Quantum Probability in a Two State Random Walk

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1 Introduction

Originally, quantum mechanics was formalized using matrices and linear algebra to assign probabilities to states. In modern research, however, Feynman's path integral formulation of quantum mechanics provides the foundation, like in Peskin and Schroder's fundamental text *An Introduction to Quantum Field Theory* [1] or Glimm and Jaffe's *Quantum Physics: A Functional Integral Point of View* [2]. However, the path integral, an example of which is in equation 1, is not generally convergent and there is not strong mathematical rigor around the use of them. Right now, much of the work requires analytic continuation to "imaginary time," and while this works; the overall goal is to create a more solid foundation.

$$\psi\left(x,t\right)=\frac{1}{Z}\int_{\vec{x}(0)=x}e^{iS\left(\vec{x},\dot{\vec{x}}\right)}\psi_0\left(\vec{x}\left(t\right)\right)\mathcal{D}\vec{x} \tag{1}$$

This paper does not attempt to solve this problem. While creating a measure over the space of all functions is still an unsolved problem in mathematics, we do not quite investigate this. One of the best attempts at such a measure was developed by Weiner [3]. Gudder and Sorkin in [4]use an adaptation of the Weiner measure.

Quantum mechanics adds even more trouble to this problem. The main issue is that for two disjoint events, A, B, the probability of the disjoint union is no longer what we expect it to be. The probability of rolling a 1 (A) or rolling a 2 (B) on a six sided die should follow equation 2 (for μ the probability of the events)

$$\mu\left(A\sqcup B\right)-\mu\left(A\right)-\mu\left(B\right)=0 \tag{2}$$

However, in Young's double-slit experiment, shown in Figure 1, it is shown experimentally that the probability of a photon or electron landing in one location on the screen (a) after passing through the two slits (b) and is not the sum of the probabilities of going through either slit (g). The quantum mechanic formulation of this problem ends up implying that the normal, classical rules of probability no longer apply.

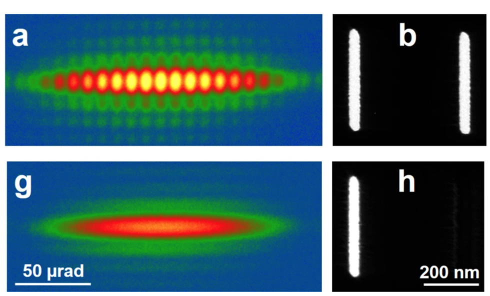

Figure 1: Young-Feynman Double Slit Experiment[5]

However, it turns out that, by both theoretical and experimental confirmation, that taking three disjoint events, like a triple slit experiment, satisfies the following property

$$\mu\left(A \sqcup B \sqcup C\right)-\mu\left(A \sqcup B\right)-\mu\left(B \sqcup C\right)-\mu\left(A \sqcup C\right)+\mu\left(A\right)+\mu\left(B\right)+\mu\left(C\right)=0 \tag{3}$$

A basic formulation of quantum mechanics, which we will not delve too much into here, is that the square of the wave function gives the probability distribution for the state. However, the effect of squaring the complex number comes into play here, no longer following the basic Kolmogorov theory of probability. We will show in this paper that for a space of finite sets, $\nu\left(A\right)=\left|A\right|^2$ always satisfies equation 3 and is an example of what we will call a quadratic measure. One interesting consequence of equation 3 is that sets of measure 0 are no longer necessarily small, like in a normal measure. There can be disjoint sets A, B, C that have the property

$$\mu\left(A \sqcup B\right)+\mu\left(B \sqcup C\right)+\mu\left(A \sqcup C\right)=\mu\left(A\right)+\mu\left(B\right)+\mu\left(C\right)\neq 0$$

This would mean that $\mu\left(A \cup B \cup C\right)=0$, even though the individual sets non-zero measures.

In this paper, we attempt to investigate the example introduced by Gudder and Sorkin in [4]. We discuss some possible formalizations of a measure theory that satisfies quantum mechanical axioms, in a similar way to the clear structure of classical probability that we know. Very few of these definitions are entirely formalized or accepted among those working on this problem, so we will see the rigor of the mathematics decrease as we get further into the subject.

We follow Gudder and Sorkin's paper very closely. In section 2, we will introduce the finite path example and prove some interesting results. Section 3 introduces basic measure theory definitions that we then try to adapt to quantum measures in section 4. Then section 5 tries to extend what we have worked on to infinite paths, in a similar method to the Weiner measure. A problem that comes up in this section is then addressed in section 6, which leads to some large results. Section 7 is a conclusion, introducing some further topics and possible extensions.

2 Formulation of Example

We have a system with only two possible states, 0 and 1, or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We will generally refer to the states as 0 and 1, but the matrices are helpful for the beginning. The system moves in discrete time steps, $t=0,1,2,\ldots$, and at each time step, the system stays in its current state or moves to the other with equal probability. We assume every walk we examine starts at the same location: the system starts in state 0 at $t=0$. The transition probabilities are given by

$$U=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

This is a unitary matrix, since the conjugate transpose $U^\dagger$ is its inverse

$$U^\dagger=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}, \qquad UU^\dagger=U^\dagger U=\boldsymbol{I}_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We use a unitary matrix to preserve probability– it "rotates" an orthonormal basis (our states 0, 1) to another orthonormal basis. This rotation is a superposition of each state which collapses after a measurement.

In quantum mechanics, the probability of measuring the system to be in some state ψ when we know it is currently in ϕ is given by

$$\left|\psi^\dagger\phi\right|^2$$

Then, taking the state $\chi = 0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as an example, after a single time step, it undergoes a transition

$$U\chi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

So then the probability it is still in state 0 after one transition is

$$P\left(0|0\right) = \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\dagger} U\chi \right|^2 = \frac{1}{2} \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} \right|^2 = \frac{1}{2}$$

And similarly for state 1 after transition:

$$P\left(1|0\right) = \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\dagger} U\chi \right|^2 = \frac{1}{2} \left| \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} \right|^2 = \frac{1}{2}$$

The probabilities when starting in state 1 are similar: $P\left(0|1\right) = \frac{1}{2} = P_1\left(1|1\right)$. So these are our two states and they have an equal probability of transition. This is the quantum analog of a transition matrix of a Markov process. Now we formalize for a random walk.

Definition 1. An ***n*-path** ω of the particle is a string of $n+1$ states, $0's$ or $1's$, of the form

$$\omega = \alpha_0 \alpha_1 \cdots \alpha_n, \qquad \alpha_i \in \{0, 1\}, \alpha_0 = 0$$

The sample space of n-paths is Ω_n. Then we can write

$$\Omega_n = \{\omega_0, \omega_1, \ldots, \omega_{2^n - 1}\}$$

Making ω_i correspond to i in binary notation,

$$0 = \omega_0 = 0 \cdots 0, 1 = \omega_1 = 0 \cdots 01 \ldots, 2^n - 1 = \omega_{2^n-1} = 011 \cdots 1$$

Then

$$\Omega_n = \{0, 1, \ldots, 2^n - 1\}$$

This allows us to directly correspond paths in Ω_n to a number, but the string in binary is what we are referencing.

The following definition will make some notation cleaner.

Definition 2. The number of **position changes** of an n-path $\omega \in \Omega_n$, $c_n\left(\omega\right)$, is the number of switches between 0 and 1 in the path.

We can see that for a path ω,

$$c_n\left(\omega\right) = \sum_{i=1}^{n} \left|\alpha_i - \alpha_{i-1}\right|$$

We can now formulate the main construction of our example.

Definition 3. The **joint amplitude** (or "Schwinger amplitude") between two n-paths $\omega, \omega' \in \Omega_n$ is

$$D^n\left(\omega, \omega'\right) = \frac{1}{2^n} i^{|\alpha_1 - \alpha_0|} \ldots i^{|\alpha_n - \alpha_{n-11}|} i^{-\left|\alpha'_1 - \alpha'_0\right|} \ldots i^{\left|\alpha'_n - \alpha'_{n-1}\right|} \delta_{\alpha_n \alpha'_n}$$

It follows from this equation and definition 2 that

$$D^n\left(\omega, \omega'\right) = \frac{1}{2^n} i^{\left[c_n(\omega) - c_n\left(\omega'\right)\right]} \delta_{\alpha_n \alpha'_n}$$

Definition 4. The ***n*-decoherence matrix** is the $2^n \times 2^n$ matrix with element i, j defined as

$$D^n_{i,j} = D^n(\omega_i, \omega_j) = D^n(i, j)$$

Where $\omega_i, \omega_j \in \Omega_n$ and correspond to i, j in binary.

Let's make sense of these definitions.

Example 1. Take $n = 2$, so $\Omega_2 = \{0, 1, 2, 3\} = \{000, 001, 010, 011\}$. Then

$$c_2(0) = 0, \qquad c_2(1) = 1, \qquad, c_2(2) = 2 \qquad, c_2(3) = 1$$

And

$$D^2(i, i) = \frac{1}{4}, i \in \{0, 1, 2, 3\}$$

Since α_n, α'_n must be equal for $D^n(\omega, \omega')$ to be non-zero, then we have the relation

$$D^2(i, j) = 0, i \not\equiv j \mod 2$$

In other words, i and j must have the same parity (both even or both odd). Thus, from definition 4, we get

$$D^2 = \frac{1}{4}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

From this pattern, we can generalize the following information about all decoherence matrices.

Proposition 1. *For D^n the n-decoherence matrix,*

$$D^n_{ij} = \begin{cases} \frac{1}{2^n} & c_n(i) \equiv c_n(j) \mod 4, i \equiv j \mod 2 \\ -\frac{1}{2^n} & c_n(i) \not\equiv c_n(j) \mod 4, i \equiv j \mod 2 \\ 0 & i \not\equiv j \mod 2 \end{cases}$$

Proof. We know in binary representation, an odd number will have the last digit 1 and an even number will have last digit 0. Thus, from definition 3, we see that $\delta_{\alpha_n \alpha'_n} = 0$ when the i, j are of different parity.

When they are of the same parity, $\delta_{\alpha_n \alpha'_n} = 1$, and that means

$$c_n(i) - c_n(j) \equiv 0, 2 \mod 4$$

But then, accordingly,

$$i^{[c_n(i) - c_n(j)]} = \pm 1$$

Which was to be shown. □

3 Measure Theory

To understand where we are going, in this section, we will briefly develop some measure theory concepts. Most of these formalizations are from [6].

