Problems in Differential and Algebraic Topology. Seattle Conference, 1963

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Introduction

In the summer of 1963 a conference on Differential and Algebraic Topology was held under the auspices of the American Mathematical Society at the University of Washington, Seattle, Washington. It was financed by a grant from the National Science Foundation.

The emphasis of the conference was on differential topology, and we have attempted to give the general background and statements of results for recent developments in the field. A glance at these notes will show that it is impossible to separate developments in differential topology from those in algebraic topology. But in order to keep these notes a manageable size, it was necessary to limit the discussion to just a few areas in differential topology. We have chosen immersion and imbedding theory and combinatorial and differentiable structures because of their rapid development in the past couple of years, and because they coincide with present interests of the editor.

At the instigation of Norman E. Steenrod, who directed the conference, participants presented problems of current interest, for which we hope the discussion serves as motivation and elucidation.

For background material for other recent developments of differential and algebraic topology, we refer the reader to the following summaries.

(1) J. F. Adams, *Cohomology operations* (Seattle Conference)

(2) M. G. Barratt, *Homotopy operations and homotopy groups* (Seattle Conference)

(3) F. Hirzebruch, *Lectures on K-theory* (Seattle Conference)

(4) S. Smale, *A survey of some recent developments in differential topology*, Bull. Amer. Math. Soc. 69 (1963), pp. 131–145.

Since the original draft of this report, considerable progress has been made on many of the problems presented. These results have been inserted wherever sufficient information was available. R. Lashof.

1. Immersions and imbeddings

Definition 1.0. An *immersion* of one smooth manifold M^k in a second

X^n, of dimensions k and n respectively, is a C^1-map $f: M \to X$ with the property that, for each $p \in M$ in some coordinate system (and hence all) about p and $f(p)$, the jacobian matrix of f has rank k. An *imbedding* is an immersion which is also a homeomorphism onto its image.

DEFINITION 1.1. An isotopy between two immersions (imbeddings) $f, g: M \to X$, is a C^1-map $F: M \times I \to X$, such that $F_t: M \to X$ is an immersion (imbedding) for each t.

The fundamental problem of immersion (imbedding) theory is: Given manifolds M and X, find the isotopy classes of immersions (imbeddings) of M in X. This includes in particular the problem of whether M can be immersed (imbedded) in X at all.

The first theorem of this type was based on *general position* arguments and proved by Whitney [58] in 1936.

THEOREM 1.0. *Given manifolds M^k, X^n, any two immersions (imbeddings) $f, g: M \to X$ which are homotopic are isotopic provided $n \geqq 2k + 2$ ($n \geqq 2k + 3$). If $n \geqq 2k$ ($n \geqq 2k + 1$), there exists an immersion (imbedding) of M^k in X^n.*

In 1944, Whitney [59] improved the existence theorem as follows:

THEOREM. *Every k-manifold can be immersed in E^{2k-1}, $k > 1$, and imbedded in E^{2k}.*

The next general advance was the work of Smale [46] and Hirsch [19] on the theory of immersions (1957–1959).

THEOREM 1.1. *Assume $k \geqq n$ if any component of M^k is closed. If a map $f: M^k \to X^n$ is covered by a bundle monomorphism $\varphi: \tau_M \to \tau_X$ of the tangent bundles, there is an immersion $g: M \to X$ homotopic to f such that the differential $dg: \tau_M \to \tau_X$ is homotopic to φ through bundle monomorphisms. Any two such immersions g are regularly homotopic. If $n > k$, g can be chosen so as to approximate f as closely as desired.*

COROLLARY 1. *M^k immerses in E^n if and only if there exists a bundle ν^{n-k} over M such that $\tau_M \oplus \nu^{n-k}$ is trivial.*

This beautiful result of Hirsch reduces the immersion problem to a homotopy problem. Still one wishes to reduce this problem further to obtain a pure number theoretic expression for the least possible dimension for immersing a manifold in euclidean space. A great deal of work has been done on the special case of projective spaces by many authors. A good many (but not all) of the authors take this theorem as the starting point [12], [25], [44], for proving the existence of immersions. The non-existence theorems are largely proved by using characteristic classes expressed as cohomology operations, including K-theory operations [43], [53], [33], [45]. These results are too complicated to reproduce here. (See Hirsch's excellent summary [20]).

On the other hand, Hirsch's theorem completely solves the existence question for immersions of π-manifolds (i.e., manifolds which can be imbedded with trivial normal bundles in sufficiently large euclidean spaces). Namely,

THEOREM 1.2 (Hirsch). *A π-manifold M^k is immersible in E^{k+1}. If none of the components of M^k are closed, M^k is immersible in E^k.*

The situation for imbeddings is not nearly so good. In the so-called *meta-stable* range, however, the results are somewhat better than for immersions, in the sense that they include a more explicit classification theory. These results are due to Haefliger [15], [16], who greatly extended the techniques of Whitney using ideas of Thom, Wu, and Shapiro.

We first state his general theorem: Let M^k be compact, and $n \geqq 3/2(k+1)$.

THEOREM 1.3 (Haefliger). *Let f: $M^k \to X^n$ be a map. Considering f as a map of Δ_X into Δ_M (the diagonals in $M \times M$ and $X \times X$ resp.), suppose that f is extendable to a map θ: $M \times M \to X$ satisfying*

(a) *θ is equivariant with respect to the involutions $(x, y) \to (y, x)$,*

(b) $\theta^{-1}(\Delta_X) \subset \Delta_M$.

Then f is homotopic to an imbedding g: $M \to X$ such that $g \times g$ is homotopic, preserving properties (a) *and* (b) *to θ. If $n > 3/2(k+1)$, then any two such imbeddings g are isotopic.*

COROLLARY 1. *If M is topologically imbeddable in X, then M is differentiably imbeddable.*

THEOREM 1.3′. *If f: $M \to X$ is a map such that $f_\#$: $\pi_i(M) \to \pi_i(X)$ is an isomorphism for $i \leqq 2k-n$ and onto for $i = 2k-n+1$, then f is homotopic to an imbedding. If $f_\#$ is an isomorphism for $i \leqq 2k-n+1$ and onto for $i = 2k-n+2$, and $n > 3/2(k+1)$, then two imbeddings which are homotopic to f are isotopic.*

COROLLARY 2. *If M^k is r-co-connected* (i.e., *$(k-r)$-connected*), *$r \geqq (k+3)/2$, M is imbeddable in E^{k+r}. If $r > (k+1)/2$, then two imbeddings in E^{k+r+1} are isotopic.*

REMARK. Although for immersions in the meta-stable range the analogue of Theorem 1.3 but using only a neighborhood of the diagonal [18]; no direct analogue of Theorem 1.3′ has been given.

Another approach to imbeddings in the meta-stable range has been given by J. Levine (Bull. Amer. Math. Soc., 59 (1963), 806–809); based on constructions of Browder and Novikov [7], [41], [31], together with Corollary 2 above.

THEOREM 1.4. *Suppose* $2n \geqq 3(k+1)$ *and* $\mathfrak{S}$ *is a* $(n-k)$*-plane bundle over a simply connected closed manifold* M^k, *stably equivalent to the the normal bundle of* M, *such that Thom space* $T(\mathfrak{S})$ *is reducible* (i.e., $\pi_n(T(\mathfrak{S})) \to H_n(T(\mathfrak{S}))$ *is onto*). *Then there is an imbedding* f *of* M *in* S^n *such that the normal bundle* ν_f *is fibre homotopically equivalent to* $\mathfrak{S}$:

(a) *over* M *if* $n = 6$, 14 *or* $n \not\equiv 2 \bmod 4$

(b) *over* M*-open disc, if* $n \equiv 2 \bmod 4$.

