On the Homology Theory of Modules.

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§ 0. Introduction.

Let Λ be a ring with unit. In generalizing the notions of torsion products and extension groups of abelian groups, Cartan and Eilenberg have defined a set of abelian groups $\mathrm{Tor}_n{}^\Lambda(A, B)$, $\mathrm{Ext}_\Lambda{}^n(A, B)$ $(n=0, 1, \cdots)$ for any two Λ-modules A, B. These groups are in a deep connection with various homology and cohomology theories of groups, of associative algebras, of Lie algebras, etc.[1] The present paper attempts a general study of the groups $\mathrm{Tor}_n{}^\Lambda(A, B)$ and $\mathrm{Ext}_\Lambda{}^n(A, B)$.

The definition of these groups can be sketched as follows. Take a Λ-free module X_0 with an epimorphism[2] $X_0 \to A_0$, of which we denote the kernel with A_1. Take next a Λ-free module X_1 with an epimorphism $X_1 \to A_1$, of which the kernel is denoted by A_2. Repeating this, we obtain a sequence X_* of Λ-free modules and Λ-homomorphisms:

$$(X_*) \qquad \cdots\cdots \to X_n \to X_{n-1} \to \cdots\cdots \to X_1 \to X_0 \to 0,$$

where $X_n \to X_{n-1}$ is defined as $X_n \to A_n \to X_{n-1}$. This sequence, called a *free resolution* of A, is acyclic with respect to the augmentation $X_0 \to A \to 0$. $\mathrm{Tor}_n{}^\Lambda(A, B)$, the n-th torsion product of A and B, is then defined as the n-th homology group of the lower sequence $X_* \otimes_\Lambda B$, and $\mathrm{Ext}_\Lambda{}^n(A, B)$, the n-th extension group of B by A, as the n-th cohomology group of the upper sequence $\mathrm{Hom}_\Lambda(X_*, B)$. Both are independent of the special choice of the free resolution X_* of A.

These groups $\mathrm{Tor}_n{}^\Lambda(A, B)$, $\mathrm{Ext}_\Lambda{}^n(A, B)$—unless any confusion is likely to occur we shall omit the letter Λ in the following— may be considered as giving homology and cohomology theories of module A with coefficient module B, because of the following properties.

I. Any homomorphism f: $A \to A'$ induces homomorphisms

$$f_*: \ \mathrm{Tor}_n(A, B) \to \mathrm{Tor}_n(A', B) \qquad (n=0, 1, \cdots)$$

and

$$f^*: \ \mathrm{Ext}^n(A', B) \to \mathrm{Ext}^n(A, B) \qquad (n=0, 1, \cdots),$$

satisfying:

I-1) i_*, i^* are identities if i is the identity mapping.

I-2) $(g \circ f)_* = g_* \circ f_*$, $(g \circ f)^* = f^* \circ g^*$.

II. If (A) $0 \to \dot{A} \xrightarrow{i} A \xrightarrow{j} \bar{A} \to 0$ is exact, then homomorphisms

$$\partial_*: \ \mathrm{Tor}_n(\bar{A}, B) \to \mathrm{Tor}_{n-1}(\dot{A}, B) \qquad (n=1, 2, \cdots),$$

$$\delta^*: \ \mathrm{Ext}^n(\dot{A}, B) \to \mathrm{Ext}^{n+1}(\bar{A}, B) \qquad (n=0, 1, \cdots)$$

1) H. Cartan-S. Eilenberg, Satellites des foncteurs de module; H. Cartan-S. Eilenberg, Homological algebra; to appear soon. This book will be referred to as [C–E] in the sequel. H. Cartan, Seminaire de topologie algébrique, 1950–51.

2) By an *epimorphism* we mean a 'homomorphism onto'; 'isomorphism into' will be called a *monomorphism*; and the word *isomorphism* will be used to mean an 'isomorphism onto'.

are defined so that

II-1) the sequences

$$\cdots\cdots\to \mathrm{Tor}_n(\dot{A}, B)\xrightarrow{i_*}\mathrm{Tor}_n(A, B)\xrightarrow{j_*}\mathrm{Tor}_n(\overline{A}, B)\xrightarrow{\partial_*}\mathrm{Tor}_{n-1}(A, B)$$
$$\to \mathrm{Tor}_{n-1}(A, B)\to\cdots\cdots\to \mathrm{Tor}_0(\dot{A}, B)\to \mathrm{Tor}_0(A, B)\to \mathrm{Tor}_0(\overline{A}, B)\to 0,$$
$$0\to \mathrm{Ext}^0(\overline{A}, B)\to \mathrm{Ext}^0(A, B)\to \mathrm{Ext}^0(\dot{A}, B)\to \mathrm{Ext}^1(\overline{A}, B)\to\cdots\cdots$$
$$\to \mathrm{Ext}^n(\overline{A}, B)\xrightarrow{j^*}\mathrm{Ext}^n(A, B)\xrightarrow{i^*}\mathrm{Ext}^n(\dot{A}, B)\xrightarrow{\delta^*}\mathrm{Ext}^{n+1}(\overline{A}, B)\to\cdots\cdots$$

are both exact. These exact sequences will be denoted by $\mathrm{Tor}^\Lambda(\boldsymbol{A}, \boldsymbol{B})$ and by $\mathrm{Ext}_\Lambda(\boldsymbol{A}, \boldsymbol{B})$ respectively.

II-2) Let two sequences

$$(\boldsymbol{A})\ 0\to\dot{A}\to A\to\overline{A}\to 0,\quad (\boldsymbol{A}')\ 0\to\dot{A}'\to A'\to\overline{A}'\to 0$$

be exact, and let $\boldsymbol{f}$: $\boldsymbol{A}\to\boldsymbol{A}'$ be a homomorphism of the sequence $\boldsymbol{A}$ into $\boldsymbol{A}'$, i.e., a triple of homomorphisms $\dot{f}:\dot{A}\to\dot{A}'$, $f:A\to A'$, $\overline{f}:\overline{A}\to\overline{A}'$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & \dot{A} & \to & A & \to & \overline{A} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \dot{A}' & \to & A' & \to & \overline{A}' & \to & 0 \end{array}$$

is commutative. Then $\boldsymbol{f}_*=\{\dot{f}_*, f_*, \overline{f}_*\}$ and $\boldsymbol{f}^*=\{\dot{f}^*, f^*, \overline{f}^*\}$ give homomorphisms of the exact sequences

$$\boldsymbol{f}_*:\ \mathrm{Tor}^\Lambda(\boldsymbol{A}, B)\to\mathrm{Tor}^\Lambda(\boldsymbol{A}', B),$$
$$\boldsymbol{f}^*:\ \mathrm{Ext}_\Lambda(\boldsymbol{A}', B)\to\mathrm{Ext}_\Lambda(\boldsymbol{A}, B).$$

III. $\mathrm{Tor}_n{}^\Lambda(A, B)=\mathrm{Ext}_\Lambda{}^n(A, B)=0$ $(n>0)$, if A is Λ-free.

IV. $\mathrm{Tor}_0{}^\Lambda(A, B)=A\otimes_\Lambda B$, $\mathrm{Ext}_\Lambda{}^0(A, B)=\mathrm{Hom}_\Lambda(A, B)$, and for $f:A\to A'$, we have $f_*=f\otimes i$[3] on $\mathrm{Tor}_0{}^\Lambda(A, B)$ and $f^*=\mathrm{Hom}(f, i)$[3] on $\mathrm{Ext}_\Lambda{}^0(A', B)$, where i is the identity mapping of B.

These four properties are characteristic for Tor^Λ and Ext_Λ.

If we consider the second entry, the coefficient module in $\mathrm{Tor}^\Lambda(A, B)$ or in $\mathrm{Ext}_\Lambda(A, B)$ as variable, then we have the following analogues of I$\sim$IV.

I′. Any homomorphism $f:B\to B'$ induces homomorphisms

$$_*f:\ \mathrm{Tor}_n(A, B)\to\mathrm{Tor}_n(A, B')\qquad (n=0, 1, \cdots\cdots),$$
$$^*f:\ \mathrm{Ext}^n(A, B)\to\mathrm{Ext}^n(A, B')\qquad (n=0, 1, \cdots\cdots).$$

These satisfy

I′-1) $_*i$, $^*i$ are identities if i is the identity mapping.

I′-2) $_*(g\circ f)={}_*g\circ{}_*f$, $^*(g\circ f)={}^*g\circ{}^*f$.

II′. If $(\boldsymbol{B})\ 0\to\dot{B}\xrightarrow{i}B\xrightarrow{j}\overline{B}\to 0$ is exact, then homomorphisms

$$_*\partial:\ \mathrm{Tor}_n(A, \overline{B})\to\mathrm{Tor}_{n-1}(A, \dot{B})\qquad (n=1, 2, \cdots\cdots),$$
$$^*\delta:\ \mathrm{Ext}^n(A, \overline{B})\to\mathrm{Ext}^{n+1}(A, \dot{B})\qquad (n=0, 1, \cdots\cdots)$$

are defined so as to satisfy:

II′-1) The sequences

$$\cdots\cdots\to\mathrm{Tor}_n(A, \dot{B})\xrightarrow{*i}\mathrm{Tor}_n(A, B)\xrightarrow{*j}\mathrm{Tor}_n(A, \overline{B})\xrightarrow{*\partial}\mathrm{Tor}_{n-1}(A, \dot{B})$$

3) See S. Eilenberg-N. E. Steenrod, Foundations of algebraic topology, pp. 141, 147. This book will be referred to as [E-S] in the sequel.

$$\to \mathrm{Tor}_{n-1}(A, B) \to \cdots \to \mathrm{Tor}_0(A, \dot{B}) \to \mathrm{Tor}_0(A, B) \to \mathrm{Tor}_0(A, \overline{B}) \to 0,$$
$$0 \to \mathrm{Ext}^0(A, \dot{B}) \to \mathrm{Ext}^0(A, B) \to \mathrm{Ext}^0(A, \overline{B}) \to \mathrm{Ext}^1(A, \dot{B}) \to \cdots$$
$$\to \mathrm{Ext}^n(A, \dot{B}) \xrightarrow{*i} \mathrm{Ext}^n(A, B) \xrightarrow{*j} \mathrm{Ext}^n(A, \overline{B}) \xrightarrow{*s} \mathrm{Ext}^{n+1}(A, \dot{B}) \to \cdots$$

are both exact. These exact sequences will be denoted by $\mathrm{Tor}^{\Lambda}(A, \boldsymbol{B})$ and by $\mathrm{Ext}_{\Lambda}(A, \boldsymbol{B})$ respectively.

II′-2) Let f: $\boldsymbol{B} \to \boldsymbol{B}'$ be a homomorphism of exact sequences:

$$\begin{array}{ll} (\boldsymbol{B}) & 0 \to \dot{B} \to B \to \overline{B} \to 0 \\ \downarrow f & \qquad\;\; \downarrow \quad \downarrow \quad \downarrow \\ (\boldsymbol{B}') & 0 \to \dot{B}' \to B' \to \overline{B}' \to 0. \end{array}$$

Then f induces homomorphism of the exact homology and cohomology sequences

$$_*f\colon\ \mathrm{Tor}^{\Lambda}(A, \boldsymbol{B}) \to \mathrm{Tor}^{\Lambda}(A, \boldsymbol{B}'),$$
$$^*f\colon\ \mathrm{Ext}_{\Lambda}(A, \boldsymbol{B}) \to \mathrm{Ext}_{\Lambda}(A, \boldsymbol{B}').$$

III′. $\mathrm{Tor}_n{}^{\Lambda}(A, B) = 0$ $(n > 0)$ if B is Λ-free, and $\mathrm{Ext}_{\Lambda}{}^n(A, B) = 0 (n > 0)$ if B is Λ-injective[4].

IV′. $\mathrm{Tor}_0{}^{\Lambda}(A, B) = A \otimes_{\Lambda} B$, $\mathrm{Ext}_{\Lambda}{}^0(A, B) = \mathrm{Hom}_{\Lambda}(A, B)$, and for $f\colon B \to B'$, we have $_*f = i \otimes f$ on $\mathrm{Tor}_0{}^{\Lambda}(A, B)$, $^*f = \mathrm{Hom}(i, f)$ on $\mathrm{Ext}_{\Lambda}{}^0(A, B)$, where i is the identity mapping of A.

These properties I′∼IV′ are again characteristic for Tor^{Λ} and Ext_{Λ}. Also we have

V. Homomorphisms f_*, f^* in I commute with the homomorphisms $_*f$, $^*f$ in I′.

Now, it happens in various cohomology theories that 1-cohomology groups have a close relation with extension theories. In the present cohomology theory, the elements of our 1-cohomology group $\mathrm{Ext}_{\Lambda}{}^1(A, B)$ are in a 1−1 correspondence with the equivalence classes of module extensions of B by A. Thereby, two extensions

$$(E)\ \ 0 \to B \to E \to A \to 0, \qquad (E')\ \ 0 \to B \to E' \to A \to 0$$

are called equivalent if there is an isomorphism $E \to E'$ such that

$$\begin{array}{l} 0 \to B \to E \to A \to 0 \\ \qquad\ \ \| \quad\ \downarrow \quad \| \qquad\qquad (=\text{: identity mapping}) \\ 0 \to B \to E' \to A \to 0 \end{array}$$

is commutative ([C-E]). In this paper we shall define a certain equivalence relation among the set of all exact sequences of the form

$$(E_n)\ \ 0 \to B \to E_{n-1} \to \cdots \to E_0 \to A \to 0$$
$$(E_0, \cdots, E_{n-1}\text{: arbitrary } \Lambda\text{-modules})$$
$$(n\text{: any fixed integer } \geq 1),$$

which we call n-fold extensions of B by A and prove that there is a certain 1–1 correspondence between the equivalence classes of n-fold extensions of B by A and the elements of the group $\mathrm{Ext}_{\Lambda}{}^n(A, B)$. This will be done in § 3.

In § 4 we shall introduce a bilinear multiplication

$$\mathrm{Ext}_{\Lambda}{}^p(A, B) \times \mathrm{Ext}_{\Lambda}{}^q(B, C) \to \mathrm{Ext}_{\Lambda}{}^{p+q}(A, C),$$

and give some results concerning this including the following theorem: The

4) See for definition § 2 of this paper.

coboundary homomorphism

$$\delta^*:\quad \mathrm{Ext}_\Lambda{}^n(\dot{A}, B)\to\mathrm{Ext}_\Lambda{}^{n+1}(\overline{A}, B) \qquad (n=0, 1, \cdots)$$

with respect to the exact sequence (A) $0\to\dot{A}\to A\to\overline{A}\to 0$ coincides with the left multiplication by the element in $\mathrm{Ext}_\Lambda{}^1(\overline{A}, \dot{A})$ represented by the extension A, while the coboundary homomorphism

$$^*\delta:\quad \mathrm{Ext}_\Lambda{}^n(A, \overline{B})\to\mathrm{Ext}_\Lambda{}^{n+1}(A, \dot{B}) \qquad (n=0, 1, \cdots)$$

with respect to the exact sequence (B) $0\to\dot{B}\to B\to\overline{B}\to 0$ coincides with the right multiplication by the element in $\mathrm{Ext}_\Lambda{}^1(\overline{B}, \dot{B})$ represented by the extension B.

Also some relations between this cohomology theory and the cohomology theory of groups will be given in that section.

Professor S. Eilenberg has kindly communicated to me some of the results stated above on Tor^Λ and Ext_Λ, and engaged me in this investigation, for which I wish to express my hearty thanks to him. I publish here however the complete proof of all the results, as no details about this homology theory seem to have been hitherto published. I wish to thank also Professor Chevalley, who induced me to work on the subject of § 4.

In what follows, we shall use a certain way of denomination of mappings appearing in diagrams, different from the usual one, as we shall explain in § 1. We hope that this way of denomination, as well as the notions of the translation and the translation category introduced also in § 1 facilitates the handling with the diagrams.

§ 1. Categories and functors of diagrams[5].

1. Denomination of mappings in a diagram. Let A, B be two vertices in a diagram, and let a mapping of A into B be given as $A\to B$ in the diagram. Usually such a mapping is denoted by a letter like f; indicated in the diagram as $A\xrightarrow{f}B$; and f is considered as a left operator on A. To indicate this mapping, we shall write now AfB or, if there is no fear of confusion, simply AB. If namely there is only one arrow from A to B, we have not to name the mapping by a letter like f; it is sufficiently clear to write simply $A\to B$, and name the mapping AB. If, on the contrary, there are two or more arrows from A to B, then we shall write like $A\underset{2}{\overset{1}{\rightrightarrows}}B$, and denote the mappings with $A1B$, $A2B$.

Mapping AfB, AB, or $A1B$ will be considered as a right operator on A, so that $(a)AfB=b$ will mean $f(a)=b$ in the usual notation. The composite of several mappings $A\to B$, $B\to C$, and $C\to D$ is denoted by $AB\cdot BC\cdot CD$, or more simply by $ABCD$. Thus the commutativity of the square diagram

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & D \end{array}$$

is written as $ABD=ACD$.

2. Categories of diagrams. Let $\mathscr{C}$ be a category in the sense of [E-S][3]. A diagram $D=\{A, B, \cdots; AB, \cdots\}$ consisting of vertices A, B, $\cdots$ which represent some objects in $\mathscr{C}$, and arrows AB, $\cdots$ which represent some map-

5) For definitions of categories and functors, see [E-S, Chap. IV].

pings in $\mathscr{C}$ is called a diagram in $\mathscr{C}$. In the widest sense any subcategory of $\mathscr{C}$ is a diagram in $\mathscr{C}$ but we shall confine ourselves to diagrams which are connected as 1-dimensional complexes of vertices and edges. Two diagrams $\boldsymbol{D}=\{A, B, \cdots; AB, \cdots\}$, $\boldsymbol{D}'=\{A', B', \cdots; A'B', \cdots\}$ are said to be *isomorphic* if there is a 1–1 correspondence between $\boldsymbol{D}$ and $\boldsymbol{D}'$, vertex-to-vertex, arrow-to arrow, such that, if $A'B' \in \boldsymbol{D}'$ corresponds to $AB \in \boldsymbol{D}$, then A' corresponds to A and B' to B.

Let $\boldsymbol{D}=\{A, B, \cdots; AB \cdots\}$ and $\boldsymbol{D}'=\{A', B', \cdots; A'B', \cdots\}$ be isomorphic diagrams such that A corresponds to A', B to B', etc. A set of mappings $f=\{AA', BB', \cdots\}$ in $\mathscr{C}$ of which the set of domains coincides with the set of objects in $\boldsymbol{D}$ is called a *translation* of $\boldsymbol{D}$ into $\boldsymbol{D}'$ (notation $f: \boldsymbol{D}\to\boldsymbol{D}'$) if each square of the form

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ A' & \to & B' \end{array} \qquad (A, B \in \boldsymbol{D}, A, B' \in \boldsymbol{D}')$$

is commutative.

