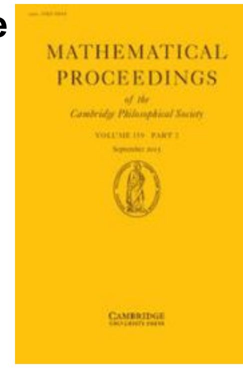

A canonical operad pair

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A canonical operad pair

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1. *Introduction.* The purpose of this paper is to construct an operad $\mathscr{H}_\infty$ with the good properties of both the little convex bodies partial operad $\mathscr{K}_\infty$ and the little cubes operad $\mathscr{C}_\infty$ used in May's theory of E_∞ ring spaces or multiplicative infinite loop spaces ((6), chapter VII). In (6) $\mathscr{H}_\infty$ can then be used instead of $\mathscr{K}_\infty$ and $\mathscr{C}_\infty$, and the theory becomes much simpler; in particular all partial operads can be replaced by genuine ones. The method used here is a modification of that which May suggests on (6), page 170, but cannot carry out.

We shall use various spaces associated to a finite-dimensional real inner product space V. First, let $\mathscr{A}V$ be the space of all topological embeddings of V in itself. Next, for $k = 0, 1, 2, \ldots$, let $\mathscr{D}_V(k)$ be the subspace of $(\mathscr{A}V)^k$ consisting of k-tuples of embeddings with disjoint images; by convention $\mathscr{D}_V(0) = *$, a point. Finally, let $F(V, k)$ be the kth configuration space of V; it consists of k-tuples of distinct points of V, with $F(V, 0) = *$ also.

The pair $(\mathscr{A}, \mathscr{D})$ has a complicated structure: $\mathscr{A}V$ is a monoid under composition, there is a direct product map from $\mathscr{A}V \times \mathscr{A}W$ to $\mathscr{A}(V \oplus W)$ inducing a map from $\mathscr{D}_V(k) \times \mathscr{D}_W(l)$ to $\mathscr{D}_{V\oplus W}(kl)$, and so on. We shall sum this up by saying that $(\mathscr{A}, \mathscr{D})$ is an object in a category Φ, to be defined in Section 2 below.

The symmetric group Σ_k acts on $F(V, k)$ and $\mathscr{D}_V(k)$; in Section 3 we shall construct a Σ_k-equivariant map

$$\theta: F(V, k) \to \mathscr{D}_V(k).$$

For $(v_1, \ldots, v_k)$ a point of $F(V, k)$ we shall have

$$\theta(v_1, \ldots, v_k) = (c_1, \ldots, c_k),$$

where c_r embeds V onto some open ball with centre v_r in an orientation-preserving way; the complete definition of θ does not add anything important to this information.

We can now state the main theorem of this paper.

THEOREM. *There is an object $(\mathscr{E}, \mathscr{H})$ of the category Φ, a morphism $\pi: (\mathscr{E}, \mathscr{H}) \to (\mathscr{A}, \mathscr{D})$, and maps $\phi: F(V, k) \to \mathscr{H}_V(k)$ making $F(V, k)$ Σ_k-equivariantly homotopy equivalent to $\mathscr{H}_V(k)$ such that*

$$\pi\phi = \theta: F(V, k) \to \mathscr{D}_V(k).$$

The proof is given in Section 4.

The difference from the programme suggested on (6), page 170 is that $\mathscr{E}V$ is not a subspace of $\mathscr{A}V$. This does not affect the applications to iterated loop space theory; see Section 5.

2. *The category* Φ. Let $\mathscr{I}_*$ be the category of finite-dimensional real inner product spaces and isometric linear isomorphisms. An object $(\mathscr{A}, \mathscr{D})$ of Φ then consists of the following:

(*a*) a continuous functor $\mathscr{A}$ from $\mathscr{I}_*$ to topological monoids;

(*b*) subfunctors $\mathscr{D}_{(-)}(k)$ of $\mathscr{A}(-)^k$ for $k = 0, 1, 2, \ldots$, with $\mathscr{D}_V(0) = *$ for all V;

(*c*) a continuous commutative and associative natural transformation

$$(c, d) \mapsto c \times d \colon \mathscr{A}V \times \mathscr{A}W \to \mathscr{A}(V \oplus W)$$

of functors from $\mathscr{I}_* \times \mathscr{I}_*$ to topological monoids.