Now suppose f imbeds M in S^n, then the Thom construction defines an element $\alpha_f \in \pi_n(T(\nu_f))$. If f, g are isotopic imbeddings of M in S^n, it is easy to show that there is a bundle map $\varphi: \nu_f \to \nu_g$ over the identity such that $\varphi_*: \pi_n(T(\nu_f)) \to \pi_n(T(\nu_g))$ sends α_f into α_g. We say that φ induces an equivalence between the normal invariants of f and g. Then Levine proves:

THEOREM 1.5. *If* $2n > 3(k+1)$, *then two imbeddings of* M *in* S^n *are isotopic if and only if they have equivalent normal invariants.*

In special cases, depending on dimension and assumptions concerning characteristic classes, these results have been pushed a bit further (Haefliger and Hirsch, *Existence and classification of differentiable imbeddings*, Topology (2) (1963), 129–136). Also a number of authors [20] have obtained special results for projective spaces. Although it is hoped that results on projective spaces will show the way to obtain general results for imbeddings below the meta-stable range, the results so far are too fragmentary and complicated to reproduce here.

The first general results for imbeddings below the meta-stable range have recently been obtained for π-manifolds by J. Minkus and R. De Sapio [11]. In particular, their results partially establish a conjecture of M. Hirsch, (see Corollary 2 below).

Problem 1. Is every parallelizable n-manifold imbeddable in euclidean $n + [(n+1)/2]$-space?

DEFINITION 1.2. A closed k-manifold M^k, is *almost differentiably imbeddable* in E^n, if there is a smooth submanifold $M_1 \subset E^n$, and a homeomorphism $h: M \to M_1$ which is differentiable except at one point.

THEOREM 1.6. *Let* M^k *be an* r*-co-connected* (i.e., $(k-r)$-connected) *closed* k*-dimensional* π*-manifold,* $k \geqq 5$, $r \geqq k/2 + 1$. *Then*

(a) *if* M *bounds a* π*-manifold,* M *is differentiably imbeddable in* E^{2r-1} *with trivial normal bundle*;

(b) *if* $k \not\equiv 2 \bmod 4$ *or* $k \equiv 2 \bmod 4$ *and the* Arf *invariant of* M *is zero,* M *is almost differentiably imbeddable in* E^{2r-1}.

COROLLARY 1. *An* $(l - 1)$*-connected* $(2l + 1)$ *manifold,* $l \geqq 2$, *is almost differentiably imbeddable in* E^{2l+3} *if and only if it is a* π*-manifold, and is differentiably imbeddable if and only if it is the boundary of a* π*-manifold.*

COROLLARY 2. *Let* $m = m(k)$, *be the smallest dimensional euclidean space in which all homotopy* k*-spheres are differentiably imbeddable. Then, if* M^k *is a* π*-manifold, which is at least* $k-(m + 1)/2$*-connected, it is differentiably imbeddable in* E^m, *with fibre homotopically trivial normal bundle.*

REMARK 1. De Sapio has shown that for certain coconnectivities and dimensions, the theorem is best possible.

REMARK 2. An example of Hsiang and Szczarba [22] shows that some π-manifolds M^k, which bound π-manifolds, can be imbedded in E^{2r-1} with normal bundles which are not fibre homotopically trivial, and further give two imbeddings which are not isotopic.

Problem 2. (a) Is Theorem 1.6 best possible for all dimensions and co-connectivities?

(b) Give a classification theory for imbeddings of π-manifolds.

The following question of Steenrod poses the imbedding problem in a somewhat different form:

Problem 3. For each integer $n \geqq 1$, does there exist a compact $(n + 1)$-manifold without boundary in which each n-manifold is imbeddable? There is one problem for each of the categories, topological piecewise linear, differentiable ($n = 1$ is trivial, $n = 2$ is given in Stiefel's thesis). N. E. Steenrod.

Problem 4. If $k > n/2$, and M^n immerses in E^{n+k}, does M imbed in E^{n+k+r}, r not too big? M. W. Hirsch.

Problem 4′. Let n and m be the least integers such that a differentiable manifold M is immersible in n-space and imbeddable in m-space. Write divergence $M = m - n$. When is div $M = 0$? Examples are the ordinary spheres, complex and quaternionic spaces a dimension a power of 2. Is div $CP_n =$ div $HP_n = 0$ all n? B. J. SANDERSON.

Imbeddings of spheres

Theorem 1.3′ of Haefliger establishes and classifies the imbeddings of homotopy spheres in euclidean space for the meta-stable range. Theorem 1.6 of De Sapio establishes the imbeddings of homotopy spheres which are boundaries of π-manifolds, in codimension two (a result originally due to Hirsch). The problem of imbedding homotopy spheres below the meta-stable range will be further discussed in § 2.

In the meta-stable range, any two imbeddings of a homotopy sphere are isotopic, and hence the normal bundle is uniquely determined. Consequently, the ordinary sphere, or one which is a π-boundary must have a trivial normal bundle. Massey [34] established that the normal (sphere) bundle of a homotopy sphere in any euclidean space must be fibre homotopically trivial. Kervaire-Milnor [28] and Adams [2] established that the stable normal bundle is trivial. On the other hand, Hsiang, Levine, and Szczarba exhibited a sphere imbedded in the meta-stable range with a non-trivial normal bundle. (*On the normal bundle to a homotopy sphere embedded in euclidean space*, to appear in Topology). In general they proved:

THEOREM 1.7. *A sphere bundle over an n-dimensional sphere may be the normal bundle of a homotopy sphere imbedded in the meta-stable range if and only if it is fibre homotopically trivial and stably trivial* ($n \neq 4k+2$, *if the* Arf *invariant of a closed manifold is not zero*).

Problem 5. Classify the normal bundles of homotopy spheres in the meta-stable range.

The above theorem reduces part of this problem to a homotopy question. One such specific homotopy problem is described below.

Problem 6. Let i_n be a generator of $\pi_n(S^n)$ and γ_n a generator of $\pi_{n+3}(S_n)$. If $n = 13$, the Whitehead product $[i_n, \gamma_n] = 0$. If this could be shown to be the case for $n > 13$, $n \equiv 5 \bmod 8$, it would follow that the element of $\Delta_{\#}\gamma_n$ in $\pi_{n+2}(\mathrm{SO}(n))$ [$\Delta_{\#}$ is the connecting homomorphism in the homotopy sequence of the fibering $\mathrm{SO}(n) \subset \mathrm{SO}(n+1) \to S^n$] represents an $(n-1)$-sphere bundle over S^{n+3} which is fibre homotopically trivial. Since the bundle is also stably trivial, it can be realized as the normal bundle of an imbedding of an exotic $(n+3)$-sphere in E^{2n+3}; so in particular, $\theta_{8q} \neq 0$ (θ_{8q} the group of homotopy $8q$ spheres).

Hsiang and Szczarba.

Other problems connected with the imbeddings of spheres in manifolds are:

Problem 7. Let $N(k, n)$ denote the set of differentiable or combinatorial knots $S^k \subset E^{n+k}$. Is there a stability theorem $N(k, n) \cong N(k+1, n)$ for large k? Such a theorem is suggested by Kervaire's result on the group of knots $N(k, 2)$.

A. Dold.

This appears to be answered negatively by recent results of J. Levine.

Problem 8. Which classes in $H_2(S^2 \times S^2)$ are representable by smoothly imbedded 2-spheres?

Conjecture 1. Only classes $px + qy$ where p or q is 0 or 1 (pessimist).

Conjecture 2. The above together with classes $px + qy$ where p and q are relatively prime (optimist).

REMARKS. When p or $q = 0, 1, -1$ there exists a smooth sphere. When $p \equiv q \equiv 2 \pmod 4$, there exists no sphere. First unsolved case $2x + 3y$. In $(S^2 \times S^2) \# (S^2 \times S^2)$ any primitive class can be represented. Kervaire-Milnor [29] have shown all classes are represented by simplicial imbeddings.

C. T. C. Wall.

Problem 9. Let $x \in H_2(M^4)$ be primitive, M a 1-connected closed smooth manifold. Consider imbedded smooth 2-spheres representing x, with 1-connected complement. Are they all diffeotopic? C. T. C. Wall.

Problems on projective spaces

Problem 10. The existence problem for immersions of real projective spaces RP_n in euclidean space is solved for $n < 12$; $n = 2^q + s$, $0 \leqq s \leqq 3$; $n = 2^q + 2^r + 3$, $q > r$. Can one find an algebraic formula which fits the known results?

