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CONNECTIVE FIBERINGS OVER *BU* and *U*†

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§0. INTRODUCTION

EILENBERG AND MOORE [7] have developed a spectral sequence converging to the cohomology of the total space of an induced fibration. L. Smith [13] has recently developed methods by which this spectral sequence can be computed, in the special case of a fibration induced by an H-map from the path space fibration over a $K(Z, n)$. Using these methods we have computed the cohomology rings $H^*(BU(2n, \ldots, \infty), Z_p)$, and $H^*(U(2n+1, \ldots, \infty), Z_p)$, p an arbitrary prime. (We use the symbol $X(n, \ldots, \infty)$ to denote the $n-1$ connective fibering over X.) The work is thus an extension of the work of Adams [1] (who computed the stable groups: $H^{2n+k}(BU(2n, \ldots, \infty), Z_p)$ for $k < 2n$) and of Stong [16] (who determined the ring $H^*(BU(2n, \ldots, \infty), Z_2)$). We summarize our results.

If p is an odd prime and M a graded Z_p-module, denote by $F[M]$ the free Z_p-algebra generated by M. Let $Op[\beta P^1 i_n]$ denote the Hopf sub-algebra of $H^*(K(Z, n), Z_p)$ generated over the Steenrod algebra by the single element $\beta P^1 i_n$, and define graded Z_p-modules M_n in such a way that $F[M_n] \cong Op[\beta P^1 i_n]$. Finally, if n is an integer it can be written uniquely in the form $n = \sum_{i=0}^{k} a_i p^i$, with $a_i < p$. Set $\sigma_p(n) = \sum_{i=0}^{k} a_i$. We find: if p is an odd prime there exist indecomposable cohomology classes $\theta_{2i} \in H^{2i}(BU, Z_p)$; $\mu_{2i+1} \in H^{2i+1}(U, Z_p)$ such that

$$H^*(BU(2n, \ldots, \infty), Z_p) = \frac{H^*(BU, Z_p)}{Z_p[\theta_{2i} \mid \sigma_p(i-1) < n-1]} \otimes \prod_{t=0}^{p-2} F[M_{2n-3-2t}]$$

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$$H^*(U(2n+1,\ldots,\infty), Z_p) = \frac{H^*(U, Z_p)}{E[\mu_{2i+1} \mid \sigma_p(i) < n]} \otimes \prod_{t=0}^{p-2} F[M_{2n-2-2t}] \otimes E[\omega_{2ip^k+1} \mid {}^{\sigma_p(i-1)=n-t-2}_{k>0}]$$

as tensor products of Hopf algebras.

Our methods also permit determination of the cohomology of each stage of the Postnikov tower of U. If p is an odd prime, then:

$$H^*(U(1,\ldots,2n-1), Z_p) = E[\mu_{2i+1} \mid \sigma_p(i) < n] \otimes \prod_{t=0}^{p-2} F[M_{2n-1-2t}]$$

as a tensor product of Hopf algebras.

We obtain similar answers for the case $p = 2$. These agree with the work of Stong [16], Hirsch [8], and Vastersavendts [17].

We also derive divisibility conditions for Chern classes in integral cohomology. Let $s_n: BU(2n,\ldots,\infty) \to BU$ be the standard map, and let $Qs_n^*: Q^*H(BU, Z) \to Q^*H(BU(2n,\ldots,\infty), Z)$ be the induced map of indecomposables. Let $c_k \varepsilon H^{2k}(BU, Z)$ be the Chern class, and $Qc_k \varepsilon Q^{2k}H(BU, Z)$ its image in the module of indecomposables. Then $Qs_n^*(Qc_k)$ is divisible by a certain positive integer $\lambda_{n,k}$ and by no greater number. In §9 of this paper (Theorem 9.7) we determine $\lambda_{n,k}$.

ADDED IN PROOF. The author has recently computed cohomology of the Postnikov system of BU. The result is given at the end of §8.

In a forthcoming paper we will discuss more thoroughly the integral cohomology of the spaces $BU(2n,\ldots,\infty)$ and $U(2n+1,\ldots,\infty)$. In particular, we will use a certain splitting of BU into a Cartesian product [2], and some results of W. Browder [4], to calculate the Bockstein spectral sequences. We thus determine completely the groups $H^*(BU(2n,\ldots,\infty), Z)$. We can also find some of the ring structure: a typical result is that $H^*(BU(2n,\ldots,\infty)Z)$/Torsion is a polynomial algebra.

This paper is the author's PhD. thesis written at Princeton University with the guidance of J. C. Moore. It is a pleasure to thank Professor Moore for such useful advice; particularly for the suggestion that the computations for BU and U be done simultaneously. From Larry Smith I learned the use of the spectral sequence. I have had several conversations with Bob Stong, and was guided by the results of his paper [16].

§1. HOPF ALGEBRAS

A. Primitives and indecomposables

The reader is assumed familiar with the paper of Milnor and Moore [9]. We state here the results we will need.

We denote by the symbol $\mathscr{H}_c/Z_p$ the category of positively graded, locally finite, connected, bicommutative Hopf algebras over Z_p, p prime. If $A \in Ob(H_c/Z_p)$ we write $P(A)$ and $Q(A)$ for the graded vector spaces of primitives and indecomposables, respectively.

P and Q are in fact functors from the category $\mathscr{H}_c/Z_p$ to the category $\mathscr{M}/Z_p$ of positively graded vector spaces. The reader will recall the definition of the natural transformation of functors $l: P \to Q$. If $l: P(A) \to Q(A)$ is onto we say that A is primitively generated. The "Frobenius map" $\zeta: A \to A$ is given by $\zeta(x) = x^p$. The image of ζ, written ζA, is a Hopf sub-algebra of A, and the sequence of vector spaces:

$$0 \to P(\zeta A) \to P(A) \xrightarrow{l} Q(A) \tag{1.1}$$

is exact. ([9], Prop. 4.21). That is, the only decomposable primitives are p'th powers. If A is primitively generated this result can be strengthened to read that the only decomposable primitives are p'th powers of primitives. In this case we write (1.1) in the form:

$$0 \to \zeta P(A) \to P(A) \xrightarrow{l} Q(A) \to 0 \tag{1.2}$$

The category $\mathscr{H}_c/Z_p$ is abelian, with product given by the tensor product over Z_p. For the definitions of kernel and cokernel in $\mathscr{H}_c/Z_p$ the reader is referred to ([9], pp. 223–4). If $f: A \to B$ is a map in the category, the actual set of elements that f sends to zero is the ideal generated by the Hopf sub-algebra $\ker f \subset A$. Let $Z_p \to A' \xrightarrow{i} A \xrightarrow{\pi} A'' \to Z_p$ be an exact sequence in H_c/Z_p. Then the induced sequences in $\mathscr{M}/Z_p$:

$$0 \longrightarrow P(A') \xrightarrow{P(i)} P(A) \xrightarrow{P(\pi)} P(A'') \tag{1.3}$$

$$Q(A') \xrightarrow{Q(i)} Q(A) \xrightarrow{Q(\pi)} Q(A'') \longrightarrow 0 \tag{1.4}$$

are exact. ([9], 3.11,3.12). We will need a stronger version of (1.3) which is due to Moore and Smith [11].

PROPOSITION 1.1. *Let p be an odd prime. Then the map $P(\pi)$ of* (1.3) *is onto in odd dimensions. $P(\pi)$ is onto in even dimensions, except possibly those divisible by a number of the form $2mp$, where $P^{2m}(A') \neq 0$.*

Proof. See ([11], Prop. 3.3 and Cor. 4.5).

We will also need the following lemma.

LEMMA 1.2. *Suppose given a map $f: B \to A$ in the category $\mathscr{H}_c/Z_p$, where A is free commutative as a Z_p algebra. Let Qf be the map $Qf: Q(B) \to Q(A)$; let $\mathscr{Q}f$ be the map $\mathscr{Q}f: Q(B) \to Q(\text{im } f)$, and suppose that* $\ker Qf \subset \ker \mathscr{Q}f$. *Then the natural maps*

$$Q(\ker f) \to \ker(Qf)$$

$$Q(\text{im } f) \to \text{im}(Qf)$$

are isomorphisms.

Proof. The Borel structure theorem ([9], 7.8 and 7.11) implies that $\text{im} f$, being a Hopf sub-algebra of a free commutative algebra, is itself free commutative. Consequently the map $Q(\ker f) \to Q(B)$ is monic. Our hypothesis that $\ker Qf \subset \ker \mathscr{Q}f$ implies that the map $Q(\text{im } f) \to Q(A)$ is monic. The lemma now follows easily from the diagram overleaf, in which horizontal and vertical sequences are exact.

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \rightarrow & Q(\ker f) & \rightarrow & Q(B) & \xrightarrow{\mathcal{Q}f} & Q(\operatorname{im} f) & \rightarrow & 0 \\ & & & & & \searrow^{Qf} & \downarrow & & \\ & & & & & & Q(A) & & \end{array}$$

Recall that if A is an object of $\mathscr{H}_c/Z_p$ then its dual $D(A) = \operatorname{Hom}(A, Z_p)$ can also be regarded, in a canonical way, as an object of $\mathscr{H}_c/Z_p$. One has natural isomorphisms $P(D(A)) \cong D(Q(A))$, and $Q(D(A)) = D(P(A))$. In this paper we are concerned with the Hopf algebra $H^*(BU, Z_p)$, which has the peculiar property that both it and its dual are polynomial algebras. This leads us to:

Definition 1.3. We say A is a bipolynomial Hopf algebra if:

(1) $A \in \mathscr{H}_c/Z_p$.

(2) The algebra structure of A is that of a polynomial algebra.

(3) The algebra structure of $D(A)$ is that of a polynomial algebra.

If V is a graded vector space, we write $\rho_t[V]$ for its Poincaré series: $\rho_t[V] = \sum_{n=0}^{\infty} (\operatorname{rank} V^n)t^n$.

PROPOSITION 1.4. *Let A be a bipolynomial Hopf algebra. Then:*

$$\rho_t[P(A)] = \rho_t[Q(A)].$$

Proof. One has $\rho_t[P(A)] = \rho_t[D(P(A))] = \rho_t[Q(D(A))]$, so it suffices to prove that

$$\rho_t[Q(A)] = \rho_t[Q(D(A))] \tag{1.5}$$

But $\rho_t[A] = \rho_t[D(A)]$, and both A and $D(A)$ are connected polynomial algebras. (1.5) now follows by induction on dimension, and the proposition is proved.

Remark 1.5. Let A be a bipolynomial Hopf algebra. Suppose given an epimorphism in $\mathscr{H}_c/Z_p$:

$$A \xrightarrow{\pi} A'' \rightarrow Z_p \tag{1.6}$$

and suppose further that A'' is a polynomial algebra. Taking duals in (1.6) we learn that $D(A'')$ is a Hopf sub-algebra of the polynomial algebra $D(A)$. Then it follows from the Borel structure theorem ([9], 7.8 and 7.11) that $D(A'')$ is itself a polynomial algebra. Then A'' is bipolynomial, and we can use Prop. 1.4 to conclude that $\rho_t[P(A'')] = \rho_t[Q(A'')]$.

B. Hopf algebras over the Steenrod algebra

Let $\mathscr{A}(p)$ be the mod.p Steenrod algebra, and let $A \in Ob(\mathscr{H}_c/Z_p)$ be given. Suppose that A is an unstable $\mathscr{A}(p)$ module. ([15], p. 27). Then we can use the co-multiplication in $\mathscr{A}(p)$ to make $A \otimes A$ into an unstable $\mathscr{A}(p)$ module. We say that A is a Hopf algebra over $\mathscr{A}(p)$ iff $\mu : A \otimes A \rightarrow A$ and $\Delta : A \rightarrow A \otimes A$ are morphisms of unstable $\mathscr{A}(p)$ modules. If A is a Hopf algebra over $\mathscr{A}(p)$, then both $P(A)$ and $Q(A)$ are unstable $\mathscr{A}(p)$ modules in a natural way. If $f : A \rightarrow B$ is a map of Hopf algebras over $\mathscr{A}(p)$, then the Hopf sub-algebra $\ker f \subset A$ is an $\mathscr{A}(p)$ sub-algebra.

Suppose given an exact sequence in $\mathscr{H}_c/Z_p$:

$$Z_p \to A' \xrightarrow{i} A \xrightarrow{\pi} A'' \to Z_p \tag{1.7}$$

and suppose that A' and A are Hopf algebras over $\mathscr{A}(p)$. Then there is exactly one action of $\mathscr{A}(p)$ on A'' that makes π a map of $\mathscr{A}(p)$ algebras. We refer to (1.7) as an exact sequence of Hopf algebras over $\mathscr{A}(p)$. We will find the following result useful in solving extension problems.

PROPOSITION 1.6. *Suppose given an exact sequence* (1.7) *of Hopf algebras over* $\mathscr{A}(p)$, *p an odd prime, and suppose that A'' is free commutative as a Z_p-algebra. Let* $\chi : Q(A'') \to Z_p \underset{\mathscr{A}(p)}{\otimes} Q(A'')$ *be the canonical epimorphism, and suppose that the composition* $\chi lP(\pi)$:

$$P(A) \xrightarrow{P(\pi)} P(A'') \xrightarrow{l} Q(A'') \xrightarrow{\chi} Z_p \underset{\mathscr{A}(p)}{\otimes} Q(A'') \tag{1.8}$$

is onto. Then A'' is primitively generated, and

$$A \cong A' \otimes A'' \tag{1.9}$$

as a tensor product of Hopf algebras.

Proof. Choose elements $\{a_i''\} \in Q(A'')$ in such a way that the set $\{\chi a_i''\}$ is a vector space basis for $Z_p \underset{\mathscr{A}(p)}{\otimes} Q(A'')$. Choose elements $P^{J_{i,t}} \in \mathscr{A}(p)$ in such a way that the set $\{p^{J_{i,t}}a_i''\}$ is a vector space basis for $Q(A'')$. For each i choose $a_i \in P(A)$ such that $\chi lP(\pi)(a_i) = \chi a_i''$. Then the elements $\{\pi P^{J_{i,t}}a_i\}$ form a set of primitive generators for A''. Since A'' is free commutative we can define a map of algebras $k : A'' \to A$ by setting $k(\pi P^{J_{i,t}}a_i) = P^{J_{i,t}}a_i$. Since k carries primitive generators to primitives it is actually a map of Hopf algebras that splits the exact sequence (1.7). The proposition follows.

Let V be an unstable $\mathscr{A}(p)$ module and let $W \subset V$ be a vector subspace. We say that the subspace W is pseudo singly generated over $\mathscr{A}(p)$ by the element w_0, iff every element of W can be written $P^J w_0$ for some $P^J \in \mathscr{A}(p)$. It is *not* required that W be an $\mathscr{A}(p)$ subspace of V.

Let A be a Hopf algebra over $\mathscr{A}(p)$ and let $B \subset A$ be a primitively generated Hopf sub-algebra. We say that B is pseudo singly generated over $\mathscr{A}(p)$ as a Hopf sub-algebra of A, iff $P(B)$ is pseudo singly generated over $\mathscr{A}(p)$ as a vector subspace of $P(A)$.

Remark. In the situation envisioned in Prop. 1.6, suppose that B'' is a Hopf sub-algebra of A'' that is pseudo singly generated over $\mathscr{A}(p)$. Then the proof of Prop. 1.6 shows that we can choose the splitting (1.9) in such a way that B'' is pseudo singly generated over $\mathscr{A}(p)$ as a Hopf sub-algebra of A.

