HOMOTOPY ASSOCIATIVITY OF H-SPACES. I

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1. **Introduction.** The concept of an H-space arose as a generalization of that of a topological group. The essential feature which is retained is a continuous multiplication with a unit. There is a significant class of spaces which are H-spaces but not topological groups. Some of the techniques which apply to topological groups can be applied to H-spaces, but not all. From the point of view of homotopy theory, it is not the existence of a continuous inverse which is the important distinguishing feature [6; 15], but rather the associativity of the multiplication.

For example, if we regard S^0, S^1, S^3 and S^7 as the real, complex, quaternionic and Cayley numbers of unit norm, these spaces possess continuous multiplications, which in the first three cases are associative. Now it is possible to define real, complex and quaternionic projective spaces of arbitrarily large dimension, but this is not possible for the Cayley numbers. From the point of view of homotopy theory, we can investigate the "mechanism" which relates the associativity of the multiplication to the possible existence of projective spaces.

First we consider the construction of the classical projective space as generalized by Milnor [8] for an arbitrary topological group (and further generalized by Dold and Lashof [3] for an arbitrary associative H-space). Given a topological group G, Milnor constructs fibre bundles $p_i: E_i \to B_i$ with fibre G, the total space E_i being the i-fold join of G with itself. If $G = S^{d-1}$, $d = 1, 2, 4$, this gives the standard fibring of S^{id-1} onto the corresponding projective space of dimension $i-1$. In the case of the Cayley numbers, only the fibrings of S^7 onto a point and of S^{15} onto S^8 can be constructed. It seems reasonable to ask whether something weaker than associativity might permit more but not all of these fibrings to be constructed. Sugawara [14] has shown that a variant of Milnor's construction can be carried one step further than for an arbitrary H-space if the multiplication is at least homotopy associative; that is, if m: $X \times X \to X$ is the multiplication then the two maps of $X \times X \times X$ into X given by the two ways of associating are homotopic, i.e., the diagram

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$$\begin{array}{ccc} X \times X \times X & \xrightarrow{m \times 1} & X \times X \\ {\scriptstyle 1 \times m}\downarrow & & \downarrow{\scriptstyle m} \\ X \times X & \xrightarrow{m} & X \end{array}$$

is homotopy commutative. What should the next step be?

From the work of Sugawara and Dold-Lashof, it is clear that a fibre bundle is too restrictive a concept. The features of the Milnor construction which it is important to retain are embodied in the following definition, but first let us stipulate that all spaces we consider will be of the homotopy type of countable CW-complexes with base points and all maps and homotopies will respect base points.

DEFINITION 1. An A_n-*structure* on a space X consists of an n-tuple of maps

$$\begin{array}{ccccccccc} X & = & E_1 & \subset & E_2 & \subset & \cdots & \subset & E_n \\ & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & & & \downarrow{\scriptstyle p_n} \\ * & = & B_1 & \subset & B_2 & \subset & \cdots & \subset & B_n \end{array}$$

such that $p_{i*}\colon \pi_q(E_i, X) \to \pi_q(B_i)$ is an isomorphism for all q, together with a contracting homotopy $h\colon C\,E_{n-1} \to E_n$ such that $h(C\,E_{i-1}) \subset E_i$.

The Milnor or Dold and Lashof construction shows that there are spaces which admit A_n-structures for all values of n.

(For the purposes of homotopy theory, we can think of $X \to E_i \xrightarrow{p_i} B_i$ as a fibring in the light of

PROPOSITION 2. *Given* $X \subset E$ *and a map* $p\colon (E, X) \to (B, *)$ *such that* $p_*\colon \pi_q(E, X) \to \pi_q(B)$ *is an isomorphism for all* q, *there exists a homotopy equivalent fibring* $F \to \bar{E} \xrightarrow{\bar{p}} B$ *such that* F *has the homotopy type of* X.

Proof. Any map is equivalent to a fibring so that we have

$$\begin{array}{ccc} E & \rightarrow & \bar{E} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \bar{p}} & \\ & B & \end{array}$$

[11, Lemma 13]. Since $p(X) = *$, X is mapped into $F = \bar{p}^{-1}(*)$. From the induced map of the exact sequences of the pairs (E, X) and $(\bar{E}, F)$, we conclude that F has the homotopy type of X, all spaces having the homotopy type of CW-complexes.)

To study spaces which admit A_n-structures, we can work directly with the maps p_i. In the case of a topological group, this amounts to working only with the classifying bundle and never mentioning group operations. This would be an exercise in rectitude of thought of which it would be pointless to countenance the austerity, for not only would it eliminate a useful perspective on the subject, but, by disguising its own main point, it would place the reader beneath a cloud of unknowing. A similar remark can be made about A_n-spaces. We shall see that an A_n-structure on a space X is equivalent to an "A_n-form," that is, a sequence of maps $M_2, \cdots, M_n$ where each $M_i: I^{i-2} \times X^i \to X$ is appropriately defined on $\partial I^{i-2} \times X^i$ in terms of M_j for $j < i$. (In particular, as indicated above, an A_2-space is equivalent to an H-space, an A_3-space to a homotopy associative H-space [15; 14].)

Our study of A_n-spaces depends strongly on the interplay between A_n-structures and A_n-forms. In particular, using both view points we are able to provide an example for each prime p of a space which admits A_{p-1}- but not A_p-structures. (S^7 is a good example for $p = 3$.)

The main theorem, Theorem 5, was strongly suggested by the work of Sugawara. The A_n-forms we will discuss are a greatly simplified version of the appropriate part of his conditions for a group-like space [14]. Where our proofs are suggested by his, we have attempted extensive simplification. The work of Dold and Lashof [3] also had a deep influence on the development of this subject; it is particularly apparent in the proof of Theorem 10. This paper represents in part joint work with J. F. Adams, whose inspiration has permeated this entire effort, though he should not be held responsible for the present exposition. The examples of Theorem 17 are entirely due to him, and his comments were most helpful in the treatment of the complexes K_i and construction 8.

I would also like to express my gratitude to Professor J. C. Moore for suggesting a problem which led to the present paper and for his continuing advice and encouragement while supervising my thesis for Princeton University, from which much of this material is drawn, to Dr. I. M. James for contributing his work on "retractile" subsets [6] at a most opportune time and especially to Dr. S. Y. Husseini, to whom fell the thankless task of reading my preliminary attempts at exposition of this subject.

2. **A_n-forms.** Before defining A_n-forms explicitly, we introduce for each $i \geqq 2$ a special cell complex K_i which is homeomorphic to I^{i-2}. The reader is on friendly terms with the standard simplices Δ^i and the standard cubes I^i. He should think of the standard cells K_i as similar objects, also having faces and degeneracies and suitable for use as models for a singular homology theory. He should also keep in mind the important differences that

(1) the index i does not refer to the dimension of the cell but rather to the number of factors of X with which K_i will be significantly associated later,

(2) K_i has i degeneracy operators $s_1, \cdots, s_i$ defined on it, and

(3) K_i has $i(i-1)/2 - 1$ faces.

We see already that the complexes K_i are more complicated than simplices or cubes. Even to index the faces of K_i is not straightforward; the following description of this indexing is the only one we know of which has some intuitive content. Consider a word with i letters, and all meaningful ways of inserting one set of parentheses. To each such insertion except for $(x_1 \cdots x_i)$, there corresponds a cell of L_i, the boundary of K_i. If the parentheses enclose x_k through x_{k+s-1}, we regard this cell as the homeomorphic image of $K_r \times K_s$ $(r + s = i + 1)$ under a map which we call $\partial_k(r,s)$. Two such cells intersect only on their boundaries and the "edges" so formed correspond to inserting two sets of parentheses in the word. Thus we have the relations

3(a) $\partial_j(r, s + t - 1)(1 \times \partial_k(s,t)) = \partial_{j+k-1}(r + s - 1, t)(\partial_j(r,s) \times 1)$,

(b) $\partial_{j+s-1}(r + s - 1, t)(\partial_k(r,s) \times 1) = \partial_k(r + t - 1, s)(\partial_j(r,t) \times 1)(1 \times T)$

where $T: K_s \times K_t \to K_t \times K_s$ permutes the factors.