B. J. Sanderson.

Problem 11. The following is known: HP_n immersible in $(4n + k)$-space implies RP_{4n+3} immersible in $(4n + 3 + k)$-space.

Is the converse true? (It is known when n is a power of two or the sum of two powers of two.) B. J. Sanderson.

The following is suggested by results on projective spaces.

Problem 12. Let $\alpha(n)$ be the number of 1's in the dyadic expansion of n. Are all n-dimensional differentiable manifolds embeddable in $(2n - \alpha(n) + 1)$-space? B. J. Sanderson.

2. Combinatorial and Differentiable Structures on Manifolds

In § 1, we discussed problems concerned with the category of smooth manifolds and smooth maps. One has similar problems for the category of piecewise linear (or combinatorial) manifolds and piecewise linear maps, and the category of topological manifolds and continuous maps. (There are also various in-between objects, such as homotopy manifolds, homology manifolds, Lipschitz manifolds, C^r-manifolds; and stronger structures such as real analytic, complex analytic, and algebraic. These will largely be ignored here). Even more interesting is the relation between these categories.

There is now a fair body of knowledge concerning combinatorial manifolds and their relation with smooth manifolds.

DEFINITION 2.1. A function $f: K \to L$ between locally finite simplicial complexes is *piecewise linear* (PL) if there exists a rectilinear subdivision K' of K so that f maps each simplex of K' linearly into a simplex of L.

REMARKS. (a) Any open subset of a locally finite simplicial complex can be triangulated so that the inclusion map is piecewise linear.

(b) Given any PL map $f: K \to L$ between finite complexes, there exists rectilinear subdivisions K' of K and L' of L, such that f is simplicial with respect to these subdivisions.

DEFINITION 2.2. A simplicial complex M will be called a PL *manifold* if each point has a neighborhood U which is PL homeomorphic to R^n.

REMARKS. (a) Every combinatorial manifold in the sense of Whitehead [57] is a PL manifold.

(b) Every compact PL manifold is a combinatorial manifold.

We begin with combinatorial analogues of Haefliger's imbedding theorems (1.3′).

Let M and Q be closed PL manifolds.

THEOREM 2.1 (Irwin [24]). *Suppose* $\dim Q - \dim M = r \geqq 3$, *where M is r-co-connected and Q $(2r-1)$-co-connected. Then, given a continuous map $f: M \to Q$, there exists an imbedding $g: M \subset Q$ homotopic to f.*

DEFINITION 2.3. Let Q be a PL manifold. A PL isotopy of Q is a piecewise linear homeomorphism $h: Q \times I \to Q \times I$ such that

(1) h is level preserving, i.e., commutes with projection onto I.

(2) h starts with the identity, i.e. $h_0 = 1$.

A PL manifold M is said to be *unknotted* in Q if for any two imbeddings $f, g: M \subset Q$ which are homotopic, there exist a PL isotopy h of Q such that $h_1 f = g$.

THEOREM 2.2 (Unknotting theorem of Zeeman [61], [62]). *Suppose* $\dim Q - \dim M = r + 1 \geqq 3$. *Then M unknots in Q, provided that M is r-co-connected and Q is $2r$-co-connected.*

COROLLARY. *S^n unknots in S^{n+r}, $r \geqq 3$.*

Various examples of Zeeman and Hudson [23] show that this Theorem is best possible in general. In particular, there are combinatorial n-spheres that knot in S^{n+2}.

The above corollary contrasts sharply with the differentiable case, where

Haefliger [17] has shown that there exist smooth submanifolds M^{4k-1} in S^{6k}, such that M^{4k-1} is diffeomorphic to S^{4k-1}, but the pair (S^{6k}, M^{4-1}) is not diffeomorphic to the standard pair (S^{6k}, S^{4k-1}).

This raises the question as to the relation between combinatorial and differential structures, and between PL-imbeddidgs and smooth imbeddings. This theory has been developed by S. S. Cairns and J. H. C. Whitehead [57]; and by J. Munkres (*Obstructions to extending diffeomorphisms*, Proc. Amer. Math. Soc., April 1964) using a somewhat different approach. Before stating some recent results in the theory, we need

DEFINITION 2.4. A *smoothing* of a PL-manifold K^n is a pair (M^n, f), where $f: K \to M$ is a piecewise regular homeomorphism (i.e., for each simplex σ of some rectilinear subdivision of K, $f \mid \sigma$ is differentiable and the jacobian is of maximum rank).

Two smoothings (M_1, f_1), (M_2, f_2) are called *concordant* if there exists a smoothing of $K \times I$, which induces the same differentiable structure on $K \times (0)$ and $K \times (1)$ as (M_1, f_1) and (M_2, f_2), respectively. Concordance implies diffeomorphism (although not trivially), but the converse is known to be false.

THEOREM 2.3 (Cairns-Hirsch [35]). *Let K be a PL-manifold without boundary, and let (f, M) be a smoothing of $K \times R^m$. Then there exists a smoothing (g, N) of K, unique up to concordance with the following property*:

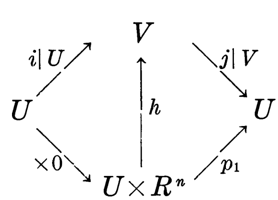
There is a piecewise-regular isotopy H: $K \times R^m \times I \to M \times I$ with $H_0 = f$ and $H_1 = \varphi \circ (g \times identity)$, where φ is a smooth imbedding of $N \times R^n$ in M.

This theorem has been generalized from products to vector bundles by Lashof-Rothenberg [32]. But we need Milnor's [37] concept of microbundles to state the result.

DEFINITION 2.5. A PL-*microbundle* $\mathfrak{E}$ of dimension n is a diagram

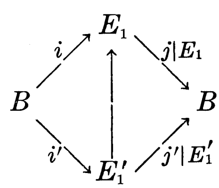
$$B \xrightarrow{i} E \xrightarrow{j} B\ ,$$

where B and E are locally finite simplicial complexes and i, j are PL maps satisfying: For each $b \in B$, there exists neighborhood U of b and a PL homeomorphism h of $U \times R^n$ onto a neighborhood V of $i(b)$ so that the diagram



is commutative. Here p_1 is projection onto the first factor and $\times 0$ is the map $u \to (u, 0)$. (Note this implies $ji\colon B \to B$ is the identity). B is called the *base space*, E the *total space*, i the *injection*, and j the *projection* of $\mathfrak{S}$.

A second PL-microbundle $\mathfrak{S}'\colon B \xrightarrow{i'} E \xrightarrow{j'} B$ over the same base space is *isomorphic* to $\mathfrak{S}$ (written $\mathfrak{S}' \simeq \mathfrak{S}$) if there exist neighborhoods E_1 of $i(B)$ and E_1' of $i'(B)$, and a PL-homeomorphism $E_1 \to E_1'$ so that the diagram



is commutative.

Example 1. The *trivial bundle* $\mathfrak{S}^n\colon B \xrightarrow{\times 0} B \times R^n \xrightarrow{p_1} B$.

Example 2. The *tangent microbundle* π_M of a PL-manifold M

$$\tau_M\colon M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M .$$

DEFINITION 2.6. Consider PL-manifolds $M \subset N$ with PL-inclusion map $i\colon M \to N$. M has a *normal microbundle* ν in N if there exists a neighborhood U of M in N and a retraction $j\colon U \to M$ so that the diagram

$$\nu\colon M \xrightarrow{i} U \xrightarrow{j} M$$

is a PL-microbundle over M.

REMARK. All these definitions which have been given in the category of PL-manifolds and maps, make sense in the category of topological manifolds and continuous maps. See [38].

Just as for vector bundles one may define *Whitney sum* of microbundles and hence *stable equivalence* of microbundles (the definitions are formally the same). Stably, normal microbundles behave just like normal vector bundles. This is summarized in the stable tubular neighborhood theorem for PL-manifolds [32].

