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C_2-equivariant James splitting and C_2-EHP sequences

Uğur Yiğit [1]

Istanbul Medeniyet University, Istanbul, Turkey

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ABSTRACT

In this paper, we prove the equivariant James splitting theorem, and we give the generalizations of EHP sequences in the classical homotopy theory to the C_2-equivariant case.

Contents

Notation. We provide here notation used in this paper for convenience.

- $V = r\sigma + s$, a real orthogonal representation of C_2, which is a sum of r-copy of the sign representation σ and s-copy of the trivial representation 1.
- $\rho = \sigma + 1$, the regular representation of C_2.
- $RO(C_2)$, the real representation ring of C_2.
- S^V, the equivariant sphere which is the one-point compactification of V.
- $\pi_V^{C_2}(X)$, the V-th C_2-equivariant homotopy group of a topological C_2-space X.
- $\boldsymbol{\pi}_V^G(X)$, the V-th G-equivariant homotopy group of a topological G-space X as a Mackey functor.
- $H_m^K(-)$ is the reduced $RO(G)$-graded equivariant ordinary homology with Burnside ring coefficients.

E-mail address: ugur.yigit@medeniyet.edu.tr.
[1] *Current address:* Department of Mathematics, Istanbul Medeniyet University, H1-20, Istanbul, TURKEY 34700.

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- $\boldsymbol{H}_m^K(-)$ is the reduced $RO(G)$-graded equivariant ordinary homology with Burnside ring coefficients as a Mackey functor.
- $\pi_{r\sigma+s}^S$, the C_2-equivariant stable homotopy groups of spheres.
- $J^{C_2}(X)$, the equivariant reduced product space for C_2-space X.
- $\Sigma^\sigma(X)$, the σ-th suspension of X.
- $\Omega^\sigma(X)$, all continuous functions from S^σ to X.

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1. Preliminaries

The absence of systematic tools like the EHP sequences for computations (specially unstable) in the C_2-equivariant homotopy theory inspired me to work on the C_2-equivariant stable and unstable homotopy groups of spheres, which also give information about the classical and motivic case. The main purpose of this paper is to give the generalizations of EHP sequences in the classical homotopy theory to the C_2-equivariant case.

The n-th homotopy group $\pi_n(X)$ of a topological space X is the set of the homotopy classes of maps from n-sphere S^n into X preserving base points. To determine the homotopy groups $\pi_n(S^k)$ of spheres is the central problem in homotopy theory. In [5], Freudenthal showed that there exists an homomorphism

$$E : \pi_{n+k}(S^n) \longrightarrow \pi_{n+k+1}(S^{n+1})$$

which is an isomorphism for $k < n - 1$. This theorem provides the stable group

$$\pi_k^S = \lim_{n\to\infty} \pi_{n+k}(S^n).$$

As a corollary, the groups $\pi_{n+k}(S^n)$ are called **stable** if $n > k + 1$, and **unstable** if $n \leq k + 1$.

In 1951, Serre [11] proved that the homotopy groups of spheres are all finite except for those of the form $\pi_n(S^n)$ or $\pi_{4n-1}(S^{2n})$ for $n > 0$, when the group is the product of the infinite cyclic group with a finite abelian group. In particular, the homotopy groups are determined by their p-components for all primes p, where 2-components are hardest to calculate.

The C_2-equivariant stable homotopy groups of the equivariant spheres are discussed by Bredon [3], [4] and by Landweber [8].

In this section we will give the main tools that are used the rest of the article. Let X be a G-space, where $G = C_2$ is a cyclic group with generator γ such that $\gamma^2 = e$. The group C_2 has two irreducible real representations, namely the trivial representation denoted by 1 (or $\mathbb{R}$) and the sign representation denoted by σ (or $\mathbb{R}_-$). The regular representation is isomorphic to $\rho_{C_2} = 1 + \sigma$ (it is denoted by ρ if there is no confusion). Thus the representation ring $RO(C_2)$ is free abelian of rank 2, so every representation V can be expressed as $V = r\sigma + s$. The equivariant sphere S^V is defined as the one-point compactification of V. The V-th C_2-equivariant homotopy group $\pi_V^{C_2}(X)$ of a topological C_2-space is $[S^V, X]_{C_2}$, the set of the homotopy classes of base points preserving C_2-maps. The C_2-equivariant stable homotopy groups of spheres are defined as

$$\pi_{r\sigma+s}^S = \lim_{V\to\infty} [S^V \wedge S^{r\sigma+s}, S^V]_{C_2}.$$

The precise computations are not published except a few examples by Bredon and Landweber. The C_2-equivariant stable homotopy groups were computed in a range by Araki and Iriye [1], but the method of computation is difficult to handle.

