - 1\right| < \omega_0,$$

for yet another $\tilde{C} > 0$ absolute constant. By integrating this estimate over the interval $[r_n, r_n(1+\omega_0)]$, we obtain

$$r_n^2(1+\omega_0)^2 Q(r_n(1+\omega_0)) - r_n^2 Q(r_n) \geq \tilde{C}\omega_0 > 0,$$

which obviously leads to a contradiction when paired with (11). With this, the proof of the boundedness for F in a neighborhood of 0 is finished and the whole argument for Theorem 1.1 is concluded.

3. Proof of Theorem 1.2

For this argument, we follow up on our findings from the previous section and **we focus first on the end behavior of F when** $r \to \infty$. We recall that

$$\lim_{r\to\infty} F := F_\infty \in \mathbb{R} \qquad \text{and} \qquad \lim_{r\to\infty} rF' = 0.$$

If we use these facts and we take $r \to \infty$ in (2), we derive that

$$\lim_{r\to\infty}\left(\frac{1}{4}r^2 + 2\sin^2 F\right)F'' = \sin(2F_\infty).$$

Thus, for proving (3), which is equivalent to $\sin(2F_\infty) = 0$, it is enough to argue that

$$\liminf_{r\to\infty} r^2|F''| = 0$$

and we do this by contradiction. The alternative is that

$$\liminf_{r\to\infty} r^2|F''| > 0$$

and, hence, there exist A_1 and $C_1 > 0$ such that

$$r^2|F''(r)| > C_1, \qquad (\forall)\, r > A_1.$$

By integration, it follows that

$$|F'(r)| = \left|\int_r^\infty F''(s)\,ds\right| = \int_r^\infty |F''(s)|\,ds \geq \frac{C_1}{r}, \qquad (\forall)\, r > A_1,$$

which contradicts the fact that $F' = o(r^{-1})$ as $r \to \infty$.

Next, **we investigate the behavior of F when** $r \to 0$ and we remember from the analysis for Theorem 1.1 that

$$\lim_{r\to 0} F := F_0 \in \mathbb{R} \qquad \text{and} \qquad \lim_{r\to 0} r^2 Q = a.$$

If $\sin F_0 = 0$, we are done. Otherwise, we can use (13) to infer that

$$F' = O(r^{-1}) \quad \text{as} \quad r \to 0.$$

Jointly with (5) and (11), this fact implies that

$$\lim_{r\to 0} r^2 (F')^2 = \frac{a + \sin^4 F_0}{2\sin^2 F_0}.$$

We argue by contradiction that $a + \sin^4 F_0 = 0$ and, hence,

$$F' = o(r^{-1}) \quad \text{as} \quad r \to 0. \tag{16}$$

The alternative is that there exist A_2 and $C_2 > 0$ such that

$$r|F'(r)| > C_2, \qquad (\forall)\, 0 < r < A_2,$$

which leads by integration to

$$|F(A_2) - F_0| = \left| \int_0^{A_2} F'(s)\, ds \right| = \int_0^{A_2} |F'(s)|\, ds = \infty.$$

This is obviously false. Now, we multiply (2) by r^2, take $r \to 0$, and use (16) to deduce

$$\lim_{r\to 0} r^2 F'' = \frac{1}{2}\sin(2F_0).$$

We claim that this implies $\sin(2F_0) = 0$, which yields (4), thus finishing the proof for the whole theorem. This is settled by contradiction, which would yield

$$\lim_{r\to 0} r^2 F'' \neq 0.$$

Through an argument similar to previous ones, one deduces by integration that

$$|F'(2r) - F'(r)| \geq \frac{C_3}{r} \tag{17}$$

if r is sufficiently small and $C_3 > 0$ is an absolute constant. However, this violates (16) and we are done.

4. Proof of Theorem 1.3

In section we will start using the 1.4 $O(\cdot)$ notation defined in the introduction. Each case of theorem 1.3 uses this notation extensively in their respective subsections.

As for the subsection themselves, in general they follow a simple format. First they start off by reformulating (2) using the substitutions introduced in section 3. In cases 1, 2, and 4 this is then used to bound the integral of this formulation up to the limit. In turn, another substitution involving a function G is introduced. Finally, by estimating the order of G, insight into solution behavior near the limit is gained. Case 3 differs in that neither type of substitution is used, rather it opts to directly bound terms and infer behavior from there.