THEOREM 2.4. *Let $f_i\colon K \to V$, $i = 0, 1$, be* PL-*imbeddings, where K and V are combinatorial manifolds without boundary, then*

(a) *$f_i(K)$ has a normal microbundle μ_i in $V \times R^n$, n sufficiently large*; i.e., *there exists* PL-*homeomorphisms $F_i\colon E(\mu_i) \to V \times R^n$ such that F_i restricted to the zero section is $f_i\colon K \to V \times 0$.*

(b) *If $h\colon K \times I \to V \times I$ is any* PL-*isotopy between f_0 and f_1, there exists a* PL-*isotopy $H\colon E(\mu_0) \times R^k \times I \to V \times R^{n+K} \times I$, k sufficiently large, covering*

h, such that $H_0 = F_0 \times 1_k$, and H_1 is a microbundle equivalence of $E(\mu_0) \times R^k$ into $F_1(E(\mu_1)) \times R^k$.

Problem 13. Do normal microbundles always exist in some stable or metastable range?

According to Zeeman [62], this would follow if any PL-homeomorphism $\varphi: S^p \times S^q \to S^p \times S^q$, sufficiently close to the identity is isotopic to the identity.

The existence of at least one normal microbundle for S^n in S^{n+k}, $k > 2$, follows from Zeeman's unknotting theorem.

Problem 14. Is there some range of codimensions for which an isotopy between two imbeddings of M in N may be covered by an isotopy between their normal microbundles, assuming such exist?

Problem 15. Does there exist a non-trivial normal microbundle for a sphere in a sphere (or euclidean space)?

DEFINITION 2.7. A *piecewise differentiable structure* on an n-plane bundle $\mathfrak{S}$ over a locally finite simplicial complex K, is a presentation of the total space $E(\mathfrak{S})$ as the union of $\sigma \times R^n$, $\sigma \in K$, with the coordinate transformations from σ to $\partial_i \sigma$ being differentiable over each simplex of a rectilinear subdivision of σ.

A vector bundle, together with a piecewise differentiable structure is called a *piecewise differentiable vector bundle* (pd-vector bundle).

It follows from the definition, that there exists a rectilinear subdivision K_1 of K, such that $\pi^{-1}(\sigma)$, $\sigma \in K_1$, has a well defined differentiable structure. We will say that $\mathfrak{S}$ is *differentiable* over K_1.

Two pd-bundles $\mathfrak{S}_1$ and $\mathfrak{S}_2$ over K are called *equivalent* if there exists a vector bundle map $\varphi: E(\mathfrak{S}_1) \to E(\mathfrak{S}_2)$ such that $\varphi \mid \pi_1^{-1}(\sigma): \pi_1^{-1}(\sigma) \to \pi_2^{-1}(\sigma)$ is differentiable over each simplex of a rectilinear subdivision of σ.

It is shown in [32], that two pd-vector bundles $\mathfrak{S}_1$ and $\mathfrak{S}_2$ over K are equivalent if and only if they are equivalent as vector bundles.

DEFINITION 2.8. *Triangulation* of pd-*vector bundles*. A triangulation of a pd-vector bundle $\mathfrak{S}$ over K, is a PL-microbundle μ over K and a fibrewise map $\varphi: E(\mu) \to E(\mathfrak{S})$, preserving the zero section, and such that φ is a piecewise regular homeomorphism over each simplex σ of a rectilinear subdivision K_1 of K where $\mathfrak{S}$ is differentiable over K_1. We denote such a triangulation by the pair (μ, φ).

It is shown in [32] that, if (μ_i, φ_i), $i = 1, 2$, are triangulations of $\mathfrak{S}$, there exists a microbundle equivalence $\psi: E(\mu_1) \to E(\mu_2)$ such that $\varphi_2\psi$ is isotopic to φ_1 through triangulations.

THEOREM 2.5 (Smoothing of imbeddings). *Let $i: K \to L$ be a* PL-*imbedding where K and L are* PL-*manifolds without boundary. Let $f: L \to V$ be a smooth triangulation. Then there exists a piecewise regular homeomorphism $h: V \to V$* (i.e., *hf is piecewise regular*) *such that $hfi(K)$ is a smooth submanifold with normal vector bundle $\mathcal{E}$, if and only if $i(K)$ has a normal microbundle μ which triangulates $\mathcal{E}$.*

COROLLARY (Milnor [37]). *K is smoothable if and only if its stable normal microbundle (in euclidean space) triangulates a vector bundle.*

In particular, an exact sequence of Lashof and Rothenberg using the above theorem, largely reduces the question of classifying smoothly imbedded homotopy spheres in spheres to the existence and classification of normal microbundles for spheres in spheres.[1]

In [37] Milnor has defined a semi-simplicial group complex PL_n which plays the role of a structural group for n-dimensional microbundles. A k-simplex in PL_n being a germ of a PL-microbundle equivalence $f: \Delta_k \times R^n \to \Delta_k \times R^n$, Δ_k the standard k-simplex. In order to compare microbundles with vector bundles, it is useful to introduce an ss-analogue O_n of the orthogonal group which acts as a structural group for pd-vector bundles. A k-simplex in O_n is a pd-vector bundle equivalence $f: \Delta_k \times R^n \to \Delta_k \times R^n$. O_n is an ss-group complex of the same homotopy type as the orthogonal group on n-dimensional euclidean space. Finally, one introduces the ss-complex (not a group complex) PD_n; a k-simplex in PD_n being a germ of a topological microbundle equivalence $f: \Delta_k \times R^n \to \Delta_k \times R^n$, where f is a piecewise regular homeomorphism with respect to the product triangulation of $\Delta_k \times R^n$. Then $\mathrm{PL}_n \subset \mathrm{PD}_n$, and O_n acts freely in the right of PD_n. It is shown in [32], that $i: \mathrm{PL}_n \to \mathrm{PD}_n$ is a homotopy equivalence. Let PL, PD, O, be the direct limits of PL_n, PD_n, O_n, under the natural inclusions $\mathrm{PL}_n \to \mathrm{PL}_{n+1}$, $\mathrm{PD}_n \to \mathrm{PD}_{n+1}$, $O_n \to O_{n+1}$. Then PD is the same homotopy type as PL, and O acts freely in the right of PD.

The following theorem has been announced by Mazur, a proof is given in [32].

THEOREM 2.6. *Let K be a* PL-*manifold which admits a smoothing. Then the concordance classes of smoothings of K are in one to one correspondence to the homotopy classes $[\tilde{K}, \mathrm{PD}/O]$ of* ss-*maps of $\tilde{K}$ into* PD/O, *where $\tilde{K}$ is the associated* ss-*complex to K.*

In [32] it is shown that PD/O is a homotopy commutative H-space, and that

[1] J. Levine has given a complete theory of imbeddings of spheres in spheres for codimension greater than 2, using differentiable methods only.

the concordance classes of smoothing of K form an abelian group with any prechosen smoothing as unit class. In particular, $\pi_i(\text{PD}/O) \simeq \Gamma_i$, the group of differentiable structures on S^i. By an argument of Milnor, $\pi_i(O)$ maps monomorphically into $\pi_i(\text{PD})$. Hence,

THEOREM 2.7 (Hirsch-Mazur). *There is an exact sequence*:

$$0 \longrightarrow \pi_i(O) \longrightarrow \pi_i(\text{PL}) \longrightarrow \Gamma_i \longrightarrow 0 \ .$$

REMARK. J. Cerf [10] has recently proved $\Gamma_4 = 0$ and hence $\Gamma_i = 0$, $i < 7$.

Problems 11 and 12 for spheres may be stated as a homotopy problem.

Problem 16. For each i, is there some m, such that $\pi_i(\text{PL}_n) \rightarrow \pi_i(\text{PL})$ is an isomorphism, $n \geqq m$?

Problem 17. If a k-dimensional microbundle μ^k has a non-zero cross-section, is μ^k the Whitney sum of a microbundle ν^{k-1} and a trivial bundle?

M. Hirsch.

Problem 18. Let M be a topological or combinatorial manifold with vanishing Euler characteristic. Does the tangent microbundle of M have a non-zero cross-section?

This problem has been solved by R. Brown and E. Fadell: *Non-singular path fields on compact topological manifolds* (to appear).

Problem 19. Let θ_n act on the diffeomorphism classes of manifolds by connected sum. What determines this action? Does it depend on more than the tangential homotopy equivalence class at the manifold? Similar questions for the action of the subgroup $\theta^n(\partial\pi)$ of smooth homotopy spheres which are boundaries of parallelizable manifolds. W. Browder.