Definition 5. A ***σ*-algebra** $\mathcal{S}$ of a set X is a collection of subsets of X such that the following three properties are satisfied

1. $\emptyset \in \mathcal{S}$
2. $E \in \mathcal{S} \implies X \backslash E \in \mathcal{S}$, where $X \backslash E$ is the complement of E with respect to X.

3. If $E_1, E_2, \ldots$ is an infinite sequence of elements of $\mathcal{S}$, then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$

This definition directly implies some useful properties.

Proposition 2. *If $\mathcal{S}$ is a σ-algebra on X, then*

1. $X \in \mathcal{S}$
2. $D, E \in \mathcal{S} \Longrightarrow D \cup E, D \cap E, D \backslash E \in \mathcal{S}$
3. *If $E_1, E_2, \ldots$ is an infinite sequence of elements of $\mathcal{S}$, then* $\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}$

Proof. From definition 5, 1 tells us $\emptyset \in \mathcal{S}$, and 2 means that $X \backslash \emptyset = X \in \mathcal{S}$.
Then if $D, E \in \mathcal{S}$:

- Take the sequence $D, E, \emptyset, \emptyset, \ldots$ and use 3, so

$$\bigcup_{k=1}^{\infty} E_k = D \cup E \in \mathcal{S}$$

- Noting that $X \backslash D, X \backslash E \in \mathcal{S}$ by 2, then the above point gives

$$(X \backslash D) \cup (X \backslash E) \in \mathcal{S}$$

  And then De Morgan's law and 2 tells us that

$$X \backslash (D \cap E) \in \mathcal{S} \Longrightarrow X \backslash (X \backslash (D \cap E)) = D \cap E \in \mathcal{S}$$

- Noting $D \cap (X \backslash E) = D \backslash E$, then the previous point gives $D \backslash E \in \mathcal{S}$

Then finally, for $E_1, E_2, \ldots$ is an infinite sequence of elements of $\mathcal{S}$, 2 gives us each of their complements in $\mathcal{S}$, and De Morgan's law again tells us

$$\bigcup_{k=1}^{\infty} (X \backslash E_k) = X \backslash \left(\bigcap_{k=1}^{\infty} E_k \right) \in \mathcal{S}$$

And similar to above,

$$X \backslash \left(X \backslash \left(\bigcap_{k=1}^{\infty} E_k \right) \right) = \bigcap_{k=1}^{\infty} E_k \in \mathcal{S}$$

□

If we have a collection of subsets $\mathcal{A} \subseteq 2^X$, then we can talk about the smallest σ-algebra containing this collection $\mathcal{A}$. The power set, the collection of all subsets of X, is a σ-algebra, so $\mathcal{A}$ is contained in at least one σ-algebra. Then every σ-algebra that contains $\mathcal{A}$ must contain the empty-set, the complement of every set $A \in \mathcal{A}$, and the union of any sequence of sets $A_i \subseteq \mathcal{A}$. But then the intersection of all σ-algebras that contain $\mathcal{A}$ satisfy the definition, so the intersection itself is a σ-algebra. This must be the smallest one, as any σ-algebra that contained $\mathcal{A}$ must be larger than the intersection of all of them.

Now with this definition and properties, we can define the normal measure.

Definition 6. For a set X and a σ-algebra $\mathcal{S}$, a **measure** is a function $\mu : \mathcal{S} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and for every disjoint sequence $E_1, E_2, \ldots$ of sets in $\mathcal{S}$,

$$\mu \left(\bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \mu(E_k)$$

A **measure space** is the triple $(X, \mathcal{S}, \mu)$

We notice right away that if we have any finite collection of disjoint sets $E_1, \ldots, E_n$ in $\mathcal{S}$, then by taking the sequence $E_1, \ldots, E_n, \emptyset, \emptyset, \ldots$, we get

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu\left(E_k\right)$$

Then we see that if a finite space $X = \{a_1, a_2, a_3, \ldots, a_m\}$ and some $\sigma-$algebra $\mathcal{S}$ on X, then if we have the measure μon each singleton, $\mu\left(a_1\right), \mu\left(a_2\right), \ldots \mu\left(a_m\right)$ (assuming they are in $\mathcal{S}$), then we have the measure of any set $A \in \mathcal{S}$ by taking the sum of the measures of the elements.

The finiteness of X is not necessary, either, though the singleton method is more common in finite spaces.

One more definition that will help us later.

Definition 7. An **algebra** is a collection of sets $\mathcal{A}$ of a set X such that

1. $\emptyset \in \mathcal{A}$
2. $E \in \mathcal{A} \Longrightarrow X \backslash E \in \mathcal{A}$
3. If $E, F \in \mathcal{A}$ then $E \cup F \in \mathcal{A}$

Notice that σ-algebras are algebras since the first two requirements are the same and proposition 2 gives us 3. The reverse is not true.

Now that we have a common foundation of measure theory, we can see what applies and doesn't apply in our example.

4 Quantum Measures

Recall the n-decoherence functional, defined in 4 and with added properties 1. This matrix gives us relations between any two paths $i, j \in \Omega_n$. But based on our work in section 3, we want to have a measure on sets of these paths. The easiest σ-algebra of Ω_n to use is the power set, 2^{Ω_n}, the set of all subsets.

Now we define the measure on this space as follows.

Definition 8. The **n-truncated q-measure** $\mu_n : 2^{\Omega_n} \to \mathbb{R}^+$ is

$$\mu_n\left(A\right) = \sum_{\omega_i, \omega_j \in A} D_{ij}^n$$

To make this somewhat easier, we define $D_n : 2^{\Omega_n} \times 2^{\Omega_n} \to \mathbb{R}^+$, the **decoherence functional**, by

$$D_n\left(A, B\right) = \sum_{\omega_i \in A} \sum_{\omega_j \in B} D_{ij}^n$$

In this sense

$$\mu_n\left(A\right) = D_n\left(A, A\right)$$

We define this as a q-measure, because, as we will see, it is not always a measure as defined in 6.

Proposition 3. *The decoherence functional satisfies the following properties:*

1. *Additivity* $D_n\left(A \sqcup B, C\right) = D_n\left(A, C\right) + D_n\left(B, C\right)$
2. *Hermiticity* $D_n\left(A, B\right) = \overline{D_n\left(B, A\right)}$

3. *Strong positivity: for any finite collection of sets* A_i $1 \leq i \leq n$*, the (hermitian) matrix* $N_{ij} = D_n(A_i, A_j)$ *is positive semi-definite.*

4. $|D_n(A,B)|^2 \leq D_n(A,A)\, D_n(B,B)$

Proof. 1 is satisfied since

$$D_n(A \sqcup B, C) = \sum_{\omega_i \in A \sqcup B} \sum_{\omega_j \in C} D^n_{ij} = \sum_{\omega_i \in A} \sum_{\omega_j \in C} D^n_{ij} + \sum_{\omega_i \in B} \sum_{\omega_j \in C} D^n_{ij}$$

And since definition 3 tells us

$$D^n(\omega_i, \omega_j) = \frac{1}{2^n} i^{[c_n(\omega_i) - c_n(\omega_j)]} \delta_{\alpha_n \alpha'_n} = \overline{\frac{1}{2^n} i^{[c_n(\omega_j) - c_n(\omega_i)]} \delta_{\alpha_n \alpha'_n}} = \overline{D^n(\omega_j, \omega_i)}$$

We have 2.

The proof of 3 requires linear algebra manipulations that are not related to the rest of this paper, so to not too get sidetracked, we will relegate the proof to Appendix A.

Once we have 3, however, we can prove 4, as we know the matrix $N_{ij} = D_n(A_i, A_j)$, $A_1 = A, A_2 = B$ is a positive semi-definite Hermitian matrix, so its eigenvalues will all be non-negative. We know the determinant of N is the product of its eigenvalues, so $\det N \geq 0$. But

$$\det N = D_n(A,A)\, D_n(B,B) - |D_n(A,B)|^2 \geq 0$$

So we have 4. □

Example 2. Looking at Ω_1, $2^{\Omega_1} = \{\emptyset, \{0\}, \{1\}, \Omega_1\}$, and using 3, we get the decoherence matrix to be

$$D^1 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$\mu_1(\emptyset) = 0 \qquad \mu_1(\{0\}) = \frac{1}{2} \qquad \mu_1(\{1\}) = \frac{1}{2} \qquad \mu_1(\Omega_1) = 1$$

So this is a true measure.

Looking back to example 1, however, we see that with the decoherence matrix

$$D^2 = \frac{1}{4} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Then the diagonal entries give us $\mu_2(\{i\}) = \frac{1}{4}$ for $i \in \{0,1,2,3\}$, but we notice that, for example,

$$\mu_2(\{0,2\}) = D^2_{00} + D^2_{02} + D^2_{20} + D^2_{22} = \frac{1}{4}(1 - 1 - 1 + 1) = 0$$

If μ_2 were a measure, then we would have

$$0 = \mu_2(\{0,2\}) = \mu(\{0\}) + \mu(\{2\}) = \frac{1}{2}$$

So μ_2 is not a normal measure. Another example is determining $\mu_2(\{1,2,3\})$

$$\mu_2(\{1,2,3\}) = \sum_{i=1}^{3} \sum_{j=1}^{3} D^2_{ij} = \frac{1}{4}(2 + 1 + 2) = \frac{5}{4}$$

While

$$\mu_2(\Omega_2) = \sum_{i=0}^{3} \sum_{j=0}^{3} D^2_{ij} = \frac{1}{4}(0 + 2 + 0 + 2) = 1$$

If μ_2 were a measure, then we would have

$$1 = \mu_2(\Omega_2) = \mu(\{0\}) + \mu(\{1,2,3\}) = \frac{3}{2}$$

There is *interference* here, something that we don't see in the classical world but in quantum mechanics is demonstrated by the famous double-slit experiment described in section 1.

Definition 9. The **interference** of two paths $i, j \in \Omega_n$ (for $i \neq j$) is

$$I^n_{ij} = \mu_n(\{i,j\}) - \mu_n(\{i\}) - \mu_n(\{j\})$$

If $I^n_{ij} = 0$ means i and j do **not interfere**, $\boldsymbol{inj}$. If the term is positive, they interfere **constructively** $\boldsymbol{icj}$, and if $I^n_{ij} < 0$, i and j interfere **destructively**, $\boldsymbol{idj}$.

