Uniqueness of a three-dimensional stochastic differential equation

Carl Mueller and Giang Truong

(Communicated by Amarjit Singh Budhiraja)

In order to extend the study of the uniqueness property of multidimensional systems of stochastic differential equations, we look at the following three-dimensional system of equations, of which the two-dimensional case has been well studied:
\[ dX_t = Y_t \, dt, \quad dY_t = Z_t \, dt, \quad dZ_t = |X_t|^\alpha \, dB_t. \]
We prove that if \((X_0, Y_0, Z_0) \neq (0, 0, 0)\) and \(\frac{3}{4} < \alpha < 1\), then the system of equations has a unique solution in the strong sense.

1. Introduction and main results

The uniqueness of ordinary differential equations (ODEs) has been extensively studied; see for example [Hartman 1964]. In particular, if \(F(u)\) is Lipschitz continuous, then
\[ u'(t) = F(u(t)), \quad u(0) = u_0, \]
has a unique solution for all \(t \geq 0\). In the case above, \(F, u(t), \) and \(u_0\) take values in \(\mathbb{R}^d, \ d \geq 1\). The stochastic differential equation (SDE) realm, on the contrary, has different criteria for uniqueness of solutions; see for example [Protter 1990]. One of the most well-known results regarding strong uniqueness of SDEs is due to [Watanabe and Yamada 1971]. The result states that if \(f(x)\) is locally Hölder continuous with index \(\alpha \in \left[\frac{1}{2}, 1\right]\) and with linear growth, then
\[ dX = f(X) \, dW, \quad X_0 = x_0, \]
has a unique strong solution for all times \(t \geq 0\). Yamada and Watanabe’s theory essentially focuses on one-dimensional SDEs.

MSC2010: primary 60H10; secondary 34F05.

Keywords: white noise, stochastic differential equations, uniqueness.

Mueller was supported by a Simons grant.
One motivation for studying higher-dimensional SDEs comes from the wave equation

\[ \partial_{tt}u = \Delta u, \]

\[ u(0, x) = u_0(x), \]

\[ \partial_t u(0, x) = u_1(x). \]

In this equation, we have

\[ \partial_{tt}u = \partial_{xx}u = \Delta u. \]

If we let

\[ v = \partial_t u, \]

then we can rewrite the wave equation as the following system of equations:

\[ \partial_t u = v, \]

\[ \partial_t v = \Delta u. \]

The original wave equation includes no noise. However, many physical systems are affected by noise. Hence, a modification of the wave equation which includes white noise is also studied:

\[ \partial_{tt}u = \Delta u + f(u)\dot{W}, \]

\[ u(0, x) = u_0(x), \]

\[ \partial_t u(0, x) = u_1(x). \]

Note \( x \in \mathbb{R} \) and \( \dot{W} = \dot{W}(t, x) \) is white noise.

One well-known point is that Lipschitz continuity is sufficient for the uniqueness of SDEs. Thus, many mathematicians have studied whether Hölder continuity can still ensure the uniqueness property of SDEs. Gomez, Lee, Mueller, Neuman, and Salins [Gomez et al. 2017] studied the uniqueness property of the following two-dimensional model of SDEs:

\[ dX = Y \, dt, \]

\[ dY = |X|^\alpha \, dB, \]

\[ (X_0, Y_0) = (x_0, y_0). \]

The results focused on \( f(x) = |x|^\alpha \) since it is a prototype of an equation with Hölder continuous coefficients. Moreover, (2) is a version of (1) when we drop the dependence on \( x \), which allows us to study the modified wave equation with more simplicity. Notice that if we take the differential \( dY \) of the first derivative of \( X \), which is \( Y \) in the system of equations, it resembles the second derivative in time in the stochastic wave equation.
It was proven in [Gomez et al. 2017] that if $\alpha > \frac{1}{2}$ and $(x_0, y_0) \neq (0, 0)$, then (2) has a unique solution in the strong sense up to the time $\tau$ at which $(X_t, Y_t)$ first hits the origin $(0, 0)$.

Since much is still unknown about higher-dimensional SDEs, we wish to continue the study of the uniqueness property of (2) in the three-dimensional case, which is

$$
\begin{align*}
dX &= Y \, dt, \\
dY &= Z \, dt, \\
dZ &= |X|^\alpha dB,
\end{align*}
$$

(3)

$(X_0, Y_0, Z_0) = (x_0, y_0, z_0)$.