The following axioms must hold:

(1) the maps $c \mapsto c \times 1 \colon \mathscr{A}V \to \mathscr{A}(V \oplus \{0\}) = \mathscr{A}V$ are identity maps;

(2) the maps $c \mapsto c \times 1 \colon \mathscr{A}V \to \mathscr{A}(V \oplus W)$ are closed inclusions;

(3) the spaces $\mathscr{D}_V(k)$ are invariant under the action of Σ_k on $(\mathscr{A}V)^k$;

(4) $1 \in \mathscr{D}_V(1)$ for all V;

(5) if $(c_r \colon 1 \leqslant r \leqslant k) \in \mathscr{D}_V(k)$ and $(d_{rs} \colon 1 \leqslant s \leqslant j_r) \in \mathscr{D}_V(j_r)$ for $1 \leqslant r \leqslant k$, then

$$(c_r d_{rs} \colon 1 \leqslant r \leqslant k,\ 1 \leqslant s \leqslant j_r) \in \mathscr{D}_V(j_1 + \ldots + j_r);$$

(6) if $(c_r \colon 1 \leqslant r \leqslant k) \in \mathscr{D}_V(k)$ and $(d_s \colon 1 \leqslant s \leqslant l) \in \mathscr{D}_W(l)$, then

$$(c_r \times d_s \colon 1 \leqslant r \leqslant k,\ 1 \leqslant s \leqslant l) \in \mathscr{D}_{V \oplus W}(kl).$$

The morphisms of Φ are natural transformations of $\mathscr{A}$ inducing natural transformations of the $\mathscr{D}_{(-)}(k)$ and preserving all the structure.

Given (*a*) and (*c*), the axioms (1) and (2) say that $\mathscr{A}$ is an $\mathscr{I}_*$-monoid in the sense of (5), 1·1. By the method described in (6), I·1, one can extend $\mathscr{A}$, and with it the $\mathscr{D}_{(-)}(k)$, to the category $\mathscr{I}$ of finite- or countable-dimensional real inner product spaces and linear isometries (which need not be surjective). First, if $f \colon V \to W$ is an isometry between finite-dimensional spaces, then W is an orthogonal direct sum,

$$W = fV \oplus X$$

say, and f induces an isomorphism $f' \colon V \to fV$. We define $\mathscr{A}f \colon \mathscr{A}V \to \mathscr{A}W$ by

$$(\mathscr{A}f)(c) = (\mathscr{A}f')(c) \times 1 \quad \text{for} \quad c \in \mathscr{A}V.$$

For a countable-dimensional space V we then set

$$\mathscr{A}V = \operatorname{colim}_W \mathscr{A}W,$$

where W runs through the finite-dimensional subspaces of V. Similarly we extend the $\mathscr{D}_{(-)}(k)$ and the direct product natural transformation of (*c*) by colimits. Axioms (1)–(6) still hold for these extended structures.

It is easy to see that the pair $(\mathscr{A}, \mathscr{D})$ of Section 1, consisting of embeddings and embeddings with disjoint images, is an object of Φ. We use composition to make $\mathscr{A}V$ a monoid; if $f \colon V \to W$ is an isometric isomorphism, then $(\mathscr{A}f)(c) = fcf^{-1}$ for $c \in \mathscr{A}V$; the natural transformation of (*c*) is given by the direct product of functions; and the axioms are easily verified.

3. *The map* $\theta \colon F(V, k) \to \mathscr{D}_V(k)$. In this section, as in Section 1, V is a finite-dimensional real inner product space, $F(V, k)$ is the kth configuration space of V, and $\mathscr{D}_V(k)$

is the space of k-tuples of embeddings of V in V with disjoint images. We construct a Σ_k-equivariant map $\theta: F(V,k) \to \mathscr{D}_V(k)$ as follows.

Given $(v_1, \ldots, v_k)$ in $F(V,k)$, let ρ be

$$\tfrac{1}{2}\min\{\|v_r - v_s\| : r \neq s\}$$

(taken as ∞ if $k = 1$), so ρ is in $(0,\infty]$ and depends continuously on $(v_1, \ldots, v_k)$. Define a continuous function $e_\rho: V \to V$ by

$$e_\rho x = \frac{\rho x}{\rho + \|x\|} \quad \text{for} \quad x \in V.$$

Then $\theta(v_1, \ldots, v_k)$ is to be $(c_1, \ldots, c_k)$ where $c_r: V \to V$ is given by

$$c_r(x) = v_r + e_\rho x \quad \text{for} \quad x \in V.$$

It is obvious that $\|e_\rho x\| < \rho$, so $c_r(x)$ is within ρ of v_r for all x. Since distinct v_r are at least 2ρ apart, the c_r must have disjoint images.