C. Properties of Tor

We introduce some notation, following [10]. Let $\mathscr{H}_c^2/Z_p$ be the bigraded version of $\mathscr{H}_c/Z_p$; we demand that the bigrading satisfy:

$$A^{-u,v} = 0 \quad \text{unless} \begin{cases} u \geqslant 0 & (1.10a) \\ \text{and} \\ v \geqslant 2u & (1.10b) \end{cases}$$

for each $A \in Ob(\mathscr{H}_c{}^2/Z_p) \cdot -u$ is called the homological degree of $A^{-u,v}$, v the complementary degree, $-u+v$ the total degree. Let $\mathscr{T}\mathscr{H}_c/Z_p$ be the category of "open triangles" of Hopf algebras; that is, diagrams in $\mathscr{H}_c/Z_p$ of the form:

$$\begin{array}{ccc} & & B \\ & & \uparrow g \\ A & \xleftarrow{f} & \Lambda \end{array} \tag{1.11}$$

We demand further that Λ be simply connected. $(\Lambda^0 = Z_p, \Lambda^1 = 0)$. A morphism in $\mathscr{T}\mathscr{H}_c/Z_p$ is the obvious thing. The tensor product over Z_p induces a product on the categories $\mathscr{H}_c{}^2/Z_p$ and $\mathscr{T}\mathscr{H}_c/Z_p$. We now regard Tor as a product-preserving functor: Tor $\mathscr{T}\mathscr{H}_c/Z_p \to \mathscr{H}_c{}^2/Z_p$. The bigrading on $\mathrm{Tor}_\Lambda[A, B]$ satisfies (1.10a) because we index projective resolutions on the negative integers; it satisfies (1.10b) because we have required that Λ be simply connected.

Let $\Lambda \in Ob\mathscr{H}_c/Z_p$. We recall that by using a specific resolution of Z_p as a Λ-module—say, the bar resolution—one can define a natural map s of graded vector spaces:

$$s : Q^*(\Lambda) \to P^{-1,*}(\mathrm{Tor}_\Lambda[Z_p, Z_p]) \tag{1.12}$$

called the suspension.

D. Calculation of Tor

The following result of L. Smith [13] permits calculation of $\mathrm{Tor}_\Lambda[A, Z_p]$.

PROPOSITION 1.7. *Let $f:\Lambda \to A$ be a given map in the category $\mathscr{H}_c/Z_p$. Then:*

$$\mathrm{Tor}_\Lambda[A, Z_p] = A/\!/_{\mathrm{im}\, f} \otimes \mathrm{Tor}_{\ker f}[Z_p, Z_p] \tag{1.13}$$

as a tensor product of Hopf algebras. Here ker$f$ *is the kernel in the category $\mathscr{H}_c/Z_p$. We specify the bigrading:*

$$\mathrm{Tor}_\Lambda^{0,*}[A, Z_p] = (A/\!/_{\mathrm{im}\, f})^* \tag{1.14}$$

Proof. [13], Prop. 1.5.

It remains to calculate $\mathrm{Tor}_\Lambda[Z_p, Z_p]$. We assemble some notation in order to state concisely a standard result. If $V \in \mathscr{M}/Z_p$ we define $V^+ \in \mathscr{M}/Z_p$ by setting $(V^+)^k = V^k$ if k is even, $(V^+)^k = 0$ if k is odd. Similarly, we define V^- to be the odd dimensional part of V. Denote by $\mathscr{M}^2/Z_p$ the category of bigraded vector spaces over Z_p with the bigrading satisfying (1.10). Given $V \in \mathscr{M}^2/Z_p$, define $V^+ \in \mathscr{M}^2/Z_p$ by setting $(V^+)^{u,v} = V^{u,v}$ if $u+v$ is even, and $(V^+)^{u,v} = 0$ if $u+v$ is odd. Similarly, define V^- to consist of those parts of V of odd total degree. Given V in either $\mathscr{M}/Z_p$ or $\mathscr{M}^2/Z_p$, set:

$Z_p[V]$ = polynomial algebra over Z_p generated by V^+
$E[V]$ = exterior algebra over Z_p generated by V^-
$\bar{Z}_p[V]$ = truncated polynomial algebra of height p generated by V^+
$\Gamma[V]$ = algebra of divided powers generated by V^+

We make each of the above algebras into Hopf algebras by declaring the first three to be primitively generated, and writing for the fourth:

$$\Delta(\gamma_k(x)) = \sum_{i+j=k} \gamma_i(x) \otimes \gamma_j(x).$$

Then $Z_p[\ \ ]$, $E[\ \ ]$, etc., are functors from $\mathcal{M}/Z_p$ to $\mathcal{H}_c/Z_p$ (or from $\mathcal{M}^2/Z_p$ to $\mathcal{H}_c^{\,2}/Z_p$). We define two more functors of this sort by setting:

$$F[V] = Z_p[V^+] \otimes E[V^-]$$
$$D(F[V]) = \Gamma[V^+] \otimes E[V^-]$$

Finally, we define a functor $s : \mathcal{M}/Z_p \to \mathcal{M}^2/Z_p$ by setting $(sV)^{-1,v} = V^v$; $(sV)^{u,v} = 0$ if $u \neq -1$. If $x \in V$ we write $s(x)$ for the corresponding element of sV. We can now state a well known result: if $V \in \mathcal{M}/Z_p$ then:

$$\mathrm{Tor}_{F[V]}[Z_p, Z_p] \cong D(F[sV]) \tag{1.15}$$

where the isomorphism is in $\mathcal{H}_c^{\,2}/Z_p$. The suspension (1.12) is indeed given by the correspondence $x \to sx$, $\forall x \in V$. Finally, if $g : V \to W$ is a morphism in $\mathcal{M}/Z_p$, the induced map:

$$\mathrm{Tor}\, g : \mathrm{Tor}_{F[V]}[Z_p, Z_p] \to \mathrm{Tor}_{F[W]}[Z_p, Z_p] \tag{1.16}$$

is given by Tor $g(sx) = sg(x)$, together with the rule that Tor g commutes with divided powers.

§2. THE EILENBERG–MOORE SPECTRAL SEQUENCE

A. Basic properties

We collect here the results of [7]; our notation is taken from [10].

We define a category $\mathcal{HFS}$ of "Hopf fiber squares." An object of $\mathcal{HFS}$ is a commutative diagram of spaces and maps;

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & E_0 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_0} \\ B & \xrightarrow[f]{} & B_0 \end{array} \tag{2.1}$$

in which the following conditions are satisfied.

(1) All spaces are homotopy commutative and homotopy associative H-spaces, and all maps are H-maps.

(2) $E_0 \xrightarrow[p_0]{} B_0$ is a fibration, and $E \xrightarrow[p]{} B$ is the fibration over B induced by f.

(3) B_0 is simply connected.

(4) The mod p-cohomology rings of B, B_0, E_0, E, are of finite type.

A morphism in the category $\mathcal{HFS}$ is a map between two fiber squares (it could be diagrammed as a commutative cube) which preserves all H-space structures.

We define a category $\mathcal{S}[\mathcal{H}_c/Z_p]$ of spectral sequences of Hopf algebras. An object of $\mathcal{S}[\mathcal{H}_c/Z_p]$ consists of the following.

(1) For each $m \geqslant 2$ a differential Hopf algebra (E_m, d_m); E_m an object of $\mathcal{H}_c^{\,2}/Z_p$ and:

$$\deg d_m = (m, 1 - m) \tag{2.2}$$

(2) $E_{m+1} = H(E_m)$; and $E_m^{-u,v} = E_{m+1}^{-u,v} = \cdots = E_\infty^{-u,v}$

for sufficiently large m.

The main result of [7] may be stated as follows.

THEOREM 2.1. *There is a contravariant functor* $\mathcal{E} : \mathcal{HFS} \to \mathcal{S}[\mathcal{H}_c/Z_p]$ *with the following properties.*

(1) *For each fiber square* (2.1) *there is a decreasing filtration* $\{F^{-n}H^*(E, Z_p)\}$ *on the Hopf algebra* $H^*(E, Z_p)$ *that is natural with respect to morphisms of fiber squares.*

(2) $F^0H^*(E, Z_p)$ *is the Hopf sub-algebra of* $H^*(E, Z_p)$ *generated by* $\operatorname{im} \bar{f}^*$ *and* $\operatorname{im} p^*$, *and* $F^{-n}H^*(E, Z_p) = Z_p$ *for* $n < 0$.

(3) *If* (E_m, d_m) *is the spectral sequence assigned by the functor* $\mathcal{E}$ *to the fiber square* (2.1), *there is a functorial isomorphism:*

$$E_2 = \operatorname{Tor}_{H^*(B_0, Z_p)}[H^*(B, Z_p), H^*(E, Z_p)] \tag{2.3}$$

also a functorial isomorphism:

$$E_\infty = {}_0H^*(E, Z_p) \tag{2.4}$$

where $E_0H^*(E, Z_p)$ *denotes the bigraded Hopf algebra associated to the filtration on* $H^*(E, Z_p)$.

Proof. [7] concerns itself with a homology spectral sequence. For explicit proofs for the case of cohomology the reader is referred to [3].

We call attention to the special case of (2.1) in which the space B consists of a single point. Then the total space E of the induced bundle is the fiber F_0 of the original bundle, and Theorem 2.1 gives a spectral sequence converging to $H^*(F_0, Z_p)$. We refer to this as the spectral sequence of the fibration $F_0 \to F_0 \to B_0$.

B. Path space fibrations

We consider the spectral sequence of the fibration $\Omega X \to PX \to X$, where X is a simply connected H-space. The E_2 term is $\operatorname{Tor}_{H^*(X, Z_p)}(Z_p, Z_p)$, and the map s of (1.12) gives

$$s : Q^{*+1}(H(X, Z_p)) \to P^{-1, *+1}(E_2) \tag{2.5}$$

Since $\deg d_r = (r, 1 - r)$, everything in $E^{-1,*}$ is an infinite cycle, so there is defined an epimorphism $E_2^{-1,*} \to E_\infty^{-1,*}$. Composing this epimorphism with the map (2.5) we obtain a map $\bar{s} : Q^{*+1}H(X, Z_p) \to P^{-1,*+1}(E_\infty)$. Now from Theorem 2.1, #2 we know that $F^0H^*(\Omega X, Z_p)$ is zero in dimensions greater than zero; consequently there is an isomorphism $\bar{\pi} : F^{-1}P^*H(\Omega X, Z_p) \to P^{-1,*+1}(E_\infty)$. Finally, let σ denote the cohomology suspension associated with the path space fibration over X: $\sigma : Q^{*+1}H(X, Z_p) \to P^*H(\Omega X, Z_p)$. Then we have the following result:

$$\sigma = \bar{\pi}^{-1}\bar{s} \tag{2.6}$$

For a proof the reader is referred to [14].

We now study the spectral sequence of the fibration:

$$K(Z, n-1) \to L(Z, n) \to K(Z, n) \tag{2.7}$$

Coefficients are taken in Z_p, p an *odd* prime. We begin by singling out a certain "elementary" chain complex that will occur in this spectral sequence.

Definition 2.2. An elementary chain complex A is a bigraded differential Hopf algebra of the form $A = \Gamma(x) \otimes E(y)$, with tot. deg. $(y) = p \cdot$ [tot. deg. (x)] + 1, and the differential determined by the rules $d(\gamma_p(x)) = y$; $dx = dy = 0$.

One sees easily that

$$H(A) = \bar{Z}_p[x]. \tag{2.8}$$

We now recall Cartan's result [5] for $H^*(K(Z, n), Z_p)$, p an odd prime. Let V_n be the graded vector space over Z_p generated freely by objects $P^I i_n$, where the P^I are admissible monomials in the Steenrod algebra subject to the following conditions.

$$\begin{aligned} &(1)\ P^I = \beta^{\varepsilon_1} P^{a_1} \cdots \beta^{\varepsilon_k} P^{a_k} \quad \text{with} \quad a_k \neq 0. \\ &(2)\ \text{excess } I < n \quad \text{if} \quad \varepsilon_1 = 0. \\ &(3)\ \text{excess } I \leq n \quad \text{if} \quad \varepsilon_1 = 1. \end{aligned} \tag{2.9}$$

Then (Cartan) there is an isomorphism of Hopf algebras:

$$H^*(K(Z, n), Z_p) = F[V_n]. \tag{2.10}$$

V_n now appears as $QH(K(Z, n), Z_p)$.

Let

$$\sigma_{\text{E.M.}} : V_n^* \to P^{*-1}H(K(Z, n-1), Z_p) \tag{2.11}$$

be the suspension associated with the fibration (2.7). Another result of Cartan states that ker $\sigma_{\text{E.M.}}$ is that subspace of V_n spanned by all vectors $P^I i_n$, P^I admissible and satisfying (2.9), and excess $I = n$. We define a map of vector spaces

$$\beta\mathscr{P}^t : V_n^- \to \ker \sigma_{\text{E.M.}} \tag{2.12}$$

by assigning to each basis element $P^I i_n$ of V_n^- the element $\beta P^t P^I i_n$ of ker $\sigma_{\text{E.M.}}$, where $2t + 1 = \deg[P^I i_n]$. It is not hard to show this map an isomorphism. In particular, ker $\sigma_{\text{E.M.}}$ is concentrated in even dimensions.

We are ready to discuss the spectral sequence of the fibration (2.7). The E_2 term is given by (2.3), (2.10), and (1.15):

$$F_2 = \text{Tor}_{H^*(K(Z,n),Z_p)}[Z_p, Z_p] = D(F[sV_n]) = K_n \otimes L_n \tag{2.13}$$

where we have set $K_n = \Gamma[sV_n^-] \otimes E[s\beta\mathscr{P}^t V_n^-]$, and $L_n = E[s(V_n^-/\beta\mathscr{P}^t V_n^-)$. Since $\beta\mathscr{P}^t V_n^-$ suspends to zero we know from equation (2.6) that the factor $E[s\beta\mathscr{P}^t V_n^-]$ must be killed somewhere in passage from E_2 to E_∞. This is in fact all that happens in this spectral sequence. The precise statement is due to Smith [13].

PROPOSITION 2.3. *Let p be an odd prime. In the spectral sequence of the fibration* (2.7), *all differentials vanish except d_{p-1}. The action of d_{p-1} on the chain complex $E_{p-1} = E_2$ is given by the following rules.*

(1) *E_{p-1} splits as a tensor product of complexes K_n and L_n.*

(2) d_{p-1} *vanishes on* L_n *(as it must, since each element of* $s(V_n^+/\beta\mathscr{P}^t V_n^-)$ *has homological degree* -1).

(3) K_n *splits further as a tensor product of elementary chain complexes* $\Gamma[s\alpha] \otimes E[s\beta\mathscr{P}^t\alpha]$, *where* $\{\alpha\}$ *is any vector space basis for* V_n^-.

(4) *Consequently (Kunneth formula and* (2.8)) *we have an isomorphism of bigraded Hopf algebras:*

$$E_\infty = {}_p = \bar{Z}_p[\bar{s}V_n^-] \otimes \left[\bar{s}\,\frac{V_n^+}{\beta\mathscr{P}^t V_n^-}\right] \tag{2.14}$$

Proof. See ([13], Theorem 3.10). Smith discusses the path space fibration over a $K(Z_p, n)$. Thus he imposes on admissible monomials only requirements (2.9 (2), (3)). We get a proof of the current proposition by adding (2.9(1)) to each stage of Smith's argument.

We will refer to the complex $D(F[sV_n])$ of (2.13), with differential given by Prop. 2.3, as the canonical complex $D(F[sV_n])$.

If M_n is an $\mathscr{A}(p)$ sub-module of V_n, then $D(F[sM_n])$ is a sub-Hopf algebra of $D(F[sV_n])$ closed under action of the differential. We refer to the sub-complex $D(F[sM_n]) \subset D(F[sV_n])$ as the canonical complex $D(F[sM_n])$.