**Theorem 1.** If $\frac{3}{4} < \alpha < 1$ and $(X_0, Y_0, Z_0) \neq (0, 0)$, then (3) has a unique strong solution, up to the time $\tau$ at which the solution $(X_t, Y_t, Z_t)$ first hits the value $(0, 0, 0)$ or blows up.

Moreover, we say the solution $(X_t, Y_t, Z_t)$ blows up in finite time, with positive probability, if there is a random time $\tau < \infty$ such that $P\left(\lim_{t \uparrow \tau} |(X_t, Y_t, Z_t)|_\infty = \infty\right) > 0$.

2. Proof of Theorem 1

Let $(X^1_i, Y^1_i, Z^1_i), i = 1, 2$, be two solutions to (3) with $(x_0, y_0, z_0) \neq (0, 0, 0)$; in other words, $(X^1_i, Y^1_i, Z^1_i), i = 1, 2$, have the same initial condition $(X_0, Y_0, Z_0) \neq (0, 0, 0)$, Note that since $(X^1_i, Y^1_i, Z^1_i), i = 1, 2$, have the same initial conditions, from now on, we use $X_0, Y_0, Z_0$ instead of $X^1_0, Y^1_0, Z^1_0$.

Let $\tau$ be the first time $\tau$ that either $(X^1_1, Y^1_1, Z^1_1)$ or $(X^2_1, Y^2_1, Z^2_1)$ hits the origin $(0, 0, 0)$ or blows up. We let $\tau$ be infinity if there is no such time.

First, if $X_0 \neq 0$, then (3) will have Lipschitz continuity up to the time that $X_t = 0$, and thus enables uniqueness to hold. Suppose after a certain amount of time $X_t$ hits zero, where Lipschitz continuity no longer holds; then due to the strong Markov property, we begin the process again with $X_0 = 0$. So our goal is to prove pathwise uniqueness between excursions of $X$ up to the time $\tau$ starting from $X_0 = 0$.

For any fixed $n$, let $\tau_n$ be the first time that either

$$
|X^1_1, Y^1_1, Z^1_1|_\infty \land |X^2_1, Y^2_1, Z^2_1|_\infty \leq 2^{-n}
$$

or

$$
|X^1_1, Y^1_1, Z^1_1|_\infty \lor |X^2_1, Y^2_1, Z^2_1|_\infty \geq 2^n.
$$

Here, the infinity norm (also known as the $L_\infty$-norm, $l_\infty$-norm, max norm, or uniform norm) of a vector $\vec{v}$ is denoted by $|\vec{v}|_\infty$ and is defined as the maximum of...
the absolute values of its components,

$$|\vec{v}|_{\infty} = \max\{|v_i| : i = 1, 2, \ldots, n\},$$

and

$$a \wedge b = \min(a, b),$$

$$a \vee b = \max(a, b).$$

If there is no such time, we let $\tau_n$ be infinity. Note that $\lim_{n \to \infty}(\tau_n) = \tau$.

Now, for each fixed $n$, we will show uniqueness up to time $\tau_n$ in the system of equations

$$dX_t^{i,n} = Y_t^{i,n} dt,$$

$$dY_t^{i,n} = Z_t^{i,n} dt,$$

$$dZ_t^{i,n} = |X_t^{i,n}|^{\alpha} 1_{[0, \tau_n]}(t) dB_t,$$

$$(X_0, Y_0, Z_0) = (x_0, y_0, z_0).$$

In other words, after the time $\tau_n$, we have $dZ_t^{i,n} = 0$, which makes $Z_t^{i,n}$ become constant. Specifically, given $m, n \in N$, we need

$$(X_t^n, Y_t^n, Z_t^n) = (X_t^m, Y_t^m, Z_t^m)$$

for all $t \leq \tau_n \wedge \tau_m$.

Now, before continuing the proof of uniqueness, by the method of contradiction, we show that the times that $X_t$ hits zero do not accumulate before the time $\tau_n$, almost surely.

For each $n$, let $A_n$ be the event on which the times that $X_t^{i,n} = 0$, $i = 1$ or $i = 2$, accumulate before $\tau_n$, and assume $P(A_n) > 0$. Then, on $A_n$, suppose $\sigma_n$ is an accumulation point of the times $t$ at which $X_t^{i,n} = 0$; i.e., there exists a sequence of times $\rho_{1,n} < \rho_{2,n} < \cdots$ that converges to $\sigma_n$, and $X_t^{i,n} = 0$. Hence, on $A_n$,

$$\lim_{k \to \infty} \rho_{k,n} = \sigma_n.$$

We have $X_t^{i,n}$ is almost surely continuous, and that $X_t^{i,n} = 0$ on $A_n$, so

$$\lim_{\rho_{k,n} \to \sigma_n} X_t^{i,n} = X_t^{i,n} = 0$$

on $A_n$.