The fact that this definition exists shows the already weird behavior of quantum mechanics and why μ_n may not always be a measure: since i, j are not equal, a true measure would have $I^n_{ij} = 0$ for all $i \neq j$. We see that in $n = 2$, $0d2$; they interfere destructively.

Then μ_2 is not a measure but it does satisfy some other properties and we will show that we named these q-measures correctly.

The next definition is written with a σ-algebra, but this will soon be limited to a special type of structure, quadratic algebras. For now, we can deal with the same collection of sets. We also only take the codomain to be $\mathbb{R}^+$. This is an example, as noted in the introduction, where the theory is less formalized, so adjusted definitions like the one below, or even non-equivalent definitions are more commonplace.

Definition 10. For X a set $\mathcal{S}$ a σ-algebra, a ***q*-measure** is a function $\mu : \mathcal{S} \to \mathbb{R}^+$ such that $\mu(\emptyset) = 0$, for disjoint sets $A, B, C \in \mathcal{S}$ such that $A \sqcup B, A \sqcup C, B \sqcup C \in \mathcal{S}$

$$\mu(A \sqcup B \sqcup C) = \mu(A \sqcup B) + \mu(B \sqcup C) + \mu(A \sqcup C) - \mu(A) - \mu(B) - \mu(C)$$

And is regular: when $A \cap B = \emptyset$: if $\mu_n(A) = 0$, then

$$\mu_n(A \cup B) = \mu_n(B)$$

and if $\mu_n(A \cup B) = 0$

$$\mu_n(A) = \mu_n(B)$$

The second property above is called grade-2 additivity and is the key difference between this and other measures. Notice the difference in the definitions of a measure and a q-measure. While measures generally have a co-domain of $[0, \infty]$, according to Gudder and Sorkin, here we only have q-measures with co-domain $[0, \infty)$. This has to do with wanting it to be a probability measure in addition to the definition containing subtraction, which leads us to want to ignore infinity. Further, we ignore the properties with an infinite number of sets, for now, and deal with only a finite number of sets.

In the future, I hope we devise a definition for a q-measure that is more similar to the definition of a measure as in definition 6.

Then we have the following major result.

Theorem 1. *The n-truncated q-measure μ_n is a q-measure, namely, it satisfies grade-2 additivity for every n, and is also regular.*

Proof. We showed in proposition 3 that the function D_n has four key properties. From additivity and Hermiticity,

$$\begin{aligned}
D_n(A, B \sqcup C) &= \overline{D_n(B \sqcup C, A)} \\
&= \overline{D_n(B, A) + D_n(C, A)} \\
&= \overline{D_n(B, A)} + \overline{D_n(C, A)} \\
&= D_n(A, B) + D_n(A, C)
\end{aligned}$$

So we have additivity in both inputs. Then we can prove grade-2 additivity of the measure:

$$
\begin{aligned}
\mu_n(A \sqcup B \sqcup C) &= D_n(A \sqcup B \sqcup C, A \sqcup B \sqcup C) \\
&= D_n(A \sqcup B, A \sqcup B \sqcup C) + D_n(C, A \sqcup B \sqcup C) \\
&= D_n(A \sqcup B, A \sqcup B) + D_n(A \sqcup B, C) + D_n(C, A \sqcup C) + D_n(C, B) \\
&= \mu_n(A \sqcup B) + \underline{D_n(A, C)} + \underbrace{D_n(B, C)} + \underleftarrow{D_n(C, C)} + \underleftarrow{D_n(C, A)} + \underleftrightarrow{D_n(C, B)} \\
&+ \underline{D_n(A, A)} + \underbrace{D_n(B, B)} + \underleftrightarrow{D_n(C, C)} - \mu(A) - \mu(B) - \mu(C) \\
&= \mu_n(A \sqcup B) + \underline{D_n(A, A \sqcup C)} + \underleftarrow{D_n(C, A \sqcup C)} \\
&+ \underbrace{D_n(B, B \sqcup C)} + \underleftrightarrow{D_n(C, B \sqcup C)} - \mu_n(A) - \mu_n(B) - \mu_n(C) \\
&= \mu_n(A \sqcup B) + D_n(A \sqcup C, A \sqcup C) + D_n(B \sqcup C, B \sqcup C) \\
&- \mu_n(A) - \mu_n(B) - \mu_n(C) \\
&= \mu_n(A \sqcup B) + \mu_n(A \sqcup C) + \mu_n(B \sqcup C) - \mu_n(A) - \mu_n(B) - \mu_n(C)
\end{aligned}
$$

Then using the fourth property of 3, we have

$$
(\mathrm{Re} D_n(A, B))^2 \leq (\mathrm{Re} D_n(A, B))^2 + (\mathrm{Im} D_n(A, B))^2 = |D_n(A, B)|^2 \leq D_n(A, A) D_n(B, B)
$$

$$
\mathrm{Re} D_n(A, B) \leq \sqrt{\mu_n(A) \mu_n(B)}
$$

This allows us to prove regularity: when $A \cap B = \emptyset$ and $\mu_n(A) = 0$, then

$$
\mu_n(A \cup B) = D_n(A \cup B, A \cup B) = D_n(A, A) + D_n(A, B) + D_n(B, A) + D_n(B, B)
$$

$$
= \mu_n(A) + D_n(A, B) + \overline{D_n(A, B)} + \mu_n(B) = \mu_n(A) + 2\mathrm{Re}(D_n(A, B)) + \mu_n(B)
$$

And since $\mu_n(A) = 0$, the first two terms are 0, and we are left with $\mu_n(B)$.

The second condition of regularity, that when $A \cap B = \emptyset$ and $\mu_n(A \cup B) = 0$, then $\mu_n(A) = \mu_n(B)$, involves some linear algebra that does not have much relevance to the rest of the work in this paper, so we also relegate it to Appendix A. □

Now that we have grade-2 additivity, we can decompose the measure of any set $A \in 2^{\Omega_n}$ into a combination of the measures of the singletons and pairs, as shown here.

Corollary 1. *For $3 \leq m \leq n$ and for any set $\{i_1, \ldots, i_m\} \in 2^{\Omega_n}$*

$$
\mu_n(\{i_1, \ldots, i_m\}) = \sum_{j<k=1}^{m} \mu_n(\{i_j, i_k\}) - (m-2) \sum_{l=1}^{m} \mu_n(i_l)
$$

Proof. By induction on m. Take $n \geq 3$. Then for the base case $m = 3$, we know from theorem 1 that by taking A, B, C to be our three singletons, we have

$$
\mu_n(\{i_1, i_2, i_3\}) = \mu(\{i_1, i_2\}) + \mu(\{i_1, i_3\}) + \mu(\{i_2, i_3\}) - \mu(i_1) - \mu(i_2) - \mu(i_3)
$$

$$
= \sum_{j<k=1}^{3} \mu_n(\{i_j, i_k\}) - \sum_{l=1}^{3} \mu_n(i_l)
$$

Assuming this formula is correct for all x, $3 \leq x \leq m-1 < n$, we will prove for $x = m \leq n$

$$
\mu_n(\{i_1, \ldots, i_m\}) = \mu_n(\{i_1, \ldots, i_{m-2}\} \sqcup \{i_{m-1}\} \sqcup \{i_m\})
$$

We note μ_n is defined on 2^{Ω_n}, so all of these sets have a defined measure. This is divided using grade-2 additivity for up to $m-1$ elements, our inductive hypothesis, which then gives

$$
\begin{aligned}
\mu_n(\{i_1, \ldots, i_m\}) = \mu_n(\{i_1, \ldots, i_{m-1}\}) + \mu_n(\{i_1, \ldots, i_{m-2}, i_m\}) + \mu_n(\{i_{m-1}, i_m\}) \\
- \mu_n(\{i_1, \ldots, i_{m-2}\}) - \mu_n(\{i_{m-1}\}) - \mu_n(\{i_m\})
\end{aligned}
$$

Looking at the sets with more than 3 elements,

$$\mu_n\left(\{i_1,\ldots,i_{m-1}\}\right) = \underbrace{\sum_{j<k=1}^{m-1}\mu_n\left(\{i_j,i_k\}\right)} - (m-3)\left(\sum_{l=1}^{m-2}\mu_n\left(i_l\right) + \underleftarrow{\mu_n\left(i_{m-1}\right)}\right)$$

$$\begin{aligned}\mu_n\left(\{i_1,\ldots,i_{m-2},i_m\}\right) = &\sum_{j<k=1}^{m}\mu_n\left(\{i_j,i_k\}\right) - \underbrace{\sum_{o=1}^{m-1}\mu_n\left(\{i_o,i_{m-1}\}\right)}\\ &-\mu_n\left(\{i_{m-1},i_m\}\right) - (m-3)\left(\sum_{l=1}^{m}\mu_n\left(i_l\right) - \underleftarrow{\mu_n\left(i_{m-1}\right)}\right)\end{aligned}$$

$$-\mu_n\left(\{i_1,\ldots,i_{m-2}\}\right) = -\underbrace{\sum_{j<k=1}^{m-2}\mu_n\left(\{i_j,i_k\}\right)} + (m-4)\sum_{l=1}^{m-2}\mu_n\left(i_l\right)$$

Then we can plug these in, cancel the terms under-braced, and go whole hog on the remaining terms:

$$\begin{aligned}&= -\underline{(m-3)\left(\sum_{l=1}^{m-2}\mu_n\left(i_l\right)\right)} + \sum_{j<k=1}^{m}\mu_n\left(\{i_j,i_k\}\right) - \cancel{\mu_n\left(\{i_{m-1},i_m\}\right)} - (m-3)\left(\sum_{l=1}^{m}\mu_n\left(i_l\right)\right)\\ &\quad + \underline{(m-4)\sum_{l=1}^{m-2}}\mu_n\left(i_l\right) + \cancel{\mu\left(\{i_{m-1},i_m\}\right)} - \mu\left(\{i_{m-1}\}\right) - \mu\left(\{i_m\}\right)\\ &= (m-4-m+3)\underbrace{\sum_{l=1}^{m-2}\mu_n\left(i_l\right)} + \sum_{j<k=1}^{m}\mu_n\left(\{i_j,i_k\}\right) - (m-3)\left(\sum_{l=1}^{m}\mu_n\left(i_l\right)\right)\\ &\quad \underbrace{-\mu\left(\{i_{m-1}\}\right) - \mu\left(\{i_m\}\right)}\\ &= -\underline{\sum_{l=1}^{m}\mu_n\left(i_l\right)} + \sum_{j<k=1}^{m}\mu_n\left(\{i_j,i_k\}\right) - (m-3)\underline{\left(\sum_{l=1}^{m}\mu_n\left(i_l\right)\right)}\\ &= \sum_{j<k=1}^{m}\mu_n\left(\{i_j,i_k\}\right) - (m-2)\sum_{l=1}^{m}\mu_n\left(i_l\right)\end{aligned}$$

Which was to be shown. $\square$

This is now the quantum analog for determining the measure of finite sets. In normal measure theory, for any finite set A, if we have the measures of the elements $a_1,\ldots,a_n \in A$, then $\mu(A) = \sum_i \mu(a_i)$ as discussed in section 3. For a q-measure, this property tells us we need the measure of the singletons as well as the pairs of elements. A similar process can be done with countably infinite sets, but we mention the finite ones here to ignore dealing with limits.