This is enough to obtain K_i by induction. Start with K_2 as a point, $*$. Given K_2 through K_{i-1}, construct L_i by fitting together copies of $K_r \times K_s$ as indicated by the above conditions. Take K_i to be the cone on L_i.

PROPOSITION 3. *K_i is homeomorphic to I^{i-2}. Degeneracy maps $s_j: K_i \to K_{i-1}$ for $1 \leqq j \leqq i$ can be defined so that the following relations hold:*

3(c) $s_j s_k = s_k s_{j+1}$ *for* $k \leqq j$,

(d) $s_j \partial_k(r,s) = \partial_{k-1}(r - 1, s)(s_j \times 1)$ *for* $j < k$ *and* $r > 2$,

(e) $s_j \partial_k(r,s) = \partial_k(r, s - 1)(1 \times s_{j-k+1})$ *for* $s > 2$, $k \leqq j < k + s$, $s_j \partial_k(i - 1, 2) = \pi_1$ *for* $1 < j = k < i$ *and* $1 < j = k + 1 \leqq i$, $s_1 \partial_2(2, i - 1) = \pi_2$ *and* $s_i \partial_1(2, i - 1) = \pi_2$(*where* π_m *for* $m = 1,2$ *is projection onto the mth factor*),

(f) $s_j \partial_k(r,s) = \partial_k(r - 1, s)(s_{j-s+1} \times 1)$ *for* $k + s \leqq j$.

We will prove Proposition 3 later in this paper by explicitly constructing the complexes K_i as subsets of I^{i-2}.

REMARK 4. The above relations are reminiscent of the usual ones between the face and degeneracy operators of a semi-simplicial complex. Because the semi-simplicial operators correspond, for example, to "$\bar{\partial}_j(\Delta^q)$ is the jth face of Δ^q" rather than "$\bar{\partial}_j$ imbeds Δ^{q-1} as the jth face of Δ^q," they compose in the opposite direction. With this change, the semi-simplical operators (call them $\bar{\partial}_j$ and $\bar{s}_j$) satisfy 3(a)–(f) if we replace $\partial_{j+1}(i - 1, 2)$ by $\bar{\partial}_j$ and s_{j+1} by $\bar{s}_j$.

The complexes K_i are important because of their role in the following theorem.

THEOREM 5. *A space X admits an A_n-structure if and only if there exist maps $M_i: K_i \times X^i \to X$ for $2 \leqq i \leqq n$ such that*

(1) $M_2(*, e, x) = M_2(*, x, e) = x$ *for* $x \in X, * = K_2$,

(2) *for* $\rho \in K_r$, $\sigma \in K_s$, $r + s = i + 1$, *we have*

$$M_i(\partial_k(r,s)(\rho,\sigma), x_1, \cdots, x_i) =$$
$$M_r(\rho, x_1, \cdots, x_{k-1}, M_s(\sigma, x_k, \cdots, x_{k+s-1}), x_{k+s}, \cdots, x_i),$$

(3) *for* $\tau \in K_i$ *and* $i > 2$, *we have*

$$M_i(\tau, x_1, \cdots, x_{j-1}, e, x_{j+1}, \cdots, x_i) = M_{i-1}(s_j(\tau), x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_i).$$

We call such a set of maps an A_n-*form* on X and the pair $(X, \{M_i\})$ an A_n-*space*. [Using conditions 3(a)–(f), the reader may readily check for himself that 5(2) and (3) are consistent; that is, they give a well-defined map of $K_i \times X^{[i]} \cup L_i \times X^i$ into X. (Here $X^{[i]}$ is the subset of X^i consisting of points with at least one coordinate being e.)]

REMARK 6. Notice that an A_2-space is just an H-space. We will often write xy for $M_2(*, x, y)$. Formula 5(2) is a bit opaque, but least so when $s = 2$ in which case it reduces to

$$M_i(\partial_k(i-1,2),(\rho,*), x_1, \cdots, x_i) = M_{i-1}(\rho, x_1, \cdots, x_k x_{k+1}, \cdots, x_i).$$

Now for $i = 3, K_3$ is homeomorphic to I and 5(2) says that $M_3: I \times X^3 \to X$ is a homotopy between $M_2(M_2 \times 1)$ and $M_2(1 \times M_2)$, that is, between $(xy)z$ and $x(yz)$. Thus M_3 is an *associating homotopy*; M_2 is a homotopy associative multiplication. For the case $i = 4$, consider the five ways of associating a product of four factors. If the multiplication is homotopy associative, these five products are related by the following string of homotopies:

$$x(y(zt)) \cong x((yz)t) \cong (x(yz))t \cong ((xy)z)t \cong (xy)(zt) \cong x(y(zt)).$$

Thus we have defined a map of $S^1 \times X^4$ into X; the map M_4 can be regarded as an extension of this map to $I^2 \times X^4$.

Of course any associative H-space admits A_n-forms for any n; we need only define $M_i(\tau, x_1, \cdots, x_i) = x_1 x_2 \cdots x_i$. We call this a *trivial* A_n-form.

Condition 5(3) is technically very useful, but actually is no restriction; that is,

LEMMA 7. *Suppose* $\{M_i, i < n\}$ *is an* A_{n-1}-*form and that* $M'_n: K_n \times X^n \to X$ *satisfies* 5(2), *then there exists* $M_n: K_n \times X^n \to X$ *satisfying* 5(2) *and* (3).

This follows from [6] as we shall indicate in more detail in an Appendix.

3. **Derived A_n-structures.** We relate A_n-forms to A_n-structures by a specific construction. We are grateful to J. F. Adams for a suggestion which has greatly simplified the construction we originally developed. Milnor defines the n-fold join $X * n * X$ by means of certain identifications on $\Delta^{n-1} \times X^n$. We will construct A_n-structures $p_i: \mathscr{E}_i \to \mathscr{B}_i$ in which the space $\mathscr{E}_i$ will also have the homotopy type of $X * i * X$ but we will use $K_{i+1} \times X^i$ instead of $\Delta^{i-1} \times X^i$ and will add further identifications which correspond to the reduced join.

CONSTRUCTION 8. *The* A_n-*structure derived from an* A_n-*form* $\{M_i\}$ *on* X.

Let $R = L_{i+1} \times X^i \cup K_{i+1} \times X \times X^{[i-1]}$. Define spaces $\mathscr{E}_i$ for $1 \leqq i \leqq n$ by means of relative homeomorphisms

$$(K_{i+1} \times X^i, R) \xrightarrow{\alpha_i} (\mathscr{E}_i, \mathscr{E}_{i-1})$$

where $\alpha_i | R$ is defined by

$$\begin{aligned}(9)\quad &\alpha_i(\partial_k(r,s)(\rho,\sigma),x_1,\cdots,x_i)=\alpha_{r-1}(\rho,x_1,\cdots,M_s(\sigma,x_k,\cdots,x_{k+s-1}),\cdots,x_i),\\ &\alpha_i(\tau,x_1,\cdots,x_{j-1},e,x_{j+1},\cdots,x_i)=\alpha_{i-1}(s_j(\tau),x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_i),\quad j>1,\end{aligned}$$

with the convention that the undefined expression $M_s(\sigma,x_k,\cdots,x_{i+1})$ is to be omitted. [$\mathscr{E}_1$ is just X. If we identify K_3 with I, then we can regard α_2 as identifying $0\times X^2$ with X by the map $(0,x,y)\to x$; $1\times X^2$ with X by the map $(1,x,y)\to xy$; and $I\times(X\vee X)$ with X in the obvious way.]

