ISSN 0001-4346, Mathematical Notes, 2008, Vol. 84, No. 5, pp. 710–717.

Bott's Periodicity Theorem and Differentials of the Adams Spectral Sequence of Homotopy Groups of Spheres

V. A. Smirnov*

Moscow Pedagogical State University

Received May 8, 2007

Abstract—Bott's periodicity theorem is applied to calculate higher-order differentials of the Adams spectral sequence of homotopy groups $\pi_*(SO)$. The resulting formulas are used to find higher-order differentials of the Adams spectral sequence of homotopy groups of spheres.

DOI: 10.1134/S0001434608110126

Key words: *Bott's periodicity theorem, differentials of the Adams spectral sequence, homotopy groups of spheres, stable homotopy group, exterior algebra, loop space.*

One of the main methods for calculating homotopy groups of topological spaces, including those of spheres, is based on spectral sequences. The first such sequence was constructed by Adams in [1]. It converges to the stable homotopy groups of a topological space, and its initial term is expressed in terms of the Steenrod algebra and the homology of this space. Later, Massey with Peterson [2] and Bousfield with Kan [3] constructed instable analogs of the Adams spectral sequence converging to homotopy groups of a topological space.

The initial term E^1 of these spectral sequences is described by using the algebra Λ [4]. This is the differential graded algebra defined by generators λ_i of dimension $i \geq 0$ and the relations

$$\lambda_{2i+n+1}\lambda_i = \sum_j \binom{n-j-1}{j} \lambda_{2i+j+1}\lambda_{i+n-j}.$$

Here and in what follows, we assume the base ring to be the field $\mathbb{Z}/2$.

As a graded module, the algebra Λ is generated by admissible sequences $\lambda_{i_1}\dots\lambda_{i_k}$, that is, by sequences with $i_1 \leq 2i_2, \dots, i_{k-1} \leq 2i_k$.

At the generators, the differential is defined by

$$d(\lambda_i) = \sum_j \binom{i-j}{j} \lambda_{j-1}\lambda_{i-j}.$$

The algebra Λ is isomorphic to the term E^1 of the Adams spectral sequence of stable homotopy groups of spheres, and its homology with respect to the differential d is isomorphic to the term E^2 of this sequence.

For the graded module M, let $\Lambda(M)$ denote the submodule of the tensor product $\Lambda \otimes M$ generated by all elements of the form $\lambda_{i_1}\dots\lambda_{i_k} \otimes x_n$, where x_n is the n-dimensional generator in M and $\lambda_{i_1}\dots\lambda_{i_k}$ is an admissible sequence with $i_k < n$.

A graded module M is called a Λ-*comodule* if a map $\varphi\colon M \to \Lambda(M)$ reducing dimension by one and satisfying the relation $d(\varphi) + \varphi \cup \varphi = 0$ is defined.

For a Λ-comodule M, we can define a new differential d_φ on $\Lambda(M)$ by setting $d_\varphi = d + \varphi\cap$. We denote the corresponding complex by $\Lambda_\varphi(M)$.

*E-mail: V.Smirnov@ru.net

In [2], Massey and Peterson proved that if the homology of a simply connected topological space is isomorphic to the exterior algebra over the graded module M, then there is a spectral sequence converging to the homotopy groups of this space whose term E^1 is isomorphic to $\Lambda(M)$. Moreover, the Steenrod operations on the homology of a topological space determine the structure of a Λ-comodule on M, for which the homology of the complex $\Lambda_\varphi(M)$ is isomorphic to the term E^2 of the spectral sequence.

In particular, the homology of the n-sphere S^n have the trivial structure of a Λ-comodule; therefore, the homology of the complex $\Lambda(S^n)$ is isomorphic to the term E^2 of the Adams spectral sequence of homotopy groups of the n-spheres.

The problem of finding higher-order differentials of the Adams spectral sequence is extremely difficult. It has been attacked by both geometric and algebraic methods. Initially, higher-order differentials were determined geometrically by means of higher-order homology operations on the homology of the topological space under consideration [5]. However, the geometric construction of higher-order homology operations is very complicated and can be used for calculations only in certain special cases.