Note that $dX_t^{i,n} = Y_t^{i,n} dt$, and $Y_t^{i,n}$ is almost surely continuous. So if $Y_t^{i,n} \neq 0$ on $A_n$, then there exists a random interval $[\sigma_n(\omega) - \epsilon(\omega), \sigma_n(\omega)]$ of positive length for which $X_t^{i,n} \neq 0$ on $[\sigma_n(\omega) - \epsilon(\omega), \sigma_n(\omega)]$. This contradicts the hypothesis of $\rho_{k,n}$ converging to $\sigma_n$.

If $Y_t^{i,n} = 0$, there are two cases,

$$\sigma_n \geq \tau_n \quad \text{and} \quad \sigma_n < \tau_n.$$

If $\sigma_n \geq \tau_n$, then it means that the times at which $X_t^{i,n}$ hit zero do not accumulate before $\tau_n$. 

$$\text{PROOFS - PAGE NUMBERS ARE TEMPORARY}$$

CARL MUELLER AND GIANG TRUONG
If $\sigma_n < \tau_n$, then with $X^{i,n}_n = 0$ and $Y^{i,n}_n = 0$, we have $|Z^{i,n}_n| > 2^{-n}$. Now suppose $Z^{i,n}_n > 2^{-n}$, as the case $Z^{i,n}_n < 2^{-n}$ is approached similarly due to symmetry. If $Z^{i,n}_n > 2^{-n}$, since $Z_t$ is almost surely continuous, there exists a time interval $\tau_t \in [\sigma_n(\omega) - \epsilon(\omega), \sigma_n(\omega)]$ on which $Z^{i,n}_t > 2^{-n}/2$. Thus for all $t \in [\sigma_n(\omega) - \epsilon(\omega), \sigma_n(\omega)]$, we almost surely have

$$Y^{i,n}_t = Y^{i,n}_0 - \int_0^t Z^{i,n}_s \, ds < -\frac{2^{-n}}{2}(\sigma_n - t).$$

Hence, integrating over $X^{i,n}_t$ for $t \in [\sigma_n - \epsilon', \sigma_n]$, we have

$$X^{i,n}_t = X^{i,n}_0 - \int_0^t Y^{i,n}_s \, ds = -\int_0^t Y^{i,n}_s \, ds > \int_0^t \frac{2^{-n}}{2}(\sigma_n - s) \, ds$$

$$= \frac{2^{-n}}{2}(\sigma_n - s^2/2)|^\sigma_n_t = \frac{2^{-n}}{4}(\sigma_n - t)^2 > 0$$

as $t < \sigma_n$. Hence, this contradicts the hypothesis of $\rho_{k,n}$ converging to $\sigma_n$ on $A_n$.

So in conclusion, $P(A_n) = 0$, which means the times at which $X_t$ hits zero do not accumulate before the time $\tau_n$.

One more point we need to address before continuing with the proof of uniqueness is the existence of solutions. This problem is resolved in Theorems 21.7 and 21.8 and Lemma 21.17 of [Kallenberg 2002], which prove that for all times $t \geq 0$, solutions of multidimensional SDEs exist with probability 1 provided the coefficients are continuous and bounded.

Specifically, in our problem, since $\alpha \in (0, 1)$, the coefficients of system (3) are continuous and bounded by $(2^n)^{\alpha} \leq 2^n = 2^n$ up to the $\tau_n$ for each $n$, which satisfies the condition stated in the existence theory in [Kallenberg 2002]. Hence, existence of solution holds for all $t \leq \tau_n$ for all $n$. Therefore, up to the time $\tau = \sup \tau_n$, existence of solutions is ensured.

From now on, $X^{i,n}_t$ means $X^{i,n}_1$ and $X^{i,n}_2$. We define $Y^{i,n}_t$ and $Z^{i,n}_t$ similarly. Back to the proof of uniqueness, we have, for all $t \in [0, \tau_n]$,

$$|Z^{i,n}_t| + |Y^{i,n}_t| \leq 2^n.$$

So

$$|Y^{i,n}_t| = |Y_0 + \int_0^t Z^{i,n}_s \, ds| \leq |Y_0| + \int_0^t |Z^{i,n}_s| \, ds \leq 2^n + 2^n t.$$

Therefore,

$$|X^{i,n}_t| \leq |X_0 + \int_0^t Y^{i,n}_s \, ds| \leq \int_0^t |Y^{i,n}_s| \, ds \leq \int_0^t (2^n + 2^n s) \, ds = 2^n \left( t + \frac{t^2}{2} \right).$$

