Mackey Functors In Equivariant Homotopy and Cohomology Theory

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Equivariant Cohomology Theories

Definition

A G-complex X is a CW-complex with an action of G so that X^H for any $H \leq G$ is a subcomplex.

We would like to give the cohomology of a G-complex so that information regarding the action of G is incorporated.

Definition

An equivariant cohomology theory is a sequence of contravariant functors H^n_G : G-complexes $\to Ab$

Equivariant Cohomology Theories

Definition

The orbit category O_G is the category consisting of objects G/H for $H \leq G$ and morphisms $G/H \to G/K$ whenever $g^{-1}Hg \subset K$ for some $g \in G$.

Since G-complexes are built from the orbits G/H using equivariant maps $G/H \to G/K$, any ECT should include groups $H^n(G/H)$ and homomorphisms $H^n(G/K) \to H^n(G/H)$.

Coefficient Systems

Definition

A coefficient system $\underline{M}$ is a contravariant functor from O_G, the orbit category, to Ab.
The collection of coefficient systems forms a category $\mathcal{C}_G$.

In equivariant <u>ordinary</u> cohomology:

$$H^*(G/H; \underline{M}) = H^0(G/H; \underline{M}) = \underline{M}(G/H)$$

for any $\underline{M} \in \mathcal{C}_G$

Coefficient Systems

Example

A Mackey functor $\underline{M}$ is a pair of functors

$$M^* : O_G^{op} \to Ab \text{ and } M_* : O_G \to Ab$$

such that

$$M^*(X) = M_*(X) = \underline{M}(X)$$

and which send disoint unions to direct sums and satisfy certain commutativity relations.

Notation: For $f : G/H \to G/K$ we call $M^*(f)$ a restriction and $M_*(f)$ a transfer.

Bredon Cohomology

If X is a G-complex, define the chains on X by:

$$\underline{C_*(X)}(G/H) = C_*(X^H)$$

Then for $\underline{M} \in \mathcal{C}_G$ define the cochains by:

$$C^n_G(X; \underline{M}) = Hom_{\mathcal{C}_G}(\underline{C_n(X)}, \underline{M})$$

and so we define Bredon cohomology to be

$$H^n_G(X; \underline{M}) = H^n(C^*_G(X; \underline{M}))$$

(Nonstable) Equivariant Homotopy Groups

An alternative definition for Bredon cohomology can be given since it is, in fact, representable. To give this, we must have definitions for equivariant homotopy groups.

Definition

Let X be a G-space. For each $H \leq G$ the equivariant homotopy groups of X are given by

$$\pi_n^H(X) = [S^n \wedge G/H_+, X]_G$$

Stable Equivariant Homotopy Groups

Definition

A G-spectrum X is a collection of G-spaces X_k together with equivariant maps $\Sigma X_k \to X_{k+1}$ (or equivalently $X_k \to \Omega X_{k+1}$)

Definition

The equivariant homotopy groups of the G-spectrum X are given by

$$\pi_n^H(X) = [\Sigma^\infty S^n \wedge (G/H)_+, X]_G$$

Stable Equivariant Homotopy Groups

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$$\pi_n^H(X) = [\Sigma^\infty S^n \wedge (G/H)_+, X]_G$$

or equivalently

$$\pi_n^H(X) = \operatorname*{colim}_k \pi_n^H(X_k)$$

Note: These homotopy groups form a Mackey functor:

$$\underline{\pi_n(X)}(G/H) = [\Sigma^\infty S^n \wedge G/H_+, X]_G = \pi_n^H(X)$$

Equivariant Homotopy Group Mackey Functor

$$
\begin{aligned}
\underline{\pi_k(X)}(G/H) &= [G/H_+ \wedge S^k, X]_G \\
&= [G_+ \wedge_H S^k, X]_G \\
&= [S^k, X]_H \\
&= \pi_k(X^H)
\end{aligned}
$$

Restriction Map $\quad \pi_k(X^K) \to \pi_k(X^H)$

Induced from inclusion of fixed points $X^K \to X^H$

Transfer Map $\quad \pi_k(X^H) \to \pi_k(X^K)$

Induced from

$$
\begin{aligned}
X^H &\to X^K \\
x &\to \sum_{gH \in K/H} g \cdot x
\end{aligned}
$$

Cohomology Theories from G-Spectra

Let X be a G-space and Y be a G-spectrum.

The groups $[\Sigma^{k-n}X, Y_k]_G$ form a direct system:

$$[\Sigma^{k-n}X, Y_k]_G \to [\Sigma^{k-n+1}X, \Sigma Y_k]_G \to [\Sigma^{k-n+1}X, Y_{k+1}]_G$$

So we can define cohomology:

$$\begin{aligned}\widetilde{Y}_G^n(X) &= \operatorname*{colim}_k [\Sigma^{k-n}X, Y_k]_G \\ &= \operatorname*{colim}_k \pi_{k-n}(F(X, Y_k)^G) \\ &= \pi_{-n}(\underline{F}(X, Y)^G)\end{aligned}$$

Equivalence of Categories

From which G-spectrum do we obtain Bredon cohomology?