4.1. **Case 1:** $\lim_{r\to\infty} F(r) = k\pi$

First we want to change the form of (2) to make it **easier to work with**. To do this, we multiply (2) by 4 to get:

$$r^2F'' + 2rF' - \sin(2F) + 8\sin^2(F)F'' + 4\sin(2F)(F')^2 - 4\frac{\sin^2(F)}{r^2}\sin(2F) = 0$$

The last two terms in the equation above have orders of $O\left(\sin(2F)(F')^2\right)$, $O\left(\frac{\sin^2(F)}{r^2}\sin(2F)\right)$, being just scalings of those functions. However, we are able to say something more substantive about the order of the third term.

Comparing (7) and (8), we see that both r^2F'' and rF' approach a constant in the limit as $r \to \infty$, and so $F'' = O\left(\frac{F'}{r}\right)$ in the limit. Using this order relation, it follows that the fourth term has order of $O\left(\frac{F'\sin^2(F)}{r}\right)$ so (2) can be written as

$$r^2F'' + 2rF' - \sin(2F) + O\left(\frac{F'\sin^2(F)}{r}\right)$$
$$+O\left(\sin(2F)(F')^2\right) + O\left(\frac{\sin^2(F)}{r^2}\sin(2F)\right) = 0$$

Recall that in the previous section, we showed $rF' \to 0$, meaning that F' is ultimately smaller than r. As a result, when we factor out a $\frac{1}{r}$ from the last three terms, only the final term does not decay to zero. This gives us:

$$r^2F'' + 2rF' - \sin(2F) + \frac{1}{r^2}O(\sin^2(F)\sin(2F)) = 0 \tag{18}$$

Utilizing a **change of variables** $x = \ln r$, recall from the previous section that $rF' = \frac{dF}{dx}$. The above equation can then be written as

$$\frac{d^2F}{dx^2} + \frac{dF}{dx} - \sin(2F) + O\left(e^{-2x}\right) = 0 \tag{19}$$

Our equation is almost in a usable form. To get it in the final form we intend to take advantage of, we must take a small detour. The idea now is to **bound the integrands** of slightly manipulated (19) equations and then **use order approximations to make conclusions about the order functions**.

The first such integrand is achieved by multiplying with $\frac{dF}{dx}$ and then integrating from some point a to ∞:

$$\int_a^\infty \frac{dF}{dx}\left[\frac{d^2F}{dx^2} + \frac{dF}{dx} - \sin(2F) + O\left(e^{-2x}\right)\right]dx$$
$$= \frac{1}{2}\left(\frac{dF}{dx}\right)^2\Big|_a^\infty + \int_a^\infty \left(\frac{dF}{dx}\right)^2 dx + \frac{\cos(2F)}{2}\Big|_a^\infty$$
$$+ \int_a^\infty (F')O\left(\frac{1}{r^2}\right)dr = 0$$

The first and third terms are finite. As for the fourth term, we know that $o(F') = (r^{-1})$, so then:

$$\int_a^\infty (F')O\left(\frac{1}{r^2}\right)dr < \int_a^\infty O\left(\frac{1}{r}\right)dr < \infty$$

When $a > 0$. Since the first, third, and fourth terms are finite and sum with the second term to 0, this means that the second term, $\int_a^\infty \left(\frac{dF}{dx}\right)^2 dx$, must also be finite.

For the second integrand: if we multiply (18) by $\sin(2F)$ and integrate we get

$$\begin{aligned} 0 &= \int_a^\infty \sin(2F)\left[\frac{d^2F}{dx^2} + \frac{dF}{dx} - \sin(2F) + O\left(e^{-2x}\right)\right]dx \\ &< \int_a^\infty \sin^2(2F)dx + \frac{dF}{dx}\Big|_a^\infty + F\Big|_a^\infty + p\sqrt{\pi} \end{aligned}$$

For some number p. So it should be clear from above that $\int_a^\infty \sin^2(2F)dx$ is finite since the terms other than it are also finite (see above).