Now let

$$t_{0,n} = \frac{2^{-2n}}{2}. \tag{5}$$
Then
\[ t_{0,n} + \frac{t_0^2}{2} = \frac{2^{-2n}}{2} + \frac{2^{-2n}}{8} \leq \frac{2^{-2n}}{2} + \frac{2^{-2n}}{2} = 2^{-2n}. \]

So
\[ 2^n \left( t_{0,n} + \frac{t_0^2}{2} \right) \leq 2^n \cdot 2^{-2n} = 2^{-n}. \]

Since the quadratic function \( 2^n(t + \frac{t^2}{2}) \) is increasing when \( t \geq 0 \), we have
\[ |X_{t,n}^i| \leq 2^n \left( t + \frac{t^2}{2} \right) \leq 2^{-n} \]
for all \( t \in [0, t_{0,n}] \). Since \( |X_{t,n}^1| \) and \( |X_{t,n}^2| \) belong in \([0, 2^{-n}]\) for \( t \in [0, t_{0,n}] \), based on the definition of \( \tau_n \) above, either
\[ |Y_0| \geq 2^{-n} \tag{6} \]
or
\[ |Z_0| \geq 2^{-n} \tag{7} \]
for each fixed \( n \). This is due to the fact that the solutions have the same initial condition \((X_0, Y_0, Z_0)\) and for each time \( t \in [0, t_{n,0}] \), either
\[ |Y_{t,n}^i| \geq 2^{-n} \quad \text{or} \quad |Z_{t,n}^i| \geq 2^{-n}. \]

First, we deal with \( Y_0 > 0 \). Due to symmetry, we can deal with the case \( Y_0 < 0 \) with similar methods and thus omit the proof.

Now, with \( Y_0 > 0 \), we look at other subcases based on \( Z_{t,n}^i \).

Case I: \( Y_0 > 0 \), \( |Z_0| \leq 2^{-n} \). If \( |Z_0| \leq 2^{-n} \), then (6) takes place. We are looking at the case \( Y_0 > 0 \), and thus \( Y_0 > 2^{-n} \). Also, note that \( dZ_{t,n}^i = 0 \) for all \( t > \tau_n \) and \( |Z_{t,n}^i| \leq 2^n \) for all \( t \in [0, \tau_n] \). Hence, \( |Z_{t,n}^i| \leq 2^n \) for all \( t \), which means \( Z_{t,n}^i \geq 2^{-n} \) for all \( t \). Next, we have
\[ Y_{t,n}^i = Y_0 + \int_0^t Z_s \, ds \geq 2^{-n} - \int_0^t 2^n \, ds = 2^{-n} - 2^n t. \]

If
\[ 0 < t < t_{0,n} = \frac{2^{-2n}}{2}, \]
where \( t_{0,n} \) is defined as in (5), then
\[ 2^n t < \frac{2^{-n}}{2}. \]
Thus
\[ 2^{-n} - 2^n t > \frac{2^{-n}}{2} \]
for all \( t \in [0, t_{0,n}] \). In other words, \( Y_{t,n}^i > 2^{-n}/2 \) for \( t \in [0, t_{0,n}] \). So, for all \( t \in [0, t_{0,n}] \), we have
\[ Y_{t,n}^i \geq \frac{2^{-n}}{2}. \]
Hence
\[ X^{i,n}_t \geq X^{i,n}_0 + \int_0^t \frac{2^n - 2}{2} ds \geq \frac{2^n - 2}{2} t. \quad (8) \]

Furthermore, based on (8) and \( |X^{i,n}_t| \leq 2^{-n} \) for all \( t \in [0, t_{0,n}] \), it leads to \( t_{0,n} \leq 2 \), otherwise \( X^{i,n}_t > 2^{-n} \), which means that \( t > t_{0,n} \), a contradiction.