Proposition

There is an equivalence of categories between the category of Mackey functors and the homotopy category consisting of G-spectra X such that $\underline{\pi_i(X)} = 0$ *for* $i \neq 0$*.*

In particular, for any Mackey functor $\underline{M}$, we have an associated Eilenberg-MacLane spectrum $H\underline{M}$ satisfying:

$$\underline{\pi_k(H\underline{M})} = \begin{cases} \underline{M} & \text{k=0} \\ 0 & \text{otherwise} \end{cases}$$

Bredon Cohomology

Now for any Mackey functor $\underline{M}$, we may obtain Bredon Cohomology from $H\underline{M}$ as follows:

$$
\begin{aligned}
\widetilde{H}^n_G(X;\underline{M}) &= \operatorname*{colim}_k [\Sigma^{k-n}X, (H\underline{M})_k]_G \\
&= [\Sigma^{-n}X, H\underline{M}]_G \\
&= \pi_{-n}(F(X, H\underline{M})^G) \\
&= \pi^G_{-n}(F(X, H\underline{M})) \\
&= \underline{\pi_{-n}(F(X, H\underline{M}))}(G/G)
\end{aligned}
$$

Note: For a group G, Bredon Cohomology is the image of G/G under a Mackey functor.

RO(G)-grading

In working with equivariant theories, we want to consider spheres with nontrivial G-action. In particular, we will look at linear spheres arising from representations of G.

Definition

For a group G and a vector space V, we will say a representation of G is a homomorphism $\rho : G \to O(V)$

Definition

For a representation space V we will write S^V to denote the one-point compactification of V

RO(G)-graded Homotopy Groups

If $V \in RO(G)$ then it is also an H-representation for any $H \leq G$ so we have RO(G)-graded homotopy groups:

$$\pi_V^H(X) = [S^V, X]_H = [G_+ \wedge_H S^V, X]_G$$

Note: Our original $\mathbb{Z}$-graded homotopy groups $\pi_n^H(X)$ are the homotopy groups associated to the trivial representation $n \in RO(G)$ where n stands for $\mathbb{R}^n$.

RO(G)-graded Cohomology

In addition to usual $\mathbb{Z}$-suspensions we have:

$$\Sigma^V X = X \wedge S^V$$

for any $V \in RO(G)$

So extending the usual suspension axiom

$$\sigma_n : H^n(X) \to H^{n+1}(\Sigma X)$$

we obtain RO(G)-graded cohomology groups:

$$H_G^\alpha(X) \cong H_G^{\alpha+V}(\Sigma^V X)$$

for $\alpha, V \in RO(G)$

Why is the RO(G)-grading important?

A few examples:

- (Lewis) Let X be a $\mathbb{Z}/p$-complex constructed from even dimensional unit disks of real G-representations. The $H^*_G(X)$ is a free $RO(G)$-graded module over the equivariant ordinary cohomology of a point.
- (Lewis) Let V be a complex G-representation and P(V) the associated complex projective space. Then all generators of $H^*_G(P(V))$ live in dimensions corresponding to nontrivial representations of G.
- (tom Dieck) RO(G)-graded cohomology theories admit important splitting theorems.

When can we extend?

In the $RO(G)$-graded setting we have transfer maps

$$\tau(G/H) : S^V \to (G/H)_+ \wedge S^V$$

These induce transfer homomorphisms

$$\begin{array}{c} \widetilde{H}^n_H(X;\underline{M}) \cong \widetilde{H}^{V+n}_G(\Sigma^V(G/H_+ \wedge X);\underline{M}) \\ \downarrow \\ \widetilde{H}^{V+n}_G(\Sigma^V X;\underline{M}) \cong \widetilde{H}^n_G(X;\underline{M}) \end{array}$$

If $n = 0$ and $X = S^0$ we get a transfer map

$$\underline{M}(G/H) \to \underline{M}(G/G)$$

When Can We Extend?

If this argument is elaborated a bit we get that the coefficient system $\underline{M}$ must actually be a Mackey functor.

Additionally it can be shown that this necessary condition is also sufficient:

Theorem

(May, Waner) The ordinary $\mathbb{Z}$-graded cohomology theory $\widetilde{H}_G^*(-;\underline{M})$ extends to an RO(G)-graded theory if and only if $\underline{M}$ extends to a Mackey functor.

Mackey Functor Valued Cohomology

We may additionally think of our Equivariant Cohomology Theory as being Mackey functor valued:

$$H_G^{\alpha}(X; \underline{M}) = \underline{\pi_{-k}(X)}(G/G)$$

and

$$H_H^{\alpha}(X; \underline{M}) = \underline{\pi_{-k}(X)}(G/H)$$

In general we have

$$\underline{H_G^{\alpha}(X; \underline{M})}(G/H) = H_G^{\alpha}(G/H_+ \wedge X; \underline{M})$$

An Example

$$\begin{array}{ccc} \underline{H^{\alpha}_{C_p}(X;\underline{M})}(C_p/C_p) = H^{\alpha}_{C_p}(X;\underline{M}) \\ \pi^* \downarrow \quad \uparrow \pi_! \\ \underline{H^{\alpha}_{C_p}(X;\underline{M})}(C_p/e) = H^{\alpha}_{C_p}(C_p \times X;\underline{M}) \end{array}$$

π^* is induced from the projection $\pi : C_p \times X \to X$

$\pi_!$ is the transfer map arising from regarding the projection π as a covering space.

Note: $H^{\alpha}_{G}(G \times X;\underline{M}) \cong H^{|\alpha|}(X;\underline{M}(G/e))$