Setting $F - F(\infty) = e^{-\frac{x}{2}}G$, then

$$\lim_{t\to\infty} \frac{\sin^2\left(2F(t) - 2F(\infty)\right)}{4\left(F(t) - F(\infty)\right)^2} = 1$$

Meaning that $\int_a^\infty \sin^2(2F)dx$ finite is equivalent to saying that $\int_a^\infty \left(F(t) - F(\infty)\right)^2 dx$ is finite since they are of the same order. Now we use G to rewrite (19):

$$\frac{dF}{dx} = -\frac{1}{2}e^{-\frac{x}{2}}G + e^{-\frac{x}{2}}\frac{dG}{dx}$$

$$\frac{d^2F}{dx^2} = \frac{1}{4}e^{-\frac{x}{2}}G - e^{-\frac{x}{2}}\frac{dG}{dx} + e^{-\frac{x}{2}}\frac{d^2G}{dx^2}$$

So then (19) can be written as:

$$\frac{d^2G}{dx^2} - \frac{1}{4}G = O(e^{-3x/2}) + \frac{\sin(2F)}{e^{-x/2}}$$

Subtracting $2G\cos(F(\infty))$ from each side yields

$$\begin{aligned} \frac{d^2G}{dx^2} - \frac{1}{4}G - 2G\cos(F(\infty)) &= O(e^{-3x/2}) + \sin(2F) - 2G\cos(F(\infty)) \\ &= O(e^{-3x/2}) + G\left(\frac{\sin(2F)}{F - F(\infty)} - 2\cos(2F)\right) \\ &= O(e^{-3x/2}) + G\left(\frac{\sin(2F)}{F - F(\infty)} + 2\sin^2(F) - 1\right) \\ &= O(e^{-3x/2}) + G\left(O(1) - O(1) + O(\sin^2(F))\right) \end{aligned} \tag{20}$$

This is the relevant equation we intend to use for **cases 1 and 2**. For case 1, $F(\infty) = k\pi$, which tells us that $\cos(2F(\infty)) = 1$, allowing us to write:

$$\frac{d^2G}{dx^2} - \frac{9}{4}G = O(e^{-3x/2}) + O\left(G(F - F(\infty))^2\right) \tag{21}$$

Looking at (2), if F is close to $k\pi$ then whenever $F' = 0$ we have

$$(\frac{1}{4}r^2 + 2\sin^2(F))F'' = \frac{1}{4}\sin(2F) + \frac{\sin^2(F)}{r^2}\sin(2F)$$

Meaning F'' and $\sin(2F) = F - k\pi$ share the same sign. So if F is approaching $k\pi$ then it is monotone. For example, F approaching $k\pi$ from below implies $F' \geq 0$. If ever $F' = 0$ and $F \neq k\pi$ are both true, then $F'' < 0$ so F' will then become negative and F will stop approaching $k\pi$. Now that it has been established that F has a monotone approach to $k\pi$, we will use this fact to determine that **G has a limit** and what that limit is.

Monotonicity tells us without loss of generality we can assume that $F - k\pi > 0$, meaning that $G > 0$. As a result,

$$\frac{d^2G}{dx^2} + O(e^{-3x/2}) = \frac{9}{4}G + O\left(G(F - F(\infty))^2\right). \tag{22}$$

Using

$$G > 0$$

means that

$$\frac{d^2G}{dx^2} + O(e^{-3x/2}) > 0,$$

so as a result, $\frac{dG}{dx} + O(e^{-3x/2})$ must be increasing and must have a limit L as $x \to \infty$. Now for **proof by contradiction** that $L = 0$:

If $L < 0$ then, from (22), $G < 0$ so we arrive at a contradiction.

If $L > 0$, then since $e^{-3x/2}, G(F - F(\infty))^2 \to 0$, we have

$$\frac{d^2G}{dx^2} \sim \frac{9}{4}G$$

Multiplying each side by $\frac{dG}{dx}$ and integrate then we get

$$\left(\frac{dG}{dx}\right)^2 \sim \frac{9}{4}G^2$$

or

$$\frac{1}{G}\left(\frac{dG}{dx}\right) \sim \frac{3}{2}.$$

Which upon integration yields

$$\ln G \sim \frac{3}{2}x,$$

contradicting that $G = o(e^{\frac{x}{2}})$. Hence L must equal 0. For large enough x, the fact that $\frac{dG}{dx} + O(e^{-3x/2})$ increasing implies

$$\frac{dG}{dx} \leq O(e^{-\frac{3x}{2}})$$

Which means that G is bounded. Using the fact that G is bounded allows us to **express** G **as an improper integral**, which can be verified by (21). The ultimate goal is to use this integral to **estimate the order** G and then **feed the estimated order back into the integral** in order to get a more accurate picture of the behavior of G.