5 Cylinder Sets

Now we try to extend our measures of finite paths into infinite paths. We do this by adding one extra digit to the end of a path and once we have the properties of such extensions, we can start to see how limits work.

For any path $\omega \in \Omega_n$, we can identify two paths $(\omega,0)$ and $(\omega,1) \in \Omega_{n+1}$, adding 0 or 1 to the right end of the path. For example, $0101 \in \Omega_4$ can be identified with 01010 and $01011 \in \Omega_5$.

We can further identify a path $\omega \in \Omega_n$ with both of its extensions as defined in the above paragraph, so

$$\omega \times \{0,1\} = \{(\omega,0),(\omega,1)\} \in 2^{\Omega_{n+1}}$$

And continuing, we can extend any set of paths $A \in 2^{\Omega_n}$ to all of their extensions

$$A \times \{0,1\} = \bigcup_{\omega \in A} \{(\omega, 0), (\omega, 1)\} \in 2^{\Omega_{n+1}}$$

We have a nice result that follows, though it takes a little work.

Proposition 4. *If $A \in 2^{\Omega_n}$, then*

$$\mu_{n+1}(A \times \{0,1\}) = \mu_n(A)$$

Proof. We start with the definition

$$\begin{aligned}
\mu_{n+1}(A \times \{0,1\}) =& \sum \left\{D^{n+1}(\omega, \omega') : \omega, \omega' \in A \times \{0,1\}\right\} \\
=& \sum \left\{D^{n+1}(\omega 0, \omega' 0) : \omega, \omega' \in A\right\} \qquad \text{due to the } \delta_{ij} \text{ term} \\
&+ \sum \left\{D^{n+1}(\omega 1, \omega' 1) : \omega, \omega' \in A\right\} \\
=& \frac{1}{2^{n+1}} \left[\sum \left\{i^{\left[c_n(\omega 0) - c_n\left(\omega' 0\right)\right]} : \omega, \omega' \in A\right\}\right. \\
&\left. + \sum \left\{i^{\left[c_n(\omega 1) - c_n\left(\omega' 1\right)\right]} : \omega, \omega' \in A\right\}\right]
\end{aligned} \tag{4}$$

Then we look at the number of switches in the extended paths $(\omega 0, \ldots)$ as opposed to ω and ω' to determine the sums. The number of switches increases only if the term on to the end of ω is different from α_n, and we compare the changes in switches for both paths.

$$i^{\left[c_n(\omega 0) - c_n\left(\omega' 0\right)\right]} = \begin{cases} i^{\left[c_n(\omega) - c_n\left(\omega'\right)\right]} & \alpha_n = \alpha_n' \\ i^{\left[c_n(\omega) - c_n\left(\omega'\right)\right] + 1} & \alpha_n = 1, \alpha_n' = 0 \\ i^{\left[c_n(\omega) - c_n\left(\omega'\right)\right] - 1} & \alpha_n = 0, \alpha_n' = 1 \end{cases}$$

And the opposite for $\omega 1, \omega' 1$

$$i^{\left[c_n(\omega 1) - c_n\left(\omega' 1\right)\right]} = \begin{cases} i^{\left[c_n(\omega) - c_n\left(\omega'\right)\right]} & \alpha_n = \alpha_n' \\ i^{\left[c_n(\omega) - c_n\left(\omega'\right)\right] - 1} & \alpha_n = 1, \alpha_n' = 0 \\ i^{\left[c_n(\omega) - c_n\left(\omega'\right)\right] + 1} & \alpha_n = 0, \alpha_n' = 1 \end{cases}$$

Plugging into the first sum of 4, we get

$$\begin{aligned}
& \sum \left\{i^{\left[c_n(\omega 0) - c_n\left(\omega' 0\right)\right]} : \omega, \omega' \in A\right\} \\
=& \sum \left\{i^{\left[c_n(\omega) - c_n\left(\omega'\right)\right]} : \omega, \omega' \in A, \alpha_n = \alpha_n'\right\} \\
&+ i \sum \left\{i^{\left[c_n(\omega) - c_n\left(\omega'\right)\right]} : \omega, \omega' \in A, \alpha_n = 1, \alpha_n' = 0\right\} \\
&- i \sum \left\{i^{\left[c_n(\omega) - c_n\left(\omega'\right)\right]} : \omega, \omega' \in A, \alpha_n = 0, \alpha_n' = 1\right\}
\end{aligned}$$

And we see the second sum gives terms that cancel out exactly with the second two above. Thus, we are only left with 2 of the first term, giving us

$$\begin{aligned}
\mu_{n+1}(A \times \{0,1\}) &= \frac{1}{2^n} \sum \left\{i^{\left[c_n(\omega) - c_n\left(\omega'\right)\right]} : \omega, \omega' \in A, \alpha_n = \alpha_n'\right\} \\
&= \sum \left\{D^n(\omega, \omega') : \omega, \omega' \in A\right\} = \mu_n(A)
\end{aligned}$$

Which was to be shown. □

A corollary based on induction follows.

Corollary 2. *If $A \in 2^{\Omega_n}$, then $\mu_{n+m}\left(A \times \{0,1\}^m\right) = \mu_n\left(A\right)$, where $\{0,1\}^m = \{0,1\} \times \cdots \times \{0,1\}$ for m factors.*

Now that we have a way to extend paths, infinite paths could be defined in this way. Since we have already defined n-paths, a general path is just an infinite string, however, we no longer have the correspondence to binary, as we did earlier. Let Ω be the space of all infinite paths.

Using cylinder set notation, we have that

$$\Omega = \{0\} \times \{0,1\} \times \{0,1\} \times \cdots$$

Remembering all of our paths must start in state 0. This is equivalent to

$$\Omega = \Omega_n \times \{0,1\} \times \cdots$$

For any n. We can take subsets of Ω, and those sets of the above form are what we study here.

Definition 11. A **cylinder set** is a subset $A \in 2^{\Omega}$ if for some n there exists a $B \in 2^{\Omega_n}$ such that

$$A = B \times \{0,1\} \times \{0,1\} \times \cdots$$

The **elementary cylinder set** of an n-path $\omega \in \Omega_n$ is

$$\text{cyl}\left(\omega\right) = \{\omega\} \times \{0,1\} \times \cdots$$

For an n-path $\omega \in \Omega_n$, an **extension** of ω is an m-path $\omega' \in \Omega_m$, $m \geq n$ such that

$$\omega' = \omega\alpha_{n+1}\cdots\alpha_m$$

It is clear that the elementary cylinder sets of two paths are contained if and only if one is an extension of the other:

$$\omega' = \omega\alpha_{n+1}\cdots\alpha_m \Longleftrightarrow \omega' \times \{0,1\} \times \cdots \subseteq \{\omega\} \times \{0,1\} \times \cdots \Longleftrightarrow \text{cyl}\left(\omega'\right) \subseteq \text{cyl}\left(\omega\right)$$

Additionally, the intersection of two elementary cylinder sets is empty if and only if the paths are not extensions of each other, $\text{cyl}\left(\omega\right) \cap \text{cyl}\left(\omega'\right) = \emptyset$.

Combining the above facts gives

Proposition 5. *Any two elementary cylinder sets are either disjoint or nested.*

We will not use elementary cylinder sets much beyond this fact.

These cylinder sets are a pretty good way to extend our finite paths, so let's denote $\mathcal{C}\left(\Omega\right) = \mathcal{C}$ the collection of all cylinder sets. We know some sets that are in Ω are not in the collection of cylinder sets, most apparently for any path $\omega \in \Omega$, $\{\omega\} \notin \mathcal{C}$.

However, for any cylinder set $A \in \mathcal{C}$, we have already shown in Corollary 2 that for the $B \in 2^{\Omega_n}$ such that

$$A = B \times \{0,1\} \times \{0,1\} \times \cdots$$

$$\mu_n\left(B\right) = \mu_{n+m}\left(B \times \{0,1\} \times \cdots \times \{0,1\}\right), \forall m$$

Then it seems natural to define a q-measure on these sets.

Definition 12. Let $\mu : \mathcal{C} \to \mathbb{R}^+$ such that when $A = B \times \{0,1\} \times \{0,1\} \times \cdots$ for $B \in 2^{\Omega_n}$, $\mu\left(A\right) = \mu_n\left(B\right)$. This is a q-measure.

To make sure this is well defined, take the above example of $A \in \mathcal{C}, B \in 2^{\Omega_n}$, and suppose $\exists B_1 \in 2^{\Omega_m}$ such that

$$A = B_1 \times \{0,1\} \times \{0,1\} \times \cdots$$

If $m = n$, then $B_1 = B$, so we are finished. Otherwise, assume $m < n$, or otherwise, relabel. But then

$$B_1 \times \{0,1\} \times \{0,1\} \times \cdots = B \times \{0,1\} \times \{0,1\} \times \cdots$$

$$\Longrightarrow B = B_1 \times \{0,1\}^{n-m}$$

And Corollary 2 gives that $\mu(A)$ is well defined.

