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ON THE TOPOLOGICAL CYCLIC HOMOLOGY OF THE INTEGERS

By Stavros Tsalidis

Abstract. This article provides a computation of the mod p homotopy groups of the fixed points of the Topological Hochschild Homology of the ring of integers under the action of any finite subgroup of the cirle group whose order is a power of an odd prime p. This leads to a computation of the Topological Cyclic Homology groups of the ring of integers, and determines also the p-adic completion of the algebraic K-theory of the p-adic integers.

1. Introduction. The topological Hochschild homology $THH(R)$ is a spectrum (or infinite loop space) associated functorially to each algebraic ring R, or, more generally, to each strictly associative ring spectrum $\mathcal{R}$. It was introduced originally by Bökstedt in [B]. The spectrum $THH(R)$ comes equipped with a natural action of the circle group $\mathbb{T}$. The topological cyclic homology of R, $TC(R)$ is the homotopy inverse limit of a diagram of spectra with vertices the fixed point spectra $THH(R)^C$ of $THH(R)$ under the action of the finite cyclic subgroups C of the circle. The maps in this diagram are either inclusions of fixed points or certain "Frobenius" maps particular to the topological Hochschild homology construction (see [G]). The topological Hochschild and cyclic homology of R are both related to the algebraic K-theory of the ring R; their relation can be depicted by a commutative diagram of spectra

$$\begin{tikzcd} & TC(R) \arrow[d, "pr"] \\ K(R) \arrow[ur, "tr"] \arrow[r, "B"] & THH(R) \end{tikzcd}$$

where tr denotes the cyclotomic trace map of [BHM], and the composition $B = pr \circ tr$ is the Bökstedt trace map. A result of McCarthy [Mc] and computations of [HM] imply that the cyclotomic trace map is a homotopy equivalence after p-adic completion for a number of rings R including the cases $R = \mathbb{Z}/p^n$ and $R = \hat{\mathbb{Z}}_p$, the p-adic integers.

The main result of this paper is the computation of the mod p homotopy groups $\pi_*(THH(\mathbb{Z})^{C_{p^n}}; \mathbb{F}_p)$ for each $n \geq 1$. This is achieved using a result of our previous work [T2, Theorem 3.8] which we recall as

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THEOREM 1.1. *Let* (n) *denote the standard canonical map (see 3.1) relating fixed points to homotopy fixed points:*

$$(n)\colon\ THH(R)^{C_{p^n}} \longrightarrow THH(R)^{hC_{p^n}}.$$

If (1) *induces isomorphisms on mod p (resp. on p-adically completed) homotopy groups in all nonnegative degrees, then the same is true for* (n) *for each n* 2.

In this article we show (Theorem 5.3) that, when $R = \mathbb{Z}$, and p is an odd prime, then the hypothesis in the above theorem is satisfied, that is, the map (1) induces isomorphisms on mod p homotopy groups in all nonnegative degrees. Therefore, Theorem 1.1 applies, and we can conclude that

$$(THH(\mathbb{Z})^{C_{p^n}};\mathbb{F}_p) = (THH(\mathbb{Z})^{hC_{p^n}};\mathbb{F}_p)$$

for each n 1 and 0. Now, for each n 1, the groups $(THH(\mathbb{Z})^{hC_{p^n}};\mathbb{F}_p)$ are the abutment of the mod p homotopy fixed points spectral sequence $\{E^r(n)\}$ described in 3.3. We will use an inductive scheme, which was expounded in our thesis [T1], to determine completely the structure of differentials in the spectral sequences $\{E^r(n)\}$ for each n 1. Those differentials were conjectured by Bökstedt and Madsen in [BM1, Conjecture 4.3] where the authors tried to justify their conjecture by comparing with the corresponding spectral sequences for the "homotopy" ring QS^0 [BM1, Assertion 5.5]. We establish this conjecture here by a different approach based on the result of [T2] mentioned above. This gives a complete computation of the groups $(THH(\mathbb{Z})^{C_{p^n}};\mathbb{F}_p)$, thus answering a question of Carlsson's [C, Problem IV]. Moreover, by [BM1, Section 7], and [Mc], these computations determine also the mod p algebraic K-theory groups of the p-adic integers since

$$K(\widehat{\mathbb{Z}}_p;\mathbb{F}_p) = TC(\widehat{\mathbb{Z}}_p;\mathbb{F}_p) = (TC(\mathbb{Z});\mathbb{F}_p)$$

for each 0, and each odd prime p.

This paper is closely related and overlaps with the papers [BM1] and [BM2]: Conjecture 4.3 of [BM1] mentioned above was the motivation for proving Theorem 1.1, and a description of the calculations in this paper using the argument we just described was given in [BM2]. We believe that this article provides a more direct and complete account of these results.

Finally, I would like to thank Tom Goodwillie for reading the manuscript and suggesting corrections.

2. Topological Hochschild homology. There is now a number of alternative constructions for topological Hochschild homology resulting from the recent constructions of categories of spectra endowed with an associative smash product of [EKMM] and [S]. We will be using here the original construction of Bökstedt's [B], [G], [T2], which we now recall briefly.

Let $s.\mathcal{S}$ denote the category of based simplicial sets. A *functor with smash product* (FSP for short) is a functor $\mathcal{F}\colon\ s.\mathcal{S} \longrightarrow s.\mathcal{S}$, together with two natural transformations: the *multiplication* $\ \colon\ \mathcal{F}(X\ ) \wedge \mathcal{F}(Y\ ) \longrightarrow \mathcal{F}(X\ \ \wedge Y\ )$ of the FSP $\mathcal{F}$, which is *associative*, (i.e., $\ (\mathrm{id} \wedge\ ) = \ (\ \wedge \mathrm{id})$), and a *unit* $1\colon\ X\ \ \longrightarrow \mathcal{F}(X\ )$ for ; this means that the composites

$$S^0 \wedge \mathcal{F}(X) \xrightarrow{1 \wedge \mathrm{id}} \mathcal{F}(S^0) \wedge \mathcal{F}(X) \longrightarrow \mathcal{F}(S^0 \wedge X)$$

and

$$\mathcal{F}(X) \wedge S^0 \xrightarrow{\mathrm{id} \wedge 1} \mathcal{F}(X) \wedge \mathcal{F}(S^0) \longrightarrow \mathcal{F}(X \wedge S^0),$$

are both the identity transformation.

Examples. (i) Each unitary ring R defines an FSP $\mathcal{R}\colon\ s.\mathcal{S} \longrightarrow s.\mathcal{S}$ by means of the formula $\mathcal{R}(X\ ) = R[X\ ]/R[\ ]$, where $R[X\ ]$ denotes the simplicial free R-module generated by $X\ $, and the base point of $X\ $. The unit and the multiplication of the FSP $\mathcal{R}$ are defined by using the unit and multiplication of the ring R in the evident way. (ii) Any based simplicial monoid $M\ $ defines an FSP $\mathcal{M}$ by $\mathcal{M}(X\ ) = M\ \ \wedge X\ $. Again, the FSP structure maps are evident.

An FSP $\mathcal{F}$ determines a *"strictly associative"* ring spectrum $n \mapsto F_n = |\mathcal{F}(S^n)|$, with unit $1_n\colon\ S^n = |S^n| \xrightarrow{|1|} |\mathcal{F}(S^n)| = F_n$, and multiplication

$$_{m,n}\colon\ F_n \wedge F_m = |\mathcal{F}(S^n)| \wedge |\mathcal{F}(S^m)| = |\mathcal{F}(S^n) \wedge \mathcal{F}(S^m)| \xrightarrow{|\ |} |\mathcal{F}(S^{n+m})| = F_{n+m},$$

where $|\ \ |$ denotes geometric realization. The structure maps of the spectrum F are defined to be the composites

$$^n\colon\ S^1 \wedge F_n \xrightarrow{1_1 \wedge \mathrm{id}} F_1 \wedge F_n \xrightarrow{\ _{1,n}} F_{n+1}.$$

Clearly, the spectrum associated to the FSP $\mathcal{R}$ is an Eilenberg-McLane spectrum HR, whereas the spectrum associated to $\mathcal{M}$ is the suspension spectrum $\Sigma^\infty|M|$.

Let now $\mathcal{I}$ denote the category whose objects are finite sets of the form $\mathbf{n} = \{1, 2, \ldots, n\}$, and whose morphisms are all injections of such sets. The topological Hochschild homology of the FSP $\mathcal{F}$, which is denoted by $THH(\mathcal{F})$, is the geometric realization of a cyclic spectrum $THH\ (\mathcal{F})$ which is defined by the formula

$$THH_q(\mathcal{F}; S^m) = \underset{\mathbf{n} \in \mathcal{I}^{q+1}}{\mathrm{ho}\varinjlim}\, \mathrm{Map}\,(S^{n_0} \wedge \qquad \wedge S^{n_q}, S^m \wedge F_{n_0} \wedge \qquad \wedge F_{n_q}),$$

where $\mathbf{n} = (n_0, n_1, \ldots, n_q)$ runs through the objects of the product category $\mathcal{I}^{q+1}$. For the cyclic structure of $THH(\mathcal{F})$ see [T2, §3]. The spectrum structure maps

$$^{m}\colon\ S^1 \wedge THH(\mathcal{F}; S^m) \longrightarrow THH(\mathcal{F}; S^{m+1})$$

are induced by the canonical map $X \wedge \mathrm{Map}(Y, Z) \longrightarrow \mathrm{Map}(Y, X \wedge Z)$, using the fact that the smash product "commutes" with homotopy colimits.