As in the classical case, we have a combinatorial model for the twisted loop and twisted suspension of a space which is called the C_2-equivariant reduced product space due to [10]. **The equivariant reduced product space** $J^{C_2}(X)$ for C_2-space X is the colimit of

$$J_n^{C_2}(X) = \amalg_{k=0}^{n} X^{\times k} / \sim .$$

The elements of the space $J^{C_2}(X)$ will be written in the form $x_1 \cdots x_k$, where an action of C_2 is

$$x_1 x_2 \cdots x_k \longrightarrow \bar{x}_k \cdots \bar{x}_1$$

where $\bar{x}_n := \gamma.x_n$ means the image of x_k under the action of the nontrivial element γ of C_2. This action is called the twisted action. Here, $\sim$ is the equivalence relation which omits the base point in any coordinate (one can look [7, Definition 4.1.] for the definition, which is due to [10]).

Definition 1. [9]

(i) A function ν^* from the set of conjugacy classes of subgroups of G to the integers is called a **dimension function**. The value of ν^* on the conjugacy class of $K \subset G$ is denoted by ν^K. Let ν^* and μ^* be two dimension functions. If $\nu^K \geq \mu^K$ for every subgroup K, then $\nu^* \geq \mu^*$. Associated to any G-representation V is the dimension function $|V^*|$ whose value at K is the real dimension of the K-fixed subspace V^K of V. The dimension function with constant integer value n is denoted n^* for any integer n.

(ii) Let ν^* be a non-negative dimension function. If for each subgroup K of G, the fixed point space Y^K is ν^K-connected, then a G-space Y is called **G-ν^*-connected**. If A G-space Y is G-0^*-connected, then it is called **G-connected**. Also, if it is G-1^*-connected, it is called **simply G-connected**. A G-space Y is **homologically G-ν^*-connected** if, for every subgroup K of G and every integer m with $0 \leq m \leq \nu^K$, the homology group $H_m^K(Y)$ is zero, where $H_m^K(-)$ is the reduced $RO(G)$-graded equivariant ordinary homology with Burnside ring coefficients.

(iii) Let ν^* be a non-negative dimension function and let $f : Y \longrightarrow Z$ be a G-map. If, for every subgroup K of G,

$$(f^K)_* : \pi_m(Y^k) \longrightarrow \pi_m(Z^K)$$

is an isomorphism for every integer m with $0 \leq m < \nu^K$ and an epimorphism for $m = \nu^K$, then f is called **G-ν^*-equivalence**. A G-pair (Y, B) is said to be **G-ν^*-connected** if the inclusion of B into Y is a G-ν^*-equivalence. The notions of **homology G-ν^*-equivalence** and of **homology G-ν^*-connectedness** for pairs are defined similarly, but with homotopy groups replaced by homology groups.

(iv) Let V be a G-representation. For each subgroup K of G, let $V(K)$ be the orthogonal complement of V^K; then $V(K)$ is a K-representation. If $\pi^K_{V(K)+m}(Y)$ is zero for each subgroup K of G and each integer m with $0 \leq m \leq |V^K|$, the G-space Y is called **G-V-connected**. Similarly, if $H^K_{V(K)+m}(Y)$ is zero for each subgroup K of G and each integer m with $0 \leq m \leq |V^K|$, then the G-space Y is called **homologically G-V-connected**.

(v) Let V be a G-representation. A G-0^*-equivalence $f : Y \longrightarrow Z$ is said to be a **G-V-equivalence** if, for every subgroup K of G, the map

$$f_* : \pi^K_{V(K)+m}(Y) \longrightarrow \pi^K_{V(K)+m}(Z)$$

is an isomorphism for every integer m with $0 \leq m < |V^K|$ and an epimorphism for $m = |V^K|$. A **homology G-V-equivalence** is defined similarly. A G-pair (Y, B) is called **G-V-connected** (respectively, **homologically G-V-connected**) if the inclusion of B into Y is a G-V-equivalence (respectively, homology G-V-equivalence).

Before giving the C_2-equivariant Freudenthal suspension theorem, we will state equivariant Hurewicz theorems:

Theorem 2. *[9] (Equivariant relative Hurewicz theorem) Let (Y, B) be a based G-CW pair with both Y and B simply G-connected and let V be a G-representation such that $|V^G| \geq 2$. Then the following two conditions are equivalent:*

(a) (Y, B) is $(V - 1)$-connected.
(b) (Y, B) is homologically $(V - 1)$-connected.

Moreover, either of these conditions implies that, for any G-representation W with $2^* < |W^*| < |V^*|$, $\boldsymbol{\pi}^G_W(Y, B)$ is a W-Mackey functor (instead of just a $(W - 1)$-Mackey functor) and the map

$$\widetilde{h}\colon s_*\boldsymbol{\pi}^G_W(Y, B) \longrightarrow \boldsymbol{H}^G_W(Y, B)$$

is an isomorphism, where s_* is the functor associated to an inclusion of W into a complete G-universe. If $|W^*| < |V^*|$ and (Y, B) is $(V - 1)$-connected, then both $\boldsymbol{\pi}^G_W(Y, B)$ and $\boldsymbol{H}^G_W(Y, B)$ are zero.