The other, algebraic, way consists in employing the structure of a E_∞-coalgebra, which exists on the homology of a topological space. In [6], it was shown that this structure determines the weak homotopy type of a topological space and can be used for calculating higher-order differentials of the Adams spectral sequence. Moreover, by using this structure, we can define a differential $\widetilde{d}$ on $\Lambda(M)$ so that it complements the differential d_φ and the homology with respect to it is isomorphic to the term E^∞ of the Adams spectral sequence. On this basis, in [7], [8], some calculations were performed; however, this way involves serious difficulties as well, which prevent the calculations from being completed.

Although the problem of finding higher-order differentials of the Adams spectral sequence has not been solved so far, it is possible to completely calculate the homotopy groups of some topological spaces. Such spaces include the classical topological groups SO, U, and Sp; calculations for these groups use Bott's periodicity [9] and yield

$$\pi_i(SO) \cong \pi_{i+8}(SO), \qquad \pi_i(U) \cong \pi_{i+2}(U), \qquad \pi_i(Sp) \cong \pi_{i+8}(Sp).$$

These groups are interesting in that their homotopy groups are calculated without applying the Adams spectral sequence. Moreover, the attempts to calculate homotopy groups for them by using the Adams spectral sequence were unsuccessful. Thus, so far, the terms E^2 of the corresponding spectral sequences have not been calculated, much less the higher-order differentials.

In this paper, we apply Bott's periodicity to calculate the higher-order differentials of the Adams spectral sequence for the homotopy groups $\pi_*(SO)$ and use the resulting formula to determine the higher-order differentials of the Adams spectral sequence of homotopy groups of spheres.

Recall [10] that the homology $H_*(SO)$ of the special orthogonal group SO is isomorphic to the exterior algebra over a graded module X generated by elements x_n of dimension $n \geq 1$. Moreover, the graded module X itself is the homology module of the real projective space RP^∞. The homology $H_*(U)$ of the unitary group U is isomorphic to the exterior algebra over a graded module Y generated by elements y_{2n-1} with $n \geq 1$ of dimension $2n-1$. The homology $H_*(SU)$ of the special unitary group SU is isomorphic to the exterior algebra over a graded module Y' generated by elements y_{2n+1} with $n \geq 1$ of dimension $2n+1$. The homology $H_*(Sp)$ of the symplectic group Sp is isomorphic to the exterior algebra over a graded module Z generated by elements z_{4n-1} with $n \geq 1$ of dimension $4n-1$.

It follows from the above considerations that, for the homotopy groups $\pi_*(SO)$, $\pi_*(U)$, and $\pi_*(Sp)$, there arise spectral sequences whose initial terms are isomorphic to the complexes $\Lambda(X)$, $\Lambda(Y)$, and $\Lambda(Z)$, respectively. At the generators $x_n \in X$, $y_{2n-1} \in Y$, and $z_{4n-1} \in Z$, the differential d_φ is defined by

$$d_\varphi(x_n) = \sum_i \binom{n-i}{i} \lambda_{i-1} x_{n-i},$$

$$d_\varphi(y_{2n-1}) = \sum_i \binom{2n-1-i}{i} \lambda_{i-1} y_{2n-1-i} = \sum_i \binom{n-i-1}{i} \lambda_{2i-1} y_{2(n-i)-1},$$

$$d_\varphi(z_{4n-1}) = \sum_i \binom{4n-1-i}{i}\lambda_{i-1}z_{4n-1-i} = \sum_i \binom{n-i-1}{i}\lambda_{4i-1}z_{4(n-i)-1}.$$

Moreover, in the complexes $\Lambda(X)$, $\Lambda(Y)$, and $\Lambda(Z)$, new differentials $\widetilde{d}$ can be introduced so that the homology with respect to these differentials are isomorphic to the terms E^∞ of the corresponding Adams spectral sequences.