Being the realization of a cyclic spectrum, $THH(\mathcal{F})$ is a spectrum with an action of the circle group $\mathbb{T}$; this means that, for each $m \quad 0$, $THH(\mathcal{F}; S^m)$ is a $\mathbb{T}$-space, and the structure maps $\ ^{m}$ are $\mathbb{T}$-equivariant. Moreover, $THH(\mathcal{F})$ can be "lifted" to a genuine $\mathbb{T}$-spectrum $TH_{\mathbb{T}}(\mathcal{F})$ [T2, §4]: for each $\mathbb{T}$-representation V the $\mathbb{T}$-space $TH_{\mathbb{T}}(\mathcal{F}; S^V)$ is defined to be the geometric realization of the cyclic space

$$[q] \mapsto \underset{\mathbf{n} \in \mathcal{I}^{q+1}}{\underrightarrow{\mathrm{holim}}}\, \mathrm{Map}(S^{n_0} \wedge \quad \wedge S^{n_q}, S^V \wedge F_{n_0} \wedge \quad \wedge F_{n_q}),$$

with the $\mathbb{T}$-action which combines the action of $\mathbb{T}$ on S^V and the cyclic structure of the above simplicial space. $TH_{\mathbb{T}}(\mathcal{F})$ is a "lifting" of $THH(\mathcal{F})$ in the sense that the restriction of $TH_{\mathbb{T}}(\mathcal{F})$ to trivial $\mathbb{T}$-representations is equivalent as a naive C_a-spectrum to $THH(\mathcal{F})$ for each finite cyclic subgroup C_a of $\mathbb{T}$.

We let $TH(\mathcal{F}) = \boldsymbol{LK}TH_{\mathbb{T}}(\mathcal{F})$, where $\boldsymbol{L}$ and $\boldsymbol{K}$ are the functors of [M]. That is, $\boldsymbol{L}$ is a left adjoint to the forgetful functor from G-spectra to G-prespectra, whereas $\boldsymbol{K}$ is the cylinder functor described in [M, ch. XII, Construction 9.6]. Thus $TH(\mathcal{F})$ is a $\mathbb{T}$-spectrum in the sense of [LMS] (i.e., the stucture maps are G-homeomorphisms) and of the same $\mathbb{T}$-homotopy type as the $\mathbb{T}$-prespectrum $TH_{\mathbb{T}}(\mathcal{F})$ defined above. When $\mathcal{F}$ is a commutative FSP (i.e., when $\quad (\quad \wedge \mathrm{id}) = \quad (\mathrm{id} \wedge \quad)$), then $TH(\mathcal{F})$ is a commutative ring $\mathbb{T}$-spectrum [HM, Proposition 1.7.1], and it is also a lifting of $THH(\mathcal{F})$. In other words, for each cyclic subgroup C $\mathbb{T}$, the fixed point spectra $THH(\mathcal{F})^C$ and $TH(\mathcal{F})^C$ are homotopy equivalent.

For the FSP $\mathcal{R}$ associated to an algebraic ring R we will write $THH(R)$ and $TH(R)$ instead of $THH(\mathcal{R})$ and $TH(\mathcal{R})$; these are both functors from the category of rings to the categories of naive and genuine $\mathbb{T}$-spectra respectively.

3. Homotopy orbits, homotopy fixed points, and Tate spectra. In this section we recall some of the theory of the homotopy fixed points, homotopy orbits, and Tate spectra associated to a G-spectrum K. The main reference for this material is [GM], which the reader should consult for more details. This theory works for any compact Lie group G. In our applications G will be either the circle group $\mathbb{T}$, or a finite cyclic subgroup C thereof. So, our presentation will be directed towards groups of that type.

3.1. Given any compact Lie group G, one can consider the *universal space* EG of G; this is a free contractible G-CW complex. The homotopy cofiber of the based G-map : $EG_+ \longrightarrow S^0$ with $(+) = 0$ and $(EG) = 1$ in $S^0 = \{0, 1\}$, is denoted by $\tilde{E}G$. Let $F(EG_+, K)$ denote the function spectrum of [LMS, ch. II, 3.1]. Then, the map induces a map of G-spectra

$$K = F(S^0, K) \longrightarrow F(EG_+, K)$$

which is a nonequivariant equivalence, since this is true for the map . Smashing the G-cofibration sequence

$$EG_+ \longrightarrow S^0 \longrightarrow \tilde{E}G$$

with the G-spectra K and $F(EG_+, K)$, and then taking G-invariants one gets the following commutative diagram of spectra whose columns are cofibration sequences of spectra

(D)
$$\begin{array}{ccc}
(EG_+ \wedge K)^G & \xrightarrow{\ \ {}'\ \ } & (EG_+ \wedge F(EG_+, K))^G \\
\downarrow & & \downarrow \\
K^G & \longrightarrow & F(EG_+, K)^G \\
\downarrow & & \downarrow \\
(\tilde{E}G \wedge K)^G & \xrightarrow{\ \ \tilde{}\ \ } & (\tilde{E}G \wedge F(EG_+, K))^G.
\end{array}$$

The three horizontal maps in this diagram are induced by ; namely, $= (\ \ )^G$, $' = (1_{EG_+} \wedge \ \ )^G$, and $\tilde{}\ = (1_{\tilde{E}G} \wedge \ \ )^G$. Since is a non-equivariant equivalence, the map $1_{EG_+} \wedge$ is an equivariant equivalence by the G-Whitehead theorem [LMS, ch. II, Theorem 5.10], and, therefore, $'$ is an equivalence of non-equivariant spectra. Moreover, the Adams' isomorphism theorem [LMS, ch. II, Theorem 7.1.] provides an equivalence of spectra

$$\Sigma^{Ad(G)}(EG_+ \underset{G}{\wedge} i\ \ K) \xrightarrow{\simeq} (EG_+ \wedge K)^G$$

where $i\ \ K$ denotes the naive G-spectrum associated to K, and $Ad(G)$ the adjoint representation of G. The spectrum

$$K_{hG} = (EG_+ \underset{G}{\wedge} i\ \ K) \simeq \Sigma^{-Ad(G)}(EG_+ \wedge K)^G$$

is called the *homotopy orbits* spectrum of K, while the spectrum $K^{hG} = F(EG_+, K)^G$ is called the *homotopy fixed points* spectrum of K. Finally, the G-spectrum $t(K) = \tilde{E}G \wedge F(EG_+, K)$ is called the *Tate* spectrum of K. There are spectral sequences abutting to the homotopy groups of the spectra K_{hG}, K^{hG}, and $t(K)^G$ which we now describe briefly.

3.2. The homotopy orbits spectral sequence $\{E^r(K_{hG})\}$. Let EG be a filtration of EG by finite G-CW subcomplexes so that

$$EG^p/EG^{p-1} = G_+ \wedge \bigvee S^p$$

where $\vee S^p$ is a finite wedge sum of p-spheres. When G is finite, one can take EG^p to be the p-skeleton of EG, i.e., $EG^p = EG^{(p)} = |\,\mathrm{sk}^p\, EG\,|$ where EG is the standard simplicial set with $EG_q = G^{q+1}$. When $G = \mathbb{T}$ the circle group, then one can take $E\mathbb{T} = \varinjlim S(q\mathbb{C})$, where $S(q\mathbb{C})$ is the unit sphere in the vector space $q\mathbb{C} = \mathbb{C}^{\;q}$ on which $\mathbb{T}$ acts by multiplication. The following string of cofibrations gives rise to the homotopy orbits spectral sequence

$$(st^G)\qquad \begin{array}{ccccccc} (EG^0_+ \wedge K)^G & \longrightarrow & \quad\longrightarrow (EG^{p-1}_+ \wedge K)^G & \longrightarrow & (EG^p_+ \wedge K)^G & \longrightarrow \\ & & & & \big\downarrow & \\ & & & & (G_+ \wedge \vee S^p \wedge K)^G. & \end{array}$$

Now let G be a finite group. Then [GM]

$$E^1_{p,q}(K_{hG}) = \ {}_{p+q}(G_+ \wedge (\vee S^p) \wedge K)^G = \ {}_qK \underset{\mathbb{Z}[G]}{\ } H_p(G_+ \wedge \vee S^p).$$

The groups $H_p(G_+ \wedge \vee S^p) = H_0(G_+) \quad H_p(\vee S^p) = \mathbb{Z}[G] \quad H_p(\vee S^p)$ together with the respective boundary maps ∂_p make up a resolution of $\mathbb{Z}$ by relatively injective $\mathbb{Z}[G]$-modules, and it follows that

$$E^2_{p,q}(K_{hG}) = H_p(G;\ \ {}_qK).$$

By the Adams isomorphism mentioned in 3.1, the string of cofibrations (st^G) which gives rise to the spectral sequence $\{E^r(K_{hG})\}$ is equivalent to the following one

$$(st_G)\qquad \begin{array}{ccccccc} EG^0_+ \underset{G}{\wedge} i\ \ K & \longrightarrow & \quad\longrightarrow EG^{p-1}_+ \underset{G}{\wedge} i\ \ K & \longrightarrow & EG^p_+ \underset{G}{\wedge} i\ \ K & \longrightarrow \\ & & & & \big\downarrow & \\ & & & & G_+ \underset{G}{\wedge} (\vee S^p \wedge i\ \ K). & \end{array}$$

Of course, we are still assuming that G is finite, so that $Ad(G) = \{0\}$.

For any subgroup $H \quad G$ one can consider K as an H-spectrum by forgetting the additional structure. The space EG is also a model for EH and one can use the same filtration $EH^p = EG^p$. The spectral sequence $\{E^r(K_{hH})\}$ is then induced by the string

$$(st_H)\qquad \begin{tikzcd} EG^0_+ \underset{H}{\wedge} i\ K \arrow[r] & \ \arrow[r] & EG^{p-1}_+ \underset{H}{\wedge} i\ K \arrow[r] & EG^p_+ \underset{H}{\wedge} i\ K \arrow[r] \arrow[d] & \ \\ & & & G_+ \underset{H}{\wedge} (\vee S^p \wedge i\ K). & \end{tikzcd}$$

The canonical quotient map $L/H \longrightarrow L/G$ which is defined for each naive G-spectrum L induces a map of strings $(st_H) \longrightarrow (st_G)$, and therefore it also induces a map of spectral sequences

$$:\ \{E^r(K_{hH})\} \longrightarrow \{E^r(K_{hG})\}.$$

The map will be called *corestriction*, because on E^2-terms

$$:\ E^2_{p,q}(K_{hH}) = H_p(H;\ \ _qK) \longrightarrow H_p(G;\ \ _qK) = E^2_{p,q}(K_{hG})$$

is the classical corestriction homomorphism in group homology [Se, ch. II, §5].