Note that
\[ X^{i,n}_t = X_0 + \int_0^t Y^{i,n}_s ds, \]
\[ Y^{i,n}_s = Y_0 + \int_0^s Z^{i,n}_k dk, \]
\[ Z^{i,n}_k = Z_0 + \int_0^k |X^{i,n}_{\tau}| \alpha 1_{[0,\tau]}(t) dB_r. \]

Thus
\[ X^{i,n}_t = X_0 + Y_0 t + \int_0^t \int_0^s \left( Z_0 + \int_0^k |X^{i,n}_{\tau}| \alpha 1_{[0,\tau]}(t) dB_r \right) dk ds \]
\[ = X_0 + Y_0 t + \int_0^t \int_0^s Z_0 dk ds + \int_0^t \int_0^s \int_0^k |X^{i,n}_{\tau}| \alpha 1_{[0,\tau]}(t) dB_r dk ds \]
\[ = X_0 + Y_0 t + Z_0 t^2 + \int_0^t \int_0^s \int_0^k |X^{i,n}_{\tau}| \alpha 1_{[0,\tau]}(t) dB_r dk ds. \]

Hence
\[ (X^{1,n}_t - X^{2,n}_t)^2 = \left( \int_0^t \int_0^s \int_0^k (|X^{1,n}_{\tau}| - |X^{2,n}_{\tau}|) \alpha 1_{[0,\tau]}(t) dB_r dk ds \right)^2. \]

Apply the Cauchy-Schwarz inequality twice, we get
\[ (X^{1,n}_t - X^{2,n}_t)^2 \leq t \int_0^t \left( \int_0^s \int_0^k (|X^{1,n}_{\tau}| - |X^{2,n}_{\tau}|) \alpha 1_{[0,\tau]}(r) dB_r dk ds \right)^2 ds \]
\[ \leq t \int_0^t s \int_0^k \left( \int_0^s (|X^{1,n}_{\tau}| - |X^{2,n}_{\tau}|) \alpha 1_{[0,\tau]}(r) dB_r \right)^2 dk ds. \]

Thus
\[ E[(X^{1,n}_t - X^{2,n}_t)^2] \leq t E \int_0^t s \int_0^k \left( \int_0^s (|X^{1,n}_{\tau}| - |X^{2,n}_{\tau}|) \alpha 1_{[0,\tau]}(r) dB_r \right)^2 dk ds. \]

By Itô’s isometry,
\[ E[(X^{1,n}_t - X^{2,n}_t)^2] \leq t E \int_0^t s \int_0^k \left( (|X^{1,n}_{\tau}| - |X^{2,n}_{\tau}|)^2 \alpha 1_{[0,\tau]}(r) \right)^2 dr dk ds \]
\[ \leq t E \int_0^t s \int_0^k (|X^{1,n}_{\tau}| - |X^{2,n}_{\tau}|)^2 dr dk ds. \]
At this stage we apply Gronwall’s lemma:

\[ 2^{1/2} \leq t^2 E \int_0^t \int_0^r \int_0^k (|X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha)^2 \, dr \, dk \, ds \]

\[ 3 \leq t^2 E \int_0^t \int_0^r \int_0^k (|X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha)^2 \, dr \, dk \, ds \]

\[ 4 \leq t^4 E \int_0^t (|X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha)^2 \, dr. \]

Now we apply the mean value theorem for the function \( f(x) = x^\alpha, \) \( 0 < \alpha < 1, \) and \( a < b: \)

\[ b^\alpha - a^\alpha = \alpha c^{\alpha-1} (b - a) \leq \alpha a^{\alpha-1} (b - a) \]

for \( c \in (a, b). \) Then for \( r \in [0, t_0,n], \) where \( t_0 \) is determined in (5), we apply (8):

\[ ||X_r^{1,n}|^\alpha - |X_r^{2,n}|^\alpha| \leq \alpha \left( \frac{2-n}{2} \right)^{\alpha-1} ||X_r^{1,n}| - |X_r^{2,n}||. \]  

(9)

Now let

\[ D_t = E[(|X_t^{1,n}| - |X_t^{2,n}|)^2]. \]

Since \( t_0,n \leq 2, \) we have for all \( t \in [0, t_0,n] \)

\[ D_t \leq E[(X_t^{1,n} - X_t^{2,n})^2] \leq C_n \int_0^t r^{2\alpha-2} D_r \, dr. \]

for some \( C_n \) depending on \( n. \) Since \( \alpha > \frac{3}{4}, \) we have \( r^{2\alpha-2} \) is integrable on \([0, t_0,n]. \)

At this stage we apply Gronwall’s lemma:

**Lemma.** Let \( I \) denote an interval of the real line of the form \([a, \infty)\) or \([a, b]\) or \([a, b)\) with \( a < b. \) Let \( \beta \) and \( u \) be real-valued continuous functions defined on \( I. \) If \( u \) is differentiable on the interior \( I^o \) of \( I \) (the interval \( I \) without the endpoint \( a \) and possibly \( b \)) and satisfies the differential inequality

\[ u'(t) \leq \beta(t) u(t), \quad t \in I^o, \]

then \( u \) is bounded by the solution of the corresponding differential equation \( v'(t) = \beta(t) v(t): \)

\[ u(t) \leq u(a) \exp \left( \int_a^t \beta(s) \, ds \right) \]

for all \( t \in I. \)

