Introduction to Conformal Field Theory in Two Dimensions

With Application to the Free Fermion

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Abstract

We discuss the basics of conformal field theory in two dimensions. We define the conformal group and the Virasoro algebra it facilitates, discuss the energy-momentum tensor, and develop the radial coordinate formulation, which we then quantize. We also develop the theory of operator-product expansions. Finally, we apply these concepts to the case of the free fermion, deriving the central charge $c = 1/2$.

Introduction

Since the early 1970s, conformal field theory has been a rich area of research in quantum field theory and statistical mechanics. It was originally invented to study the latter; specifically, to study physics of materials near their critical points and nth-order phase transitions. But it has seen a wide application in quantum field theory as well, because it turns out that scale invariance is common in in many simple cases of various situations, such as the massless Dirac equation $\gamma^\mu \partial_\mu \psi = 0$. Of course, the masslessness is a critical condition, as a mass would introduce a scale into the problem, thus making CFT inapplicable!

But again, conformal field theory was originally developed for studying critical phenomena, and there it has been very successful. Critical points are configurations of a thermodynamic system where a continuous phase transition occurs (such as water to vapor, or near the Curie temperature of a ferromagnet). Generally, to make a system approach a critical point, we need to vary some external parameter (pressure, temperature, etc.). We would usually like to know how various thermodynamic quantities scale with this parameter as we move the system closer and closer to the critical point. It turns out they tend to scale with respect to some *critical exponent*, which is closely related to the conformal dimension that we will study. But doesn't this introduce a length scale? Well, actually no! As we increase our parameter, fluctuations in the material become large enough that they are correlated at large distances, and a quantity measuring this called the *correlation length* goes to infinity. Thus, practically speaking, we have scale invariance!

Mathematically, though, CFT has some difficulties. We will only discuss the theory in two dimensions here, which is a much simpler case than even three dimensions. This is because the algebra describing the theory is infinite-dimensional, and the condition for an infinitesimal transformation to be conformal reduces to just the Cauchy-Riemann equations, allowing us to bring in the whole edifice of complex analysis to bear in working out the details. We can use the "conformal bootstrap method", where we can work out much of the behavior of the theory just using the symmetries.

Most material is adapted from Blumenhagen and Plauschinn 2009, with supplementation from Qualls 2015 and Belavin, Polyakov, and Zamolodchikov 1984.

Conformal Symmetry

Preliminaries

Consider d-dimensional spacetime with p time-like coordinates and q space-like coordinates, which we will call $\mathbb{R}^{p,q}$ (clearly $p+q=d$), with the metric $g_{\mu\nu}=\eta_{\mu\nu}=\mathrm{diag}(-1,\ldots,-1,1,\ldots 1)$ (we will work entirely in flat spacetime, and may use η and g interchangeably). We will call a differentiable map ϕ *conformal* if it transforms the metric to $g'_{\mu\nu}=\Lambda(x)g_{\mu\nu}$, for some function $\Lambda(x)$ which we call the *scale factor*. More explicitly, using the transformation rule for the metric,

$$\eta_{\rho\sigma}\,\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}=\Lambda(x)\,\eta_{\mu\nu}. \tag{1}$$

To find the conditions under which a transformation is conformal, we evaluate (1) under an infinitesimal coordinate transformation $x'^{\mu}=x^{\mu}+\varepsilon^{\mu}(x)+\mathcal{O}(\varepsilon^2)$:

$$\begin{aligned}\eta_{\rho\sigma}\,\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}}&=\eta_{\rho\sigma}\left(\delta^{\rho}_{\mu}+\frac{\partial\varepsilon^{\rho}}{\partial x^{\mu}}+\mathcal{O}(\varepsilon^2)\right)\left(\delta^{\sigma}_{\nu}+\frac{\partial\varepsilon^{\sigma}}{\partial x^{\nu}}+\mathcal{O}(\varepsilon^2)\right)\\&=\eta_{\mu\nu}+\eta_{\mu\sigma}\frac{\partial\varepsilon^{\sigma}}{\partial x^{\nu}}+\eta_{\rho\nu}\frac{\partial\varepsilon^{\rho}}{\partial x^{\mu}}+\mathcal{O}(\varepsilon^2)\\&=\eta_{\mu\nu}+\left(\frac{\partial\varepsilon_{\mu}}{\partial x^{\nu}}+\frac{\partial\varepsilon_{\nu}}{\partial x^{\mu}}\right)+\mathcal{O}(\varepsilon^2).\end{aligned}$$

Enforcing the conformal condition (1), we have that up to first order in ε, $\partial_\nu\varepsilon_\mu+\partial_\mu\varepsilon_\nu=\eta_{\mu\nu}(\Lambda(x)-1)=L(x)\,\eta_{\mu\nu}$ for some function L (where we introduced the notation $\partial_\alpha=\partial/\partial x^\alpha$). We can find L by tracing this equation with $\eta^{\mu\nu}$:

$$\begin{aligned}\eta^{\mu\nu}\left(\partial_\mu\varepsilon_\nu+\partial_\nu\varepsilon_\mu\right)=\partial^\nu\varepsilon_\nu+\partial^\mu\varepsilon_\mu&=L(x)\,\eta^{\mu\nu}\eta_{\mu\nu}\\2\partial^\mu\varepsilon_\mu&=L(x)\,d.\end{aligned}$$

Thus, for this infinitesimal transformation to be conformal, we need

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d}(\partial^\sigma \varepsilon_\sigma)\,\eta_{\mu\nu} \tag{2}$$

We can see now that $\Lambda(x) = 1 + (2/d)(\partial^\mu \varepsilon_\mu) + \mathcal{O}(\varepsilon^2)$.

Letting $d = 2$

Now that we have the primary condition for a transformation being conformal, we can work in the space we're interested in, namely $d = 2$. We will work primarily with $p = 0$, $q = 2$ with the knowledge that we can Wick rotate to $\eta_{\mu\nu} = \mathrm{diag}(-1, 1)$ by letting $x^0 \to ix^0$. So, if $d = 2$, then (2) becomes (for $\eta_{\mu\nu} = \mathrm{diag}(1, 1)$)

$$\begin{aligned} \partial_0\,\varepsilon_0 + \partial_0\,\varepsilon_0 &= \partial_0\,\varepsilon_0 + \partial_1\,\varepsilon_1 \\ \implies \partial_0\,\varepsilon_0 &= \partial_1\,\varepsilon_1. \end{aligned}$$

$$\begin{aligned} \partial_0\varepsilon_1 + \partial_1\varepsilon_0 &= \partial_0\varepsilon_0 + \partial_1\varepsilon_1 \\ \implies \partial_0\varepsilon_1 &= -\partial_1\varepsilon_0. \end{aligned}$$

In other words, in two dimensions the infinitesimal conformal condition becomes the Cauchy-Riemann equations for ε! Thus, if we work with complex numbers, we'll get a lot of powerful theory for free. Make the following definitions:

$$\begin{aligned} z &= x^0 + ix^1 & \varepsilon &= \varepsilon^0 + i\varepsilon^1 & \partial_z &= \frac{1}{2}(\partial_0 - i\partial_1) \\ \overline{z} &= x^0 - ix^1 & \overline{\varepsilon} &= \varepsilon^0 - i\varepsilon^1 & \partial_{\overline{z}} &= \frac{1}{2}(\partial_0 + i\partial_1). \end{aligned}$$

Then the above Cauchy-Riemann equations say $\varepsilon(z)$ is analytic in some open set, in which case $f(z) \equiv z + \varepsilon(z)$ is as well, and thus f is also a conformal transformation (for any point in this open set). Thus, we have the general result that *infinitesimal analytic functions of the form $f(z) = z + \varepsilon(z)$ are exactly the infinitesimal conformal transformations*. We can see that the metric transforms under f as

$$\begin{aligned} ds^2 &= (dx^0)^2 + (dx^1)^2 \\ &= (dx^0 + idx^1)(dx^0 - idx^1) \\ &= dz d\overline{z} \quad \to \quad \frac{\partial f}{\partial z}\frac{\partial \overline{f}}{\partial \overline{z}}\, dz d\overline{z}. \end{aligned}$$

Thus, the scale factor is just $|\partial f/\partial z|^2$.