There is a relationship between the groups $\pi_*(SO)$ and the stable homotopy groups of the spheres σ_*, which comes from the J-homomorphism $J\colon \pi_*(SO) \to \sigma_*$ [4]. On the term E^1 of the Adams spectral sequence, J is defined by

$$J(\lambda_{i_1}\dots\lambda_{i_k}\otimes x_n) = \lambda_{i_1}\dots\lambda_{i_k}\lambda_n;$$

in particular, $J(x_n) = \lambda_n$.

This implies that, in order to calculate the differential $\widetilde{d}$ at the elements λ_n of the complex Λ, it sufficient to evaluate it at the elements $x_n \in X = H_*(RP^\infty)$ of the complex $\Lambda(X)$.

Proposition 1. *If the topological space under consideration is the special orthogonal group SO, then the differential $\widetilde{d}$ on $\Lambda(X)$ can be defined so that the higher-order differentials at the elements x_{2m} and x_{4m+1} vanish.*

PROOF. First, let us show that the differential $\widetilde{d}$ can be chosen so that the higher-order differentials at the elements x_{2m} vanish.

Let $\widetilde{d}$ be a differential on $\Lambda(X)$ such that the homology with respect to $\widetilde{d}$ is isomorphic to the homotopy groups $\pi_*(SO)$. We determine a differential d' on $\Lambda(X)$ by setting

$$d'(\lambda_{i_1}\dots\lambda_{i_k}\otimes x_n) = \begin{cases}\lambda_{i_1}\dots\lambda_{i_k}\lambda_{m-1}\otimes x_m, & n=2m,\\ 0, & n=2m-1.\end{cases}$$

It is easy to see that the homology of $\Lambda'(X)$ with respect to this differential is generated by the elements $\lambda_{i_1}\dots\lambda_{i_k}\otimes x_{2m-1}$, where $\lambda_{i_1}\dots\lambda_{i_k}$ is an admissible sequence with $i_k < 2m-2$.

There is a chain equivalence between the complex $\Lambda(X)$ with differential d' and the graded module $\Lambda'(X)$ considered as a complex with zero differential. The corresponding maps

$$\xi'\colon \Lambda'(X)\to\Lambda(X) \qquad \text{and} \qquad \eta'\colon \Lambda(X)\to\Lambda'(X)$$

and the homotopy $h'\colon \Lambda(X)\to\Lambda(X)$ are defined by

$$\xi'(\lambda_{i_1}\dots\lambda_{i_k}\otimes x_{2m-1}) = \lambda_{i_1}\dots\lambda_{i_k}\otimes x_{2m-1},$$

$$\eta'(\lambda_{i_1}\dots\lambda_{i_k}\otimes x_n) = \begin{cases}\lambda_{i_1}\dots\lambda_{i_k}\otimes x_n, & n=2m-1,\\ 0, & n=2m,\end{cases}$$

$$h'(\lambda_{i_1}\dots\lambda_{i_{k-1}}\lambda_{i_k}\otimes x_n) = \begin{cases}\lambda_{i_1}\dots\lambda_{i_{k-1}}\otimes x_{2n}, & i_k=n-1,\\ 0, & i_k<n-1.\end{cases}$$

It follows from perturbation theory [11] that, on $\Lambda'(X)$, we can define a differential $\widetilde{d}'$, maps

$$\widetilde{\xi}'\colon \Lambda'(X)\to\Lambda(X) \qquad \text{and} \qquad \widetilde{\eta}'\colon \Lambda(X)\to\Lambda'(X),$$

and a homotopy $\widetilde{h}'\colon \Lambda(X)\to\Lambda(X)$ so that the complex $\Lambda'(X)$ with this differential is chain equivalent to the complex $\Lambda(X)$ with differential $\widetilde{d}$:

$$\widetilde{d}' = \eta' t\xi' + \eta' th' t\xi' + \eta' th' th' t\xi' + \cdots,$$

$$\widetilde{\xi}' = \xi' + h' t\xi' + h' th' t\xi' + \cdots,$$

$$\widetilde{\eta}' = \eta' + \eta' t h' + \eta' t h' t h' + \cdots ,$$
$$\widetilde{h}' = h' + h' t h' + h' t h' t h' + \cdots ,$$

where $t = \widetilde{d} - d'$.