3.3. The *homotopy fixed points* spectral sequence $\{E^r(K^{hG})\}$ is induced by the following tower of fibrations of spectra

$$(to^G)\qquad \begin{tikzcd} F(EG^0_+, K)^G \arrow[r, dash] & \ \arrow[r, dash] & F(EG^{p-1}_+, K)^G \arrow[r, dash] & F(EG^p_+, K)^G \arrow[r, dash] & \ \\ & & & F(G_+ \vee S^p, K)^G. \arrow[u] & \end{tikzcd}$$

When G is finite one has [GM, (9.3)]

$$E_1^{p,q}(K^{hG}) = \ \ _{p+q}F(G_+ \wedge \vee S^p, K)^G = \mathrm{Hom}_{\mathbb{Z}[G]}\left(H_p(G_+ \wedge \vee S^p),\ \ _{-q}K\right)$$

and $E_2^{p,q}(K^{hG}) = H^p(G;\ \ _{-q}K)$. For a subgroup $H \quad G$, the inclusion of fixed points $L^G \quad L^H$ defines a map of towers $(to^G) \longrightarrow (to^H)$, and, therefore, it induces a map of spectral sequences

$$:\ \{E^r(K^{hG})\} \longrightarrow \{E^r(K^{hH})\}.$$

The map will be called *restriction* because on E_2-terms

$$:\ E_2^{p,q}(K^{hG}) = H^p(G;\ \ _{-q}K) \longrightarrow H^p(H;\ \ _{-q}K) = E_2^{p,q}(K^{hG})$$

is the classical restriction map in group cohomology [Se, ch. VII, §5].

3.4. The Tate spectral sequence $\{E^r(t(K)^G)\}$. This is induced by a suitable "filtration" of the suspension G-spectrum of $\tilde{E}G = S^0 \quad C(EG_+)$ where $C(X)$ denotes the cone on the based space X. Using the same symbol for a space and its suspension spectrum, this "filtration" is given, when G is finite, by [GM, 9.5]:

$$\tilde{E}G^p = \begin{cases} S^0 \quad C(EG_+^{p-1}) & \text{for} \quad p \quad 0 \\ S^0 & \text{for} \quad p = 0 \\ D(\tilde{E}G^{-p}) & \text{for} \quad p \quad 0, \end{cases}$$

where $D(X) = F(X, S_G)$ is the Spanier-Whitehead G-dual spectrum of X [LMS]. The following sequence of cofibrations, then, gives rise to the Tate spectral sequence $\{E^r(t(K)^G)\}$:

$$(Seq^G) \qquad \begin{array}{ccccc} \longrightarrow (\tilde{E}G^{p-1} \wedge K)^G & \longrightarrow & (\tilde{E}G^p \wedge K)^G & \longrightarrow \\ & & \big\downarrow & \\ & & (G_+ \wedge \vee S^p \wedge K)^G. & \end{array}$$

When $G = \mathbb{T}$ one can take $\tilde{E}\mathbb{T} = \varinjlim S^{q\mathbb{C}}$, with $\tilde{E}\mathbb{T}^{2p} = \tilde{E}\mathbb{T}^{2p-1} = S^{q\mathbb{C}}$ for each $p \in \mathbb{Z}$. In both cases the successive cofibers are of the form $G_+ \wedge \vee S^p$. When G is finite, then

$$E^2_{p,q}(t(K)^G) = \widehat{H}^{-p}(G; \quad {}_qK)$$

where $\widehat{H}\ (G; A)$ denotes the Tate cohomology of the group G with coefficients in the G-module A. When $G = \mathbb{T}$, then $\mathbb{T}$ acts trivially on K and

$$E^2_{p,q}(t(K)^G) = \mathbb{Z}[t, t^{-1}] \qquad K.$$

as an algebra where $\deg(t) = -2$.

3.5. When K is a connective spectrum (as it will be in the case of our computations), then $\{E^r(t(K)^G)\}$ is an upper half plane spectral sequence, and it "contains" the homotopy fixed points spectral sequence in its left quadrant. For the precise relationship between these two spectral sequences see [BM1, Theorem 2.15]. In other words there is a map of spectral sequences $: \{E^r(K^{hG})\} \longrightarrow \{E^r(t(K)^G)\}$ which on E^2-terms is the standard surjection of the group cohomology onto the Tate cohomology. On the other hand, the right quadrant of the Tate spectral sequence "contains" a shifted copy of the homotopy orbits spectral sequence.

We now consider the case of the $\mathbb{T}$-spectrum $K = TH(\mathcal{F})$. The next lemma gives the fundamental equivariant property of the $\mathbb{T}$-spectrum $TH(\mathcal{F})$:

LEMMA 3.6. [T2, Proposition 5.9] *There are equivalences of spectra*

$$\begin{aligned}(\tilde{E}C_{p^n} \wedge TH(\mathcal{F}))^{C_{p^n}} &\simeq (\Phi^{C_p} TH(\mathcal{F})\ {}_{C_{p^n}})^{C_{p^{n-1}}} \simeq (TH(\mathcal{F})\ {}_{C_{p^{n-1}}})^{C_{p^{n-1}}} \\ &\simeq TH(\mathcal{F})^{C_{p^{n-1}}}\end{aligned}$$

for each n 1. *Moreover, these equivalences are functorial in $\mathcal{F}$.*

Here, $K\ {}_H$ denotes the G-spectrum K considered as an H-spectrum, H being a subgroup of G. The diagram (D) of 3.1 with $K = TH(\mathcal{F})$, and the above lemma imply the following:

PROPOSITION 3.7. *For each n* 1, *there is a homotopy cartesian diagram of spectra*

$$\begin{tikzcd} TH(\mathcal{F})^{C_{p^n}} \arrow[r] \arrow[d] & TH(\mathcal{F})^{hC_{p^n}} \arrow[d, "\omega"] \\ TH(\mathcal{F})^{C_{p^{n-1}}} \arrow[r, "\sim"] & t(TH(\mathcal{F}))^{C_{p^n}} \end{tikzcd}$$

whose vertical fibers are equivalent to the homotopy orbits spectrum $TH(\mathcal{F})_{hC_{p^n}}$.

As we mentioned in the end of §2, when $\mathcal{F}$ is a commutative FSP, then $TH(\mathcal{F})$ is a commutative ring $\mathbb{T}$-spectrum [HM], and $TH(\mathcal{F})\ {}_{C_a}$ is a commutative C_a-spectrum. It then follows [GM, Proposition 3.5] that the diagram of Proposition 3.7 is a diagram of ring spectra and ring maps, and that the respective homotopy fixed points and Tate spectral sequences are spectral sequences of differential algebras.

4. Some permanent cycles. For the remaining part of this paper p will always denote an *odd* prime number. We will let $\{E^r(n)\}$ denote the mod p homotopy fixed points spectral sequence for $TH(\mathbb{Z})^{hC_{p^n}}$ for each n 1, and $\{E^r(\infty)\}$ the corresponding spectral sequence for $TH(\mathbb{Z})^{h\mathbb{T}}$. Furthermore, $\{\hat{E}^r(n)\}$ will denote the mod p Tate spectral sequence for $t(TH(\mathbb{Z}))^{C_{p^n}}$, and $\{\hat{E}^r(\infty)\}$ the mod p Tate for $t(TH(\mathbb{Z}))^{\mathbb{T}}$.

The homotopy type of $THH(\mathbb{Z})$ was determined in [B]: $THH(\mathbb{Z})$ is a generalized Eilenberg-MacLane spectrum with

$$THH(\mathbb{Z}) \simeq H(\mathbb{Z}) \vee \bigvee_{n=1}^{\infty} \Sigma^{2n-1} H(\mathbb{Z}/n)$$

and $(TH\mathbb{Z}; \mathbb{F}_p) = \mathbb{F}_p(e)$ $\mathbb{F}_p[f]$ with $\deg(e) = 2p-1$ and $\deg(f) = 2p$. For any ring R, we will use the standard convention that $R[x, y, ..]$ is the polynomial algebra and $R(x, y, \ldots)$ is the exterior algebra over R (generated by the variables $x, y, \ldots$).

The spectral sequences $\{E^r(\infty)\}$ and $\{\hat{E}^r(\infty)\}$ have E^2-terms:

$$E^2_{\ ,\ }(\infty) = H^{-}\ (B\mathbb{T};\quad (T\,H\,\mathbb{Z};\mathbb{F}_p)) = \mathbb{F}_p[t] \quad \mathbb{F}_p[\,f\,] \quad \mathbb{F}_p(e)$$

and

$$\hat{E}^2_{\ ,\ }(\infty) = \mathbb{F}_p[t,t^{-1}] \quad \mathbb{F}_p[\,f\,] \quad \mathbb{F}_p(e)$$

with $\deg(t) = -2$, $\deg(\,f\,) = 2p$, and $\deg(e) = 2p-1$.