Our bounding of G tells us, by referring to its definition, that

$$F - F(\infty) = O(e^{-\frac{x}{2}})$$

Meaning that (21) becomes:

$$\frac{d^2G}{dx^2} - \frac{9}{4}G = O(e^{-3x/2}) + O\left(Ge^{-x}\right)$$

Then G can be solved for as:

$$\begin{aligned} G = Ae^{-\frac{3x}{2}} - \frac{2}{3}e^{\frac{3x}{2}} \int_x^\infty e^{-\frac{3t}{2}} \left(O(e^{-\frac{3t}{2}}) + O(Ge^{-t})\right) dt \\ -\frac{2}{3}e^{-\frac{3x}{2}} \int_{x_0}^x e^{\frac{3t}{2}} \left(O(e^{-\frac{3t}{2}}) + O(Ge^{-t})\right) dt \end{aligned} \tag{23}$$

Which can be **verified** by taking the double derivative of (23) as such:

$$\begin{aligned} \frac{dG}{dx} = -\frac{3}{2}Ae^{-\frac{3x}{2}} - e^{\frac{3x}{2}} \int_x^\infty e^{-\frac{3t}{2}} \left(O(e^{-\frac{3t}{2}}) + O(Ge^{-t})\right) dt \\ +\frac{2}{3}\left(O(e^{-\frac{3x}{2}}) + O(Ge^{-x})\right) + e^{-\frac{3x}{2}} \int_{x_0}^x e^{\frac{3t}{2}} \left(O(e^{-\frac{3t}{2}}) + O(Ge^{-t})\right) dt \\ -\frac{2}{3}\left((O(e^{-\frac{3x}{2}}) + O(Ge^{-x})\right) \end{aligned}$$

$$\begin{aligned} \frac{d^2G}{dx^2} = \frac{9}{4}Ae^{-\frac{3x}{2}} - \frac{3}{2}e^{\frac{3x}{2}} \int_x^\infty e^{-\frac{3t}{2}} \left(O(e^{-\frac{3t}{2}}) + O(Ge^{-t})\right) dt \\ -\frac{3}{2}e^{-\frac{3x}{2}} \int_{x_0}^x e^{\frac{3t}{2}} \left(O(e^{-\frac{3t}{2}}) + O(Ge^{-t})\right) dt + O(e^{-\frac{3x}{2}}) + O(Ge^{-x}) \end{aligned}$$

And adding $\frac{d^2G}{dx^2}$ to $-\frac{9}{4}G$ cancels out every term except $O(e^{-\frac{3x}{2}}) + O(Ge^{-x})$, which is exactly the right side of (21).

Back to (23): since integration does not change the order of an exponential function, the largest term in G is of order $O(e^{-x})$. So we can say $G = O(e^{-x})$ and plug this into (23). Again we only have exponentials in the integrand, so this sorts out the same way, leaving us with

$$G(x) \sim Be^{-\frac{3x}{2}}$$

Where B is some constant. Then

$$F - k\pi \sim Be^{-2x}$$

Which is what we were to show.

4.2. **Case 2:** $\lim_{r\to\infty} F(r) = (k + 1/2)\pi$

In this case, $\cos(2F(\infty)) = -1$, so (20) becomes

$$\frac{d^2G}{dx^2} + \frac{7}{4}G = O(e^{-3x/2}) + O\left(G(F - F(\infty))^2\right) \tag{24}$$

Now we intend to **rewrite** G **in a form that agrees with** (24) and that allows us to **bound it**.