To justify this construction, we must show that the c_r are embeddings, or equivalently that e_ρ is an embedding. For use in the next section we give a more general result.

LEMMA. *Let a and b be non-negative real numbers with $a+b = 1$ and define $f: V \to V$ by*

$$f(x) = ae_\rho x + bx \quad \textit{for} \quad x \in V.$$

Then f is a distance-reducing embedding.

Here distance-reducing means that $\|f(x) - f(y)\| \leqslant \|x-y\|$ for x and y in V.

Proof. We use differential calculus. By computation, f is differentiable everywhere, and

$$\begin{aligned} Df(x)(y) &= \left[a\left(\frac{\rho}{\rho+\|x\|}\right) + b\right] y \quad \text{if} \quad \langle x,y\rangle = 0, \\ &= \left[a\left(\frac{\rho}{\rho+\|x\|}\right)^2 + b\right] y \quad \text{if} \quad x \text{ and } y \text{ are linearly dependent.} \end{aligned}$$

So $Df(x)$ is diagonalizable with respect to an orthonormal base and its eigenvalues lie in $(0,1]$. Since $Df(x)$ is everywhere non-singular, f is an embedding by Rolle's theorem; since $\|Df(x)\| \leqslant 1$ everywhere, f is distance-reducing by the mean-value theorem. This completes the proof.

4. *Proof of the theorem.* Let V be a finite-dimensional real inner product space. Then $\mathscr{E}V$ is to be the space of maps

$$h: [0,1] \to \mathscr{A}V$$

such that $h(t)$ is a distance-reducing embedding for all t and such that $h(1)$ is the identity. For $k \geqslant 1$, $\mathscr{H}_V(k)$ is to be the space of k-tuples $(h_1, \ldots, h_k)$ in $(\mathscr{E}V)^k$ such that the $h_r(0)$ have disjoint images. The various products in $\mathscr{E}V$ are to be induced by the corresponding products in $\mathscr{A}V$; it is then easy to see that $(\mathscr{E}, \mathscr{H})$ is an object of Φ. And $\pi: \mathscr{E}V \to \mathscr{A}V$ is to be given by

$$\pi(h) = h(0);$$

it is easy to see that $\pi: (\mathscr{E}, \mathscr{H}) \to (\mathscr{A}, \mathscr{D})$ is a morphism of Φ.

It remains to show that $F(V, k)$ is Σ_k-equivariantly homotopy equivalent to $\mathscr{H}_V(k)$ by a map $\phi: F(V, k) \to \mathscr{H}_V(k)$ such that $\pi\phi = \theta$. All the constructions in what follows are easily checked to be equivariant and we shall not mention the point again. It may help the reader to think of $\mathscr{H}_V(k)$ as the mapping path space of θ, although this is not quite correct.

We define ϕ as follows. Given $(v_1, \ldots, v_k)$ in $F(V, k)$, let ρ be

$$\tfrac{1}{2}\min\{\|v_r - v_s\| : r \neq s\} \in (0, \infty],$$

as in Section 3. Then $\phi(v_1, \ldots, v_k)$ is to be $(h_1, \ldots, h_k)$, where

$$h_r(t)(x) = (1-t)(v_r + e_\rho x) + tx \quad \text{for} \quad 0 \leqslant t \leqslant 1 \quad \text{and} \quad x \in V$$

(e_ρ is as in Section 3). From the Lemma we see that the $h_r(t)$ are distance-reducing embeddings; and the definition of e_ρ shows that $\|e_\rho x\| < \rho$ for all x, hence that the $h_r(0)$ have disjoint images. It follows that $(h_1, \ldots, h_k)$ is in $\mathscr{H}_V(k)$. And it is obvious that $\pi\phi = \theta$.

The map homotopy inverse to ϕ will be $\psi: \mathscr{H}_V(k) \to F(V, k)$, given by

$$\psi(h_1, \ldots, h_k) = (h_1(0)(0), \ldots, h_k(0)(0)).$$

Clearly $\psi\phi = 1$. To construct a homotopy from 1 to $\phi\psi$, let $(h_1, \ldots, h_k)$ be a typical point of $\mathscr{H}_V(k)$. Write v_r for $h_r(0)$ (0) and let $\rho \in (0, \infty]$ be

$$\tfrac{1}{2}\min\{\|v_r - v_s\| : r \neq s\}.$$

Then the homotopy $H: \mathscr{H}_V(k) \times [0, 1] \to \mathscr{H}_V(k)$ from 1 to $\phi\psi$ is to be given by

$$H(h_1, \ldots, h_k; \tau) = (H_1(\tau), \ldots, H_k(\tau)) \quad \text{for} \quad \tau \in [0, 1],$$

where the value of $H_r(\tau)(t)$ at a point x of V is as indicated by Figure 1.