One can compare these spectral sequences with the corresponding ones for the spectrum $T\,H(\mathcal{I})$, where $\mathcal{I}$ denotes the identity FSP with $\mathcal{I}(X) = X$ for each X in $s.\mathcal{S}$. The spectrum associated to the FSP $\mathcal{I}$ is clearly the sphere spectrum S. Now, the inclusion

$$S^V \longrightarrow T\,H\,H(\mathcal{I})(S^V) = \ [q] \mapsto \underrightarrow{\mathrm{holim}}\,\mathrm{Map}\,(S^{n_0} \wedge \qquad \wedge S^{n_q}, S^V \wedge S^{n_0} \wedge \qquad \wedge S^{n_q})$$

defines a map of $\mathbb{T}$-spectra $: S_{\mathbb{T}} \longrightarrow T\,H(\mathcal{I})$, where $S_{\mathbb{T}}$ denotes the $\mathbb{T}$-equivariant sphere spectrum. The map is a non-equivariant equivalence since the underlying non-equivariant spectra of $S_{\mathbb{T}}$ and $T\,H(\mathcal{I})$ are both equivalent to the sphere spectrum S. It follows that induces homotopy equivalences $S_{\mathbb{T}}^{h\mathbb{T}} \xrightarrow{\simeq} T\,H(\mathcal{I})^{h\mathbb{T}}$, and $S_{\mathbb{T}}^{hC} \xrightarrow{\simeq} T\,H(\mathcal{I})^{hC}$ for each cyclic group $C \quad \mathbb{T}$. The unit of the FSP $\mathcal{Z}$ associated to the ring of integers $\mathbb{Z}$ defines a map of FSP's $\mathcal{I} \longrightarrow \mathcal{Z}$, and consequently a map of $\mathbb{T}$-spectra $T\,H(\mathcal{I}) \longrightarrow T\,H(\mathbb{Z})$. Composing with one gets a map of $\mathbb{T}$-spectra $S_{\mathbb{T}} \longrightarrow T\,H(\mathbb{Z})$. Now $S_{\mathbb{T}}$ is a *split* spectrum in the sense that

$$S_{\mathbb{T}}^{h\mathbb{T}} = \mathrm{Map}\,(E\mathbb{T}_+, S_{\mathbb{T}})^{\mathbb{T}} \simeq \mathrm{Map}\,(B\mathbb{T}_+, S),$$

where S denotes the (non-equivariant) sphere spectrum. By choosing a base point in $B\mathbb{T}$, and evaluating maps $B\mathbb{T}_+ \longrightarrow S$ at , one gets a map $\mathrm{Map}\,(B\mathbb{T}_+, S) \longrightarrow S$ which is split by the map $S \longrightarrow \mathrm{Map}\,(B\mathbb{T}_+, S)$ sending each point $x \in S$ to the constant map $f_x\colon\ B\mathbb{T}_+ \longrightarrow S$ with $f_x(t) = x$ for each $t \in B\mathbb{T}$. This implies that the zero-th column in the mod p homotopy fixed point spectral sequence for $S_{\mathbb{T}}^{h\mathbb{T}}$ (and the same is true for $S_{\mathbb{T}}^{hC_a}$ for each $C_a \quad \mathbb{T}$) consists of permanent cycles. In particular, the generator v_1 of $\ _{2p-2}(S;\mathbb{F}_p) = \mathbb{F}_p$ is a permanent cycle in the spectral sequence for $S_{\mathbb{T}}^{h\mathbb{T}}$ and represents a nonzero class in its E^∞-term.

LEMMA 4.1. [BM1, Lemma 5.4] *The map* $S \longrightarrow S_{\mathbb{T}}^{h\mathbb{T}} \longrightarrow T\,H(\mathbb{Z})^{h\mathbb{T}}$ *sends the homotopy class* $v_1 \in \ _{2p-2}(S;\mathbb{F}_p)$ *to a nonzero class in* $\ _{2p-2}(T\,H(\mathbb{Z})^{h\mathbb{T}};\mathbb{F}_p)$ *which represents the class tf in the* E^∞*-term of the spectral sequence* $\{E^r(\infty)\}$. *Consequently, the class tf and all its powers* $(tf)^n$ *are permanent cycles in* $\{E^r(\infty)\}$, *they survive to the* E^∞*-term, and they are mapped on by the classes* v_1^n.

Remark 4.2. Using the restriction maps $: \{E^r(\infty)\} \longrightarrow \{E^r(n)\}$ one gets that tf and its multiples are permanent cycles in all spectral sequences $\{E^r(n)\}$

for $n \quad 1$. Moreover, under the spectral sequence maps

$$: \{E^r(n)\} \longrightarrow \{\hat{E}^r(n)\}, \qquad 1 \quad n \quad \infty,$$

the classes t, f and e in $E^2(n)$ are mapped to the corresponding classes with the same name in $\hat{E}^2(n)$. It follows that tf and all its powers are permanent cycles in $\{\hat{E}^r(n)\}$ for each $1 \quad n \quad \infty$.

Let $\{E^r\}$ denote the (integral) homotopy fixed points spectral sequence for $TH(\mathbb{Z})^{h\mathbb{T}}$. It has

$$E^2_{k,\ell} = H^{-k}(B\mathbb{T}; \ {}_\ell\, TH(\mathbb{Z}))$$

and so $E^2_{\ ,0} = \mathbb{Z}[t]$ and for each $n \quad 1, E^2_{\ ,2n} = 0$, and $E^2_{\ ,2n-1} = (\mathbb{Z}/n)[t]$ with $\deg(t) = -2$. The map of $\mathbb{T}$-spectra $TH(\mathbb{Z}) \longrightarrow TH(\mathbb{Z})/p$ induces a map of spectral sequences $\varphi^r\colon \{E^r\} \longrightarrow \{E^r(\infty)\}$ under which the generators $\underline{t} \in E^2_{-2,0} = \mathbb{Z}$ and $\underline{e} \in E^2_{0,2p-1} = \mathbb{F}_p$ are mapped on the classes t and e in $E^2(\infty)$, i.e., $\varphi \ (\underline{t}) = t$, and $\varphi \ (\underline{e}) = e$. The first possibly nonzero differential in $\{E^r\}$ is d^{2p}. Since $d^{2p}(\underline{t}^{\ell p}) = \ell p t^{\ell p-1} d\underline{t} = 0$ in $\mathbb{F}_p$, for each $\ell \quad 0$, the classes $\underline{t}^{\ell p}\underline{e}$ are not hit by the differential d^{2p}, and the same is true for dimensional reasons for the classes $\underline{t}^m\underline{e}$ with $0 \quad m \quad p$. It follows that the classes $\underline{t}^{\ell p}\underline{e}$ and $\underline{t}^m\underline{e}$ for $\ell \quad 0$ and $0 \quad m \quad p$ survive and represent permanent cycles in the spectral sequence $\{E^r\}$, because all differentials on these classes are zero for dimensional reasons—all classes in $E^2_{\ ,\ }$, except the ones on the base line, are in odd total degree.

LEMMA 4.3. *For each $\ell \quad 0$, and each $0 \quad m \quad p$ the classes $t^{\ell p}e$ and $t^m e$ are permanent cycles in the spectral sequences $\{E^r(n)\}$ for each $1 \quad n \quad \infty$. Therefore, the corresponding classes $t^{\ell p}e$ and $t^m e$ are also permanent cycles in $\{\hat{E}^r(n)\}$ for each $\ell \in \mathbb{Z}$ and $1 \quad n \quad \infty$.*

Proof. The statement for $\{E^r(\infty)\}$ is shown by using the map of spectral sequences $\varphi\colon \{E^r\} \longrightarrow \{E^r(\infty)\}$, since $\varphi(\underline{t}^m\underline{e}) = t^m e$ for each $m \quad 0$. The statement for $\{E^r(n)\}$ then follows by using the restriction maps

$$: \{E^r(\infty)\} \longrightarrow \{E^r(n)\}$$

since $\quad (t^m e) = t^m e$, and the one for $\{\hat{E}^r(n)\}$ using the maps $\{E^r(n)\} \longrightarrow \{\hat{E}^r(n)\}$.

5. The Tate spectral sequence $\{\hat{E}^r(1)\}$. In this section we determine the differentials in the Tate spectral sequence $\{\hat{E}^r(1)\}$, and compute the mod p homotopy groups $t(TH\mathbb{Z})^{C_p}$. Actually, from this point on all homotopy groups will be assumed to be with mod p coefficients, and $\quad K$ will mean the mod p homotopy of K, unless it is explicitly stated otherwise. The E^2-term of the spectral sequence

$\{\hat{E}^r(1)\}$, as an $\mathbb{F}_p$-algebra, is given by

$$\hat{E}^2_{\ ,} \ = \mathbb{F}_p[t, t^{-1}] \quad \mathbb{F}_p(u_1) \quad \mathbb{F}_p[f] \quad \mathbb{F}_p(e)$$

with $\deg(t) = -2, \deg(f) = 2p$, and $\deg(e) = 2p-1$. The first possibly nonzero differential is d^{2p}. However, we first get some information about the next differential d^{2p+1}.