Contributions to this problem have been make by Tamura [52], C.T.C. Wall [55], W. Browder [8], and Kosinski. Also at the Seattle Conference, Munkres proved the following:

THEOREM 2.7. *Consider the pairing* $\pi_{m-1}(\text{SO}(n-m)) \otimes \Gamma_{n-m+1} \longrightarrow \Gamma_n$. (See discussion of diffeomorphisms of S^n below). *Let Σ^n be an exotic sphere lying in the image under this pairing, say $\Sigma^n = \alpha \otimes \gamma$. Let M^n contain a smoothly imbedded S^n with α the characteristic class of the normal bundle. Then $M \# \Sigma$ is diffeomorphic to M.*

In fact, $M \# \Sigma$ is concordant to M, in the following sense: If $f\colon K \rightarrow M$ is a smoothing of K, there is a smoothing $g\colon K \rightarrow M \# \Sigma$ such that the induced differentiable structures on K agree outside a combinatorial ball; the assignment $(M, f) \rightarrow (M \# \Sigma, g)$ defines an action of $\theta_n = \Gamma_n$ on concordance classes. Under the preceding hypotheses, the element Σ acts trivially on the concordance class of (M, f).

Problem 20. Is $\theta_n \cong \theta_n(\partial\pi) \oplus \theta_n/\theta_n(\partial\pi)$. C. T. C. Wall.

REMARK. The group Γ of smooth combinatorial n spheres (under connected sum) is isomorphic to the group of diffeomorphisms of S^{n-1} modulo those which can be extended over B^n. The group θ_n is the group of h-cobordism classes of smooth homotopy n-spheres. $\theta_n \simeq \Gamma_n$, $n \neq 3, 4$.

Problem 21. Is an orientation preserving diffeomorphism of S^n, which is extendable to a diffeomorphism of the ball B^{n+1}, diffeotopic to the identity?

S. Smale.

This has been answered affirmatively for $n \geqq 8$ by J. Cerf.

Problem 22. Let Σ^n be a homotopy sphere, bounding M^{n+1}, M^{n+1} not parallelizable but, for example, highly connected. How can one calculate the class of Σ^n in $\theta_n/\theta_n(\partial\pi) \subset \operatorname{Coker} J_n$? C. T. C. Wall.

Topological manifolds and imbeddings

Very little is known about classifying topological manifolds. However, the following weak analogue of the corollary to Theorem 2.5 follows from the results of Browder [7].

THEOREM 2.7. *Let M be a closed topological n-manifold, and suppose that its tangent microbundle is stably equivalent to a vector bundle $\mathfrak{E}$. If M is the homotopy type of a finite complex, then:*

(1) *if n is odd, M is the same homotopy type as a smooth closed n-manifold;*

(2) *if $n = 4k$, and if the index of M is equal to the Hirzebruch number defined by the Pontrjagin classes of $\mathfrak{E}$, M is the same homotopy type as a smooth closed n-manifold;*

(3) *if $n = 4k + 2$, then M is the same homotopy type as a closed combinatorial n-manifold.* (*Which is differentiable except possibly at one point.*)

We are equally ignorant as to the classification of topological imbeddings, even of smooth manifolds (cf. corollary of Theorem 1.4). However, in the case of locally flat *imbeddings* the situation is somewhat better.

DEFINITION 2.8. A topological imbedding i: $M^m \to N^n$ is called *locally flat* if, for each $x \in M \subset N$, there exists a neighborhood U of x in M, and a homeomorphism h: $U \to R^n$ such that $h/U \cap M \subset R^m \subset R^n$.

REMARKS. (1) It follows from Zeeman's unknotting theorem that, if i is a combinatorial imbedding of a combinatorial manifold, it will be locally flat provided the codimension is at least three.

(2) The existence of a topological normal microbundle implies the imbed-

ding is locally flat (cf. Def. 2.6). Hence Milnor's theorem on the existence of a topological normal microbundle stably [38], implies that $i: M^m \to N^n \subset N \times R^s$ is locally flat if s is sufficiently large.

Stallings unknotting theorem [49]

THEOREM 2.8. *A locally flat imbedding of S^n in S^{n+k} is topologically unknotted provided $k \geqq 3$, or if $k = 2$, and the complement of S^n in S^{n+k} has the homotopy type of S^1.*

In the stable range we have Greathouse [14], Gluck [13].

THEOREM 2.9. *Let f be a locally flat imbedding of the closed combinatorial manifold M^k into the combinatorial manifold V^n. If $2k + 2 \leqq n$, then for each $\varepsilon > 0$ there is an ε-push h of $(V^n, f(M^k))$ such that $hf: M^k \to V^n$ is piecewise linear.*

COROLLARY. *Any topological imbedding of a closed combinatorial manifold in euclidean space is stably unknotted.*

Milnor has given the following as his candidates for the toughest and most important problems in geometric topology.

Problem 23. Let M^3 be a homology 3-sphere with $\pi_1 \neq 0$. Is the double suspension of M^3 homeomorphic to S^5?

Problem 24. Is simple homotopy type a topological invariant?

Problem 25. Can rational Pontrjagin classes be defined as topological invariants?

Problem 26 (Hauptvermutung). If two PL-manifolds are homeomorphic, does it follow that they are PL-homeomorphic?

Problem 27. Can topological manifolds be triangulated?

Problem 28. The Poincaré hypothesis in dimensions 3, 4.

Problem 29. (The annulus conjecture). Is the region bunded by two locally flat n-spheres in $(n + 1)$-space necessarily homeomorphic to $S^n \times [0, 1]$?

Discussion of Milnor's problems

Problem 23. Since the double suspension of M^3 is the same homotopy type as S^5, and since any combinatorial 5-manifold the homotopy type of S^5 is homeomorphic to S^5 [47], [62]; the problem reduces to whether the double suspension is a combinatorial manifold. (Also note that it has been shown that $SM^4 = S^5$ for any smooth homotopy 4-sphere.—Hirsch)

Problem 24. In the simply connected case, simple homotopy type is the same as homotopy type. By definition it is a combinatorial invariant. Hence for combinatorial manifolds, an affirmative answer to Problem 26 would imply that homeomorphic combinatorial manifolds are of the same simple homotopy type.

Problem 25. Milnor [39] has given an example which shows that the answers to problems 25, 26, and the Hurewicz conjecture (Problem 31) cannot all be affirmative.

Problem 26. Milnor has given an example to show that the Hauptrermutung is not true for simplicial complexes. Smale and Barden have shown it is true for 1-connected closed PL 5-manifolds.

Problem 27. Wall [56] has shown that not every CW-complex which is dominated by a finite complex is homotopy equivalent to a finite complex. However, the following question still remains open, in particular for topological manifolds.

Problem 30. Is a compact ANR the homotopy type of a finite complex? (Milnor has demonstrated this in the simply connected case.)

Problem 28. Smale and Stallings [47], [50] have proved the Poincaré hypothesis, i.e., that every combinatorial n-manifold of the same homotopy type as an n-sphere is homeomorphic to S^n, $n \geqq 5$; in fact PL-homeomorphic.

Problem 29. For smooth spheres, smoothly imbedded, Smale's h-cobordism theorem [47] gives a homeomorphism for $n \geqq 5$ which is actually a diffeomorphism. Similarly, Mazur's combinatorial h-cobordism theorem [36], gives a PL-homeomorphism, $n \geqq 5$, for combinatorially imbedded spheres. Stallings has outlined a proof for combinatorially imbedded spheres, n arbitrary but the homeomorphism not necessarily combinatorial, using results of M.H.A. Newman.

Problem 31 (Hurewicz conjecture). Are two simply connected closed smooth manifolds of the same homotopy type homeomorphic?

The following is closely related to Problem 31.

Problem 32. If M and M' are compact differentiable manifolds, and if there exists a homotopy equivalence $h\colon M \sim M'$ which is compatible with the tangent bundles (i.e., $h^*\tau_{M'} \simeq \tau_M$), is $M * M - \Delta \sim M' * M' - \Delta'$, where $*$ denotes the symmetric square, and Δ, Δ' the diagonals?