The Virasoro Algebra

The Witt Algebra

It is reasonable to assume that $\varepsilon(z)$ is meromorphic outside the open set in which it is analytic, and thus it may have isolated singularities outside this open set. Thus, we can perform a Laurent expansion of $\varepsilon(z)$ around $z = 0$, and in general we find that an infinitesimal conformal transformation can be written

$$z' = z + \varepsilon(z) = z + \sum_{n\in\mathbb{Z}} \varepsilon_n(-z^{n+1})$$
$$\overline{z}' = \overline{z} + \overline{\varepsilon}(\overline{z}) = \overline{z} + \sum_{n\in\mathbb{Z}} \overline{\varepsilon}_n(-\overline{z}^{n+1})$$

using the constant parameters $\{\varepsilon_n\}$ and $\{\overline{\varepsilon}_n\}$. To be clear, since we are assuming all singularities are poles, there is a finite number of (but arbitrarily many) non-zero terms in the principal part; we have just set the bound at infinity to be the most general.

The generators corresponding to a transformation for a particular n are

$$\ell_n = -z^{n+1}\partial_z \qquad \overline{\ell}_n = -\overline{z}^{n+1}\partial_{\overline{z}}.$$

We can quickly verify this for an infinitesimal parameter ε_n :

$$e^{\varepsilon_n \ell_n} z = \left(1 - \varepsilon_n z^{n+1}\partial_z\right) z = z + \varepsilon_n \left(-z^{n+1}\right)$$

with similar steps possible for $\overline{\ell}_n$. Since the sum is over all $n \in \mathbb{Z}$, there are an *infinite* number of infinitesimal conformal transformations for $d = 2$. This turns out to be a fundamental fact about this case.

We can determine the algebra corresponding to these transformations by evaluating the commutators. For an arbitrary $f = f(z)$,

$$\begin{aligned}
[\ell_m, \ell_n]f &= -z^{m+1}\partial_z \left\{\left(-z^{n+1}\right)(\partial_z f)\right\} + z^{n+1}\partial_z \left\{\left(-z^{m+1}\right)(\partial_z f)\right\} \\
&= (n+1)z^{m+n+1}\partial_z f - (m+1)z^{m+n+1}\partial_z f + z^{m+n+2}\partial_z^2 f - z^{m+n+2}\partial_z^2 f \\
&= -(m-n)z^{m+n+1}\partial_z = (m-n)\ell_{m+n} f \\
\implies [\ell_m, \ell_n] &= (m-n)\ell_{m+n}.
\end{aligned}$$

A similar calculation with $\overline{\ell}_m$, $\overline{\ell}_n$, and $f = f(\overline{z})$ shows that

$$[\overline{\ell}_m, \overline{\ell}_n] = (m-n)\overline{\ell}_{m+n}.$$

Finally,

$$[\ell_m, \overline{\ell}_n] = -z^{m+1}\partial_z \left\{\left(-\overline{z}^{n+1}\right)\partial_{\overline{z}}\right\} + \overline{z}^{n+1}\partial_{\overline{z}} \left\{\left(-z^{m+1}\right)\partial_z\right\} = 0.$$

The first two relations define the *Witt algebra*, of which these generators give us two commuting copies. Since the generators are infinite-dimensional, this tells us that *the* *algebra of infinitesimal conformal transformations in Euclidean 2D space is infinite-dimensional.*

We will from now on treat z and $\overline{z}$ as independent variables, dealing with the two independent copies of the Witt algebra. Thus, we're actually working in $\mathbb{C}^2$, rather than $\mathbb{C}$.

Global Conformal Transformations

These generators are not well-defined on the Euclidean plane $\mathbb{R}^2 \simeq \mathbb{C}$; specifically, $\ell_n = -z^{n+1}\partial_z$ blows up at $z = 0$ for $n \leq -2$. The situation doesn't improve if we move to the Riemann sphere $S^2 \simeq \mathbb{C} \cup \{\infty\}$, for we have problems at $z = \infty$ as well. Perform the change of variables $z = -1/w$. Then because $\partial_z = \partial/\partial(-1/w) = (1/w^2)\partial_w$,

$$\ell_n = -\left(-\frac{1}{w}\right)^{n+1} \partial_z = -\left(-\frac{1}{w}\right)^{n-1} \partial_w.$$

Letting $z \to \infty$ is equivalent to letting $w \to 0$, and we see that ℓ_n is only well-defined for $n \leq 1$. This tells us that the only *globally-well-defined* transformations on the Riemann sphere are generated by $\{\ell_{-1}, \ell_0, \ell_1\}$.

The Conformal Group

Now we can determine the group of global conformal transformations by examining their generators $\{\ell_{-1}, \ell_0, \ell_1\}$. Begin with $\ell_{-1} = -\partial_z$:

$$e^{b\ell_{-1}} z = (1 - b\,\partial_z)\, z = z - b.$$

So ℓ_{-1} generates translations. Next, $\ell_0 = -z\partial_z$:

$$e^{a\ell_0} z = \left(1 - az\partial_z + a^2 \frac{1}{2} (z\partial_z)^2 - \dots\right) z = e^{-a} z.$$

To expand on this further, write $\partial_z = (1/2)(\partial_x - i\partial_y) = (e^{-i\theta}/2)[\partial_r - (i/r)\partial_\theta]$, which means $\ell_0 = -z\partial_z = -(r/2)\partial_r + (i/2)\partial_\theta$, and thus

$$\ell_0 + \overline{\ell}_0 = -r\partial_r \qquad i(\ell_0 - \overline{\ell}_0) = -\partial_\theta. \tag{3}$$

So we see that $\ell_0 + \overline{\ell}_0$ generates dilations and $i(\ell_0 - \overline{\ell}_0)$ generates rotations.

Finally, $\ell_1 = -z^2\partial_z$:

$$\begin{aligned} e^{c\ell_1} z &= \left(1 - cz^2\partial_z + c^2 \frac{1}{2} \left(z^2\partial_z\right)^2 - \dots\right) z \\ &= \left(1 - cz + c^2 z^2 - \dots\right) z = \frac{z}{1 - cz}, \end{aligned}$$

We can summarize the information above by saying that $\{\ell_{-1}, \ell_0, \ell_1\}$ generate transformations of the form

$$z \to \frac{az+b}{cz+d}.$$

where $a, b, c, d \in \mathbb{C}$. For this transformation to be invertible, we need for $ad - bc \neq 0$, and assuming this is the case, we can rescale the constants such that $ad - bc = 1$. Also, note that the transformation is invariant under $(a, b, c, d) \to (-a, -b, -c, -d)$.

If we arrange the constants in a matrix,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

then we can work out the group these transformations facilitate based on the above restrictions. The invertibility condition would give us $SL(2, \mathbb{C})$. But the fact that it's invariant under multiplication by -1 lets us remove half the group, giving us $SL(2, \mathbb{C})/\mathbb{Z}_2$. This group, the *Möbius group*, is in fact the group of conformal transformations on the Riemann sphere.

The Virasoro Algebra

The Witt algebra admits a *central extension*, characterized by the following commutation relations:

$$\begin{aligned} [\tilde{x}, \tilde{y}]_{\tilde{\mathfrak{g}}} &= [x, y]_{\mathfrak{g}} + c\, p(x, y) \\ [\tilde{x}, c]_{\tilde{\mathfrak{g}}} &= 0 \\ [c, c]_{\tilde{\mathfrak{g}}} &= 0 \end{aligned}$$

where $\tilde{\mathfrak{g}}$ is the new algebra with the central extension, $\mathfrak{g}$ is the Witt algebra, $\tilde{x}, \tilde{y} \in \tilde{\mathfrak{g}}$, $x, y \in \mathfrak{g}$, $p : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a bilinear function, and $c \in \mathbb{C}$ is a fixed constant called the *central charge* that characterizes the new algebra.