Note that Lewis proved this equivariant relative Hurewicz theorem for compact Lie groups in [9]. Because of this reason, he introduced the W-Mackey functors, which are the generalization of Mackey functors. However, in our case, the group is C_2, so one can consider these Mackey functors as usual Mackey functors. Also, one can check [9] for details.

Theorem 3. *[9] Let Y and Z be G-connected G-CW complexes, $f : Y \longrightarrow Z$ be a G-map, and V and W be G-representations with $|W^*| < |V^*|$. If f is a V-equivalence, then f is also a W-equivalence and a homology W-equivalence. Moreover, if Y and Z are simply G-connected and f is a homology V-equivalence, then f is a V-equivalence.*

Now, we will state Freudenthal suspension theorem for C_2-spaces: For example, one can find it in [2]:

Theorem 4. *Let X be a pointed C_2-space.*
***(i)** Suppose that the underlying space of X is m-connected ($m \geq 1$), and X^{C_2} is n-connected ($n \geq 1$), then for $p + q \leq 2m$ and $q \leq 2n$*

$$\Sigma : \pi_U(X) \longrightarrow \pi_{U+1}(S^1 \wedge X)$$

is isomorphic, and epimorphic if $p + q \leq 2m + 1$ and $q \leq 2n + 1$, where $U = p\sigma + q$,
***(ii)** Suppose that the underlying space of X is m-connected, and X^{C_2} is path connected, then for $p+q \leq 2m$ and $q < m$*

$$\Sigma^\sigma : \pi_U(X) \longrightarrow \pi_{U+\sigma}(S^\sigma \wedge X)$$

is isomorphic, and it is epimorphic if $p + q \leq 2m + 1$ and $q \leq m$, where $U = p\sigma + q$.

For $X = S^V$, this theorem is given by Bredon in [3] before:

Theorem 5. *For the representations $U = p\sigma + q$, $V = r\sigma + s$ and the suspension and twisted suspension homomorphisms*

$$\pi_{U+1}(S^{V+1}) \overset{\Sigma}{\longleftarrow} \pi_U(S^V) \overset{\Sigma^\sigma}{\longrightarrow} \pi_{U+\sigma}(S^{V+\sigma})$$

Σ is epimorphism when $p+q \leq 2(r+s)-1$ and $q \leq 2s-1$, and isomorphism if the strict inequalities hold. Similarly, Σ^σ is epimorphism when $p+q \leq 2(r+s)-1$ and $q \leq r+s-1$, and isomorphism if the strict inequalities hold.

2. C_2-equivariant James splitting

Let (X, q) be a pair consisting of a path-connected compact topological C_2-space with a basepoint $x_0 \in X$, and a continuous map $q : X \longrightarrow \mathbb{R}_+$, where $\mathbb{R}_+$ is a nonnegative real numbers such that

(i) $q^{-1}(0) = x_0$.
(ii) $q(g.x) = q(x)$ for all $g \in C_2$, and $x \in X$.

Let

$$\Gamma^V(X, q) = V \times X/\{(v, x)| \parallel v \parallel \geq q(x)\},$$

which is called C_2-Moore suspension of X. We define the action of C_2 by $g.|(v, x)| = |(g.v, g.x)|$. It is easy to see that $\Gamma^V(X, q)$ is C_2-homeomorphic to $\Sigma^V X$. We define the space $\Omega^{*V}\Gamma^V(X, q)$ as

$$\Omega^{*V}\Gamma^V(X, q) = \{(r, f) \in \mathbb{R}_+ \times Map(V, \Gamma^V(X, q))|\forall v \in V \ \|v\| \geq r \Rightarrow f(v) = x_0\}$$

with the action $g.(r, f) = (r, g.f)$ [10]. The space $\Omega^{*V}\Gamma^V(X, q)$ is called C_2-Moore loops on $\Gamma^V(X, q)$. Rybicki [10, Lemma 1.1.] showed that $\Omega^{*V}\Gamma^V(X, q)$ is homotopy equivalent to $\Omega^V\Sigma^V X$.

Now, we define a continuous C_2-map $\bar{\lambda} : X \longrightarrow \Omega^{*V}\Gamma^V(X, q)$ by $\bar{\lambda}(x) = (q(x), \lambda_x(-))$, where $\lambda_x(v) = |(v, x)|$. The map $\bar{\lambda}$ extends to a continuous C_2-map

$$\lambda : J^{C_2}(X) \longrightarrow \Omega^{*V}\Gamma^V(X, q) \tag{2.1}$$

defined by $\lambda(x_1 \cdots x_k) = \bar{\lambda}(x_1) \cdots \bar{\lambda}(x_k)$, which is given by Rybicki in [10].