Let us define a differential $\widehat{d'}$ on $\Lambda(X)$ by setting

$$\widehat{d'} = d' + \xi' \widetilde{d'} \eta'.$$

A direct calculation shows that $\widehat{d'}$ is a differential on $\Lambda(X)$ whose higher-order components vanish at the elements x_{2m}.

The next step is to correct the differential $\widehat{d'}$ so that its higher-order components vanish at the elements x_{4m+1} too.

Let us define a differential d'' on $\Lambda'(X)$ by setting

$$d''(\lambda_{i_1} \dots \lambda_{i_k} \otimes x_n) = \begin{cases} \lambda_{i_1} \dots \lambda_{i_k} \lambda_{2m-1} \otimes x_{2m+1}, & n = 4m+1, \\ 0, & n = 4m+3. \end{cases}$$

It is easy to see that the homology of $\Lambda''(X)$ with respect to this differential is generated by the elements $\lambda_{i_1} \dots \lambda_{i_k} \otimes x_{4m+3}$, where $\lambda_{i_1} \dots \lambda_{i_k}$ is an admissible sequence with $i_k < 4m+1$.

There is a chain equivalence between the complex $\Lambda'(X)$ with differential d'' and the graded module $\Lambda''(X)$ considered as a complex with zero differential. The corresponding maps

$$\xi'' \colon \Lambda''(X) \to \Lambda'(X) \qquad \text{and} \qquad \eta'' \colon \Lambda'(X) \to \Lambda''(X)$$

and the homotopy $h'' \colon \Lambda'(X) \to \Lambda'(X)$ are defined by

$$\xi''(\lambda_{i_1} \dots \lambda_{i_k} \otimes x_{4m+3}) = \lambda_{i_1} \dots \lambda_{i_k} \otimes x_{4m+3},$$
$$\eta''(\lambda_{i_1} \dots \lambda_{i_k} \otimes x_n) = \begin{cases} \lambda_{i_1} \dots \lambda_{i_k} \otimes x_n, & n = 4m+3, \\ 0, & n = 4m+1, \end{cases}$$
$$h''(\lambda_{i_1} \dots \lambda_{i_{k-1}} \lambda_{i_k} \otimes x_{2n+1}) = \begin{cases} \lambda_{i_1} \dots \lambda_{i_{k-1}} \otimes x_{4n+1}, & i_k = 2n-1, \\ 0, & i_k < 2n-1. \end{cases}$$

It follows from perturbation theory [11] that, on $\Lambda''(X)$, we can define a differential $\widetilde{d''}$, maps

$$\widetilde{\xi}'' \colon \Lambda''(X) \to \Lambda'(X) \qquad \text{and} \qquad \widetilde{\eta}'' \colon \Lambda'(X) \to \Lambda''(X),$$

and a homotopy $\widetilde{h}'' \colon \Lambda'(X) \to \Lambda'(X)$ so that the complex $\Lambda''(X)$ with this differential is chain equivalent to the complex $\Lambda'(X)$ with differential $\widetilde{d'}$.

Expressions for $\widetilde{d''}$, $\widetilde{\xi}''$, $\widetilde{\eta}''$, and $\widetilde{h}''$ are similar to the corresponding expressions for $\widetilde{d'}$, $\widetilde{\xi}'$, $\widetilde{\eta}'$, and $\widetilde{h}'$.

Let us define a differential $\widehat{d}$ on $\Lambda(X)$ by setting

$$\widehat{d} = d' + \xi' d'' \eta' + \xi' \xi'' \widetilde{d''} \eta'' \eta'.$$

A direct verification shows that $\widehat{d}$ is the required differential on $\Lambda(X)$.

Let us find the consequences of Bott's periodicity $\pi_i(U) \cong \pi_i(\Omega^2 SU)$ for the term E^1 and the differentials of the Adams spectral sequence. First, we calculate the homology $H_*(\Omega^2 SU)$.