Hence, with \( D_0 = 0, \) we have \( D_t = 0 \) for all \( t \in [0, t_0,n]. \) Therefore, (3) has unique strong solution in \([0, t_0,n]. \)

Since \( \alpha \geq \frac{3}{4}, \) we have \( 2\alpha - 2 \geq -1. \) Hence, \( r^{2\alpha-2} \) is integrable on \([0, t_0,n]. \) Note that in this case since \( X_t \geq 2^{-n}, \) \( \eta \leq 1. \) Applying Gronwall’s lemma, with \( D_0 = 0, \) we have \( D_t = 0 \) for all \( t \in [0, t_0,n]. \) Therefore, (3) has a unique solution in strong sense up to time \( t_0,n. \)
Note that for all $t \in [0, t_0, n]$, we have $Y_{i, n}^t > 2^{-n}/2 > 0$. Hence $X_t^{i, n}$ is strictly increasing, which leads to $X_{0, n}$ strictly positive. Therefore, by the strong Markov property, we have uniqueness until the next time $X$ next hits zero.

**Case II:** $Y_0 > 0$, $Z_0 < -2^{-n}$. Since $Z_0$ starts negative, $Y_{i, n}^t$ decreases for an amount of time. Since $Y_0$ is positive, let say $Y_0 = \beta > 0$. Note that for all $t$, we have $|Z_{i, n}^t| < 2^n$, which means $Z_{i, n}^t < -2^n$. First, we have

$$Y_{i, n}^t = \beta + \int_0^t Z_s \, ds \geq \beta - \int_0^t 2^n \, ds = \beta - 2^n t.$$  

Let

$$t_{0, n}^i = \frac{\beta}{2^{n+1}}. \quad (10)$$

If

$$0 < t < t_{0, n}^i = \frac{\beta}{2^{n+1}},$$

then

$$2^n t < \frac{\beta}{2};$$

thus

$$\beta - 2^n t > \frac{\beta}{2}$$

for all $t \in [0, t_{0, n}^i]$. In other words, $Y_{i, n}^t > \beta/2$ for all $t \in [0, t_{0, n}^i]$.

So, for all $t \in [0, t_0, n \wedge t_{0, n}^i]$, where $t_0, n$ and $t_{0, n}^i$ are determined in (5) and (10) respectively, we have

$$Y_{i, n}^t \geq \frac{\beta}{2}$$

Hence

$$X_{i, n}^t \geq X_{0, n}^t + \int_0^t \frac{\beta}{2} \, ds \geq \frac{\beta}{2} t.$$

Applying the same method (9) above, we use the mean value theorem for the new lower bound of $X_{i, n}^t$:

$$|X_r^{1, n}| - |X_r^{2, n}| \leq \alpha \left( \frac{\beta}{2} r \right)^{a-1} |X_r^{1, n}| - |X_r^{2, n}|.$$  

Hence

$$D_t \leq E[(X_t^{1, n} - X_t^{2, n})^2] \leq C_n \int_0^t r^{2a-2} \, dr.$$  

Again, applying Gronwall’s lemma, with $D_0 = 0$, we have $D_t = 0$ for all $t \in [0, t_0, n \wedge t_{0, n}^i]$. Therefore, (3) has a unique strong solution in $[0, t_0, n \wedge t_{0, n}^i]$. As in the previous cases, we have $Y_{i, n}^t > \beta/2 > 0$ for all $t \in [t_0, n \wedge t_{0, n}^i]$, which makes $X_{i, n}^t$ strictly increasing. So $X_{0, n} \wedge t_{0, n}^i$ is strictly positive. Thus, by the strong Markov property, we have uniqueness until the next time $X$ hits zero.
Case III: $Y_0 > 0$, $Z_0 > 2^{-n}$. Now we let $T_n$ be the first time that either $Z_{1,n}^i$ or $Z_{2,n}^i$ hits the value $2^{-n}/2$. Since both $Z_{1,n}^i$ and $Z_{2,n}^i$ are continuous, we have $T_n > 0$ with probability 1. So we now prove uniqueness up to the time $t_{0,n} \wedge T_n$, where $t_{0,n}$ is defined in (5).

Then for all $t$ in $[0, t_{0,n} \wedge T_n]$, we have

$$Z_{1,n}^i \geq \frac{2^{-n}}{2},$$

therefore

$$Y_{1,n}^i \geq Y_0 + \int_0^t \frac{2^{-n}}{2} ds \geq \frac{2^{-n}}{2} t$$

since $Y_{1,n}^i \geq 0$.