PROPOSITION 2.4. *Let* M_n *be an* $\mathscr{A}(p)$ *sub-module of* V_n. *Then the cohomology of the canonical complex* $D(F[sM_n])$ *is given by :*

$$H(D(F[sM_n])) = \bar{Z}_p[\bar{s}M_n^-] \otimes E\left[\bar{s}\,\frac{M_n^+}{M_n^+ \cap \beta\mathscr{P}^t V_n^-}\right] \otimes E\left[\bar{s}\,\frac{M_n^+ \cap \beta\mathscr{P}^t V_n^-}{\beta\mathscr{P}^t M_n^-}\right] \tag{2.15}$$

an isomorphism of bigraded Hopf algebras.

Proof. Follows easily from Prop. 2.3 and the definition of the complex $D(F[sM_n])$.

In the canonical complex $D(F[sV_n])$ all elements of $s(\beta\mathscr{P}^t V_n^-)$ are boundaries, but in the complex $D(F[sM_n])$ only the elements of $s(\beta\mathscr{P}^t M_n^-)$ are boundaries. For this reason the factor at the far right of (2.15) has no analogue in the expression (2.14). The injection $D(F[sM_n]) \to D(F[sV_n])$ induces a map of homology:

$$H(D(F[sM_n])) \to H(D(F[sV_n])) \tag{2.16}$$

This map injects the first two factors of (2.15) into the two factors of (2.14), and carries the third factor of (2.15) to zero.

Let M_n be an $\mathscr{A}(p)$ sub-module of V_n. Consider the sub-algebra

$$A[\sigma M_n^1] \subset H^*(K(Z, n-1), Z_p) \tag{2.17}$$

generated by the elements $\sigma_{\text{E.M.}}(x)$, $x \in M_n$. $A[\sigma M_n]$ is primitively generated, therefore is a Hopf sub-algebra of $H^*(K(Z, n-1), Z_p)$, and therefore, by the Borel structure theorem, is free commutative.

We assemble the results of this section into the following theorem.

THEOREM 2.5. *Let* p *be an odd prime. Suppose given a Hopf fiber square of the form:*

$$\begin{array}{ccc} K(Z, n-1) & = & K(Z, n-1) \\ {\scriptstyle i}\downarrow & & \downarrow \\ E & \longrightarrow & L(Z, n) \\ {\scriptstyle q}\downarrow & & \downarrow \\ B & \xrightarrow[f]{} & K(Z, n) \end{array} \tag{2.18}$$

in which E is n-connected, and suppose that $H^*(B, Z_p)$ is free commutative as a Z_p-algebra. Suppose also that the $\mathscr{A}(p)$ Hopf sub-algebra $\ker f^* \subset H^*(K(Z, n), Z_p)$ *is generated over $\mathscr{A}(p)$ by a single element b_0 satisfying*

$$\dim b_0 < np - 1. \tag{2.19}$$

Then the following statements hold.

There exists a sub-module $\overline{M}_n \subset P^*H(K(Z, n), Z_p)$ satisfying

$$\ker f^* = F[\overline{M}_n] \tag{2.20}$$

If $\overline{M}_n$ is such a sub-module, the map $l_n : P^*H(K(Z, n), Z_p) \to Q^*H(K(Z, n), Z_p)$ carries $\overline{M}_n$ isomorphically to an $\mathscr{A}(p)$ sub-module $M_n \subset V_n$, and

$$H^*(E, Z_p) = H^*(B, Z_p)/\!/\operatorname{im} f^* \otimes A[\sigma M_n] \otimes E\left[s\, \frac{M_n{}^+ \cap \beta\mathscr{P}^t V_n{}^-}{\beta\mathscr{P}^t M_n{}^-}\right] \tag{2.21}$$

as a tensor product of Hopf algebras. Here

$$\operatorname{im} q^* = H^*(B, Z_p)/\!/\operatorname{im} f^* \tag{2.22}$$

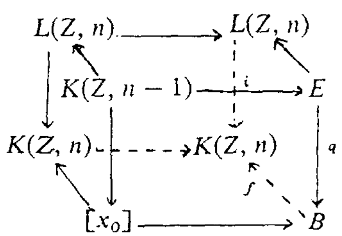
$$\operatorname{im} i^* = A[\sigma M_n] \tag{2.23}$$

$$\ker i^* = H^*(B, Z_p)/\!/\operatorname{im} f^* \otimes E\left[s\, \frac{M_n{}^+ \cap \beta\mathscr{P}^t V_n{}^-}{\beta\mathscr{P}^t M_n{}^-}\right] \tag{2.24}$$

Finally the splitting (2.21) can be chosen in such a way that $A[\sigma M_n]$ is pseudo singly generated over $\mathscr{A}(p)$ as a Hopf sub-algebra of $H^*(E, Z_p)$.

Proof. Kernel f^* is a primitively generated Hopf sub-algebra of the free commutative algebra $H^*(K(Z, n), Z_p)$, so it clearly has the form (2.20) (although there is no unique choice of $\overline{M}_n$. Since $H^*(B, Z_p)$ is free commutative so is $\operatorname{im} f^*$. Therefore $l_n/\overline{M}_n$ is monic. $M_n = \operatorname{im}(l_n/\overline{M}_n)$ is independent of the choice of $\overline{M}_n$.

Consider the commutative diagram of Hopf fiber squares:



(2.25)

Here $[x_0]$ is a space consisting of a single point. Denote by (E_m, d_m) the spectral sequence of the fiber square (2.18), and by $(\bar{E}_m, \bar{d}_m)$ the spectral sequence of the path space fibration over $K(Z, n)$. E_2 and $\bar{E}_2$ are given by (2.3). Since f is an H-map f^* is a map of Hopf algebras and Prop. 1.7 is applicable to computing E_2. We obtain:

$$E_2 = \frac{H^*(B, Z_p)}{\operatorname{im} f^*} \otimes \operatorname{Tor}_{\ker f^*}[Z_p, Z_p] \tag{2.26}$$

$$\bar{E}_2 = \operatorname{Tor}_{H^*(K(Z,n), Z_p)}[Z_p, Z_p] \tag{2.27}$$

The map of E_2 terms induced by (2.25) carries the first factor of (2.26) to zero, and when restricted to the second factor gives a map $\operatorname{Tor}_{\ker f^*}[Z_p, Z_p] \to \operatorname{Tor}_{H^*(K(Z,n),Zp)}[Z_p, Z_p]$ induced by the injection $\ker f^* \to H^*(K(Z, n), Z_p)$. We use 2.20 for $\ker f^*$, and appeal to (1.15) to write out all Tor terms:

$$E_2 = \frac{H^*(B, Z_p)}{\operatorname{im} f^*} \otimes D(F[sM_n]) \tag{2.28}$$

$$_2 = D(F[sV_n]) \tag{2.29}$$

We now use in conjunction the following facts:

(1) The map $E_2 \to \bar{E}_2$ commutes with differentials.

(2) The spectral sequence $(\bar{E}_m, \bar{d}_m)$ is given by Prop. 2.3.

(3) (E_m, d_m) is a spectral sequence of Hopf algebras, and $\deg(d_m = (m, 1 - m)$.

to deduce that:

(1) $d_2 = \dots = d_{p-2} = 0$; $E_2 = E_{p-1}$.

(2) The tensor product of (2.28) is a splitting of the chain complex (E_{p-1}, d_{p-1}).

(3) d_{p-1} vanishes on the first factor of (2.28), and the second factor is the canonical complex $D(F[sM_n])$. Then we have by Prop. 2.4:

$$E_p = \frac{H^*(B)}{\operatorname{im} f^*} \otimes \bar{Z}_p[\bar{s}M_n^-] \otimes E\left[\bar{s}\frac{M_n^+}{M_n^+ \cap \beta\mathscr{P}^t V_n^-}\right] \otimes E\left[\bar{s}\frac{M_n^+ \cap \beta\mathscr{P}^t V_n^-}{\beta\mathscr{P}^t M_n^-}\right] \tag{2.30}$$

Since all indecomposables of (2.30) have homological degree $\geqslant -1$ all higher differentials vanish:

$$E_p = E_\infty \tag{2.31}$$

We can now use Theorem 2.1, #2, and equation (1.14), to deduce one part of our proposition $\operatorname{im} q^* = F^0H^*(E, Z_p) = E_\infty^{0,*} = H^*(B, Z_p)/\!/\operatorname{im} f^*$, and we have proved (2.22).

We must now solve the extension problem posed by (2.30) and (2.31). Let A be the Hopf algebra $H^*(E, Z_p)$, and let A' be the sub-Hopf algebra $\operatorname{im} q^* \cong H^*(B, Z_p)/\!/\operatorname{im} f^*$. Then (2.30), (2.31) imply that there is a filtration on the quotient $A/\!/A'$, with the associated bigraded Hopf algebra given by:

$$E_\infty(A/\!/A') = \bar{Z}_p[\bar{s}M_n^-] \otimes E\left[\bar{s}\frac{M_n^+}{M_n^+ \cap \beta\mathscr{P}^t V_n^-}\right] \otimes E\left[\bar{s}\frac{M_n^+ \cap \beta\mathscr{P}^t V_n^-}{\beta\mathscr{P}^t M_n^-}\right] \tag{2.32}$$

We invoke once more the functorial property of the spectral sequence in conjunction with (2.25) to deduce a commutative diagram:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
F^{-1}(A/\!/A')^* & \xrightarrow{\quad i^* \quad} & F^{-1}H^*(K(Z, n-1), Z_p) \\
\downarrow \pi & & \downarrow \bar{\pi} \\
E_\infty^{-1,*+1} = \bar{s}M_n^- \otimes \bar{s}\,\dfrac{M_n^+}{M_n^+ \cap \beta\mathcal{P}^t V_n^-} \otimes \bar{s}\,\dfrac{M_n^+ \cap \beta\mathcal{P}^t V_n^-}{\beta\mathcal{P}^t M_n^-} & \xrightarrow[E(i^*)]{} & \bar{E}_\infty^{-1,*+1} = \bar{s}V_n^- \otimes \bar{s}\,\dfrac{V_n^+}{\beta\mathcal{P}^t V_n^-} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array} \tag{2.33}$$

Here we have used (2.14) for $\bar{E}_\infty$. π and $\bar{\pi}$ are isomorphisms in dimensions greater than zero because $F^0(A/\!/A')$ and $F^0H^*(K(Z, n-1), Z_p)$ are trivial except in dimension zero. $\bar{\pi}$ is given by (2.6). The map $E(i^*)$ is a restriction of the map (2.16): it is monic on the first two summands of $E_\infty^{-1,*+1}$ and sends the third summand to zero.

Since $i^*(A') = Z_p$ there is a well defined map of Hopf algebras

$$i^* : A/\!/A' \to H^*(K(Z, n-1), Z_p).$$

Im i^* is free commutative; consequently, the map $Q(\ker \bar{i}^*) \to Q(A/\!/A')$ is monic. To determin $\ker \bar{i}^*$ we first use (2.32) to deduce that all indecomposables of $A/\!/A'$ live in $F^{-1}(A/\!/A')$. On the other hand, the elements of

$$\bar{s}\,\frac{M_n^+ \cap \beta\mathcal{P}^t V_n^-}{\beta\mathcal{P}^t M_n^-} \subset F^{-1}(A/\!/A')$$

are odd dimensional primitives of $A/\!/A'$; therefore (by (1.1)) they are indecomposable in $A/\!/A'$. It now follows easily from (2.33) that

$$\ker \bar{i}^* = E\left[\bar{s}\,\frac{M_n^+ \cap \beta\mathcal{P}^t V_n^-}{\beta\mathcal{P}^t M_n^-}\right] \tag{2.34}$$

A second, immediate consequence of (2.33) is that

$$\operatorname{im} \bar{i}^* = \operatorname{im} i^* = A[\sigma M_n] \tag{2.35}$$

this proves (2.23).

Consider the following exact sequence of Hopf algebras over $\mathcal{A}(p)$:

$$Z_p \to \ker \bar{i}^* \to A/\!/A' \to \operatorname{im} \bar{i}^* \to Z_p \tag{2.36}$$

Using (2.19) and our assumption that E is n-connected we find from Prop. 1.1 that the composition $P[A/\!/A'] \to P[\operatorname{im} \bar{i}^*] \to Q[\operatorname{im} \bar{i}^*] \to Z_p \otimes_{\mathcal{A}(p)} Q[\operatorname{im} \bar{i}^*]$ is onto. Therefore the sequence (2.36) satisfies the hypotheses of Prop. 1.6, and we invoke that proposition to conclude that

$$A/\!/A' \cong \operatorname{im} \bar{i}^* \otimes \ker \bar{i}^* = A[\sigma M_n] \otimes E\left[s\,\frac{M_n^+ \cap \beta\mathcal{P}^t V_n^-}{\beta\mathcal{P}^t M_n^-}\right] \tag{2.37}$$

as a tensor product of Hopf algebras. From the remark following Prop. 1.6 it follows that we can choose the splitting (2.37) in such a way that $A[\sigma M_n]$ is pseudo singly generated over $\mathcal{A}(p)$ as a Hopf sub-algebra of $A/\!/A'$.

In exactly the same way we can now apply Prop. 1.6 to the exact sequence $Z_p \to A' \to A \to A/\!/A' \to Z_p$. We conclude that

$$A \cong A' \otimes A/\!/A' = \frac{H^*(B, Z_p)}{\operatorname{im} f^*} \otimes A[\sigma M_n] \otimes E\left[\bar{s}\, \frac{M_n^+ \cap \beta\mathscr{P}^t V_n^-}{\beta\mathscr{P}^t M_n^-}\right] \tag{2.38}$$

as a tensor product of Hopf algebras, and that the splitting (2.38) can be chosen in such a way that $A[\sigma M_n]$ is pseudo singly generated over $\mathscr{A}(p)$ as a Hopf sub-algebra of A. We thus obtain (2.21). (2.24) follows from (2.34), and the proof of Theorem 2.5 is complete.

We now define particular sub-modules $\overline{M}_n \subset P^*H(K(Z, n), Z_p)$; $M_n \subset V_n$. Let $Op[\beta P^1 i_n]$ be that sub-algebra of $H^*(K(Z, n), Z_p)$ generated over $\mathscr{A}(p)$ by the single element $\beta P^1 i_n$. Then $Op[\beta P^1 i_n]$ is a primitively generated Hopf sub-algebra of $H^*(K(Z, n), Z_p)$ and therefore has the form $F[\overline{M}_n]$ for some $\overline{M}_n \subset P^*H(K(Z, n), Z_p)$. There is no natural way to choose $\overline{M}_n$, but we assume some choice has been made. Let l_n be the map $l_n : P^*H(K(Z, n), Z_p) \to Q^*H(K(Z, n), Z_p) = V_n$, and denote its restriction to $\overline{M}_n$ by $l_n \mid \overline{M}_n$. Set $M_n = \operatorname{im}(l_n \mid \overline{M}_n)$. Then M_n is an $\mathscr{A}(p)$ submodule of V_n, and is independent of the choice of $\overline{M}_n$. We record the relations:

$$Op[\beta P^1 i_n] = F[\overline{M}_n]$$
$$Op[\beta P^1 i_{n-1}] = A[\sigma M_n]$$

The symbols M_n, $\overline{M}_n$ will have throughout the rest of the paper the meanings we have just assigned.

It will appear in the course of the inductive calculation of §5 that $Op[\beta P^1 i_n]$ is the kernel of a map from $H^*(K(Z, n), Z_p)$ to a free commutative algebra, and consequently that $M_n \cong \overline{M}_n$.