Now to show μ is a q-measure on $\mathcal{C}$, take three disjoint sets $A, B, C \in \mathcal{C}$. These correspond to $A_1 \in 2^{\Omega_m}, B_1 \in 2^{\Omega_n}, C_1 \in 2^{\Omega_o}$, for example,

$$A = A_1 \times \{0,1\} \times \cdots$$

Taking $M = \max\{m, n, o\}$, we can represent the union of all three, labeling $A_M = A_1 \times \{0,1\}^{M-m} \in \Omega_M$ and similarly for B, C:

$$\left(A_1 \times \{0,1\}^{M-m}\right) \sqcup \left(B_1 \times \{0,1\}^{M-n}\right) \sqcup \left(C_1 \times \{0,1\}^{M-o}\right)$$

$$= A_M \sqcup B_M \sqcup C_M =: X \in 2^{\Omega_M}$$

Then we have, by definition of μ, since $A \sqcup B \sqcup C = X \times \{0,1\} \times \cdots$

$$\mu(A \sqcup B \sqcup C) = \mu_M(X)$$

But we know μ_M satisfies grade-2 additivity, so

$$\mu_M(X) = \mu_M(A_M \sqcup B_M) + \mu_M(B_M \sqcup C_M) + \mu_M(A_M \sqcup C_M) - \mu_M(A_M) - \mu_M(B_M) - \mu_M(C_M)$$

But then each of the above works as expected, that is,

$$\mu(A \sqcup B) = \mu_M(A_M \sqcup B_M)$$

And similarly for the rest of the unions, so we have grade-2 additivity

$$\mu(A \sqcup B \sqcup C) = \mu(A \sqcup B) + \mu(B \sqcup C) + \mu(A \sqcup C) - \mu(A) - \mu(B) - \mu(C)$$

Now, we need to verify regularity. If $\mu(A) = 0$, then for any B such that $A \cap B = \emptyset$, taking M as the same convention as above,

$$\mu(A \cup B) = \mu_M(A_M \cup B_M) = \mu_M(B_M) = \mu(B)$$

Where the first and third equalities follow from the definition of μ and the second from the regularity of μ_M. Similarly, if $\mu(A \cup B) = 0$, then

$$\mu_M(A_M \cup B_M) = 0 \Longrightarrow \mu_M(A_M) = \mu_M(B_M) \Longrightarrow \mu(A) = \mu(B)$$

So we have that μ as defined above is a q-measure on the $\mathcal{C}$. Now, we want to verify that μ is a continuous q-measure on $\mathcal{C}$.

Proposition 6. *As defined above on $\mathcal{C}$, μ is a continuous q-measure, namely, for a decreasing sequence $A_1 \supseteq A_2 \supseteq \cdots$ of cylinder sets with $\bigcap A_i \in \mathcal{C}$ and an increasing sequence $B_1 \subseteq B_2 \subseteq \cdots$ with $\bigcup B_i \in \mathcal{C}$, then*

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcap_{n\geq 1} A_n\right) \qquad \lim_{n\to\infty} \mu(B_n) = \mu\left(\bigcup_{n\geq 1} B_n\right)$$

Proof. We will show in the proof of theorem 2 that Ω is compact and every cylinder set in $\mathcal{C}$ is also compact. Then each A_i, B_i is compact, as are the union and intersection, since they are also cylinder sets. We also know for two cylinder sets X, Y, $X \setminus Y \in \mathcal{C}$, so the sets

$$A_i \setminus \bigcap_{n \geq 1} A_n \in \mathcal{C}$$

And thus also compact. Then

$$A_i \setminus \bigcap_{n \geq 1} A_n \supseteq A_{i+1} \setminus \bigcap_{n \geq 1} A_n \qquad \forall i$$

As they will follow the same inequalities as A_i. But since

$$\bigcap_{i \geq 1} \left(A_i \setminus \bigcap_{n \geq 1} A_n \right) = \left(\bigcap_{i \geq 1} A_i \right) \setminus \bigcap_{n \geq 1} A_n = \emptyset$$

Then by Cantor's theorem, there must be some $j \geq 1$ such that

$$A_m \setminus \bigcap_{n \geq 1} A_n = \emptyset \qquad \forall m \geq j$$

But since the intersection must be contained in each A_m, we then have

$$A_m = \bigcap_{n \geq 1} A_n \qquad \forall m \geq j$$

Which gives

$$\lim_{n \to \infty} \mu \left(A_n \right) = \mu \left(\bigcap_{n \geq 1} A_n \right)$$

Taking the complements of this work will give the second half of this result, so we find μ is continuous. $\square$

However, $\mathcal{C}$ is not a σ-algebra (due to the requirement of infinite unions), and we want to create a q-measure space with a continuous measure. We discussed the smallest σ-algebra containing a set in section 3, and let $\mathcal{S}$ be the smallest σ-algebra containing the cylinder sets. Unfortunately, we can show that μ cannot be extended to a continuous q-measure on $\mathcal{S}$. An extension would be a q-measure $\hat{\mu} : \mathcal{S} \to \mathbb{R}^+$ such that $\hat{\mu}|_{\mathcal{C}} = \mu$.

Proposition 7. *If μhad an extension to $\mathcal{S}$, it would not be continuous.*

Proof. Take the set $B_1 = \{0010, 0100, 0110\} = \{2, 4, 6\} \in \Omega_3$. Then

$$\mu_3 \left(B_1 \right) = \mu_3 \left(\{2, 4\} \right) + \mu_3 \left(\{4, 6\} \right) + \mu_3 \left(\{2, 6\} \right) - \sum_{i=2,4,6} \mu_3 \left(\{i\} \right)$$

From corollary 1, and using proposition 1, since we have $c_3 \left(2 \right) = c_3 \left(4 \right) = c_3 \left(6 \right) = 2$

$$\mu_3 \left(\{2, 4\} \right) = \mu_3 \left(\{4, 6\} \right) = \mu_3 \left(\{2, 6\} \right) = \frac{4}{8}$$

So

$$\mu_3 \left(B_1 \right) = \frac{9}{8}$$

Then define $B_2 = \{010, 100, 110\}$ so that $B_1 \times B_2 \in \Omega_6$. Then noting the number of switches will always be equal and they each end on a 0, the measure of each pairwise union (36 of them) will be $\frac{4}{2^6}$, and each singleton (9 of them) $\frac{1}{2^6}$, so

$$\mu_6 \left(B_1 \times B_2 \right) = 36 \left(\frac{4}{2^6} \right) - 7 \left(\frac{9}{2^6} \right) = \frac{9 \left(4 \times 4 - 7 \right)}{8^2} = \left(\frac{9}{8} \right)^2$$

Similarly,

$$\mu_9\left(B_1 \times B_2 \times B_2\right) = 351\left(\frac{4}{2^9}\right) - 25\left(\frac{27}{2^9}\right) = \frac{9\left(39 \times 4 - 25 \times 3\right)}{8^3} = \left(\frac{9}{8}\right)^3$$

And then

$$\mu_{3n}\left(B_1 \times \underbrace{B_2 \times \cdots \times B_2}_{n-1 \text{ times}}\right) = \binom{3^n}{2}\left(\frac{4}{2^{3n}}\right) - \left(3^n - 2\right)\left(\frac{3^n}{2^{3n}}\right) = \left(\frac{9}{8}\right)^n$$

Then define

$$\begin{aligned} A_1 &= B_1 \times \{0,1\} \times \{0,1\} \times \cdots \\ A_2 &= B_1 \times B_2 \times \{0,1\} \times \{0,1\} \times \cdots \\ A_3 &= B_1 \times B_2 \times B_2 \times \{0,1\} \times \{0,1\} \times \cdots \end{aligned}$$

And A_i similarly. Then each of these is a cylinder set, so from definition 2, we know $\bigcap_{i \geq 1} A_i \in \mathcal{S}$, and $A_1 \supseteq A_2 \supseteq \cdots$. But then from definition 12, we have

$$\lim_{i \to \infty} \mu\left(A_i\right) = \lim_{i \to \infty} \mu_{3i}\left(A_i\right) = \lim_{i \to \infty} \left(\frac{9}{8}\right)^i = \infty$$

Now remembering that q-measures have a co-domain of $\mathbb{R}^+$, then this means

$$\lim_{i \to \infty} \mu\left(A_i\right) = \infty = \mu\left(\bigcap_{i \geq 1} A_i\right)$$

Would not be allowed. Since these sets are in $\mathcal{S}$, then we know μ does not extend to a continuous measure on $\mathcal{S}$. $\square$

This is a disappointing result, but there are plenty of sets in $\mathcal{A} \backslash \mathcal{C}$ that μ can be extended to. We define some of them next.

First, take any path $\omega = \alpha_0 \alpha_1 \cdots \in \Omega$, and any set of paths $A \in 2^{\Omega}$. If for some $\omega' = \beta_0 \beta_1 \cdots \in A$ and for some $n \in \mathbb{N}$,

$$\alpha_i = \beta_i, 0 \leq i \leq n$$

Then we write $\omega\left(n\right) A$, and interpret as a path ω' contained in A such that ω matches with ω' for the first n terms. Then the set of paths that have this correspondence with the set A to the n^{th} degree will be labeled

$$A^{(n)} = \left\{\omega \in \Omega : \omega\left(n\right) A\right\}$$

Notice that each $A^{(n)}$ is a cylinder set, the set B such that $A^{(n)} = B \times \{0,1\} \times \cdots$ is the set of paths $\omega \in \Omega_n$ such that $\omega\left(n\right) A$.

Proposition 8. *For $A^{(n)}$ defined as above, $A^{(i+1)} \subseteq A^{(i)}$ for every $i \in \mathbb{N}$. Further, $A \subseteq \bigcap_{n \geq 0} A^{(n)}$.*

Proof. If $\omega = \alpha_0 \alpha_1 \cdots \in A^{(i+1)}$, then the definition tells us there exists some $\omega' = \beta_0 \beta_1 \cdots \in A$ such that the first $i+1$ terms match. But then the first i terms must match, so $\omega \in A^{(i)}$.

Then, for $\omega \in A$, it trivially matches with itself for all n terms, so $\omega \in A^{(n)} \forall n$, so

$$\omega \in \bigcap_{n \geq 0} A^{(n)} \implies A \subseteq \bigcap_{n \geq 0} A^{(n)}$$

$\square$

Definition 13. A **lower set** is a collection $A \in 2^{\Omega}$ such that $A = \bigcap_{n \geq 0} A^{(n)}$. The collection of lower sets is denoted $\mathcal{L}$.

A collection A is a **beneficial set** if $\hat{\mu}(A) = \lim_{n\to\infty} \mu\left(A^{(n)}\right)$ exists and is finite. Define $\hat{\mu}$ by this limit. The collection of beneficial sets is denoted $\mathcal{B}$.

The intersection of the two types of collections is denoted

$$\mathcal{B}_{\mathcal{L}} = \mathcal{B} \cap \mathcal{L}$$

The next proposition explains how these three collections (of collections) of sets interact with our collection of cylinder sets $\mathcal{C}$.