Similarly, if there is an isomorphism $\varphi\colon H^*M = H^*M'$ which maps the characteristic classes of M into those of M', is $H^*(M * M - \Delta) = H^*(M' * M' - \Delta')$? A. Dold.

C. Weber (Geneva) has some results in this direction.

Problem 33. It is known that if M and M' are homeomorphic smooth n-

manifolds and M has a tangent k-field for $k < (n-1)/2$, then so does M'. Is this true if M and M' merely have the same homotopy type? M. Hirsch.

W. A. Southerland has proven this in the cases

(a) n even

(b) $n \equiv 1$ (4), M a spin manifold.

The Arf invariant

In a number of the previous theorems and problems, the question of the Arf invariant of a manifold has arisen. We give a brief description of this invariant together with the present status of the problem.

The algebraic concept: A quadratic form in a Z_2-vector space V takes the form of a function $\varphi: V \to Z_2$ satisfying

$$\varphi(x+y) = \varphi(x) + \varphi(y) + \psi(x, y)\ , \qquad x, y \in V\ ,$$

where $\psi: V \times V \to Z_2$ is bilinear and symmetric. If ψ is non-degenerate (i.e., $\psi(x, y) = 0$ all y implies $x = 0$), then one may choose a basis $e_1, \cdots, e_n, e'_1, \cdots, e'_n$ for V, such that $\psi(e_i, e'_i) = 1$, $i = 1, \cdots, n$, and ψ for all other pairs is zero. This is called a symplectic basis. Define the Arf invariant of φ by:

$$\operatorname{Arf} \varphi = \textstyle\sum_{i=1}^{n} \varphi(e_i)\varphi(e'_i)\ .$$

Then Arf proved that two such quadratic forms over V are equivalent under change of basis if and only if they have the same Arf invariant.

For a compact $(k-1)$-connected $2k$-manifold M with or without boundary, Kervaire [27] defines $\varphi(x)$, $\varphi: H^k(M, bM) \to H^{2k}(M, bM; \pi_{2k-1}(S^k)) \simeq \pi_{2k-1}(S^k)$, as the first obstruction to the existence of a map $f: (M, bM) \to (S^k, p)$ satisfying $f^*(\gamma) = x$, γ the generator of $H^k(S^k, p)$. Then φ satisfies: $\varphi(x+y) = \varphi(x) + \varphi(y) + [i, i]x \cdot y$, where the last term stands for the image of the class $xy \in H^{2k}(M, bM; z)$ under the coefficient homeomorphism $Z \to \pi_{2k-1}(S^k)$, which carries 1 into the Whitehead product class $[i, i]$. For k odd, $k \neq 1, 3,$; the subgroup of $\pi_{2k-1}(S^k)$ generated by $[i, i]$ is identified with Z_2. Then φ may be considered as a function $\varphi: H^k(M, bM; Z_2) \to Z_2$ satisfying $\varphi(x+y) = \varphi(x) + \varphi(y) + xy$. Then Arf (M) is defined to be Arf (φ). It may be shown that the definition of Arf invariant may be extended to any framed $4n+2$ manifold so as to be an invariant of framed cobordism; i.e., there exists a homomorphism $\Phi: \Omega_{4n+2}^{\text{framed}} \to Z_2$ such that $\Phi(\{M\}) = \operatorname{Arf} M$ for a closed framed $2n$-connected manifold M.

Problem 34. Let $\Phi: \Omega_{4n+2}^{\text{framed}} \to Z_2$ be the Arf invariant as defined by Kervaire. Is $\Phi \equiv 0$?

The following argument of E. Brown (cf. Novikov [42]) may be of some help in this problem:

PROPOSITION 2.10. *Φ: $\Omega^{\text{framed}}_{8n+2} \to Z_2$ may be factored through $\Omega^{\text{spin}}_{8n+2}$.*

Outline of proof. Let $k = 4n + 1$. There is a secondary cohomology operation φ: $H^k(x) \cap \text{Ker Sq}^{k-1} \cap \text{Ker Sq}^{k-1}\,\text{Sq}^1 \to H^{2k}(x)/(\text{Sq}^2 H + \text{Sq}^1 H)$, corresponding to the relations $\text{Sq}^{k+1} = \text{Sq}^2\text{Sq}^{k-1} + \text{Sq}^1(\text{Sq}^{k-1}\text{Sq}^1) = 0$ in $H^k(x)$ such that if $\varphi(x)$ and $\varphi(y)$ are defined, $\varphi(x + y)$ is defined, and

$$\varphi(x + y) = \varphi(x) + \varphi(y) + xy .$$

If M is a 1-connected closed $2k$-manifold such that the Stiefel-Whitney class $w_2 = 0$, φ: $H^k(M) \to H^{2k}(M) \simeq Z_2$. Let Arf (M) = Arf (φ). One may show that this induces a homomorphism Φ_0: $\Omega^{\text{spin}}_{2k} \to Z_2$ which, when composed with the homomorphism $\Omega^{\text{framed}} \to \Omega^{\text{spin}}$ gives Kervaire's Φ.

Problem 35. Is $\Phi_0 = 0$? Is $\Omega^{SU}_{8n+2} \to \Omega^{\text{spin}}_{8n+2} \xrightarrow{\Phi_0} Z_2$ zero? (Ω^{SU} may be easier to calculate than Ω^{spin}).[1] E. Brown.

Kervaire has shown [27] that a closed 4-connected 10-manifold has Arf invariant zero if and only if it is smoothable. By Wall's classification of handle bodies [54], there exists an almost closed 4-connected smooth 10-manifold with Arf invariant non-zero, and hence with boundary an exotic 9-sphere. Putting a cone over the boundary of this 10-manifold we obtain a closed 4-connected combinatorial 10-manifold K^{10}. By the Cairns-Hirsch theorem, K^{10} must have a non-trivial normal microbundle.

Problem 36. Is K^{10} a combinatorial boundary? M. Hirsch.

If K^{10} is not a boundary, it must have a non-trivial mod 2 combinatorial characteristic class, according to the following theorem that Browder, Peterson, and Liulevicius obtained at the Seattle conference.

THEOREM 2.11. *The non-oriented combinatorial cobordism classes N_{PL} are determined by their* mod 2 *characteristic numbers.*

Problem 37. Give a geometric interpretation of these mod 2 characteristic classes of a PL-manifold, and compute $H^*(BPL; Z_2)$.

Browder, Liulevicius, and Peterson have shown that $H^*(BPL; Z_2)$ is free as a module over $H^*(BO; Z_2)$. More explicitly $H^*(BPL; Z_2) \simeq H^*(BO; Z_2) \otimes C$, C Hopf algebra; and $N_{\text{PL}} \simeq N \otimes C^*$ as algebras.

Automorphism groups of S^n

Considering S^n as a smooth manifold, a combinatorial manifold, or a topological manifold, we consider the groups:

G_n = Group of PL-homeomorphisms of S^n onto itself

[1] This problem has been solved affirmatively by Brown and Peterson.

H_n = Group of homeomorphisms of S^n onto itself
D_n = Group of orientation preserving diffeomorphisms of S^n onto itself.

The group of diffeomorphisms of S^n

$D = D_n$ is made into a topological group by considering D to be a subspace of the space of all smooth maps of S^n into S^n with the C^1-topology. Milnor [39] showed that D is not arcwise connected, in general. Novikov [41] announced some further results on the homotopy of D. We present here some results obtained by Munkres and Milnor at the Seattle conference in answer to a problem suggested by Munkres.

First it is easy to show that D_n is the same homotopy type as the product of so_{n+1} and $\mathrm{diff}_c\,(F^n)$ = diffeomorphisms of R^n fixed outside of some compact set.