Based on (11), for $t \in [0, t_{0,n} \wedge T_n]$

$$X_{1,n}^i \geq X_0 + \int_0^t \frac{2^{-n}}{2} s ds \geq \frac{2^{-n}}{4} t^2.$$  

Now we define

$$X_{1,n}^i = \tilde{X}_{1,n}^i, \quad Y_{1,n}^i = \tilde{Y}_{1,n}^i, \quad Z_{1,n}^i = \tilde{Z}_{1,n}^i$$

for $i = 1, 2$ and for $t \leq \tau_n \wedge T_n \wedge t_{0,n}$, where $t_{0,n}$ is defined as in (5) above.

Thus the following system of equations holds up to the stopping time $\tau_n \wedge T_n \wedge t_{0,n}$:

$$d\tilde{X}_{1,n}^i = \tilde{Y}_{1,n}^idt$$

$$d\tilde{Y}_{1,n}^i = \tilde{Z}_{1,n}^i dt$$

$$d\tilde{Z}_{1,n}^i = [\tilde{X}_{1,n}^i, \alpha] \cdot 1_{[0, \tau_n \wedge T_n \wedge t_{0,n}]}(t) dB_t,$$

with $(\tilde{X}_{1,n}^i, \tilde{Y}_{1,n}^i, \tilde{Z}_{1,n}^i) = (X_0, Y_0, Z_0)$ for $i = 1, 2$. Furthermore, using (13), $\tilde{X}_{1,n}^i$, $\tilde{Y}_{1,n}^i$, and $\tilde{Z}_{1,n}^i$ can be defined for all times.

Using Itô’s isometry as above with $\tilde{X}_{1,n}^i$, $\tilde{Y}_{1,n}^i$, and $\tilde{Z}_{1,n}^i$,
Using (12) and the mean value theorem, for \( r \in [0, \tau_n \land T_n \land t_{0,n}] \), we have
\[
\left| \tilde{X}_{t}^{1,n,0} - \tilde{X}_{t}^{2,n,0} \right| \leq \alpha \left( \frac{2^{-n}}{4} t \right)^{\alpha - 1} \left| \tilde{X}_{r}^{1,n} - \tilde{X}_{r}^{2,n} \right|.
\]

Hence
\[
E[(\tilde{X}_{t}^{1,n} - \tilde{X}_{t}^{2,n})^2] \leq t^4 \alpha^2 \left( \frac{2^{-n}}{4} \right)^{2(\alpha - 1)} E \int_{0}^{t} r^{4(\alpha - 1)} (|\tilde{X}_{r}^{1,n} - \tilde{X}_{r}^{2,n}|)^2 dr,
\]
so if we let
\[
D_t = E[(\tilde{X}_{t}^{1,n} - \tilde{X}_{t}^{2,n})^2],
\]
then
\[
D_t \leq E[(\tilde{X}_{t}^{1,n} - \tilde{X}_{t}^{2,n})^2] \leq C_n \int_{0}^{t} r^{4\alpha - 4} D_r dr.
\]

Again, applying Gronwall’s lemma, with \( D_0 = 0 \), we have \( D_t = 0 \). Note that at the time \( t_{0,n} \land T_n \), since we have \( Z_{t}^{1,n} > 0 \) for all \( t \in [0, t_{0,n} \land T_n] \), and also \( Y_0 > 0 \), it leads to \( Y_{t}^{1,n} > 0 \) for all \( t \in [0, t_{0,n} \land T_n] \). Thus \( X_{t}^{i,n} \) is strictly increasing, which means \( X_{t_{0,n} \land T_n} \) must be strictly greater than zero. Therefore, by the strong Markov property, we obtain uniqueness of the process until \( X \) next hits zero.

Case IV: \( Y_0 = 0 \). If \( Y_0 = 0 \), then based on the definition of \( \tau_n \), we have \( |Z_0| > 2^{-n} \).