Proposition 9. *As the sets are defined above, $\mathcal{C} \subseteq \mathcal{B}_{\mathcal{L}}$*

Proof. Take a cylinder set $A \in \mathcal{C}$, then there is some $B \in \Omega_n$ such that $A = B \times \{0,1\} \times \cdots$. But then

$$A = A^{(n)} = A^{(n+1)} = \cdots$$

As the elements in a cylinder set must correspond to some $\omega' \in B$, once we have the first n elements corresponding, then any $\omega \in A^{(n+i)}, i \geq 0$ is also an element of $B \times \{0,1\} \times \cdots$, since ω starts as ω', and then adds on other terms. So $\omega \in A$, and we have all of them equal.

We combine this with the result of Proposition 8 gives that $A = \bigcap_{n\geq 0} A^{(n)}$, so then $A \in \mathcal{L}$, and thus $\mathcal{C} \subseteq \mathcal{L}$. But every set $A \in C$ corresponds to its $B \in \Omega_n$ which has a finite measure, then so does A, so $C \subseteq \mathcal{B}$. Thus, $C \subseteq \mathcal{B}_{\mathcal{L}}$, and for cylinder sets A, $\hat{\mu}(A) = \mu(A)$. $\square$

We see that for any path $\omega \in \Omega$, then the set $\{\omega\} = \bigcap_{n\geq 0} \{\omega\}^{(n)}$, since ω is the only path that will match itself for every n. Then if we label $\omega^N = \omega_0\omega_1 \cdots \omega_N \in \Omega_N$ the first N terms of ω, then

$$\mu\left(\{\omega\}^{(n)}\right) = \mu_N\left(\omega^N\right) = \frac{1}{2^n}$$

So

$$\lim_{n\to\infty} \mu\left(\{\omega\}^{(n)}\right) = 0$$

This gives $\{\omega\} \in \mathcal{B}_{\mathcal{L}}$, and thus, since $\{\omega\} \notin \mathcal{C}$, $\mathcal{C} \subsetneq \mathcal{B}_{\mathcal{L}}$.

We can generalize this to any finite set $A \subseteq \Omega$, $|A| = m$: if we represent $A^{(n)} = B_n \times \{0,1\} \times \{0,1\} \times \cdots$, then $|B_n| \leq m$, and

$$\mu\left(A^{(n)}\right) = \mu_N(B_n) = \sum\left\{D^n(\omega,\omega') : \omega,\omega' \in B_n\right\} \leq \frac{m^2}{2^n}$$

Where we use the properties of D^n from proposition 1, so $A \in \mathcal{B}_{\mathcal{L}}$. There are more beneficial lower sets than there are cylinder sets.

6 Quadratic Algebras

We have seen that μ cannot be extended to a continuous measure on the smallest σ-algebra that contains the cylinder sets. However, there is a more general type of algebra, based on grade-2 additivity, that will lead to better results.

Definition 14. A **quadratic algebra** is a collection of sets $\mathcal{Q}$ of a set X if $\emptyset, X \in \mathcal{Q}$ and for any three mutually disjoint sets $A, B, C \in \mathcal{Q}$ such that the pairwise unions are also in the collection, $A \cup B, B \cup C, A \cup C \in \mathcal{Q}$, then

$$A \cup B \cup C \in \mathcal{Q}$$

Similar to σ-algebras, if we have two quadratic algebras $\mathcal{Q}$,$\mathcal{R}$, then the intersection $\mathcal{Q} \cap \mathcal{R}$ will also be a quadratic algebra, since for any disjoint sets $A, B, C, A \cup B, B \cup C, A \cup C \in \mathcal{Q} \cap \mathcal{R}$ then

$$A \cup B \cup C \in \mathcal{Q}, \mathcal{R}$$

So $A \cup B \cup C$ will be in the intersection. This would allow us to look at the smallest quantum algebra that contains a set.

Let's look at an example of how quantum algebras work.

Example 3. For the set $X = \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}$, let $\mathcal{Q} \subseteq 2^X$ contain $\emptyset, X$, and sets $A \neq \emptyset, X$ such that element types a, b, c have different cardinalities, $0, 1, 2, 3$. For example,

$$\{a_1, b_1, b_2\}, \{a_2, c_1, c_2, c_3\} \in \mathcal{Q}$$

Since we have $1, 2, 0$ and $1, 0, 3$ as the cardinalities of each element type. These are also disjoint sets, but the union has cardinalities $2, 2, 3$, which cannot be in $\mathcal{Q}$. Since we do not have finite unions, this cannot be an algebra or a σ-algebra, according to definition 7 and proposition 2.

We can show that this is a quadratic algebra. Following the definition, take $A, B, C \in \mathcal{Q}$ mutually disjoint such that $A \cup B, B \cup C, A \cup C \in \mathcal{Q}$. If any of A, B, C are empty, then the union of all three is the union of two, which we know is in $\mathcal{Q}$. If each is non-empty, then we know the minimum number of elements in a set must be 3, since the smallest cardinalities would be $0, 1, 2$. But also, since X only has 9 elements, if all the sets are disjoint, they must have a maximum of 3; there are not enough elements for 3 disjoint sets with at least one with more than 3 elements. Then 3 disjoint sets with 3 elements each means that $A \cup B \cup C = X$, which we know is in $\mathcal{Q}$, so this is a quadratic algebra.

Example 4. For the set $X = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ for odd n, any m, let

$$\mathcal{Q} = \{A \subseteq S : |\{x_i : x_i \in A\}| \text{ is odd or } 0\}$$

So only sets with 0 or an odd number of $x's$ are in this collection. Clearly $\emptyset$ (0 $x's$) and X (n $x's$) are in $\mathcal{Q}$. Now take three sets $A, B, C \in \mathcal{Q}$ that satisfy the required properties. Then since the pairwise unions are in $\mathcal{Q}$ each must have 0 or an odd number of $x's$. If more than one of A, B or C had an odd number of $x's$, then their union would contain a (non-zero) even number of $x's$, which cannot happen. Thus, at most one of them has an odd number of $x's$, which means the union of all three sets has the same number of $x's$. Since this is odd, we know $A \cup B \cup C \in \mathcal{Q}$.

Now that we have a few examples of quadratic algebras, we revamp definition 10.

Definition 15. (10 version 2) For a set X and a quadratic algebra $\mathcal{Q}$, a ***q*-measure** is a function $\nu : \mathcal{Q} \to \mathbb{R}^+$ such that ν is regular and for disjoint sets $A, B, C \in \mathcal{Q}$ such that $A \sqcup B, A \sqcup C, B \sqcup C \in \mathcal{Q}$, then

$$\nu(A \sqcup B \sqcup C) = \nu(A \sqcup B) + \nu(B \sqcup C) + \nu(A \sqcup C) - \nu(A) - \nu(B) - \nu(C)$$

A **quadratic measure space** is the triple $(X, \mathcal{Q}, \nu)$

Looking at example 3, defining

$$\nu(A) = \begin{cases} 0 & A = \emptyset \\ \frac{1}{6} & |A| = 3 \\ \frac{1}{2} & |A| = 6 \\ 1 & A = X \end{cases}$$

Constitutes a valid q-measure: if one set is empty, label it C, then $\nu(A \sqcup B \sqcup C) = \nu(A \sqcup B)$, and since C is empty,

$$\nu(A \sqcup B) + \nu(B \sqcup C) + \nu(A \sqcup C) - \nu(A) - \nu(B) - \nu(C)$$
$$= \nu(A \sqcup B) + \nu(B) + \nu(A) - \nu(A) - \nu(B) - \nu(C) = \nu(A \sqcup B)$$

Similarly for two (or three) empty sets. Then as explained above, we know that three disjoint non-empty sets must each have 3 elements and their union is the whole space X. But then $\nu(A \sqcup B \sqcup C) = \nu(X) = 1$, and each pairwise union has 6 elements, so

$$\nu(A \sqcup B) + \nu(B \sqcup C) + \nu(A \sqcup C) - \nu(A) - \nu(B) - \nu(C) = \frac{3}{2} - \frac{3}{6} = 1$$

This measure is also regular, as when $\nu(A) = 0$ and $A \cap B = \emptyset$, then $|A \cup B| = |B|$, so $\nu(A \cup B) = \nu(B)$. Then if $\nu(A \cup B) = 0$,

$$|A \cup B| = \emptyset \implies |A| = \emptyset = |B| \implies \nu(A) = 0 = \nu(B)$$

In example 4, $\nu(A) = |A|^2$ is a valid measure, and even further, it will always be a valid measure.

Proposition 10. *For any finite set X and a quadratic algebra $\mathcal{Q}$, the measure $\nu(A) = |A|^2$ for $A \in \mathcal{Q}$ is a valid q-measure.*

Proof. All sets $A \in \mathcal{Q}$ are finite, so $|A|^2$ makes sense. Then taking disjoint sets, the cardinality is additive, so

$$\begin{aligned}\nu(A \sqcup B \sqcup C) = \nu(A \sqcup B) + \nu(B \sqcup C) + \nu(A \sqcup C) - \nu(A) - \nu(B) - \nu(C)\\ (|A| + |B|)^2 + (|B| + |C|)^2 + (|A| + |C|)^2 - \left(|A|^2 + |B|^2 + |C|^2\right)\\ = |A|^2 + |B|^2 + |C|^2 + 2|A||B| + 2|B||C| + 2|A||C|\\ = (|A| + |B| + |C|)^2\end{aligned}$$

And similarly to above, we have the regularity condition: when $\nu(A) = 0$ and $A \cap B = \emptyset$, then $|A \cup B| = |B|$, so

$$\nu(A \cup B) = |A \cup B|^2 = |B|^2 = \nu(B)$$

And when $\nu(A \cup B) = 0$,

$$|A \cup B| = \emptyset \implies |A| = \emptyset = |B| \implies \nu(A) = 0 = \nu(B)$$

□

This result could be extended, hopefully, to a measure $\nu(\psi) = |\psi|^2$ on the space of complex functions, so the measure is the magnitude square of the wave function.

Now that we understand how quadratic algebras and measures work, let's return to $\mathcal{L}, \mathcal{B}_{\mathcal{L}}$ with a major result helping us extend μ. The proof relies on some topology results, namely Tychonov's theorem. The proof of that theorem is outside the scope of this paper, but it is equivalent to the Axiom of Choice.