Lemma 5.1. *The differential d^{2p+1} is nonzero on the class u_1. That is, $d^{2p}u_1 = 0$, and $d^{2p+1}u_1 = t^{p+1}f$ up to multiplication by a nonzero unit in $\mathbb{F}_p$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "="] & TH(\mathbb{Z})^{C_p} \arrow[r] \arrow[d] & TH(\mathbb{Z})^{hC_p} \arrow[d, "\omega"] \\
S \arrow[r, "\sim"] & TH(\mathbb{Z}) \arrow[r, "\sim"] & t(TH(\mathbb{Z}))^{C_p}.
\end{tikzcd}$$

As we mentioned in Lemma 4.1, v_1 is a nonzero class in $_{2p-2}TH(\mathbb{Z})^{hC_p}$ which represents tf in the E^2-term (and also in the E^∞-term) of the spectral sequence $\{E^r(1)\}$. So ω v_1 is a class in $_{2p-2}t(TH(\mathbb{Z}))^{C_p}$ represented in $\hat{E}^2_{-2,2p}(1)$ by tf. But ω $v_1 = \tilde{\ } \tilde{\ } v_1 = 0$ since $\tilde{\ } v_1 \in {}_{2p-2}TH(\mathbb{Z}) = 0$. Since tf is a permanent cycle in $\{\hat{E}^r(1)\}$, this means that tf must be hit by a differential. This can only happen for d^{2p+1} and so $d^{2p+1}(t^{-p}u) = tf$. Since

$$d^{2p}(t^{-p}u) = d^{2p}(t^{-p})u + t^{-p}d^{2p}u = -pt^{-p-1}u + t^{-p}d^{2p}u = t^{-p}d^{2p}u \in \mathbb{F}_p$$

and $t^{-p} \neq 0$ one gets that $d^{2p}u \neq 0$. One now checks easily that $d^{2p+1}(t^{-p}) = 0$ independently of the behavior of the differential d^{2p}: if $d^{2p}(t) = 0$, then t survives to $\hat{E}^{2p+1}(1)$ and $d^{2p+1}(t^{-p}) = -pt^{-p-1} = 0 \in \mathbb{F}_p$; whereas, if $d^{2p}(t) \neq 0$, then $d^{2p+1}(t^{-p}) = 0$ for dimensional reasons. Since

$$d^{2p+1}(t^{-p}u) = d^{2p+1}(t^{-p})u + t^{-p}d^{2p+1}u = t^{-p}d^{2p+1}u$$

it follows that $d^{2p+1}u \neq 0$.

We now show that $d^{2p}t \neq 0$. Consider the commutative diagram of $\mathbb{T}$-spectra

$$\begin{tikzcd}
TH(\mathbb{Z}) \arrow[r, "p"] \arrow[d] & TH(\mathbb{Z}) \arrow[r] \arrow[d, "p"] & TH(\mathbb{Z})/p \arrow[d] \\
TH(\mathbb{Z}) \arrow[r, "p^2"] \arrow[d] & TH(\mathbb{Z}) \arrow[r] \arrow[d] & TH(\mathbb{Z})/p^2 \arrow[d] \\
{} \arrow[r] & TH(\mathbb{Z})/p \arrow[r, "="] & TH(\mathbb{Z})/p \arrow[d] \\
 & & \Sigma TH(\mathbb{Z})/p
\end{tikzcd}$$

in which all rows and columns are cofibration sequences and $X \xrightarrow{k} X$ denotes the multiplication by k map. Recall that $\ K$ denotes the mod p homotopy of K, in other words, the integral homotopy of K/p. We have $\ \Sigma TH(\mathbb{Z}) = \Sigma\ \ TH(\mathbb{Z})$, and we denote by Σx the class in $\ \Sigma TH(\mathbb{Z})$ which corresponds to the class $x \in\ \ TH(\mathbb{Z})$.

The map induces a map of Tate spectral sequences with $\ (f) = \Sigma e$. So $\ (d^{2p+1}u) =\ (ft^{p+1}) = \Sigma t^{p+1}e$. But

$$(d^{2p+1}u) = d^{2p+1}\ \ (u) = d^{2p+1}0 = 0.$$

This means that the class $\Sigma t^{p+1}e$ is zero in the E^{2p+1}-term of the mod p Tate spectral sequence for $\Sigma TH(\mathbb{Z})$ with respect to the group C_p. It follows that $\Sigma t^{p+1}e$ must be hit by the differential d^{2p}. Since the Tate spectral sequence for $\Sigma TH(\mathbb{Z})$ is just the suspension of the corresponding Tate for $TH(\mathbb{Z})$ we get that $d^{2p}(t) = t^{p+1}e$ modulo multiplication by units. We have therefore shown

THEOREM 5.2. *In the Tate spectral sequence $\{E^r(1)\}$ all differentials are determined by $d^{2p}(t) = t^{p+1}e$ and $d^{2p+1}u_1 = t^{p+1}f$. One then computes*

$$E^{2p+1}_{*,*}(1) = \mathbb{F}_p[t^p, t^{-p}] \quad \mathbb{F}_p(u) \quad \mathbb{F}_p[tf] \quad \mathbb{F}_p(e)$$

and

$$t(TH(\mathbb{Z}))^{C_p} = E^{\infty}_{*,*}(1) = E^{2p+2}_{*,*}(1) = \mathbb{F}_p[t^p, t^{-p}] \quad \mathbb{F}_p(e).$$

Consider the map ˜ : $\ \ TH(\mathbb{Z}) \longrightarrow\ \ t(TH(\mathbb{Z}))^{C_p}$. Notice that in nonnegative degrees the homotopy groups of the domain and range of ˜ are isomorphic, and that in order to show that ˜ is an isomorphism in nonnegative degrees it is enough to show that $˜_{2p}$ and $˜_{2p-1}$ are nontrivial, ˜ being a ring map. Now, the map of FSP's $\mathcal{Z}(\ ) \longrightarrow \mathcal{Z}/p(\ )$ induces the following commutative diagram where all four groups involved are isomorphic to $\mathbb{F}_p$:

$$\begin{array}{ccc} {}_{2p}TH(\mathbb{Z}) & \xrightarrow{\ \tilde{}_{2p}\ } & {}_{2p}t(TH(\mathbb{Z}))^{C_p} \\ \big\downarrow & & \big\downarrow \\ {}_{2p}TH(\mathbb{Z}/p) & \xrightarrow{\ \tilde{}_{2p}\ } & {}_{2p}t(TH(\mathbb{Z}/p))^{C_p}. \end{array}$$

The left vertical map in the above diagram is an isomorphism by [B], and the lower $\tilde{}_{2p}$ is an isomorphism by [HM, Proposition 4.3]. It follows that the upper $\tilde{}_{2p}$ is also an isomorphism. Since $\tilde{}$ commutes with the Bockstein , we have

$$\tilde{}_{2p-1}(e) = \tilde{}_{2p-1}(\quad f) = \quad \tilde{}_{2p-1}(f) \neq 0.$$

Therefore, the map $\tilde{}$ is an isomorphism in nonnegative degrees. From the homotopy cartesian square of Lemma 5.1

$$\begin{array}{ccc} TH(\mathbb{Z})^{C_p} & \longrightarrow & TH(\mathbb{Z})^{hC_p} \\ \big\downarrow & & \big\downarrow \\ TH(\mathbb{Z}) & \xrightarrow{\ \tilde{}\ } & t(TH(\mathbb{Z}))^{C_p} \end{array}$$

one gets that $\ :\ TH(\mathbb{Z})^{C_p} \longrightarrow\ TH(\mathbb{Z})^{hC_p}$ is also an isomorphism for each $\ 0$. Theorem 3.8 of [T2] then implies the following:

THEOREM 5.3. *For each $n\ \ 1$, the homomorphisms*

$$:\ TH(\mathbb{Z})^{C_{p^n}} \longrightarrow\ TH(\mathbb{Z})^{hC_{p^n}},\ \textit{and}\ \tilde{}\ :\ TH(\mathbb{Z})^{C_{p^{n-1}}} \longrightarrow\ t(TH(\mathbb{Z}))^{C_{p^n}}$$

are isomorphisms for $\ 0$.

In the next section we will use Theorem 5.3 to determine the differentials in the spectral sequences $\{E^r(n)\}$ and $\{\hat{E}^r(n)\}$ for each $n\ \ 2$.

6. The spectral sequences $\{E^r(n)\}$ and $\{\hat{E}^r(n)\}$. The next theorem gives a complete description of the differential structure of the spectral sequences $\{E^r(n)\}$ and $\{\hat{E}^r(n)\}$ for each $1\ \ n\ \ \infty$. The second part of the theorem (about the structure of $\{E^r(n)\}$) follows from the first part by using the comparison map φ: $\{E^r(n)\} \longrightarrow \{\hat{E}^r(n)\}$, and it was conjectured by Bökstedt and Madsen [BM1, Conjecture 4.3]. Recall that

$$\hat{E}^2_{,}(n) = \hat{H}^-\ (C_{p^n};\ \ TH(\mathbb{Z})) = \mathbb{F}_p[t, t^{-1}]\ \ \mathbb{F}_p(u_n)\ \ \mathbb{F}_p[f]\ \ \mathbb{F}_p(e)$$

and

$$E^2_{\ ,\ }(n) = H^{-}\ (C_{p^n};\quad TH(\mathbb{Z})) = \mathbb{F}_p[t]\quad \mathbb{F}_p(u_n)\quad \mathbb{F}_p[f]\quad \mathbb{F}_p(e).$$

THEOREM 6.1. *The following formulas give the nonzero differentials in* $\{\hat{E}^r(n)\}$ *on the classes* t^{p^k} *and* u_n *up to multiplication by nonzero units in* $\mathbb{F}_p$ *:*

(i) $d^{2\ \ (k+1)}(t^{p^k}) = t^{p^k+\ \ (k+1)}ef^{\ \ (k+1)}$ *for* $k\quad n$*, and*

(ii) $d^{2\ \ (n)+1}(u_n) = t^{\ \ (n)+1}f^{\ \ (n-1)+1}$

where $\ (k) = p(\frac{p^k-1}{p-1})$*. All remaining nonzero differentials in* $\{\hat{E}^r(n)\}$ *are determined multiplicatively by the differentials (i), (ii) and the fact that the classes* tf *and* $t^{\ell p}e$ *(*$\ell \in \mathbb{Z}$*) are permanent cycles.*

Furthermore, all nonzero differentials in the homotopy fixed points spectral sequence $\{E^r(n)\}$ *are determined multiplicatively by the differentials (i), (ii) and the fact that the classes* $t^m ef^k$ *are permanent cycles for each* $0\quad m\quad p$ *and* $k\quad 0$*.*

Theorem 5.2 is case $n = 1$ of the above theorem. The proof of Theorem 6.1 is then by induction on n. We first describe the inductive argument in the case $(n = 1)\quad (n = 2)$. As we will see afterwards the argument in the general case is verbatim the same.

