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The homology of matrix Lie algebras over rings and the Hochschild homology

B.L. Tsygan

1. To begin with we introduce certain functors from the category of associative algebras to the category of vector spaces. The connection between these functors and the homology of Lie algebras is the main content of this note.

Let k be a commutative ring with identity, and A an associative k-algebra with identity. We consider the standard Hochschild resolution of the bimodule A:

$$R_{n-1} = \overset{n}{\otimes} A;\ d_{n-1}(a_1 \otimes \dots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n.$$

The cyclic group of order i acts on R_{i-1}: if t_i is a generator of this group, we set $t_i(a_1 \otimes \dots \otimes a_i) = a_2 \otimes \dots \otimes a_i \otimes a_1$; $\tilde{t}_i = (-1)^{i-1} t_i$.

Let Q_{i-1} be the invariants of this action: $Q_{i-1} = \{x \in \overset{i}{\otimes} A : \tilde{t}_i x = x\}$.

It turns out that the Q_i form a subcomplex in R. We denote its i-th homology group by $K_i^+(A)$.

We consider the tensor algebra $\mathcal{T}$ of the space $\bigotimes_{i=0}^{\infty} K_i^+(A)$. If to the elements of $K_{i-1}^+(A)$ we assign the degree i, then $\mathcal{T}$ becomes a graded algebra. The quotient algebra of $\mathcal{T}$ by the ideal generated by the elements of the form

$$ab - (-1)^{\deg a \cdot \deg b} ba,$$

is denoted by $\mathcal{H}(A)$.

On $\mathcal{H}(A)$ there is a unique comultiplication whose set of primitive elements is the same as $\bigoplus_{i=0}^{\infty} K_i^+(A)$, which turns $\mathcal{H}(A)$ into a Hopf algebra.

***Theorem* 1.** *Let k be a field of characteristic zero. Then the co-algebras $\mathcal{H}(A)$ and $H_*(\mathfrak{gl}(n, A); k)$ are isomorphic in dimensions $0 \leqslant i \leqslant n$.*

***Corollary* 1.** *There is an isomorphism of co-algebras*

$$H_*(\mathfrak{gl}(\infty, A); k) \simeq \mathcal{H}(A). \tag{1}$$

For the case $A = k[\xi]$ Theorem 1 is a special case of the standard isomorphism $H_*(\mathfrak{g}[\xi]; k) \simeq H_*(\mathfrak{g}, k)$, where $\mathfrak{g}$ is a semisimple Lie algebra. The case $A = k[\xi, \xi^{-1}]$ is contained in [1].

The proof of Theorem 1 uses the fact that the homology of the Lie algebra $\mathfrak{gl}(n, A)$ with coefficients in the trivial module is the same as that of the subcomplex of chains that are invariant under $\mathfrak{gl}(n, k)$. The terms of this subcomplex in dimensions $i \leqslant n$ can be written out explicitly, starting from the theory of invariants. In the space of invariant i-forms on $\mathfrak{gl}(n, k)$ there is a standard basis whose elements carry over under the isomorphism $\mathfrak{gl}(n, k) \simeq (k^n)^* \otimes k^n$ to the forms

$$\Phi_\sigma(\varphi_1 \otimes x_1, \dots, \varphi_i \otimes x_i) = \varphi_1(x_{\sigma 1}) \dots \varphi_i(x_{\sigma i}), \quad \varphi_j \in (k^n)^*, \quad x_j \in k^n, \quad \sigma \in S_i.$$

We consider the basis $\{e_\sigma : \sigma \in S_i\}$ of the space of invariants in $\overset{i}{\otimes}\mathfrak{gl}(n, k)$, which is dual to the basis $\{\Phi_\sigma\}$. Any invariant i-dimensional chain of the complex for the computation of the homology $H_*(\mathfrak{gl}(n, A); k)$ has the form

$$\omega = \sum_\sigma a_\sigma \otimes e_\sigma, \tag{2}$$

where $a_\sigma \in \overset{i}{\otimes} A$ and $s a_\sigma = (-1)^{l(s)} a_\sigma$, $s(a_1 \otimes \dots \otimes a_i) = a_{s1} \otimes \dots \otimes a_{si}$ for any s commuting with σ. It can be shown that chains in the expansion (2) of which only those basis invariants that correspond to cyclic permutations of σ occur form a subcomplex. Moreover, it turns out that this subcomplex is exactly isomorphic to Q, while the whole complex of invariant chains is a free cocommutative co-algebra having the above subcomplex as the complex of primitive elements.