Theorem 2. *Both $\mathcal{L}$ and $\mathcal{B}_{\mathcal{L}}$ are quadratic algebras, and $\hat{\mu}$ is a q-measure on $\mathcal{B}_{\mathcal{L}}$ that extends μ to $\mathcal{C}$.*

Proof. Examine the set $\{0, 1\}$ under the discrete topology. This means that every subset of $\{0, 1\}$ is open, and thus, since every set's complement is open, also closed. Then since there are only two elements, any open cover of $\{0, 1\}$ must have at most two open sets that cover it, so any open cover has a finite cover. Thus $\{0, 1\}$ is compact.

Tychonov's theorem then gives us $\Omega = \{0\} \times \{0, 1\} \times \{0, 1\} \times \cdots$ is compact in the product topology, and since every set is both open and closed in $\{0, 1\}$, then any cylinder set is the product of closed and open sets. This means in the product topology that the cylinder sets are both closed and open, and closed sets in a compact space are compact, so cylinder sets are compact.

Now take any two lower sets $A, B \in \mathcal{L}$ such that $A \cap B = \emptyset$. Since they are lower sets, by definition 13, $A = \bigcap\limits_{n \geq 0} A^{(n)}$, and similarly for B, which gives that

$$\bigcap_{n \geq 0} \left(A^{(n)} \cap B^{(n)}\right) = \left(\bigcap_{n \geq 0} A^{(n)}\right) \cap \left(\bigcap_{n \geq 0} B^{(n)}\right) = A \cap B = \emptyset$$

These are decreasing based on proposition 8. But then any decreasing sequence of non-empty compact sets has a non-empty intersection. Since the intersection above is empty, we then must have some n where $A^{(n)} \cap B^{(n)} = \emptyset$, and since they are nested, then all the sets afterward must be empty,

$$A^{(m)} \cap B^{(m)} = \emptyset \forall m \geq n$$

This is a property we get for any two disjoint lower sets.

To prove $\mathcal{L}$ is a quadratic algebra, take mutually disjoint $A, B, C \in \mathcal{L}$ such that the pairwise unions are also in $\mathcal{L}$. The above work gives us three possibly different $n's$, and by taking the maximum and relabeling, we know $A^{(m)}, B^{(m)}, C^{(m)}$ are mutually disjoint $\forall m \geq n$. But then

$$\bigcap_{n\geq 0} \left[(A \cup B \cup C)^{(n)} \right] = \bigcap_{n\geq 0} \left[A^{(n)} \cup B^{(n)} \cup C^{(n)} \right]$$

As if ω corresponds to an ω' in the union for the first n terms, $\omega(n)[A \cup B \cup C]$, then ω' must be in at least one of A, B, or C. Then by distributivity,

$$= \bigcap_{n\geq 0} \left[A^{(n)} \cup B^{(n)} \cup C^{(n)} \right] = \left[\bigcap_{n\geq 0} A^{(n)} \right] \cup \left[\bigcap_{n\geq 0} B^{(n)} \right] \cup \left[\bigcap_{n\geq 0} C^{(n)} \right]$$
$$= A \cup B \cup C$$

This is the definition of a lower set, so $A \cup B \cup C \in \mathcal{L}$. Then since $\emptyset^{(n)} = \emptyset$ and $\Omega^{(n)} = \Omega$, both $\emptyset, \Omega \in \mathcal{L}$, and $\mathcal{L}$ is a quadratic algebra.

Now for $\mathcal{B}_{\mathcal{L}}$, take three sets $A, B, C \in \mathcal{B}_{\mathcal{L}}$ that satisfy the desired properties. We must show the limit of the measure of the union exists and is finite, since we already know it is a lower set. But

$$\lim_{n\to\infty} \mu \left[(A \cup B \cup C)^{(n)} \right] = \lim_{n\to\infty} \mu \left[A^{(n)} \cup B^{(n)} \cup C^{(n)} \right]$$

By the same reasoning as above, and since μ is a q-measure on cylinder sets, which we know each of these $A^{(n)}$ are, then

$$= \lim_{n\to\infty} \left[\mu \left(A^{(n)} \cup B^{(n)} \right) + \mu \left(A^{(n)} \cup C^{(n)} \right) + \mu \left(B^{(n)} \cup C^{(n)} \right) \right.$$
$$\left. - \mu \left(A^{(n)} \right) - \mu \left(B^{(n)} \right) - \mu \left(C^{(n)} \right) \right]$$

And since all of the limits exist, since each of the pairwise unions are in $\mathcal{B}_{\mathcal{L}}$, we then get

$$= \hat{\mu}(A \cup B) + \hat{\mu}(A \cup C) + \hat{\mu}(B \cup C) - \hat{\mu}(A) - \hat{\mu}(B) - \hat{\mu}(C)$$

So the limit exists, and thus $A \cup B \cup C \in \mathcal{B}$. Combining our results, we then know $\mathcal{B}_{\mathcal{L}}$ is a quadratic algebra. But from this work, we have

$$\hat{\mu} \left[(A \cup B \cup C)^{(n)} \right] = \lim_{n\to\infty} \mu \left[(A \cup B \cup C)^{(n)} \right]$$
$$= \hat{\mu}(A \cup B) + \hat{\mu}(A \cup C) + \hat{\mu}(B \cup C) - \hat{\mu}(A) - \hat{\mu}(B) - \hat{\mu}(C)$$

and $\hat{\mu}$ extends μ on $\mathcal{C}$. □

Definition 16. An **upper set** is a collection $A \in 2^{\Omega}$ if $A = \bigcup_{n\geq 0} \left((A')^{(n)} \right)'$.

This may appear to be a weird definition. However, A' still satisfies proposition 8, so $(A')^{(n)}$ is still a decreasing sequence of cylinder sets. Since the complement of a cylinder set is still a cylinder set, and taking the complement of a decreasing sequence of sets creates an increasing sequence, then an upper set A is the union of an increasing sequence of cylinder sets.

The intersection of the upper and lower sets is very convenient.

Theorem 3. *$\mathcal{L} \cap \mathcal{U} = \mathcal{C}$, and if $A, A' \in \mathcal{L}$, then $A \in \mathcal{C}$.*

Proof. First for any $A \in \mathcal{L} \cap \mathcal{U}$... we know that

$$\bigcap_{n\geq 0} A^{(n)} = A = \bigcup_{n\geq 0} \left((A')^{(n)}\right)'$$

Where both $A^{(n)}$ and $\left((A')^{(n)}\right)'$ are cylinder sets for each n. The first equality above gives that each $A^{(n)} \supseteq A$, as otherwise, the intersection is also smaller, and similarly for the other equality, $\left((A')^{(n)}\right)' \subseteq A$. But then notice that

$$A' = \left(\bigcup_{n\geq 0} \left((A')^{(n)}\right)'\right)' = \bigcap_{n\geq 0} \left((A')^{(n)}\right)$$

So

$$\begin{aligned} \emptyset = A \cap A' &= \left(\bigcap_{n\geq 0} A^{(n)}\right) \cap \left(\bigcap_{n\geq 0} \left((A')^{(n)}\right)\right) \\ &= \bigcap_{n\geq 0} \left(A^{(n)} \cap \left((A')^{(n)}\right)\right) \end{aligned}$$

We showed in the proof of theorem 2 that each cylinder set is a compact set. Then, since the intersection is empty, one of the sets $A^{(n)} \cap \left((A')^{(n)}\right)$ must be empty for some n. But since

$$\left((A')^{(n)}\right)' \subseteq A \subseteq A^{(n)}$$

Then

$$A^{(n)} \cap (A')^{(n)} = \emptyset \implies A^{(n)} = \left((A')^{(n)}\right)' = A$$

But since each of the first two sets are cylinder sets, then so is A. So $A \in \mathcal{C}$.

Now if $A, A' \in \mathcal{L}$, we have

$$A = \bigcap_{n\geq 0} A^{(n)}, A' = \bigcap_{n\geq 0} (A')^{(n)}$$

But then the second equality gives

$$A = \left(\bigcap_{n\geq 0} (A')^{(n)}\right)' = \bigcup_{n\geq 0} \left((A')^{(n)}\right)'$$

This means that $A \in \mathcal{U}$, so by the first part of the theorem, $A \in \mathcal{C}$. $\square$

Now we get a theorem similar to the result for lower sets.

Theorem 4. *$\mathcal{U}$ is a quadratic algebra.*

Proof. We recall that $\Omega^{(n)} = \Omega$ and $\emptyset^{(n)} = \emptyset$, so

$$\emptyset = \bigcup_{n\geq 0} \left((\emptyset')^{(n)}\right)' \qquad \Omega = \bigcup_{n\geq 0} \left((\Omega')^{(n)}\right)'$$

So both are a part of $\mathcal{U}$. Now suppose we have three mutually disjoint sets $A, B, C \in \mathcal{U}$, and we want to show that

$$A \cup B \cup C = \bigcup_{n\geq 0} \left(((A \cup B \cup C)')^{(n)}\right)'$$

Which we do by double inclusion. Firstly, for any element $\omega \in \left(\left((A \cup B \cup C)'\right)^{(n)}\right)'$, then $\omega \notin \left((A \cup B \cup C)'\right)^{(n)}$, so there is no path $\omega' \in (A \cup B \cup C)'$ such that ω matches with ω' for the first n terms, which means that $\omega \notin (A \cup B \cup C)'$ (because they would match). Thus,

$$\left(\left((A \cup B \cup C)'\right)^{(n)}\right)' \subseteq A \cup B \cup C$$

For the other inclusion: first, since $(A \cup B \cup C)' = A' \cap B' \cap C'$,

$$\left((A \cup B \cup C)'\right)^{(n)} = (A' \cap B' \cap C')^{(n)}$$

Then the right side are those paths ω that match with paths in A', B', and C' up to the first n terms, then

$$(A' \cap B' \cap C')^{(n)} \subseteq (A')^{(n)} \cap (B')^{(n)} \cap (C')^{(n)}$$

As this right side are the paths ω such that it matches with a path in A' up to the first n terms and similarly for B', C', a larger set. Taking complements, then, we have

$$\left(\left((A \cup B \cup C)'\right)^{(n)}\right)' \supseteq \left((A')^{(n)} \cap (B')^{(n)} \cap (C')^{(n)}\right)' = \left((A')^{(n)}\right)' \cup \left((B')^{(n)}\right)' \cup \left((C')^{(n)}\right)' \tag{5}$$

Now since $A, B, C, \in \mathcal{U}$, then

$$A \cup B \cup C = \left(\bigcup_{n \geq 0} \left((A')^{(n)}\right)'\right) \cup \left(\bigcup_{n \geq 0} \left((B')^{(n)}\right)'\right) \cup \left(\bigcup_{n \geq 0} \left((C')^{(n)}\right)'\right)$$
$$= \bigcup_{n \geq 0} \left(\left((A')^{(n)}\right)' \cup \left((B')^{(n)}\right)' \cup \left((C')^{(n)}\right)'\right)$$

Now combining this with equation 5, we get

$$A \cup B \cup C \subseteq \bigcup_{n \geq 0} \left(\left((A \cup B \cup C)'\right)^{(n)}\right)'$$

Thus, we have equality, which means $A \cup B \cup C \in \mathcal{U}$, so $\mathcal{U}$ is a quadratic algebra. □

Note that in this proof, we didn't need that the pairwise unions were in $\mathcal{U}$.