Let $\boldsymbol{D}$ be a diagram in $\mathscr{C}$, and $\mathfrak{D}=\{\boldsymbol{D}, \boldsymbol{D}', \boldsymbol{D}'', \cdots\}$ be a set of diagrams in $\mathscr{C}$ isomorphic to $\boldsymbol{D}$. For $\boldsymbol{D}', \boldsymbol{D}'' \in \mathfrak{D}$, there may be a translation from $\boldsymbol{D}'$ to $\boldsymbol{D}''$. $\mathfrak{D}$ together with all such possible translations forms a category $\tilde{\mathfrak{D}}$, which will be called the *translation category* over $\mathfrak{D}$.

Throughout this papper, Λ will denote a ring with unit, not necessarily commutative, and unless otherwise stated, a Λ-module will mean a unitary left Λ-module. Following [E–S] we denote the category of Λ-modules and Λ-homomorphisms by $\mathscr{G}_\Lambda$ (Z will always denote the ring of rational integers, and we write $\mathscr{G}$ for $\mathscr{G}_Z$). Diagrams in $\mathscr{G}_\Lambda$ are also called diagrams *over* Λ.

A translation $f=\{AA', BB', \cdots\}$ of $\boldsymbol{D}=\{A, B, \cdots; AB, \cdots\}$ into $\boldsymbol{D}'=\{A', B', \cdots; A'B', \cdots\}$ will be called *epimorphic*, *monomorphic*, or *isomorphic* if every one of $AA', BB', \cdots$ is an epimorphism, monomorphism, or an isomorphism[2]. In what follows we shall often consider diagrams in the translation category $\tilde{\mathfrak{D}}$ over $\mathfrak{D}$, $\mathfrak{D}$ being a set of diagrams in $\mathscr{G}_\Lambda$ isomorphic to a certain diagram $\boldsymbol{D}$, as

$$\begin{array}{ccccccc} \cdots\cdots\to & \boldsymbol{D} & \to & \boldsymbol{D}' & \to\cdots\cdots \\ & \downarrow & & \downarrow & \\ \cdots\cdots\to & \boldsymbol{D}_1 & \to & \boldsymbol{D}_1' & \to\cdots\cdots \end{array}.$$

In such diagrams each vertex represents a diagram in $\mathscr{G}_\Lambda$ isomorphic to $\boldsymbol{D}$, and each arrow a translation. We use 0 in such a diagram to mean a trivial diagram consisting of 0's and 0-homomorphisms isomorphic to $\boldsymbol{D}$. A sequence

$$(*) \qquad \boldsymbol{D}\xrightarrow{f}\boldsymbol{D}'\xrightarrow{g}\boldsymbol{D}''$$

of translations $f=\{AA', BB', \cdots\}$, $g=\{A'A'', B'B'', \cdots\}$ is said to be *exact* if the sequences $A\to A'\to A''$, $B\to B'\to B''$, $\cdots$ are exact. (*) is called a 0-sequence if $A\to A'\to A''$, $B\to B'\to B''$, $\cdots$ are 0-sequences (i.e., $AA'A''=0$, $BB'B''=0, \cdots$).

Let $\mathfrak{E}(\Lambda)$ be the set of all exact sequences over Λ of the form

$$(A) \quad 0\to\dot{A}\to A\to\bar{A}\to 0.$$

The translation category $\tilde{\mathfrak{E}}(\Lambda)$ over $\mathfrak{E}(\Lambda)$ will play an important rôle in the following.

We shall denote with $\mathfrak{B}(\Lambda)$ the set of all 'one-arrow diagrams', namely diagrams over Λ isomorphic to $A \to B$; and with $\mathfrak{C}(\Lambda)$ the set of 'two-arrow 0-sequences', namely diagrams over Λ isomorphic to $A \to B \to C$, A, B, C being Λ-modules and ABC being 0. The translation categories $\tilde{\mathfrak{B}}(\Lambda)$, $\tilde{\mathfrak{C}}(\Lambda)$ will also play certain rôles in the sequel

3. Functors on translation categories of diagrams. We note here some fundamental lemmas giving functors on translation categories of diagrams in $\mathscr{G}_\Lambda$. In what follows we shall identify two diagrams D, D' over Λ if there exists an *isomorphic* translation $D \to D'$.

Lemma 1.1. *Let*

(1)
$$\begin{tikzcd} N \arrow[r] & A \arrow[r] \arrow[d] & Q \arrow[r] & 0 \\ & B & & \end{tikzcd}$$

be a diagram over Λ such that the sequence N-A-Q-0 is exact and $NAB=0$. Then there is a unique mapping $QB \in \mathscr{G}_\Lambda$

(1′)
$$Q \to B$$

satisfying $AQB=AB$. Any translation in the translation category over the set $\mathfrak{D}_1(\Lambda)$ of all diagram in $\mathscr{G}_\Lambda$ of the form (1) *induces a translation in $\tilde{\mathfrak{B}}(\Lambda)$, and thus the assignment* $(1) \Longrightarrow (1')$ *defines a functor $\tilde{\mathfrak{D}}_1(\Lambda) \Longrightarrow \tilde{\mathfrak{B}}(\Lambda)$.*

Proof. The unique existence of QB is obvious. We have only to prove that if

$$\begin{tikzcd} & N \arrow[r] \arrow[ddl] & A \arrow[r] \arrow[d] \arrow[ddl] & Q \arrow[r] \arrow[ddl] & 0 \\ & & B \arrow[ddl] & & \\ N' & A' \arrow[l] \arrow[r] \arrow[d] & Q' \arrow[r] & 0 & \\ & B & & & \end{tikzcd}$$

is a commutative diagram representing a translation in $\mathfrak{D}_1(\Lambda)$, and if QB, $Q'B'$ are so defined that $AQB=AB$, $A'Q'B'=A'B'$, then we have

$$QBB'=QQ'B'.$$

Since AQ is an epimorphism, it is sufficient to show $AQBB'=AQQ'B'$, which is done as

$$AQBB'=ABB'=AA'B'=AA'Q'B'=AQQ'B'.$$

In a similar way we can prove also

Lemma 1.2. *Let*

(2)
$$\begin{tikzcd} & & B \arrow[d] & \\ 0 \arrow[r] & N \arrow[r] & A \arrow[r] & Q \end{tikzcd}$$

be a diagram over Λ such that the sequence 0-N-A-Q is exact and $BAQ=0$. Then there is a unique mapping BN

(2′)
$$B \to N$$

satisfying $BNA=BA$. The assignment $(2) \Longrightarrow (2')$ *defines a functor of the translation category over the set of all diagrams in $\mathscr{G}_\Lambda$ of the form* (2).

Lemma 1.3. *From a 'one-arrow diagram'*

(3)
$$A \xrightarrow{f} B$$

over Λ we obtain in the obvious way, a commutative diagram

(3′)
$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & \underset{(\mathrm{Ker}\, f)}{N} \arrow[r] & A \arrow[r] \arrow[dr] & M \arrow[r] \arrow[d, "f"', "(\mathrm{Im}\, f)"] & 0 \\
 & & & B \arrow[d] & \\
 & & & Q\ (\mathrm{Coker}\, f) \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

in which the sequences 0-N-A-M-0, 0-M-B-Q-0 *are in* $\mathfrak{E}(\Lambda)$ (*i.e., exact*). *Any translation of* (3)

(3′′)
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
A' \arrow[r, "f'"] & B'
\end{tikzcd}$$

can be extended uniquely to a translation of (3′)

(3′′′)
$$\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
 & & 0 \arrow[r] & N \arrow[r] \arrow[ddll] & A \arrow[r] \arrow[dr] \arrow[ddll] & M \arrow[r] \arrow[d] \arrow[ddll] & 0 \\
 & & & 0 \arrow[d] & & B \arrow[d] \arrow[ddll] & \\
0 \arrow[r] & N' \arrow[r] & A' \arrow[r] \arrow[dr] & M' \arrow[r] \arrow[d] & 0 & Q \arrow[d] \arrow[ddll] & \\
 & & & B' \arrow[d] & & 0 & \\
 & & & Q' \arrow[d] & & & \\
 & & & 0' & & &
\end{tikzcd}$$

and in this way the assignment (3) $\Longrightarrow$ (3′) *defines a functor from the translation category* $\mathfrak{B}(\Lambda)$ *into the translation category over the set of all diagrams in* $\mathscr{G}_\Lambda$ *of the form* (3′).

Proof. We have only to show the unique existence of mappings NN', MM', QQ' such that $NN'A'=NAA'$, $AMM'=AA'M'$, $MM'B'=MBB'$, and $BQQ'=BB'Q$. Now since $NAA'\cdot A'B'=NABB'=0$, and since the sequence 0-N'-A'-B' is exact, there exists uniquely a mapping $NN'\in\mathscr{G}_\Lambda$ such that $NN'\cdot N'A=NAA'$, i.e., $NN'A'=NAA'$ (Lemma 1.2). $NAA'B'=0$ implies also $NAA'M'=0$. Therefore by Lemma 1.1 there exists uniquely a mapping $MM'\in\mathscr{G}_\Lambda$ such that $AMM'=AA'M'$. To prove $MM'B'=MBB'$ for this MM', it is sufficient to show $AMM'B'=AMBB'$, which is done as follows

$$AMM'B'=AA'M'B'=AA'B'=ABB'=AMBB'.$$

Finally we have $MBB'Q'=MM'B'Q'=0$. Thus by Lemma 1.1 there is a unique mapping QQ' satisfying $BQQ'=BB'Q'$. This completes the proof of the lemma.

In short, Lemma 1.3 states that Ker, Im, and Coker are functors from $\tilde{\mathfrak{B}}(\Lambda)$ into $\mathscr{G}_\Lambda$. Therefore we may speak of kernels, images and cokernels of translations.

Let

(4) $$A \to B \to C$$

be a 0-sequence over Λ. We call the factor module $\mathrm{Ker}BC/\mathrm{Im}\,AB$ *homology factor* of the sequence A-B-C and denote it by $H(A\text{-}B\text{-}C)$. Since the assignment (4) $\Longrightarrow H(A\text{-}B\text{-}C)$ can be composed by the functors Ker, Im, and Coker, we have clearly

Lemma 1.4. The assignment (4) $\Longrightarrow$ $H(A\text{-}B\text{-}C)$ defines a functor from $\tilde{\mathfrak{S}}(\Lambda)$ into $\mathscr{G}_\Lambda$.

In the sequel we shall denote this functor by H.

Lemma 1.5. *In the commutative diagram*

$$\text{(8)}\qquad \begin{tikzcd} & & B_0 \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & 0 \\ & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & C_2 & \\ 0 \arrow[r] & A_3 \arrow[r] & B_3 & & \end{tikzcd}$$

let B_0-B_1-B_2, C_0-C_1-C_2, A_1-A_2-A_3, B_1-B_2-B_3 *belong to* $\mathfrak{S}(\Lambda)$, *and let* B_0-C_0-0, A_1-B_1-C_1-0, 0-A_2-B_2-C_2, 0-A_3-B_3 *be exact. Then a mapping*

$$\text{(8}'\text{)}\qquad \Delta:\quad H(C_0\text{-}C_1\text{-}C_2)\to H(A_1\text{-}A_2\text{-}A_3)$$

in $\mathscr{G}_\Lambda$ *is induced in such a way that the diagram*

$$\text{(8}''\text{)}\qquad \begin{tikzcd} \mathrm{Ker}\, B_1C_1C_2 \arrow[r, "(B_1C_1)"] \arrow[d, "(B_1B_2)"] & \mathrm{Ker}\, C_1C_2 \arrow[r] & H(C_0\text{-}C_1\text{-}C_2) \arrow[d, "\Delta"] \\ \mathrm{Ker}\, B_2C_2\cap \mathrm{Ker}\, B_2B_3 \cong \mathrm{Ker}\, A_2A_3 \arrow[rr] & & H(A_1\text{-}A_2\text{-}A_3) \end{tikzcd}$$

is commutative. The assignment (8) $\Longrightarrow$ (8′) *defines a functor from the translation category over the set of all diagrams in* $\mathscr{G}_\Lambda$ *of the form* (8) *into* $\tilde{B}(\Lambda)$.

Proof. The kernel of the epimorphism

$$\text{(8}'''\text{)}\qquad \mathrm{Ker}\, B_1C_1C_2\to H(C_0\text{-}C_1\text{-}C_2)\to 0$$

is the inverse image of $\mathrm{Im}\, C_0C_1=\mathrm{Im}\, B_0C_0C_1=\mathrm{Im}\, B_0B_1C_1$ under the map B_1C_1, i.e. $\mathrm{Im}\, B_0B_1\cup \mathrm{Ker}\, B_1C_1=\mathrm{Im}\, B_0B_1\supset \mathrm{Im}\, A_1B_1$. But we have $(\mathrm{Im}\, B_0B_1\cup \mathrm{Im}\, A_1B_1)B_1B_2$ $=(\mathrm{Im}\, A_1B_2)B_1B_2=\mathrm{Im}\, A_1B_1B_2=\mathrm{Im}\, A_1A_2B_2$, which is in $\mathrm{Ker}\, B_2C_2\cap\mathrm{Ker}\, B_2B_3$ and vanishes if carried over to $H(A_1\text{-}A_2\text{-}A_3)$. Thus the kernel of (8‴) vanishes if mapped into $H(A_1\text{-}A_2\text{-}A_3)$. Therefore there exists a unique mapping Δ such that (8‴) is commutative. Obviously the assignment (8) $\Longrightarrow$ (8″) defines a functor, and so also does the assignment (8) $\Longrightarrow$ (8′).

4. Lemma 田 and Lemma 曲. The following two lemmas are of fundamental importance in the diagram system.

Lemma 田. *Let* $A\xrightarrow{f}B\xrightarrow{g}C$ *be a sequence of translations of* 0-*sequences in* $\tilde{\mathfrak{S}}(\Lambda)$:

$$\text{(9)}\qquad \begin{array}{ccccc} \boldsymbol{A} & \to & \boldsymbol{B} & \to & \boldsymbol{C} \\ A_0 & \to & B_0 & \to & C_0 \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \to & B_1 & \to & C_1 \\ \downarrow & & \downarrow & & \downarrow \\ A_2 & \to & B_2 & \to & C_2. \end{array}$$

If, in (9), B_0C_0 *is an epimorphism and* A_1-B_1-C_1 *is exact, and if* A_2B_2 *is a monomorphism, then the induced sequence*

$$\text{(9}'\text{)}\qquad H(A_0\text{-}A_1\text{-}A_2)\xrightarrow{H(f)}H(B_0\text{-}B_1\text{-}B_2)\xrightarrow{H(g)}H(C_0\text{-}C_1\text{-}C_2)$$

is exact.

Proof. It is obvious that (9′) is a 0-sequence. Consider now an element $b_1 \in \operatorname{Ker} B_1B_2$ such that $(b_1)B_1C_1 = c_1$ is in $\operatorname{Im} C_0C_1$. Since B_0C_0 is an epimorphism, there is an element $b_0 \in B_0$ such that $(b_0)B_0C_0C_1 = c_1$, and we have

$$(b_1-(b_0)B_0B_1)B_1C_1=(b_1)B_1C_1-(b_0)B_0B_1C_1=c_1-(b_0)B_0C_0C_1=0.$$

Therefore, from the exactness of A_1-B_1-C_1 follows the existence of an element $a_1 \in A_1$ such that $(a_1)A_1B_1 = b_1-(b_0)B_0B_1$, and we have

$$(a_1)A_1A_2B_2=(a_1)A_1B_1B_2=(b_1-(b_0)B_0B_1)B_1B_2=(b_0)B_0B_1B_2=0.$$

Thus, by assumption on A_2B_2, a_1 is in Ker A_1A_2 and represents an element in $H(A_0\text{-}A_1\text{-}A_2)$ which is mapped onto the element in $H(B_0\text{-}B_1\text{-}B_2)$ represented by b_1. This proves the exactness of (9′).

Lemma 曲. *The sequence of homology factors*

$$\text{(10)}\qquad H(B_0\text{-}B_1\text{-}B_2) \xrightarrow{H(g)} H(C_0\text{-}C_1\text{-}C_2) \xrightarrow{\Delta} H(A_1\text{-}A_2\text{-}A_3) \xrightarrow{H(f)} H(B_1\text{-}B_2\text{-}B_3)$$

obtained from the diagram (8) *in Lemma* 1.5 *is exact, where f, g denote the translations*

$$f:\ \begin{array}{ccc} A_1 & \to & B_1 \\ \downarrow & & \downarrow \\ A_2 & \to & B_2 \\ \downarrow & & \downarrow \\ A_3 & \to & B_3 \end{array} \qquad\qquad g:\ \begin{array}{ccc} B_0 & \to & C_0 \\ \downarrow & & \downarrow \\ B_1 & \to & C_1 \\ \downarrow & & \downarrow \\ B_2 & \to & C_2 \end{array}$$

in (8).

Proof. Let $b_1 \in \operatorname{Ker} B_1B_2$ represent $\beta \in H(B_0\text{-}B_1\text{-}B_2)$. Then $H(g)$ image γ of β is represented by $(b_1)B_1C_1$. From the commutativity of (8″) follows then that $\Delta\gamma=0$. Conversely, let $c_1 \in \operatorname{Ker} C_1C_2$ represent $\gamma \in H(C_0\text{-}C_1\text{-}C_2)$ such that $\Delta\gamma=0$. Then there exist $a_1 \in A_1$, $b_1 \in B_1$ such that $(b_1)B_1C_1=c_1$ and $(a_1)A_1A_2B_2=(b_1)B_1B_2$. For the element $b_1-(a_1)A_1B_1 \in B_1$, we have now

$$\begin{aligned}(b_1-(a_1)A_1B_1)B_1B_2&=(b_1)B_1B_2-(a_1)A_1B_2B_2\\ &=(b_1)B_1B_2-(a_1)A_1A_2B_2=0\end{aligned}$$

and also

$(b_1-(a_1)A_1B_1)B_1C_1=(b_1)B_1C_1-(a_1)A_1B_1C_1=(b_1)B_1C_1=c_1$. This shows that γ is the $H(g)$ image of the element in $H(B_0\text{-}B_1\text{-}B_2)$ represented by $(b_1-(a)A_1B_1)$ in Ker B_1B_2, and the exactness of $H(g)\text{-}\Delta$ in (10) is proved.

Now, from the commutativity of (8″) follows that $H(f)\cdot\Delta=0$. Therefore it remains only to prove that $\operatorname{Im}\Delta \supset \operatorname{Ker} H(f)$. Let $a_2 \in \operatorname{Ker} A_2A_3$ represent an element α in $H(A_1\text{-}A_2\text{-}A_3)$ which is annihilated by $H(f)$. Then there is an element $b_1 \in B_1$ such that $(b_1)B_1B_2=(a_2)A_2B_2$. For this b_1 we have

$(b_1)B_1C_1C_2=(b_1)B_1B_2C_2=(a_2)A_2B_2C_2=0$. Thus $(b_1)B_1C_1=c_1$ represents an element in $H(C_0\text{-}C_1\text{-}C_2)$ which is mapped by Δ onto α. This completes the proof.