We define spaces $\mathscr{B}_i$ for $1\leqq i\leqq n$ similarly in terms of relative homeomorphisms

$$(K_{i+1}\times X^{i-1},S)\xrightarrow{\beta}(\mathscr{B}_i,\mathscr{B}_{i-1})$$

where $S=L_{i+1}\times X^{i-1}\cup K_{i+1}\times X^{[i-1]}$. The restriction $\beta_i|S$ is defined by the formulas (9) except that we replace α by β throughout and omit x_1 and all terms $M_s(\sigma,x_1,\cdots,x_s)$. [$\mathscr{B}_1$ is a point and $\mathscr{B}_2$ can be recognized as SX.] Notice that the definition of $\mathscr{B}_i$ uses the maps M_j only for $j<i$, hence even if X admits only A_n-forms, $\mathscr{B}_{n+1}$ can still be constructed, although $\mathscr{E}_{n+1}$ cannot.

Let $\rho_i:K_{i+1}\times X^i\to K_{i+1}\times X^{i-1}$ be defined by omitting x_1, i.e., $\rho_i(\tau,x_1,\cdots,x_i)=(\tau,x_2,\cdots,x_i)$. We see that ρ_i induces $\mathfrak{p}_i:\mathscr{E}_i\to\mathscr{B}_i$. [Notice that $\beta_i|S$ can be defined by $\beta_i(\tau,x_2,\cdots,x_i)=\mathfrak{p}_{i-1}\alpha_i(\tau,e,x_2,\cdots,x_i)$. Hence, by induction, $\mathfrak{p}_i$ is well defined. Of course, in the above discussion we have used implicitly the fact that $\alpha_i|R$ and $\beta_i|S$ are well-defined maps. This can be proved in much the same way that the reader proved $M_i|K_i\times X^{[i]}\cup L_i\times X^i$ was well defined.]

In order to show that we have in fact constructed an A_n-structure on X, we need

THEOREM 10. *If X is arc-connected, $\mathfrak{p}_{i*}:\pi_q(\mathscr{E}_i,X)\to\pi_q(\mathscr{B}_i)$ is an isomorphism for all q.*

(In fact we will show that $\mathfrak{p}_i$ is a quasifibring [4].)

We also must show that $\mathscr{E}_{i-1}$ is contractible in $\mathscr{E}_i$; we prove a stronger result.

THEOREM 11. *If X is arc-connected, then $(\mathscr{E}_n,\cdots,\mathscr{E}_1)$ has the homotopy type of $(X*n*X,\cdots,X)$.*

Corresponding to this in the base we have

THEOREM 12. *$(\mathscr{B}_{i+1},\mathscr{B}_i)$ has the homotopy type of $(C\mathscr{E}_i\cup_{\mathfrak{p}_i}\mathscr{B}_i,\mathscr{B}_i)$.*

(For any map $f:X\to Y$, the mapping cone $CX\cup_f Y$ is defined as the space obtained from the disjoint union of CX and Y by identifying X as the base of the cone with its image under f in Y.)

The base spaces $\mathscr{B}_i$ will be very useful invariants associated with an A_n-space; we give them a special name.

DEFINITION 13. The *X-projective i-space $XP(i)$*, $i\leqq n$, associated with an

A_n-space is the base space $\mathscr{B}_{i+1}$ of the derived A_n-structure. ($\mathscr{B}_{n+1}$ can be defined even when $\mathfrak{p}_{n+1}$ cannot; it has the homotopy type of $C\mathscr{E}_n \cup_{\mathfrak{p}_n} \mathscr{B}_n$.)

The justification of this terminology arises from considering the classical fibrings of $S^{d-1} * i * S^{d-1} = S^{di-1}$ by S^{d-1} for $d = 1,2,4$. $S^{d-1}P(i)$ is respectively real, complex, or quaternionic projective space of i-dimensions, and the fibrings can be identified with our construction by using the strictly associative multiplication on S^{d-1}.

Convention. If X admits some A_n-form, we will refer to the X-projective i-space $XP(i)$ without emphasizing the particular A_i-form to which $XP(i)$ corresponds.

4. **Proof of Theorem 5.** Theorems 11 and 12 show that the existence of an A_n-form implies the existence of an A_n-structure. To prove the converse, we first observe that it is sufficient to construct an A_n-form $\{M_i\}$ on some space F of the same homotopy type as X, for we can then define an A_n-form $\{N_i\}$ on X by suitable deformations of the maps

$$K_i \times X^i \xrightarrow{1 \times j^i} K_i \times F^i \xrightarrow{M_i} F \xrightarrow{s} X$$

where $j: X \to F$ and $s: F \to X$ are homotopy inverses. Thus, in light of Proposition 2, we might as well assume that $X \to E_i \xrightarrow{p_i} B_i$ is a fibring.

Assume by induction that M_j is defined (and $\mathfrak{p}_j:\mathscr{E}_j \to \mathscr{B}_j$ is constructed) for $j < i$, and that we have commutative diagrams

(14)
$$\begin{array}{ccc} \mathscr{E}_j & \xrightarrow{d_j} & E_j \\ {\scriptstyle \mathfrak{p}_j}\downarrow & & \downarrow{\scriptstyle p_j} \\ \mathscr{B}_j & \xrightarrow[b_j]{} & B_j \end{array}$$

such that $d_j | \mathscr{E}_{j-1} = d_{j-1}$, etc. The induction begins with the commutative diagram

$$\begin{array}{ccccccc} \mathscr{E}_1 & = & X & = & X & = & E_1 \\ & & \downarrow & & \downarrow & & \\ & & * & = & * . & & \end{array}$$

Let $J =$ interior of $\partial_1(2,i)(K_2 \times K_i)$. Note that on all of R except for $J \times (X^i - X^{[i]})$, α_i is defined without using M_i. There is no difficulty in extending from this subset to $\gamma: (L_{i+1} - J) \times X^i \cup K_{i+1} \times X^{[i]} \to C\mathscr{E}_{i-1}$. Let $k: CE_{i-1} \to E_i$ be the contraction. Define $j: K_{i+1} \times X^{i-1} \to B_i$ by

$$j(\tau, x_2, \cdots, x_i) = k \circ Cd_{i-1} \circ \gamma(\tau, e, x_2, \cdots, x_i)$$

so that $j \circ \rho_i = p_i \circ k \circ Cd_{i-1} \circ \gamma$ over the whole domain of γ. Thus j induces an extension $b_i: \mathscr{B}_i \to B_i$ of b_{i-1}. Since $L_{i+1} - J$ is a deformation retract of K_{i+1}, $k \circ Cd_{i-1} \circ \gamma$ can be extended to a map $d: K_{i+1} \times X^i \to E_i$ covering $j \circ \rho_i$. Thus

$d(\partial_1(2,i)(K_2 \times K_i) \times X^i) \subset X$ and the desired map M_i can be defined by $M_i(\tau, x_1, \cdots, x_i) = d(\partial_1(2,i)(*,\tau), x_1, \cdots, x_i)$. The space $\mathscr{E}_i$ can now be constructed, and we see that d induces a map $d_i : \mathscr{E}_i \to E_i$ such that

$$\begin{array}{ccc} \mathscr{E}_i & \xrightarrow{d_i} & E_i \\ \mathfrak{p}_i \downarrow & & \downarrow p_i \\ \mathscr{B}_i & \xrightarrow[b_i]{} & B_i \end{array}$$

is commutative.

Notice that Theorem 5 has the following corollaries (which are already known [15; 14]).

Corollary 15. *A space X admits a multiplication if and only if there is a map $\mathfrak{p}:(X * X, X) \to (SX, *)$ such that $\mathfrak{p}_* : \pi_q(X * X, X) \to \pi_q(SX)$ is an isomorphism for all q.*

Corollary 16. *A space X admits a homotopy associative multiplication if and only if there is a map $\bar{\mathfrak{p}}:(X * X * X, X * X, X) \to (C(X * X) \cup_{\mathfrak{p}} SX, SX, *)$ such that $\mathfrak{p} = \bar{\mathfrak{p}} \mid X * X$ and $\mathfrak{p}_* : \pi_q(X * X, X) \to \pi_q(SX)$ is an isomorphism for all q and $\bar{\mathfrak{p}}_* : \pi_q(X * X * X, X) \to \pi_q(C(X * X) \cup_{\mathfrak{p}} SX)$ is an isomorphism for all q.*

5. **Examples.** Theorem 5 also shows the relation between the nonexistence of a homotopy associative multiplication on S^7 and the impossibility of fibring S^{23} by S^7 over the Cayley projective plane. This suggests that the nonexistence of a homotopy associative multiplication on S^7 can be shown using Steenrod operations. In fact, this approach can be generalized to indicate more fully the nontrivial distinctions involved in the A_n-classification.