Recall that if the homology of a simply connected space is isomorphic to the exterior algebra over a graded module M on which comultiplication is trivial, then the homology of the loop space over this space is isomorphic to the polynomial algebra over the desuspension $S^{-1}M$ [5].

This implies, in particular, that the homology $H_*(\Omega SU)$ of the loop space over the special unitary group is isomorphic to the polynomial algebra with generators $x_{2n} = s^{-1} y_{2n+1}$ of dimension $2n$.

To calculate the homology $H_*(\Omega^2 SU)$ of the double loop space over SU, we represent the polynomial algebra with generators x_{2n} as the exterior algebra with generators $x_{2n}^{2^k}$.

On the elements x_{2n}, comultiplication is not trivial; it is defined by

$$\nabla(x_{2n}^{2^k}) = \sum_i x_{2i}^{2^k} \otimes x_{2(n-i)}^{2^k}.$$

Considering the spectral sequence for the homology $H_*(\Omega^2 SU)$, we see that the term E^1 of this spectral sequence is isomorphic to the polynomial algebra with generators $s^{-1}(x_{2n}^{2^k})$. It can be represented as the exterior algebra with generators $(s^{-1}(x_{2n}^{2^k}))^{2^i}$ as well.

The differential d^1 on the generators is defined by

$$d^1(s^{-1}(x_{2n}^{2^k})) = \begin{cases} (s^{-1}(x_n^{2^k}))^2, & n = 2i, \\ 0, & n = 2i+1. \end{cases}$$

This implies that the term E^2 of the spectral sequence is isomorphic to the exterior algebra with generators $s^{-1}(x_{4i+2}^{2^k})$. Moreover, since the higher-order differentials of this spectral sequence vanish, the homology $H_*(\Omega^2 SU)$ is isomorphic to this exterior algebra.

Thus, the following assertion is valid.

Proposition 2. *The term E^1 of the spectral sequence for the homology $H_*(\Omega^2 SU)$ is isomorphic to the exterior algebra with generators $(s^{-1}(x_{2n}^{2^k}))^{2^i}$. The homology itself is isomorphic to the exterior algebra with generators $s^{-1}(x_{4i+2}^{2^k})$.*

Note that any odd positive integer $2n-1$ has a unique representation in the form $2^k(4i+2)-1$; hence there is a one-to-one correspondence between the generators y_{2n-1} of the homology $H_*(U)$ and the generators $s^{-1}(x_{4i+2}^{2^k})$ of the homology $H_*(\Omega^2 SU)$.

This correspondence generates a homology isomorphism $H_*(U) \cong H_*(\Omega^2 SU)$. Bott showed in [9] that it induces a continuous map $U \to \Omega^2 SU$ and, therefore, induces an isomorphism of homotopy groups

$$\pi_*(U) \to \pi_*(\Omega^2 SU) \cong \pi_{*+2}(U);$$

the latter implies the periodicity of the homotopy groups $\pi_*(U)$.

Let us rewrite the generators of the homology $H_*(\Omega^2 SU)$ specified above by using elements of the Dyer–Lashof algebra acting on the homology of the iterated loop spaces [12].

Recall that the Dyer–Lashof algebra $\mathscr{R}$ is defined by generators Q^i of dimension $i \geq 0$ and relations

$$Q^{2i+n+1}Q^i = \sum_j \binom{n-j-1}{j} Q^{2i+j+1}Q^{i+n-j}.$$

As a graded module, the Dyer–Lashof algebra is generated by admissible sequences $Q^{i_1}\dots Q^{i_k}$ ("admissible" means that $i_1 \leq 2i_2, \dots, i_{k-1} \leq 2i_k$).

Recall that the *redundancy* of an admissible sequence $Q^{i_1}\dots Q^{i_k}$ is defined as $i_1 - \dots - i_k$.