Let W and X be C_2-spaces. This part is the equivariant analogue of the work of George W. Whitehead in the book [12] on James splitting theorem. Let $f : (J_m^{C_2}(W), J_{m-1}^{C_2}(W)) \longrightarrow (X, *)$ be a C_2-map. We will construct an extension $g : J^{C_2}(W) \longrightarrow J^{C_2}(X)$ which is called combinatorial extension of f. Note that all the actions on the cartesian, smash and wedge products of G-spaces are twisted actions, where the twisted action means that the action of the nontrivial element γ of C_2 is reversing the order of the element on the product. For example, we are given the action of C_2 on the space $X \wedge Y$ by

$$g.(x \wedge y) = \begin{cases} (g.y) \wedge (g.x) & \text{for } g = \gamma \in C_2 \\ x \wedge y & \text{for } g = 1 \in C_2. \end{cases}$$

We use the notation $N^*(-)$ to remind the reader each time that the action on the products is the twisted one.

Remark 6. Let $h_m : N^*(W^m) \longrightarrow X$ be a sequence of C_2-maps such that $h_m \circ i_k = h_{m-1}$ for $k = 1, \cdots, m$, where $i_k : N^*(W^{m-1}) \longrightarrow N^*(W^m)$ is the map defined by

$$i_k(w_1, \cdots, w_{m-1}) = (w_1, \cdots, w_{k-1}, e, w_k, \cdots, w_{m-1}).$$

Then there is a map $h : J^{C_2}(W) \longrightarrow X$ such that $(h|J_m^{C_2}(W)) \circ \pi_m = h_m$ for $m = 1, 2, 3, \cdots$, where $\pi_m : N^*(W^m) \longrightarrow J_m^{C_2}(W)$ is the natural map defined by

$$\pi_m(w_1, \ldots, w_m) = w_1 \cdots w_m.$$

By Remark 6, it is enough to construct a sequence of C_2-maps $f_n : N^*(W^n) \longrightarrow J^{C_2}(X)$ $(n = 1, 2, \cdots)$ such that

$$f_n \circ i_k = f_{n-1}, \quad (k = 1, \cdots, n)$$
$$f_m = f \circ \pi_m.$$

f_n is a constant map for all $n < m$. For $n \geq m$, let P_n be the set of all strictly increasing m-termed subsequences of $(1, \cdots, n)$ with lexicographical order from the right; that is, $\alpha < \beta$ if and only if there exists $j (1 \leq j \leq m)$ such that $\alpha_i = \beta_i$ for $i > j$ and $\alpha_j < \beta_j$. Let $\alpha_1, \cdots, \alpha_N$ be the $N = \binom{n}{m}$ elements of P_n. For each $r (1 \leq r \leq N)$, define $g_r : N^*(W^n) \longrightarrow J^m(W)$ by

$$g_r(a_1, \cdots, a_n) = \pi_m(a_{\alpha_r}).$$

Then we will define a map $f_n : N^*(W^n) \longrightarrow J_N^{C_2}(X) \subset J^{C_2}(X)$ by

$$f_n(x) = \pi_N(fg_1(x), \cdots, fg_N(x)).$$

The combinatorial extension $g : J^{C_2}(W) \longrightarrow J^{C_2}(X)$ of f is defined by the condition

$$(g|J^n(W)) \circ \pi_n = f_n, \quad (n = 1, 2, \cdots).$$

In particular, let $N^*(W^{(n)})$ be the n-fold smash product with the twisted action. Then the natural projection $p_n : N^*(W^n) \longrightarrow N^*(W^{(n)})$ induces a map $f_n : (J_n^{C_2}(W), J_{n-1}^{C_2}(W)) \longrightarrow (N^*(W^{(n)}), *)$. Let

$$g_n : J^{C_2}(W) \longrightarrow J^{C_2}(N^*(W^{(n)}))$$

be the combinatorial extension of f_n. Let $X = \bigvee_{n=1}^{\infty} N^*(W^{(n)})$ and $i_n : N^*(W^{(n)}) \longrightarrow X$ be the inclusion, so $i'_n = J^{C_2}(i_n) : J^{C_2}(N^*(W^{(n)})) \longrightarrow J^{C_2}(X)$. If $x \in J_m^{C_2}(W)$, define $\theta_m(x) = \prod_{n=1}^m i'_n(g_n(x))$.

If $x \in J_{m-1}^{C_2}(W)$, then $g_m(x) = e$; hence, $\theta_m|J_{m-1}^{C_2}(W) = \theta_{m-1}$. Therefore, the maps θ_m together define a map $\theta : J^{C_2}(W) \longrightarrow J^{C_2}(X)$. Recall that $\Sigma^\sigma(X)$ and $\Gamma^\sigma(X, q)$ are C_2-homeomorphic. Let $\widetilde{\theta}$: $\Sigma^\sigma J^{C_2}(W) \longrightarrow \Sigma^\sigma(X)$ be the adjoint to the composite map

$$J^{C_2}(W) \xrightarrow{\theta} J^{C_2}(X) \xrightarrow{\lambda} \Omega^{*\sigma}\Gamma^\sigma(X, q) \xrightarrow{\Psi} \Omega^\sigma\Sigma^\sigma(X)$$

where λ was defined earlier (2.1), and $\Omega^{*\sigma}\Gamma^\sigma(X, q) \xrightarrow{\Psi} \Omega^\sigma\Sigma^\sigma(X)$ is a homotopy equivalence [10, Lemma 1.1.].