G can be written as

$$
\begin{aligned}
G &= A\cos(\frac{\sqrt{7}}{2}x) + B\sin(\frac{\sqrt{7}}{2}x) \\
&+\frac{2}{\sqrt{7}}\int_{x_0}^{x}\sin(\frac{1}{2}\sqrt{7}(x-t))\times\Big(O(e^{-\frac{3t}{2}})+O((F-F(\infty))^2G)\Big)\,dt
\end{aligned}
\tag{25}
$$

A fact that can be verified with (24) by calculating $\frac{d^2G}{dx^2}$ as such:

$$
\begin{aligned}
\frac{dG}{dx} &= -\frac{\sqrt{7}}{2}A\sin(\frac{\sqrt{7}}{2}x) + \frac{\sqrt{7}}{2}B\cos(\frac{\sqrt{7}}{2}x) \\
&-\frac{2}{\sqrt{7}}\sin(\frac{1}{2}\sqrt{7}(x-x_0))\times\Big(O(e^{-\frac{3x_0}{2}})+O((F-F(\infty))^2G)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2G}{dx^2} &= -\frac{7}{4}A\cos(\frac{\sqrt{7}}{2}x) - \frac{7}{4}B\sin(\frac{\sqrt{7}}{2}x) \\
&-\cos(\frac{\sqrt{7}}{2}(x-x_0))\times\Big(O(e^{-\frac{3x_0}{2}})+O((F-F(\infty))^2G)\Big)
\end{aligned}
$$

So then in accordance with the left hand side of (24):

$$
\begin{aligned}
\frac{d^2G}{dx^2}+\frac{7}{4}G &= \frac{\sqrt{7}}{2}\int_{x_0}^{x}\sin(\frac{1}{2}\sqrt{7}(x-t))\times\Big(O(e^{-\frac{3t}{2}})+O((F-F(\infty))^2G)\Big)\,dt \\
&\quad -\cos(\frac{\sqrt{7}}{2}(x-x_0))\times\Big(O(e^{-\frac{3x_0}{2}})+O((F-F(\infty))^2G)\Big) \\
&= -\int_{x_0}^{x}\cos(\frac{1}{2}\sqrt{7}(x-t))\times\frac{d}{dt}\Big(O(e^{-\frac{3t}{2}})+O((F-F(\infty))^2G)\Big)\,dt \\
&= O(e^{-3x/2})+O\left(G(F-F(\infty))^2\right)
\end{aligned}
$$

Where the last equality is due to cosine being bounded. So it is then clear that (25) satisfies (24).

In order to bound G, we can choose x_0 such that

$$
\int_{x_0}^{\infty}(F-F(\infty))^2 < \frac{\sqrt{7}}{4}
$$

Where for $x > x_0$ we can define

$$
M(x) = \sup_{x_0\le t\le x}|G(t)|
$$

So then using (25) we can say

$$
\begin{aligned}
M &\le A+B+\frac{2}{\sqrt{7}}\Big|\int_{x_0}^{x}O(e^{-\frac{3t}{2}})dt\Big| + \frac{1}{2}\Big|\int_{x_0}^{x}\sin(\frac{1}{2}\sqrt{7}(x-t))O(G)dt\Big| \\
&\le A+B+C+\frac{1}{2}\Big|G-\int_{x_0}^{x}\cos(x-t)\frac{dG}{dx}dt\Big| \le A+B+C+\frac{1}{2}M
\end{aligned}
$$

So at last we have the bound of M as

$$
M \le 2A+2B+2C,
$$

where C is the constant associated with the $O(e^{-\frac{3x}{2}})$ term. So then G **is bounded**.

The conclusions for case 2 follow thusly: $F - (k\pi + \frac{1}{2}) = Ge^{-\frac{x}{2}}$, so given the oscillatory terms in G, we see that the solution oscillates about $k\pi + \frac{1}{2}$. Also, G bounded means $F - (k\pi + \frac{1}{2}) = O(e^{-\frac{x}{2}})$.

4.3. **Case 4:** $\lim_{r\to 0} F(r) = (l + 1/2)\pi$

This case has a similar structure to case 2. First, **in order to get a more usable function**, we rewrite (2) as

$$r^2F'' + \frac{\frac{1}{2}r^3F'}{\frac{1}{4}r^2 + 2\sin^2(F)} + \frac{\sin(2F)}{\frac{1}{4}r^2 + 2\sin^2(F)}(r^2F'^2 - \frac{1}{4}r^2 - \sin^2(F)) = 0$$