For a graded module M, let $\mathscr{R}(M)$ denote the submodule $\mathscr{R} \otimes M$ generated by all elements of the form $Q^{i_1}\dots Q^{i_k} \otimes x_n$, where x_n is the n-dimensional generator of M and $Q^{i_1}\dots Q^{i_k}$ is an admissible sequence with redundancy at least n. By $\mathscr{R}_m(M)$ we denote the submodule $\mathscr{R}(M)$ generated by elements of the same form for which $i_k < n+m$.

The generators of the term E^1 of the spectral sequence for the homology $H_*(\Omega^2 SU)$ determined above can be rewritten in the form

$$\begin{aligned} s^{-1}(x_{2n}^{2^k}) &= Q^{2^k n}\dots Q^{2n} \otimes s^{-2}y_{2n+1}, \\ (s^{-1}(x_{2n}))^{2^i} &= Q^{2^i n - 2^{i-1}}\dots Q^{2n-1} \otimes s^{-2}y_{2n+1}, \\ (s^{-1}(x_{2n}^{2^k}))^{2^i} &= Q^{2^{k+i}n-2^{i-1}}\dots Q^{2^{k+1}n-1}Q^{2^k n}\dots Q^{2n} \otimes s^{-2}y_{2n+1}. \end{aligned}$$

At the generators, the differential is defined by

$$d(Q^{2^k n}\dots Q^{2n}\otimes s^{-2}y_{2n+1})=\begin{cases}(Q^{2^k i}\dots Q^{2i}\otimes s^{-2}y_{2i+1})^2, & n=2i,\\ 0, & n=2i+1.\end{cases}$$

For the generators of the homology $H_*(\Omega^2SU)$, we have

$$s^{-1}(x_{4m+2}^{2^k})=s^{-1}Q^{2^{k-1}(4m+2)}\dots Q^{4m+2}\otimes s^{-1}y_{4m+3}.$$

Thus, Proposition 2 can be reformulated as follows.

Proposition 2′. *The term E^1 of the spectral sequence for the homology $H_*(\Omega^2SU)$ is isomorphic to the exterior algebra over the module $\mathscr{R}_2(S^{-2}Y)$. The homology itself is isomorphic to the exterior algebra over the module $S^{-1}\mathscr{R}_1(S^{-1}Z)$.*

Let us rewrite the generators found above in terms of elements algebra of the Λ. Recall that the redundancy of an admissible sequence $\lambda_{i_1}\dots\lambda_{i_k}$ is the number $i_1-\dots-i_k$. Let $\Lambda_m(M)$ denote the submodule $\Lambda(M)$ generated by the elements $\lambda_{i_1}\dots\lambda_{i_k}\otimes x_n$, where x_n is the n-dimensional generator of M and $\lambda_{i_1}\dots\lambda_{i_k}$ is an admissible sequence with redundancy at least $n-m$.

The algebras $\mathscr{R}$ and Λ isomorphic; the elements Q^i of the algebra $\mathscr{R}$ correspond to the elements λ_i of the algebra Λ. For the graded module M and $m\ge 1$, this correspondence determines an isomorphism

$$\mathscr{R}(S^{-m}M)\cong S^{-m}\Lambda_m(M).$$

The generators of the term E^1 of the spectral sequence for the homology $H_*(\Omega^2SU)$ specified above can be rewritten as

$$\begin{aligned}s^{-1}(x_{2n}^{2^k})&=s^{-2}\lambda_{2^k n}\dots\lambda_{2n}\otimes y_{2n+1},\\ (s^{-1}(x_{2n}))^{2^i}&=s^{-2}\lambda_{2^i n-2^{i-1}}\dots\lambda_{2n-1}\otimes y_{2n+1},\\ (s^{-1}(x_{2n}^{2^k}))^{2^i}&=s^{-2}\lambda_{2^{k+i}n-2^{i-1}}\dots\lambda_{2^{k+1}n-1}\lambda_{2^k n}\lambda_{2n}\otimes y_{2n+1}.\end{aligned}$$

For the generators of the homology $H_*(\Omega^2SU)$, we have

$$s^{-1}(x_{4m+2}^{2^k})=s^{-2}\lambda_{2^{k-1}(4m+2)}\dots\lambda_{4m+2}\otimes y_{4m+3}.$$

Thus, Proposition 2′ can be reformulated as follows.