§3. CONNECTIVE FIBERINGS AND BOTT PERIODICITY

Given any space X and integer $n \geqslant 1$ one can construct a fibration over X, $s: X(n, \ldots, \infty) \to X$, with the following two properties.

(1) $\pi_i(X(n, \ldots, \infty)) = 0$ for $i < n$.

(2) The map s induces isomorphisms in homotopy in dimensions $\geqslant n$.

For details of the construction the reader is referred to the paper [18] of G. Whitehead. The space $X(n, \ldots, \infty)$ is called the $n - 1$ connective fibering over X. If X is a CW-complex one can choose $X(n, \ldots, \infty)$ to have the homotopy type of a CW-complex. The pair $(X(n, \ldots, \infty), s)$ satisfying (1) and (2) is then unique up to homotopy type. We note further that for any $k > 0$ one can regard $X(n + k, \ldots, \infty)$ as the $n + k - 1$ connective fibering over $X(n, \ldots, \infty)$.

If one takes loop spaces of the connective fiberings over X, one obtains connective fiberings over ΩX. This follows straight from the definitions:

$$\Omega(X(n, \ldots, \infty)) \simeq (\Omega X)(n - 1, \ldots, \infty) \tag{3.1}$$

In this paper we are concerned with the connective fiberings over BU and U. Since BU is the classifying space for U there is a homotopy equivalence:

$$\varphi : U \xrightarrow{\simeq} \Omega(BU) \tag{3.2}$$

φ can be taken to be an H-map.

We can go further. Bott periodicity asserts the existence of a homotopy equivalence:

$$\psi : \Omega(SU) \xrightarrow{\simeq} BU \tag{3.3}$$

where SU is the special unitary group. ψ can be taken to be an H-map (see [6]).

We now combine both (3.2) and (3.3) with (3.1) to obtain homotopy equivalences:

$$U(2n-1, \ldots, \infty) \simeq \Omega(BU(2n, \ldots, \infty)) \tag{3.4}$$

$$BU(2n, \ldots, \infty) \simeq \Omega(U(2n+1, \ldots, \infty)) \tag{3.5}$$

(3.4) and (3.5) will be useful in our inductive calculation of cohomology.

Any loop space has the structure of a homotopy commutative, homotopy associative H-space, so we can use (3.4) and (3.5) to impose H-space structures on $U(2n-1, \ldots, \infty)$ and $BU(2n, \ldots, \infty)$ respectively. For $n = 1$ we get by this method structures which are equivalent to the usual ones on U and BU (since φ and ψ are H-maps).

Let us consider the map $q_n : BU(2n, \ldots, \infty) \to BU(2n-2, \ldots, \infty)$ as a fibration. Application of the exact homotopy sequence shows that the fiber is a $K(Z, 2n-3)$. Therefore the fibration is induced from the path space fibration over a $K(Z, 2n-2)$. We obtain a fiber square:

$$\begin{array}{ccc} K(Z, 2n-3) & \longrightarrow & K(Z, 2n-3) \\ \downarrow{\scriptstyle i_n} & & \downarrow \\ BU(2n, \ldots, \infty) & \longrightarrow & L(Z, 2n-2) \\ \downarrow{\scriptstyle q_n} & & \downarrow \\ BU(2n-2, \ldots, \infty) & \xrightarrow[f_{n-1}]{} & K(Z, 2n-2) \end{array} \tag{3.6n}$$

The horizontal maps in (3.6n) induce a map from the exact homotopy sequence of the fibration over $BU(2n-2, \ldots, \infty)$ to the exact homotopy sequence of the fibration over $K(Z, 2n-2)$ and one deduces that f_{n-1} must carry $\pi_{2n-2}(BU(2n-2, \ldots, \infty))$ isomorphically to $\pi_{2n-2}(K(Z, 2n-2))$. If we choose canonical generators for these homotopy groups, then the homotopy class of f_{n-1} is well determined.

Similar remarks apply to the fibration by which one obtains $U(2n+1, \ldots, \infty)$ from $U(2n-1, \ldots, \infty)$. In the fiber square:

$$\begin{array}{ccc} K(\ , 2n-2) & \longrightarrow & K(Z, 2n-2) \\ \downarrow{\scriptstyle j_n} & & \downarrow \\ U(2n+1, \ldots, \infty) & \longrightarrow & L(Z, 2n-1) \\ \downarrow{\scriptstyle r_n} & & \downarrow \\ U(2n-1, \ldots, \infty) & \xrightarrow[g_{n-1}]{} & K(Z, 2n-1) \end{array} \tag{3.7n}$$

g_{n-1} induces an isomorphism of homotopy in dimension $2n-1$.

The fiber square (3.6n) can be obtained by "looping" the fiber square (3.7n); and (3.7n) can be obtained by "looping" (3.6, $n+1$). Consequently both (3.6n) and (3.7n) are Hopf fiber squares, and the theory of §2 is applicable.

Our goal in §§4 and 5 is to compute inductively the cohomology of the connective fiberings over BU and U. We will assume known $H^*(BU(2n-2, \ldots, \infty), Z_p)$, $H^*(U(2n-1, \ldots, \infty), Z_p)$, and the images and kernels of f_{n-1}^* and g_{n-1}^*. Application of Theorem 2.5 to the fiber squares (3.6n) and (3.7n) will then give us $H^*(BU(2n, \ldots, \infty), Z_p)$ and $H^*(U(2n+1, \ldots, \infty), Z_p)$. In order to complete the induction we must determine images and kernels of f_n^* and g_n^*. To this end we will first compute the suspensions (see (3.4), (3.5)):

$$\sigma_{2n} : Q^*H(BU(2n, \ldots, \infty), Z_p) \to P^{*-1}H(U(2n-1, \ldots, \infty), Z_p) \tag{3.8n}$$

$$\sigma_{2n+1} : Q^*H(U(2n+1, \ldots, \infty), Z_p) \to P^{*-1}H(BU(2n, \ldots, \infty), Z_p) \tag{3.9n}$$

Then the commutative diagram:

$$\begin{array}{ccc} P^{*-1}H(K(Z, 2n-1), Z_p) & \xrightarrow{Pg_{n-1}^*} & P^{*-1}H(U(2n-1, \ldots, \infty), Z_p) \\ \uparrow{\scriptstyle \sigma_{E.M.}} & & \uparrow{\scriptstyle \sigma_{2n}} \\ V_{2n}^* & \xrightarrow{Qf_n^*} & Q^*H(BU(2n, \ldots, \infty), Z_p) \end{array} \tag{3.10n}$$

will enable us to determine im f_n^* and ker f_n^*. We will then invoke the diagram:

$$\begin{array}{ccc} P^{*-1}H(K(Z, 2n), Z_p) & \xrightarrow{Pf_n^*} & P^{*-1}H(BU(2n, \ldots, \infty), Z_p) \\ \uparrow{\scriptstyle \sigma_{E.M.}} & & \uparrow{\scriptstyle \sigma_{2n+1}} \\ V_{2n+1} & \xrightarrow{Qg_n^*} & Q^*H(U(2n+1, \ldots, \infty), Z_p) \end{array} \tag{3.11n}$$

in order to determine im g_n^* and ker g_n^*. Our task will be slightly complicated by the fact that $\sigma_{E.M.}$ is not monic.

§4. STATEMENT OF THE MAIN THEOREM

Let p be a prime and n a non-negative integer. Then n can be written uniquely in the form $n = \sum_i a_i p^i$, with $0 \leqslant a_i < p$. Define an arithmetic function σ_p by setting $\sigma_p(n) = \sum_i a_i$. If p^k is the largest power of p dividing n, we set $\nu_p(n) = k$. The following relations are evident.

$$\sigma_p(n \cdot p^j) = \sigma_p(n), j \geqslant 0 \tag{4.1}$$

$$\sigma_p(n) = \sigma_p(n-1) + 1 - \nu_p(n) \cdot (p-1) \tag{4.2}$$

$$\sigma_p(n \cdot p^j - 1) = \sigma_p(n-1) + j \cdot (p-1) \tag{4.3}$$

$$\nu_p(n!) = \frac{n - \sigma_p(n)}{p-1} \tag{4.4}$$

The rest of this section consists of the statement of Theorem 4.1.

THEOREM 4.1. *Let p be an odd prime. There exist indecomposable cohomology classes:*

$$\theta_{2i} \in H^{2i}(BU, Z_p) \text{ for each } i \text{ satisfying } \sigma_p(i-1) < n \tag{4.5}$$

such that

$$H^*(BU(2n, \ldots, \infty), Z_p) \cong \frac{H^*(BU, Z_p)}{Z_p[\theta_{2i} \mid \sigma_p(i-1) < n-1]} \otimes \prod_{t=0}^{p-2} F[M_{2n-3-2t}] \tag{4.6n}$$

as a tensor product of Hopf algebras. Each sub-Hopf algebra $F[M_{2n-E-2t}]$ is pseudo singly generated over $\mathscr{A}(p)$. Recall the maps $q_n : BU(2n, \ldots, \infty) \to BU(2n-2, \ldots, \infty)$, $i_n : K(Z, 2n-3) \to BU(2n, \ldots, \infty)$, $f_n : BU(2n, \ldots, \infty) \to K(Z, 2n)$ defined in §3. We list for the induced maps of cohomology images and kernels in the category $\mathscr{H}_c/Z_p$ (or only images when the kernel is obvious).

$$\operatorname{im} q_n^* = \frac{H^*(BU, Z_p)}{{}_p[\theta_{2i} \mid \sigma_p(i-1) < n-1]} \otimes \prod_{t=1}^{p-2} F[M_{2n-3-2t}] \tag{4.7n}$$

$$\operatorname{im} i_n^* = Op[\beta P^1 i_{2n-3}] \tag{4.8n}$$

$$\ker i_n^* = \operatorname{im} q_n^* \tag{4.9n}$$

$$\operatorname{im} f_n^* = Z_p[\theta_{2i} \mid \sigma_p(i-1) = n-1] \otimes F[M_{2n-2p+1}] \tag{4.10n}$$

$$\ker f_n^* = Op[\beta P^1 i_{2n}] \tag{4.11n}$$

We will deduce from (4.10n) and (4.11n) information about the module M_{2n}:

$$M_{2n} = \overline{M}_{2n} \tag{4.12n}$$

$$\frac{M_{2n}^{+} \cap \beta\mathscr{P}^t V_{2n}^{-}}{\beta\mathscr{P}^t M_{2n}^{-}} = 0 \tag{4.13n}$$

We now list the corresponding results for the connective unitary group.

$$H^*(U(2n+1, \ldots, \infty), Z_p)$$

$$\cong \frac{H^*(U, \ _p)}{E[\mu_{2i+1} \mid \sigma_p(i) < n]} \otimes \prod_{t=0}^{p-2} F[M_{2n-2-2t}] \otimes E[\omega_{2ip^k+1} \mid {}^{\sigma_p(i-1)=n-t-2}_{k>0}] \tag{4.14n}$$

as a tensor product of Hopf algebras. Each sub-Hopf algebra $F[M_{2n-2-2t}]$ is pseudo singly generated over $\mathscr{A}(p)$. Recall the maps of §3: $r_n : U(2n+1, \ldots, \infty) \to U(2n-1, \ldots, \infty)$, $j_n : K(Z, 2n-2) \to U(2n+1, \ldots, \infty)$, $g_n : U(2n+1, \ldots, \infty) \to K(Z, 2n+1)$. We list for the induced maps of cohomology images and kernels in the category $\mathscr{H}_c/Z_p$.

$$\operatorname{im} r_n^* = \frac{H^*(U, Z_p)}{[\mu_{2i+1} \mid \sigma_p(i) < n]} \otimes \prod_{t=1}^{p-2} F[M_{2n-2-2t}] \otimes E[\omega_{2ip^k+1} \mid {}^{\sigma_p(i-1)=n-t-2}_{k>0}] \tag{4.15n}$$

$$\operatorname{im} j_n^* = Op[\beta P^1 i_{2n-2}] \tag{4.16n}$$

$$\ker j_n^* = \operatorname{im} r_n^* \otimes E[\omega_{2ip^k+1} \mid {}^{\sigma_p(i-1)=n-2}_{k>0}] \tag{4.17n}$$

$$\operatorname{im} g_n^* = E[\mu_{2i+1} \mid \sigma_p(i) = n] \otimes F[M_{2n-2p+2}] \otimes E[\omega_{2ip^k+1} \mid {}^{\sigma_p(i-1)=n-p}_{k>0}] \tag{4.18n}$$

$$\ker g_n^* = Op[\beta P^1 i_{2n+1}] \tag{4.19n}$$

We will deduce from (4.18n) and (4.19n) information about the module M_{2n+1}:

$$M_{2n+1} = \overline{M}_{2n+1} \tag{4.20n}$$

$$\left(\frac{M^+_{2n+1} \cap \beta\mathscr{P}^t V^-_{2n+1}}{\beta\mathscr{P}^t M^-_{2n+1}}\right)^* \cong \text{Span}^{*-1}[\omega_{2ip^k+1} \mid {}^{\sigma_p(i-1)=n-1}_{k>0}] \tag{4.21n}$$

This completes the statement of Theorem 4.1.

§5. PROOF OF THE MAIN THEOREM: INDUCTIVE STEP

We assume true statements (4.5k–4.21k) for all values of $k < n$. We are going to establish (4.5n–4.21n).

A. Calculation of $H^*(BU(2n, \ldots, \infty), Z_p)$

Using inductive assumptions (4.6, $n-1$) and (4.11, $n-1$) we see that the fiber square (3.6n) satisfies the hypotheses of Theorem 2.5. Therefore we need only plug our inductive assumptions into (2.21) in order to obtain $H^*(BU(2n, \ldots, \infty), Z_p)$. Specifically, we use (4.6, $n-1$) for $H^*(B, Z_p)$, (4.10, $n-1$) for $\operatorname{im} f^*$, and (4.11, $n-1$) for $\ker f^*$. Observe that $A[\sigma M_{2n-2}] = Op(\beta P^1 i_{2n-3}) \cong F[M_{2n-3}]$ (by (4.20, $n-2$)). We use (4.13, $n-1$) to set the factor on the far right of (2.21) equal to the trivial algebra Z_p. The result for $H^*(BU(2n, \ldots, \infty), Z_p)$, namely, (4.6$n$), now follows immediately from (2.21). (4.7n), (4.8n), (4.9n) follow from (2.22), (2.23), and (2.24), respectively.

B. The commutative diagram (3.10n)

Our purposes require us to consider only a portion of the diagram (3.10n).