6.2. First, using Theorem 5.2 and the comparison map $\{E^r(1)\} \longrightarrow \{\hat{E}^r(1)\}$ one computes the differentials in $\{E^r(1)\}$. We have

$$E^2_{\ ,\ }(1) = H^{-}\ (C_p;\quad TH(\mathbb{Z})) = \mathbb{F}_p[t]\quad \mathbb{F}_p(u_1)\quad \mathbb{F}_p[f]\quad \mathbb{F}_p(e).$$

The first nonzero differential is d^{2p} with $d^{2p}t \neq 0$, and $d^{2p}u_1 = 0$. One computes

$$\begin{aligned} E^{2p+1}_{\ ,\ }(1) &= \mathbb{F}_p[t^p]\quad \mathbb{F}_p(u_1)\quad \mathbb{F}_p[tf]\quad \mathbb{F}_p(e) \\ &\quad \bigoplus \mathbb{F}_p[f^p]\quad \mathbb{F}_p(e)\quad \mathbb{F}_p(u_1)\quad \mathbb{F}_p[tf] \\ &\quad \bigoplus_{\ ,\ell,m} \left\langle u_1 t^{\ell} ef^m \middle|\ \ = 0, 1,\ m = kp + l,\ k \in \mathbb{N},\ \text{and } \ell\quad l \right\rangle \end{aligned}$$

where $\langle v_1, v_2, \ldots\rangle$ denotes the vector space over $\mathbb{F}_p$ with basis the vectors $v_1, v_2, \ldots$.

The next differential d^{2p+1} is determined by multiplicativity and its values $d^{2p+1}u_1 \neq 0$, and $d^{2p+1}(te) = 0$. The spectral sequence collapses at this stage, and one gets

$$\begin{aligned} E^{\infty}_{\ ,\ }(1) &= E^{2p+2}_{\ ,\ }(1) \\ &= \mathbb{F}_p[f^p]\quad \mathbb{F}_p(e)\quad (\mathbb{F}_p[tf]/(tf)^{p+1}) + \mathbb{F}_p[t^p]\quad \mathbb{F}_p(e) \\ &\quad \bigoplus_{\ ,\ell,m} \left\langle u_1 t^{\ell} ef^m \middle|\ \ = 0, 1,\ m = kp + l,\ k \in \mathbb{N},\ \text{and } \ell\quad l \right\rangle. \end{aligned}$$

From the above E^{∞}-term and Theorem 5.3 one can read off the mod p homotopy groups

$$ {}_{2r}TH(\mathbb{Z})^{C_p} = {}_{2r}TH(\mathbb{Z})^{hC_p} = \begin{array}{ll} \mathbb{F}_p & \text{if} \quad p^2 \quad r \\ \mathbb{F}_p \quad \mathbb{F}_p & \text{if} \quad p^2 | r, \end{array} $$

and

$$ {}_{2r-1}TH(\mathbb{Z})^{C_p} = {}_{2r-1}TH(\mathbb{Z})^{hC_p} = \begin{array}{ll} \mathbb{F}_p & \text{if} \quad p^2 \quad r \\ \mathbb{F}_p \quad \mathbb{F}_p & \text{if} \quad p^2 | r. \end{array} $$

6.3. We now turn now the computation of the differential structure of the Tate spectral sequence $\{\hat{E}^r(2)\}$. We have

$$ \hat{E}^2_{\ ,}(2) = \hat{H}^-(C_{p^2}; \quad TH(\mathbb{Z})) = \mathbb{F}_p[t, t^{-1}] \quad \mathbb{F}_p(u_2) \quad \mathbb{F}_p[f] \quad \mathbb{F}_p(e), $$

and the first, possibly nonzero differential, is again d^{2p}. The spectral restriction map $\hat{\ }$: $\{\hat{E}^r(2)\} \longrightarrow \{\hat{E}^r(1)\}$ is given on E^2-terms by $(t) = t$, $(u_2) = 0$, $(f) = f$, and $(e) = e$. It follows that $d^{2p}t \neq 0$ also in the spectral sequence $\{\hat{E}^r(2)\}$.

To see that $d^{2p}(u_2) = 0$ in $\{\hat{E}^r(2)\}$, we look at the homotopy orbits spectral sequences $\{\tilde{E}^r(i)\}$ for $i = 1, 2$. From Theorem 5.2 one gets that $d^{2p}(u_1 t^{-kp}) = 0$ in $\{\hat{E}^r(1)\}$ for each $k \in \mathbb{Z}$, and therefore $d^{2p}(u_1 t^{-kp}) = 0$ in $\{\tilde{E}^r(1)\}$ for each $k \quad 1$ since these differentials do not cross the fiber line (that is the vertical line consisting of the groups $H^0(C_{p^2}; \quad TH(\mathbb{Z}))$). By the corestriction map of spectral sequences $: \{\tilde{E}^r(1)\} \longrightarrow \{\tilde{E}^r(2)\}$ which has $(u_1 t^{-kp} e\, f^m) = u_2 t^{-kp} e\, f^m$ for each $k, m \quad 0$ and $= 0, 1$, one gets that $d^{2p}(u_2 t^{-kp}) = 0$ in $\{\tilde{E}^r(2)\}$ for each $k \quad 1$. It follows that $d^{2p}(u_2 t^{-kp}) = 0$ for each $k \quad 1$ also in $\{\hat{E}^r(2)\}$ since these differentials do not cross the fiber line. Since $\{\hat{E}^r(2)\}$ is a spectral sequence of algebras, and since $d^{2p}(t^{kp}) = 0$ in $\mathbb{F}_p$ we get

$$ d^{2p}(u_1) = d^{2p}(t^{kp} \quad u_1 t^{-kp}) = t^{kp} d^{2p}(u_2 t^{-kp}) = 0 $$

in $\{\hat{E}^r(2)\}$. One now computes $\hat{E}^{2p+1}_{\ ,}(2) = \mathbb{F}_p[t^p, t^{-p}] \quad \mathbb{F}_p[u_2] \quad \mathbb{F}_p[tf] \quad \mathbb{F}_p(e)$.

6.4. The next differential d^{2p+1} is trivial on t^p for dimensional reasons. If we had $d^{2p+1}(u_2) \neq 0$, then the spectral sequence $\{\hat{E}^r(2)\}$ would collapse, and we would have

$$ \hat{E}^{\infty}_{\ ,}(2) = \hat{E}^{\infty}_{\ ,}(1) = \mathbb{F}_p[t^p, t^{-p}] \quad \mathbb{F}_p(e). $$

But this does not give the right abutment, because in this case the map

$$ \tilde{\ } : \quad TH(\mathbb{Z})^{C_p} \longrightarrow \quad t(TH(\mathbb{Z}))^{C_{p^2}} $$

can not be an isomorphism in nonnegative dimensions. It follows that $d^{2p+1}(u_2) = 0$ and $\hat{E}^{2p+2}_{,}(2) = \hat{E}^{2p+1}_{,}(2)$. The next differential in $\{\hat{E}^r(2)\}$ that can possibly be nontrivial is the differential d^{2p^2+1}. It is $d^{2p^2+1}(u_2) = 0$ for dimensional reasons and

LEMMA 6.5. $d^{2p^2+1}(t^p) = 0$.

Proof. Consider the restriction map $\hat{\ }: \{\hat{E}^r(\infty)\} \longrightarrow \{\hat{E}^r(2)\}$. It has $\hat{\ }(t^p) = t^p$ and $\hat{\ }(tf) = tf$. Let d^r_∞ denote the differential in $\{\hat{E}^r(\infty)\}$. Then $d^{2p^2+1}_\infty t = 0$ for dimensional reasons. It follows that

$$d^{2p^2+1}(t^p) = d^{2p^2+1}\,\hat{\ }\,(t^p) = \hat{\ }\,d^{2p^2+1}_\infty(t^p) = \hat{\ }\,(0) = 0 \quad \text{in } \{\hat{E}^r(2)\}.$$

The next two differentials in $\{\hat{E}^r(2)\}$ that can possibly be nontrivial are the differentials $d^{2\ (2)}$ and $d^{2\ (2)+1}$ which are described by Theorem 6.1. For their computation we need the following lemma which is due to M. Bökstedt:

LEMMA 6.6. $d^{2\ (2)+1}(u_2) = t^{p^2+p+1}f^{p+1}$ *up to multiplication by nonzero units in* $\mathbb{F}_p$.