Proposition 2″. *The term E^1 of the spectral sequence for the homology $H_*(\Omega^2SU)$ is isomorphic to the exterior algebra over the module $S^{-2}\Lambda_2(Y)$. The homology $H_*(\Omega^2SU)$ itself is isomorphic to the exterior algebra over the module $S^{-2}\Lambda_1(Z)$.*

The isomorphism $H_*(U)\to H_*(\Omega^2SU)$ induces the map

$$f\colon \Lambda(Y)\to\Lambda(S^{-2}\Lambda_2(Y))$$

defined at the generators by

$$\begin{aligned}f(\lambda_{i_1}\dots\lambda_{i_k}\otimes y_{4m+1})&=\lambda_{i_1}\dots\lambda_{i_k}\otimes s^{-2}y_{4m+3},\\ f(\lambda_{i_1}\dots\lambda_{i_k}\otimes y_{2^k(2m+1)-1})&=\lambda_{i_1}\dots\lambda_{i_k}\otimes s^{-2}\lambda_{2^{k-1}(2m+1)}\dots\lambda_{2(2m+1)}\otimes y_{4m+3}.\end{aligned}$$

Note that, for the graded module M and $m\ge1$, there is an isomorphism

$$g_m\colon \Lambda(S^{-m}\Lambda_m(M))\cong S^{-m}\Lambda(M),$$

which is defined at the generators by

$$g_m(\lambda_{i_1}\dots\lambda_{i_k}s^{-m}\lambda_{j_1}\dots\lambda_{j_l}\otimes x_n)=s^{-m}\lambda_{i_1}\dots\lambda_{i_k}\lambda_{j_1}\dots\lambda_{j_l}\otimes x_n.$$

This implies the following assertion.

Proposition 3. *The map* $\nu\colon \Lambda(Y) \to \Lambda(Y)$*, which increases dimension by two, is the composition of the maps* f *and* g_2 *and the double suspension and defined at the generators by*

$$\nu(\lambda_{i_1} \dots \lambda_{i_k} \otimes y_{4m+1}) = \lambda_{i_1} \dots \lambda_{i_k} \otimes y_{4m+3},$$
$$\nu(\lambda_{i_1} \dots \lambda_{i_k} \otimes y_{2^k(2m+1)-1}) = \lambda_{i_1} \dots \lambda_{i_k} \lambda_{2^{k-1}(2m+1)} \dots \lambda_{2(2m+1)} \otimes y_{4m+3},$$

commutes with the differentials and induces an isomorphism of the spectral sequences of the homotopy groups $\pi_*(U)$ *and* $\pi_{*+2}(U)$*.*

Summarizing the above considerations, we obtain the following theorem.

Theorem. *The differential on the complex* $\Lambda(X)$*, where* $X = H_*(RP^\infty)$*, the homology with respect to which is isomorphic to the term* E^∞ *for the homotopy groups* $\pi_*(SO)$*, can be chosen so that the higher-order differentials* δ *vanish at the generator* x_{2m} *and* x_{4m+1} *and are defined by*

$$\delta(x_{4m+3}) = \sum_{i\geq 0} \binom{m-i-1}{i} \lambda_{4i+1}\nu(x_{4(m-i)-1}),$$

where

$$\nu(x_{2^k(2l+1)-1}) = \lambda_{2^{k-1}(2l+1)} \dots \lambda_{2(2l+1)} x_{4l+3},$$

at the generators x_{4m+3}*.*

Proof. By Proposition 1, the differential $\widetilde{d}$ on $\Lambda(X)$ can be chosen so that the higher-order differentials δ vanish at the generators x_{2m} and x_{4m+1}. The generators x_{4m+3} are the images of x_{4m+1} under the map ν defined by $\nu(x_{4m+1}) = x_{4m+3}$. Since the map ν commutes with the differentials, it follows that

$$\widetilde{d}(x_{4m+3}) = \nu(d(x_{4m+1})) = \sum_{i} \binom{2m-i}{i} \lambda_{2i-1}\nu(x_{4m-2i+1}).$$