Theorem 17. *For each prime p, there exist spaces which admit A_{p-1}-structures but not A_p-structures.*

Proof. The examples we give are due to J. F. Adams, as are the techniques used in the proof. These spaces are constructed quite explicitly in [1], where further details of the assertions below also can be found. (The above theorem is the part of our joint work referred to there.)

Let p be an odd prime and Q^p, the group of all those fractions which, in lowest terms, have denominators prime to p. Let X be a Moore space with one nonvanishing homology group Q^p in dimension $2n-1$. Since Q^p has no p-torsion and is infinitely divisible by all other primes, $H_*(X; Z_q) = 0$ if q is prime to p and $H_i(X; Z_p) = 0$ except for $i = 0$ or $2n-1$, while $H_{2n-1}(X; Z_p) = Z_p$. [This can be seen by mapping Q^p onto Z_p by $a/b \to ab'(p)$ where $bb' \equiv 1 \bmod p$.]We can use $\mathscr{C}$-theory [12] to deduce that for $i > 2n-1$, $\pi_i(X)$ is isomorphic to the p-primary component of $\pi_i(S^{2n-1})$.

Let us imbed X in the space $Z = \Omega^2 S^2 X$ in the usual way. Since Z is a loop space, it admits an A_n-structure for any n; call the corresponding maps $N_i: K_i \times Z^i \to Z$ and let $M_i': K_i \times X^i \to Z$ denote the restrictions. We would like to deform the maps M_i' into X so as to obtain A_{p-1}-forms on X. By induction, assume the image of M_j lies in X for $j < i$. Let $T = L_i \times X^i \cup K_i \times X^{[i]}$. M_i' is defined on T in terms of M_j for $j < 1$, so $M_i(T) \subset X$. The obstructions to deforming M_i into X rel T appear as classes in $H^q(K_i \times X^i, T; \pi_q(Z, X))$ which, since (K_i, L_i) is isomorphic to $(I^{i-2}, \dot{I}^{i-2})$, is isomorphic to $H^{q+2-i}(X^{(i)}; \pi_q(Z, X))$. Since the p-primary component of $\pi_q(\Omega^2 S^{2n+1}, S^{2n-1}) = 0$, for $q < 2pn - 2$ [10], we can conclude that $\pi_q(Z, X) = 0$ for $q < 2pn - 2$. Since X has nontrivial cohomology only in dimension $2n - 1$, $H^q(X^{(i)}; G) = 0$ for any coefficient group G if $q \neq i(2n-1)$. Thus the obstructions we have considered lie in trivial groups unless $i(2n-1) + i - 2 \geqq 2pn - 2$, i.e., $i \geqq p$. There are no obstructions to obtaining A_{p-1}-forms on X.∎

Now suppose X did admit an A_p-form (equivalently, an A_p-structure). Consider the corresponding maps $\mathfrak{p}_i: \mathscr{E}_i \to \mathscr{B}_i$. According to Proposition 2, we can replace $\mathfrak{p}_i$ by a homotopy equivalent map $\bar{\mathfrak{p}}_i$ which is a fibring in the sense of Serre. Therefore in the argument below we can assume without loss of generality that $\mathfrak{p}_i$ is a fibring. Thus we know that the Thom-Grysin sequence [2, Exposé 8] applies to the "Thom Space" $C\mathscr{E}_i \cup_{\mathfrak{p}_i} \mathscr{B}_i = XP(i)$. This sequence can be used to compute the cohomology ring of real or complex projective space [9, Theorem 23]. In exactly the same way, since $H^*(X; Z_p) \approx H^*(S^{2n-1}; Z_p)$, we find that $H^*(XP(i); Z_p)$ is a truncated polynomial algebra on a generator $u \in H^{2n}(XP(i); Z_p)$ with $u^i \neq 0$. But if we choose n prime to $p(p-1)$, the Adem relations on the Steenrod operations mod p imply that u^p must vanish; therefore X cannot admit A_p-structures.

(For $X = S^7$, the above argument shows that S^7 admits *no* homotopy associative multiplication [5, Theorem (1.4)].)

Finally, we remark that if the A_n-form is trivial, our construction 8 reduces to that of Dold and Lashof.

In that case and more generally whenever X admits a structure $\{p_i; i = 1, 2, \cdots\}$ satisfying the usual conditions but for all positive integers i, then X has the homotopy type of ΩB_∞ where B_∞ is the limit of the base spaces B_i [15, Lemma 10]. We will investigate this relationship more fully in *Homotopy associativity of H-spaces*. II, a sequel to the present paper.

6. **Complexes K_i.**

Proof of Proposition 3. We exhibit particular representatives of the complexes K_i as subsets of I^{i-2}. Figure 18 pictures the cases $i = 2, 3, 4, 5$; K_i being heavily outlined as a subset of I^{i-2}.

Let K_i be the subset of I^{i-2} consisting of points $(t_1, \cdots, t_{i-2})$ such that $2^j t_1 \cdots t_j \geqq 1$ for $1 \leqq j \leqq i-2$. L_i, the boundary of K_i, consists of the point of

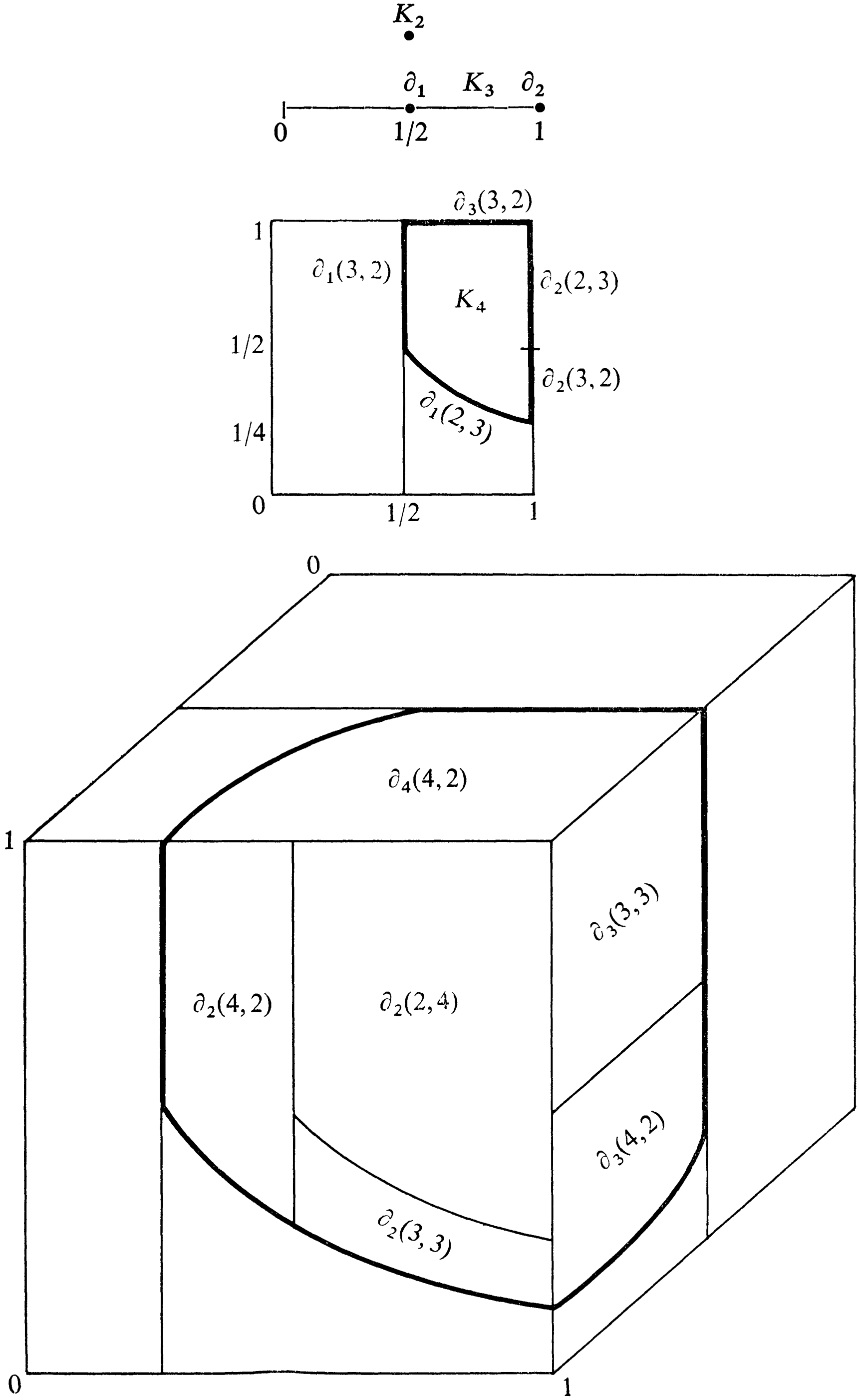
K_2
∂_1 K_3 ∂_2
0 1/2 1
$\partial_3(3,2)$
$\partial_1(3,2)$
K_4
$\partial_2(2,3)$
$\partial_2(3,2)$
$\partial_1(2,3)$
1
1/2
1/4
0 1/2 1
0
$\partial_4(4,2)$
1
$\partial_3(3,3)$
$\partial_2(4,2)$
$\partial_2(2,4)$
$\partial_3(4,2)$
$\partial_2(3,3)$
0 1