Now, we will give the splitting theorem:

Theorem 7. *If W is C_2-connected, and $(W)^{C_2}$ is simply C_2-connected, then the map $\widetilde{\theta}$ is a weak C_2-homotopy equivalence.*

Proof. Since W is C_2-connected, $J^{C_2}(W)$ and $X = \bigvee_{n=1}^{\infty} N^*(W^{(n)})$ are C_2-connected, so $\Sigma^\sigma J^{C_2}(W)$ and $\Sigma^\sigma X$ are simply C_2-connected. Therefore, it is enough to show that $\widetilde{\theta}$ is a homology C_2-equivalence. Let $X_m = \bigvee_{n=1}^{m} N^*(W^{(n)})$, so $\widetilde{\theta}(\Sigma^\sigma J_m^{C_2}(W)) \subset \Sigma^\sigma X_m$ for every m. Therefore $\widetilde{\theta}$ induces

$$\widetilde{\theta}_m : \Sigma^\sigma J_m^{C_2}(W)/\Sigma^\sigma J_{m-1}^{C_2}(W) \longrightarrow \Sigma^\sigma X_m/\Sigma^\sigma X_{m-1}.$$

But,

$$\Sigma^\sigma J_m^{C_2}(W)/\Sigma^\sigma J_{m-1}^{C_2}(W) = \Sigma^\sigma(J_m^{C_2}(W)/J_{m-1}^{C_2}(W)) \cong \Sigma^\sigma N^*(W^{(m)})$$

as is $\Sigma^\sigma X_m/\Sigma^\sigma X_{m-1}$. It follows that $\widetilde{\theta}|\Sigma^\sigma J_m^{C_2}(W) : \Sigma^\sigma J_m^{C_2}(W) \longrightarrow \Sigma^\sigma X_m$ is naturally G-homeomorphism, so it is a homology C_2-equivalence. However, the homology groups of the filtered spaces $\Sigma^\sigma J^{C_2}(W)$ and $\Sigma^\sigma X$ are the direct limits of those of the subspaces $\Sigma^\sigma J_m^{C_2}(W)$ and $\Sigma^\sigma X_m$, respectively and therefore $\widetilde{\theta}$ is a homology C_2-equivalence. □

Our main interest is representation spheres $W = S^V$, so we have the following result.

Corollary 8. *$\widetilde{\theta}$: $\Sigma^\sigma J^{C_2}(S^V) \longrightarrow \bigvee_{k=1}^{\infty} S^{|V|k+\sigma}$ is a weak G-homotopy equivalence, so $\Sigma^\sigma\Omega^\sigma\Sigma^\sigma S^V$ is weak G-homotopy equivalent to $\bigvee_{k=1}^{\infty} S^{|V|k+\sigma}$.*

From this splitting, by collapsing all the appropriate factors of the wedge and then taking the adjoint, we get the Hopf invariant map

$$H^\sigma : \Omega^\sigma S^{V+\sigma} \longrightarrow \Omega^\sigma S^{V\otimes\rho+\sigma}.$$

There is also the map

$$E^\sigma : S^V \longrightarrow \Omega^\sigma S^{V+\sigma}$$

adjoint to the identity $S^{V+\sigma} \longrightarrow S^{V+\sigma}$, which induces the suspension homomorphism on the homotopy groups. Now, we will give C_2-EHP sequences.

3. C_2-equivariant EHP sequences

In order to construct the fibration we use the fact that $J^{C_2}(S^n) \simeq \Omega^\sigma\Sigma^\sigma S^n$, which is given in [10]. We have that $J_2^{C_2}(S^n) = S^n \times S^n/(x,e) \sim (e,x)$. This identification gives a copy of S^n in $J_2^{C_2}(S^n)$, and the quotient is $J_2^{C_2}(S^n)/J_1^{C_2}(S^n) = J_2^{C_2}(S^n)/S^n$ is $S^{n\rho}$. As denoting the quotient map $J_2^{C_2}(S^n) \longrightarrow S^{n\rho}$ by $x_1x_2 \longrightarrow \overline{x_1x_2}$, we will define $f : J^{C_2}(S^n) \longrightarrow J^{C_2}(S^{n\rho})$ by

$$f(x_1x_2\cdots x_k) = \overline{x_1x_2}\ \overline{x_1x_3}\cdots\overline{x_1x_k}\ \overline{x_2x_3}\ \overline{x_2x_4}\cdots\overline{x_2x_k}\cdots\overline{x_{k-1}x_k}.$$