Now define pairings

(a) $\pi_p(\mathrm{so}_q) \otimes \pi_0\,\mathrm{diff}_c\,R^q \to \pi_0(\mathrm{diff}_c\,R^{p+q})$

(b) $\pi_p(\mathrm{so}_q) \otimes \pi_q(\mathrm{so}_p) \to \pi_0(\mathrm{diff}_c\,R^{p+q})$

as follows: Represent

$\alpha \in \pi_p(\mathrm{so}_q)$ by $f\colon R^p \to \mathrm{so}_q$, f smooth with compact support

$\beta \in \pi_q(\mathrm{so}_p)$ by $g\colon R^q \to \mathrm{so}_p$, g smooth with compact support

$\gamma \in \pi_0\,\mathrm{diff}_c\,R^q$ by $h\colon R^q \to R^q$, h diffeomorphism with compact support.

Write

$F(x, y) = (x, f(x), y)$, $x \in R^p$, $y \in R^q$

$G(x, y) = (g(y)x, y)$

$1 \times h(x, y) = (x, h(y))$.

Then F, G, $1 \times h$ are diffeomorphisms of $R^p \times R^q$ onto itself, and

(a_0) $F(1 \times h)F^{-1}(1 \times h)^{-1}$

(b_0) $FGF^{-1}G^{-1}$

give the above pairings.

The above pairings in turn induce pairings

(a′) $\pi_p(\mathrm{so}_q) \otimes \Gamma_{q+1} \to \Gamma_{p+q+1}$

(b′) $\pi_p(\mathrm{so}_q) \otimes \pi_q(\mathrm{so}_p) \to \Gamma_{p+q+1}$.

These pairings in turn can be shown to correspond to composition in the stable homotopy groups of spheres. And it follows, for example, that:

$$\pi_1(\mathrm{so}_7) \otimes \Gamma_8 \to \Gamma_9\,, \qquad \pi_3(\mathrm{so}_{11}) \otimes \Gamma_{14} \to \Gamma_{17}$$
$$\pi_1(\mathrm{so}_8) \otimes \Gamma_9 \to \Gamma_{10}\,, \qquad \pi_1(\mathrm{so}_{13}) \otimes \Gamma_{14} \to \Gamma_{15}\,,$$

are non-trivial.

The diffeomorphisms a_0 and b_0 above may also be considered as representing an element of $\pi_p\,\mathrm{diff}_c\,R_q$, and for example (a) may be factored:

$$\pi_p(\mathrm{SO}_q) \otimes \pi_0 \operatorname{diff}_c R^q \longrightarrow \pi_p \operatorname{diff}_c R^q \longrightarrow \pi_0 \operatorname{diff}_c R^{p+q} .$$

Hence the above give non-trivial elements of the higher homotopy groups of D_{p+q} not coming from SO_{p+q+1}.

The group of homeomorphisms of S^n

H is made into a topological group by giving it the compact open topology. Since Kister [30] and Mazur have shown that microbundles are equivalent to fibre bundles, it follows that stably any topological microbundle is equivalent to a sphere bundle with group H. Consequently, the question (*Problem* 25) as to whether rational Pontrjagin classes are topological invariants is equivalent to

Problem 38. Is $i^* : H^*(H_n, Q) \longrightarrow H^*(\mathrm{SO}_{n+1}, Q)$ onto, when Q is the rationals and $i : \mathrm{SO}_{n+1} \longrightarrow H_n$ the inclusion?

For recent results concerning the group of homeomorphisms of S^n and other manifolds, see M. Brown and Gluck [9].

The PL-automorphism groups

Let Σ be a fixed finite simplicial complex (for example a simplex boundary), and let G denote the group of all piecewise linear homeomorphisms of Σ. Then G can be topologized in three different ways, as follows, thus yielding three different spaces G_1, G_2, G_3:

First topology (Stasheff). Let G_1 denote G with the compact open topology.

Second topology (Wall). For each integer k, let $G^{(k)}$ denote the subspace of G_1 consisting of those PL-homeomorphisms of Σ which are simplicial with respect to rectilinear subdivisions of Σ having at most k-vertices. Now topologize G as the direct limit of the $G^{(k)}$, and call the result G_2.

Third topology (Milnor). It is believed that G can be given the structure of an infinite simplicial complex G_3 with the fine topology in such a way that a map $f : K \longrightarrow G_3$ is piecewise linear if and only if the associated map

$$(x, y) \longrightarrow \bigl(x, f(x)y\bigr)$$

from $K \times \Sigma$ to itself is peicewise linear. (Here K denotes an arbitrary simplicial complex). Furthermore, it is believed that this simplicial structure in G is essentially unique.

Problem 39. Do the natural maps $G_3 \longrightarrow G_2 \longrightarrow G_1$ provide homotopy equivalence?

Problem 40. It is conjectured that G_2, G_3 are topological groups.

Besides the above automorphism groups, the H-space F_n of maps of degree one of S^n onto itself (with the compact open topology) has played an important role in differential topology. For fibre spaces with fibre a homotopy sphere, it plays the role that a structural group does for fibre bundles. Letting $F =$ inj Lim F_n, the inclusion $O \to F$ induces a map $BO \to BF$ of the universal base spaces, which on homotopy groups is essentially the J-homomorphism.

Although the cohomology of F mod p has been computed, very little is known about the cohomology of BF except that $H^*(BF; Z_2) \simeq H^*(BO; Z_2) \otimes D$; $H^*(BF; Z_p)$ is free as a right module over $A/(\delta^*)$.

Problem 41. Compute the Hopf algebra structures of H_*F_n, H_*F, or better of $H_*(BF; Z_p)$ and the induced homomorphism $H^*(BF; Z_p) \to H^*(BO; Z_p)$, p a prime. J. Milnor.

Problem 42. Milnor claims that $H^{11}(BF; Z_3) \simeq Z_3$. The ordinary methods for constructing fibre homotopy invariants do not appear to construct this characteristic class. How can we construct it? J. F. Adams.

H-spaces

The study of Lie groups and H-spaces has been strongly motivated by problems in differential topology. Conversely, the study of finite dimensional H-spaces has been a motivating factor in differential topology. W. Browder proved [7]:

THEOREM 2.12. *A 1-connected H-space X, which is the same homotopy type as a finite complex, is the homotopy type of a closed smooth manifold.* (*Except possibly when the homological dimension is* $4k + 2$).

Problem 43. If a 1-connected manifold M is the homotopy type of a topological group is it homeomorphic to a Lie group? W. Browder.

REMARK. A. D. Wallace has shown that a smooth manifold with an associative differentiable multiplication is a Lie group.

Problem 44. It would be interesting to get some fresh examples of H-manifolds; i.e., manifolds which can be given the structure of an H-space. The known examples are the Lie groups, the 7-sphere, the real projective 7-space, and combinations of these. How can one construct H-manifolds which are different from these either in the topological sense or in the sense of homotopy type (cf. 39 above)? (There are also problems in differential topology that arise.)

One possibility is to look at the total spaces of sphere bundles over spheres. These have been considered by J. F. Adams [3] and I. M. James

[25]. Among the questions left open by this work are whether any 7-sphere bundles over S^{11} or S^{15} can be H-spaces. Two examples stand out. One is the Stiefel manifold of 2-frames in quaternionic 3-space, which is a 7-sphere bundle over S^{11}. The other is the Stiefel manifold of 2-frames in octonionic 2-space, which is a 7-sphere bundle over S^{15}. They do not have the same homotopy type as any known examples of H-spaces.

Again consider the twelve different classes of 3-sphere bundles over S^7 which have cross-sections. Of these twelve, four have trivial fibre homotopy type and so are the same homotopy type as $S^3 \times S^7$, and consequently are H-manifolds. One is the product bundle, homeomorphic to $S^3 \times S^7$. Are the other three homeomorphic to $S^3 \times S^7$? (Two of the classes are weakly equivalent and therefore homeomorphic to each other.) I. M. James.

The results of Browder and other authors have shown a strong connection between multiplicative properties of an H-space and properties of its homology and cohomology rings.

Problem 45. Does existence of p-torsion (p an odd prime) in the cohomology of an H-space G imply $H_*(G; Z_p)$ is not commutative? This has been checked for Lie groups G except $H_*(E_8, Z_5)$. W. Browder.

Problem 46. What can be said of the action of the mod p Steenrod algebra on the mod p cohomology of topological groups. E. Thomas.