To be more precise, Figure 1 shows the value of $H_r(\tau)(t)$ for τ or t equal to 0 or 1 or $\tau = \frac{1}{2}$ or $\tau + t = 1$; two formulae are given for diagonal points, which agree because $h_r(1)$ is the identity, and the several formulae given for each vertex are also easily seen to agree. The trapezium on the left is filled in by an expression of the form

$$h_r(\lambda_{\tau,t})(\alpha_{\tau,t} e_\rho x + (1 - \alpha_{\tau,t}) x),$$

where λ and α are continuous functions from the trapezium to [0, 1] taking the values given by the figure on the boundary; such functions exist because [0, 1] is contractible. Similarly the triangle on the left is filled in by an expression of the form

$$\alpha_{\tau,t} e_\rho x + (1 - \alpha_{\tau,t}) x$$

with α a continuous function from the triangle to [0, 1], the trapezium on the right by an expression of the form

$$\alpha_{\tau,t} e_\rho x + (1 - \alpha_{\tau,t}) x + \kappa_{\tau,t}$$

with α a continuous function from the trapezium to [0, 1] and κ a continuous function from the trapezium to V (note that V is contractible), and the triangle on the right by an expression of the form

$$h_r(\lambda_{\tau,t})(e_\rho x) + \kappa_{\tau,t}$$

with λ a continuous function from the triangle to [0, 1] and κ a continuous function from the triangle to V.

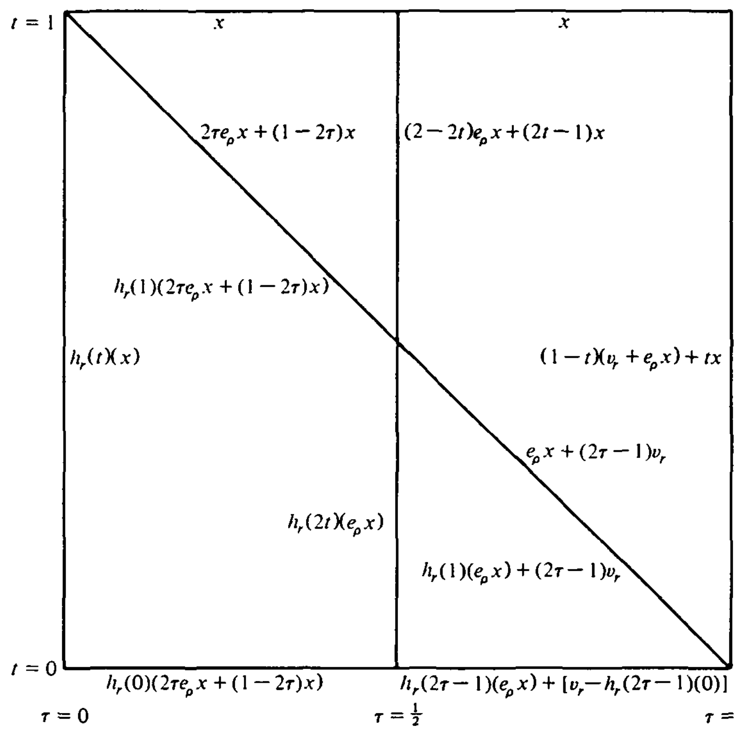

Fig. 1

Now the Lemma in Section 3 shows that $H_r(\tau)(t)$ is always a distance-reducing embedding. For $\tau \geqslant \frac{1}{2}$ we have $\|H_r(\tau)(0)(x) - v_r\| < \rho$ for all x, since $\|e_\rho x\| < \rho$ and $h_r(2\tau - 1)$ is distance-reducing, so the $H_r(\tau)(0)$ for fixed $\tau \geqslant \frac{1}{2}$ have disjoint images. The same holds for $\tau \leqslant \frac{1}{2}$, because the $h_r(0)$ have disjoint images. Therefore the k-tuple $(H_1(\tau), \ldots, H_k(\tau))$ is in $\mathscr{H}_V(k)$ for all τ and gives a homotopy from $(h_1, \ldots, h_k)$ to $\phi\psi(h_1, \ldots, h_k)$.

This completes the proof.

5. *Applications to iterated loop spaces.* This section gives the properties which $\mathscr{H}_V$ shares with the little cubes operad or the little convex bodies operad. The results are analogous to those in (6), VII·1–2. Some of them are extended from infinite loop spaces to finitely many times iterated loop spaces.