Corollary. *Let*

$$\begin{array}{ll} (A) & \cdots\cdots \to A_{-1} \to A_0 \to A_1 \to A_2 \to \cdots\cdots \\ (B) & \cdots\cdots \to B_{-1} \to B_0 \to B_1 \to B_2 \to \cdots\cdots \\ (C) & \cdots\cdots \to C_{-1} \to C_0 \to C_0 \to C_2 \to \cdots\cdots \end{array}$$

be infinite 0*-sequences over* Λ, *and suppose that an exact sequence of translations*

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_0 & \to & B_0 & \to & C_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_1 & \to & B_1 & \to & C_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

is given. Then the infinite sequence

$$\cdots\cdot \to H(A_{-1}\text{-}A_0\text{-}A_1) \to H(B_{-1}\text{-}B_0\text{-}B_1) \to H(C_{-1}\text{-}C_0\text{-}C_1)$$
$$\xrightarrow{\Delta} H(A_0\text{-}A_1\text{-}A_2) \to H(B_0\text{-}B_1\text{-}B_2) \to \cdots\cdot$$

is exact.

Corollary. *If either two of the above sequences* $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ *are exact, then so also is the rest.*

Remark. Obviously both assignments (9) $\Longrightarrow$ (9′), and (8) $\Longrightarrow$ (10) define functors.

5. Tensor products and groups of homomorphisms. As is stated in [E-S], tensor product $\otimes_\Lambda A$(or $A\otimes_\Lambda$) for a fixed $A \in \mathscr{G}_\Lambda$ is a covariant functor $\mathscr{G}_\Lambda \to \mathscr{G}_\Lambda$. Therefore we may consider $\otimes_\Lambda A$(or $A\otimes_\Lambda$) also as a functor of diagrams over Δ. Namely, if $\boldsymbol{D}$ is a diagram over Δ, then $\boldsymbol{D}\otimes_\Lambda A$(or $A\otimes_\Lambda \boldsymbol{D}$) is a diagram over Δ which is isomorphic to $\boldsymbol{D}$. It should be noted that, for $\boldsymbol{A} \in \mathfrak{E}(\Delta)$, $\boldsymbol{A}\otimes_\Lambda B$ is not necessarily in $\mathfrak{E}(\Delta)$. As will be seen in § 2, Λ-modules P of a special class called *Λ-projective* modules have the property that the functor $\otimes_\Lambda P$(or $P\otimes_\Lambda$) is *exact*, i.e. maps $\mathfrak{E}(\Delta)$ info $\mathfrak{E}(\Delta)$. We shall here note the following

Lemma 1.6. *Let*

(11) $$0 \to A \xrightarrow{f} B \xrightarrow{h} C \to 0$$

be an exact sequence over Λ, and let $G \in \mathscr{G}_\Lambda$. Then the induced sequence

(11′) $$A\otimes_\Lambda G \xrightarrow{f'} B\otimes_\Lambda G \xrightarrow{h'} C\otimes_\Lambda G \to 0$$

is also exact. If (11) *is direct, i.e., if* Im f *is a direct summand of* B, *then the sequence*

(11″) $$0 \to A\otimes_\Lambda G \to B\otimes_\Lambda G \to C\otimes_\Lambda G \to 0$$

is exact and direct. (11″) *is exact whenever G is Λ-free.*

Proof of this lemma is given in [E-S, p. 142, Lemma 9.8] except for the last statement. The last statement is obvious in case where G has a finite base. It is easily seen that the proof for the general case can be reduced to this special case. So we omit the proof of this lemma.

Given two Λ-modules A,B, denote by $\mathrm{Hom}_\Lambda(A, B)$ the additive group of all homomorphisms

$$\varphi:\quad A \to B$$

with addition $\varphi_1+\varphi_2$ defined by

$$(\varphi_1+\varphi_2)(a)=\varphi_1(a)+\varphi_2(a).$$

If further Λ is commutative, then $\mathrm{Hom}_\Lambda(A, B)$ has the structure of a Λ-module with the product $\lambda\varphi$ defined by

$$(\lambda\varphi)(a)=\lambda(\varphi(a))=\varphi(\lambda a).$$

For a Λ-homomorphism $A \to A'$ written as AA' we denote by $(AA')^\#$ the homomorphism

$$\mathrm{Hom}_\Lambda(A', B) \to \mathrm{Hom}_\Lambda(A, B)$$

induced by AA', while we denote by $(BB')_\#$ the homomorphism

$$\mathrm{Hom}_\Lambda(A, B) \to \mathrm{Hom}_\Lambda(A, B')$$

induced by BB': $B \to B'$. We shall consider $(AA')^\#$ as a left operator and $(BB')_\#$ as a right operator.

$\mathrm{Hom}_\Lambda(\ ,\ )$ is a functor $\mathscr{G}_\Lambda \times \mathscr{G}_\Lambda \to \mathscr{G}$ ($\mathscr{G}_\Lambda$ if Λ is commutative) contravariant in the first argument and covariant in the second. Similarly to the case of tensor products we have the following

Lemma 1.7. *Let*

$$(12) \qquad 0 \to A \to B \to C \to 0$$

be an exact sequence over Λ. Then the induced sequences

$$(12_\#) \qquad 0 \to \mathrm{Hom}_\Lambda(G, A) \xrightarrow{(AB)_\#} \mathrm{Hom}_\Lambda(G, B) \xrightarrow{(AB)_\#} \mathrm{Hom}_\Lambda(G, C),$$

$$(12^\#) \qquad 0 \to \mathrm{Hom}_\Lambda(C, G) \xrightarrow{(BC)^\#} \mathrm{Hom}_\Lambda(B, G) \xrightarrow{(AB)^\#} \mathrm{Hom}_\Lambda(A; G)$$

are exact. If (12) *is direct, then the sequences*

$$(12_{\#\#}) \qquad 0 \to \mathrm{Hom}_\Lambda(G, A) \to \mathrm{Hom}_\Lambda(G, B) \to \mathrm{Hom}_\Lambda(G, C) \to 0$$

$$(12^{\#\#}) \qquad 0 \to \mathrm{Hom}_\Lambda(C, G) \to \mathrm{Hom}_\Lambda(B, G) \to \mathrm{Hom}_\Lambda(A, G) \to 0$$

are exact and direct. $(12_{\#\#})$ *is exact whenever G is Λ-free.*

This lemma follows from [E-S, p. 148, Lemma 10.8] (The commutativity of Λ is not needed in the proof of this lemma).

Those Λ-modules G for which every sequence $(12^{\#\#})$ is exact form a special class of Λ-modules called Λ-injective modules. Those Λ-modules G for which every sequence $(12_{\#\#})$ is exact, form another special class of Λ-modules called Λ-projective modules. These special classes of Λ-modules will be treated in the next section.

§ 2. Projective modules and injective modules.

1. Definitions. A Λ-module P is called *Λ-projective* if every diagram

$$\begin{array}{c} P \\ \downarrow \\ B \to C \to 0 \end{array}$$

over Λ with exact B-C-0 is supplemented by a Λ-homomorphism PB to a commutative diagram

$$\begin{tikzcd} & P \arrow[dl] \arrow[d] & \\ B \arrow[r] & C \arrow[r] & 0\,. \end{tikzcd}$$

A Λ-module Q is called *Λ-injective* if every diagram

$$\begin{array}{c} 0 \to A \to B \\ \downarrow \quad \\ Q \quad \end{array}$$

over Λ with exact 0-A-B is supplemented by a Λ-homomorphism BQ to a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl] \\ & Q & \end{tikzcd}$$

Clearly a Λ-module G is Λ-projective (Λ-injective) if and only if (12_{**}) $((12^{**}))$ is always exact.

2. **Λ-projective modules.** It is obvious that every Λ-free module is Λ-projective. Since any Λ-module can be represented as a factor module of a Λ-free module, we have

Theorem 2.1 *Any Λ-module can be represented as a factor module of a Λ-projective module.*

We shall call an exact sequence over Λ

(13) $$0 \to A_1 \to X_0 \to A \to 0$$

a *projective representation* of A if X is Λ-projective. If in the exact sequence over Λ

(14) $$0 \to H \to G \to X \to 0$$

X is Λ-projective, then by definition of Λ-projectivity there exists $XG \in \mathscr{G}_\Lambda$ such that XGX is the identity of X. Therefore the exact sequence (14) is direct. In particular if (14) is a representation of X as a factor module of a Λ-free module G, then X is a direct factor of G. This proves that every Λ-projective module is a direct factor of some Λ-free module. Conversely every direct factor of a Λ-projective module is Λ-projective. In fact, let X' be a direct factor of a Λ-projective module X, and let XX', $X'X$ be the projection and injection respectively; $X'XX'=$identity of X'. Given a diagram

(15) $$\begin{tikzcd} & X' \arrow[d] & \\ B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

over Λ with exact B-C-0, supplement this by X, XX', and $X'X$ to

(15′) $$\begin{tikzcd} & X \arrow[d, shift left] & \\ & X' \arrow[u, shift left] \arrow[d] & \\ B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Since X is Λ-projective (15′) can be again supplemented by XB to

(15″) $$\begin{tikzcd} & X \arrow[d, shift left] \arrow[ddl] & \\ & X' \arrow[u, shift left] \arrow[d] & \\ B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

so that $XBC=XX'C$, If we define $X'B$ by $X'B=X'XB$, then we have $X'BC=X'XBC=X'XX'C=X'C$. This shows that (15) is supplemented by $X'B$ to a commutative diagram, and thus proves our assertion.

Lemma 2.1. (i) *If a Λ-projective module is represented as a factor module of some Λ-module, then it is a direct factor.*

(ii) *Any direct factor of a Λ-projective module is Λ-projective.*

(iii) *A Λ-module is Λ-projective if and only if it is a direct factor of a Λ-free module.*

(iv) *The direct sum of Λ-projective modules is Λ-projective.*

(v) *The tensor product of Λ-projective modules is Λ-projective.*

(vi) *The sequence* (11″) *in Lemma 1.6 is exact whenever G is Λ-projective.*

Proof. We have already proved (i), (ii), and 'only if' part of (iii); also 'if' part of (iii) is clear from (ii). (iv) and (v) are obvious if 'Λ-projective' is replaced by '*Λ-free*'. So we have them as easy consequences of (iii), if we represent each summand of the direct sum or each factor of the tensor product as a direct summand of a Λ-free module.

To prove (vi) we represent G as a direct summand of a Λ-free module F to obtain the translation

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & A\otimes_\Lambda B & \rightarrow & B\otimes_\Lambda G & \rightarrow & C\otimes_\Lambda G & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & A\otimes_\Lambda F & \rightarrow & B\otimes_\Lambda F & \rightarrow & C\otimes_\Lambda F & \rightarrow & 0. \end{array} \tag{16}$$

In (16) every sequence in a straight line is exact by Lemma 1.6, Therefore $A\otimes_\Lambda G\rightarrow B\otimes_\Lambda G$ must be a monomorphism.

Remark. It is an open question whether the converse of (vi) is true or not.

3. Λ-injective modules. As an example of an injective module we first note the Z-module T of real numbers mod 1. As analogue of Theorem 2.1. we have

Theorem 2.2. *Any Λ-module can be represented as a sub-module of a Λ-injective module.*

The proof of the theorem requires some preliminaries. Let M be a left Λ-module. The additive group $\mathrm{Hom}_Z(M, T)$ of homomorphisms of M into the group T of real numbers mod 1 can be given the structure of a right Λ-module if we define the multiplication $\varphi\cdot\lambda$ ($\varphi\in\mathrm{Hom}(M, T)$, $\lambda\in\Lambda$) by

$$(\varphi\cdot\lambda)(m)=\varphi(\lambda m) \qquad (m\in M).$$

In fact we have only to check $\varphi\cdot(\lambda\mu)=(\varphi\cdot\lambda)\cdot\mu$:

$$(\varphi\cdot(\lambda\mu))(m)=\varphi(\lambda\mu m)=\varphi(\lambda(\mu m))=(\varphi\cdot\lambda)(\mu m)=((\varphi\cdot\lambda)\cdot\mu)(m).$$

$\mathrm{Hom}(M, T)$ taken with this structure will be denoted by M°. If M is a right Λ-module, then $\mathrm{Hom}(M, T)$ has the structure of a left Λ-module quite analogously to the above case. This we denote also by M°.

The assignment $M \Longrightarrow M^\circ$ defines in the obvious manner a contravariant functor from the category of left Λ-modules and their Λ-homomorphisms to the category of right Λ-modules and their Λ-homomorphisms, or a functor in the opposite direction. This functor is exact, i.e., if

$$0\rightarrow A\rightarrow B\rightarrow C\rightarrow 0$$

is exact, then so is the induced sequence

$$0\rightarrow C^\circ\rightarrow B^\circ\rightarrow A^\circ\rightarrow 0,$$

because T is Z-injective.

We now prove

Lemma 2.2. *Any $m\in M$ can be considered as $\in M^{\circ\circ}$ in the following manner*

$$m(\varphi)=\varphi(m) \qquad (\varphi\in M^\circ).$$

In this way M is imbedded into $M^{\circ\circ}$ as a submodule.

Proof. Clearly M is imbedded into $M^{\circ\circ}$ as a subgroup. So it is sufficient to show that this imbedding commutes with the operation of Λ, we may assume M is a left Λ-module. Then we have

$$(\lambda\cdot m)(\varphi)=m(\varphi\cdot\lambda)=(\varphi\cdot\lambda)(m)=\varphi(\lambda m)=(\lambda m)(\varphi),$$

where m is considered as $\in M^{\circ\circ}$ in the first expression and as $\in M$ in the last. This proves the lemma.

Without any modification we may speak of Λ-projective right modules and Λ-injective right Λ-modules.

Lemma 2.3. *If P is Λ-projective, then P° is Λ-injective.*

Proof. Assuming P to be a Λ-projective right Λ-module we shall prove that, for any exact sequence

(17) $$0 \to A \to B \to C \to 0$$

over Λ, the induced sequence

(17′) $$0 \to \mathrm{Hom}_\Lambda(C, P^\circ) \to \mathrm{Hom}_\Lambda(B, P^\circ) \to \mathrm{Hom}_\Lambda(A, P^\circ) \to 0$$

is exact.

In general, for any left Λ-module M and any right Λ-module N we have

(18) $$\mathrm{Hom}_\Lambda(M, N^\circ) \cong \mathrm{Hom}_\Lambda(N, M^\circ),$$

where the isomorphism is obtained by the correspondence

$$f \Longleftrightarrow f' \quad (f \in \mathrm{Hom}_\Lambda(M, N^\circ),\ f' \in \mathrm{Hom}_\Lambda(N, M^\circ))$$

defined by

(18′) $$(f(m))(n) = (f'(n))(m) \qquad (m \in M,\ n \in N).$$

Clearly by (18′) is obtained the isomorphism of the groups of Z-homomorphisms

$$\mathrm{Hom}_Z(M, N^\circ) \cong \mathrm{Hom}_Z(N, M^\circ).$$

If $f \in \mathrm{Hom}(M, N^\circ)$ is a Λ-homomorphism, then the corresponding f' is also a Λ-homomorphism and *vice versa*, for we have

$$f'(n\lambda)(m) = (f(m))(n\lambda) = (\lambda \cdot (f(m))(n),$$
$$((f'(n)) \cdot \lambda)(m) = f'(n)(\lambda m) = (f(\lambda m))(n).$$

This proves (18).

The isomorphism (18) is natural in the sense that for a Λ-homomorphism μ: $M_1 \to M_2$ inducing μ°: $M_2{}^\circ \to M_1{}^\circ$, the diagram

(19) $$\begin{array}{ccc} \mathrm{Hom}_\Lambda(M_2, N^\circ) & \cong & \mathrm{Hom}_\Lambda(N, M_2{}^\circ) \\ \downarrow \mu^\# & & \downarrow \mu^\circ{}_\# \\ \mathrm{Hom}_\Lambda(M_1, N^\circ) & \cong & \mathrm{Hom}_\Lambda(N, M_1{}^\circ) \end{array}$$

is commutative. In fact, let $f_2 \in \mathrm{Hom}_\Lambda(M_2, N^\circ)$. Then we have

$$(\mu^\# f_2)'(n)(m_1) = (\mu^\# f_2)(m_1)(n) = f_2(\mu m_1)(n) = f_2'(n)(\mu m_1)$$
$$= \mu^\circ(f_2'(n))(m_1) = \mu_\#^\circ f_2'(n)(m_1).$$

This shows the commutativity of (19).

Now by (17) is induced the exact sequence

(17°) $$0 \to C^\circ \to B^\circ \to A^\circ \to 0,$$

and for this (17°) the sequence

(17″) $$0 \to \mathrm{Hom}_\Lambda(P, C^\circ) \to \mathrm{Hom}_\Lambda(P, B^\circ) \to \mathrm{Hom}_\Lambda(P, A^\circ) \to 0$$

is exact since P is Λ-projective. The above consideration shows now that (17″) is translation-isomorphic to (17′). So the exactness of (17′) is proved.

Now we come to the proof of Theorem 2.2. Let M be an arbitrary left Λ-module, M° the derived right Λ-module. Represent M° as a factor module of a Λ-projective right Λ-module P as

(20) $$0 \to R \to P \to M^\circ \to 0.$$

From (20) we obtain an exact sequence

(20°) $$0 \to M^{\circ\circ} \to P^\circ \to R^\circ \to 0.$$

M, being a submodule of $M^{\circ\circ}$ by Lemma 2.2, is a submodule of P° which is Λ-injective because of Lemma 2.3. This completes the proof of the theorem.

We shall call an exact sequence over Λ

(21) $$0 \to A \to X^0 \to A^1 \to 0$$

an injective representation of A if X^0 is Λ-injective.

To obtain an analogue of Lemma 2.1 we make the following consideration. If in the exact sequence over Λ

(22) $$0 \to X \to G \to H \to 0$$

X is Λ-injective then the identity mapping of X can be extended to a Λ-homomorphism GX. Therefore the exact sequence (22) is direct. Also as in the case of Λ-projective modules we can prove easily that any direct summand of a Λ-injective module is Λ-injective.

Now, if we consider Λ as a Λ-free right module, then Λ° is a Λ-injective left module. And if F is a right Λ-free module, then the Λ-injective left module F° is a direct product of Λ°'s:

$$F^\circ = \Pi \Lambda^\circ.$$

Λ-modules of this type may be classified as the analogue of Λ-free modules. We shall call them Λ-modules of type Λ°. Then it can be seen from the proof of Theorem 2.2 that every Λ-module is a submodule of some Λ-module of type Λ°. Thus we have proved

Lemma 2.4. (i) *If a Λ-injective module is represented as a submodule of some Λ-module, then it is a direct summand.*

(ii) *Any direct factor of a Λ-injective module is Λ-injective.*

(iii) *A Λ-module is Λ-injective if and only if it is a direct summand of some Λ-module of type Λ°.*

(iv) *The direct product of Λ-injective modules is Λ-injective.*

Remark. No information has been obtained about the tensor product of Λ-injective modules.