Problem 47. What can be said of the action of the Steenrod squares on the mod 2 cohomology of (finite dimensional) H-spaces with non-primitively generated mod 2-cohomology (Exs. E_6, E_7, E_8). (For the primitive case see E. Thomas [53]). E. Thomas.

3. ALGEBRAIC TOPOLOGY

Although the Seattle Conference emphasized geometric topology, a number of papers in algebraic topology were presented, and a number of problems were raised. In any case, it is abundantly clear from the preceding discussion that it is impossible to divorce these two aspects of topology. We make no attempt to summarize the situation in algebraic topology. For the most recently developed branch of the theory, i.e., K-theory, we refer the reader to the excellent summaries presented at the conference by F. Hirzebruch and J. F. Adams.

Homology and fibrations

Problem 48. Relate occurrence of torsion in the homology of fibre, base, and total space for a fibration of H-spaces. In particular, if $H^*(B)$(or $H^*(F)$) has no torsion and $H^*(E) = 0$, does this imply that $H^*(F)$(or $H^*(B)$) has torsion of at most order p for each prime p? Assume B, F, E connected. This is equivalent to $E_2 = E_\infty$ in the Bockstein spectral sequence.

For example, the loop space of a Lie group has no torsion, and the (simply connected) Lie group has no p^2 torsion. The loop space of a sphere has no torsion, and the double loop space has no p^2 torsion. W. Browder.

REMARK. Adams has given a counter-example to Browder's conjecture in the case that it is not a fibration of H-spaces.

Problem 49. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibre space, and suppose we consider cohomology with Z_q coefficients (q a prime). Then $H^*(E)$ is a graded commutative algebra over the Steenrod algebra A_q (in the sense of Steenrod), and also over $R = H^*(B)/\text{Ker}\, p^*$ (in the usual sense). Suppose that $H^*(F) = U(X)$, the free graded commutative algebra over A_q generated by an A_q-submodule X of $H^*(F)$, and the elements of X are transgressive.

The problem is to give sufficient conditions that imply $H^*(E) = U_R(N)$, the free graded commutative $A_q - R$ algebra generated by an $A_q - R$ submodule N of $H(E)$. Massey and Peterson have shown that the result holds in two special cases with $q = 2$. W. Massey and F. Peterson.

Problem 50. Let S_n denote the symmetric group on n-perameters. Let A_n denote the n-fold join of A_n with itself, with S_n acting by permutation. What invariants of A are necessary to determine the homology of the orbit space of A_n under the action of S_n? I. James.

Problem 51. Let π be a finite group of automorphisms of a finitely generated group G, and let n be a positive integer. Does there exist a finite connected complex K such that $H_n(K; Z) \simeq G$, $H_q(K; Z) = 0$ for $q \neq n$, and π acts as an automorphism group of K so as to induce the prescribed automorphisms in $H_n = G$? A related problem is obtained by requiring π to act freely in K. (In this case the Lefschetz fixed point formula imposes the condition: for each $\alpha \in \pi$, $\alpha \neq 1$, we must assume the trace of the action induced by α in $G \otimes Q$ is $(-1)^{n+1}$. N. E. Steenrod.

The case where Ext $(G, G \otimes Z_2) = 0$ is trivial (Wall).

Homotopy theory

Problem 52. W. D. Barcus [6] has some results on the stable homotopy groups of Eilenberg-MacLane spaces (stable in the sense of S-theory). What can be said about the stable homotopy groups of other spaces whose ordinary homotopy groups are known? For example, the stable (in the other sense) classical groups? I. James.

REMARK. Since the infinite classical groups are infinite loop spaces, it follows that their ordinary homotopy maps monomorphically into their stable homotopy.

Problem 53. Let X, Y be complexes which are of the same homotopy n-type for all n. Are X and Y the same homotopy type? This is true if $\pi_n(X)$ is finite for each n, and can be false if $\pi_n(X)$ is infinitely generated (Adams [1]). Is it true if $\pi_n(X)$ is finitely generated for each n? In particular, consider the case where X and Y are H-spaces. J. F. Adams.

Problem 54. Can any thing general be said about $J: \pi_q(SO_n) \to \pi_{q+n}(S^n)$ on Samelson products? M. G. Barratt.

Stasheff pointed out the solution of Problem 54 on the basis of results of B. Steer (Thesis, Oxford): *Extensions of mappings into H-spaces*, Proc. Amer. Math. Soc., 13 (1963), 219–272, See Theorem 5.76.

Problem 55. Can anything general be said about $Q: \pi_{q+1}(B_G) \to \pi_q(G)$ on composition elements? M. G. Barratt.

Problem 56. Let ν_n generate $\pi_{n+3}(S_n)$, $n \geqq 5$; $P_n = [i_n, \nu_n]$ does not appear to show periodic behavior mod 8 as a function of n. Is it always zero for $n \equiv 5, 7 \bmod 8$? Is it alternately zero and of order 12 when n is large? M. G. Barratt.

Problem 57. Let K_p denote the stable p-primary components of the homotopy groups of spheres of stems >0. This is a commutative graded ring. Which if any of the following is true?

(a) K_p is commutative as an ungraded ring.

(b) K_p is nilpotent (or at least every element is).

(c) The product between even stems is zero in K_2.

(d) The cokernel $K_p/\mathrm{Im}\, J$ is a Z_{p^f}-module for some f depending on p. (The image of J is never an ideal in K_p). M. G. Barratt.

Problem 58. Examine periodicity phenomena in $\pi_{2n+k}(U_n)$, $1 \leqq k < 2n$. This has something to do with the order of a line bundle over a complex projective space in the Grothenieck group J of a vector bundle under fibre homotopy equivalence. (cf. [43]). M. Rothenberg.

K-theory

Problem 59. The *cannabalistic characteristic classes* θ_k or ρ_k appear to be the strongest currently known fibre homotopy invariants of spin $8n$-bundles. Can one prove that they determine the Stiefel-Whitney classes $w_4, w_5, \cdots$? Can one give a formula exhibiting $w_4, w_5, \cdots$ as a function of the classes θ_k or ρ_k? J. F. Adams.

Problem 60. What relations does the tangent bundle of a smooth manifold M satisfy in $KO(M)$, $KU(M)$, etc? R. Bott.

Problem 61. (a) Let G be a compact Lie group and H a closed subgroup. Calculate $K^*(G \mid H)$ and $KO^*(G \mid H)$.

(b) Is the following conjecture true? Let G be a compact connected Lie group. Suppose that $H_1(G; Z)$ has no torsion. Let U be a closed connected subgroup of G of maximal rank. Then the natural homomorphism $R(U) \to K^*(G \mid U)$ is surjective.

(Compare Atiyah-Hirzebruch, *Vector bundles and homogeneous spaces*, AMS, Symposia Pure Math. vol. 3, 1961) F. Hirzebruch.

Problem 62. Let M be a compact oriented $4k$-manifold. Suppose α is in the kernel of the homomorphism $R(\mathrm{so}(4k)) \to R(\mathrm{so}(4k-1))$. Following the Atiyah-Singer index theorem we can introduce the rational number

$$I(M, \alpha) = \frac{ch(\alpha)}{x_1 \cdots x_{2k}} \sum_{i=1}^{2k} \left(\frac{x_i/2}{\sinh x_i/2}\right)^2 [M]$$

which is in fact an integer, interpretable as the index of an elliptic operator. $I(M, \alpha)$ can be expressed in terms of Pontrjagin numbers and the Euler number. Can all relations between Pontrjagin numbers be obtained in this way? We have the analogous problem for weakly almost complex manifolds where one can use the Riemann-Roch formula. F. Hirzebruch.

Problem 63. Let $\mathfrak{x}$ be a function which attaches to each partition $[k_1, \cdots, k_j]$ of n an integer. For which $\mathfrak{x}$ does there exist a *connected* n-dimensional projective algebraic manifold X whose Chern numbers are given by the formula

$$(c_{k_1} \cdots c_{k_j})[X] = \mathfrak{x}([k_1, \cdots, k_j]) .$$

Same problem for projective algebraic replaced by complex or almost complex or weakly almost complex. F. Hirzebruch.

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