Let $s_n : BU(2n, \ldots, \infty) \to BU$ and $t_{n-1} : U(2n-1, \ldots, \infty) \to U$ be the canonical maps. It follows easily from our inductive assumptions and the newly established (4.7n) that:

$$\operatorname{im} s_n{}^* = \frac{H^*(BU, Z_p)}{Z_p[\theta_{2i} \mid \sigma_p(i-1) < n-1]} \tag{5.1n}$$

$$\operatorname{im} t^*_{n-1} = \frac{H^*(U, Z_p)}{E[\mu_{2i-1} \mid \sigma_p(i-1) < n-1]} \tag{5.2, n-1}$$

also that

$$\operatorname{im}[H^*(BU(2(n-p+2), \ldots, \infty), Z_p) \to H^*(BU(2n, \ldots, \infty), Z_p)] = \operatorname{im} s_n{}^* \otimes F[M_{2n-2p+1}] \tag{5.3n}$$

$$\operatorname{im}[H^*(U(2(n-p+2)-1, \ldots, \infty), Z_p) \to H^*(U(2n-1, \ldots, \infty), Z_p)]$$
$$= \operatorname{im} t^*_{n-1} \otimes F[M_{2n-2p}] \otimes E[\omega_{2ip^k+1} \mid {}^{\sigma_p(i-1)=n-p-1}_{k>0}] \tag{5.4, n-1}$$

From these last two equations it follows that we can restrict the domain of

$$\sigma_{2n} : Q^*H(BU(2n, \ldots, \infty), Z_p) \to P^{*-1}H(U(2n-1, \ldots, \infty), Z_p)$$

to obtain a map $\bar{\sigma}_{2n}$:

$$\bar{\sigma}_{2n} : Q^*[\operatorname{im} s_n{}^*] \oplus M_{2n-2p+1} \to P^{*-1}[\operatorname{im} t^*_{n-1}]$$
$$\oplus P^{*-1}F[M_{2n-2p}] \oplus \text{Span}^{*-1}[\omega_{2ip^k+1} \mid {}^{\sigma_p(i-1)=n-p-1}_{k>0}] \tag{5.5n}$$

We now consider the map $f_n : BU(2n, \ldots, \infty) \to K(Z, 2n)$. f_n induces an isomorphism of homotopy in dimension $2n$, therefore an isomorphism of cohomology in that dimension, so that $f_n^* i_{2n} = \beta P^1 i_{2n-2p+1} \in F[M_{2n-2p+1}]$. Since the cohomology of $K(Z, 2n)$ is singly generated over $\mathscr{A}(p)$ it follows from (5.3n) that $\operatorname{im} f_n^* \subset \operatorname{im} s_n^* \otimes F[M_{2n-2p+1}]$. We already know from inductive assumption (4.18, $n-1$) that

$$\operatorname{im} g_{n-1}^* \subset \operatorname{im} t_{n-1}^* \otimes F[M_{2n-2p}] \otimes E[\omega_{2ip^k+1} |^{\sigma_p(i-1)=n-p-1}_{k>0}].$$

We conclude that we can write down the following restriction of the diagram (3.10n):

$$\begin{array}{ccc} 0 & & \\ \uparrow & & \\ P^{*-1}H(K(Z, 2n-1), Z_p) & \xrightarrow{Pg_{n-1}^*} & P^{*-1}[\operatorname{im} t_{n-1}^*] \oplus P^{*-1}F[M_{2n-2p}] \oplus \operatorname{Span}^{*-1}[\omega_{2ip^k+1} |^{\sigma_p(i-1)=n-p-1}_{k>0}] \\ \uparrow{\scriptstyle \sigma_{E.M.}} & & \uparrow \\ V_{2n}^* & \xrightarrow[Qf_n^*]{} & Q^*[\operatorname{im} s_n^*] \oplus M_{2n-2p+1} \end{array} \tag{5.6n}$$

The diagram (5.6n) will be our principal tool in sections C, D, and E.

C. Kernel $\bar{\sigma}_{2n}$

In order to use (5.6n) for the determination of $\operatorname{im} f_n^*$ and $\ker f_n^*$, we first discuss $\bar{\sigma}_{2n}$. On restricting $\bar{\sigma}_{2n}$ to $Q^*[\operatorname{im} s_n^*]$ we obtain an isomorphism of $Q^*[\operatorname{im} s_n^*]$ with P^{*-1} $[\operatorname{im} t_{n-1}]$:

$$0 \to Q^*\left[\frac{H(BU, Z_p)}{Z_p[\theta_{2i} \mid \sigma_p(i-1) < n-1]}\right] \to P^{*-1}\left[\frac{H(U, Z_p)}{E[\mu_{2i+1} \mid \sigma_p(i-1) < n-1]}\right] \to 0 \tag{5.7n}$$

In fact, the map $\sigma_2: Q^*[H(BU, Z_p)] \to P^{*-1}[H(U, Z_p)]$ is an isomorphism, and (5.7n) follows from the relation $t_{n-1}^* \sigma_2 = \sigma_{2n} s_n^*$. We claim next that $\bar{\sigma}_{2n}$ is onto. This follows easily from (5.7n), (5.6n), and (4.18, $n-1$).

We can now find the kernel of $\bar{\sigma}_{2n}$, and begin with the assertion:

$$\beta\mathscr{P}^t M_{2n-2p+1}^- \subset \ker \bar{\sigma}_{2n} \tag{5.8n}$$

This is a consequence of the vanishing of the Bockstein operator on pth powers. (5.8n) can be strengthened to read:

$$\beta\mathscr{P}^t M_{2n-2p+1}^- = \ker \bar{\sigma}_{2n} \tag{5.9n}$$

In fact, since $\bar{\sigma}_{2n}$ is *onto* one has the following relation among Poincaré series:

$$\rho_t[Q[\operatorname{im} s_n^*]] + \rho_t[M_{2n-2p+1}] - \rho_t[\ker \sigma_{2n}]$$
$$= t \cdot [\rho_t P[\operatorname{im} t_{n-1}^*] + \rho_t PF[M_{2n-2p}] + \rho_t \operatorname{Span}[\omega_{2ip^k+1} |^{\sigma_p(i-1)=n-p-1}_{k>0}]] \tag{5.10n}$$

But $\sigma_{E.M.}$ carries $M_{2n-2p+1}$ onto $PF[M_{2n-2p}]$, so that:

$$\rho_t[M_{2n-2p+1}] - \rho_t[M_{2n-2p+1}^+ \cap \beta\mathscr{P}^t V_{2n-2p+1}^-] = t\rho_t[PF[M_{2n-2p}]] \tag{5.11n}$$

From inductive assumption (4.21, $n-p$):

$$\rho_t[M_{2n-2p+1}^+ \cap \beta\mathscr{P}^t V_{2n-2p+1}^-] - \rho_t[\beta\mathscr{P}^t M_{2n-2p+1}^-] = t\rho_t \operatorname{Span}[\omega_{2ip^k+1} |^{\sigma_p(i-1)=n-p-1}_{k>0}] \tag{5.12n}$$

and from (5.7n):

$$\rho_t Q[\operatorname{im} s_n^*] = t\rho_t P[\operatorname{im} t_{n-1}^*] \tag{5.13n}$$

Adding the last three equations and substituting into (5.10n) we learn that $\rho_t[\ker \bar{\sigma}_{2n}] = \rho_t[\beta\mathscr{P}^t M^-_{2n-2p+1}]$. Combining this result with (5.8n) we obtain (5.9n).

D. Kernel f_n^*

We claim

$$f_n^*(\beta P^1 i_{2n}) = 0 \tag{5.14n}$$

In fact, $f_n^*(\beta P^1 i_{2n})$ is primitive, so that $f_n^*(\beta P^1 i_{2n}) \in P[\operatorname{im} s_n^*] \oplus PF[M_{2n-2p+1}]$. But $f_n^* i_{2n} = \beta P^1 i_{2n-2p+1} \in PF[M_{2n-2p+1}]$, and $\beta P^1 \beta P^1 i_{2n-2p+1} = 0$. So we must have $f_n^*(\beta P^1 i_{2n}) \in P[\operatorname{im} s_n^*]$. But $\beta P^1 i_{2n}$ is odd dimensional and im s_n^* is concentrated in even dimensions, so we have proved (5.14n). In particular,

$$M_{2n} \subset \ker(Qf_n^*) \tag{5.15n}$$

We must now prove a converse to (5.15n), and begin with the assertion:

$$\ker(Qf_n^*)^- \subset M^-_{2n} \tag{5.16n}$$

Indeed, suppose given $x \in V^-_{2n}$ with $Qf_n^*(x) = 0$. Then from (5.6n), $Pg^*_{n-1}(\sigma_{\text{E.M.}} x) = 0$, and from the inductive assumption (4.19, $n-1$) it follows that $\sigma_{\text{E.M.}}(x) = P^J \beta P^1 i_{2n-1}$ for some $P^J \in \mathscr{A}(p)$. But $\sigma_{\text{E.M.}}$ is monic on V^-_{2n}, so $x = P^J \beta P^1 i_{2n} \in M^-_{2n}$. We have proved (5.16$n$). We next claim that:

$$\frac{\ker(Qf_n^*)^+ \cap \beta\mathscr{P}^t V^-_{2n}}{\beta\mathscr{P}^t(\ker(Qf_n^*)^-)} = 0 \tag{5.17n}$$

For suppose given $x \in V^-_{2n}$ with $Qf_n^*((\beta\mathscr{P}^t x)) = \beta\mathscr{P}^t Qf_n^*(x) = 0$. Since $\beta\mathscr{P}^t$ is monic on $M^-_{2n-2p+1}$ we must have $Qf_n^*(x) \in Q[\operatorname{im} S_n^*]$. But $Q[\operatorname{im} s_n^*]$ is empty in odd dimensions: $Qf_n^*(x) = 0$, and we have shown (5.17n). We can now prove the full converse of (5.15n):

$$\ker Qf_n^* \subset M_{2n} \tag{5.18n}$$

Let $x \in V_{2n}$ be given with $Qf_n^*(x) = 0$. As before we use (5.6n) to establish $\sigma_{\text{E.M.}}(x) = P^J \beta P^1 i_{2n-1}$ for some $P^J \in \mathscr{A}(p)$, and therefore $x = P^J \beta P^1 i_{2n} + (\text{element of } \beta\mathscr{P}^t V^-_{2n})$. Using (5.15$n$) we strengthen this to read $x = P^J \beta P^1 i_{2n} + (\text{element of } \ker Qf_n^* \cap \beta\mathscr{P}^t V^-_{2n})$. Invoking (5.16$n$) and (5.17$n$) we have finally $x \in M_{2n}$, and we have proved (5.18n). (5.18n) and (5.15n) imply

$$\ker Qf_n^* = M_{2n} \tag{5.19n}$$

E. Image f_n^*

We now use (5.6n) to determine $\operatorname{im}(Qf_n^*)$. By our inductive assumptions $M_{2n-2p+1}$ is psuedo singly generated over $\mathscr{A}(p)$ as a subspace of $Q^*H(BU(2n, \ldots, \infty), Z_p)$. But $f_n^* i_{2n}$ is equal to the generator of $M_{2n-2p+1}$; therefore $M_{2n-2p+1} \subset \operatorname{im} Qf_n^*$. It remains to determine $\operatorname{im} Qf_n^* \cap Q[\operatorname{im} s_n^*]$. Let j be an integer satisfying $\sigma_p(j-1) = n-1$. By (4.18, $n-1$), $\exists\, y \in P^*H(K(Z, 2n-1), Z_p)$ such that $Pg^*_{n-1}(y) = \mu_{2j-1}$. Choose $x \in V_{2n}$ with $\sigma_{\text{E.M.}}(x) = y$. Then from (5.6$n$) we have $\bar{\sigma}_{2n} Qf_n^*(x) = \mu_{2j-1}$. Choose $\theta_{2j} \in Q^{2j}[\operatorname{im} s_n^*]$ such that $\bar{\sigma}_{2n}(\bar{\theta}_{2j}) = \mu_{2j-1}$. Using (5.9$n$) we now see that $Qf_n^*(x) = \bar{\theta}_{2j} + \text{element of } \beta\mathscr{P}^t M^-_{2n-2p+1}$. But $\beta\mathscr{P}^t M^-_{2n-2p+1} \subset \operatorname{im} Qf_n^*$, and we have proved: $\operatorname{Span}[\bar{\theta}_{2j} \mid \sigma_p(j-1) = n-1] \subset \operatorname{im} Qf_n^*$.

The above argument is easily run backwards: given $\theta_{2j} \in \operatorname{im} Qf_n^*$ we use (5.6n) and (4.18, $n-1$) to deduce that $\sigma_p(j-1) = n-1$. We conclude:

$$\operatorname{im} Qf_n^* = \operatorname{Span}[\bar{\theta}_{2j} \mid \sigma_p(j-1) = n-1] \oplus M_{2n-2p+1} \tag{5.20n}$$

We must now pass from our results (5.19n), (5.20n), to the kernel and image of f_n^* itself. Observe that f_n^* satisfies the hypotheses of Lemma 1.2: that $\ker Qf_n^* \subset \ker \mathscr{Q}f_n^*$ follows from (5.14n) and (5.19n). Invoking that lemma we establish isomorphisms $Q(\ker f_n^*) \simeq \ker(Qf_n^*)$; $Q(\operatorname{im} f_n^*) \simeq \operatorname{im}(Qf_n^*)$. From the first of these isomorphisms and (5.14n) we obtain $M_{2n} = \bar{M}_{2n}$ and $\ker f_n^* = Op[\beta P^1 i_{2n}] \cong F[M_{2n}]$; we have proved (4.11$n$) and (4.12$n$). (4.13$n$) follows from (5.17$n$) and (5.19$n$), and we have completed our discussion of $\ker f_n^*$. To determine $\operatorname{im} f_n^*$, note that our inductive hypotheses say that $F[M_{2n-2p+1}]$ is pseudo singly generated over $\mathscr{A}(p)$ as a Hopf sub-algebra of $H^*(BU(2n, \ldots, \infty), Z_p)$. Therefore $F[M_{2n-2p+1}] \subset \operatorname{im} f_n^*$. For each j satisfying $\sigma_p(j-1) = n-1$ choose an element

$$\theta_{2j} \in P^{2j}\left(\frac{H(BU, Z_p)}{Z_p[\theta_{2i} \mid \sigma_p(i-1) < n-1]}\right)$$

whose image in the space of indecomposables is θ_{2j}. That such a θ_{2j} exists follows from (5.20n): since $H^*(K(Z, 2n), Z_p)$ is primitively generated so is $\operatorname{im} f_n^*$. That θ_{2j} is unique follows from Remark 1.5: $P[\operatorname{im} s_n^*]$ can have rank no greater than one in any given dimension. We now use the isomorphism $Q(\operatorname{im} f_n^*) \simeq \operatorname{im}(Qf_n^*)$ to deduce from (5.20n) that $\operatorname{im} f_n^* = Z_p[\theta_{2j} \mid \sigma_p(j-1) = n-1] \otimes F[M_{2n-2p+1}]$, and we have proved (4.10n).

The reader has noticed that the classes θ_{2j} for $\sigma_p(j-1) = n-1$ have been well defined only in the quotient algebra $H^*(BU, Z_p)/\!/Z_p[\theta_{2i} \mid \sigma_p(i-1) < n-1]$. One can choose in a number of ways corresponding elements (not necessarily primitive) in $H^*(BU, Z_p)$, so as to satisfy (4.5n). Our purposes do not require us to make this choice specific.

F. Calculation of $H^*(U(2n+1, \ldots, \infty), Z_p)$

Our procedure here is analogous to that of §5A. We apply Theorem 2.5 to the fiber square (3.7n), plugging in our inductive assumptions (4.12, $n-1$), (4.14, $n-1$), (4.18,–4.21, $n-1$). Equations (4.14n), (4.15n), (4.16n), (4.17n) are immediate consequences.