4. Lemmas on projective and injective representations.

Lemma 2.5. *Let*

(X) $$0 \to A_1 \to X \to A \to 0$$

be a projective representation of $A \in \mathscr{G}_\Lambda$ and

(B) $$0 \to \dot{B} \to B \to \overline{B} \to 0$$

an exact sequence over Λ. Then any Λ-homomorphism $A\overline{B}$ can be extended to a translation

(23) $$\begin{array}{ccccccccc} (X) & & 0 \to & A_1 & \to & X & \to & A & \to 0 \\ \downarrow & & & \downarrow & & \downarrow & & \downarrow & \\ (B) & & 0 \to & \dot{B} & \to & B & \to & \overline{B} & \to 0. \end{array}$$

If (23) *is supplemented by AB with $AB\overline{B} = A\overline{B}$, then it can be further supplemented by $X\dot{B}$ with $X\dot{B}B = XB - XAB$, and $A_1X\dot{B} = A_1\dot{B}$.*

Proof. Given $AB \in \mathscr{G}_\Lambda$, there is a Λ-homomorphism XB such that $XA\overline{B} = XA\overline{B}$, for, X is Λ-projective. Since $A_1XB\overline{B} = A_1XA\overline{B} = 0$, existence of $A_1\dot{B}$ is clear. If we have $AB\overline{B} = A\overline{B}$, then $XB - XAB$ is annihilated by $B\overline{B}$. In fact

$$(XB - XAB)\cdot B\overline{B} = XB\overline{B} - XAB\overline{B} = XB\overline{B} - XA\overline{B} = 0.$$

Thus, by Lemma 1.2, there is a Λ-homomorphism $X\dot{B}$ satisfying $X\dot{B}B = XB -$

XAB. For this $X\dot{B}$ we have

$$(A_1X\dot{B}-A_1\dot{B})\cdot\dot{B}B=A_1X\dot{B}B-A_1\dot{B}B=A_1XB-A_1XAB-A_1\dot{B}B$$
$$=A_1XB-A_1\dot{B}B=0,$$

which implies that

$$A_1X\dot{B}=A_1\dot{B},$$

and the lemma is proved.

Lemma 2.6. *Let*

$$(A) \qquad 0\to\dot{A}\to A\to\overline{A}\to 0$$

be an exact sequence over Λ, and

$$(Y) \qquad 0\to B\to Y\to B^1\to 0$$

an injective representation of $B\in\mathcal{G}_\Lambda$. Then any Λ-homomorphism $\dot{A}B$ can be extended to a translation

$$(24) \qquad \begin{array}{ll} (A) & 0\to\dot{A}\to A\to\overline{A}\to 0 \\ & \qquad\downarrow\quad\ \downarrow\quad\ \downarrow \\ (Y) & 0\to B\to Y\to B^1\to 0. \end{array}$$

If (24) *is supplemented by AB with $\dot{A}AB=\dot{A}B$, then it can be further supplemented by $\overline{A}Y$ with $A\overline{A}Y=AY-ABY$, and $\overline{A}YB^1=\overline{A}B^1$.*

The proof of this lemma is quite similar to that of the preceding lemma, and so it is omitted.

Lemma 2.7. *Let*

$$(X_0) \qquad 0\to A_1\to X_0\to A\to 0$$

be a projective representation of $A\in\mathcal{G}_\Lambda$, and

$$(Y^0) \qquad 0\to B\to Y^0\to B^1\to 0$$

an injective representation of $B\in\mathcal{G}_\Lambda$. Then

(i) *the two sub-groups*

$$(A_1X_0)^\sharp\,\mathrm{Hom}_\Lambda(X_0,B^1),\ \mathrm{Hom}_\Lambda(A_1,Y^0)(Y^0B^1)_\sharp$$

coincide and

(ii) *we have a natural isomorphism*

$$(25)\quad \mathrm{Hom}_\Lambda(A,B^1)/\mathrm{Hom}_\Lambda(A,Y^0)(Y^0B^1)_\sharp\cong\mathrm{Hom}_\Lambda(A_1,B)/(A_1X_0)^\sharp\mathrm{Hom}_\Lambda(X_0,B)$$

Proof. *Ad*(i): Given $X_0B^1\in\mathrm{Hom}_\Lambda(X_0,B^1)$, there exists $X_0Y^0\in\mathcal{G}_\Lambda$ such that $X_0Y^0B^1=X_0B^1$, for, X_0 is Λ-projective. Therefore we have

$$A_1X_0B^1=A_1X_0Y^0B^1\in\mathrm{Hom}_\Lambda(A_1,Y^0)(Y^0B^1)_\sharp.$$

Conversely, let $A_1Y^0\in\mathrm{Hom}_\Lambda(A_1,Y^0)$. Since Y^0 is Λ-injective, there is a Λ-homomorphism A_0Y^0 such that $A_1X^0Y^0=A_1Y^0$. So we obtain

$$A_1Y^0B^1=A_1X_0Y^0B^1\in(A_1X_0)^\sharp\mathrm{Hom}_\Lambda(X_0,B^1),$$

which proves (i).

Ad (ii): By Lemmas 2.5, 2.6, Both groups $\mathrm{Hom}_\Lambda(A,B^1)$ and $\mathrm{Hom}_\Lambda(A_1,B)$, and accordingly both of the factor groups in (25), can be considered as factor groups of the group $\mathbf{Hom}_\Lambda(X_0,Y^0)$ of all translations $X_0\to Y^0$ in which addition is defined in the obvious manner. The kernel of the epimorphism

$$\mathbf{Hom}_\Lambda(X_0,Y^0)\to\mathrm{Hom}_\Lambda(A,B^1)/\mathrm{Hom}_\Lambda(A,Y^0)(Y^0B^1)_\sharp$$

is obviously consisting of those translations $X_0\to Y^0$ which can be supplemented by AY^0 with $AY^0B^1=AB^1$, while the kernel of the epimorphism

$$\mathbf{Hom}_\Lambda(X_0,Y^0)\to\mathrm{Hom}_\Lambda(A_1,B)/(A_1X_0)^\sharp\,\mathrm{Hom}_\Lambda(X_0,B)$$

is consisting of those translations $X_0\to Y^0$ which can be supplemented by X_0B

with $A_1X_0B=A_1B$. Lemmas 2.5, 2.6 now assure exactly that these kernels coincide, and the lemma is proved.

§ 3. The groups Tor_n^Λ (A, B), Ext_Λ^n (A, B).

1. Resolutions. Let A be a Λ-module. A lower sequence of Λ-projective modules

$$(X_*) \qquad \cdots\cdot \to X_2 \to X_1 \to X_0 \to 0$$

with augmentation $X_0 \to A$ is called a projective resolution of A if X_* is acyclic with respect to the augmentation X_0A, i.e. if the augmented sequence

$$(X_* \to A \to 0) \qquad \cdots\cdot \to X_2 \to X_1 \to X_0 \to A \to 0$$

is exact.

An upper sequence of Λ-injective modules

$$(X^*) \qquad \cdots\cdot\ 0 \to X^0 \to X^1 \to X^2 \to \ \cdots\cdot$$

with co-augmentation $A \to X^0$ is called an injective resolution of A if X^* is acyclic with respect to the augmentation AX^0, i.e., if the augmented sequence

$$(0 \to A \to X^*) \quad 0 \to A \to X^0 \to X^1 \to X^2 \to \ \cdots\cdot$$

is exact.

If we take a series of projective representations

$$(26) \qquad \begin{array}{l} 0 \to A_1 \to X_0 \to A \to 0\,, \\ 0 \to A_2 \to X_1 \to A_1 \to 0\,, \\ \quad\vdots \qquad\qquad\qquad\qquad , \\ 0 \to A_n \to X_{n-1} \to A_{n-1} \to 0\,, \\ \quad\vdots \end{array}$$

and if we define X_nX_{n-1} by $X_nA_n \cdot A_nX_{n-1}$, $(n \geq 1)$, then the lower sequence

$$(26') \qquad (X_*) \quad \cdots\cdot \to X_2 \to X_1 \to X_0 \to 0$$

with the augmentation X_0A is clearly a projective resolution. Conversely any projective resolution (26) of A is decomposed into a series of projective representations (26). Similarly any injective resolution of A can be composed by, and decomposed into, a series of injective representations

$$(27) \qquad \begin{array}{l} 0 \to A \to X^0 \to A^1 \to 0\,, \\ 0 \to A^1 \to X^1 \to A^2 \to 0\,, \\ \quad\vdots \qquad\qquad\qquad\qquad , \\ 0 \to A^{n-1} \to X^{n-1} \to A^n \to 0\,, \\ \quad\vdots \qquad\qquad\qquad\qquad . \end{array}$$

Let now

$$\begin{array}{l} 0 \to B_1 \to E_0 \to B \to 0\,, \\ 0 \to B_2 \to E_1 \to B_1 \to 0\,, \\ \quad\vdots \qquad\qquad\qquad\qquad , \\ 0 \to B_n \to E_{n-1} \to B_{n-1} \to 0\,, \\ \quad\vdots \end{array}$$

be a series of exact sequences over Λ, and let a Λ-homomorphism AB be given. Then by repeated application of Lemma 2.5, we obtain a series of translations

$$(29)\qquad \begin{array}{ll} & 0 \to A_1 \to X_0 \to A \to 0 \\ f_0 & \quad\ \downarrow \qquad \downarrow \qquad \downarrow \\ & 0 \to B_1 \to E_0 \to B \to 0\,, \\ & 0 \to A_2 \to X_1 \to A_1 \to 0 \\ f_1 & \quad\ \downarrow \qquad \downarrow \qquad \downarrow \\ & 0 \to B_2 \to E_1 \to B_1 \to 0\,, \\ \vdots & \qquad\qquad \vdots \qquad\qquad , \\ & 0 \to A_n \to X_{n-1} \to A_{n-1} \to 0 \\ f_{n-1} & \quad\ \downarrow \qquad \downarrow \qquad \downarrow \\ & 0 \to B_n \to E_{n-1} \to B_{n-1} \to 0\,, \\ \vdots & \qquad\qquad \vdots \qquad\qquad . \end{array}$$

Let another series of translations g_0, g_1, ... of the same exact sequences as in (29) be given with the sole condition that $AfB=AgB$. Then the difference $X_0fE_0-X_0gE_0$ is annihilated by E_0B, so that there is X_0B_1 with $X_0B_1E_0=X_0fE_0-X_0gE_0$. For this X_0B_1 we prove

$$(30)\qquad A_1X_0B_1=A_1fB_1-A_1gB_1\,.$$

In fact we have

$$\begin{aligned}(A_1fB_1-A_1gB_1-A_1X_0B_1)B_1E_0&=A_1fB_1E_0-A_1gB_1E_0-A_1X_0B_1E_0\\&=A_1X_0fE_0-A_1X_0gE_0-A_1X_0fE_0+A_1X_0gE_0=0\end{aligned}$$

Since X_0 is Λ-projective there is also X_0E_1 such that $X_0E_1B_1E_0=X_0fE_0-X_0gE_0$. The Λ-homomorphism

$$X_1fE_1-X_1gE_1-X_1A_1X_0E_1$$

is then annihilated by E_1B_1, for we have

$$\begin{aligned}(X_1fE_1-X_1gE_1-X_1A_1X_0E_1)E_1B_1E_0&=X_1fE_1B_1E_0-X_1gE_1B_1E_0-X_1A_1X_0E_1B_1E_0\\&=X_1A_1fB_1E_0-X_1A_1gB_1E_0-X_1A_1X_0B_1E_0\\&=X_1A_1(A_1fB_1-A_1gB_1-A_1X_0B_1)B_1E_0=0\end{aligned}$$

Therefore there is X_1E_2 with

$$X_1E_2B_1E_1=X_1fE_1-X_1gE_1-X_1A_1X_0E_1\,,$$

for which we have also

$$(31)\qquad A_2X_1E_2B_2=A_2fB_2-A_2gB_2\,.$$

Thus we obtain inductively a series of Λ-homomorphisms

$$X_0E_1\,,\ X_1E_2\,,\ \ldots,\ X_nE_{n+1}\,,\ \ldots$$

such that

$$\begin{aligned}X_0E_1B_1E_0&=X_0fE_0-X_0gE_0\,,\\X_nE_{n+1}B_{n+1}E_n-X_nA_nX_{n-1}E_n&=X_nfE_n-X_ngE_n\qquad(n>0)\end{aligned}$$

This proves the following

Theorem 3.1. *Let*

$$(X_*\to A\to 0)\qquad \cdots\cdot\to X_2\to X_1\to X_0\to A\to 0$$

be an augmented projective resolution of A, and

$$(\boldsymbol{E})\qquad \cdots\cdot\to E_2\to E_1\to E_0\to B\to 0$$

an exact sequence over Λ. Then any Λ-homomorphism AB can be extended to a translation

$$\begin{array}{ccccccccc}\cdots\cdot\to & X_2 & \to & X_1 & \to & X_0 & \to & A & \to 0\\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots\cdot\to & E_2 & \to & E_1 & \to & E_0 & \to & B & \to 0\end{array}$$

Any two such translations extending AB are chain homotopic[6].

Similarly we can prove the following

Theorem 3.2. *Let*

$$(E)\qquad 0\to A\to E^0\to E^1\to E^2\to \cdots\cdot$$

be an exact sequence over Λ, and

$$(0\to B\to Y^*)\qquad 0\to B\to Y^0\to Y^1\to Y^2\to \cdots\cdot$$

an augmented injective resolution of $B\in\mathscr{G}_\Lambda$. Then any Λ-homomorphism AB can be extended to a translation

$$\begin{array}{ccccccccc} 0 & \to & A & \to & E^0 & \to & E^1 & \to & E^2 \to \cdots\cdot \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & B & \to & Y^0 & \to & Y^1 & \to & Y^2 \to \cdots\cdot \end{array}$$

Any two such translations f, g extending AB are cochain homotopic, i.e., there exists a series of Λ-homomorphisms

$$E^1Y^0,\ E^2Y^1,\ \ldots\ldots,\ E^{n+1}Y^n,\ \ldots\cdot$$

such that

$$E^0E^1Y^0=E^0fY^0-E^0gY^0\ ,$$
$$E^nE^{n+1}Y^n-E^nY^{n-1}Y^n=E^nfY^n-E^ngY^n\qquad (n>0)\ .$$

2. Tor_n^Λ (A, B), Ext_Λ^n (A, B) and their characteristic properties. Let X^* be a projective resolution of $A\in\mathscr{G}_\Lambda$. We denote by $H_n(X_*\otimes_\Lambda B)$ the n-th homology factor of the 0-sequence $X_*\otimes_\Lambda B$, and by $H^n(\mathrm{Hom}_\Lambda(X^*, B))$ the n-th homology factor of the upper 0-sequence $\mathrm{Hom}_\Lambda(X_*, B)$ over Z (over Λ, if Λ is commutative). If X_*' is also a projective resolution of $A'\in\mathscr{G}_\Lambda$, and if $AA'\in\mathscr{G}_\Lambda$ is given, then by Th. 3.1. we have unique homomorphisms (or Λ homomorphisms as the case may be)

(32) $\quad (AA')_*:\ \ H_n(X_*\otimes_\Lambda B)\to H_n(X_*'\otimes_\Lambda B$ (considered as a right operator) ,

$\qquad (AA')^*:\ \ H^n(\mathrm{Hom}_\Lambda(X_*, B))\to H^n(\mathrm{Hom}_\Lambda(X_*, B))$ (considered as a left operator)

satisfying the following conditions

(i) $(AA)_*:\ \ H_n(X_*\otimes_\Lambda B)\to H_n(X_*\otimes_\Lambda B)$,

$\quad (AA)^*:\ \ H^n(\mathrm{Hom}_\Lambda(X_*, B))\to H^n(\mathrm{Hom}_\Lambda(X_*, B))$

are the identities if AA is the identy.

(ii) For $(AA')_*:\quad H_n(X_*\otimes_\Lambda B)\to H_n(X_*\otimes_\Lambda B)$,

$\qquad (AA')^*:\quad H^n(\mathrm{Hom}_\Lambda(X_*', B))\to H^n(\mathrm{Hom}_n(X_*, B))$,

$\qquad (A'A'')_*:\quad H_n(X_*'\otimes_\Lambda B)\to H_n(X_*''\otimes_\Lambda B)$,

$\qquad (A'A'')^*:\quad H^n(\mathrm{Hom}_\Lambda(X_*'', B))\to H^n(\mathrm{Hom}_\Lambda(X_*', B))$,

and

$\qquad (AA'A'')_*:\quad H_n(X_*\otimes_\Lambda B)\to H_n(X_*''\otimes_\Lambda B)$,

$\qquad (AA'A'')^*:\quad H^n(\mathrm{Hom}_\Lambda(X_*'', B))\to H^n(\mathrm{Hom}_\Lambda(X_*, B))$,

we have

$$(AA')_*\cdot(A'A'')_*=(AA'A'')_*\ ,$$
$$(AA')^*\circ(A'A'')^*=(AA'A'')^*\ .$$

Thus we may consider $H_n(X_*\otimes_\Lambda B)$ and $H^n(\mathrm{Hom}_\Lambda(X_*, B))$ as invariants of the pair (A, B), and we write $\mathrm{Tor}_n^\Lambda(A, B)$ for $H_n(X_*\otimes_\Lambda B)$, $\mathrm{Ext}_\Lambda^n(A, B)$ for $H^n(\mathrm{Hom}_\Lambda(X_*, B))$. We now verify the characteristic properties I, II, III, IV, I′, II′, III′, IV′ and V listed in the introduction of this paper. The homomorphisms $(AA')_*$, $(AA')^*$ can be regarded as giving homomorphisms

6) Cf. H. Cartan, Seminaire de topologie algèbrique. 1950–51.

$$(AA')_*:\quad \mathrm{Tor}_n^\Lambda(A,\ B)\to\mathrm{Tor}_n^\Lambda(A',\ B)$$
$$(AA')^*:\quad \mathrm{Ext}_\Lambda^n(A',\ B)\to\mathrm{Ext}_\Lambda^n(A,\ B)\ ,$$

satisfying:

I–1) $(AA)_*$, $(AA)^*$ are the identities of AA is the identity map.

I–2) $(AA')_*\cdot(A'A'')_*=(AA'A'')_*$, $(AA')^*\circ(A'A'')^*=(AA'A'')^*$.

On the other hand, a Λ-homomorphism BB' induces translations

$$X_*\otimes_\Lambda B\to X_*\otimes B'$$
$$\mathrm{Hom}_\Lambda(X_*,\ B)\to\mathrm{Hom}_\Lambda(X_*,\ B)$$

and thus induces homomorphisms

$$(33)\qquad {}_*(BB'):\quad H_n(X_*\otimes_\Lambda B)\to H_n(X_*\otimes_\Lambda B')$$
$$^*(BB'):\quad H^n(\mathrm{Hom}_\Lambda(X_* B))\to H^n(\mathrm{Hom}_\Lambda(X_*,\ B'))$$

Clearly those homomorphisms in (33) commute with the homomorphisms in (32) and therefore they can be regarded as giving homomorphisms

$${}_*(BB'):\quad \mathrm{Tor}_n^\Lambda(A,\ B)\to\mathrm{Tor}_n^\Lambda(A,\ B')$$
$$^*(BB'):\quad \mathrm{Ext}_n^\Lambda(A,\ B)\to\mathrm{Ext}_n^\Lambda(A,\ B')$$

satisfying:

I′–1) ${}_*(BB)$, ${}_*(BB)$ are the identities of BB is the identity map.