We will first deal with the case \( Z_0 > 2^{-n} \), and the case \( Z_0 < 2^{-n} \) is approached the same way due to symmetry. As in Case III, let \( T_n \) be the first time that either \( Z_{t}^{1,n} \) or \( Z_{t}^{2,n} \) hits the value \( 2^{-n}/2 \). Due to the continuity of \( Z_{t}^{1,n} \) and \( Z_{t}^{2,n} \), we have \( T_n > 0 \) with probability 1. So with for all \( t \in [0, t_{0,n} \land T_n] \), where \( t_{0} \) is determined in (5), we have
\[
Y_{t}^{1,n} \geq Y_0 + \int_{0}^{t} \frac{2^{-n}}{2} ds = \frac{2^{-n}}{2} t.
\]

Then
\[
X_{t}^{1,n} \geq X_0 + \int_{0}^{t} \frac{2^{-n}}{2} ds \geq \frac{2^{-n}}{4} t^2.
\]

We now apply the same method as in Case III by looking at \( \tilde{X}_{t}^{i,n} \), \( \tilde{Y}_{t}^{i,n} \), and \( \tilde{Z}_{t}^{i,n} \), which are defined as
\[
X_{t}^{i,n} \equiv \tilde{X}_{t}^{i,n}, \quad Y_{t}^{i,n} \equiv \tilde{Y}_{t}^{i,n}, \quad Z_{t}^{i,n} \equiv \tilde{Z}_{t}^{i,n}
\]
for \( i = 1, 2 \) and for \( t \leq \tau_n \land T_n \land t_{0,n} \), as \( t_{0,n} \) defined as in (5) above.

Thus the following system of equations holds up to the stopping time \( \tau_n \land T_n \land t_{0,n} \):
\[
\begin{align*}
\frac{d}{dt} \tilde{X}_{t}^{1,n} &= \tilde{Y}_{t}^{1,n} dt \\
\frac{d}{dt} \tilde{Y}_{t}^{1,n} &= \tilde{Z}_{t}^{1,n} dt \\
\frac{d}{dt} \tilde{Z}_{t}^{1,n} &= |\tilde{X}_{t}^{1,n}|^{\alpha} 1_{[0, \tau_n \land T_n \land t_{0,n}]}(t) dB_t,
\end{align*}
\]

\[1^{1/2} \]
with \((\tilde{X}^{i,n}_0, \tilde{Y}^{i,n}_0, \tilde{Z}^{i,n}_0) = (X_0, Y_0, Z_0)\) for \(i = 1, 2\). Furthermore, using (13), \(\tilde{X}^{i,n}_t, \tilde{Y}^{i,n}_t, \tilde{Z}^{i,n}_t\) can be defined for all time.

Again, using the same strategy in Case III and the mean value theorem, we have

\[\left| |\tilde{X}^{1,n}_i|^{\alpha} - |\tilde{X}^{2,n}_i|^{\alpha} \right| \leq \alpha \left( \frac{2^{-n} \tau}{4} \right)^{\alpha - 1} |\tilde{X}^{1,n}_i| - |\tilde{X}^{2,n}_i|.
\]

Hence, if we let

\[D_t = E[(|\tilde{X}^{1,n}_i| - |\tilde{X}^{2,n}_i|)^2],\]

then

\[D_t \leq E[(\tilde{X}^{1,n}_i - \tilde{X}^{2,n}_i)^2] \leq C_n \int_0^t r^{4\alpha - 4} D_r \, dr.
\]

Gronwall’s lemma with \(D_0 = 0\) yields \(D_t = 0\), completing the proof of Theorem 1.

In this case, we also have \(Y^{i,n}_t > 2^{-n}/2t > 0\) for all \(t \in [0, t_0, n \wedge T}\). Hence \(X^{i,n}_t\) is strictly increasing, which yields \(X_{0,n \wedge T}\) strictly positive. So, by the strong Markov property, we have uniqueness up to the time \(X\) next hits zero.

Now with uniqueness proved, we actually can even strengthen the proof by showing that \(\tau^{1}_n = \tau^{2}_n\) for all \(n\), where \(\tau^{1}_n\) and \(\tau^{2}_n\) respectively stand for the stopping times at the critical values for \(X^{1}_t\) and \(X^{2}_t\). Without loss of generality, suppose \(\tau^{2}_n > \tau^{1}_n\). So at the time \(\tau^{1}_n\), \(X^{2}_t\) has not yet reached the critical values, which are \(2^{-n}\) or \(2^n\), as stated above. But since we have uniqueness up to \(\tau^{1}_n \wedge \tau^{2}_n\), this implies \(X^{1}_t\) has also not reached the critical value at the time \(\tau^{1}_n\), which is a contradiction to the definition of \(\tau^{1}_n\). Hence, \(\tau^{1}_n\) and \(\tau^{2}_n\) must be equal.

References


[6] Received: 2019-10-03 Revised: 2020-02-18 Accepted: 2020-05-23

carl.e.mueller@rochester.edu Department of Mathematics, University of Rochester, Rochester, NY, United States

gtruong@u.rochester.edu Department of Mathematics, University of Rochester, Rochester, NY, United States