2. We now consider the relations between the groups $K_i^+(A)$ and the Hochschild homology. Note that, on the one hand, $\overset{i+1}{\otimes} A$ is the i-th term of the standard resolution of the bimodule A and, on the other hand, it is the i-th term of the complex C for the computation of the homology $H^n(A, A)$:

$$\delta_{i-1}(a_1 \otimes \ldots \otimes a_i)=\sum_{j=1}^{i-1}(-1)^j a_1 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_i + a_2 \otimes \ldots \otimes a_{i-1} \otimes a_i a_1.$$

Let β_{i-1} and γ_{i-1} be maps from $\overset{i}{\otimes} A$ to $\overset{i}{\otimes} A$: $\quad \beta_{i-1}=\tilde{t}_i-\mathrm{id}, \quad \gamma_{i-1}=\mathrm{id}+\tilde{t}_i+\ldots+\tilde{t}_i^{i-1}$.

***Proposition* 1.** *The sequence of complexes* $\ldots \xrightarrow{\gamma} R \xrightarrow{\beta} C \xrightarrow{\gamma} R \xrightarrow{\beta} C \xrightarrow{\gamma} \ldots$ *is exact.*

Proof. Let $\xi_i\colon \overset{i}{\otimes} A \to \overset{i-1}{\otimes} A$; $\xi_i(a_1 \otimes \ldots \otimes a_i) = a_2 \otimes \ldots \otimes a_{i-1} \otimes a_i a_1$. Then

$$\delta_{i-1}=\sum_{j=0}^{i-1} \tilde{t}_{i-1}^{-j}\xi_i\tilde{t}_i^j; \qquad d_{i-1}=\sum_{j=1}^{i-1} \tilde{t}_{i-1}^{-j}\xi_i\tilde{t}_i^j,$$

$$\gamma_{i-1}\delta_i=\sum_{j=0}^{i-1}\tilde{t}_i^j\sum_{j=0}^{i}\tilde{t}_i^{-j}\xi_{i+1}\tilde{t}_{i+1}^j=\sum_{j=0}^{i-1}\tilde{t}_i^j\xi_{i+1}\sum_{j=0}^{i}t_{i+1}^j=d_i\gamma_i;$$

$\beta_{i-1}d_i=$

$$=\sum_{j=1}^{i}\tilde{t}_i^{1-j}\xi_{i+1}\tilde{t}_{i+1}^j-\sum_{j=0}^{i}\tilde{t}_i^{-j}\xi_{i+1}\tilde{t}_{i+1}^j+\xi_{i+1}=\sum_{j=0}^{i}\tilde{t}_i^{-j}\xi_{i+1}\tilde{t}_{i+1}^{j+1}-\sum_{j=0}^{i}\tilde{t}_i^{-j}\xi_{i+1}\tilde{t}_{i+1}^j=\delta_i\beta_i.$$

***Corollary* 2.** *The sequence* $\ldots \longrightarrow H_i(A, A) \longrightarrow K_i^+(A) \longrightarrow K_{i-2}^+(A) \longrightarrow H_{i-1}(A, A) \longrightarrow \ldots$ *is exact.*

Proof. We consider the exact sequence

$$\ldots \longrightarrow H_i(A, A) \longrightarrow H_i(C/\ker\gamma) \longrightarrow H_{i-1}(\ker\gamma) \longrightarrow H_{i-1}(A, A) \longrightarrow \ldots$$

$$H_i(C/\ker\gamma) \simeq H_i(\mathrm{im}\,\gamma) \simeq K_i^+(A);$$

$$H_{i-1}(\ker\gamma) \simeq H_{i-1}(R/\ker\beta) \simeq H_{i-2}(\ker\beta) \simeq K_{i-2}^+(A)$$

(The last isomorphism but one follows from the fact that R is acyclic.)

***Corollary* 3.** *Let* $H_i(A, A) = 0$ *for* $i \geqslant n$. *Then in dimensions* $i \geqslant n-1$ *the groups* K_i^+: $K_i^+(A) \simeq K_{i+2}^+(A)$ *are periodic mod 2.*

3. *Example.* Let $\mathcal{F}(X)$ be the free algebra with space of generators X.

$$H_i(\mathcal{F}(X), \mathcal{F}(X)) = 0, \quad i > 1; \quad H_0(\mathcal{F}(X), \mathcal{F}(X)) \simeq \mathcal{F}(X)/[\mathcal{F}(X), \mathcal{F}(X)];$$

$$H_1(\mathcal{F}(X), \mathcal{F}(X)) = \ker d, \quad d\colon X \otimes \mathcal{F}(X) \to \mathcal{F}(X), \ d(x \otimes f) = xf - fx.$$

According to Corollary 3, $K_i^+(\mathcal{F}(X)) \simeq K_{i+2}^+(\mathcal{F}(X))$, $i \geqslant 1$; it can be shown that $K_2^+(\mathcal{F}(X)) \simeq k$, $K_1^+(\mathcal{F}(X)) \simeq 0$; hence, $K_i^+(\mathcal{F}(X)) = 0$, $i \equiv 1 \pmod 2$; $K_i^+(\mathcal{F}(X)) \simeq k$, $i > 0$, $i \,\vdots\, 2$, $K_0^+(\mathcal{F}(X), \mathcal{F}(X)) \simeq \mathcal{F}(X)/[\mathcal{F}(X), \mathcal{F}\ (X)]$.

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