Now it would be natural for us to try to look at $\mathcal{B}_{\mathcal{U}}$, the beneficial upper sets. However, Sorkin and Gudder were unable to prove this result, and I have not been able to do so, either.

7 Future Work and Conclusion

We have tried to formalize the measure of infinite paths in the two state quantum random walk, though we see that there are some issues, namely, the work in Appendix A showing that μ cannot be extended to a continuous measure (the only type of measure we want) on the smallest σ-algebra containing the cylinder sets $\mathcal{C}$. This led us to create quadratic algebras and we found some fairly significant sets that satisfied this property. Still, there is some work to do, especially with the upper beneficial sets $\mathcal{B}_{\mathcal{U}}$ discussed at the end of section 6.

Two natural extension that Sorkin and Gudder briefly discuss are expectation values and integrals. For a regular measure space $(X, \mathcal{S}, \mu)$, the integral of a function $f : X \to \mathbb{R}$ is built up from the integral of characteristic functions: taking $A \in \mathcal{S}$, we have

$$\chi_A : X \to \mathbb{R}, \chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{else} \end{cases}, \qquad \int_X \chi_A d\mu = \mu(A)$$

Connecting this to probability theory, we would then say $\int_X \chi_A d\mu$ is the probability of event A occurring. However, interference makes this interpretation incorrect in quantum mechanics.

One way to create a more foundational q-integral would be to take, for a random variable $g : \Omega_n \to \mathbb{R}^+$

$$\int_{\Omega_n} g d\mu_n = \sum_{i,j=0}^{2^n-1} \min\left[g\left(i\right), g\left(j\right)\right] D_{ij}^n$$

And a general random variable $f : \Omega_n \to \mathbb{R}$ split in the way of ordinary measure theory, $f = f^+ - f^-$ where $f^+, f^- \geq 0$, splitting up the positive and negative parts and being 0 elsewhere. Then this integral is

$$\int_{\Omega_n} f d\mu_n = \int_{\Omega_n} f^+ d\mu_n - \int_{\Omega_n} f^- d\mu_n$$

If for the random variable $f : \Omega_n \to \mathbb{R}$ we treat $\hat{f}_{ij} = \min\left[f\left(i\right), f\left(j\right)\right]$ as a $2^n \times 2^n$ matrix, then it can be shown that

$$\int f d\mu_n = \mathrm{tr}\left(\hat{f} D^n\right)$$

More work can be done in this area.

Another subject to look into is continuous paths. Taking the interval between time steps to 0 and towards continuous states, we can hopefully extend our theory to measures, and therefore integrals and other mathematics, of continuous paths. It would be interesting physically to have a rigorous theory of the Feynman path integral, which would be the continuous extension of the paths we have dealt with. Sorkin and Gudder are almost certainly working on this.

An interesting note here is a consequence of the Uncertainty Principle: the product of the uncertainties of position and momentum is bounded below:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

If we have a continuous path of states (locations), then we know the position with exact precision. That would mean the uncertainty in momentum must be very large. But since momentum is proportional to the derivative of position, if there is a derivative of the path, then we would know the momentum. This means that we expect the probability that the path is continuous to be 1, while the probability of differentiability is 0.

A thought that Sorkin is likely working on is a further generalization of this work. Classically, we follow equation 2, where the probability of two disjoint events is the sum of the two. In this paper and quantum mechanics, we have generalized to equation 3, where three events are necessary. The events that satisfy the classical probability rule certainly satisfy the quantum probability rule. It could be possible that a further generalization is necessary for, possibly, a theory of quantum gravity that connects quantum mechanics with relativity. While we would like the theory around grade-2 additivity to be more rigorous before we move onto further generalizations, the prospect of a potential solution to one of the largest open questions in physics is enticing.

Another possible application of this work, and the final one we mention, is quantum computing. While not all research in this area would necessarily lead to advances in quantum computing, dealing with quantum mechanics and $0's$ and $1's$ naturally relates itself to qubits, and the formalizations in this area will surely lead to new revelations.

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A Conditions of D_n, proof of regularity

Claim 1. Strong positivity: for any finite collection of sets A_i $1 \leq i \leq m$, the (hermitian) matrix $N_{ij} = D_n(A_i, A_j)$ is positive semi-definite.

Proof. The Schur product theorem gives that the Hadamard product of two positive semi-definite matrices is also positive semi-definite. Now recall that

$$D^n_{jk} = \frac{1}{2^n} i^{[c_n(j)-c_n(k)]} \delta_{\alpha_{j,n}\alpha_{k,n}}$$

With j, k as the binary form of the paths $\omega_j, \omega_k \in \Omega_n$. But then $\delta_{\alpha_{j,n}\alpha_{k,n}} = p_{jk}$, where p is the parity operator, 1 if j, k have the same parity and 0 if not. Then we can define two matrices for each n

$$P = [p_{jk}], \qquad C = \left[i^{[c_n(j)-c_n(k)]}\right]$$

Then $D^n = \frac{1}{2^n} C \circ P$, the Hadamard product. These are both positive semi-definite. C is clear, and for P, the rank of P is 2; the matrix is made up of two column vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Since P is a Hermitian matrix, then its distinct eigenvalues have linearly independent eigenvectors. Since

$$Pv_1 = 2^{n-1} v_1 \qquad Pv_2 = 2^{n-1} v_2$$

Then we have the two vectors that form a basis for the range of P, then any other eigenvalue must have a eigenvector $v \perp v_1, v_2$. But then $Pv = 0$, since $v \cdot v_1, v_2 = 0$, thus, 2^{n-1} and 0 are the only eigenvalues of P, so it is positive semi-definite.

With the Schur product theorem, we have D^n is a positive semi-definite matrix.

Now we refer to a theorem proved in [7], where for any positive semi-definite matrix D^n of size 2^n, there exists a complete Hilbert space H and a spanning set of vectors $e_i \in E, 1 \leq i \leq 2^n$, such that $D^n_{ij} = \langle e_i, e_j \rangle$. This then implies that for $\mathcal{E} : 2^{\Omega_n} \to H$, $\mathcal{E}(A) = \sum \{e_i : \omega_i \in A\}$, then

$$D_n(A, B) = \langle \mathcal{E}(A), \mathcal{E}(B) \rangle$$

But this shows that for sets $A_i \in 2^{\Omega_n}$, $1 \leq i \leq m$, $N_{ij} = D_n(A_i, A_j)$ is positive semi-definite, since for any vector $\psi \in \mathbb{C}^m$,

$$\psi^\dagger N \psi = \sum_{i,j=1}^{n} \psi_j^* D_n(A_i, A_j) \psi_i = \sum_{i,j=1}^{n} \psi_j^* \langle \mathcal{E}(A_i), \mathcal{E}(A_j) \rangle \psi_i$$

$$= \left\langle \sum_{i=1}^{n} \psi_i \mathcal{E}(A_i), \sum_{j=1}^{n} \psi_j \mathcal{E}(A_j) \right\rangle$$

Where the last equality comes from the hermiticity and linearity of the measure on a Hilbert space. But the inner product on a Hilbert space by definition is non-negative, so

$$\psi^\dagger N \psi \geq 0$$

Which gives us the result. □

Claim 2. If $A \cap B = \emptyset$ and $\mu_n(A \cup B) = 0$, then $\mu_n(A) = \mu_n(B)$,

Proof. We have that D^n is a Hermitian matrix and is positive semi-definite, so linear algebra tells us there are only non-negative (real) eigenvalues, and we can diagonalize D^n through a unitary transformation Ψ. That is, for the diagonal matrix Λ with the eigenvalues λ_α of D^n on the diagonal,

$$D^n = \Psi \Lambda \Psi^\dagger$$

So $D^n_{ij} = \sum_\alpha \psi_{i\alpha} \lambda_\alpha \psi^*_{j\alpha}$. Then since

$$0 = \mu_n(A \cup B) = D_n(A \cup B, A \cup B) = \sum_{\omega_i \in A \cup B} \sum_{\omega_j \in A \cup B} D^n_{ij}$$

$$= \sum_{\omega_i \in A \cup B} \sum_{\omega_j \in A \cup B} \left[\sum_\alpha \psi_{i\alpha} \lambda_\alpha \psi^*_{j\alpha} \right]$$

Let us label

$$\phi_\alpha := \sum_{\omega_i \in A} \psi_{i\alpha} \qquad \chi_\alpha := \sum_{\omega_j \in B} \psi_{j\alpha}$$

Since we have that $A \cap B = \emptyset$, then no ω_i is in both, so

$$= \sum_{\omega_i \in A} \sum_{\omega_j \in B} \left[\sum_\alpha \psi_{i\alpha} \lambda_\alpha \psi^*_{j\alpha} \right]$$

$$= \sum_\alpha (\phi_\alpha + \chi_\alpha) \lambda_\alpha (\phi^*_\alpha + \chi^*_\alpha)$$

But then this is just a λ_α norm in a vector space, $= |\phi + \chi|^2_\lambda$. This equaled 0 from the start, so we know that $|\phi + \chi|^2_\lambda = 0 \implies \phi + \chi = 0$, from the properties of the vector space. But then this means

$$\phi_\alpha = -\chi_\alpha \text{ whenever } \lambda_\alpha > 0$$

But noticing from our definition,

$$\mu_n(A) = \sum |\phi_\alpha|^2 \lambda_\alpha \qquad \mu_n(B) = \sum |\chi_\alpha|^2 \lambda_\alpha$$

And since $\lambda_\alpha \geq 0$, this means that $\mu_n(A) = \mu_n(B)$. □