For ease of use, denote members of $\tilde{\mathfrak{g}}$ with L_n, $n \in \mathbb{Z}$, and write the commutation relation

$$[L_m, L_n] = (m-n)L_{m+n} + c\, p(m, n).$$

We need to determine the form of $p(m, n)$ for the central extension of the Witt algebra. First, we know that $p(m, n) = -p(n, m)$ to ensure the Lie bracket is antisymmetric. We can also ensure $p(-1, 1) = p(n, 0) = 0$ by redefining the generators:

$$\hat{L}_n = L_n + \frac{cp(n,0)}{n} \quad (n \neq 0), \qquad \hat{L}_0 = L_0 + \frac{cp(1,-1)}{2}.$$

Since we are merely adding constants, this doesn't modify the algebra; they drop out under commutation. Then,

$$
\begin{aligned}
[\hat{L}_n, \hat{L}_0] &= nL_n + c\,p(n,0) &&= n\left(\hat{L}_n - \frac{cp(n,0)}{n}\right) + c\,p(n,0) = n\hat{L}_n \\
[\hat{L}_1, \hat{L}_{-1}] &= 2L_0 + c\,p(1,-1) &&= 2\left(\hat{L}_0 - \frac{cp(1,-1)}{2}\right) + c\,p(1,-1) = 2\hat{L}_0.
\end{aligned}
$$

We also know that the algebra must satisfy the following Jacobi identity:

$$
\begin{aligned}
&[[L_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n] \\
&\qquad = (m-n)[L_{m+n}, L_0] + n[L_n, L_m] - m[L_m, L_n] \\
&\qquad = (m-n)\left\{(m+n)L_{m+n} + c\,p(m+n,0)\right\} \\
&\qquad + n\left\{(n-m)L_{m+n} + c\,p(n,m)\right\} \\
&\qquad - m\left\{(m-n)L_{m+n} + c\,p(m,n)\right\} \\
&\qquad = (m-n)c\,p(m+n,0) + nc\,p(n,m) - mc\,p(m,n) \\
&\qquad = (m+n)p(n,m) = 0.
\end{aligned}
$$

Thus, if $m + n \neq 0$, $p(n,m) = 0$. The algebra must also satisfy this Jacobi identity:

$$
\begin{aligned}
&[[L_{-n+1}, L_n], L_{-1}] + [[L_n, L_{-1}], L_{-n+1}] + [[L_{-1}, L_{-n+1}], L_n] \\
&\qquad = (-2n+1)[L_1, L_{-1}] + (n+1)[L_{n-1}, L_{-n+1}] + (n-2)[L_{-n}, L_n] \\
&\qquad = (-2n+1)\left\{2L_0 + c\,p(1,-1)\right\} \\
&\qquad + (n+1)\left\{2(n-1)L_0 + c\,p(n-1,-n+1)\right\} \\
&\qquad + (n-2)\left\{-2nL_0 + c\,p(-n,n)\right\} \\
&\qquad = (-2n+1)c\,p(1,-1) + (n+1)c\,p(n-1,-n+1) + (n-2)c\,p(-n,n) \\
&\qquad = (n+1)c\,p(n-1,-n+1) + (n-2)c\,p(-n,n) \\
&\qquad = 0.
\end{aligned}
$$

This lets us set up the following recurrence relation:

$$
\begin{aligned}
p(n,-n) &= \frac{n+1}{n-2}\,p(n-1,-n+1) \\
&= \frac{(n+1)!}{3!(n-2)!}\,p(2,-2) \\
&= \frac{(n-1)n(n+1)}{6}\frac{1}{2} \\
&= \frac{1}{12}(n^3 - n),
\end{aligned}
$$

where we chose $p(2,-2) = 1/2$, following standard convention (notice also that $(n+1)!/3!(n-2)! = \binom{n+1}{3}$). Thus, the commutation relation for the centrally-extended

Witt algebra is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

This is the *Virasoro algebra* Vir_c, with central charge c. Notice that the central extension doesn't affect the cases $n = -1, 0, 1$, and thus doesn't change the global subalgebra $\{L_{-1}, L_0, L_1\}$; they still generate their respective transformations from before.

We have performed this analysis in two-dimensional Euclidean space, but we can perform similar steps in two-dimensional flat spacetime as well. Define the light-cone coordinates $u = -t + x$ and $v = t + x$, which gives that

$$ds^2 = -dt^2 + dx^2 = du\, dv.$$

Conformal transformations are again given by $u \to f(u)$ and $v \to g(v)$ for some smooth functions f, g, which means $ds'^2 = \partial_u f \partial_v g\, du\, dv$, similarly to the Euclidean case. This means the algebra of infinitesimal conformal transformations is again infinite-dimensional, letting us define the Witt and Virasoro algebras just as before.

What is the point of adding this extra term to the commutation relations? Centrally-extending the Witt algebra into the Virasoro algebra allows projective representations of the conformal group to become true representations. In quantum theory, we're generally using projective representations of groups, because states are physically indistinguishable from their scalar multiples.

Primary Fields

Before moving on, we need a few definitions for two-dimensional conformal Euclidean field theories. Again, we identify $\mathbb{R}^2 \simeq \mathbb{C}$ and let $z = x^0 + ix^1$ and $\overline{z} = x^0 - ix^1$. It's again convenient to consider z and $\overline{z}$ as independent variables (though at some point, we will need to identify them as complex conjugates). So, we write fields on Euclidean space as $\phi(z, \overline{z})$.

Definition 1. Fields depending only on z are called *chiral* fields, and fields depending only on $\overline{z}$ are called *anti-chiral* fields.

Definition 2. If a field $\phi(z, \overline{z})$ transforms under scalings $z \to \lambda z$ for $\lambda \in \mathbb{C}$ as

$$\phi(z, \overline{z}) \to \phi'(z, \overline{z}) = \lambda^h \overline{\lambda}^{\overline{h}} \phi\left(\lambda z, \overline{\lambda}\overline{z}\right)$$

for some $h \in \mathbb{C}$, we say it has *conformal dimension* $(h, \overline{h})$.

Definition 3. If a field $\phi(z, \overline{z})$ transforms under conformal transformations $z \to f(z)$ as

$$\phi(z, \overline{z}) \to \phi'(z, \overline{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \phi\left(f(z), \overline{f}(\overline{z})\right).$$

for some $h \in \mathbb{C}$, we say it is a *primary field* of conformal dimension $(h, \overline{h})$. If this only holds for $f \in SL(2,\mathbb{C})/\mathbb{Z}_2$, ie only if f is a global conformal transformation, then ϕ is called a *quasi-primary field.*

Note that a field being primary implies it is quasi-primary, but not the reverse. Also, not all fields in a conformal field theory are even quasi-primary; such fields are called *secondary fields* (but we will not overmuch concern ourselves with them).

How does a primary field behave under infinitesimal conformal transformations? Let $\phi(z,\overline{z})$ be such a field, and $f(z) = z + \varepsilon(z)$ be such a transformation. Then

$$(\partial_z f)^h = (1 + \partial_z \varepsilon(z))^h = 1 + h\,\partial_z\varepsilon(z) + \mathcal{O}\left(\varepsilon^2\right).$$

$$\phi\left(z + \varepsilon(z), \overline{z}\right) = \phi(z) + \varepsilon(z)\,\partial_z\phi(z,\overline{z}) + \mathcal{O}\left(\varepsilon^2\right).$$

By the definition of primary field, ϕ transforms as

$$\begin{aligned}\phi\left(z,\overline{z}\right) \to \phi'(z,\overline{z}) = &\left(1 + h\,\partial_z\varepsilon(z)\right)\left(1 + \overline{h}\,\partial_{\overline{z}}\overline{\varepsilon}(z)\right)\\ &\times\left(\phi + \varepsilon(z)\,\partial_z\phi + \overline{\varepsilon}(\overline{z})\,\partial_{\overline{z}}\phi\right)\\ = &\,\phi + \left(\varepsilon\partial_z + \overline{\varepsilon}\partial_{\overline{z}} + h\partial_z\varepsilon + \overline{h}\partial_{\overline{z}}\overline{\varepsilon}\right)\phi.\end{aligned}$$

In other words,

$$\delta_{\varepsilon,\overline{\varepsilon}}\,\phi = \phi' - \phi = \left(\varepsilon\partial_z + \overline{\varepsilon}\partial_{\overline{z}} + h\partial_z\varepsilon + \overline{h}\partial_{\overline{z}}\overline{\varepsilon}\right)\phi. \tag{4}$$

The Energy-Momentum Tensor

Fundamental to much of physics work is the energy-momentum tensor, from which we can derive many properties of its associated theory. Thus it will be productive to explore its properties in two-dimensional conformal field theories. Since the algebra of infinitesimal conformal transformations is infinite-dimensional, we have strong constraints on the form of the theory, and in fact we can study the theory only by working out the behavior of the energy-momentum tensor under conformal transformations.

By Noether's theorem, to every continuous symmetry we have a conserved current j_μ, ie $\partial^\mu j_\mu = 0$. We can apply our infinitesimal conformal symmetry $x^\mu \to x^\mu + \varepsilon^\mu(x)$, which yields a conserved current of the form $j_\mu = T_{\mu\nu}\varepsilon^\nu$, where $T_{\mu\nu}$ is the energy-momentum tensor. Note that $T_{\mu\nu}$ is symmetric. This represents the change in x^μ under an infinitesimal conformal transformation. If we assume that ε^μ is constant, the conservation of the current gives us that

$$\partial^\mu j_\mu = 0 = \partial^\mu(T_{\mu\nu}\varepsilon^\nu) = (\partial^\mu T_{\mu\nu})\varepsilon^\nu.$$

which means $\partial^\mu T_{\mu\nu} = 0$.