Figure 18

K_i such that for some j either $2^j t_1 \cdots t_j = 1$ or $t_j = 1$. We give the face operators as follows:

$k = 1: \partial_1(r,s)(K_r \times K_s)$ is to be the subset of K_i such that $2^{s-1}t_1 \cdots t_{s-1} = 1$. Let $\rho = (t_1, \cdots, t_{r-2}) \in K_r$ and $\sigma = (u_1, \cdots, u_{s-2}) \in K_s$ then

$$\partial_1(r,s)(\rho,\sigma) = (u_1, \cdots, u_{s-2}, u, t_1, \cdots, t_{r-2})$$

with $u = 1/2^{s-1}u_1 \cdots u_{s-2}$.

$k > 1: \partial_k(r,s)(K_r \times K_s)$ is the subset of the "face" $t_{k-1} = 1$ such that $(t_k, \cdots, t_{k+s-3}) \in K_s$ and $(t_1, \cdots, t_{k-2}, 2^{s-1}t_k \cdots t_{k+s-2}, t_{k+s-1}, \cdots, t_{i-2}) \in K_r$. For ρ, σ as above we have

$$\partial_k(r,s)(\rho,\sigma) = (t_1, \cdots, t_{k-2}, 1, u_1, \cdots, u_{s-2}, \bar{t}_{k-1}, t_k, \cdots, t_{r-2})$$

where $\bar{t}_{k-1} = t_{k-1}/2^{s-1}u_1 \cdots u_{s-2}$.

Relations 3(a) and (b) can be verified directly from the definitions. For example with $k,\ j > 1$ and

$$\rho = (t_1, \cdots, t_{r-2}), \quad \sigma = (u_1, \cdots, u_{s-2}), \quad \tau = (v_1, \cdots, v_{t-2})$$

we have

$$\partial_j(r, s+t-1)(1 \times \partial_k(s,t))(\rho,\sigma,\tau)$$
$$= (t_1, \cdots, t_{j-2}, 1, u_1, \cdots, u_{k-2}, 1, v_1, \cdots, v_{t-2}, \bar{u}_{k-1}, u_k, \cdots, u_{s-2}, \bar{t}_{j-1}, t_j, \cdots, t_{r-2})$$

with

$$\bar{u}_{k-1} = u_{k-1}/2^{t-1}v_1 \cdots v_{t-2} \text{ and } \bar{t}_{j-1} = t_{j-1}/2^{s-1}u_1 \cdots u_{s-2}.$$

On the other hand

$$\partial_{j+k-1}(r+s-1,t)(\partial_j(r,s) \times 1)(\rho,\sigma,\tau)$$
$$= (t_1, \cdots, t_{j-2}, 1, u_1, \cdots, u_{k-2}, 1, v_1, \cdots, v_{t-2}, \bar{u}_{k-1}, u_k, \cdots, u_{s-2}, \bar{\bar{t}}_{j-1}, t_j, \cdots, t_{r-2})$$

where

$$\bar{\bar{t}}_{j-1} = t_{j-1}/2^{s+t-2}u_1 \cdots u_{k-2}v_1 \cdots v_{t-2}\bar{u}_{k-1}u_k \cdots u_{s-2}.$$

Expanding $\bar{u}_{k-1}$ we see that relation 3(a) holds.

The degeneracy maps $s_j: K_{i+1} \to K_i$ can be defined on L_{i+1} using 3(d)–(f), since the latter are compatible with 3(a) and (b). It also follows easily that 3(c) will be satisfied on L_{i+1}. The map s_j on all of K_{i+1} can now be obtained "by taking the cone." That is, represent K_{i+1} as pairs (t,τ) with $\tau \in L_{i+1}$ and similarly for K_i. If $s_j(\tau) = (s,\tau')$, $\tau' \in L_i$ then define $s_j(t,\tau)$ as (ts,τ'). With this definition it is easy to verify 3(c) on all of K_{i+1}, while s_j was constructed so as to satisfy 3(d)–(f). The relations are important in that they make the conditions of Theorem 5 consistent.

7. **Proof of Theorem 10.** This will follow from:

THEOREM 19. *$\mathfrak{p}_i:\mathscr{E}_i \to \mathscr{B}_i$ is a quasifibring (with fibre X).*

Proof. (Cf. [3].) For $i = 1$, $\mathfrak{p}_1$ is trivially a quasifibring since $\mathscr{B}_1$ is a point. By induction assume $\mathfrak{p}_{i-1}$ is a quasifibring. We will decompose $\mathfrak{p}_i$ as the union of two sub-quasifibrings.

Let $U = \mathscr{B}_i - \mathscr{B}_{i-1}$ and $P = \mathfrak{p}_i^{-1}(U) = \mathscr{E}_i - \mathscr{E}_{i-1}$.

LEMMA 20. *$\mathfrak{p}_i|P$ is a quasifibring.*

Proof. Since α_i^{-1} is a well-defined homeomorphism on P and β_i^{-1} is a well-defined homeomorphism on U, we see that $\mathfrak{p}_i|P$ is equivalent to the map $X \times U \to U$ which is projection onto the second factor, trivially a quasifibring.

Now recall that R denotes the set $L_{i+1} \times X^i \cup K_{i+1} \times X \times X^{[i-1]}$ and S denotes the set $L_{i+1} \times X^{i-1} \cup K_{i+1} \times X^{[i-1]}$.

LEMMA 21. *There exists a neighborhood $\mathscr{R}$ in $K_{i+1} \times X^i$ of R of which R is a strong deformation retract.*

Proof. Take a "tubular" neighborhood N' of L_{i+1} in K_{i+1}, i.e., there is a homeomorphism of N' onto $L_{i+1} \times (0,1]$ which maps L_{i+1} identically onto $L_{i+1} \times 1$. Let N be the inverse image of $L_{i+1} \times (1/2, 1]$. There exists a deformation $h_s: K_{i+1} \to K_{i+1}$ rel L_{i+1} such that h_1 is the identity, $h_0(N) \subset L_{i+1}$ and h_s is constant outside N'.

Since X has the homotopy type of a CW-complex, there exists an open neighborhood N_e of e and a deformation $k_s: X \to X$ such that k_1 is the identity, $k_0(N_e) = e$ [16, (M), p. 230]; also cf. [7, Theorem (8.3)].