G. The commutative diagram (3.11n)

We argue as in §5B. Let $t_n : U(2n+1, \ldots, \infty) \to U$ be the canonical map. Then $\operatorname{im} t_n^* = H^*(U, Z_p)/\!/E[\mu_{2i+1}/\sigma_p(i) < n]$. By restricting σ_{2n+1} to $\operatorname{im}[H^*(U(2(n-p+2)+1, \ldots, \infty), Z_p) \to H^*(U(2n+1, \ldots, \infty), Z_p)]$ we get a map $\bar{\sigma}_{2n+1}$ whose image lies in $\operatorname{im}[H^*(BU(2(n-p+2), \ldots, \infty), Z_p) \to H^*(BU(2n, \ldots, \infty), Z_p)]$. We obtain a diagram analogous to (5.6n).

$$\begin{array}{ccc}
0 & & \\
\uparrow & & \\
P^{*-1}H(K(Z, 2n), Z_p) & \xrightarrow{Pf_n^*} & P^{*-1}[\operatorname{im} s_n^*] \oplus P^{*-1}F[M_{2n-2p+1}] \\
\uparrow{\scriptstyle \sigma\ \text{E.M.}} & & \uparrow{\scriptstyle \sigma_{2n+1}} \\
V_{2n+1}^* & \xrightarrow[Qg_n^*]{} & Q^*[\operatorname{im} t_n^*] \oplus \operatorname{Span}^*[\omega_{2jp^k+1} \mid {}^{\sigma_p(j-1)=n-p}_{k>0}] \oplus M_{2n-2p+2}^*
\end{array} \tag{5.21n}$$

H. Kernel $\bar{\sigma}_{2n+1}$

The argument here is analogous to that of §5C. We claim first that on restricting $\bar{\sigma}_{2n+1}$ to $Q^*[\text{im } t_n^*]$ we obtain a monomorphism into $P^{*-1}[\text{im } s_n^*]$:

$$0 \to Q^*\left(\frac{H(U, Z_p)}{E[\mu_{2i+1} \mid \sigma_p(i) < n]}\right) \to P^{*-1}\left(\frac{H(BU, Z_p)}{Z_p[\theta_{2i} \mid \sigma_p(i-1) < n-1]}\right) \qquad (5.22n)$$

In fact, if we denote by σ_3 the map $\sigma_3\colon Q^*(H(SU, Z_p)) \to P^{*-1}(H(BU, Z_p))$, then $\sigma_3(\mu_{2j+1}) = \lambda.\ (c_l)^{p^{v_p(j)}}$. Here $l = j/p^{v_p(j)}$, c_l is the Chern class in dimension $2l$, and $\lambda \neq 0 \in Z_p$. To show the map (5.22n) monic we need only observe that if $\sigma_p(j) \geqslant n$ then by (4.2), (4.3), $\sigma_p\left(\frac{j}{p^{v_p(j)}}\right) - 1 \geqslant n - 1$; thus the Chern class c_l is non-zero in the quotient algebra $H^*(BU, Z_p)/\!/Z_p[\theta_{2i} \mid \sigma_p(i-1) < n-1]$.

We can now discuss ker $\bar{\sigma}_{2n+1}$. Clearly

$$\beta\mathscr{P}^t M^-_{2n-2p+2} \subset \ker \bar{\sigma}_{2n+1} \qquad (5.23n)$$

We want to show:

$$\beta\mathscr{P}^t M^-_{2n-2p+2} = \ker \bar{\sigma}_{2n+1} \qquad (5.24n)$$

Define sets of numbers: $A = \{2j+1 \mid \sigma_p(j) \geqslant n\}$; $B = \{2jp^k + 1 \mid {}^{\sigma_p(j-1) = n-p}_{k>0}\}$. Use of (4.2) and (4.3) shows that A and B are disjoint. Further, given j satisfying $\sigma_p(j-1) = n-p$, then $\sigma_p(pj-1) = n-1$, so that by the newly established (4.10n) we have $(\theta_{2jp})^{p^{k-1}} \in \text{im } Pf_n^*$ for any $k > 0$: $\sum_{M \in B-1} P^m[\text{im } s_n^*] \subset \text{im } Pf_n^*$. Again referring to (4.10n) we strengthen this to read: $\sum_{M \in B-1} P^m[\text{im } s_n^*] \oplus PF[M_{2n-2p+1}] \subset \text{im } Pf_n^*$, and diagram (5.21$n$) now implies $\sum_{M \in B-1} P^m[\text{im } s_n^*] \oplus PF[M_{2n-2p+1}] \subset \text{im } \bar{\sigma}_{2n+1}$. Finally, using (5.22$n$) and the fact that A and B are disjoint we have $\sum_{M \in A-1} P^m[\text{im } s_n^*] \oplus \sum_{M \in B-1} P^m[\text{im } s_n^*] \oplus PF[M_{2n-2p+1}] \subset$ im $\bar{\sigma}_{2n+1}$; therefore:

$$\rho_t Q[\text{im } t_n^*] + \rho_t \text{Span}[\omega_{2jp^k+1} \mid {}^{\sigma_p(j-1)=n-p}_{k>0}] + \rho_t M_{2n-2p+2} - \rho_t \ker \bar{\sigma}_{2n+1}$$
$$\geqslant t\rho_t \sum_{m \in A-1} {}^m[\text{im } s_n^*] + t\rho_t \sum_{m \in B-1} P^m[\text{im } s_n^*] + t\rho_t PF[M_{2n-2p+1}] \qquad (5.25n)$$

Now, the indecomposables of im t_n^* are in 1-1 correspondence with the set A, and the generators of $\text{Span}[\omega_{2jp^k+1} \mid {}^{\sigma_p(j-1)=n-p}_{k>0}\}$ are in 1–1 correspondence with the set B, so (5.25n) implies $\rho_t M_{2n-2p+2} - \rho_t \ker \bar{\sigma}_{2n+1} \geqslant t\rho_t PF[M_{2n-2p+1}]$. Finally we observe that $\sigma_{\text{E.M.}}$ maps $M_{2n-2p+2}$ onto $PF[M_{2n-2p+1}]$ with kernel $M^+_{2n-2p+2} \cap \beta\mathscr{P}^t V^-_{2n-2p+2} = \beta\mathscr{P}^t M^-_{2n-2p+2}$; we have used inductive assumption (4.13, $n-p+1$). We now have $\rho_t[\beta\mathscr{P}^t M^-_{2n-2p+2}] \geqslant \rho_t[\ker \bar{\sigma}_{2n+1}]$, and combining this equation with (5.23n) we obtain (5.24n).

I. Image g_n^*

Since $g_n^*(i_{2n+1}) = \beta P^1 i_{2n-2p+2} \in M_{2n-2p+2}$ we have

$$M_{2n-2p+2} \subset \text{im } Qg_n^* \qquad (5.26n)$$

We assert also that

$$\text{Span}[\omega_{2jp^k+1} \mid {}^{\sigma_p(j-1)=n-p}_{k>0}] \subset \text{im } Qg_n^* \tag{5.27n}$$

In fact, let j, k be given, with $\sigma_p(j-1) = n-p$; $k > 0$. One finds easily from inductive assumptions $(4.8, n-p+2)$, $(4.17, n-p+2)$, that $\bar{\sigma}_{2n+1}(\omega_{2jp^k+1}) \subset P[\text{im } s_n^*]$. (i.e., $\bar{\sigma}_{2n+1}(\omega_{2jp^k+1})$ has no component in the direct summand $PF[M_{2n-2p+1}]$) (5.24n) tells us that $\bar{\sigma}_{2n+1}(\omega_{2jp^k+1}) \neq 0$. But by Remark 1.5 $P[\text{im } s_n^*]$ has rank no greater than one in any given dimension, and since by (4.3) $\sigma_p(pj-1) = n-1$ we know from §5E that θ_{2pj} is primitive. We conclude that $\bar{\sigma}_{2n+1}(\omega_{2jp^k+1}) = \lambda(\theta_{2jp})^{p^{k-1}}$, $\lambda \neq 0 \in Z_p$. Since $(\theta_{2jp})^{p^{k-1}} \in \text{im } Pf_n^*$, we can now combine (5.21n), (5.24n), and (5.26n), to establish (5.27n).

We claim next that

$$\text{Span}[\mu_{2j+1} \mid \sigma_p(j) = n] \subset \text{im } Qg_n^* \tag{5.28n}$$

In fact, let j be given with $\sigma_p(j) = n$. Let $l = j/p^{\nu_p(j)}$. Then $\sigma_p(l-1) = n-1$ so that $(\theta_{2l})^{p^{\nu_p(j)}} \in \text{im } Pf_n^*$. But we have also $\bar{\sigma}_{2n+1}(\mu_{2j+1}) = \lambda \cdot (\theta_{2l})^{p^{\nu_p(j)}}$, $\lambda \neq 0 \in Z_p$. (5.28n) now follows easily from (5.21n), (5.24n), (5.26n).

We now claim the converse: $\text{im } Qg_n^* \cap \text{im } t_n^* \subset \text{Span}[\mu_{2j+1} \mid \sigma_p(j) = n]$. Suppose given $\mu_{2j+1} \in \text{im } Qg_n^*$. Then $\sigma_p(j) \geqslant n$. From (5.21n), (5.22n) we find that $\bar{\sigma}_{2n+1}(\mu_{2j+1})$ is a non-zero primitive of $\text{im } f_n^*$. But $\text{im } f_n^*$ is primitively generated, so that by (1.2) and (4.10n) we find $\bar{\sigma}_{2n+1}(\mu_{2j+1}) = \lambda(\theta_{2j'})^{p^k}$, $k \geqslant 0$ and $\sigma_p(j'-1) = n-1$. Then $j = j'p^k$, and from (4.1), (4.2), $\sigma_p(j) \leqslant n$. Therefore $\sigma_p(j) = n$.

We conclude:

$$\text{im } Qg_n^* = \text{Span}[\mu_{2j+1} \mid \sigma_p(j) = n] \oplus \text{Span}[\omega_{2jp^k+1} \mid {}^{\sigma_p(j-1)=n-p}_{k>0}] \oplus M_{2n-2p+2} \tag{5.29n}$$

J. Kernel g_n^*

Clearly

$$g_n^*(\beta P^1 i_{2n+1}) = 0 \tag{5.30n}$$

In fact, $\beta P^1 \beta P^1 i_{2n2p+2} = 0$, and by (1.1) the exterior algebra $\text{im } t_n^* \otimes E[\omega_{2jp^k+1} \mid {}^{\sigma_p(j-1)=n-p}_{k>0}]$ has no even dimensional primitives. (5.30n) implies:

$$M_{2n+1} \subset \ker Qg_n^* \tag{5.31n}$$

We must prove a converse to (5.31n), and claim first that

$$\ker(Qg_n^*)^- \subset M_{2n+1}^- \tag{5.32n}$$

This follows from (4.11n) and (5.21n) in the same way that (5.16n) follows from $(4.19, n-1)$ and (5.6n).

Our next step is to establish an isomorphism of degree +1 of graded vector spaces:

$$\rho : \text{Span}[\omega_{2ip^k+1} \mid {}^{\sigma_p(i-1)=n-1}_{k>0}] \xrightarrow{\approx} \frac{\ker(Qg_n^*)^+ \cap \beta\mathscr{P}^t V_{2n+1}^-}{\beta\mathscr{P}^t \ker(Qg_n^*)^-} \tag{5.33n}$$

We define ρ in the following way. Suppose given i, k with $\sigma_p(i-1) = n-1$ and $k > 0$. If $\nu_p(i) = 0$ then by (4.1) and (4.2) $\sigma_p(i) = \sigma_p(i \cdot p^{k-1}) = n$, and by (5.29$n$) there exists

$P^J i_{2n+1} \in V_{2n+1}^-$ with $Qg_n^*(P^J i_{2n+1}) = \mu_{2ip^{k-1}+1}$. Clearly, $Qg_n^*(\beta\mathscr{P}^t P^J i_{2n+1}) = 0$, for by (1.1) there are no even dimensional primitives in im t_m^*. We set $\rho(\omega_{2ip^k+1}) = \beta\mathscr{P}^t P^J i_{2n+1}$. Regarded as an element of the quotient space (5.33n), $\beta\mathscr{P}^t P^J i_{2n+1}$ is evidently independent of the choice of P^J to satisfy $Qg_n^*(P^J i_{2n+1}) = 0$; it is also non-zero. If $v_p(i) > 0$ then by (4.3), $\sigma_p(\frac{i}{p} - 1) = n - p$, and by (5.29$n$) there exists $P^J i_{2n+1} \in V_{2n+1}^-$ such that

$$Qg_n^*(P^J i_{2n+1}) = \omega_{2ip^k+1}.$$

Now, the subspace

$$P[\text{im } t_n^*] \oplus \text{Span}[\omega_{2jp^k+1} |^{\sigma_p(j-1)=n-p}_{k>0}] \subset P[H(U(2n+1, \ldots, \infty), Z_p)]$$

is closed under the action of $\mathscr{A}(p)$; this follows easily from our inductive assumption (4.17, $n - p + 2$). Since there are no even dimensional primitives in an exterior algebra it follows that $Qg_n^*(\beta\mathscr{P}^t P^J i_{2n+1}) = 0$, and we set $\rho(\omega_{2ip^k+1}) = \beta\mathscr{P}^t P^J i_{2n+1}$. This completes the definition of ρ.

ρ is clearly $1-1$. To show it onto suppose given $x \in V_{2n+1}^-$ with $Qg_n^*(x) \neq 0$ and $\beta\mathscr{P}^t Qg_n^*(x) = 0$. Since $\beta\mathscr{P}^t$ is monic on $M_{2n-2p+2}^-$, $Qg_n^*(x)$ must lie in $Q[\text{im } t_n^*] \oplus \text{Span } [\omega_{2jp^k+1} |^{\sigma_p(j-1)=n-p}_{k>0}]$. Therefore ρ is onto.

We are now ready to prove:

$$\ker(Qg_n^*)^+ \subset M_{2n+1}^- \tag{5.34n}$$

We induct on dimension. Suppose given $x \in \ker(Qg_n^*)^+$ with dim $x < 2np + 2$. Then from (5.21n), $(Pf_n^*)\sigma_{E.M.}(x) = 0$, and so by (4.11$n$), $\sigma_{E.M.}(x) = P^J \beta P^1 i_{2n}$ for some $P^J \in \mathscr{A}(p)$. Since $\sigma_{E.M.}$ is monic in dimensions $< 2np + 2$, we conclude that $x = P^J \beta P^1 i_{2n+1} \in M_{2n+1}^+$. Now suppose that (5.34$n$) has been established for all dimensions $< 2m$, and let $x \in$ (ker Qg_n^*) be given with dim $x = 2m$. Then as usual $\sigma_{E.M.}(x) = P^J \beta P^1 i_{2n}$ for some $P^J \in \mathscr{A}(p)$. If $m \not\equiv 1$ (mod p) we can immediately conclude $x = P^J \beta P^1 i_{2n+1} \in M_{2n+1}^+$, and are done. If $m = tp + 1$, then the best we can say is $x = P^J \beta P^1 i_{2n+1} + \beta P^t y$ for some $y \in V_{2n+1}^-$, dim $y = 2t + 1$. Our induction will be complete if we can show:

$$\beta P^t y \in M_{2n+1}^+ \tag{5.35n}$$

If $y \in \ker(Qg_n^*)^-$ then (5.32n) implies (5.35n), so it suffices to assume that $\beta P^t y$ is non-zero in the quotient space of (5.33n). Then

$$Qg_n^*(y) = \lambda\mu_{2t+1}, \quad \lambda \neq 0 \in Z_p \tag{5.36n}$$

(Here μ_{2t+1} is to be interpreted as either a μ or an ω.) The proof now divides into two cases.

Case 1. $t \equiv 1$ (mod p). Then we can solve the Adem relation

$$P^1 \beta P^{t-1} = P^t \beta - (p-1)(t-1)\beta P^t$$

for βP^t, and write:

$$\beta P^t y = \lambda_1 P^t \beta y + \lambda_2 P^1 \beta P^{t-1} y \tag{5.37n}$$

$\lambda_1, \lambda_2 \in Z_p$. But im $t_n^* \otimes E[\omega_{2ip^k+1} |^{\sigma_p(i-1)}_{k>0}]$ is closed under $\mathscr{A}(p)$ action and contains no even-dimensional primitives, so it is clear from (5.36n) that $Qg_n^*(\beta y) = 0$ and $Qg_n^*(\beta P^{t-1} y) = 0$. Since dim$(\beta y) < 2m$ and dim $(\beta P^{t-1} y) < 2m$ it follows from the inductive assumption that βy and $\beta P^{t-1} y$ are in M_{2n+1}; hence from (5.37b) that $\beta P^t y \in M_{2n+1}$.