It is easy to check that $f(x_1x_2\cdots x_k) = f(x_1x_2\cdots\widehat{x_i}\cdots x_k)$ if $x_i = e$ since $\overline{xe} = \overline{ex}$ is the identity element of $j_2^{C_2}(S^{n\rho})$, so it is well defined and also C_2-equivariant. Let F denote the homotopy fiber of f, so we have a G-fibration

$$F \longrightarrow J^{C_2}(S^n) \longrightarrow J^{C_2}(S^{n\rho}).$$

If we apply fixed point functor to f, we get

$$f^{C_2} : (J^{C_2}(S^n))^{C_2} \longrightarrow (J^{C_2}(S^{n\rho}))^{C_2}.$$

By Miguel Xicotencatl [13], we know that

$$(J^{C_2}(X))^{C_2} \simeq (\Omega^\sigma \Sigma^\sigma X)^{C_2} \simeq (\Omega\Sigma X) \times X^{C_2}.$$

Then we have $f^{C_2} : \Omega\Sigma S^n \times S^n \xrightarrow{H\times \mathbf{1}} \Omega\Sigma S^{2n} \times S^n$, where H is a classical Hopf map. Thus we get $F \simeq S^n$, so we have a G-fibration

$$S^n \xrightarrow{E^\sigma} \Omega^\sigma S^{n+\sigma} \xrightarrow{H^\sigma} \Omega^\sigma S^{n\rho+\sigma}.$$

We would like to generalize this G-fibration to S^V, where C_2-representations $V = r\sigma + s$. We can define the same map

$$f : J^{C_2}(S^V) \longrightarrow J^{C_2}(S^{V\otimes\rho}).$$

If we apply again fixed points functor we get

$$f^{C_2} : \Omega\Sigma S^{r+s} \times S^s \longrightarrow \Omega\Sigma S^{2r+2s} \times S^{r+s}.$$

It is not easy to determine what the fiber of this map is.

However, we can use the long exact sequence of the G-pair $(J^{C_2}(S^V), S^V)$ to construct the C_2-EHP sequences. For C_2-representations $V = r\sigma + s$, $U = p\sigma + q$, and C_2-pair $(J^{C_2}(S^V), S^V)$, the long exact sequence is

$$\cdots \longrightarrow \pi_U(S^V) \xrightarrow{E^\sigma} \pi_U(J^{C_2}(S^V)) \longrightarrow \pi_U(J^{C_2}(S^V), S^V) \longrightarrow \pi_{U-1}(S^V) \longrightarrow \cdots \tag{3.1}$$

By using the fact that $J^{C_2}(X) \simeq \Omega^\sigma\Sigma^\sigma X$, we get that

$$\cdots \longrightarrow \pi_U(S^V) \xrightarrow{E^\sigma} \pi_{U+\sigma}(S^{V+\sigma}) \longrightarrow \pi_U(J^{C_2}(S^V), S^V) \longrightarrow \pi_{U-1}(S^V) \longrightarrow \cdots \tag{3.2}$$

Now, before proceeding, I need to prove some results. All the following lemmas are valid for nonequivariant case, so these are also valid on fixed point spaces, so they are also true in equivariant case.

Lemma 9. *Suppose that (Y, B, B') is a G-triple such that (Y, B) and (B, B') are ν-connected. Then (Y, B') is ν-connected.*

Proof. By assumption, we know that the inclusions $j : B' \longrightarrow B$ and $k : B \longrightarrow Y$ are ν-equivalences, so $i = k \circ j : B' \longrightarrow Y$ is also ν-equivalence. □

Corollary 10. *Let $\{X_d\}$ be a filtration of a G-space X. If each of the pairs (X_{q+1}, X_q) is ν-connected, then (X, X_0) is ν-connected.*

It can be proved that by induction on m:

Lemma 11. *If G-space X is ν-connected, then $J_m^{C_2}(X)$ is simply G-connected, and $(J_{m+1}^{C_2}(X), J_m^{C_2}(X))$ is $((m+1)\nu + m)$-connected.*

Then by Corollary 10, it follows that

Lemma 12. *If G-space X is ν-connected, then $(J^{C_2}(X), J_m^{C_2}(X))$ is $((m+1)\nu + m)$-connected.*

In particular, $(J^{C_2}(X), J_2^{C_2}(X))$ is $(3\nu + 2)$-connected. Therefore, the injection $i : (J_2^{C_2}(X), X) \longrightarrow (J^{C_2}(X), X)$ is an $(3\nu + 2)$-equivalence. For $X = S^V$ which is $|(V-1)^*|$-connected, $i : (J_2^{C_2}(S^V), S^V) \longrightarrow (J^{C_2}(S^V), S^V)$ is an $(3|(V-1)^*| + 2)$-equivalence. By the lemma 1.2. of G. Lewis in [9], it is $(3(V-1)+2)$-equivalence. Thus $i_* : \pi_U(J_2^{C_2}(S^V), S^V) \longrightarrow \pi_U(J^{C_2}(S^V), S^V)$ is an isomorphism for $p + q < 3(r + s) - 1$ and $q < 3s - 1$.

Now, I will state equivariant Blakers-Massey theorem which is proved by Hauschild in [6] and one application equivariant homotopy excision theorem.