Change of variables $r = e^x$ i.e. $\ln r = x$. Recall from section 3 that $rF' = \frac{dF}{dx} \to 0$ as $r \to 0 \iff x \to -\infty$. This gives us the facts:

$$\begin{aligned} r^2F'' &= \frac{d^2F}{dx^2} - \frac{dF}{dx} \\ \frac{1}{4}r^2 + 2\sin^2(F) &\sim 2 \\ r^3F' &\sim 0 \end{aligned}$$

so we can rewrite (2) as:

$$\begin{aligned} &\frac{d^2F}{dx^2} - \frac{dF}{dx} + O(r^2) - \frac{\sin^2(F)\sin(2F)}{\frac{1}{4}r^2 + 2\sin^2(F)} + \frac{r^2F'\sin(2F)}{\frac{1}{4}r^2 + 2\sin^2(F)} \\ &= \frac{d^2F}{dx^2} - \frac{dF}{dx} + O(e^{2x}) - \frac{1}{2}\sin(2F) + \frac{1}{2}r^2F'^2\sin(2F) = 0 \end{aligned} \tag{26}$$

In order to rewrite our equation, we will do what we did before and **show the improper integrals are finite** and then **introduce a function** G whose behavior is tied to this fact.

If we multiply (26) by $\frac{dF}{dx}$ and integrate from x to $-\infty$ then

$$\begin{aligned} \int_{-\infty}^{x_0} \left(\frac{dF}{dx}\right)^2 dx &= \int_{-\infty}^{x_0} \frac{dF}{dx}\left(\frac{d^2F}{dx^2} + O(e^{2x}) - \frac{1}{2}\sin(2F) + \frac{1}{2}r^2F'^2\sin(2F)\right)dx \\ &\le \left(\frac{dF}{dx}\right)^2\Big|_{-\infty}^{x_0} + O(1)F\Big|_{-\infty}^{x_0} - \frac{1}{2}F\Big|_{-\infty}^{x_0} + \frac{1}{2}(a+1)F\Big|_{-\infty}^{x_0} + O(1)F\Big|_{-\infty}^{x_0} \le \infty \end{aligned}$$

Where the last 2 terms are a result of limit taken in section 2. We will then use the finiteness of this integral for when multiplying (26) by $\sin(2F)$ and integrating over the same domain:

$$\begin{aligned} \int_{-\infty}^{x_0} \sin^2(2F) &= \int_{-\infty}^{x_0} r^2F'^2\sin^2(2F) + O(e^{2x})\sin(2F) + 2\sin(2F)\left(\frac{d^2F}{dx^2} - \frac{dF}{dx}\right)dx \\ &\le (a+1)\Big|_{-\infty}^{x_0} + O(1) + O(1)\cos(2F)\Big|_{-\infty}^{x_0} - 2F\Big|_{-\infty}^{x_0} + 2\frac{dF}{dx}\Big|_{-\infty}^{x_0} \le \infty \end{aligned}$$

Defining

$$F - F(0) = e^{x/2}G$$

Allows us to rewrite (26):

$$e^{x/2}\left(\frac{d^2G}{dx^2}-\frac{1}{4}G\right)+O(e^{2x})-\frac{1}{2}\sin(2F)+\frac{1}{2}\sin(2F)\left(\frac{dF}{dx}\right)^2=0$$

Divide by $e^{x/2}$:

$$\frac{d^2G}{dx^2}-\frac{1}{4}G+O(e^{\frac{3}{2}x})=\frac{1}{2}\frac{\sin(2F)}{e^{x/2}}-\frac{1}{2}\frac{\sin(2F)}{e^{x/2}}\left(\frac{dF}{dx}\right)^2$$

Subtract $G\cos(2F(0))$:

$$\begin{aligned}\frac{d^2G}{dx^2}+\frac{3}{4}G+O(e^{\frac{3}{2}x})&=\frac{1}{2}\frac{\sin(2F)}{e^{x/2}}-\frac{1}{2}\frac{\sin(2F)}{e^{x/2}}\left(\frac{dF}{dx}\right)^2-G\cos(2F(0))\\&=O(G(F-F(0))^2)-\frac{1}{2}\frac{\sin(2F)-\sin(2F(0))}{e^{x/2}}\left(\frac{dF}{dx}\right)^2\\&=O(G(F-F(0))^2)+O\left(\frac{F-F(0)}{e^{x/2}}\right)\left(\frac{dF}{dx}\right)^2\end{aligned}$$

Yielding the form:

$$\frac{d^2G}{dx^2}+\frac{3}{4}G=O(e^{\frac{3}{2}x})+O(G(F-F(0))^2)+O\left(G\left(\frac{dF}{dx}\right)^2\right) \tag{27}$$

This is the final expression of (2) relevant to this case. As for the next step, we shall **rewrite G and verify it using** (27).