Case 2. $t \equiv 1 \pmod p$. In this case the right hand side of (5.36n) is really a μ, so that $\sigma_p(t) = n$. Since $v_p(t) = 0$ we have $\sigma_p(t-1) = n-1$. Write $t = ap^l + 1$, with $l > 0$ and $v_p(a) = 0$. Then $\sigma_p(a-1) = n-2$, so that θ_{2a} is non-zero in $H^*(BU(2n-2, \ldots, \infty), Z_p)$. We have:

$$\theta_{2a}^{p^l} = P^{ap^{l-1}}\theta_{2a}^{p^{l-1}} \tag{5.38n}$$

Since Steenrod powers commute with suspension we can deduce from (5.22, $n-1$) and (5.38n) a relation in $H^*(U(2n-1, \ldots, \infty), Z_p)$: $\mu_{2ap^l+1} = \lambda \cdot P^{ap^{l-1}}\mu_{2ap^{l-1}+1}$ + decomposables; $\lambda \neq 0 \in Z_p$. But since $\mathscr{A}(p)$ carries primitives to primitives we can strengthen this to read:

$$\mu_{2ap^l+1} = \lambda \quad {}^{ap^{l-1}}\mu_{2ap^{l-1}+1} \tag{5.39n}$$

Now $\sigma_p(ap^{l-1}) = n-1$ so that $\theta_{2ap^{l-1}+2}$ is a non-zero primitive of $H^*(BU(2n, \ldots, \infty), Z_p)$. Then we get from (5.7$n$) and (5.39$n$) a relation in $H^*(BU(2n, \ldots, \infty), Z_p)$: $\theta_{2ap^l+2} = \lambda P^{ap^{l-1}}\theta_{2ap^{l-1}+2}$. Carrying this argument one step further we obtain finally a relation in $H^*(U(2n+1, \ldots, \infty), Z_p)$:

$$\mu_{2t+1} = \lambda P^{ap^{l-1}}\mu_{2ap^{l-1}+3} \tag{5.40n}$$

Here $\mu_{2ap^{l-1}+3}$ is either a "μ" or an "ω". In either case it is in im g_n^*, since $\theta_{2ap^{l-1}+2}$ is in im f_n^*. Choose $v \in V_{2n+1}$ with $Qg_n^*(v) = \mu_{2ap^{l-1}+3}$. Then $Qg_n^*(P^{ap^{l-1}}v) = \lambda\mu_{2t+1}$, and from (5.32$n$) and (5.36$n$) we see that y differs from $P^{ap^{l-1}}v$ by at most an element of M_{2n+1}^-. So to show $\beta P^t y \in M_{2n+1}^+$ it suffices to show that

$$\beta P^t P^{ap^{l-1}} v \in M_{2n+1}^- \tag{5.41n}$$

To this end we establish:

LEMMA 5.1. *Let X be a space, and suppose given $v \in H^{2j+3}(X, Z_p)$. Then*

$$\beta P^{pj+1}P^j v = P^{pj}\beta P^{j+1} v \tag{5.42n}$$

Proof. The monomial $P^{pj}\beta P^{j+1}$ is not admissable. If we write down the appropriate Adem relation, and use the rule $P^k x = 0$ if $2k > \dim x$, we obtain (5.42n).

Applying Lemma 5.1 we find:

$$\beta P^t P^{ap^{l-1}} v = P^{ap^l}\beta P^{ap^{l-1}+1} v \tag{5.43n}$$

Now $Qg_n^*(\beta P^{ap^{l-1}+1}v) = \beta P^{ap^{l-1}+1}\mu_{2ap^{l-1}+3} = 0$, since there are no even dimensional primitives in an exterior algebra. By the inductive assumption on $\ker(Qg_n^*)^+$ it follows that $\beta P^{ap^{l-1}+1}v \in M_{2n+1}^+$. (5.41$n$) now follows from (5.43$n$). Thus we have established (5.35n), and our inductive proof of (5.34n) is complete.

Combining (5.31n), (5.32n), (5.34n) we obtain

$$\ker(Qg_n^*) = M_{2n+1} \tag{5.44n}$$

Mimicking the end of §5E we now use Lemma 1.2 to deduce im g_n^*, ker g_n^*, from our results (5.29n), (5.44n). In this way we establish (4.18–4.20n). (4.21n) follows from substituting (5.44n) into (5.33n), and we have completed the inductive part of the proof of Theorem 4.1.

§6. PROOF OF THE MAIN THEOREM: LOW DIMENSIONS

The statements (4.5n–4.21n) for p an odd prime are valid in low dimensions if we read the trivial algebra Z_p for $F[M_k]$ whenever $M_k = 0$. We proceed to show this.

Observe that $BU \cong K(Z, 2) \times BU(4, \ldots, \infty)$, and that the map $f_1: BU \to K(Z, 2)$ is the projection of product onto factor. Setting θ_2 equal to the Chern class in dimension two we find $\operatorname{im} f_1^* = Z_p[\theta_2]$, $\ker f_1^* = Z_p$, establishing (4.10, 1) and (4.11, 1). The statements (4.6, 1–4.9, 1) and (4.12, 1–4.13, 1) are all trivial.

Observe that $U \cong S^1 \times U(3, \ldots, \infty)$. Therefore $H^*(U(3, \ldots, \infty), Z_p) = H^*(U, Z_p)/E[\mu_1]$; this is (4.14, 1). Statements (4.15, 1–4.17, 1) are trivial. We consider now the map $g_1^*: H^*(K(Z, 3), Z_p) \to H^*(U(3, \ldots, \infty), Z_p)$. By Cartan's calculation,

$$H^*(K(Z, 3), Z_p) = Z_p[\beta P^{p^n} P^{p^{n-1}} \cdots P^1 i_3 \mid n = 0, 1 \cdots] \otimes E[P^{p^{n-1}} \cdots P^1 i_3 \mid n = 0, 1, \cdots] \tag{6.1}$$

Using diagram (3.11,1) we easily find $g_1^*(P^{p^{n-1}} \cdots P^1 i_3) = \mu_{2p^n+1}$ and $g_1^*(\beta P^{p^n} \cdots P^1 i_3) = 0$. Thus $\operatorname{im} g_1^* = E[\mu_{2i+1} \mid \sigma_p(i) = 1]$; this is (4.18, 1). Using the Adem relations $P^{p^n} \beta P^{p^{n-1}} = \beta P^{p^n} P^{p^{n-1}} (n = 1, 2, \ldots)$ one sees that $\beta P^{p^n} P^{p^{n-1}} \cdots P^1 i_3 = P^{p^n} P^{p^{n-1}} \cdots \beta P^1 i_3$; therefore:

$$\ker g_1^* = Z_p[\beta P^{p^n} P^{p^{n-1}} \cdots P^1 i_3 \mid n = 0, 1, \cdots] \subset Op[\beta P^1 i_3] \tag{6.2}$$

But the general argument of §5 shows that $Op[\beta P^1 i_3] \subset \ker g_1^*$, and we have established (4.19, 1). The general argument also suffices to prove (4.20, 1) and (4.21, 1).

In order to prove equations (4.5n–4.21n) for $1 < n \leqslant p$ we need make just one adjustment in the general argument of §5. Note that for $n > p$ one has $f_n^*(i_{2n}) = \beta P^1 i_{2n-2p+1} \in F[M_{2n-2p+1}]$ and $g_n^*(i_{2n+1}) = \beta P^1 i_{2n-2p+2} \in F[M_{2n-2p+2}]$. But for small values of n the following relations hold:

$$\begin{aligned} f_n^*(i_{2n}) &= \lambda \theta_{2n}, \qquad n \leqslant p \\ g_n^*(i_{2n+1}) &= \lambda \mu_{2n+1}^*, \qquad n < p \\ g_p^*(i_{2p+1}) &= \lambda \omega_{2p+1} \end{aligned} \tag{6.3}$$

(with $\lambda \neq 0 \in Z_p$). Thus f_n^*, g_n^* do not kill the factors $F[M_k]$ until $p - 1$ of them have accumulated. The statements of Theorem (4.1) and the arguments of §5 go through, however, even without change of wording. We need only set $M_k = 0$ for $k < 3$. We have completed the proof of Theorem 4.1.

§7. THE CASE $p = 2$

The calculations of $H^*(BU(2n, \ldots, \infty), Z_p)$ and $H^*(U(2n + 1, \ldots, \infty), Z_p)$ for $p = 2$ are somewhat different from the case in which p is odd. We indicate the changes.

By the results of Serre and Cartan [12], [5], $H^*(K(Z, n), Z_2)$ is a pure polynomial algebra on a certain graded Z_2-module V_n. Suppose given a map $f^*: H^*(K(Z, n), Z_2) \to A$

in the category $\mathscr{H}_c/Z_2$, and suppose we have chosen $\overline{M}_n \subset P^*H(K(Z, n), Z_2)$ to satisfy $\ker f^* = Z_2[\overline{M}_n]$. Let l_n be the map $l_n: P^*H(K(Z, n), Z_2) \to Q^*H(K(Z, n), Z_2)$. If $\operatorname{im} f^*$ contains odd dimensional elements x for which $x^2 = 0$ we cannot conclude (as we did in Theorem 2.5) that $l_n | \overline{M}_n$ is monic; for the odd-dimensional generators of $H^*(K(Z, n), Z_2)$ have infinite height. The version of Theorem 2.5 appropriate to the case $p = 2$ runs as follows.

Theorem 7.1. *Suppose given a Hopf fiber square of the form* (2.18). *Choose a submodule* $\overline{M}_n \subset P^*H(K(Z, n), Z_2)$ *satisfying* $\ker f^* = Z_2[\overline{M}_n]$. *If either*

a.) $\ker l_n \cap \overline{M}_n = 0$

or

b.) *the algebra* $H^*(B, Z_2)/\!/_{\operatorname{im} f^*}$ *is generated by odd dimensional classes,*

then

$$H^*(E, Z_2) = \frac{H^*(B, Z_2)}{\operatorname{im} f^*} \otimes A[\sigma M_n] \otimes E[s(\ker 1_n \cap \overline{M}_n)] \tag{7.1}$$

as a tensor product of algebras. Here $\operatorname{im} q^* = H^*(B, Z_2)/\!/\operatorname{im} f^*$; $\operatorname{im} i^* = A[\sigma M_n]$; $\ker i^* = H^*(B, Z_2)/\!/\operatorname{im} f^* \otimes E[s(\ker(l_n | \overline{M}_n))]$. *The splitting* (7.1) *can be chosen in such a way that* $A[\sigma M_n]$ *is a Hopf sub-algebra of* $H^*(E, Z_2)$ *that is pseudo-singly generated over* $\mathscr{A}(2)$.

To prove Theorem 7.1 one uses the Eilenberg–Moore spectral sequence for the fiber square (2.18). One find $E_2 = H^*(B, Z_2)/\!/\operatorname{im} f^* \otimes \operatorname{Tor}_{\ker f_*}[Z_2, Z_2] = H^*(B, Z_2)/\!/\operatorname{im} f^* \otimes E[s\overline{M}_n]$. Since all indecomposables of this E_2 term have homological degree $\geqslant -1$ the spectral sequence collapses. The remainder of the proof is analogous to the proof of Theorem 2.7.

Let $Op[Sq^3 i_n]$ be that Hopf sub-algebra of $H^*(K(Z, n), Z_2)$ generated over the Steenrod algebra by the single element $Sq^3 i_n$. In applying Theorem 7.1 to the computation of $H^*(BU(2n, \ldots, \infty), Z_2)$ and $H^*(U(2n+1, \ldots, \infty), Z_2)$ one chooses $\overline{M}_n$ so that $Z_2[\overline{M}_n] = Op[Sq^3 i_n]$. The reader can easily supply the details of the computation by analogy with §5. One finds:

$$H^*(BU(2n, \ldots, \infty), Z_2) = \frac{H^*(BU, Z_2)}{Z_2[\theta_{2i} | \sigma_2(i-1) < n-1]} \otimes Z_2[\overline{M}_{2n-3}] \tag{7.2}$$

as a tensor product of Hopf algebras;

$$\ker f_n^* = Op[Sq^3 i_{2n}] \tag{7.3}$$

Also,

$$H^*(U(2n+1, \ldots, \infty), Z_2) = \frac{H^*(U, Z_2)}{E[\mu_{2i+1} | \sigma_2(i) < n]} \otimes Z_2[\overline{M}_{2n-2}] \otimes E[\omega_{2^{k-1}i+1} | {}^{\sigma_2(i-1)=n-2}_{k>0}] \tag{7.4}$$

as a tensor product of Hopf algebras;

$$\ker g_n^* = Op[Sq^3 i_{2n+1}] \tag{7.5}$$

The equation analogous to (4.12n–4.13n) is:

$$\ker l_{2n} \cap \overline{M}_{2n} = 0 \tag{7.6}$$

and the equation analogous to $(4.20n-4.21n)$ is

$$(\ker l_{2n+1} \cap \overline{M}_{2n+1})^* = \text{Span}^{*-1}[\omega_{i\cdot 2^{k+1}-1} \mid {}^{\sigma_2(i-1)=n-2}_{k>0}] \tag{7.7}$$

(7.2), (7.3), and (7.6) were first obtained by Stong [6], using methods different from ours.

§8. POSTNIKOV SYSTEM OF THE UNITARY GROUP

If X is a space and n an integer one can construct a new space $X(1, 2, \ldots, n)$ and a map $\alpha: X \to X(1, 2, \ldots, n)$ such that

(1) $\pi_i(X(1, 2, \ldots, n)) = 0$ for $i > n$

(2) α induces isomorphisms of homotopy in dimensions $\leqslant n$.

If X is a CW-complex one can choose $X(1, 2, \ldots, n)$ to be a CW-complex. Under these circumstances, requirements (1), (2) determine $(X(1, 2, \ldots, n), \alpha)$ up to homotopy type. The space $X(1, 2, \ldots, n)$ is known as a stage of the Postnikov tower of X.

Regard the map $s_{n+1}: BU(2n+2, \ldots, \infty) \to BU$ as a fibration. A simple argument shows that the fiber has the homotopy type $U(1, \ldots, 2n-1)$. We consider the Eilenberg–Moore spectral sequence of the fibration:

$$U(1, \ldots, 2n-1) \to BU(2n+2, \ldots, \infty) \xrightarrow{s_{n+1}} BU \tag{8.1}$$

for cohomology with coefficients in Z_p, p odd. From (1.13), (2.3), and $(4.6, n+1)$ we find:

$$\begin{aligned} E_2 &= H^*(BU(2n+2, \ldots, \infty), Z_p) /\!/ \operatorname{im} s_{n+1}^* \otimes \operatorname{Tor}_{\ker s_{n+1}^*}[Z_p, Z_p] \\ &= \prod_{t=0}^{p-2} F[M_{2n-1-2t}] \otimes \operatorname{Tor}_{Z_p[\theta_{2i}|\sigma_p(i-1)<n]}[Z_p, Z_p] \\ &= \prod_{t=0}^{p-2} F[M_{2n-1-2t}] \otimes E[\mu_{2i-1} \mid \sigma_p(i-1) < n] \end{aligned} \tag{8.2}$$

Since all indecomposables of (8.2) have homological degree $\geqslant -1$ the spectral sequence collapses. Since $E_2 = E_\infty$ is free commutative as an algebra and primitively generated as a Hopf algebra the extension problem of Hopf algebras is trivial, and we find:

$$H^*(U(1, 2, \ldots, 2n-1), Z_p) = E[\mu_{2i+1} \mid \sigma_p(i) < n] \otimes \prod_{t=0}^{p-2} F[M_{2n-1-2t}] \tag{8.3}$$

as a tensor product of Hopf algebras. The result for $p = 2$ is slightly more complicated:

$$\begin{aligned} H^*(U(1, \ldots, 2n-1), Z_2) = {}& E[\mu_{2i+1} \mid \sigma_2(i) < n-1] \otimes Z_2[\mu_{2i+1} \mid \sigma_2(i) = n-1] \\ & \otimes Z_2\left[\frac{\overline{M}_{2n-1}}{\text{Span}[\omega_{i\cdot 2^{k+1}+2} \mid {}^{\sigma_2(i-1)=n-2,\ \sigma_2(i)=n-1}_{k>0}]}\right] \end{aligned} \tag{8.4}$$

as a tensor product of algebras. (8.4) has also been obtained by Hirsch [8] and Vastersavendts [17]. I am grateful to Miss Vastersavendts for having pointed out an error in my original version of (8.4).