Take $\mathscr{R}$ as the union of $N \times X^i$ and all sets of the form $K_{i+1} \times X \times X \times \cdots \times N_e \times \cdots \times X$ except $K_{i+1} \times N_e \times X \times \cdots \times X$. Define $D_t'': K_{i+1} \times X^i \to K_{i+1} \times X^i$ by $h_t \times 1 \times (k_t)^{i-1}$. We have D_1'' as the identity, $D_0''(\mathscr{R}) \subset R$ and $D_t''(R) \subset R$. Therefore D_t'' can be deformed to give a *strong* deformation retraction D_t' of $\mathscr{R}$ onto R.

Now let $\mathscr{S}$ be obtained from $\mathscr{R}$ by omitting the first X factor, so that $\mathscr{S}$ is an open neighborhood of S. A strong deformation retraction d_t' of $\mathscr{S}$ onto S is given by $d_t'(\tau, x_2, \cdots, x_i) = \mathfrak{p}_i D_t'(\tau, e, x_2, \cdots, x_i)$.

Let $Q = \alpha_i(\mathscr{R})$ and $V = \beta_i(\mathscr{S})$, so that $\mathscr{E}_i$ is covered by $P \cup Q$ and $\mathscr{E}_{i-1} \subset Q$ while $\mathscr{B}_i$ is covered by $U \cup V$ and $\mathscr{B}_{i-1} \subset V$. Notice that $Q = \mathfrak{p}_i^{-1}(V)$. We will prove

LEMMA 22. *$\mathfrak{p}_i|Q$ is a quasifibring.*

From this and the fact that $\mathfrak{p}_i$ is a quasifibring over U and $U \cap V$, it will follow that $\mathfrak{p}_i$ is a quasifibring [4, (2.2)].

To prove Lemma 22 we will need the following criterion:

LEMMA 23 [4, (2.10)]. *$p:E\to B$ is a quasifibring if for some $B'\subset B$ and $E'=p^{-1}(B')$, $p|E'$ is a quasifibring and there exist deformations*

$$D_t:E\to E,\ d_t:B\to B$$

with

$$D_1 = 1,\qquad D_t(E')\subset E',\qquad D_0(E)\subset E',$$

$$d_1 = 1,\qquad d_t(B')\subset B',\qquad d_0(B)\subset B',\qquad pD_1=d_1p$$

and

$$D_{0*}:\pi_i(p^{-1}(x))\approx\pi_i(p^{-1}(d_0(x)))\qquad \textit{for all } x\in B,\ i\geqq 0.$$

Proof of Lemma 22. We will deform Q onto $\mathscr{E}_{i-1}=\alpha_i(R)$. Define $D_t:Q\to Q$ by $D_t\circ\alpha_i=\alpha_iD_t'$ on $Q-\mathscr{E}_{i-1}$ and as the identity on $\mathscr{E}_{i-1}$. Similarly define d_t: $V\to V$ by $d_t\circ\beta_i=\beta_id_t'$ on $V-\mathscr{B}_{i-1}$ and as the identity on $\mathscr{B}_{i-1}$. Both D_t and d_t are well defined since D_t' and d_t' are *strong* deformation retractions, i.e., constant on R and S respectively. Clearly $\mathfrak{p}_iD_t=d_t\mathfrak{p}_i$. We have $D_1=1$, $D_t|\mathscr{E}_{i-1}=1$, $D_0(Q)\subset\mathscr{E}_{i-1}$ and $d_1=1$, $d_t|\mathscr{B}_{i-1}=1$, $d_0(V)\subset\mathscr{B}_{i-1}$.

Finally, we must consider $g_z=D_0|\mathfrak{p}_i^{-1}(z):\mathfrak{p}_i^{-1}(z)\to\mathfrak{p}_i^{-1}(d_0(z))$ for any $z\in V$. If $z\in\mathscr{B}_{i-1}$, $d_0(z)=z$ and g_z is the identity, trivially a homotopy equivalence. If $z\in V-\mathscr{B}_{i-1}$, then z can be written uniquely as $\beta_i(\tau,x_2,\cdots,x_i)$ and for $d_0(z)$, there exists an r such that $d_0(z)$ can be written uniquely as $\beta_r(\mu,y_2,\cdots,y_r)$. Define a homeomorphism $f_z:X\to\mathfrak{p}_i^{-1}(z)$ by $f_z(x)=\alpha_i(\tau,x,x_2,\cdots,x_i)$ and h_z: $X\to\mathfrak{p}_i^{-1}(d_0(z))$ by $h_z(x)=\alpha_r(\mu,x,y_2,\cdots,y_r)$.

Now recall the definition of d_t'. We see that d_0 is homotopic rel $\mathscr{B}_{i-1}$ to $\bar{d}_0\beta_i(\tau,x_2,\cdots,x_i)\to\beta_i(h_0(\tau),k_0(x_2),\cdots,k_0(x_i))$. We are concerned with $(\tau,x_2,\cdots,x_i)$ if $h_0(\tau)\in L_{i+1}$ or if $k_0(x_j)=e$. Thus $\bar{d}_0(z)$ can be represented as $\beta_r(\rho,y_2,\cdots,y_r)$ where each y_j is some x_k except that at most one y_j may have the form $M_s(\sigma,x_{k+1},\cdots,x_{k+s})$. Now by means of f_z and h_z, g_z can be identified with a map of X into X given either by $x\to x$ (trivially a homotopy equivalence) or by $x\to M_s(\sigma,x,x_{k+1},\cdots,x_{k+s})$ for some *fixed* $\sigma,x_{k+1},\cdots,x_{k+s}$. The latter is homotopic to $x\to x(x_{k+1}(\cdots(x_{k+s})\cdots))$, again a homotopy equivalence since X is arc-connected (right translation is a homotopy equivalence). Since $\mathfrak{p}_{i-1}=\mathfrak{p}_i|\mathscr{E}_{i-1}$ is a quasifibring by the inductive assumption, by Lemma 23 we conclude that $\mathfrak{p}_i|Q$ is a quasifibring. This completes the proof of Theorem 10.

8. **The homotopy type of $\mathscr{E}_i$ and $\mathscr{B}_i$.** Consider the space, call it $\mathscr{D}_i$, defined by a relative homeomorphism

$$(K_{i+1}\times X^{i-1},S)\xrightarrow{\gamma_i}(\mathscr{D}_i,\mathscr{E}_{i-1}),$$

where $\gamma_i(\tau,x_2,\cdots,x_i)=\alpha_i(\tau,e,x_2,\cdots,x_i)$.

PROPOSITION 24. *$\mathscr{D}_i,\mathscr{E}_{i-1}$ has the homotopy type of $C\mathscr{E}_{i-1},\mathscr{E}_{i-1}$.*

In order to prove this, we need an auxiliary map.

PROPOSITION 25. *There exist homeomorphisms* $\eta_i: I \times K_i \to K_{i+1}$ *such that*
(1) $\eta_i(0,\tau) = \partial_2(2,i)(*,\tau)$, *and*
(2) $\eta_i(t,\partial_k(r,s)(\rho,\sigma)) = \partial_{k+1}(r+1,s)(\eta_r(t,\rho),\sigma)$.

Proof. Assume the proposition true for $j < i$. It is easy to verify that equation (2) yields a homeomorphism η_i' of $I \times L_i$ *into* K_{i+1}. The image J_i consists of all faces $\partial_k(r,s)(K_r \times K_s)$ with $k > 1$, $r > 1$. We can regard K_{i+1} as formed from $I \times J_i$ by identifying $(0,\eta_i'(t,\tau))$ with $(0,\eta_i'(t,\tau'))$ for any $\tau,\tau' \in L_i$. In this way, the face $\partial_2(2,i)(K_2 \times K_i)$ is identified with CL_i which in turn we regard as K_i. Looking at things this way, we obtain η_i satisfying (1) and (2) by setting $\eta_i(t,(s,\tau)) = (s,\eta_i'(t,\tau))$ for $\tau \in L_i, (s,\tau) \in CL_i = K_i$.