Theorem 13. *[6] (Blakers-Massey theorem) Let X_1 and X_2 be subcomplexes of the G-CW-complex X such that $X = X_1 \cup X_2$ with non-empty intersection $X_0 = X_1 \cap X_2$. If*

$$\pi_i(X_1^H, X_0^H) = 0 \;\; for \;\; 0 < i < m_H,$$
$$\pi_i(X_2^H, X_0^H) = 0 \;\; for \;\; 0 < i < n_H,$$

and $|U^H| < m_H + n_H - 2$ for all subgroups H of G, then the map induced by inclusion

$$i_U : \pi_U(X_2, X_0) \longrightarrow \pi_U(X, X_1)$$

is an isomorphism.

One important consequence of the Blakers-Massey theorem is homotopy excision theorem:

Theorem 14. *(Equivariant homotopy excision theorem) Let $f : (X, A) \longrightarrow (Y, B)$ be a map such that $f_* : H_*(X, A) \approx H_*(Y, B)$ for all ∗. Suppose that X, A, and B are simply G-connected, (X^H, A^H) is m_H-connected, and $f|A^H : A^H \longrightarrow B^H$ is n_H-connected for all subgroups H of G. Then $f_* : \pi_U(X, A) \longrightarrow \pi_U(Y, B)$ is an isomorphism for $|U^H| < m_H + n_H + 1$.*

Proof. Let Z be the mapping cylinder of f, and C be the mapping cylinder of $f|A^H : A^H \longrightarrow B^H$. There are commutative diagrams

$$\begin{array}{ccc} \pi_U^H(X, A) & \xrightarrow{\pi_U(i)} & \pi_U^H(X \cup C, C) \\ \downarrow{\scriptstyle \pi_U(f)} & & \downarrow{\scriptstyle \pi_U(j)} \\ \pi_U^H(Y, B) & \xrightarrow[\pi_U(k)]{} & \pi_U^H(Z, C) \end{array} \qquad \begin{array}{ccc} H_U^H(X, A) & \xrightarrow{H_U(i)} & H_U^H(X \cup C, C) \\ \downarrow{\scriptstyle H_U(f)} & & \downarrow{\scriptstyle H_U(j)} \\ H_U^H(Y, B) & \xrightarrow[H_U(k)]{} & H_U^H(Z, C) \end{array}$$

where i, j, k are inclusions. Since $H_U(f), H_U(i), H_U(k)$ are isomorphism for all U, so is $H_U(j)$. By exactness of the homology sequence of the triple $(Z; X \cup C, C)$, the groups $H_U^H(Z, X \cup C) = 0$ are zero for all U. However, X and C are simply G-connected, and their intersection A is simply G-connected. From Hurewicz theorem, we can deduce that $\pi_U^H(Z, X \cup C) = 0$ and therefore, $\pi_U(j)$ is an isomorphism for all U. On the other hand, we can apply the Blakers-Massey theorem to the triad $(X \cup C, X, C)$ and therefore $\pi_U(i)$ is an isomorphism for $|U^H| < m_H + n_H + 1$. Since $\pi_U(k)$ is an isomorphism, then $\pi_U(f)$ has the desired properties. □

Now, we will return the long exact sequence (3.2). There is a relative G-homeomorphism $f : (N^*(S^V \times S^V), N^*(S^V \vee S^V)) \longrightarrow (J_2^{C_2}(S^V), S^V)$. For $V = r\sigma + s$ and $s > 1$, the underlying spaces $N^*(S^V \times S^V)$ and $N^*(S^V \vee S^V)$ are 1-connected, and the pair

$$(N^*(S^V \times S^V), N^*(S^V \vee S^V))$$

is $(2(r+s)-1)$-connected. Moreover, the map $f|N^*(S^V \vee S^V) : N^*(S^V \vee S^V) \longrightarrow S^V$ is $(r+s-1)$-connected. It follows from the equivariant homotopy excision theorem that

$$f_* : \pi_U(N^*(S^V \times S^V), N^*(S^V \vee S^V)) \longrightarrow \pi_U(J_2^{C_2}(S^V), S^V)$$

is an isomorphism for $p+q \leq 3(r+s)-2$. And also, $(N^*(S^V \times S^V))^{C_2} \simeq S^s$ and $(N^*(S^V \vee S^V))^{C_2} \simeq *$ are 1-connected, and the pair

$$((N^*(S^V \times S^V))^{C_2}, (N^*(S^V \vee S^V))^{C_2})$$

is $(s-1)$-connected. Moreover, the map

$$f|(N^*(S^V \vee S^V))^{C_2} : (N^*(S^V \vee S^V))^{C_2} \longrightarrow (S^V)^{C_2}$$

is $(s-1)$-connected. It follows from the equivariant homotopy excision theorem that

$$f_* : \pi_U(N^*(S^V \times S^V), N^*(S^V \vee S^V)) \longrightarrow \pi_U(J_2^{C_2}(S^V), S^V)$$

is an isomorphism for $q \leq 2s-2$. On the other hand, the quotient map is a relative G-homeomorphism $g : (N^*(S^V \times S^V), N^*(S^V \vee S^V)) \longrightarrow (N^*(S^V \wedge S^V), *)$. The map $g|N^*(S^V \vee S^V) : N^*(S^V \vee S^V) \longrightarrow *$ is $(r+s-1)$-connected, so we can deduce that