Added in proof. Using these same methods the author has recently determined cohomology for the Postnikov system of BU as well. The result for p odd is:

$$H^*(BU(2, \ldots, 2n)Z_p) = Z_p[\theta_{2i} \mid \sigma_p(i-1) < n] \otimes \prod_{t=0}^{p-2} F[M_{2n-2t}] \tag{8.5}$$

as a tensor product of algebras. In case $p = 2$ one finds:

$$H^*(BU(2, \ldots, 2n), Z_2) = Z_2[\theta_{2i} \mid \sigma_2(i-1) < n] \otimes Z_2[\overline{M}_{2n}] \tag{8.6}$$

§9. DIVISIBILITY OF INTEGRAL CHERN CLASSES

Let $f: X \to Y$ be a map. Throughout this chapter we will use the symbol f^* to denote the induced map of cohomology with coefficients in either Z or Z_p. The precise meaning will always be clear from context. In particular, all divisibility conditions pertain to integral cohomology classes.

A. Formulation of the problem

Let $c_k \varepsilon H^{2k}(BU, Z)$ be the Chern class, and let $Qc_k \varepsilon Q^{2k}H(BU, Z)$ be its image in the module of indecomposables. Let s_n be the standard map $s_n: BU(2n, \ldots, \infty) \to BU$, and Qs_n^* the induced map $Qs_n^*: Q^*H(BU, Z) \to Q^*H(BU(2n, \ldots, \infty), Z)$. Then $Qs_n^*(Qc_k)$ is divisible by some positive integer $\lambda_{n,k}$ and by no greater number. We set ourselves the problem of determining $\lambda_{n,k}$. If $n > k$ then $\lambda_{n,k} = \infty$, so only the case $n \leqslant k$ is interesting. For $n = k$ the answer is a well known consequence of Bott periodicity:

$$\lambda_{k,k} = (k-1)! \tag{9.1}$$

B. A splitting of BU

Let p be a prime. Let C be the Serre class of abelian groups having finite orders relatively prime to p. We say a map $f: X \to Y$ is a p-homotopy equivalence iff f induces isomorphisms of homotopy groups mod C.

It has recently been observed by J. F. Adams and D. Anderson that for any prime p there exists a homotopy associative, homotopy commutative H-space W_p, with:

$$\begin{aligned} \Pi_{2j}(W_p) &= Z \bmod C \quad \text{if} \quad j \equiv 0 \pmod{p-1} \quad \text{and } j \neq 0 \\ \Pi_{2j}(W_p) &= 0 \bmod C \quad \text{otherwise} \\ \Pi_{2j+1}(W_p) &= 0 \bmod C \end{aligned} \tag{9.2}$$

further, that there are p-homotopy equivalences

$$\varphi_1 : \prod_{j=0}^{p-2} \Omega^{2j} W_p \underset{p}{\simeq} BU \tag{9.3}$$

$$\psi_1 : \prod_{j=0}^{p-2} \Omega^{2j+1} W_p \underset{p}{\simeq} U \tag{9.4}$$

φ_1 and ψ_1 are H-maps. Bott periodicity appears as a p-homotopy equivalence between the identity component of $\Omega^{2(p-1)}W_p$ and W_p itself. For a proof of these results we refer the reader to [2].

Any homotopy associative H-space has the rational homotopy type of a product of

Eilenberg–Maclane spaces. It is therefore easy to infer from (9.2) and (9.3) that for $0 \leqslant j \leqslant p-2$:

$$H^*(\Omega^{2j}W_p, Z_p) = Z_p[\hat{\theta}_{2i} \mid i = -j \,(\text{mod}\ p-1)] \tag{9.5}$$

also that the map

$$\rho_p : H^*(\Omega^{2j}W_p, Z) \to H^*(\Omega^{2j}W_p, Z_p) \tag{9.6}$$

is onto. We can go further. (9.3) implies the existence of p-homotopy equivalences:

$$\varphi_n : \prod_{j=0}^{p-2} (\Omega^{2j}W_p)(2n, \ldots, \infty) \underset{p}{\simeq} BU(2n, \ldots, \infty) \tag{9.7}$$

(9.7) allows us to interpret geometrically the occurrence of a $p-1$-fold tensor product in our result (4.6n) for $H^*(BU(2n, \ldots, \infty), Z_p)$. In fact, it is easy to deduce from Theorem 4.1 that for $0 \leqslant j \leqslant p-2$:

$$H^*((\Omega^{2j}W_p)(2n, \ldots, \infty), Z_p) = \frac{H^*(\Omega^{2j}W_p)}{Z_p\left[\theta_{2i} \,\middle|\, \begin{matrix} \sigma_p(i-1) < n-1 \\ i \equiv -j(\text{mod}\ p-1) \end{matrix}\right]} \otimes F[M_{2n-3-2t(n,j)}] \tag{9.8n}$$

as a tensor product of Hopf algebras. Here $t(n,j)$ is the unique integer satisfying both $0 \leqslant t(n,j) \leqslant p-2$ and $t(n,j) = n+j-1 \pmod{p-1}$. Denote by $q_n{}^j$ the standard map $q_n{}^j : (\Omega^{2j}W_p)(2n, \ldots, \infty) \to (\Omega^{2j}W_p)(2n-2, \ldots, \infty)$. (9.2) implies that $q^j{}_n$ is a p-homotopy equivalence for $n \not\equiv -j+1 (\text{mod}\ p-1)$. But for $n \equiv -j+1 (\text{mod}\ p-1)$ we find from Theorem 4.1 that:

$$\ker q_n{}^{j*} = Z_p[\theta_{2i} \mid \sigma_p(i-1) = n-2] \otimes F[M_{2(n-1)-3-2t(n-1,j)}] (n \equiv -j+1 (\text{mod}\ p-1)) \tag{9.9n}$$

For each integer k choose a class $\hat{c}_k \varepsilon H^{2k}[\Omega^{2j}W_p, Z]$ (where $0 \leqslant j \leqslant p-2$ and $j \equiv -k \pmod{p-1}$) such that $Q\hat{c}_k$ generates an infinite cyclic direct summand of $Q^{2k}H[\Omega^{2j}W_p, Z]$. Let $s_n{}^j$ denote the standard map $s_n{}^j : (\Omega^{2j}W_p)(2n, \ldots, \infty) \to (\Omega^{2j}W_p)$, and let the integer $\hat{\lambda}_{n,k}$ be such that $Qs_n{}^{j*}(Q\hat{c}_k) \varepsilon Q^{2k}H((\Omega^{2j}W_p)(2n, \ldots, \infty), Z)$ is divisible by $\hat{\lambda}_{n,k}$ and by no greater number. Recall the function ν_p defined at the beginning of section 4. The existence of the p-homotopy equivalence (9.3) implies that

$$\nu_p(\lambda_{n,k}) = \nu_p(\hat{\lambda}_{n,k}). \tag{9.10}$$

C. Calculation of $\lambda_{n,k}$

The exact sequence of coefficient groups $0 \to Z \xrightarrow{\times p} Z \to Z_p \to 0$ gives rise to a long exact sequence of cohomology, of which we will need only the portion:

$$H^n(X, Z) \xrightarrow{\times p} H^n(X, Z) \xrightarrow{\rho_p} H^n(X, Z_p) \tag{9.11}$$

LEMMA 9.1. *Suppose given a map $f: X \to Y$ and an element $y \varepsilon H^*(Y, Z)$ satisfying $f^*\rho_p(y) = 0$. Then $Qf^*(Qy) \varepsilon Q^*H(X, Z)$ is divisible by p.*

Proof. We have $\rho_p f^*(y) = f^*\rho_p(y) = 0$. It follows from the exact sequence (9.11) that $f^*(y)$ is divisible by p. Then so is $Q(f^*(y)) = Qf^*(Qy)$.

The map $\rho_p : H^*(Y, Z) \to H^*(Y, Z_p)$ is an algebra map. We use the symbol $Q\rho_p$ to denote the corresponding map of indecomposables.

LEMMA 9.2. *Suppose given a map $f: X \to Y$ and an element $y \varepsilon H^*(Y, Z)$ such that $Qf^*(Qy)$ is divisible by p. Then $Qf^*[Q\rho_p(Qy)] = 0$.*

Proof. The sequence of groups $QH(X, Z) \xrightarrow{xp} QH(X, Z) \xrightarrow{Q\rho_p} QH(X, Z_p)$ is not in general exact, however the composition $(Q\rho_p)(xp)$ is zero. Since $Qf^*(Qy) \varepsilon \operatorname{im}(xp)$ it follows that $0 = Q\rho_p[Qf^*(Qy)] = Qf^*[Q\rho_p(Qy)]$.

We are now ready to calculate $v_p(\hat{\lambda}_{n,k})$. We will think of k as being fixed, and watch the growth of $\hat{\lambda}_{n,k}$ as n grows from 1 to k. Throughout the discussion let j be the unique integer satisfying both $0 \leqslant j \leqslant p - 2$ and $j \equiv -k \pmod{p-1}$.

LEMMA 9.3.

$$v_p(\hat{\lambda}_{n,k}) = 0 \quad \text{for} \quad n \leqslant 1 + \sigma_p(k-1) \tag{9.12}$$

Proof. In the group $Q^*H(\Omega^{2j}W_p, Z_p)$ one has:

$$Q\rho_p(Q\hat{c}_k) = Q\hat{\theta}_{2k} \tag{9.13}$$

It is clear from (9.8n) that for values of n satisfying $\sigma_p(k-1) \geqslant n - 1$, $Qs_n{}^{j*}(Q\hat{\theta}_{2k})$ is non-zero in $Q^*[H((\Omega^{2j}W_p)(2n, \ldots, \infty))]$. It follows from Lemma 9.2 that $Qs_n{}^{j*}(Q\hat{c}_k)$ is not divisible by p. This proves Lemma 9.3.

Note that if l is any integer, then $l - \sigma_p(l)$ is divisible by $p - 1$.

LEMMA 9.4. *For values of n of the form:*

$$n = 2 + \sigma_p(k-1) + \alpha \cdot (p-1) \tag{9.14}$$

where

$$\alpha = 0, 1, 2, \ldots, \frac{(k-1) - \sigma_p(k-1)}{p-1} - 1 \tag{9.15}$$

we have

$$v_p\left(\frac{\hat{\lambda}_{n,k}}{\hat{\lambda}_{n-1,k}}\right) \geqslant 1 \tag{9.16}$$

Proof. We handle first the case $\alpha = 0$. Choose an element $\hat{c}_{k'} \varepsilon H^{2k}[\Omega^{2j}W_p, Z]$ such that $Q(\hat{c}_{k'}) = Q(\hat{c}_k)$, and such that $\rho_p(\hat{c}_{k'}) = \hat{\theta}_{2k}$. That such a $\hat{c}_{k'}$ exists follows from (9.13) and the fact that the map (9.6) is onto. For $n = 2 + \sigma_p(k-1)$ it is clear from (9.9n) that $s_n{}^j \rho_p \hat{c}_{k'} = 0$. It follows from Lemma 9.1 that $Qs_n^{j*}(Q\hat{c}_k)$ is divisible by p. But from Lemma 9.3 we know that $Qs_{n-1*}^j(Q\hat{c}_k)$ is not divisible by p. (9.16) follows.

Suppose now that $\alpha > 0$, and let n have the value (9.14). Choose an element $x \varepsilon H^{2k}((\Omega^{2j}W_p)(2(n-1), \ldots, \infty), Z)$ that satisfies

$$\hat{\lambda}_{n-1,k} Qx = Qs_{n-1*}^j(Q\hat{c}_k) \tag{9.17}$$

Since $\alpha > 0$ we have $n - 2 > \sigma_p(k-1)$; therefore the left hand factor of (9.8, $n-1$) has no indecomposable in dimension $2k$. It follows that we can choose x in such a way that $\rho_p(x)$ lies in the right hand factor of (9.8, $n-1$); that is, in $F[M_{2(n-1)-3-2t(n-1,j)}]$. (Here we are again using the fact that (9.6) is onto). From (9.9n) it now follows that $q_n{}^{j*}\rho_p(x) = 0$, so from Lemma 9.1 we see that $Qq_n{}^{j*}(Qx)$ is divisible by p. Then from (9.17) it follows that $Qs_n^{j*}(Q\hat{c}_k)$ is divisible by $p \cdot \hat{\lambda}_{n-1,k}$. (9.16) follows.

LEMMA 9.5. *For values of n of the form* (9.14), (9.15), *we have*

$$\nu_p\left(\frac{\lambda_{n,k}}{\lambda_{n-1,k}}\right) \geq 1 \tag{9.18}$$

Proof. Immediate from Lemma (9.4) and equation (9.10).

LEMMA 9.6. *For values of n of the form* (9.14), (9.15),

$$\nu_p\left(\frac{\lambda_{n,k}}{\lambda_{n-1,k}}\right) = 1 \tag{9.19}$$

and for all other values of n in the range $2 \leq n \leq k$,

$$\nu_p\left(\frac{\lambda_{n,k}}{\lambda_{n-1,k}}\right) = 0. \tag{9.20}$$

Proof.

$$\nu_p(\lambda_{k,k}) = \nu_p \prod_{n=2}^{k} \left(\frac{\lambda_{n,k}}{\lambda_{n-1,k}}\right) = \sum_{n=2}^{k} \nu_p\left(\frac{\lambda_{n,k}}{\lambda_{n-1,k}}\right).$$

But from (9.1) and (4.4) we have

$$\nu_p(\lambda_{k,k}) = \nu_p((k-1)!) = \frac{(k-1)-\sigma_p(k-1)}{p-1}.$$

Therefore

$$\frac{(k-1)-\sigma_p(k-1)}{p-1} = \sum_{n=2}^{k} \nu_p\left(\frac{\lambda_{n,k}}{\lambda_{n-1,k}}\right) \tag{9.21}$$

We now invoke Lemma 9.5, observing that there are $[(k-1)-\sigma_p(k-1)]/(p-1)$ different values of n in the range $2 \leq n \leq k$ for which (9.18) holds. (9.19) and (9.20) now follow from (9.21).

We define a function lig m:

$$\operatorname{lig} m = \begin{cases} \text{least integer} \geq m & \text{if } m \geq 0 \\ 0 & \text{if } m < 0. \end{cases}$$

THEOREM 9.7. $Qs_n{}^*(Qc_k)$ *is divisible by*

$$\lambda_{n,k} = \prod_p p^{\operatorname{lig}\left[\frac{(n-1)-\sigma_p(k-1)}{p-1}\right]} \tag{9.22}$$

and by no greater number.

Proof. Immediate from Lemma 9.6.

The reader can use (4.4) to verify that in the "stable range" $k < 2n$ our divisibility condition (9.22) agrees with the stable result of Adams [1].

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