For a general transformation $\varepsilon^\mu(x)$, the current conservation gives

$$\begin{aligned}\partial^\mu j_\mu = 0 &= (\partial^\mu T_{\mu\nu})\varepsilon^\nu + T_{\mu\nu}(\partial^\mu \varepsilon^\nu)\\ &= \frac{1}{2}T_{\mu\nu}(\partial^\mu\varepsilon^\nu + \partial^\nu\varepsilon^\mu)\\ &= \frac{1}{2}T_{\mu\nu}\frac{2}{d}\eta^{\mu\nu}(\partial^\sigma\varepsilon_\sigma)\\ &= \frac{1}{d}T^\mu_\mu(\partial^\sigma\varepsilon_\sigma)\end{aligned}$$

where we used the fact that $T_{\mu\nu}$ is symmetric and equation (2). Since this must be true for arbitrary ε^μ, this tells us that $T_{\mu\nu}$ is traceless (ie, $T^\mu_\mu = 0$).

Now we will restrict to two Euclidean dimensions, again using complex coordinates. Using the coordinate identities $x^0 = (z+\overline{z})/2$ and $x^1 = (z-\overline{z})/2i$, as well as the transformation rule for $T_{\mu\nu}$, namely $T_{\mu\nu} = (\partial x^\alpha/\partial x^\mu)(\partial x^\beta/\partial x^\nu)T_{\alpha\beta}$, we have that

$$\begin{aligned}T_{zz} &= \left(\frac{1}{2}\right)^2 T_{00} + 2\left(\frac{1}{2}\right)\left(\frac{1}{2i}\right)T_{10} + \left(\frac{1}{2i}\right)^2 T_{11} &&= \frac{1}{4}\left(T_{00} - 2i\,T_{10} - T_{11}\right)\\ T_{\overline{z}\overline{z}} &= \left(\frac{1}{2}\right)^2 T_{00} + 2\left(\frac{1}{2}\right)\left(-\frac{1}{2i}\right)T_{10} + \left(-\frac{1}{2i}\right)^2 T_{11} &&= \frac{1}{4}\left(T_{00} + 2i\,T_{10} - T_{11}\right)\end{aligned}$$

$$T_{z\overline{z}} = T_{z\overline{z}} = \frac{1}{4}(T_{00} + T_{11}) = \frac{1}{4}T^\mu_\mu = 0$$

where we used the symmetry of $T_{\mu\nu}$. Using this fact again gives that

$$\begin{aligned}T_{zz} &= \frac{1}{2}(T_{00} - iT_{10})\\ T_{\overline{z}\overline{z}} &= \frac{1}{2}(T_{00} + iT_{10}).\end{aligned}$$

Finally, we can use that $\partial^\mu T_{\mu\nu} = \partial_0 T_{00} + \partial_1 T_{10} = \partial_0 T_{01} + \partial_1 T_{11} = 0$ to get that

$$\begin{aligned}\partial_{\overline{z}} T_{zz} &= \frac{1}{4}(\partial_0 + i\partial_1)(T_{00} - iT_{10})\\ &= \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} + i\partial_1 T_{00} - i\partial_0 T_{10})\\ &= -\frac{i}{4}(\partial_1 T_{11} + \partial_0 T_{01}) = 0.\end{aligned}$$

Similarly, we can show that $\partial_z T_{\overline{z}\overline{z}} = 0$, which tells us that the two non-vanishing components of the energy-momentum tensor are a chiral and anti-chiral field: $T(z) \equiv T_{zz}(z,\overline{z})$ and $\overline{T}(\overline{z}) \equiv T_{\overline{z}\overline{z}}(z,\overline{z})$.

Radial Quantization

Continuing the exploration of conformal field theory on the flat Euclidean plane, let x^0 be the time direction and x^1 be the space direction, and compactify x^1 into a circle of radius $R = 1$. We have formed a "spacetime cylinder" of infinite length, for which we can introduce the usual complex coordinate $w = x^0 + ix^1$.

We can then perform the change of variables $z \equiv e^w = e^{x^0} e^{ix^1}$. We have thus mapped points on the cylinder to the complex plane, with x^0 "measuring" the radius and x^1 "measuring" the argument. The infinite past is then mapped to $z = \overline{z} = 0$, the infinite future at $z, \overline{z} \to \infty$, with the space at each time represented by a circle centered at the origin. Translations in time $x^0 \to x^0 + a$ become dilations $z \to e^a z$, and space translations $x^1 \to x^1 + b$ become rotations $z \to e^{ib} z$. The Hamiltonian H generates time translations, and the momentum operator P generates space translations. Therefore, from the equations (3):

$$H = L_0 + \overline{L}_0 \qquad P = i(L_0 - \overline{L}_0).$$

Asymptotic States

Let $\phi = \phi(z, \overline{z})$ be a primary field of conformal dimension $(h, \overline{h})$. To make obvious that the conformal dimension criterion is satisfied, we can offset the exponents appropriately in a Laurent expansion:

$$\phi(z, \overline{z}) = \sum_{n, \overline{m} \in \mathbb{Z}} z^{-n-h} \overline{z}^{-\overline{m}-\overline{h}} \phi_{n,\overline{m}}$$

where the $\phi_{n,\overline{m}}$ are the Laurent modes, which we have converted to operators to achieve quantization. We can define the *asymptotic in-state* as the state in the infinite past:

$$|\phi\rangle = \lim_{z,\overline{z} \to 0} \phi(z, \overline{z}) |0\rangle = \lim_{z,\overline{z} \to 0} \left(\sum_{n, \overline{m} \in \mathbb{Z}} z^{-n-h} \overline{z}^{-\overline{m}-\overline{h}} \phi_{n,\overline{m}} \right) |0\rangle .$$

In order for this state to be well-defined, we need that $\phi_{n,\overline{m}} |0\rangle = 0$ for $n > -h$ and $\overline{m} > -\overline{h}$, which means the principle part of the Laurent expansion falls away. The terms with positive exponents go to 0 in the limit, so we're left with

$$|\phi\rangle = \phi_{-h,-\overline{h}} |0\rangle .$$

To find a similar expression for an *asymptotic out-state* at $z, \overline{z} \to \infty$, we need to compute the hermitian conjugates of the Laurent mode operators. Because we have Wick-rotated the time coordinate, $x^0 = it$, so under complex conjugation, $x^0 \to -x^0$. This means $z = e^{x^0 + ix^1} \to e^{-x^0 + ix^1} = 1/\overline{z}$, where we have finally identified $\overline{z}$ as the complex conjugate of z. Thus, in line with ϕ being primary, we define the hermitian conjugate as

$$\phi^\dagger(z, \overline{z}) = \overline{z}^{-2h} z^{-2\overline{h}} \phi\left(\frac{1}{\overline{z}}, \frac{1}{z}\right) .$$

Laurent expanding,

$$\phi^\dagger(z,\overline{z}) = \overline{z}^{-2h} z^{-2\overline{h}} \sum_{n,\overline{m}\in\mathbb{Z}} \overline{z}^{n+h} z^{\overline{m}+\overline{h}} \phi_{n,\overline{m}} = \sum_{n,\overline{m}\in\mathbb{Z}} \overline{z}^{n-h} z^{\overline{m}-\overline{h}} \phi_{n,\overline{m}}.$$

Comparing term-by-term with the Laurent expansion for ϕ, we see that

$$\phi^\dagger_{n,\overline{m}} = \phi_{-n,-\overline{m}}.$$

Now we can define the asymptotic out-states, this time using the hermitian conjugate field:

$$\langle\phi| = \lim_{z,\overline{z}\to 0} \langle 0|\, \phi^\dagger(z,\overline{z}) = \lim_{\overline{z},z\to 0} \langle 0| \left(\sum_{n,\overline{m}\in\mathbb{Z}} \overline{z}^{n-h} z^{\overline{m}-\overline{h}} \phi_{n,\overline{m}} \right),$$

Once again, $\langle\phi|$ will only be well-defined if the singular terms in the Laurent expansion are eliminated, which requires that $\langle 0|\, \phi_{n,\overline{m}} = 0$ for $n < h$ and $\overline{m} < \overline{h}$. Since the terms with positive exponent go to 0 in the limit, we are left with

$$\langle\phi| = \langle 0|\, \phi_{h,\overline{h}}.$$

The Operator Product Expansion

Now we develop a bit of theory to help us multiply fields, a critical step to calculate arbitrary correlation functions.