COROLLARY 26. $s_{j+1}\eta_i(t,\tau) = \eta_{i-1}(t,s_j(\tau))$.

The verification is straightforward.

Proof of Proposition 24. Define $\phi_i: C\mathscr{E}_{i-1} \to \mathscr{D}_i$ by $\phi_i(t,\alpha_{i-1}(\tau,\chi)) = \gamma_i(\eta_i(t,\tau),\chi)$ where $\chi = (x_2,\cdots,x_i)$. To obtain an inverse for ϕ_i define $\bar{\psi}_i$: $K_{i+1} \times X^{i-1} \to C\mathscr{E}_{i-1}$ by $\bar{\psi}_i(\tau,\chi) = (t,\alpha_{i-1}(\tau',\chi))$ where $\eta_i(t,\tau') = \tau$. Now part of $\bar{\psi}_i | S$ lies in $C\mathscr{E}_{i-2} \subset C\mathscr{E}_{i-1}$, instead of in $\mathscr{E}_{i-1}$ as it should to induce a map of $\mathscr{D}_i$ into $C\mathscr{E}_{i-1}$. However $C\mathscr{E}_{i-2} \subset C\mathscr{E}_{i-1}$ is homotopic to $\phi_{i-1}(C\mathscr{E}_{i-2}) \subset \mathscr{D}_{i-1} \subset \mathscr{E}_{i-1}$ by the following argument: It is straightforward to verify that $\phi_i | \mathscr{E}_{i-1}$ is homotopic in $\mathscr{E}_{i-1}$ to the identity. It involves showing, by an inductive argument, that $s_1(\eta_i(1,\tau)): K_i, L_i \to K_i, L_i$ is homotopic to the identity in a way which is compatible with γ_i. We conclude that $C(\phi_{i-1} | \mathscr{E}_{i-2}): C\mathscr{E}_{i-2}, \mathscr{E}_{i-2} \to C\mathscr{E}_{i-2}, \mathscr{E}_{i-2}$ is homotopic to the identity. Next $C(\phi_{i-1} | \mathscr{E}_{i-2})$ is homotopic in $C\mathscr{E}_{i-1}$ to ϕ_i by:

LEMMA 27. *Let* $f: Y \to Z$ *extend to* $F: CY \to Z$ *then* $Cf: CY \to CZ$ *is homotopic* rel Y *to* F.

Proof. Define a homotopy $F_r: CY \to CZ$ for $0 \leqq r \leqq 2$ by

$$F_r(t,F(rt+1-r,y)), \qquad 0 \leqq r \leqq 1,$$

$$F_{r+1}(t(1-r)+r,F(t,y)), \qquad 0 \leqq r \leqq 1.$$

Thus $\bar{\psi}_i$ can be deformed to a map $\psi_i: K_{i+1} \times X^{i-1} \to C\mathscr{E}_{i-1}$ such that $\psi_i | S = \gamma_i | S$. It is clear that $\psi_i: \mathscr{D}_i, \mathscr{E}_{i-1} \to C\mathscr{E}_{i-1}, \mathscr{E}_{i-1}$, the map induced by ψ_i, is a homotopy inverse for ϕ_i as a map of pairs.

Proof of Theorem 12. Recall the definition of $\beta_i: (K_{i+1} \times X^{i-1}, S) \to (\mathscr{B}_i, \mathscr{B}_{i-1})$. We see directly that β_i can be factored as $\sigma_i \circ \gamma_i$ where $\sigma_i: (\mathscr{D}_i, \mathscr{E}_{i-1}) \to (\mathscr{B}_i, \mathscr{B}_{i-1})$ is also a relative homeomorphism and $\sigma_i | \mathscr{E}_{i-1} = \mathfrak{p}_{i-1}$. Since $(\mathscr{D}_i, \mathscr{E}_{i-1})$ has the homotopy type of $(C\mathscr{E}_{i-1}, \mathscr{E}_{i-1})$, this shows that $(\mathscr{B}_i, \mathscr{B}_{i-1})$ has the homotopy type of $(C\mathscr{E}_{i-1} \cup_{\mathfrak{p}_{i-1}} \mathscr{B}_{i-1}, \mathscr{B}_{i-1})$.

Proof of Theorem 11. We can map $X \times \mathscr{D}_i$ into $\mathscr{E}_i$ in the obvious way (corresponding to $X \times K_{i+1} \times X^{i-1} \to K_{i+1} \times X \times X^{i-1}$). We can regard $\mathscr{E}_i$ as obtained from $X \times \mathscr{D}_i$ by identifying

$$(x, \gamma_i(\partial_1(r,s)(\rho,\sigma), x_2, \cdots, x_i))$$

with

$$\alpha_{r-1}(\rho, M_s(\sigma, x, x_2, \cdots, x_s), x_{s+1}, \cdots, x_i).$$

The subset on which these identifications are carried out is just $X \times \phi_i(\mathscr{E}_{i-1})$. Thus $(\mathscr{E}_i, \mathscr{E}_{i-1})$ has the homotopy type of $X \times C\mathscr{E}_{i-1} \cup_\mu \mathscr{E}_{i-1}$ where $\mu: X \times \mathscr{E}_{i-1} \to \mathscr{E}_{i-1}$ is obtained by

$$X \times \mathscr{E}_{i-1} \xrightarrow{1 \times \phi_i|} X \times \mathscr{D}_i \to \mathscr{E}_i.$$

Note that $\mu \mid e \times \mathscr{E}_{i-1}$ is just $\phi_i \mid \mathscr{E}_{i-1}$ and hence homotopic to the identity.

To prove that $X \times C\mathscr{E}_{i-1} \cup_\mu \mathscr{E}_{i-1}, \mathscr{E}_{i-1}$ has the homotopy type of $X * \mathscr{E}_{i-1}, \mathscr{E}_{i-1}$, we use the Meyer-Vietoris sequence in the obvious way [13, Lemma 3.4]. [X is arc-connected and hence $X \times C\mathscr{E}_{i-1} \cup_\mu \mathscr{E}_{i-1}$ and $X * \mathscr{E}_{i-1}$ are simply connected, so that a homology argument is sufficient.] Thus it is sufficient to show

Lemma 28. *The map of $X \times \mathscr{E}_{i-1}$ into $X \times \mathscr{E}_{i-1}$ given by $q(x,z) = (x, \mu(x,z))$ is a homotopy equivalence.*

When $i=2$, this is a familiar fact about H-spaces. We mimic the proof used in that special case, cf. [15, Lemma 6].

Proof. By checking the formulas, we find that μ applied to $X \times X \subset X \times \mathscr{E}_{i-1}$ goes into X and is in fact just M_2, the multiplication on X. The induced map $\mu_*: \pi_n(X) + \pi_n(\mathscr{E}_{i-1}) = \pi_n(X \times \mathscr{E}_{i-1}) \to \pi_n(\mathscr{E}_{i-1})$ can be seen by the usual arguments for H-spaces to coincide with the usual addition in the homotopy groups, mapping $\pi_n(X)$ into a subgroup of $\pi_n(\mathscr{E}_{i-1})$ in the obvious way. Thus $q_*: \pi_n(X) + \pi_n(\mathscr{E}_{i-1}) \to \pi_n(X) + \pi_n(\mathscr{E}_{i-1})$ can be seen to be given by $q_*(\alpha,\beta) = (\alpha, \alpha+\beta)$ which is obviously an isomorphism for all n. Since in our category all spaces have the homotopy type of a CW-complex, q is a homotopy equivalence.

9. **Reduction to the Dold-Lashof construction.** We now make more precise our remark that construction 8 reduces to that given by Dold and Lashof [3] if the A_n-form is trivial. (It is necessary not only that M_2 be strictly associative, but also that M_i be given by $M_i(\tau, x_1, \cdots, x_i) = x_1 \cdots x_i$.)