I′–2) ${}_*(BB')\cdot{}_*(B'B'')={}_*(BB'B'')$, $^*(BB')\cdot{}^*(B'B'')$.

V) $(AA')_*$ and ${}_*(BB')$ commute with each other, so do also $(AA')^*$ and $^*(BB')$.

Now let

$$(B)\qquad 0\to\dot{B}\to B\to\bar{B}\to 0$$

be an exact sequence over Λ. Since each X_n in the projective resolution X_* of A is Λ-projective, the sequences

$$0\to X_n\otimes_\Lambda\dot{B}\to X_n\otimes_\Lambda B\to X_n\otimes_\Lambda\bar{B}\to 0$$
$$0\to\mathrm{Hom}_\Lambda(X_n,\ \bar{B})\to\mathrm{Hom}_\Lambda(X_n,\ B)\to\mathrm{Hom}_\Lambda(X_n,\ B)\to 0$$

are both exact, i.e. the sequences of translations

$$0\to X_*\otimes_\Lambda\dot{B}\to X_*\otimes_\Lambda B\to X_*\otimes_\Lambda\bar{B}\to 0$$
$$0\to\mathrm{Hom}_\Lambda(X_*,\ \bar{B})\to\mathrm{Hom}_\Lambda(X_*,\ B)\to\mathrm{Hom}_\Lambda(X_*\dot{B})\to 0$$

are exact. Therefore, by Lemma 田 and 曲 we obtain homomorphisms

$${}_*\partial:\quad \mathrm{Tor}_n^\Lambda(A,\ \bar{B})\to\mathrm{Tor}_{n-1}^\Lambda(A,\ \dot{B})\qquad (n=1,\ 2,\ \ldots.)\ ,$$
$$^*\delta:\quad \mathrm{Ext}_\Lambda^n(A,\ \bar{B})\to\mathrm{Ext}_\Lambda^{n+1}(A,\ \dot{B})\qquad (n=0,\ 1,\ \ldots.)\ ,$$

and exact sequences

$$\cdots\cdot\to\mathrm{Tor}_n^\Lambda(A,\ \dot{B})\ {}_*\xrightarrow{(\dot{B}B)}\mathrm{Tor}_n^\Lambda(A,B)\ {}_*\xrightarrow{(B\bar{B})}\mathrm{Tor}_n^\Lambda(A,\ \bar{B})$$
$$\mathrm{Tor}^\Lambda(A,\ \boldsymbol{B})\,{}_*\xrightarrow{\partial}\mathrm{Tor}_{n-1}^\Lambda(A,\ \dot{B})\to\mathrm{Tor}_{n-1}^\Lambda(A,\ B)\to\cdots\cdot\to\mathrm{Tor}_0^\Lambda(A,\ \dot{B})$$
$$\to\mathrm{Tor}_0^\Lambda(A,\ B)\to\mathrm{Tor}_0^\Lambda(A,\ \bar{B})\to 0$$
$$0\to\mathrm{Ext}_\Lambda^0(A,\ \dot{B})\to\mathrm{Ext}_\Lambda^0(A,\ B)\to\mathrm{Ext}_\Lambda^0(A,\ \bar{B})\to\ \cdots\cdot$$
$$\mathrm{Ext}_\Lambda(A,\ \boldsymbol{B})\to\mathrm{Ext}_\Lambda^n(A,\ \dot{B})\to\mathrm{Ext}_\Lambda^n(A,\ B)\to\mathrm{Ext}_\Lambda^n(A,\ \bar{B})\to$$
$$\mathrm{Ext}_\Lambda^{n+1}(A,\ \dot{B})\to\mathrm{Ext}_\Lambda^{n+1}(A,\ B)\to\ \cdots\cdot$$

Naturality of the functors $\boldsymbol{B}\Longrightarrow\mathrm{Tor}^\Lambda(A,\ \boldsymbol{B})$
$\boldsymbol{B}\Longrightarrow\mathrm{Ext}_\Lambda(A,\ \boldsymbol{B})$

is obvious. Thus II′ is proved. III′ is obvious since $\otimes_\Lambda B$ is an exact functor if B is Λ-projective, and since $\mathrm{Hom}_\Lambda(A,\ B)$ is an exact (contravariant) functor if B is Λ-injective. III is also clear, for if A is Λ-projective we can take as a projective resolution of A the sequence

$$\cdots\cdots \to 0 \to 0 \to A \to 0$$

for which augmentation is the identity mapping of A.

The proof of II requires some preliminary considerations. Let

$$(A) \qquad 0 \to \dot{A} \to A \to \bar{A} \to 0$$

be an exact sequence over Λ, and let

$$0 \to \dot{A}_1 \to \dot{X}_0 \to \dot{A} \to 0$$
$$0 \to \bar{A}_1 \to \bar{X}_0 \to \bar{A} \to 0$$

be projective representations of $\dot{A}$ and of $\bar{A}$. Then since $\bar{X}_0$ is Λ-projective there exists $\bar{X}_0A$ such that $\bar{X}_0A\bar{A}=\bar{X}_0\bar{A}$, Put now $X_0=\dot{X}_0\oplus\bar{X}_0$ (direct sum) and denote by $\dot{X}_0X_0$, $\bar{X}_0X_0$, $X_0\dot{X}_0$, $X_0\bar{X}_0$ the injections and the projections for that direct sum. If we define X_0A by

$$X_0A=X_0\dot{X}_0\dot{A}A+X_0\bar{X}_0A\ ,$$

then X_0A is an epimorphism and the diagram

$$(34) \qquad \begin{array}{ccccccccc} 0 & \to & \dot{X}_0 & \to & X_0 & \to & \bar{X}_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \dot{A} & \to & A & \to & \bar{A} & \to & 0 \end{array}$$

is commutative. In fact, let a be an element in A. Then there exists $\bar{x}_0$ in $\bar{X}_0$ such that $(\bar{X}_0)\bar{X}_0\bar{A}=(a)A\bar{A}$. Since

$$(a-(\bar{x}_0)\bar{X}_0A)A\bar{A}=(a)A\bar{A}-(\bar{x}_0)\bar{X}_0A\bar{A}=(a)A\bar{A}-(\bar{x}_0)\bar{X}_0\bar{A}=0$$

and since $\dot{X}_0\dot{A}$ is an epimorphism, $\dot{X}_0$ contains an element $\dot{x}_0$ such that

$$(\dot{x}_0)\dot{X}_0\dot{A}A=a-(\bar{x}_0)\bar{X}_0A\ ,$$

and so we have

$$\begin{aligned}((\dot{x}_0)\dot{X}_0\dot{X}_0+(\bar{x}_0)\bar{X}_0X_0)X_0A&=(\dot{x}_0)\dot{X}_0X_0A+(\bar{\ }_0)\bar{X}_0XA\\ &=(\dot{x}_0)\dot{X}_0X_0\dot{X}_0\dot{A}A+(\dot{x}_0)\dot{X}_0X_0\bar{X}_0A+(\bar{x}_0)\bar{X}_0X_0\dot{X}_0\dot{A}A+(\bar{x}_0)\bar{X}_0X_0\bar{X}_0A\\ &=(\dot{x}_0)\dot{X}_0\dot{A}A+(\bar{x}_0)\bar{X}_0A=a\ .\end{aligned}$$

This proves that X_0A is an epimorphism. Commutativity of the diagram (34) can be shown as follows.

$$\dot{X}_0X_0A=\dot{X}_0X_0\dot{X}_0\dot{A}A+\dot{X}_0X_0\bar{X}_0A=\dot{X}_0\dot{A}A$$
$$X_0A\bar{A}=X_0\dot{X}_0\dot{A}A\bar{A}+X_0\bar{X}_0A\bar{A}=X_0\bar{X}_0\bar{A}\ .$$

Now, if we put trivial sequences $0\to 0\to 0\to 0\to 0$ above and under (34), and if we apply the corollary to Lemmas 田, 曲, to the so obtained diagram we see that (34) can be supplemented to a commutative diagram

$$(34') \qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_1 & \to & A_1 & \to & A_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \dot{X}_0 & \to & X_0 & \to & \bar{X}_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \dot{A} & \to & A & \to & \bar{A} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where each sequence on a straight line is exact. Repeated application of the process to obtain (34′) from A will lead us to the following conclusion:

Lemma 3.1. *Given any exact sequence*

$$0\to\dot{A}\to A\to\bar{A}\to 0$$

there exist projective resolutions $\dot{X}_*$ *of* $\dot{A}$, X_* *of* A, $\bar{X}^*$ *of* $\bar{A}$, *and a sequence of translations*

$$\text{(34'')}\qquad \begin{array}{ccccccccc} 0 & \to & \dot{X}_* & \to & X_* & \to & \bar{X}_* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \dot{A} & \to & A & \to & \bar{A} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

which is exact. Since each $\bar{X}_n$ *is* Λ*-projective, each sequence* $0\to X_n\to X_n\to\bar{X}_n\to 0$ *appearing in* (34″) *is direct.*

Therefore for an arbitrary Λ-module B we obtain exact sequences of translations

$$0\to\dot{X}_*\otimes_\Lambda B\to X_*\otimes_\Lambda B\to\bar{X}_*\otimes_\Lambda B\to 0$$

$$0\to\mathrm{Hom}_\Lambda(\bar{X}_*, B)\to\mathrm{Hom}_\Lambda(X_*, B)\to\mathrm{Hom}_\Lambda(\dot{X}_*, B)\to 0\ .$$

These exact sequences and Lemmas 田, 曲 clearly prove the property II.

Finally, if we apply $\otimes_\Lambda B$ and $\mathrm{Hom}_\Lambda(\quad, B)$ to (26), then by Lemmas 1.6, 1.7, we obtain exact sequences

$$\begin{array}{l} A_1\otimes_\Lambda B\to X_0\otimes_\Lambda B\to A\otimes_\Lambda B\to 0\ , \\ A_2\otimes_\Lambda B\to X_1\otimes_\Lambda B\to A_1\otimes_\Lambda B\to 0\ , \\ \qquad\qquad\vdots\qquad\qquad\qquad\quad , \\ A_n\otimes_\Lambda B\to X_{n-1}\otimes_\Lambda B\to A_{n-1}\otimes_\Lambda B\to 0\ , \\ \qquad\qquad\vdots\qquad\qquad\qquad\quad , \end{array}$$

and

$$\begin{array}{l} 0\to\mathrm{Hom}_\Lambda(A, B)\to\mathrm{Hom}_\Lambda(X_0, B)\to\mathrm{Hom}_\Lambda(A_1, B)\ , \\ 0\to\mathrm{Hom}_\Lambda(A_1, B)\to\mathrm{Hom}_\Lambda(X_1, B)\to\mathrm{Hom}_\Lambda(A_2, B)\ , \\ \qquad\qquad\qquad\vdots\qquad\qquad\qquad\qquad\quad , \\ 0\to\mathrm{Hom}_\Lambda(A_{n-1}, B)\to\mathrm{Hom}_\Lambda(X_{n-1}, B)\to\mathrm{Hom}_\Lambda(A_n, B)\ . \end{array}$$

Therefore we have

$$\begin{aligned} \text{Image of }(X_{n+1}\otimes B\to X_n\otimes B) &= \text{Image of }(A_{n+1}\otimes B\to X_n\otimes B) \\ &= \text{Kernel of }(X_n\otimes B\to A_n\otimes B)\ , \\ \text{Kermel of }(\mathrm{Hom}_\Lambda(X_n, B)\to\mathrm{Hom}_n(X_{n+1}, 0) & \\ &= \text{Kernel of }(\mathrm{Hom}_\Lambda(X_n, B)\to\mathrm{Hom}_\Lambda(A_n, B)) \\ &= \text{Image of }(\mathrm{Hom}_\Lambda(A_{n+1}, B)\to\mathrm{Hom}_\Lambda(X_n, B)) \end{aligned}$$

This proves

$$\begin{array}{ll} \mathrm{Tor}_n^\Lambda(A, B)\cong A\otimes_\Lambda B\ , & \mathrm{Ext}_\Lambda^0(A, B)=\mathrm{Hom}_\Lambda(A, B)\ , \\ \mathrm{Tor}_n^\Lambda(A, B)\cong\text{Kernel of }(A_n\otimes_\Lambda B\to X_{n-1}\otimes_\Lambda B) & (n>0)\ , \\ \mathrm{Ext}_\Lambda^n(A, B)\cong\text{Cokernel of }(\mathrm{Hom}_\Lambda(X_{n-1}, B)\to\mathrm{Hom}_\Lambda(A_n, B)) & (n>0) \end{array}$$

Obviously, these isomorphisms are natural and the properties III, III′ are proved.

3. $\mathrm{Tor}_n^\Lambda(A, B)$, $\mathrm{Ext}_\Lambda^n(A, B)$, and resolutions of B.

Let

$$(Y_0) \qquad 0 \to B_1 \to Y_0 \to B \to 0\,,$$

be a projective representation of B. Then, in the exact sequence $\mathrm{Tor}^{\Lambda}(A, Y_0)$, every third group $\mathrm{Tor}^{\Lambda}_n(A, Y_0)$ vanishes except $\mathrm{Tor}^{\Lambda}_0(A, Y_0)=A\otimes_{\Lambda}Y_0$. Therefore we have natural isomorphisms

$$\begin{aligned}\mathrm{Tor}_1{}^{\Lambda}(A, B)&=\text{Kernel of }(\mathrm{Tor}_0{}^{\Lambda}(A, B_1)\to\mathrm{Tor}_0{}^{\Lambda}(A, Y_0))\\&=\text{Kernel of }(A\otimes_{\Lambda}B_1\to A\otimes_{\Lambda}Y_0)\\ \mathrm{Tor}^{\Lambda}_n(A, B)&=\mathrm{Tor}^{\Lambda}_{n-1}((A, B_1) \qquad (n>1)\,.\end{aligned}$$

Thus if

$$\begin{gathered}0 \to B_1 \to Y_0 \to B \to 0\\ 0 \to B_2 \to Y_1 \to B_1 \to 0\end{gathered}$$

is the series of projective representations giving a projective resolution Y_* of B, then we have

$$\begin{aligned}\mathrm{Tor}^{\Lambda}_n(A, B)&=\mathrm{Tor}^{\Lambda}_{n-1}(A, B)=\cdots\cdots=\mathrm{Tor}^{\Lambda}_1(A, B_{n-1})\\&=\text{Kernel of }(A\otimes_{\Lambda}B_n\to A\otimes_{\Lambda}Y_{n-1})\,, \qquad (n>0)\end{aligned}$$

But, the last group being naturally isomorphic to $H_n(A\otimes_{\Lambda}Y_*)=H_n(Y_*\otimes_{\Lambda}A)$ $=\mathrm{Tor}^{\Lambda}_n(B, A)$, we obtain

Theorem 3.3. $\mathrm{Tor}^{\Lambda}_n(A, B)=H_n(X_*\otimes_{\Lambda}B)=H_n(Y_*\otimes_{\Lambda}A)=\mathrm{Tor}^{\Lambda}_n(B, A)$.

Similarly we can also prove the analogue of this theorem, namely

Theorem 3.4. $\mathrm{Ext}^n_{\Lambda}(A, B)$ *is naturally isomorphic to the n-th homology factor* $H^n(\mathrm{Hom}_{\Lambda}(A, Y^*))$ *of the upper* 0*-sequence* $\mathrm{Hom}_{\Lambda}(A, Y^*)$, *where* Y^* *is an arbitrary injective resolution of* B.

For later purpose we shall give to this theorem a proof of somewhat different nature from the proof given above. This proof is based on lemma 2.5. stated in the last paragraph of § 2. As we have seen earlier at the end of the foregoing paragraph, we have natural isomorphisms

$$\mathrm{Ext}^n_{\Lambda}(A, B)=\mathrm{Hom}_n(A_n, B)/(A_nX_{n-1})^{\#}\mathrm{Hom}_{\Lambda}(X_{n-1}, B)\,,$$

but, the two conclusions in Lemma 27 give us in turn a series of natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\Lambda}(A_n, B)/A_nX_{n-1})^{\#}\,\mathrm{Hom}_{\Lambda}(X_{n-1}, B)&=\mathrm{Hom}_{\Lambda}(A_{n-1}, B^1)/\mathrm{Hom}_{\Lambda}(A_{n-1}, Y^0)(Y^0B^1)_{\#}\\&=\mathrm{Hom}_{\Lambda}(A_{n-1}, B^1)/(A_{n-1}X_{n-2})^{\#}\,\mathrm{Hom}_{\Lambda}(X_{n-2}, B^1)\\&=\mathrm{Hom}_{\Lambda}(A_{n-1}, B^2)/\mathrm{Hom}_{\Lambda}(A_{n-1}, Y^1)(Y^1B^2)_{\#}=\cdots\cdots\\&=\cdots\cdots=\mathrm{Hom}_{\Lambda}(A_{n-i}, B^i)/\mathrm{Hom}_{\Lambda}(A_{n-i}Y_{i-1})(Y_{i-1}B^i)_{\#}\\&=\mathrm{Hom}_{\Lambda}(A_{n-i}, B^i)/(A_{n-i}X_{n-i-1})^{\#}\,\mathrm{Hom}_{\Lambda}(X_{n-i-1}, B^i)\\&=\mathrm{Hom}_{\Lambda}(A_{n-i-1}, B^{i+1})/\mathrm{Hom}_{\Lambda}(A_{n-i+1}, Y^i)(Y^iB^{i+1})=\cdots\cdots\\&\quad\cdots\cdots\\&=\cdots\cdots=\mathrm{Hom}_{\Lambda}(A, B^n)/\mathrm{Hom}_{\Lambda}(A, Y^{n-1})(Y^{n-1}B^n)_{\#}\\&=H^n(\mathrm{Hom}_{\Lambda}(A, Y^*))\,.\end{aligned}$$

This proves the theorem.

Remark. From this proof it is readily seen that if

$$\begin{array}{cccccccccccc}(X_n) & 0 & \to & A_n & \to & X_{n-1} & \to\cdots\cdots\to & X_0 & \to & A & \to & 0\\ \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ (Y^n) & 0 & \to & B & \to & Y_0 & \to\cdots\cdots\to & Y^{n-1} & \to & B^n & \to & 0\end{array}$$

is a translation, then $AB^n\in\mathrm{Hom}_{\Lambda}(A, B^n)$ represents the same element in $\mathrm{Ext}^n_{\Lambda}(A, B)=\mathrm{Hom}_{\Lambda}(A, B^n)/\mathrm{Hom}_{\Lambda}(A, Y^{n-1})(Y^{n-1}B^n)_{\#}$ as A_nB does in $\mathrm{Ext}^n_{\Lambda}(A, B)$ $=\mathrm{Hom}\,(A_n, B)/(A_nX_{n-1})^{\#}\mathrm{Hom}\,(X_{n-1}, B)$.

It is not hard to see that $\mathrm{Tor}_n^{\Lambda}(A, B)$, $\mathrm{Ext}_{\Lambda}^n(A, B)$ can be also defined as the n-th homology factor of the lower 0-sequence $X_*\otimes_\Lambda Y_*$ and that of the upper 0-sequence $\mathrm{Hom}_\Lambda(X_*, Y^*)$, where X_*, Y_* are projective resolutions of A and B respectively, Y^* an injective resolution of B, and where the complexes $X_*\otimes_\Lambda Y_*$, $\mathrm{Hom}_\Lambda(Y_*, Y^*)$ are defined in the usual way. However we shall not go further in this direction.