Conserved Charges

Recall the existence of the conserved current used in the above discussion of the energy-momentum tensor: $j_\mu = T_{\mu\nu}\varepsilon^\nu$. It must have an associated conserved charge, defined (at a constant time) as

$$Q \equiv \int dx^1\, j_0. \tag{5}$$

In field theory, such a conserved charge generates symmetry transformations for an arbitrary operator A, written $\delta A = [Q, A]$, where Q and A are set to the same times.

We use the coordinate transformation from the previous section, in which we compactified the two-dimensional spacetime into a cylinder and then projected this cylinder into $\mathbb{C}$. In these coordinates, constant time x^0 corresponds to constant radius. This makes the integral in (5) a closed contour integral around a circle centered at the origin. Taking the convention that contour integrals are always in the positive direction, we generalize (5) as

$$Q = \frac{1}{2\pi i} \oint \left(dz\, T(z)\varepsilon(z) + d\overline{z}\, T(\overline{z})\varepsilon(\overline{z}) \right), \tag{6}$$

where we have indicated that we also need to integrate over the $\overline{z}$ coordinate.

We can compute the transformation of a field generated by a conserved charge using $\delta\phi = [Q, \phi]$ and (6):

$$\delta_{\varepsilon,\overline{\varepsilon}}\phi(w,\overline{w}) = \frac{1}{2\pi i}\oint dz\ [T(z)\varepsilon(z), \phi(w,\overline{w})] + \frac{1}{2\pi i}\oint d\overline{z}\ \left[\overline{T}(\overline{z})\overline{\varepsilon}(\overline{z}), \phi(w,\overline{w})\right],$$

where again, the commutators must be evaluated at equal times, and so $|z| = |w|$.

Radial Ordering

Operator multiplication such as is occurring in the above integral is only defined in the right time order; for two operators A and B, the product $A(z)B(w)$ is only valid for $z^0 > w^0$, where z^0 and w^0 are the time components of the respective variables. Operators represent observables, and of course we can only observe one thing after another, and it should have the value it does at that time! In this mapping to the complex plane, where the time coordinate is represented by the radius of circles, this is equivalent to saying that $|z| > |w|$. To ensure this condition is always satisfied, we define the *radial ordering of operators* as

$$R\left(A(z)B(w)\right) = \begin{cases} A(z)B(w), & |z| > |w| \\ B(w)A(z), & |w| > |z| \end{cases}.$$

We may drop the explicit R to reduce notational clutter, but it is always present in the following.

So we know the above contour integrals over commutators must look something like this:

$$\oint dz\,[A(z), B(w)] = \oint_{|z|>|w|} dz\, A(z)B(w) - \oint_{|z|<|w|} dz\, B(w)A(z).$$

But wait a minute: we just said that $|z| = |w|$! The solution is to nudge the contour a little bit when we get close to w. For the first integral, we go a tiny bit towards the origin, and for the second a little bit away, leaving the rest the same. Since we're subtracting them, this actually leaves a contour centered at w itself! Thus, the result is

$$\oint dz\,[A(z), B(w)] = \oint_{C(w)} dz\, R(A(z)B(w)),$$

where $C(w)$ is the contour around w.

Now we can evaluate the contour integrals we wanted:

$$\begin{aligned}\delta_{\varepsilon,\overline{\varepsilon}}\,\phi(w,\overline{w}) =& \frac{1}{2\pi i}\oint_{C(w)} dz\, R(T(z)\phi(w,\overline{w}))\,\varepsilon(z) \\ &+ \frac{1}{2\pi i}\oint_{C(w)} d\overline{z}\, R(\overline{T}(\overline{z})\phi(w,\overline{w}))\,\overline{\varepsilon}(\overline{z}).\end{aligned}$$

But recall that we have computed $\delta_{\varepsilon,\overline{\varepsilon}}\,\phi(w,\overline{w})$ before, in equation (4). We can turn that result into one about contour integrals as well, using Cauchy's integral theorem:

$$\begin{aligned}\varepsilon(w)\left(\partial_w\phi(w,\overline{w})\right) &= \frac{1}{2\pi i}\oint_{C(w)} dz\,\frac{\varepsilon(z)}{z-w}\partial_w\phi(w,\overline{w})\\ h\,\left(\partial_w\varepsilon(w)\right)\phi(w,\overline{w}) &= \frac{1}{2\pi i}\oint_{C(w)} dz\,h\,\frac{\varepsilon(z)}{(z-w)^2}\phi(w,\overline{w}),\end{aligned}$$

with similar results for the anti-chiral terms.

We can now compare the two expressions we have for $\delta_{\varepsilon,\overline{\varepsilon}}\,\phi(w,\overline{w})$. Chiral and anti-chiral terms must go together, so we get that, for a biholomorphic field ϕ,

$$R(T(z)\phi(w,\overline{w})) = \frac{h}{(z-w)^2}\phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\phi(w,\overline{w}) + \ldots,$$

where the ellipses denote non-singular terms, which we don't care about as much since they'll drop out of contour integrals. Again, we have a similar expression for the anti-chiral terms. This is the *operator-product expansion* of the operators $T(z)$ and $\phi(w,\overline{w})$.

Laurent Modes of T

We can now work out the algebra of the Laurent modes of T. First, Laurent expand $T(z)$ in the following way:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n.$$

where the $\{L_n\}$ are the Laurent modes:

$$L_n = \oint \frac{dz}{2\pi i}\,T(z)z^{n+1}.$$

Choose the infinitesimal conformal transformation $\varepsilon(z) = -\varepsilon_n z^{n+1}$ and plug that into equation (6), which we can now evaluate in terms of these modes:

$$Q_n = \oint \frac{dz}{2\pi i}\,T(z)\left(-\varepsilon_n z^{n+1}\right) = \varepsilon_n \sum_{m\in\mathbb{Z}}\oint \frac{dz}{2\pi i}\,L_m z^{n-m-1} = -\varepsilon_n L_n.$$

Thus, the Laurent mode L_n generates the conformal transformation $-\varepsilon_n z^{n+1}$. It is a member of its very own Virasoro algebra!

We can additionally compute the following commutation relation between the modes of T and the modes of a chiral primary field ϕ after (somewhat tedious) contour integration that we will leave out:

$$[L_m,\phi_n] = \left((h-1)m-n\right)\phi_{m+n}. \tag{7}$$

We can apply the above to the modes themselves by letting $\phi_n = L_n$. For $m, n \in \{-1, 0, 1\}$, then, because the modes are part of a Virasoro algebra, we have $((h-1)m - n)L_{m+n} = [L_m, L_n] = (m-n)L_{m+n}$. From this, we can conclude that $T(z)$ has conformal dimension $h = 2$, $\overline{h} = 0$.

The Two-Point Function and General OPE

We actually have enough theory already to calculate the two-point function for chiral, quasi-primary fields, which we will initially write as $\langle \phi_1(z)\phi_2(w)\rangle = g(z, w)$. Invariance under L_{-1} implies invariance under translations, which means $g(z, w) = g(z - w)$. Invariance under L_0 implies

$$\langle \phi_1(z)\phi_2(w)\rangle \to \langle \lambda^{h_1}\phi_1(\lambda z)\lambda^{h_2}\phi_2(\lambda w)\rangle = \lambda^{h_1+h_2} g(\lambda(z-w)) = g(z-w),$$

so we can write

$$g(z-w) = \frac{d_{12}}{(z-w)^{h_1+h_2}},$$

where d_{12} is the *structure constant*.

Finally, invariance under L_1 implies invariance under the transformation $z \to -1/z$, which means, by the definition of quasi-primary field,

$$\begin{aligned}\langle \phi_1(z)\phi_2(w)\rangle &\to \left\langle \frac{1}{z^{2h_1}}\frac{1}{w^{2h_2}}\phi_1\left(-\frac{1}{z}\right)\phi_2\left(-\frac{1}{w}\right)\right\rangle \\ &= \frac{1}{z^{2h_1}w^{2h_2}}\frac{d_{12}}{\left(-\frac{1}{z}+\frac{1}{w}\right)^{h_1+h_2}} \\ &= \frac{d_{12}}{(z-w)^{h_1+h_2}}.\end{aligned}$$

This expression can only be true if $h_1 = h_2$. Thus, we have the form of the two-point function:

$$\langle \phi_i(z)\phi_j(w)\rangle = \frac{d_{ij}\delta_{ij}}{(z-w)^{2h_i}}.$$

In fact, these symmetries also fix the three-point function, which we will present here without going into the derivation:

$$\langle \phi_i(z_i)\phi_j(z_j)\phi_k(z_k)\rangle = \frac{C_{ijk}}{(z_i-z_j)^{h_i+h_j-h_k}(z_j-z_k)^{h_j+h_k-h_i}(z_i-z_k)^{h_i+h_k-h_j}},$$

where again C_{ijk} is some structure constant.