Proof. Notice that the class $t^{p+1}f^{p+1}$ survives to $\hat{E}^{2\ (2)+1}_{,}(2)$, and consider the commutative diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "="] & TH(\mathbb{Z})^{C_{p^2}} \arrow[r] \arrow[d, "\varphi"] & TH(\mathbb{Z})^{hC_{p^2}} \arrow[d, "\omega"] \\
S \arrow[r, "\tilde{\ }"] & TH(\mathbb{Z})^{C_p} \arrow[r, "\tilde{\ }"] & t(TH(\mathbb{Z}))^{C_{p^2}}.
\end{tikzcd}$$

The class $v_1 \in\ S$ maps to a nonzero class $\ v_1 \in\ TH(\mathbb{Z})^{hC_{p^2}}$ which represents the class $t^{p+1}f^{p+1}$ in $\{E^\infty(2)\}$. So $\omega\ \ v_1$ is a class in $\ t(TH(\mathbb{Z}))^{C_{p^2}}$ which represents the class $(tf)^{p+1}$ in $\{E^r(2)\}$. In the commutative diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "="] & S_{\mathbb{T}}^{C_p} \arrow[r, "'"] \arrow[d] & S_{\mathbb{T}}^{hC_p} \arrow[d, "'"] \\
S \arrow[r, "\tilde{\ }"] & TH(\mathbb{Z})^{C_p} \arrow[r] & TH(\mathbb{Z})^{hC_p}
\end{tikzcd}$$

one finds that $\ \tilde{\ }\ v_1 = \ '\ '\ \ v_1 = 0$ because the class $t^{p+1}f^{p+1}$ did not survive to the $E^\infty(1)$. Since is an isomorphism in nonnegative degrees we get $\tilde{\ }\ v_1 = 0$, and therefore $\omega\ \ v_1 = \tilde{\ }\ \tilde{\ }\ v_1 = \ '(0) = 0$ in $\ t(TH(\mathbb{Z}))^{C_{p^2}}$. Since $t^{p+1}f^{p+1}$ is a permanent cycle in $\{\hat{E}^r(2)\}$ it must be hit by some differential. But the only differential that can possibly hit $t^{p+1}f^{p+1}$ is $d^{2\ (2)+1}$ and therefore we have $d^{2\ (2)+1}(t^{-p^2}u_2) = t^{p+1}f^{p+1}$ up to multiplication by nonzero units in $\mathbb{F}_p$. Notice now that $d^{2\ (2)+1}(t^{-p^2}) = 0$ independently of the value of the differential $d^{2\ (2)}(t^p)$: if $d^{2\ (2)}(t^p) = 0$, then $d^{2\ (2)+1}(t^{-p^2}) = pd^{2\ (2)+1}(t^{-p}) = 0$ in $\mathbb{F}_p$, whereas, if $d^{2\ (2)}(t^p) \neq 0$, then $d^{2\ (2)+1}(t^{-p}) = 0$ for dimensional reasons. Therefore,

$$d^{2\ (2)+1}(t^{-p^2}u_2) = t^{-p^2}d^{2\ (2)+1}(u_2) = t^{p+1}f^{p+1}$$

and $d^{2\ (2)+1}(u_2) = t^{p^2+p+1}f^{p+1}$ up to multiplication by nonzero units in $\mathbb{F}_p$.

6.7. To finish our computation of $\{\hat{E}^r(2)\}$ we still need to show that $d^{2\ (2)}(t^p) \neq 0$. One can establish this by checking both possibilities for the differential:

Case 1. $d^{2\ (2)}(t^p) \neq 0$. In this case $d^{2\ (2)}(u_2) = 0$ and one computes

$$\begin{aligned}\hat{E}^{2\ (2)+1}_{,}(2) &= \mathbb{F}_p[t^{p^2}, t^{-p^2}] \quad \mathbb{F}_p(u_2) \quad \mathbb{F}_p[tf] \quad \mathbb{F}_p(e)\\ &\bigoplus \left\langle u_2 t^{\ell p} e(tf)^m \middle| \ = 0, 1,\ p \quad \ell,\ m = 0, 1, \ldots, p-1 \right\rangle\end{aligned}$$

and since $d^{2\ (2)+1}(u_2) \neq 0$

$$\begin{aligned}\hat{E}^{\infty}_{,}(2) &= \hat{E}^{2\ (2)+1}_{,}(2)\\ &= \mathbb{F}_p[t^{p^2}, t^{p^{-2}}] \quad \mathbb{F}_p(e) \quad (\mathbb{F}_p[tf]/(tf)^{p+1})\\ &\bigoplus \left\langle u_2 t^{\ell p} e(tf)^k \middle| \ = 0, 1,\ p \quad \ell,\ k \in \mathbb{N} \right\rangle .\end{aligned}$$

So this case gives the correct abutment with

$$\begin{aligned}{}_{2r}t(TH(\mathbb{Z}))^{C_{p^2}} &= \begin{matrix} \mathbb{F}_p & \text{if} & p^2 \quad r\\ \mathbb{F}_p \quad \mathbb{F}_p & \text{if} & p^2 | r, \end{matrix} \quad \text{and}\\ {}_{2r-1}t(TH(\mathbb{Z}))^{C_{p^2}} &= \begin{matrix} \mathbb{F}_p & \text{if} & p^2 \quad r\\ \mathbb{F}_p \quad \mathbb{F}_p & \text{if} & p^2 | r. \end{matrix}\end{aligned}$$

Case 2. $d^{2\ (2)}(t^p) = 0$. In this case, since $d^{2\ (2)+1}u_2 \neq 0$, the spectral sequence collapses again at $\hat{E}^{2\ (2)+2}_{,}(2) = \hat{E}^{\infty}_{,}(2)$, but this time it does not provide the correct abutment.

6.8. Proof of Theorem 6.1. To prove the induction step in the general case we assume that the differential structures of $\{\hat{E}^r(n)\}$ and $\{E^r(n)\}$ are the ones given by Theorem 6.1, and we will show that the same is true for $\{\hat{E}^r(n+1)\}$ and $\{E^r(n+1)\}$. The proof is the same as in the special case $n = 1$. Using the assumed differential structure of $\{E^r(n)\}$ one computes $E^{\infty}_{*,*}(n)$, and therefore $\quad TH(\mathbb{Z})^{C_{p^n}} = \quad TH(\mathbb{Z})^{hC_{p^n}}$ for each $\quad 0$. By Theorem 5.3, the homomorphism

$$\tilde{\ }: \quad TH(\mathbb{Z})^{C_{p^n}} \longrightarrow \quad t(TH(\mathbb{Z}))^{C_{p^{n+1}}}$$

is an isomorphism for each $\quad 0$. This determines the size of the $\hat{E}^{\infty}_{*,*}(n+1)$. More explicitly, these computations give

$$\bigoplus_{k+\ell=2r} \hat{E}^{\infty}_{k,\ell}(n+1) = \quad {}_{2r}TH(\mathbb{Z})^{C_{p^n}} = \quad {}_{2r}TH(\mathbb{Z})^{hC_{p^n}} = \begin{array}{lll} \mathbb{F}_p{}^{\,n} & \text{if} & p^{n+1} \quad r \\ \mathbb{F}_p{}^{\,(n+1)} & \text{if} & p^{n+1}|r \end{array}$$

and

$$\bigoplus_{k+\ell=2r-1} \hat{E}^{\infty}_{k,\ell}(n+1) = \quad {}_{2r-1}TH(\mathbb{Z})^{C_{p^n}} = \quad {}_{2r-1}TH(\mathbb{Z})^{hC_{p^n}}$$
$$= \begin{array}{lll} \mathbb{F}_p{}^{\,n} & \text{if} & p^{n+1} \quad r \\ \mathbb{F}_p{}^{\,n+1} & \text{if} & p^{n+1}|r. \end{array}$$

The restriction map $\ :\ \{E^r(n+1)\} \longrightarrow \{E^r(n)\}$ has $\ (t) = t$, and it follows that in $\{E^r(n+1)\}$ the differentials $d^{2\ (k+1)}(t^{p^k}) = t^{p^k+\ (k+1)} ef^{\ (k)}$ for $k \quad n$. The same is then true for the Tate spectral sequence $\{\hat{E}^r(n+1)\}$ since these differentials do not cross the fiber line. This shows part (i) of Theorem 6.1 for $k \quad n$.

Dually, using the corestriction maps $\ :\ \{\tilde{E}^r(n)\} \longrightarrow \{\tilde{E}^r(n+1)\}$ one finds that

$$d^{2\ (k+1)}(t^{-\ell p^k} u_{n+1}) = 0$$

for each $k \quad n$, and $\ell \quad 1$. The same is true then in the Tate spectral sequence $\{\hat{E}^r(n+1)\}$ for the differentials $d^{2\ (k+1)}$ that do not cross the fiber line. Using the multiplicativity in $\{\hat{E}^r(n+1)\}$ one finds that $d^{2\ (k+1)} u_{n+1} = 0$ in $\{\hat{E}^r(n+1)\}$. Furthermore, $d^{2\ (n)+1} u_{n+1} = 0$ since, otherwise, the spectral sequence collapses and $\hat{E}^{\infty}_{*,*}(n+1) = \hat{E}^{\infty}_{*,*}(n)$ which does not have the correct size.

The next possibly nonzero differential is $d^{2\ (n+1)-2p+1}$. This differential is trivial on u_{n+1} for dimensional reasons, and it is also trivial on t^{p^n} by an argument using the restriction map $\ :\ \{E^r(\infty)\} \longrightarrow \{E^r(n+1)\}$ exactly as in Lemma 6.5.

The next two differentials that can possibly be nontrivial are the ones described by Theorem 6.1 with $n = n+1$, i.e., $d^{2\ (n+1)}$ and $d^{2\ (n+1)+1}$. Now the class $(tf)^{\ (n)+1}$ survives in $\hat{E}^{2\ (n+1)+1}_{*,*}(n+1)$ but not in $E^{\infty}_{*,*}(n)$. By the argument of Lemma 6.6 the class $(tf)^{\ (n)+1}$ must be hit by the differential $d^{2\ (n+1)+1}$, and this implies that

$$d^{2\ (n+1)+1}(u_{n+1} t^{-p^{n+1}}) = (tf)^{\ (n)+1} \quad \text{in } \{\hat{E}^r(n+1)\}.$$

By multiplicativity, it follows that $d^{2\ \ (n+1)+1}(u_{n+1}) = t^{\ \ (n+1)+1} f^{\ \ (n)+1}$.

We are once again left to show that $d^{2\ \ (n+1)}(t^{p^n}) \neq 0$. One checks that, if this is the case, then $\hat{E}^{\infty}_{,}(n+1) = \hat{E}^{2\ \ (n+1)+2}_{,}(n+1)$ and hence it has the correct size. On the other hand, if $d^{2\ \ (n+1)}(t^{p^n}) = 0$, then $d^{2\ \ (n+1)+1}(u_{n+1}) \neq 0$, and the spectral sequence collapses again at the $E^{2\ \ (n+1)+2}$-stage but this time the abutment does not satisfy (**). This concludes the proof of the inductive step of Theorem 6.1.