4. $\mathrm{Ext}_{\Lambda}^n(A, B)$ and the n-fold extensions of B by A.

Let A, B be arbitratry Λ-modules fixed once for all. We call any exact sequence over Λ of the form

$$(E_n)\qquad 0\to B\to E_{n-1}\to\cdots\cdots\to E_0\to A\to 0\qquad (n\geqq 1)$$

an *n-fold extension of B by A*, and denote the category consisting of all n-fold extensions of B by A and of all possible translations between such n-fold extensions that give identities both on A and on B by $\mathcal{E}_n(A, B)$. The set of objects in this category, i.e., the set of all n-fold extensions of B by A will be denoted by $\boldsymbol{E}_n(A, B)$. For two n-fold extensions $\boldsymbol{E}_n$, $\boldsymbol{E}_n'\in\boldsymbol{E}_n(A, B)$ we shall write $\boldsymbol{E}_n\simeq\boldsymbol{E}_n'$ if there exists either a mapping $\boldsymbol{E}_n'\boldsymbol{E}_n\in\mathcal{E}_n(A, B)$ or a mapping $\boldsymbol{E}_n\boldsymbol{E}_n'\in\mathcal{E}_n(A, B)$, $\boldsymbol{E}_n\sim\boldsymbol{E}_n'$ if there is a finite series of n-fold extensions $\boldsymbol{E}_n=\boldsymbol{E}_n^0$, $\boldsymbol{E}_n^1$, ..., $\boldsymbol{E}_n^k=\boldsymbol{E}_n$ $\in\boldsymbol{E}_n(A, B)$ such that $\boldsymbol{E}_n^i\simeq\boldsymbol{E}_n^{i+1}$ $(i=0, \ldots, k-1)$. Clearly $\sim$ is an equivalence relation, by which the elements of $\boldsymbol{E}_n(A, B)$ are classified into equivalence classes.

Let now X_* be a projective resolution of A. Then, by Theorem 3.1, the identity of A can be extended to a translation

$$\begin{array}{lccccccccccc} & (X_n) & 0\to & A_n & \to & X_{n-1} & \to\cdots\cdots\to & X_0 & \to & A & \to 0 \\ (34') & & & \downarrow & & \downarrow & & \downarrow & & \| & \\ & (E_n) & 0\to & B & \to & E_{n-1} & \to\cdots\cdots\to & E_0 & \to & A & \to 0\,. \end{array}$$

If f_1, f_2 are such translations extending the identity mapping of A, then Theorem 3.1 states further that f_1, f_2 are chain homotopic, i.e., there exist Λ-homomorphisms X_0E_1, X_1E_2, ... $X_{n-2}E_{n-1}$, $X_{n-1}B$, such that

$$X_0E_1E_0=X_01E_0-X_02E_0\,,$$
$$X_iE_{i+1}E_i+X_iX_{i-1}E_i=X_i1E_i-X_i2E_i\qquad (i=1, \ldots, n-2)\,,$$
$$X_{n-1}BE_{n-1}+X_{n-1}X_{n-2}E_{n-1}=X_{n-1}1E_{n-1}-X_{n-1}2E_{n-1}\,.$$

Now, as we have

$$A_nX_{n-1}BE_{n-1}=A_nX_{n-1}1E_{n-1}-A_nX_{n-1}2E_{n-1}-A_nX_{n-1}X_{n-2}E_{n-1}$$
$$=A_n1BE_{n-1}-A_n2BE_{n-1}\,,$$

the difference homomorphism A_n1B-A_n2B lies in the subgroup $(A_nX_{n-1})^\sharp\,\mathrm{Hom}_\Lambda(X_{n-1}, B)$ of $\mathrm{Hom}_\Lambda(A_n, B.)$ Therefore in this way we obtain a mapping

$$(35)\qquad \boldsymbol{E}_n(A, B)\to\mathrm{Hom}_\Lambda(A_n, B)/(A_nX_{n-1})^\sharp\,\mathrm{Hom}_\Lambda(X_{n-1}, B(=\mathrm{Ext}_\Lambda^n(A, B)$$

If X_*' is another projective resolution, and if $\boldsymbol{E}_n\boldsymbol{E}_n'$ is a translation $\in\mathcal{E}_n(A, B)$

$$\begin{array}{lcccccccccc} (E_n) & 0\to & B & \to & E_{n-1} & \to\cdots\cdots\to & E_0 & \to & A & \to 0 \\ \downarrow & & \| & & \downarrow & & \downarrow & & \| & \\ (E_n') & 0\to & B & \to & E'_{n-1} & \to\cdots\cdots\to & E_0' & \to & A & \to 0 \end{array}$$

of $\boldsymbol{E}_n$ into another n-fold extension $\boldsymbol{E}_n'\in\mathcal{E}_n(A, B)$, then the identity of A can be extended to a translation

$$\begin{array}{lcccccccccc} (X_n') & 0\to & A_n' & \to & X'_{n-1} & \to\cdots\cdots\to & X_0' & \to & A & \to 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & \\ (X_n) & 0\to & A_n & \to & X_{n-1} & \to\cdots\cdots\to & X & \to & A & \to 0 \end{array}$$

Then each of the translations $\boldsymbol{X}_n\boldsymbol{E}_n$ in (34′), $\boldsymbol{X}_n'\boldsymbol{X}_n\boldsymbol{E}_n$, $\boldsymbol{X}_n\boldsymbol{E}_n\boldsymbol{E}_n'$ extends the identity of A. This shows firsty that the mapping (35) does not depend on the special choice of the projective resolution X_* of A, and secondly that equivalent n-fold extension of B by A are mapped onto the same element of $\mathrm{Ext}^n_\Lambda(A, B)$.

We now prove the following

Theorem 3.5. (**Classification Theorem**) *The mapping* (35) *induces a one-to-one correspondence between the equivalence classes of the n-fold extensions* $E_n(A, B)$ *of* B *by* A, *and the elements of* $\mathrm{Ext}^n_\Lambda(A, B)$

As we have shown that equivalent n-fold extensions are mapped onto the same element of $\mathrm{Ext}^n(A, B)$, (35) defines a mapping χ_n of the set $\overline{E}_n(A, B)$ of equivalence classes of $E_n(A, B)$ into the group $\mathrm{Ext}^n_\Lambda(A, B)$. In the following paragraphs we shall prove that χ_n is onto and one-to-one.

5. Proof of the classification theorem, I—Construction.

In this paragraph we shall define a mapping

$$\gamma_n:\quad \mathrm{Hom}_\Lambda(A_n, B) \to E_n(A, B)$$

such that $\chi_n \circ \gamma_n$ is the canonical mapping $\mathrm{Hom}_\Lambda(A_n, B) \to \mathrm{Ext}^n_\Lambda(A, B)$.

Construction γ_1. Let $A_1 f B \in \mathrm{Hom}_\Lambda(A_1, B)$, put $W = X_0 \oplus B$ (direct sum), and define a Λ-homomorphism $A_1 W$ by

$$A_1 W = A_1 X_0 \oplus (-A_1 f B) = A_1 X_0 W - A_1 f B W\,.$$

Cleary $A_1 W$ is a monomorphism, and we have

$$A_1 W X_0 = A_1 X_0 W X_0 - A_1 f B W X_0 = A_1 X_0 W X_0 = A_1 X_0\,.$$

Therefore we may identify $A_1 = Im A_1 W$ to obtain the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & 0 & \to & A & = & A_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B & \to & W & \to & X_0 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ & & B & & E_f & & A & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{36}$$

where we have put $E_f = W/A_1$.

Since (36) in commutative it is supplemented uniquely by BE_f and $E_f A$ such that

$$BWE_f = BE_f\,,\qquad WX_0A = WE_fA\,.$$

Then, by the corollary to Lemmas 田 & 曲, the sequence over Λ

$$(E_f)\qquad 0 \to B \to E_f \to A \to 0$$

is exact, i.e., E_f is a 1-fold extension of B by A; and we now define γ_1: $\mathrm{Hom}_\Lambda(A_1, B) \to E_1(A, B)$ by

$$\gamma_1(f) = E_f\,.$$

Then, putting $X_0 E_f = X_0 W E_f$, we have

$$A_1 f B E_f = A_1 f B W E_f = A_1 X_0 W E_f - A_1 W E_f = A_1 X_0 W E_f = A_1 X_0 E_f\,,$$
$$X_0 E_f A = X_0 W E_f A = X_0 W X_0 A = X_0 A\,.$$

Thus we obtain a translation

$$\begin{array}{ccccccccc} 0 & \to & A_1 & \to & X_0 & \to & A & \to & 0 \\ & & \downarrow f & & \downarrow & & \| & & \\ 0 & \to & B & \to & E_f & \to & A & \to & 0 \end{array}$$

proving that $\chi_1(\gamma_1(f))$ is represented by $f \in \mathrm{Hom}_\Lambda(A_1, B)$.

Constructions γ_n. Let now $f \in \mathrm{Hom}_\Lambda(A_n, B)$. Applying the above construction we obtain the following commutative diagrams

(37)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & A_n \arrow[r, equal] \arrow[d] & A_n \arrow[d] & \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & 0 \quad \text{(direct)} \\
0 \arrow[r] & B \arrow[r] & E_f \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & 0 \, , \\
 & & 0 & 0 &
\end{tikzcd}$$

(38)
$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r] \arrow[d, "f"] & X_{n-1} \arrow[r] \arrow[d] & A_{n-1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & E_f \arrow[r] & A_{n-1} \arrow[r] & 0 \, ,
\end{tikzcd}$$

satisfying

(39)
$$\begin{gathered} A_n W = A_n X_{n-1} W - A_n f B W \\ X_{n-1} E_f = X_{n-1} W E_f \end{gathered}$$

The translation (38) can be extended now to

(40)
$$\begin{tikzcd}
 & 0 \arrow[r] & A_n \arrow[r] \arrow[d, "f"] & X_{n-1} \arrow[r] \arrow[d] & X_{n-2} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & X_0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
(E_f^n) & 0 \arrow[r] & B \arrow[r] & E_f \arrow[r] & X_{n-2} \arrow[r] & \cdots \arrow[r] & X^0 \arrow[r] & A \arrow[r] & 0 \, .
\end{tikzcd}$$

The second sequence E_f of (40) being an n-fold extension of B by A, we define γ_n: $\mathrm{Hom}_\Lambda(A_n, B) \to E_n(A, B)$ by

$$\gamma_n(f) = E_f^n \, .$$

Since (40) is a translation, $\chi_n \gamma_n(f)$ is clearly represented by $f \in \mathrm{Hom}_\Lambda(A_n, B)$.

Thus we have proved that χ_n is one-to-one ($n = 1, 2, \ldots.$)

6. Proof of the classification theorem, II. We must prove finally that the mapping χ_n is one-to-one ($n=1, 2, \ldots.$). This will be done if we prove the following

Lemma 3.2. *Let*

$$(E_n') \qquad 0 \to B \to E'_{n-1} \to E'_{n-2} \to \cdots \to E_0' \to A \to 0$$

be an n-fold extension of B by A, and let the identity mapping of A be extended to a translation

(41)
$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r] \arrow[d, "g"] & X_{n-1} \arrow[r] \arrow[d] & X_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & E'_{n-1} \arrow[r] & E'_{n-2} \arrow[r] & \cdots \arrow[r] & E_0' \arrow[r] & A \arrow[r] & 0 \, .
\end{tikzcd}$$

If $A_n f B \in \mathrm{Hom}_\Lambda(A_n, B)$ *lies in the same coset of* $(A_n X_{n-1})^* \mathrm{Hom}_\Lambda(X_{n-1}, B)$, *i.e., if there is a Λ-homomorphism* X_{n-1} B *such that*

$$A_n f B = A_n g B + A_n X_{n-1} B \, ,$$

then there exists a translation

(42)
$$\begin{tikzcd}
(E_f^n) \arrow[d] & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E_f \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \cdots \arrow[r] & X_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
(E_n') & 0 \arrow[r] & B \arrow[r] & E'_{n-1} \arrow[r] & E'_{n-2} \arrow[r] & \cdots \arrow[r] & E' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

extending the identity mapping of B and the Λ-homomorphisms $X_{n-2}E'_{n-2}, \ldots,$ X_0E_0', AA in (41), *where E_f^n is the n-fold extension $\gamma_n(f) \in \boldsymbol{E}_n(A, B)$ constructed in the preceding paragraphs.*

In short, this lemma states that if $\chi_n(\boldsymbol{E}_n') \in \mathrm{Ext}_\Lambda^n(A, B)$ is represented by a mapping $f \in \mathrm{Hom}_\Lambda(A_n, B)$ then there is a translation $\boldsymbol{E}_f{}^n\boldsymbol{E}_n' \in \mathcal{E}_u^n(A, B)$ of $\boldsymbol{E}_f{}^n = \gamma_n(f)$ into $\boldsymbol{E}_n'$. Thus if for two n-fold extension $\boldsymbol{E}_n, \boldsymbol{E}_n' \in \boldsymbol{E}_n(A, B)$ we have $\chi_n(\boldsymbol{E}_n) = \chi_n(\boldsymbol{E}_n')$, then, taking an arbitrary representation $f \in \mathrm{Hom}_\Lambda(A_n, B)$, we obtain an n-fold extension $\boldsymbol{E}_f{}^n \in \boldsymbol{E}_n(A, B)$ and two translations $\boldsymbol{E}_f{}^n\boldsymbol{E}_n, \boldsymbol{E}_f{}^n\boldsymbol{E}_n' \in \mathcal{E}_n(A, B)$, proving that $\boldsymbol{E}_n \simeq \boldsymbol{E}_f{}^n \simeq \boldsymbol{E}_n'$.

This lemma gives us more than the proof of the classification theorem, namely this proves also the following

Theorem 3.6. *Two n-fold extensions $\boldsymbol{E}_n, \boldsymbol{E}_n' \in \boldsymbol{E}_n(A, B)$ are equivalent if and only if there exist an n-fold extension $'\boldsymbol{E}_n \in \boldsymbol{E}_n(A, B)$ and two translations $'\boldsymbol{E}_n\boldsymbol{E}^u$, $'\boldsymbol{E}_n\boldsymbol{E}_n' \in \mathcal{E}_n(A, B)$.*

We shall refer later to a dual of this theorem.

Proof of Lemma 3.2. The only thing that we have to do is to define $E_fE'_{n-1}$ so that the first and the second sequence in (42) become commutative. To do this we first define WE'_{n-1} by

$$WE'_{n-1} = WBE'_{n-1} + WX_{n-1}E'_{n-1} + WX_{n-1}BE'_{n-1}\,.$$

Then we have

$$\begin{aligned} A_nWE'_{n-1} &= A_nWBE'_{n-1} + A_nWX_{n-1} + A_nWX_{n-1}BE'_{n-1} \\ &= A_nX_{n-1}WBE'_{n-1} - A_nfBWBE'_{n-1} + A_nX_{n-1}E'_{n-1} + A_nWX_{n-1}BE'_{n-1} \\ &= -A_nfBE'_{n-1} + A_ngBE'_{n-1} + A_nX_{n-1}BE'_{n-1}BE'_{n-1} \\ &= (-A_nfB + A_ngB + A_nX_{n-1}B)BE'_{n-1} = 0\,. \end{aligned}$$

Therefore, by Lemma 1.1, there exists a Λ-homomorphism $E_fE'_{n-1}$ such that $WE_fE'_{n-1} = WE'_{n-1}$. We now prove the commutativity $BE_fE'_{n-1} = BE'_{n-1}$ in the first square of (42), as follows:

$$\begin{aligned} BE_fE'_{n-1} &= BWE_fE'_{n-1} = BWE'_{n-1} \\ &= BWBE'_{n-1} + BWX_{n-1}E'_{n-1} + BWX_{n-1}BE'_{n-1} = BE'_{n-1}\,. \end{aligned}$$

Commutativity $E_fE'_{n-1}E'_{n-2} = E_fX_{n-2}E'_{n-2}$ in the second square of (42) is equivalent to $WE_fE'_{n-1}E'_{n-1} = WE_fX_{n-2}E'_{n-2}$, which is shown in the following manner:

$$\begin{aligned} WE_fE'_{n-1}E'_{n-2} &= WE'_{n-1}E'_{n-2} \\ &= WBE'_{n-1}E'_{-1} + WX_{n-1}E'_{n-1}E'_{n-2} + WX_{n-1}BE'_{n-1}E'_{n-2} \\ &= WX_{n-1}X_{n-2}E'_{n-2} \\ &= WX_{n-1}A_{n-1}X_{n-2}E'_{-2} \\ &= WE_fA_{n-1}X_{n-2}E'_{n-2} \\ &= WE_fX_{n-2}E'_{n-2}\,. \end{aligned}$$

This proves Lemma 3.2 and completes the proof of the classification theorem.

7. Redefinition of the one-to-one correspondence in the classification theorem by means of injective resolutions of B. Let Y^* be an injective resolution of B. Then by Theorem 3.2, there exists a translation

$$(43)\quad \begin{array}{cccccccccccccc} (\boldsymbol{E}_n) & 0 & \to & B & \to & E_{n-1} & \to & E_{n-2} & \to \cdots \to & E_0 & \to & A & \to & 0 \\ \downarrow & & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ (\boldsymbol{Y}^n) & 0 & \to & B & \to & Y^0 & \to & Y^1 & \to \cdots \to & Y^{n-1} & \to & B^n & \to & 0 \end{array}$$

extending the identity mapping of B. The class AB^n mod $\mathrm{Hom}_\Lambda(AY^{n-1})(Y^{n-1}B^n)_\sharp$ depends neither on the special choice of the translation $\boldsymbol{E}_n\boldsymbol{Y}^n$ nor on the special choice of the injective resolution $\boldsymbol{Y}^*$. So we obtain a mapping

$$\chi_n': \quad \boldsymbol{E}_n(A, B) \to \mathrm{Ext}^n_\Lambda(A, B) = \mathrm{Hom}_\Lambda(A, B^n)/\mathrm{Hom}_\Lambda(A, Y^{n-1})(Y^{n-1}B^n)_\sharp .$$

Now if we superpose (34′) on this translation, then we obtain a translation $\boldsymbol{X}_n\boldsymbol{E}_n\boldsymbol{Y}^n$. Thus, from what we have remarked after the proof of Theorem 3.4, follows that χ_n and χ_n' coincide.

If one develops the dual argument of the preceding paragraphs he can reprove the classification theorem, in which course he will obtain the following analogue of Theorem 3.6:

Theorem 3.7. *Two n-fold extensions $\boldsymbol{E}_n$, $'\boldsymbol{E}_n \in \boldsymbol{E}_n(A, B)$ are equivalent if and only if there exist an n-fold extension $\boldsymbol{E}_n' \in \boldsymbol{E}_n(A, B)$ and two translations $\boldsymbol{E}_n\boldsymbol{E}_n'$, $'\boldsymbol{E}_n\boldsymbol{E}_n' \in \mathscr{E}_n(A, B)$.*

Remark. For $n=1$, every translation $\boldsymbol{E}_1\boldsymbol{E}_1'$ in $\mathscr{E}_1(A, B)$ is an isomorphic translation, and so it is invertible. Therefore $\boldsymbol{E}_1 \sim \boldsymbol{E}_1'$ if and only if there exists an isomorphic translation $\boldsymbol{E}_1\boldsymbol{E}'$ which gives identity mappings on A and on B.