The General OPE

The above forms of the two and three-point functions let us derive the general operator-product expansion for two chiral quasi-primary fields in terms of other quasi-primary fields and their derivatives (the proof that such functions can be expressed in those terms is quite complicated, so we won't go over it here). We make the educated guess

$$\phi_i(z)\phi_j(w) = \sum_{k,n\geq 0} C_{ij}^k \frac{a_{ijk}^n}{n!} \frac{1}{(z-w)^{h_i+h_j-h_k-n}} \partial_w^n \phi_k(w), \tag{8}$$

where C_{ij}^k and a_{ijk}^n are some constants. Our job is to determine them.

First, take $w = 1$ in the above equation and consider it as part of a three-point function:

$$\langle (\phi_i(z)\phi_j(1))\, \phi_k(0)\rangle = \sum_{l,n\geq 0} C_{ij}^l \frac{a_{ijl}^n}{n!} \frac{1}{(z-1)^{h_i+h_j-h_l-n}} \langle (\partial_w^n \phi_l)\,(1)\phi_k(0)\rangle\,.$$

We can find the two-point function on the right-hand side using the general form we found in the previous section:

$$\langle (\partial_w^n \phi_l)\,(w)\phi_k(0)\rangle \bigg|_{w=1} = \partial_w^n \left(\frac{d_{lk}\delta_{h_l,h_k}}{w^{2h_k}}\right)\bigg|_{w=1} = (-1)^n n! \binom{2h_k+n-1}{n} d_{lk}\delta_{h_l,h_k}.$$

Plugging this back in gives

$$\langle \phi_i(z)\phi_j(1)\phi_k(0)\rangle = \sum_{l,n\geq 0} C_{ij}^l\, d_{lk}\, a_{ijk}^n \binom{2h_k+n-1}{n} \frac{(-1)^n}{(z-1)^{h_i+h_j-h_k-n}}.$$

But we also have the general expression for the three-point function from the previous section. With $z_i = z$, $z_j = 1$, and $z_k = 0$, we have

$$\frac{C_{ijk}}{(z-1)^{h_i+h_j-h_k} z^{h_i+h_k-h_j}} = \sum_{l,n\geq 0} C_{ij}^l\, d_{lk}\, a_{ijk}^n \binom{2h_k+n-1}{n} \frac{(-1)^n}{(z-1)^{h_i+h_j-h_k-n}}$$

$$\frac{C_{ijk}}{(1+(z-1))^{h_i+h_k-h_j}} = \sum_{l,n\geq 0} C_{ij}^l\, d_{lk}\, a_{ijk}^n \binom{2h_k+n-1}{n} (-1)^n (z-1)^n.$$

Then using the identity

$$\frac{1}{(1+x)^N} = \sum_{n=0}^{\infty} (-1)^n \binom{N+n-1}{n} x^n,$$

for $x = z-1$, we can compare coefficients and find that the final OPE has the constants

$$a_{ijk}^n = \binom{2h_k+n-1}{n}^{-1} \binom{h_k+h_i-h_j+n-1}{n}$$

$$C_{ij}^l = \frac{C_{ijk}}{d_{lk}}.$$

Normal Ordering

In quantum field theory, operators are generally built from juxtaposing field operators and their derivatives, evaluated at the same spacetime location. But in QFT, we need to give an ordering prescription for such products, which we call the *normal ordering*. Essentially this means we must create before we destroy: creation operators to the left, annihilation operators to the right.

First question: which are the annihilation operators, and which are the creation operators? We already know that $\phi_{n,\overline{m}}\left|0\right\rangle = 0$ for $n > -h$ or $\overline{m} > -\overline{h}$, as discussed for asymptotic states. Those operators must be annihilators.

Recall that $H = L_0 + \overline{L}_0$, which motivates us to define a "chiral energy" for the L_0 eigenvalue. For a chiral primary field, we can calculate

$$L_0\phi_n\left|0\right\rangle = (L_0\phi_n - \phi_n L_0)\left|0\right\rangle = [L_0, \phi_n]\left|0\right\rangle = -n\phi_n\left|0\right\rangle,$$

for $n \leq -h$, where we have employed equation (7) and the fact that $L_0\left|0\right\rangle = 0$, which is true because $\left|0\right\rangle$ is the vacuum state by definition! Thus,

$$\begin{aligned} &\phi_{n>-h} \quad \text{are annihilation operators, and} \\ &\phi_{n\leq -h} \quad \text{are creation operators,} \end{aligned}$$

with a similar result for $\overline{L}_0$.

Normal Ordered Product Formula

Now we are ready for the following normal ordered expression for products:

$$\phi(z)\chi(w) = (\text{singular terms}) + \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} N\left(\chi\partial^n\phi\right)(w), \tag{9}$$

where $N(\chi\phi)$ tells us to normally order operators χ and ϕ. We will verify the above for $n = 0$.

First, multiply equation (9) by $(1/2\pi i)(1/(z-w))$ and contour integrate; this picks out the $n = 0$ term, giving

$$\oint_{C(w)} \frac{dz}{2\pi i} \frac{\phi(z)\chi(w)}{z-w} = N(\chi\phi)(w). \tag{10}$$

Any first-order poles in the singular terms, when multiplied by $1/(z-w)$, become second-order, and thus disappear when contour-integrating.

We can also Laurent expand $N(\chi\phi)$, giving

$$N(\chi\phi)(w) = \sum_{n=\mathbb{Z}} w^{-n-h_\phi-h_\chi} N(\chi\phi)_n,$$

where $N(\chi\phi)_n$ is the nth Laurent mode of the normal-ordered product:

$$N(\chi\phi)_n = \oint_{C(0)} \frac{dw}{2\pi i}\, w^{n+h_\phi+h_\chi-1} N(\chi\phi)(w).$$

Plugging equation (10) into this expression for the Laurent modes gives

$$\begin{aligned} N(\chi\phi)_n &= \oint_{C(0)} \frac{dw}{2\pi i} w^{n+h_\phi+h_\chi-1} \oint_{C(w)} \frac{dz}{2\pi i} \frac{\phi(z)\chi(w)}{z-w} && (11)\\ &= \oint_{C(0)} \frac{dw}{2\pi i} w^{n+h_\phi+h_\chi-1} \left(\oint_{|z|>|w|} \frac{dz}{2\pi i} \frac{\phi(z)\chi(w)}{z-w} - \oint_{|z|<|w|} \frac{dz}{2\pi i} \frac{\chi(w)\phi(z)}{z-w} \right). && (12) \end{aligned}$$

We can evaluate the first term in parentheses:

$$\begin{aligned} \oint_{|z|>|w|} \frac{dz}{2\pi i} \frac{\phi(z)\chi(w)}{z-w} &= \oint_{|z|>|w|} \frac{dz}{2\pi i} \left\{ \frac{1}{z-w} \right\} \sum_{r,s} z^{-r-h_\phi} w^{-s-h_\chi} \phi_r \chi_s \\ &= \oint_{|z|>|w|} \frac{dz}{2\pi i} \left\{ \frac{1}{z} \sum_{p\geq 0} \left(\frac{w}{z} \right)^p \right\} \sum_{r,s} z^{-r-h_\phi} w^{-s-h_\chi} \phi_r \chi_s \\ &= \oint_{|z|>|w|} \frac{dz}{2\pi i} \sum_{p\geq 0} \sum_{r,s} z^{-r-h_\phi-p-1} w^{-s-h_\chi+p} \phi_r \chi_s. \end{aligned}$$

Only the $1/z$ term will contribute, so this integral amounts to a delta function $\delta_{r,-h_\phi-p}$. We can plug the result into the integral over w, giving

$$\oint \frac{dw}{2\pi i} \sum_{p\geq 0} w^{-s-h_\chi+p+n+h_\phi+h_\chi-1} \phi_{-h_\phi-p} \chi_s.$$

Once again, only the $1/w$ term contributes, so we get the delta function $\delta_{s,p+n+h_\phi}$. So this integral becomes

$$\sum_{p\geq 0} \phi_{-h_\phi-p}\, \chi_{h_\phi+n+p} = \sum_{k\leq -h_\phi} \phi_k\, \chi_{n-k}.$$

This is then the first term of equation (11). We can perform a similar calculation for the second term, yielding

$$N(\chi\phi)_n = \sum_{k>-h_\phi} \chi_{n-k}\, \phi_k + \sum_{k\leq -h_\phi} \phi_k\, \chi_{n-k}.$$

Indeed, this is normally-ordered! In the first term, annihilation operators χ_{n-k} are on the left of creation operators ϕ_k, and in the second term annihilation operators ϕ_k are on the left of creation operators χ_{n-k}. Thus, the regular part of the above operator-product expansion contains only normally-ordered products.