7. The mod p topological cyclic homology of the integers In this section we give a brief account of the topological cyclic homology spectrum of an FSP $\mathcal{F}$, and of how Theorem 6.1 can be used to compute the mod p topological cyclic homology in the case where $\mathcal{F} = \mathcal{Z}$. This is the content of [BM1, Section 7] which can be consulted for more details.

7.1. Diagram (D) of 3.1 with $K = TH(\mathcal{F})$ considered as a C_{p^n}-spectrum yields a diagram of spectra

$$(D_n) \qquad \begin{tikzcd} & TH(\mathcal{F})_{hC_{p^n}} \arrow[dr] \arrow[dl] & \\ TH(\mathcal{F})^{C_{p^n}} \arrow[rr, dash] \arrow[d, "\varphi"] & & TH(\mathcal{F})^{hC_{p^n}} \arrow[d, "\omega"] \\ TH(\mathcal{F})^{C_{p^{n-1}}} \arrow[rr, dash, "\tilde{}"] & & t(TH(\mathcal{F}))^{C_{p^n}} \end{tikzcd}$$

for each n $\;$ 1. The inclusion $\;$ of the $C_{p^{n+1}}$-fixed points in the C_{p^n}-fixed points defines a map of diagrams $\;$: $(D_{n+1}) \longrightarrow (D_n)$ for each n $\;$ 1. The homotopy inverse limit of the diagrams (D_n) over the inclusions $\;$ is a diagram p-adically equivalent to the following one

$$(D_\infty) \qquad \begin{tikzcd} & \Sigma TH(\mathcal{F})_{h\mathbb{T}} \arrow[dr] \arrow[dl] & \\ \operatorname{holim} TH(\mathcal{F})^{C_{p^n}} \arrow[rr, dash, "\Gamma"] \arrow[d, "\Phi"] & & TH(\mathcal{F})^{h\mathbb{T}} \arrow[d, "\Omega"] \\ \operatorname{holim} TH(\mathcal{F})^{C_{p^{n-1}}} \arrow[rr, dash, "\tilde{\Gamma}"] & & t(TH(\mathcal{F}))^{\mathbb{T}}. \end{tikzcd}$$

One then defines $TC(\mathcal{F};p)$, the *p-part of topological cyclic homology of $\mathcal{F}$*, to be the homotopy fixed set of the endomorphism Φ, in other words,

$$TC(\mathcal{F};p) = (\operatorname{holim}\ TH(\mathcal{F})^{C_{p^n}})^{h\Phi}$$
$$\text{fiber}(\Phi - 1\colon\ \operatorname{holim}\ TH(\mathcal{F})^{C_{p^n}} \longrightarrow \operatorname{holim}\ TH(\mathcal{F})^{C_{p^n}}).$$

Remark 7.2. As the name suggests $TC(\mathcal{F};p)$ is the p-part of a "global" object, the topological cyclic homology of $\mathcal{F}$ which can be constructed as follows: for each positive integer m, n with $n|m$ one can define two maps

$$,\varphi\colon\ TH(\mathcal{F})^{C_{p^m}} \longrightarrow TH(\mathcal{F})^{C_{p^n}}$$

the inclusion of fixed points , and the Frobenius φ defined by means of diagram (D) or as in [G], [BHM]. The homotopy inverse limit over all $,\varphi$, for all positive integers $n|m$ is called the topological cyclic homology of the FSP $\mathcal{F}$ and denoted by $TC(\mathcal{F})$. It turns out, however, that after p-adic completion $TC(\mathcal{F})^{\wedge}_p \simeq TC(\mathcal{F};p)^{\wedge}_p$. Since in this article we are only interested in mod p homotopy groups we only need to consider the p-part $TC(\mathcal{F};p)$. For more details on the "global" $TC(\mathcal{F})$ see [G].

7.3. When $\mathcal{F} = \mathcal{Z}$, the FSP associated to the ring of integers $\mathbb{Z}$, then by Theorem 5.3 and [BK, ch. IX, Theorem 3.1], the map Γ induces isomorphisms on mod p (resp. p-adically completed) homotopy groups

$$\Gamma :\quad \operatorname{holim}\ TH(\mathbb{Z})^{C_{p^n}} \longrightarrow \quad TH(\mathbb{Z})^{h\mathbb{T}}$$

for each 0, and the same is true for $\tilde{\Gamma}$. Using the restriction maps : $\{E^r(\infty)\} \longrightarrow \{E^r(n)\}$, (resp. $\hat{}$: $\{\hat{E}^r(\infty)\} \longrightarrow \{\hat{E}^r(n)\}$) and Theorem 6.1 one easily determines the differential structure of the spectral sequence $\{E^r(\infty)\}$ (resp. $\{\hat{E}^r(\infty)\}$) as they are described in the following:

THEOREM 7.4. *All nontrivial differentials in the spectral sequences $\{E^r(\infty)\}$ and $\{\hat{E}^r(\infty)\}$ are multiplicatively generated by the differentials from the base line*

$$d^{2\ (k+1)}(t^{p^k}) = t^{p^k+\ (k+1)}ef^{\ (k)}, \qquad k \in \mathbb{N}$$

and the fact that the classes tf and $t^{\ell p}e$ ($\ell \in \mathbb{Z}$) are permanent cycles. The E^∞-terms are given by

$$E^{\infty}_{\ ,}(\infty) = \mathbb{F}_p(e) \quad \mathbb{F}_p[tf] \quad \prod_{k=0}^{\infty}\Big\langle t^i e f^j | \ \ (k) \quad i \qquad (k+1),\ \textit{and}\ p^k|(i-j)\Big\rangle$$

$$\hat{E}^{\infty}_{\,,}(\infty) \;=\; \mathbb{F}_p(e) \quad \mathbb{F}_p[tf] \quad \prod_{k=0}^{\infty} \left\langle t^{ip^{k+1}} e(tf)^j \,|\, i \in \mathbb{Z},\ and \quad (k) \quad j \qquad (k+1) \right\rangle .$$

Notice that the only classes in even total degree that survive to the E^{∞}-term of both $\{E^r(\infty)\}$ and $\{\hat{E}^r(\infty)\}$ are the powers of tf. So

$$_{2r}\, T\,H\,(\mathbb{Z})^{h\mathbb{T}} \;=\; _{2r}\, t(T\,H\,(\mathbb{Z}))^{\mathbb{T}} \;=\; \begin{matrix} \mathbb{F}_p \;=\; (tf)^i & \text{if} \quad r = i(p-1) \\ \{0\} & \text{if} \quad (p-1) \quad r. \end{matrix}$$

On the other hand, each $_{2r-1}\, T\,H\,(\mathbb{Z})^{h\mathbb{T}} \;=\; _{2r-1}\, t(T\,H\,(\mathbb{Z}))^{\mathbb{T}}$ is an infinite dimensional $\mathbb{F}_p$-vector space. Let $\{x_{2r-1}(k)\}_{k\in\mathbb{N}}$ (resp. $\{\hat{x}_{2r-1}(k)\}_{k\in\mathbb{N}}$) be generators (ordered by fiber degree) of

$$\bigoplus_{p+q=2r-1} E^{\infty}_{p,q}(\infty) \qquad \left(\text{resp.} \quad \bigoplus_{p+q=2r-1} \hat{E}^{\infty}_{p,q}(\infty) \right)$$

and choose elements $_{2r-1}(k) \in \; _{2r-1}\, T\,H\,(\mathbb{Z})^{h\mathbb{T}}$ (resp. $\hat{}_{2r-1}(k) \in \; _{2r-1}\, t(T\,H\,(\mathbb{Z}))^{\mathbb{T}}$) representing $x_{2r-1}(k)$ (resp. $\hat{x}_{2r-1}(k)$). Then

$$\begin{aligned} _{2r-1}\, T\,H\,(\mathbb{Z})^{h\mathbb{T}} \;&=\; \prod_{k=0}^{\infty} \langle \; _{2r-1}(k) \rangle, \qquad \text{and} \\ _{2r-1}\, t(T\,H\,(\mathbb{Z}))^{\mathbb{T}} \;&=\; \prod_{k=0}^{\infty} \langle \; \hat{}_{2r-1}(k) \rangle. \end{aligned}$$

In order to conclude the calculation of the mod p topological cyclic homology groups of the integers $TC\;(\mathbb{Z};\mathbb{F}_p) = \quad TC(\mathbb{Z};p)$ it is left to determine the action of Φ on the mod p homotopy groups of $\underleftarrow{\mathrm{holim}}\; T\,H\,(\mathbb{Z})^{C_{p^n}}$, or, what is equivalent by the diagram (D_∞) of 7.1, to determine the behavior of the map

$$\Omega : \quad T\,H\,(\mathbb{Z})^{h\mathbb{T}} \longrightarrow \quad t(T\,H\,(\mathbb{Z}))^{\mathbb{T}}$$

on the classes $_{2r-1}(k)$ (it is easy to see that Ω is an isomorphism in even dimensions). This is done in [BM, §7] where it is computed that

$$TC_{2r-1}(\mathbb{Z};\mathbb{F}_p) \;=\; \begin{matrix} \mathbb{F}_p \quad \mathbb{F}_p & \text{if } r \quad 0,1 \text{ mod } (p-1) \text{ or if } r = 1 \\ \mathbb{F}_p & \text{otherwise} \end{matrix}$$

$$TC_{2r}(\mathbb{Z};\mathbb{F}_p) \;=\; \begin{cases} \mathbb{F}_p & \text{if } r = 0 \\ \mathbb{F}_p \quad \mathbb{F}_p & \text{if } (p-1)|r,\ (r \neq 0) \\ \{0\} & \text{otherwise.} \end{cases}$$

By the result of [Mc] mentioned in the introduction $TC\ (\mathbb{Z};\mathbb{F}_p) = K\ (\widehat{\mathbb{Z}}_p;\mathbb{F}_p)$, and the above groups are isomorphic to the respective algebraic K-theory groups of the p-adic integers with mod p coefficients.

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