§ 4. Product in the extension groups.

1. **Motivation.** Let A, B, C be Λ-modules. By the classification theorem the elements of $\mathrm{Ext}^p(A, B)$ $(p>0)$ can be considered as the equivalence classes of exact sequences of the form

$$(\boldsymbol{E}_p) \qquad 0 \to B \to E_{p-1} \to \cdots \to E_0 \to A \to 0 ,$$

while the elements of $\mathrm{Ext}^q(B, C)$ $(q>0)$ as the equivalence classes of exact sequences of the form

$$(\boldsymbol{F}_q) \qquad 0 \to C \to F_{q-1} \to \cdots \to F_0 \to B \to 0 .$$

Now if we put these exact sequences in a line in tying them together at B, then we get an exact sequence

$$(\boldsymbol{E}_p \bigcirc \boldsymbol{F}_q) \qquad 0 \to C \to F_{q-1} \to \cdots \to F_0 \to E_{p-1} \to \cdots \to E_0 \to A \to 0 ,$$

where we have put $F_0E_{p-1} = F_0BE_{p-1}$. In this way we obtain a pairing

$$(44) \qquad \boldsymbol{E}_p(A, B) \times \boldsymbol{E}_q(B, C) \to \boldsymbol{E}_{p+q}(A, C) ,$$

which we shall call the *composition pairing*. Obviously this composition pairing satisfies the following conditions.

(i) If $\boldsymbol{E}_p \sim \boldsymbol{E}'_p$ and $\boldsymbol{F}_q \sim \boldsymbol{F}'_q$, then ${}_p\boldsymbol{E} \bigcirc \boldsymbol{F}_q \sim \boldsymbol{E}'_p \bigcirc \boldsymbol{F}'_q$.

(ii) $\bigcirc$ is associative, i.e., for $\boldsymbol{G}_r \in \boldsymbol{E}_r(C, D)$ we have

$$(\boldsymbol{E}_p \bigcirc \boldsymbol{F}_q) \bigcirc \boldsymbol{G}_r = \boldsymbol{E}_p \bigcirc (\boldsymbol{F}_q \bigcirc \boldsymbol{G}_r) .$$

Therefore this defines an associative pairing

$$(45) \qquad \mathrm{Ext}^p(A, B) \times \mathrm{Ext}^q(B, C) \to \mathrm{Ext}^{p+q}(A, C) .$$

This composition pairing turns out to be bilinear, and the above definition can be naturally extended to admit the value $p=0$ or $q=0$, To show this we shall begin with another equivalent definition of this composition pairing.

2. **Composition product.** Let X_* be a projective resolution of A, and $\boldsymbol{U}^*$ an injective resolution of C. We have seen earlier that the following identifications are allowable.

$$\mathrm{Ext}^p(A, B) = \mathrm{Hom}_\Lambda(A_p, B)/(A_pX_{p-1})^\sharp\, \mathrm{Hom}_\Lambda(X_{p-1}, B) \qquad (p \geqq 0,\ X_{-1}=0) ,$$
$$\mathrm{Ext}^q(B, C) = \mathrm{Hom}_\Lambda(B, C^q)/\mathrm{Hom}_\Lambda(B, W^{q-1})(W^{q-1}C^q)_\sharp \qquad (q \geqq 0,\ W^{-1}=0) ,$$

$$\begin{aligned}\mathrm{Ext}^{p+q}(A, C) &= \mathrm{Hom}_\Lambda(A_{p+q}, C)/(A_{p+q}X_{p+q-1})^\# \mathrm{Hom}_\Lambda(X_{p+q-1}, C) \\ &= \mathrm{Hom}_\Lambda(A_p, C^q)/\mathrm{Hom}_\Lambda(A_p W^{q-1})(W^{q-1}C^q)_\# \\ &= \mathrm{Hom}_\Lambda(A_p, C^q)/(A_pX_{p-1})^\# \mathrm{Hom}_\Lambda(X_{p-1}, C^q) \\ &\qquad (p, q \geqq 0, \ X_{-1} = W^{-1} = 0)\end{aligned}$$

Now if we define a composition product

$$\mathrm{Hom}_\Lambda(A_p, B) \times \mathrm{Hom}_\Lambda(B, C^q) \to \mathrm{Hom}_\Lambda(A_p, C^q)$$

by the simple composition $A_pB \bigcirc BC^q = A_pBC^q$. Then this product $\bigcirc$ is clearly bilinear and we have also

$$(A_pX_{p-1})^\# \mathrm{Hom}_\Lambda(X_{p-1}, B) \bigcirc \mathrm{Hom}\,(B, C^q) \subset (A_pX_{p-1})^\# \mathrm{Hom}_\Lambda(X_{p-1}, C^q)\,,$$
$$\mathrm{Hom}_\Lambda(A_p, B) \bigcirc \mathrm{Hom}_\Lambda(B, W^{q-1})(W^{q-1}C^q)_\# \subset \mathrm{Hom}_\Lambda(A_p, W^{q-1})(W^{q-1}C^q)\,.$$

so that $\bigcirc$ defines a bilinear product

$$(45') \qquad \mathrm{Ext}^p(A, B) \times \mathrm{Ext}^q(B, C) \to \mathrm{Ext}^{p+q}(A, C) \qquad (p, q \geqq 0)\,.$$

We now show the coincidence of (45) and (45′) for p, $q>0$ in proving that

$$\chi_p(E_p) \bigcirc \chi_q(F_q) = \chi_{p+q}(E_p \bigcirc F_q)\,,$$

where χ is the mapping appearing in the classification theorem. Let

$$(46) \qquad \begin{array}{ccccccccccc} 0 & \to & A_p & \to & X_{p-1} & \to \cdots \to & X_0 & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & B & \to & E_{p-1} & \to \cdots \to & E_0 & \to & A & \to & 0\,, \end{array}$$

$$(47) \qquad \begin{array}{ccccccccccc} 0 & \to & C & \to & F_{q-1} & \to \cdots \to & F_0 & \to & B & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C & \to & W^0 & \to \cdots \to & W^{q-1} & \to & C^q & \to & 0 \end{array}$$

be translations such that A_pB represents $\chi_p(E_p)$ in

$$\mathrm{Hom}_\Lambda(A_p, B)/(A_pX_{p-1})^\# \mathrm{Hom}_\Lambda(X_{p-1}, B)$$

and BC_q represents $\chi_q(F_q)$ in

$$\mathrm{Hom}_\Lambda(B, C^q)/\mathrm{Hom}_\Lambda(B, W^{q-1})(W^{q-1}C^q)_\#\,.$$

Now, by Theorem 3.2, A_pB can be extended to a translation

$$(46') \qquad \begin{array}{ccccccccccc} 0 & \to & A_{p+q} & \to & X_{p+q-1} & \to \cdots \to & X_p & \to & A_p & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C & \to & F_{q-1} & \to \cdots \to & F_0 & \to & B & \to & 0 \end{array}$$

so that $A_{p+q}C$ represents $\chi_{p+q}(E_p \bigcirc F_q)$. Therefore it is sufficient to prove coincidence of the element in $\mathrm{Hom}_\Lambda(A_{p+q}, C)/(A_{p+q}X_{p+q-1})^\# \mathrm{Hom}_\Lambda(X_{p+q-1}, C)$ represented by $A_{p+q}C$ and the element in $\mathrm{Hom}_\Lambda(A_p, C^q)/\mathrm{Hom}_\Lambda(A_pW^{q-1})(W^{q-1}C^q)_\#$ represented by A_pBC^q, or equivalently, to prove that $A_{p+q}C$ and $A_pC^q = A_pBC^q$ can be extended to a translation

$$(48) \qquad \begin{array}{ccccccccccc} 0 & \to & A_{p+q} & \to & X_{p+q-1} & \to \cdots \to & X_p & \to & A_p & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C & \to & W^0 & \to \cdots \to & W^{q-1} & \to & C^q & \to & 0 \end{array}$$

Translation (48) is readily obtained in composing the translations (46′) and (47). Thus the coincidence of (45) and (45′) is proved, and so (45′) is independent of the special choice of resolutions.

Let Y_*, Y^* be projective resp. injective resolutions of B. If we put Y_*, Y^* in a line in tying them together at B, then the resulting sequence

$$(Y^*_*) \qquad \cdots \to Y_1 \to Y_0 \overset{(\to B \to)}{\longrightarrow} Y^0 \to Y^1 \to \cdots$$

is exact. (Y^*_* will be called a *complete resolution* of B.) By Theorems 3.1, 3.2, $A_pB \in \mathrm{Hom}_\Lambda(A_p, B)$ can be extended to a translation

$$
\begin{tikzcd}[column sep=small]
0 \arrow[r] & A_{p+q} \arrow[r] \arrow[ddd] & X_{p+q-1} \arrow[r] \arrow[ddd] & \cdots \arrow[r] & X_p \arrow[rr] \arrow[ddd] & & X_{p-1} \arrow[r] \arrow[ddd] & \cdots \arrow[r] & X_0 \arrow[r] \arrow[ddd] & A \arrow[r] \arrow[ddd] & 0 \\
& & & & & (\to A_p) \arrow[d] & & & & & \\
& & & & & (\to B \to) & & & & & \\
0 \arrow[r] & B_q \arrow[r] & Y_{q-1} \arrow[r] & \cdots \arrow[r] & Y_0 \arrow[rr] & & Y^0 \arrow[r] & \cdots \arrow[r] & Y^{p-1} \arrow[r] & B^p \arrow[r] & 0
\end{tikzcd} \tag{49}
$$

Also $BC^q \in \mathrm{Hom}_\Lambda(B, C^q)$ can be extended to a translation

$$
\begin{tikzcd}[column sep=small]
0 \arrow[r] & B_q \arrow[r] \arrow[ddd] & Y_{q-1} \arrow[r] \arrow[ddd] & \cdots \arrow[r] & Y_0 \arrow[rr] \arrow[ddd] & & Y^0 \arrow[r] \arrow[ddd] & \cdots \arrow[r] & Y^{p-1} \arrow[r] \arrow[ddd] & B^p \arrow[r] \arrow[ddd] & 0 \\
& & & & & (\to B \to) \arrow[d] & & & & & \\
& & & & & (\to C^q \to) & & & & & \\
0 \arrow[r] & C \arrow[r] & W^0 \arrow[r] & \cdots \arrow[r] & W^{q-1} \arrow[rr] & & W^q \arrow[r] & \cdots \arrow[r] & W^{q+p-1} \arrow[r] & C^{q+p} \arrow[r] & 0
\end{tikzcd} \tag{50}
$$

Since the composite of the translations (49) and (50) gives a translation $X_{(p+q)} \to U^{(q+p)}$, the three Λ-homomorphisms $A_{p+q}B_qC$, A_pBC^q, and AB^pC^{q+p} represent one and the same element in $\mathrm{Ext}^{p+q}(A, C)$.

Take now an injective resolution V^* of $D \in \mathscr{G}_\Lambda$ and let $\alpha^p \in \mathrm{Ext}^p(A, B)$ be represented by $A_pB \in \mathrm{Hom}_\Lambda(A_p, B)$, $\beta^q \in \mathrm{Ext}^q(B, C)$ by $BC^q \in \mathrm{Hom}_\Lambda(B, C^q)$, and $\gamma^r \in \mathrm{Ext}^r(C, D)$ by $CD^r \in \mathrm{Hom}_\Lambda(C, D^r)$. Then, we proceed as follows:

(i) Extend (49) leftward up to $A_{p+q+r}B_{q+r}$,

(ii) Take a complete resolution U^*_* of C and extend (50) leftward up to $B_{q+r}C_r$,

(iii) Replace BC^q by CD^r and extend it to obtain a similar translation as (50), and finally

(iv) Extend the translation obtained in (iii) rightward up to $C^{q+p}D^r$.

Thus we obtain a commutative diagram

$$
\begin{tikzcd}[column sep=tiny]
0 \arrow[r] & A_{p+q+r} \arrow[r] \arrow[dd] & X_{p+q+r-1} \arrow[r] \arrow[dd] & \cdots \arrow[r] & X_{p+q} \arrow[rr] \arrow[dd] & & X_{p+q-1} \arrow[r] \arrow[dd] & \cdots \arrow[r] & X_p \arrow[rr] \arrow[dd] & & X_{p-1} \arrow[r] \arrow[dd] & \cdots \arrow[r] & X_0 \arrow[r] \arrow[dd] & A \arrow[r] \arrow[dd] & 0 \\
& & & & & (\to A_{p+q} \to) \arrow[dd] & & & & (\to A_p \to) \arrow[dd] & & & & & \\
0 \arrow[r] & B_{q+r} \arrow[r] \arrow[dd] & Y_{q+r-1} \arrow[r] \arrow[dd] & \cdots \arrow[r] & Y_q \arrow[rr] \arrow[dd] & & Y_{q-1} \arrow[r] \arrow[dd] & \cdots \arrow[r] & Y_0 \arrow[rr] \arrow[dd] & & Y^0 \arrow[r] \arrow[dd] & \cdots \arrow[r] & Y^{p-1} \arrow[r] \arrow[dd] & B^p \arrow[r] \arrow[dd] & 0 \\
& & & & & (\to B_q \to) \arrow[dd] & & & & (\to B \to) \arrow[dd] & & & & & \\
0 \arrow[r] & C_r \arrow[r] \arrow[dd] & U_{r-1} \arrow[r] \arrow[dd] & \cdots \arrow[r] & U_0 \arrow[rr] \arrow[dd] & & U^0 \arrow[r] \arrow[dd] & \cdots \arrow[r] & U^{q-1} \arrow[rr] \arrow[dd] & & U^q \arrow[r] \arrow[dd] & \cdots \arrow[r] & U^{q+p-1} \arrow[r] \arrow[dd] & C^{q+p} \arrow[r] \arrow[dd] & 0 \\
& & & & & (\to C \to) \arrow[dd] & & & & (\to C^q \to) \arrow[dd] & & & & & \\
0 \arrow[r] & D \arrow[r] & V^0 \arrow[r] & \cdots \arrow[r] & V^{r-1} \arrow[rr] & & V^r \arrow[r] & \cdots \arrow[r] & V^{r+q-1} \arrow[rr] & & V^{r+q} \arrow[r] & \cdots \arrow[r] & V^{r+q+p-1} \arrow[r] & D^{r+q+p} \arrow[r] & 0 \\
& & & & & (\to D^r \to) & & & & (\to D^{r+q} \to) & & & & &
\end{tikzcd}
$$

From what we have seen above, it follows then:

(i) Each of $A_{p+q}B_qC$, A_pBC^q, AB^pC^{q+p} represents $\alpha^p \bigcirc \beta^q$.

(ii) Each of $B_{q+r}C_rD$, B_qCD^r, BC^qD^{r+q} represents $\beta^q \bigcirc \gamma^r$.

Thus $(\alpha^p \bigcirc \beta^q) \bigcirc \gamma^r$ is represented by each one of

$$A_{p+q+r}B_{q+r}C_r \cdot C_rD, \quad A_{p+q}B_qC \cdot CD^r, \quad AB^pC^{q+p}D^{r+q+p},$$

while $\alpha^p \bigcirc (\beta^q \bigcirc \gamma^r)$ by each one of

$$A_{p+q+r}B_{q+r} \cdot B_{q+r}C_rD, \quad A_pB \cdot BC^qD^{r+q}, \quad AB^p \cdot B^pC^{q+p}D^{r+q+p}.$$

This proves the associativity $(\alpha^p \bigcirc \beta) \bigcirc \gamma^r = \alpha^p \bigcirc (\beta \bigcirc \gamma^r)$. Summarizing, we have established the following

Theorem 4.1. *The composition product* $\bigcirc$:

$$\mathrm{Ext}^p(A, B) \times \mathrm{Ext}^q(B, C) \to \mathrm{Ext}^{p+q}(A, C) \qquad (p,\ q \geqq 0)$$

defined in (45′) *is bilinear and associative. For* $p,\ q > 0$, $\bigcirc$ *is induced by the pairing* (44).

Corollary. $\mathrm{Ext}_\Lambda(A, A)$ *has the structure of a ring in which multiplication is defined by the composition product* $\bigcirc$.

3. The effect of multiplication by the elements of Ext^0 and Ext^1.

Theorem 4.2. *Let $\alpha^0 \in \mathrm{Ext}^0(A, B)$ be represented by $AB \in \mathrm{Hom}_\Lambda(A, B)$. Then the left multiplication by α^0,*

$$\alpha^0\bigcirc:\quad \mathrm{Ext}^q(B, C) \to \mathrm{Ext}^q(A, C)$$

is identical with the induced homomorphism $(AB)^*$. If $\beta^0 \in \mathrm{Ext}^0(B, C)$ is represented by $BC \in \mathrm{Hom}_\Lambda(B, C)$, then the right multiplication by β^0

$$\bigcirc\beta^0:\quad \mathrm{Ext}^p(A, B) \to \mathrm{Ext}^p(A, C)$$

coincides with the induced homomorphism ${}^*(BC)$.

Proof. $(AB)^*$ is induced by $(A_qB_q)^\sharp$: $\mathrm{Hom}_\Lambda(B_q, C) \to \mathrm{Hom}_\Lambda(A_q, C)$ obtained in a translation

$$\begin{array}{ccccccccccc} 0 \to & A_q & \to & X_{q-1} & \to \cdots \to & X_0 & \to & A & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & B_q & \to & Y_{q-1} & \to \cdots \to & Y_0 & \to & B & \to 0 \end{array} \tag{51}$$

extending AB, while $\alpha^0\bigcirc$ by $(AB)^\sharp$: $\mathrm{Hom}_\Lambda(B, C^q) \to \mathrm{Hom}_\Lambda(A, C^q)$.
If $B_qC \in \mathrm{Hom}_\Lambda(B_q, C)$ is extended to a translation

$$\begin{array}{ccccccccccc} 0 \to & B_q & \to & Y_{q-1} & \to \cdots \to & Y_0 & \to & B & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & C & \to & U^0 & \to \cdots \to & U^{q-1} & \to & C^q & \to 0, \end{array} \tag{52}$$

then B_qC, BC^q represent the same element, say β^q, in $\mathrm{Ext}^q(B, C)$. Combining (51) and (52) we have now a commutative diagram

$$\begin{array}{ccccccccccc} 0 \to & A_q & \to & X_{q-1} & \to \cdots \to & X_0 & \to & A & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & B_q & \to & Y_{q-1} & \to \cdots \to & Y_0 & \to & B & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & C & \to & U^0 & \to \cdots \to & U^{q-1} & \to & C^q & \to 0, \end{array}$$

in which A_qB_qC represents $(AB)^*\beta^q$ and ABC^q represents $\alpha^0\bigcirc\beta^q$. Therefore we have $(AB)^*\beta^q = \alpha^0\bigcirc\beta^q$.