Some normal ordered products that will come up later are $N(\chi\partial\phi)$ and $N(\partial\chi\phi)$, so let's compute their modes. First, we can get the Laurent expansion of $\partial\phi$:

$$\partial_z\phi(z) = \partial_z \sum_n z^{-n-h}\,\phi_n = \sum_n (-n-h) z^{-n-(h+1)}\,\phi_n.$$

We can then perform essentially the same analysis as above with this modified Laurent expansion, giving

$$\begin{aligned} N(\chi\partial\phi)_n &= \sum_{k>-h_\phi-1} (-h_\phi - k)\chi_{n-k}\phi_k + \sum_{k\leq -h_\phi-1} (-h_\phi - k)\phi_k\chi_{n-k} \\ N(\partial\chi\phi)_n &= \sum_{k>-h_\phi} (-h_\chi - n + k)\chi_{n-k}\phi_k + \sum_{k\leq -h_\phi} (-h_\chi - n + k)\phi_k\chi_{n-k}. \end{aligned} \tag{13}$$

The Free Fermion

As an application of the theory we have just developed, we will study the case of a free Majorana fermion in two-dimensional Minkowski space, with metric $h_{\alpha\beta} = \mathrm{diag}(1,-1)$. Our final goal will be to compute the central charge of this theory. We are going to breeze through this somewhat; some of the details require theory that is beyond the scope of this paper. The action of such a fermion is

$$S = \frac{1}{4\pi\kappa}\int dx^0\,dx^1\,\sqrt{|h|}\,\overline{\Psi}\gamma^\alpha\partial_\alpha\Psi.$$

for some normalization constant κ. We have also defined $\overline{\Psi} = \Psi^\dagger\gamma^0$, where $\dagger$ denotes the Hermitian conjugate and the $\{\gamma^\alpha\}$ are 2×2 matrices satisfying the Clifford algebra

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\,h^{\alpha\beta}\,\mathbb{1}_2,$$

where $\mathbb{1}_2$ is the 2×2 identity matrix. We make the following choice of representation for the gamma matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

We can simplify the action using the following identity:

$$\gamma^0\gamma^\mu\partial_\mu = \gamma^0\left(\gamma^0\partial_0 + \gamma^1\partial_1\right) = \mathrm{diag}(\partial_0 + i\partial_1, \partial_0 - i\partial_1) = 2\,\mathrm{diag}(\partial_{\overline{z}}, \partial_z).$$

The Majorana spinor is equal to the combination of real fields $\Psi = \left(\psi(z,\overline{z}), \overline{\psi}(z,\overline{z})\right)^T$.

Finally, we need to compute the new metric $g_{\mu\nu}$ after changing coordinates to $z = x^0 + ix^1$:

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Thus, we can rewrite the action as

$$\begin{aligned} S &= \frac{1}{4\pi\kappa}\int dz\, d\overline{z}\, \sqrt{|g|}\, \Psi^\dagger\, \mathrm{diag}(\partial_{\overline{z}}, \partial_z)\Psi \\ &= \frac{1}{4\pi\kappa}\int dz\, d\overline{z}\, \left(\psi\, \partial_{\overline{z}}\, \psi + \overline{\psi}\, \partial_z\, \overline{\psi}\right). \end{aligned}$$

We compute the equations of motion with a variational principle:

$$\begin{aligned} 0 = \delta_\psi\, S &= \frac{1}{4\pi\kappa}\int dz\, d\overline{z}\, \left((\delta\psi)\, \partial_{\overline{z}}\, \psi + \psi\, \partial_{\overline{z}}\, (\delta\psi)\right) \\ &= \frac{1}{4\pi\kappa}\int dz\, d\overline{z}\, \left((\delta\psi)\, \partial_{\overline{z}}\, \psi - (\partial_{\overline{z}}\psi)\, \delta\psi\right) + \frac{1}{4\pi\kappa}\int dz\, (\psi\, \delta\psi)\Big|_{-\infty}^{\infty} \\ &= \frac{1}{2\pi\kappa}\int dz\, d\overline{z}\, (\delta\psi)\, \partial_{\overline{z}}\psi, \end{aligned}$$

where in the second step we integrated by parts, and in the third step used the fact that fermionic fields anticommute, so $-(\partial_{\overline{z}}\psi)\delta\psi = +\delta\psi(\partial_{\overline{z}}\psi)$, as well as assumed that $\psi(\pm\infty) = 0$. Since this equation must be satisfied for all variations $\delta\psi$, we get the equation of motion

$$\partial_{\overline{z}}\psi = 0.$$

We can similarly vary with respect to $\overline{\psi}$ and get $\partial_z\overline{\psi} = 0$. Thus, ψ and $\overline{\psi}$ are chiral and anti-chiral, respectively. Now we can show that they're primary with conformal dimensions $(1/2, 0)$ and $(0, 1/2)$ by adjusting the action and showing that it's invariant:

$$\begin{aligned} S \to S' &= \frac{1}{4\pi\kappa}\int dz\, d\overline{z}\, \left(\psi'(z,\overline{z})\partial_{\overline{z}}\psi'(z,\overline{z}) + \overline{\psi}'(z,\overline{z})\partial_z\overline{\psi}'(z,\overline{z})\right) \\ &= \frac{1}{4\pi\kappa}\int \frac{\partial z}{\partial w}\, dw\, \frac{\partial \overline{z}}{\partial \overline{w}}\, d\overline{w}\, \Bigg(\left(\frac{\partial w}{\partial z}\right)^{\frac{1}{2}} \psi(w,\overline{w})\, \frac{\partial \overline{w}}{\partial \overline{z}}\, \partial_{\overline{w}} \left(\frac{\partial w}{\partial z}\right)^{\frac{1}{2}} \psi(w,\overline{w}) \\ &+ \left(\frac{\partial \overline{w}}{\partial \overline{z}}\right)^{\frac{1}{2}} \overline{\psi}(w,\overline{w})\, \frac{\partial w}{\partial z}\, \partial_w \left(\frac{\partial \overline{w}}{\partial \overline{z}}\right)^{\frac{1}{2}} \overline{\psi}(w,\overline{w})\Bigg) \\ &= \frac{1}{4\pi\kappa}\int dw\, d\overline{w}\, \left(\psi(w,\overline{w})\, \partial_{\overline{w}}\, \psi(w,\overline{w}) + \overline{\psi}(w,\overline{w})\, \partial_w\overline{\psi}(w,\overline{w})\right). \end{aligned}$$

Finally, we can use the fact that the fields are fermions to deduce that there are two possibilities for the transformation under a 2π rotation, for the Neveu-Schwarz and Ramond sectors:

$$\psi\left(e^{i2\pi}z\right) = +\psi(z) \qquad \psi\left(e^{i2\pi}z\right) = -\psi(z). \tag{14}$$

There are similar expressions for the anti-chiral field.

We can now perform the coordinate transformation to the cylinder, and then to the complex plane, with radius measuring x^0 and argument measuring x^1. We need to modify the definition of radial ordering a bit because we are working with fermions:

$$R\left(\Psi(z)\Theta(w)\right) = \begin{cases} +\Psi(z)\Theta(w), & |z| > |w| \\ -\Theta(w)\Psi(z), & |w| > |z| \end{cases}.$$

Now we would like to compute the algebra of the field Laurent modes. We can Laurent expand a chiral field of conformal weight 1/2 as

$$\psi(z) = \sum_r \psi_r z^{-r-\frac{1}{2}},$$

with a similar expression for an anti-chiral field. But notice that, to consider both cases in (14), we must have $r \in \mathbb{Z}$ for the former and $r \in \mathbb{Z} + 1/2$ for the latter. So we have two possible Laurent expansions, depending on the sector under consideration.

We also need to work out the operator-product expansion for these fields. Since they have conformal dimension 1/2, from equation (8), we can see that

$$\psi(z)\psi(w) = \frac{\kappa}{z-w} + \ldots,$$

where the ellipses denote non-singular terms. To prove that the constant in the numerator is in fact κ can be done by computing the propagator of the action integral, but we will assert it without proof. Notice that interchanging z and w leads to an overall minus sign, consistent with the properties of fermions.