$$g_* : \pi_U(N^*(S^V \times S^V), N^*(S^V \vee S^V)) \longrightarrow \pi_U(N^*(S^V \wedge S^V))$$

is an isomorphism for $p+q \leq 3(r+s)-2$. And also, the map

$$g|(N^*(S^V \vee S^V))^{C_2} : (N^*(S^V \vee S^V))^{C_2} \longrightarrow *$$

is G-homeomorphism, so we can deduce that

$$g_* : \pi_U(N^*(S^V \times S^V), N^*(S^V \vee S^V)) \longrightarrow \pi_U(N^*(S^V \wedge S^V))$$

is an isomorphism for all q.

Also, we know that with twisted action $N^*(S^V \wedge S^V) \simeq S^{|V|\rho}$. By equivariant Freudenthal suspension theorem

$$E^\sigma : \pi_U(S^{|V|\rho}) \longrightarrow \pi_{U+\sigma}(\Sigma^\sigma(S^{|V|\rho}))$$

is an isomorphism for $p+q < 4(r+s)-1$ and $q < 2(r+s)-1$.

If we put together all the result above, we proved that the lemma for S^V:

Lemma 15. *For $p+q < 3(r+s)-2$, $q < 2(r+s)-1$, and $q < 3s-1$, $\pi_U(J^{C_2}(S^V), S^V)$ and $\pi_U(J^{C_2}(N^*(S^V \wedge S^V)))$ are isomorphic, where the action on $S^V \wedge S^V$ is again twisted, so we have*

$$\pi_U(J^{C_2}(S^V), S^V) \simeq \pi_U(J^{C_2}(S^{V \otimes \rho})).$$

Note that this lemma stated first in [10] without proof and for with different range. By inserting this result to the long exact sequence (3.2), we deduce that

$$\cdots \longrightarrow \pi_U(S^V) \xrightarrow{E^\sigma} \pi_{U+\sigma}(S^{V+\sigma}) \longrightarrow \pi_{U+\sigma}(S^{|V|\sigma+\sigma}) \longrightarrow \pi_{U-1}(S^V) \longrightarrow \cdots \tag{3.3}$$

Thus in the range $p+q < 3(r+s)-2$, $q < 2(r+s)-1$, and $q < 3s-1$, we have a G-fibration

$$S^V \xrightarrow{E^\sigma} \Omega^\sigma S^{V+\sigma} \xrightarrow{H^\sigma} \Omega^\sigma S^{V\otimes\rho+\sigma}. \tag{3.4}$$

We also know that $J(S^V) \simeq \Omega\Sigma S^V$, where the action on $J(S^V)$ and $\Omega\Sigma S^V$ are deduced from the cartesian product and quotient map and conjugation action on function spaces, respectively as usual (not twisted). We have also 2-local C_2-fibrations

$$S^V \xrightarrow{E} \Omega S^{V+1} \xrightarrow{H} \Omega S^{2V+1}. \tag{3.5}$$

To show the existence, it is enough to look underlying and fixed points fibrations of (3.5). Let $V = r\sigma + s$ and $\Omega(X) = \mathrm{Map}(S^1, X)$ be a C_2-space of all continuous maps. Fixed points of it is $(\Omega\Sigma(X))^{C_2} = \Omega\Sigma(X^{C_2})$. Then fixed points fibrations of (3.5) are

$$S^s \xrightarrow{E} \Omega S^{s+1} \xrightarrow{H} \Omega S^{2s+1}$$

which is a 2-local fibration. And underlying fibrations of (3.5) are

$$S^{r+s} \xrightarrow{E} \Omega S^{r+s+1} \xrightarrow{H} \Omega S^{2(r+s)+1}$$

which is also a 2-local fibration.

As a result, one can compute unstable C_2-homotopy groups of equivariant spheres by using these EHP sequences. For example, when $U = 2$ and $V = 3$, we have

$$\cdots \longrightarrow \pi_2(S^3) \xrightarrow{E^\sigma} \pi_{2+\sigma}(S^{3+\sigma}) \longrightarrow \pi_{2+\sigma}(S^{4\sigma}) \longrightarrow \pi_1(S^3) \longrightarrow \cdots \tag{3.6}$$

Since $\pi_2(S^3) = \pi_1(S^3) = 0$, we have

$$\pi_{2+\sigma}(S^{3+\sigma}) \cong \pi_{2+\sigma}(S^{4\sigma}).$$

Because $\pi_{2+\sigma}(S^{3+\sigma}) \cong \mathbb{Z}$ by [1], $\pi_{2+\sigma}(S^{4\sigma}) \cong \mathbb{Z}$.

One project is to compute the $RO(C_2)$-graded C_2-equivariant stable and unstable homotopy groups of C_2-equivariant spheres by using the C_2-EHP spectral sequences, and the C_2-Lambda algebra, which is constructed in the author's dissertation.

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