Similar to case 2 we can write G in the following way:

$$\begin{aligned}G=A\cos(\frac{\sqrt{3}}{2}x)+B\sin(\frac{\sqrt{3}}{2}x)+\frac{2}{\sqrt{3}}\int_{x_0}^{x}\sin(\frac{\sqrt{3}}{2}(x-t))\times\\\left(O(e^{\frac{3}{2}x})+O(G(F-F(0))^2)+O\left(G\left(\frac{dF}{dx}\right)^2\right)\right)dt\end{aligned}$$

Then $\frac{d^2G}{dx^2}$ is

$$\begin{aligned}\frac{d^2G}{dx^2}=-\frac{3}{4}A\cos(\frac{\sqrt{3}}{2}x)-\frac{3}{4}B\sin(\frac{\sqrt{3}}{2}x)-\cos(\frac{\sqrt{3}}{2}(x-x_0))\times\\\left(O(e^{\frac{3}{2}x})+O(G(F-F(0))^2)+O\left(G\left(\frac{dF}{dx}\right)^2\right)\right)\end{aligned}$$

Applying this to (27) for **verification**:

$$\frac{d^2G}{dx^2}+\frac{3}{4}G=\frac{\sqrt{3}}{2}\int_{x_0}^{x}\sin(\frac{\sqrt{3}}{2}(x-t))\times\Bigg(O(e^{\frac{3}{2}x})$$
$$+O(G(F-F(0))^2)+O\left(G\left(\frac{dF}{dx}\right)^2\right)dt-\cos(\frac{\sqrt{3}}{2}(x-x_0))\times$$
$$\left(O(e^{\frac{3}{2}x})+O(G(F-F(0))^2)+O\left(G\left(\frac{dF}{dx}\right)^2\right)\right)=\int_{x_0}^{x}\cos(\frac{\sqrt{3}}{2}(x-x_0))\times$$
$$\frac{d}{dt}\left(O(e^{\frac{3}{2}x})+O(G(F-F(0))^2)+O\left(G\left(\frac{dF}{dx}\right)^2\right)\right)dt=$$
$$O(1)\left(O(e^{\frac{3}{2}x})+O(G(F-F(0))^2)+O\left(G\left(\frac{dF}{dx}\right)^2\right)\right)$$

So now that we have established that G works we will use its form to create a bound.

We choose x_0 so that

$$\int_{-\infty}^{x_0}(F-F(0))^2<\frac{\sqrt{3}}{4}$$

And define

$$M(x)=\sup_{x\leq t\leq x_0}|G(t)|$$

Giving us

$$M\leq A+B+O(e^{\frac{3}{2}x})+\frac{1}{2}O(G)+O\left(G\left(\frac{dF}{dx}\right)^2\right)$$
$$\leq A+B+C+\frac{1}{2}M+\sigma M$$

Meaning M is bounded as such:

$$M\leq\frac{1}{(\frac{1}{2}-\sigma)}(A+B+C)$$

Where we can quantify a σ as small as we want since $\frac{dF}{dx}\to 0$ as $r\to 0$. So then specifying a $\sigma<\frac{1}{2}$ tells us that M, and by consequence G, must be bounded.

Case 4 conclusions follow: form of G tells us that F oscillates about its limit $(l+\frac{1}{2})\pi$ and $F-(l+\frac{1}{2})\pi=O(e^{x/2})=O(r^{1/2})$.

4.4. **Case 3:** $\lim_{r\to 0}F(r)=l\pi$

The general strategy of this section is to **bound rF' above and below so that we can make a statement about its order**. The first step to doing this is simplifying the problem: We can assume $l=0$ since our argument can be applied to any l.