We can pull out the Laurent modes with a contour integral:

$$\psi_r = \oint \frac{dz}{2\pi i}\, \psi(z)\, z^{r-\frac{1}{2}}.$$

Now we can calculate the algebra. Because we're dealing with fermions, we need to use anti-commutators instead of commutators.

$$\begin{aligned} \{\psi_r, \psi_s\} &= \oint \frac{dz}{2\pi i} \o \int \{\psi(z), \psi(w)\}\, z^{r-\frac{1}{2}} w^{s-\frac{1}{2}} \\ &= \oint \frac{dw}{2\pi i} w^{s-\frac{1}{2}} \left(\oint_{|z|>|w|} \frac{dz}{2\pi i} \psi(z)\psi(w) z^{r-\frac{1}{2}} - \oint_{|z|<|w|} \frac{dz}{2\pi i} - \psi(w)\psi(z) z^{r-\frac{1}{2}} \right) \\ &= \oint \frac{dw}{2\pi i}\, w^{s-\frac{1}{2}} \oint_{C(w)} \frac{dz}{2\pi i}\, R\left(\psi(z)\psi(w)\right) z^{r-\frac{1}{2}}. \end{aligned}$$

The only singular term in the OPE of the fields is the first one. So by Cauchy's integral formula,

$$\oint \frac{dz}{2\pi i}\, R\left(\psi(z)\psi(w)\right) z^{r-\frac{1}{2}} = \oint \frac{dz}{2\pi i} \frac{\kappa}{z-w} z^{r-\frac{1}{2}} = \kappa w^{r-\frac{1}{2}}.$$

Plugging back in, we get

$$\{\psi_r, \psi_s\} = \kappa \oint \frac{dw}{2\pi i} w^{r+s-1} = \kappa\, \delta_{r+s,0}.$$

We can write the *canonical* energy-momentum tensor for fields ψ_i and Lagrangian $\mathcal{L}$ as

$$T_{\mu\nu} = 8\pi\kappa\gamma \left(-\eta_{\mu\nu}\mathcal{L} + \sum_i \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \psi_i\right)}\, \partial_\nu \psi_i \right),$$

for some to-be-determined normalization constant γ. From the action, we have the Lagrangian

$$\mathcal{L} = \psi\, \overline{\partial}\, \psi + \overline{\psi}\, \partial\, \overline{\psi},$$

which yields the following energy-momentum tensor when the above is evaluated:

$$T_{zz} = \gamma\psi\partial\psi \quad T_{z\overline{z}} = -\gamma\overline{\psi}\partial\overline{\psi} \quad T_{\overline{z}z} = -\gamma\psi\overline{\partial}\psi \quad T_{\overline{z}\overline{z}} = \gamma\overline{\psi}\overline{\partial\psi}.$$

The equations of motion $\overline{\partial}\psi = \partial\overline{\psi} = 0$ means $T_{z\overline{z}} = T_{\overline{z}z} = 0$.

Let $T(z) = T_{zz}$; all following results have analogues for the anti-chiral part. If we actually want to utilize $T(z)$ in practice, we will need the normally-ordered version, $\gamma N(\psi\partial\psi)$, and specifically the Laurent modes $L_m = \gamma N(\psi\partial\psi)_m$. The analysis we performed to get equation (13) used bosonic fields, but we can perform a similar analysis to get an expression for fermionic fields. The general expression is

$$N(\psi\,\theta)_r = -\sum_{s>-h_\theta} \psi_{r-s}\theta_s + \sum_{s\leq -h_\phi} \theta_s\psi_{r-s}.$$

which results in the Laurent modes

$$L_m = \gamma \sum_{s>-\frac{3}{2}} \psi_{m-s}\psi_s \left(s + \frac{1}{2}\right) - \gamma \sum_{s\leq -\frac{3}{2}} \psi_s\psi_{m-s} \left(s + \frac{1}{2}\right), \tag{15}$$

using the fact that ψ has conformal dimension 1/2. We can now calculate the commutator of L_m and the modes of $\psi(z)$:

$$\begin{aligned}
[L_m, \psi_r] &= \gamma \sum_{s>-\frac{3}{2}} [\psi_{m-s}\psi_s, \psi_r] \left(s+\frac{1}{2}\right) - \gamma \sum_{s\leq-\frac{3}{2}} [\psi_s\psi_{m-s}, \psi_r] \left(s+\frac{1}{2}\right) \\
&= \gamma \sum_{s>-\frac{3}{2}} \left(s+\frac{1}{2}\right) (\psi_{m-s}\{\psi_s, \psi_r\} - \{\psi_{m-s}, \psi_r\}\psi_s) \\
&- \gamma \sum_{s\leq-\frac{3}{2}} \left(s+\frac{1}{2}\right) (\psi_s\{\psi_{m-s}, \psi_r\} - \{\psi_s, \psi_r\}\psi_{m-s}) \\
&= \gamma\kappa \sum_{s>-\frac{3}{2}} \left(s+\frac{1}{2}\right) (\psi_{m-s}\delta_{s,-r} - \psi_s\delta_{m-s,-r}) \\
&- \gamma\kappa \sum_{s\leq-\frac{3}{2}} \left(s+\frac{1}{2}\right) (\psi_s\delta_{m-s,-r} - \psi_{m-s}\delta_{s,-r}) \\
&= \gamma\kappa \left(\left(-r+\frac{1}{2}\right)\psi_{m+r} - \left(m+r+\frac{1}{2}\right)\psi_{m+r}\right) \\
&= \gamma\kappa\,(-m-2r)\,\psi_{m+r}.
\end{aligned}$$

Following convention, we let $\gamma\kappa = 1/2$, which means

$$[L_m, \psi_r] = \left(-\frac{m}{2} - r\right)\psi_{m+r}.$$

To simplify the following work, we choose the standard normalization $\kappa = 1$, which means $\gamma = 1/2$.

We can finally compute the central charge! First, note

$$\langle 0| L_2L_{-2} |0\rangle = \langle 0| [L_2, L_{-2}] |0\rangle = \langle 0| \left(4L_0 + \frac{c}{2}\right) |0\rangle = \frac{c}{2}.$$

where we used the fact that $L_n |0\rangle = 0$ for $n > -h$, and that $T(z)$ is of conformal dimension 2. Consider only the Neveu-Schwarz sector, which means the ψ modes are labeled by half-integers. Then by equation (15),

$$L_{-2} |0\rangle = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}} |0\rangle$$

and

$$\langle 0| L_2 = \frac{1}{2} \langle 0| \left(\psi_{\frac{3}{2}} + \psi_{\frac{1}{2}} + 2\psi_{\frac{1}{2}}\psi_{\frac{3}{2}}\right) = \frac{1}{2} \langle 0| \left(\left\{\psi_{\frac{3}{2}}, \psi_{\frac{1}{2}}\right\} + \psi_{\frac{1}{2}}\psi_{\frac{3}{2}}\right) = \frac{1}{2} \langle 0| \psi_{\frac{1}{2}}\psi_{\frac{3}{2}}.$$

Thus,

$$\begin{aligned}
\langle 0| L_2L_{-2} |0\rangle = \frac{c}{2} &= \frac{1}{4} \langle 0| \psi_{\frac{1}{2}}\psi_{\frac{3}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}} |0\rangle \\
&= \frac{1}{4} \langle 0| \psi_{\frac{1}{2}} \left\{\psi_{\frac{3}{2}}, \psi_{-\frac{3}{2}}\right\} \psi_{-\frac{1}{2}} |0\rangle - 0 \\
&= \frac{1}{4} \langle 0| \left\{\psi_{\frac{1}{2}}, \psi_{-\frac{1}{2}}\right\} |0\rangle - 0 = \frac{1}{4}.
\end{aligned}$$

So the central charge of the conformal field theory given by a real free fermion is $c = 1/2$!

Conclusion

At this point, we have done just enough of the theory to get some of the basics down of the free fermion case: we have the modes of the energy-momentum tensor (and thus the tensor itself), their algebra, the Hamiltonian, equations of motion, and of course the algebra of the field itself. But there is still much more one could do, especially since the free fermion theory actually also describes the Ising lattice model of magnetism near the critical point. This paper may present only the basics, but it is enough to get a general feel for how everything works and what sort of math we're doing. As mentioned, things get hairier in higher dimensions, and there's still ongoing work on how to solve some of their problems.

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