PROPOSITION 1. *The functor $V \to \mathscr{H}_V$ is a functor from $\mathscr{I}$ to operads. If V is infinite-dimensional, then $\mathscr{H}_V$ is an E_∞ operad.*

Proof. Given the definition of an operad ((3), 1·1 or (6), VI·1·2), one easily verifies the first statement. As for the second, the action of Σ_k on $\mathscr{H}_V(k)$ is clearly free; and $\mathscr{H}_V(k)$ is aspherical because

$$\mathscr{H}_V(k) = \operatorname{colim}_W \mathscr{H}_W(k)$$

with W running through the finite subspaces of V and the maps involved being closed inclusions (Axiom (2)), so that

$$\pi_q \mathscr{H}_V(k) \cong \operatorname{colim}_W \pi_q \mathscr{H}_W(k) \cong \operatorname{colim}_W \pi_q F(W,k) = 0,$$

since $F(W,k)$ is $(\dim(W)-2)$-connected ((3), 4·5).

Let V be a finite-dimensional vector space and $S(V)$ be its one-point compactification. If X is a space with base-point, then write $\Omega^V X$ for its V-fold loop space (that is, the function space $X^{S(V)}$) and $\Sigma^V X$ for its V-fold suspension (that is, the smash product $S(V) \wedge X$). Write QX for $\operatorname{colim}_W \Omega^W \Sigma^W X$ where W runs through the finite-dimensional subspaces of $\mathbf{R}^\infty$, and write $\mathscr{H}_\infty$ for $\mathscr{H}_V$ with $V = \mathbf{R}^\infty$.

PROPOSITION 2. *The operads $\mathscr{H}_V$ for V finite-dimensional act naturally and compatibly with suspension on V-fold loop spaces, and $\mathscr{H}_\infty$ acts naturally on the zeroth spaces of spectra.*

Here spectrum is taken in the sense of (6), chapter II: it means a coordinate-free strict Ω-spectrum. The proof is like that of (6), VII·2·1.

Write H_V and H_∞ for the monads corresponding to the operads $\mathscr{H}_V$ and $\mathscr{H}_\infty$ (see (3), 2·4). They behave just like the monads corresponding to the little cubes operads ((3), 4–5); in particular there are canonical maps $\alpha_V: H_V X \to \Omega^V \Sigma^V X$ of $\mathscr{H}_V$-spaces for V finite-dimensional and X a based space, and there is also a canonical map

$$\alpha_\infty: H_\infty X \to QX$$

of $\mathscr{H}_\infty$-spaces.

PROPOSITION 3. *The map α_V (V finite-dimensional) is a homeomorphism for $V = \{0\}$ and a group-completion for dim (V) positive. The map α_∞ is a group-completion.*

The case $V = \{0\}$ is trivial: $\mathscr{H}_{\{0\}}(0) = *$, $\mathscr{H}_{\{0\}}(1) = \{1\}$, and $\mathscr{H}_{\{0\}}(k)$ is empty for $k \geqslant 2$, so $H_{\{0\}} X = X$. The case when V is positive-dimensional is given by Segal in (8) and, for $\dim(V) \geqslant 2$, by Cohen in (2), III·3·3. (To say that a map $\alpha: A \to B$ is a group-completion means at least that B is grouplike and α induces a weak homotopy equivalence of classifying spaces. For $\dim(V) \geqslant 2$ and sometimes also for $\dim(V) = 1$ there is an equivalent homological statement; see (1), 3·2, (4), 1 and (7).) For α_∞ the result is given by May in (4), 2·2, and there is always an equivalent homological statement.

Write $\mathscr{L}$ for the linear isometries operad ((6), I·1·2).

PROPOSITION 4. *With the obvious structure $(\mathscr{H}_\infty, \mathscr{L})$ is an E_∞ operad pair.*

This is proved just like (6), VII·2·3.

For $\mathscr{G} \to \mathscr{L}$ a morphism of operads a $\mathscr{G}$-spectrum is defined as in (6), IV·1·1.

PROPOSITION 5. *If there is a morphism $(\mathscr{C}, \mathscr{G}) \to (\mathscr{H}_\infty, \mathscr{L})$ of operad pairs, then the zeroth space of a $\mathscr{G}$-spectrum is naturally a $(\mathscr{C}, \mathscr{G})$-space.*

This is proved just like (6), VII·2·4.

I am grateful to Peter May for some comments.

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