The latter statement of the theorem can be proved quite similarly.

Theorem 4.3. *Let $\alpha^1 \in \mathrm{Ext}^1(A, B)$ be represented by an exact sequence*

$$(E) \qquad 0 \to B \to E \to A \to 0.$$

Then the left multiplication by α^1,

$$\alpha^1\bigcirc:\quad \mathrm{Ext}^q(B, C) \to \mathrm{Ext}^{q+1}(A, C)$$

is identical with the coboundary homomorphism δ^* with respect to the exact sequence E. If $\beta^1 \in \mathrm{Ext}^1(B, C)$ is represented by an exact sequence

$$(F) \qquad 0 \to C \to F \to B \to 0,$$

then the right multiplication by β^1,

$$\bigcirc\beta^1:\quad \mathrm{Ext}^p(A, B) \to \mathrm{Ext}^{p+1}(A, C)$$

coincides with the coboundary homomorphism ${}^*\delta$ with respect to the exact sequence F.

Proof. If we extend the identity mapping of A to a translation

$$\begin{array}{cccccccc} (X_0) & 0 \to & A_1 & \to & X_0 & \to & A & \to 0 \\ \downarrow & & \downarrow & & \downarrow & & \| & \\ (E) & 0 \to & B & \to & E & \to & A & \to 0, \end{array}$$

Then we have $\chi(E) = \alpha^1$, so that $A_1B \in \mathrm{Hom}_\Lambda(A_1, B)$ represents α^1. On the other hand, if we denote the coboundary homomorphism $\mathrm{Ext}^p(A_1, C) \to \mathrm{Ext}^{p+1}(A, C)$

with respect to X_0 with δ^*_0. Then we have $\delta_0{}^*\circ(A_1B)^*=(AA)^*\circ\delta^*=\delta^*$. So we now prove $\alpha^1\bigcirc=\delta_0{}^*\circ(A_1B)^*$. If $\beta^q\in \mathrm{Ext}^q(B,C)$ is represented by $BC^q\in \mathrm{Hom}_\Lambda(B,C^q)$, then $(A_1B)^*\beta^q$ is represented by $A_1BC^q\in \mathrm{Hom}_\Lambda(A_1, C^q)$, and $\alpha^1\bigcirc\beta^q$ also by $A_1BC^q\in \mathrm{Hom}_\Lambda(A_1, C^q)$. Thus, $\delta_0{}^*$ being nothing other than the identification isomorphism

$$\delta_0{}^*:\quad \mathrm{Ext}^q(A_1, C)\cong \mathrm{Ext}^{q+1}(A, C)\,,$$

we have proved the first assertion $\alpha^1\bigcirc=\delta^*$ of the theorem,

Next, extend the identity mapping of C to a translation

$$\begin{array}{cccccccccc}
(F) & & 0 & \to & C & \to & F & \to & B & \to 0\\
\downarrow & & & & \| & & \downarrow & & \downarrow & \\
(U^0) & & 0 & \to & C & \to & U^0 & \to & C^1 & \to 0
\end{array}$$

to obtain BC^1 representing β^1. If $\alpha^p\in \mathrm{Ext}^p(A, B)$ is represented by $A_pB\in \mathrm{Hom}_\Lambda(A_p, B)$, then $\alpha^p\bigcirc\beta^1$ is represented by A^pBC^1. On the other hand we have $^*\delta\alpha^p={}^*\delta_0(\alpha^{p*}(BC^1))$, where $^*\delta_0$ is the coboundary homomorphism with respect to U^0. $\alpha^{p*}(BC^1)$ being represented by A_pBC^1, and $^*\delta_0$ being nothing other than the reduction isomorphism

$$^*\delta_1:\quad \mathrm{Ext}^p(B, C^1)\cong \mathrm{Ext}^{p+1}(B, C)\,,$$

we have $^*\delta\alpha^p=\alpha^p\bigcirc\beta^1$. This completes the proof.

4. φ-product in the cohomology group $\mathrm{Ext}_\Lambda(A, B)$.

As we have remarked before, the composition product $\bigcirc$ gives the structure of a ring to $\mathrm{Ext}_\Lambda(A, A)$. This is a special case of the following more general notion of φ-product in $\mathrm{Ext}_\Lambda(A, B)$.

Let φ be an arbitrary Λ-homomorphism from B into A. φ can be also considered as an element in $\mathrm{Ext}^0(B, A)$. We now fix one $\varphi\in \mathrm{Hom}_\Lambda(B. A)$. Then, φ-product in $\mathrm{Ext}_\Lambda(A, B)$ is defined as follows. Let $\alpha\in \mathrm{Ext}^p(A, B)$, $\beta\in \mathrm{Ext}^q(A, B)$. Then $\alpha\bigcirc\varphi\in \mathrm{Ext}^p(A, A)$ so that $(\alpha\bigcirc\varphi)\bigcirc\beta\in \mathrm{Ext}^{p+q}(A, B)$. On the other hand we have $\varphi\bigcirc\beta\in \mathrm{Ext}^q(B, B)$, and $\alpha\bigcirc(\varphi\bigcirc\beta)\in \mathrm{Ext}^{p+q}(A, B)$. Since $\bigcirc$ is associative these two elements $(\alpha\bigcirc\varphi)\bigcirc\beta$, $\alpha\bigcirc(\varphi\bigcirc\beta)$ equal to each other. So we define the φ-product $\alpha\veebar_\varphi\beta$ of α and β as

$$\alpha\veebar_\varphi\beta=\alpha\bigcirc\varphi\bigcirc\beta\,,$$

Then bilinearity of $\veebar_\varphi$ is clear; associativity is checked as

$$(\alpha\veebar_\varphi\beta)\veebar_\varphi\gamma=(\alpha\bigcirc\varphi\bigcirc\beta)\bigcirc\varphi\bigcirc\gamma=\alpha\bigcirc\varphi\bigcirc(\beta\bigcirc\varphi\bigcirc\gamma)=\alpha\veebar_\varphi(\beta\veebar_\varphi\gamma)\,,$$

If $f\in \mathrm{Hom}_\Lambda(A, A')=\mathrm{Ext}^0(A, A')$, then $\varphi\bigcirc f\in \mathrm{Hom}_\Lambda(B, A')$, and putting $\varphi'=\varphi\bigcirc f$ we have, by Theorem 4.2,

$$f^*(\alpha'\veebar_{\varphi'}\beta')=(f^*\alpha')\veebar_\varphi(f^*\beta')\,. \qquad (\alpha', \beta'\in \mathrm{Ext}_\Lambda(A', B))$$

On the other hand, if $g\in \mathrm{Hom}_\Lambda(B', B)=\mathrm{Ext}^0(B', B)$, then putting $\varphi'=g\bigcirc\varphi\in \mathrm{Hom}_\Lambda(B', A)$, we have

$$^*g(\alpha'\veebar_{\varphi'}\beta')=(^*g\alpha')\veebar_\varphi(^*g\beta')\,.$$

This shows the naturality of the φ-product.

5. Relation to the homology theory of associative systems.

Let Π denote a group or more generally an associative system with unit, $Z(\Pi)$ the algebra of Π over Z, and let G be a Π-group, i.e., a $Z(\Pi)$-module. As stated by Cartan and Eilenberg, $H_n(\Pi, G)$, the n-th homology group, and $H^n(\Pi, G)$, the n-th cohomology group of Π with coefficients in G can be defined as the n-th torsion product $\mathrm{Tor}_n^{Z(\Pi)}(Z, G)$ and as the n-th extension group $\mathrm{Ext}^n_{Z(\Pi)}(Z, G)$ respectively, where Z is considered as a Π-group on which each element of Π operates identically. As stated further by the same authors, the multiplicative structure of the cohomology group $H^*(\Pi, R)=\mathrm{Ext}_{Z(\Pi)}(Z, R)$ with coefficients in a ring R with unit can be introduced in the following manner. Let X_* be a projective resolution of the $Z(\Pi)$-module Z. Then $X_*^2=X_*\otimes_Z X_*$ is a projective resolution of the $Z(\Pi\times\Pi)$-module $Z\otimes_Z Z=Z$. By the diagonal injection $\Pi\to\Pi\times\Pi$, the augmented sequence $X_*^2\to Z\to 0$ can be considered as an exact sequence over $Z(\Pi)$. Therefore there exists a translation

$$\begin{array}{cccccc} & X_* & \to & Z & \to & 0 \\ \tau & \downarrow & & \| & & \\ & X_*^2 & \to & Z & \to & 0 \end{array}$$

which is unique up to a chain homotopy.

On the other hand any pair of $Z(\Pi)$-homomorphisms $X_pR\in\mathrm{Hom}_{Z(\Pi)}(X_p, R)$, $X_qR\in\mathrm{Hom}_{Z(\Pi)}(X_q, R)$ determines in a natural way a homomorphism $X_pR\otimes X_qR\in\mathrm{Hom}_{Z(\Pi)}(X_p\otimes_Z X_q, R)$, and thus we have a canonical homomorphism

(53) $$\mathrm{Hom}_{Z(\Pi)}(X_*, R)\otimes_Z \mathrm{Hom}_{Z(\Pi)}(X_*, R)\to \mathrm{Hom}_{Z(\Pi)}(X_*^2, R)\ ,$$

which is compatible with the coboundary operators. Combining (53) with the homomorphism $\tau^\sharp$: $\mathrm{Hom}_{Z(\Pi)}(X_*^2, R)$ $\mathrm{Hom}_{Z(\Pi)}(X_*, R)$, and passing to the cohomology groups, i.e., extension groups, we give now a homomorphism

(54) $$\mathrm{Ext}^p_{Z(\Pi)}(Z, R)\otimes_Z \mathrm{Ext}^q_{Z(\Pi)}(Z, R)\to \mathrm{Ext}^{p+q}_{Z(\Pi)}(Z, R)$$

to introduce in $\mathrm{Ext}_{Z(\Pi)}(Z, R)$ the structure of a ring.

Finally we shall prove the following

Theorem 4.4. *Let* $R(\Pi)$ *denote the algebra of* Π *over* R. *Then any* $R(\Pi)$-*module is automatically a* Π-*group,* and in this sense we have

(i) $H_n(\Pi, G)=\mathrm{Tor}_n^{R(\Pi)}(R, G)$,

(ii) $H^n(\Pi, G)=\mathrm{Ext}^n_{R(\Pi)}(R, G)$,

(iii) $H^*(\Pi, R)=\mathrm{Ext}_{R(\Pi)}(R, R)$,

where $\mathrm{Ext}_{R(\Pi)}(R, R)$ is provided with the ring structure defined by the composition product $\circ$.

Proof. Let X_* be a free resolution of the $Z(\Pi)$-module Z. The augmented exact sequence $X_*\to Z\to 0$ is then *direct* as an exact sequence over Z, Therefore the lower sequence

(55) $$X_*\otimes_Z R \to Z\otimes_Z R(=R)\to 0$$

is also exact. Now each X_n in X_* being $Z(\Pi)$-free, $X_n\otimes_Z R$ can be considered as an $R(\Pi)$-free module, and (55) as a free resolution of the $R(\Pi)$-module $Z\otimes R=R$. Thus $\mathrm{Tor}_n^{R(\Pi)}(R, G)$ is defined as the n-th homology factor of the lower 0-sequence

$$(X_*\otimes_Z R)\otimes_{R(\Pi)} G\ ,$$

and $\mathrm{Ext}_{R(\Pi)}(R, G)$ as the n-th homology factor of the upper 0-sequence

$$\mathrm{Hom}_{R(\Pi)}(X_*\otimes_Z R, G)\ .$$

We now show identities

$$(X_*\otimes_Z R)\otimes_{R(\Pi)} G=X_*\otimes_{Z(\Pi)} G\ ,$$
$$\mathrm{Hom}_{R(\Pi)}(X_*\otimes_Z R, G)=\mathrm{Hom}_{Z(\Pi)}(X_*, G)\ ,$$

which will prove (i) and (ii). Since X_* is $Z(\Pi)$-free, it is sufficient to obtain natural isomorphisms

$$(Z(\Pi)\otimes_Z R)\otimes_{R(\Pi)} G=Z(\Pi)\otimes_{Z(\Pi)} G\ ,$$
$$\mathrm{Hom}_{R(\Pi)}(Z(\Pi)\otimes_Z R, G)=\mathrm{Hom}_{Z(\Pi)}(Z(\Pi), G)\ .$$

These identities are both obvious, for we have natural identifications

$$(Z(\Pi)\otimes_Z R)\otimes_{R(\Pi)} G=R(\Pi)\otimes_{R(\Pi)} G=G=Z(\Pi)\otimes_{Z(\Pi)} G\ ,$$

$$\mathrm{Hom}_{R(\Pi)}(Z(\Pi)\otimes_Z R,\ G)=\mathrm{Hom}_{R(\Pi)}(R(\Pi),\ G)=G=\mathrm{Hom}_{Z(\Pi)}(Z(\Pi),\ G)\,.$$

To prove (iii) we first replace Z by R in (54) in the following way. Let X_* be a projective resolution of the $R(\Pi)$-module R. Then, quite in the same way as in the case of Z, we obtain the exact sequence over $R(\Pi)$

$$X_*^2(=X_*\otimes_R X_*)\to R\to 0\,,$$

and a translation

$$\begin{array}{cccccc} & X_* & \to & R & \to & 0 \\ \tau & \downarrow & & \downarrow & & \\ & X_*^2 & \to & R & \to & 0\,, \end{array}$$

through which a bilinear multiplication

$$\mathrm{Ext}^p_{R(\Pi)}(R,\ R)\otimes_Z \mathrm{Ext}^q_{R(\Pi)}(R,\ R)\to \mathrm{Ext}^{p+q}_{R(\Pi)}(R,\ R) \tag{54'}$$

is obtained. Obviously (54′) agrees with (54) under the identification

$$\mathrm{Ext}_{Z(\Pi)}(Z,\ R)=\mathrm{Ext}_{R(\Pi)}(R,\ R)$$

obtained in (ii). Therefore what we have to prove is the coincidence of (54′) and the composition product in the corollary to Theorem 4.1.

Now, since the product (54′) and the composition product are both independent of the special choice of a projective resolution of the $R(\Pi)$-module R, we may take as X_* the special resolution 'non-homogeneous complex of Π', namely the exact sequence

$$\cdots\cdots\to C_q\xrightarrow{\partial_q} C_{q-1}\to\cdots\cdots\to C_1\to C_0 \underset{\substack{\partial_0\\(\to 0)}}{\xrightarrow{\varepsilon}} R\to 0$$

over $R(\Pi)$, where C_q is the $R(\Pi)$-free module with base

$$[s_1,\ \ldots,\ s_q]\qquad (s_1,\ \ldots,\ s_q\in\Pi)\,,$$

and where we have put

$$\partial_q[s_1,\ \ldots,\ s_q]=s_1[s_2,\ \ldots,\ s_q]+\textstyle\sum_{i=1}^{q-1}(-1)^i[s_1,\ \ldots,\ s_is_{i+1},\ \ldots,\ s_q]$$
$$+(-1)^q[s_1,\ \ldots,\ s_{q-1}]\,,$$

$$\varepsilon[\ \ ]=1$$

Then a translation τ: $C_*\to C_*^2(=C_*\otimes_R C_*)$ extending the identity mapping of R is given as follows:

$$\tau_q[s_1,\ \ldots,\ s_q]=[\ \ ]\otimes[s_1,\ \ldots,\ s_q]+\textstyle\sum_{i=1}^{q-1}[s_1,\ \ldots,\ s_i]\otimes s_1\ldots s_i[s_{i+1},\ \ldots,\ s_q]$$
$$+[s_1,\ \ldots,\ s_q]\otimes s_1\ldots s_q[\ \ ]\,.$$

Therefore (54′) is given by the multiplication

$$\mathrm{Hom}_{R(\Pi)}(C_p,\ R)\times\mathrm{Hom}_{R(\Pi)}(C_q,\ R)\to\mathrm{Hom}_{R(\Pi)}(C_{p+q},\ R)$$

defined as

$$\begin{aligned}(f\cdot g)([s_1,\ \ldots,\ s_p,\ s_{p+1},\ \ldots,\ s_{p+q}])&\\ &=f([s_1,\ \ldots,\ s_p])\cdot g(s_1\ldots s_p[s_{p+1},\ \ldots,\ s_{p+q}])\\ &=f([s_1,\ \ldots,\ s_p])\cdot s_1\ldots s_p g([s_{p+1},\ \ldots,\ s_{p+q}])\end{aligned}$$

$$(f\in\mathrm{Hom}_{R(\Pi)}(C_p,\ R),\ g\in\mathrm{Hom}_{R(\Pi)}(C_q,\ R))$$

On the other hand, if we define $R(\Pi)$-homomorphisms f_k: $X_{p+k}\to X_k$ ($k=0$, $1,\ \ldots.$) for given $f\in\mathrm{Hom}_{R(\Pi)}(C_p,\ R)$ by

$$f_k([s_1,\ \ldots,\ s_p,\ s_{p+1},\ \ldots,\ s_{p+k}])=f([s_1,\ \ldots,\ s_p])\cdot s_1\ldots s_p[s_{p+1},\ \ldots,\ s_{p+q}]\,,$$

then it is easy to verify

$$\partial_k\circ f_k=f_{k-1}\circ\partial_{p+k}\qquad (k=1,\ 2,\ \ldots.)\,,$$
$$\varepsilon\circ f_0=f\,,$$

so that $f=\{f,\ f_0,\ f_1,\ \ldots.\}$ defines a translation

$$\begin{array}{ccccccccccc}
\cdots\cdots\to & C_{p+q} & \to\cdots\cdots\to & C_{p+1} & \to & C_p & \to & R_p & \to & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots\cdots\to & C_q & \to\cdots\cdots\to & C_1 & \to & C_0 & \to & R & \to & 0 \ ,
\end{array}$$

where $C_pR_pR=f$. Thus, for $g\in \mathrm{Hom}_{R(\Pi)}(C_q, R)$, the composition product $f_q \bigcirc g \in \mathrm{Hom}_{R(\Pi)}(C_{p+q}, R)$ which, by definition, represents the composition product of the elements in $\mathrm{Ext}_{R(\Pi)}(R, R)$ represented respectively by f and g is so obtained that

$$\begin{aligned}
(f_q \bigcirc g)([s_1, \ldots, s_p, s_{p+1}, \ldots, s_{p+q}]) \\
&= g(f_q([s_1, \ldots, s_p, s_{p+1}, \ldots, s_{p+q}])) \\
&= g(f([s_1, \ldots, s_p])\cdot s_1 \ldots s_p[s_{p+1}, \ldots, s_{p+q}]) \\
&= f([s_1, \ldots, s_p])\cdot s_1 \ldots s_p g([s_{p+1}, \ldots, s_{p+k}]) \,.
\end{aligned}$$

This shows the identity $f_q \bigcirc g = f\cdot g$ and therefore, the coincidence of the product in $H^*(\Pi, R)$ and the composition product in $\mathrm{Ext}_{R(\Pi)}(R, R)$, q.e.d.

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