Recall that Dold and Lashof defined quasifibrings $\bar{p}_i: \bar{E}_i \to \bar{B}_i$ by the following inductive procedure. Let $M: X \times X \to X$ be an associative multiplication. Let $\bar{E}_1 = X$, $\bar{B}_1 = *$. Assume by induction that $\bar{p}_i: \bar{E}_i \to \bar{B}_i$ is defined as well as an associative action $\bar{M}: X \times \bar{E}_i \to \bar{E}_i$. Using the *un*reduced cone, define $\bar{E}_{i+1}$ as $X \times C\bar{E}_i \cup_{\bar{M}} \bar{E}_i$ and $\bar{B}_{i+1}$ as $C\bar{E}_i \cup_{\bar{p}_i} \bar{B}_i$. Define $\bar{p}_{i+1}$ by $\bar{p}_{i+1}(x,t,z) = (t,z)$ for $z \in \bar{E}_i$, $\bar{p}_{i+1} \mid \bar{E}_i = \bar{p}_i$ and $\bar{M}': X \times \bar{E}_{i+1} \to \bar{E}_{i+1}$ by $\bar{M}'(x,(y,t,z)) = (xy,t,z)$ $x, y \in X$, $z \in \bar{E}_i$. It is easy to verify that the inductive hypotheses are satisfied.

Dold and Lashof sho that $\bar{p}_{i+1}$ is a quasifibring; they have constructed an A_n-structure on X for arbitrary n.

We have seen in the last section that $\mathscr{E}_i$ can be regarded as $X \times C\mathscr{E}_{i-1} \cup_\mu \mathscr{E}_{i-1}$. The use of the reduced versus the unreduced cone is irrelevant to the question of homotopy type; let us assume the Dold-Lashof construction reworked using the reduced cone.

By reversing the process used to show that $\mathscr{E}_i$ has the homotopy type of $X \times C\mathscr{E}_{i-1} \cup_\mu \mathscr{E}_{i-1}$, we can show that, up to homotopy type, $\bar{E}_i$ can be defined in terms of a relative homeomorphism

$$(\Delta^{i-1} \times X^i, \bar{R}) \xrightarrow{a_i} (\bar{E}_i, \bar{E}_{i-1})$$

where $\bar{R} = \partial\Delta^{i-1} \times X^i \cup \Delta^{i-1} \times X \times X^{[i-1]}$ and $a_i | \bar{R}$ is given as follows: Let $\tau = (t_1, \cdots, t_i) \in \Delta^{i-1}$, $\chi = (x_1, \cdots, x_i) \in X^i$.

$$a_i(\tau,\chi) = a_{i-1}(t_1, \cdots, \hat{t}_j, \cdots, t_1, x_1, \cdots, x_j x_{j+1}, \cdots, x_i), \qquad \text{if } t_j = 0, j < i,$$

$$a_i(t_1, \cdots, t_{i-1}, 0, \chi) = a_{i-1}(t_1, \cdots, t_{i-1}, x_1, \cdots, x_{i-1}),$$

$$a_i(\tau, \chi) = a_{i-1}(t_1, \cdots, t_{j-1} + t_j, \cdots, t_i, x_1, \cdots, \hat{x}_j, \cdots, x_i), \text{ if } x_j = e \text{ for } j > 1.$$

Thus to map $\mathscr{E}_i$ into $\bar{E}_i$, we need only a suitably defined homeomorphism $\mathscr{T}_{i+1}: K_{i+1} \to \Delta^{i-1}$. $\mathscr{T}_2$ is canonical (!). We take $\mathscr{T}_3$ to be the linear map determined by sending 1/2 to (0,1) and 1 to (1,0). Define $\bar{\mathscr{T}}_i: L_i \to \partial\Delta^{i-2}$ as follows: Let $\mathscr{T}_r(\rho) = (t_1, \cdots, t_{r-1}) \in \Delta^{r-2}$ for $\rho \in K_r$ then

$$\bar{\mathscr{T}}_i \partial_k(\rho, \sigma) = (t_1, \cdots, t_{k-1}, 0, \cdots, 0, t_k, \cdots, t_{r-1})$$

[$s - 2$ coordinates are set equal to zero]. This gives a well-defined map of L_i onto Δ^{i-2}. It can be described as collapsing each face which is homeomorphic to $K_r \times K_s$ onto the image of one of its axes K_r. Hence it is possible to extend to a relative homeomorphism $\mathscr{T}_i: K_i, L_i \to \Delta^{i-2}, \partial\Delta^{i-2}$.

Now map $\mathscr{E}_i$ into $\bar{E}_i$ by $\alpha_i(\tau, x_1, \cdots, x_i) \to a_i(\mathscr{T}_{i+1}(\tau), x_1, \cdots, x_i)$. Although $\mathscr{T}_{i+1}^{-1}$ is not uniquely defined, it is easy to construct an inverse to the above map, using the fact that the A_n-form is trivial, and hence $\alpha_i(\partial_k(r,s)(\rho,\sigma), x_1, \cdots, x_i)$ is independent of σ.

A similar analysis shows that $\bar{B}_i$ has the homotopy type of $\mathscr{B}_i$ and that $\bar{p}_i$ is equivalent to $\mathfrak{p}_i$.

Appendix

Retractile subspaces. In [6], I. M. James has studied a property of pairs of complexes for which we find many uses. We rework his Lemma 3.2 as a definition.

Definition A.1. A subcomplex L is *retractile* in a complex K if given any null-homotopic map $f: K \to X$ such that $f | L$ is constant then f is null-homotopic rel L.

PROPOSITION A.2 [6, Corollary 4.4]. *Let (X,m) be an H-space. If L is retractile in K, and we are given homotopic maps $f_0,f_1:K\to X$ which agree on L, then* $f_0 \simeq f_1$ rel L.

We are more interested in nontrivial homotopies on L.

PROPOSITION A.3. *L is retractile in K if and only if given any null-homotopic map $f:K\to X$ and a null-homotopy $g_t:L\to X$ such that $g_1=f|L$ then g_t extends to a null-homotopy $f_t:K\to X$ such that $f_1=f$.*

Proof. By the homotopy extension theorem, g_t extends to $f_t':K\to X$ such that $f_1'=f$. Since f_0' is null-homotopic and constant on L, f_0' is null-homotopic rel L. Thus there is a null-homotopy $\bar{f}_t:K\to X$ with $0\leqq t\leqq 2$ which restricted to L is g_t for $0\leqq t\leqq 1$ and constant for $1\leqq t\leqq 2$. It is easy to alter $\bar{f}_t$ to obtain f_t as desired again using the homotopy extension theorem.

PROPOSITION A.4. *Let (X,m) be an H-space. If L is retractile in K then given homotopic maps $f_0,f_1:K\to X$ and a homotopy $g_t:L\to X$ such that $g_i=f_i|L$ for $i=0,1$ then g_t extends to a homotopy $f_t:K\to X$.*

This is proved by reducing to the previous case, just as James did in proving his Corollary 4.4. [It is necessary to note that his Lemma 3.4 can be generalized to

LEMMA A.5. *Let $p:X\to Y$ induce isomorphisms of all homotopy groups. Let $h_0,h_1:K\to X$ and $g_t:L\to X$ such that $g_i=h_i|L$ for $i=0,1$. Suppose there exists $j_t:K\to Y$ such that $j_i=ph_i$ for $i=0,1$ and $j_t|L=pg_t$. Then g_t extends to $h_t:K\to X$.*]

Proof of Lemma 7. Corresponding to $M_n'|L_n\times X^n\to X$ we have a map of X^n into X^{L_n}. The extension to $K_n\times X^n$ corresponds to a homotopy between this map f_0 and the one given by $f_1(x_1,\cdots,x_n)(\tau)=x_1(x_2(\cdots(x_{n-1}x_n)))$. Since X^{L_n} is an H-space (because X is) and $X^{[i]}$is retractile in X^i[6, Lemma 3.1], Proposition A.4 says that there is a homotopy between f_0 and f_1 which corresponds to a map $M_n:K_n\times X^n\to X$ satisfying 5(2) and (3) as desired.

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