Moreover, if i is even, then $\nu(x_{4m-2i+1}) = x_{4m-2i+3}$. The corresponding terms are components of the differential d_1 and do not give values of the higher-order differentials δ. Discarding these terms, we obtain the sum

$$\sum_{i} \binom{2m-2i-1}{2i+1} \lambda_{4i+1}\nu(x_{4(m-i)-1}) = \sum_{i} \binom{m-i-1}{i} \lambda_{4i+1}\nu(x_{4(m-i)-1}).$$

Using the formulas for the map ν from Proposition 3, we finally obtain the required expression for the higher differentials δ. □

Corollary. *The differential on the algebra* Λ *with respect to which the homology is isomorphic to the term* E^∞ *for the stable homotopy groups of spheres can be chosen so that the higher-order differentials* δ *vanish at the generators* λ_{2m} *and* λ_{4m+1} *and are defined by*

$$\delta(\lambda_{4m+3}) = \sum_{i\geq 0} \binom{m-i-1}{i} \lambda_{4i+1}\nu(\lambda_{4(m-i)-1}),$$

where

$$\nu(\lambda_{2^k(2l+1)-1}) = \lambda_{2^{k-1}(2l+1)} \dots \lambda_{2(2l+1)} \lambda_{4l+3},$$

at the generators λ_{4m+3}*.*

In particular,

$$\delta(\lambda_7) = \lambda_1\lambda_2\lambda_3, \qquad \delta(\lambda_{11}) = \lambda_1\lambda_4\lambda_2\lambda_3, \qquad \delta(\lambda_{15}) = \lambda_1\lambda_6\lambda_7 + \lambda_5\lambda_4\lambda_2\lambda_3.$$

REFERENCES

1. J. F. Adams, "On the structure and applications of the Steenrod algebra," Comment. Math. Helv. **32** (1), 180–214 (1958).
2. W. S. Massey and F. P. Peterson, *The* mod 2 *Cohomology Structure of Certain Fibre Spaces*, in *Mem. Amer. Math. Soc.* (Amer. Math. Soc., Providence, R. I., 1967), Vol. 74.
3. A. K. Bousfield and D. M. Kan, "The homotopy spectral sequence of a space with coefficients in a ring," Topology **11** (1), 79–106 (1972).
4. G. W. Whitehead, *Recent Advances in Homotopy Theory* (Amer. Math. Soc., Providence, R. I., 1970; Mir, Moscow, 1974).
5. R. E. Mosher and M. C. Tangora, *Cohomology Operations and Applications in Homotopy Theory* (Harper&Row, New York–London; 1968; Mir, Moscow, 1970).
6. V. A. Smirnov, *Simplicial and Operad Methods in Algebraic Topology*, in *Transl. Math. Monogr.* (Amer. Math. Soc., Providence, R. I., 2001), Vol. 198.
7. V. A. Smirnov, "A_∞-structures and differentials of the Adams spectral sequence," Izv. Ross. Akad. Nauk Ser. Mat. **66** (5), 193–224 (2002) [Russian Acad. Sci. Izv. Math. **66**(5), 1057–1086 (2002)].
8. V. A. Smirnov, "Secondary Steenrod operations in the cohomology of infinite-dimensional projective spaces," Mat. Zametki **79** (3), 476–480 (2006) [Math. Notes **79** (3–4), 440–445 (2006)].
9. R. Bott, "The stable homotopy for the classical groups," Ann. of Math. (2) **70** (2), 313–337 (1959).
10. R. M. Switzer, *Algebraic Topology—Homotopy and Homology* (Springer, Berlin, 1975; Nauka, Moscow, 1985).
11. V. K. A. M. Gugenheim, L. A. Lambe and J. D. Stasheff, "Perturbation theory in differential homological algebra. II," Illinois J. Math. **35** (3), 357–373 (1991).
12. E. Dyer and R. Lashof, "Homology of iterated loop spaces," Amer. J. Math. **84** (1), 35–88 (1962).