Next we will show that F **is monotone near 0**: So $F\to 0$, meaning we can specify a sequence $\{r_n\}$ such that $r_n\to 0$ as $n\to\infty$ and where $F(r_n)>0$, $F'(r_n)>0$. We know we can specify r_n's such that the latter condition is true because if $F(r_n) > 0$ then F is approaching 0 from above so $\sin(2F) > 0$. If F' were ever to be zero then (2) tells us

$$(\frac{1}{4}r^2 + 2\sin^2(F))F'' = \frac{1}{4}\sin(2F) + \frac{\sin^2(F)}{r^2}\sin(2F) > 0$$

meaning that $F'' > 0$. **So when r is close enough to 0, F becomes monotone increasing from above**. (Similarly if we try approaching from below F will be monotone decreasing, so F monotone either way).

We are intent to show that $\frac{rF'}{F} \to 1$ as $r \to 0^+$. This shall be done via **contradiction**. First assume there exists an r such that $rF' > F$. Then if we look at (2):

$$(\frac{1}{4}r^2 + 2\sin^2(F))F'' = -\frac{1}{2}rF' + \frac{1}{4}\sin(2F) + \frac{\sin^2(F) - (rF')^2}{r^2}\sin(2F) < \frac{1}{2}(F - rF') + \frac{\sin^2(F) - F^2}{r^2}\sin(2F) < 0$$

So then $F'' < 0$. By consequence:

$$\left(\frac{rF'}{F}\right)' = \frac{rFF'' + F'(F - rF')}{r^2} \le \frac{rFF''}{r^2} \le (F')^2\frac{rF'}{F} < 0$$

So then $\frac{rF'}{F}$ increases as $r \to 0^+$. Since $rF' > F$ is true too, we have

$$lim_{r\to 0^+}\frac{rF'}{F} = c > 1 \tag{28}$$

Hence

$$lim_{r\to 0^+}F' > \frac{F}{r}$$

so by integration:

$$lim_{r_0\to 0^+}F > \int_0^{r_0}\frac{F}{r}.$$

This means that $\frac{F}{r} \to 0$. Looking at (2):

$$F'' = \frac{1}{(\frac{1}{4}r^2 + 2\sin^2(F))}\left(-\frac{1}{2}rF' - \sin(2F)(F')^2 + \frac{1}{4}\sin(2F) + \frac{\sin^2(F)}{r^2}\sin(2F)\right) \sim -\frac{F'}{r},$$

and as a consequence we can say

$$F'' < -m\frac{F'}{r} \sim -m\frac{F}{r^2}$$

for some $m > 0$. Integrating the final relation close to $F = l\pi$ means F is monotone and so the integral can be bounded i.e. for some M:

$$\int_0^{r_0}\frac{-M}{r^2} = -\infty > \int_0^{r_0}F'' = F'(r_0) - F'(0)$$

But then $F' \to \infty$ as $F \to l\pi$, **contradicting** (28)**. Thus only $rF' \le F$ is possible.** Now that we have bounded rF' above we will show it is bound below using **contradiction** again:

Suppose $rF' < \frac{1}{2}\sin(2F)$. Then (2) can be turned into the inequality:

$$-F'' < \sin(2F)(F'^2 - \frac{\sin^2(F)}{r^2}) < \sin(2F)\left(\frac{1}{4}\frac{\sin^2(2F) - 4\sin^2(F)}{r^2}\right)$$
$$= \sin(2F)\left(\frac{(\cos(F)-1)(\sin^2(F))}{r^2}\right) < 0$$

So $F'' > 0$. Then

$$\left(\frac{\sin(2F)}{rF'}\right)' = \frac{2F'\cos(2F)}{rF'} - \frac{\sin(2F)(F' + rF'')}{(rF')^2}$$
$$< \frac{2\cos(2F)F' - 2F' - 2rF''}{rF'} < \frac{-2F''}{F'} < 0$$

Meaning that $\frac{\sin(2F)}{rF'}$ is increasing as $r \to 0^+$, which is a **contradiction**. Hence, the assumption that $rF' < \frac{1}{2}\sin(2F)$ was incorrect. So then we know

$$\frac{1}{2}\sin(2F) < rF' < F$$

Considering the Taylor expansion of $\sin(2F)$, we can write $rF' = F + O(F^3)$.

To confirm the conclusions for case 3: assume the form $F = ar + O(r^3)$ (for some $a > 0$). We can easily check that it satisfies the equation

$$\frac{dF}{dr} = a + O(r^2),$$

meaning that close to $r = 0$, $F \sim ar$.

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