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Strong cofibrations and fibrations in enriched categories

By

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Abstract. We define strong cofibrations and fibrations in suitably enriched categories using the relative homotopy extension resp. lifting property. We prove a general pairing result, which for topological spaces specializes to the well-known pushout-product theorem for cofibrations. Strong cofibrations and fibrations give rise to cofibration and fibration categories in the sense of homotopical algebra. We discuss various examples; in particular, we deduce that the category of chain complexes with chain equivalences and the category of categories with equivalences are symmetric monoidal proper closed model categories.

0. Introduction. One of the most useful technical results for cofibrations of topological spaces is the "pushout-product theorem": if $i : A \to X$ and $j : B \to Y$ are cofibrations and one of them is closed then

$$(i \times Y, X \times j) : A \times Y \cup X \times B \longrightarrow X \times Y$$

is a cofibration. This result is false for arbitrary cofibrations. By introducing strong cofibrations which are equivalent to closed cofibrations in the category of topological spaces and can abstractly be defined by a relative homotopy lifting property (2.4), we can give a formal proof of this theorem (Pairing theorem 2.7(1)). The dual concept of a strong fibration allows to prove a dual result (Pairing theorem 2.7(2)).

We will use the pairing result to derive fibration and cofibration structures in the sense of [1] on various categories of (based) topological spaces and of (module and ring) spectra in the sense of [7] having genuine homotopy equivalences as weak equivalences. We obtain symmetric monoidal proper closed model category structures (for a definition see [3] and [15]) on the category of (unbounded) chain complexes with chain equivalences as weak equivalences (see also [4]) and on the category of small categories with equivalences as weak equivalences. These structures will be used in a subsequent paper for the development of relative universal homological algebra in suitable non-additive categories.

We prove our results in the setting of enriched categories which are complete and cocomplete, tensored and cotensored. The latter assumptions are stronger than strictly necessary, but they simplify the statements of the results.

Though simplicially enriched categories are not addressed in this paper, similar results can be proved, but the lack of a good cylinder object (see Definition 2.1) complicates the arguments and additional fibrancy conditions are required. For details see [14].

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1. Enriched categories. To specify notations we recall the basic concepts of enriched category theory from [9].

Let $\mathscr{V} = (\mathscr{V}_0, \Box, \Phi)$ be a symmetric monoidal category with unit object Φ. We assume $\mathscr{V}$ to be *closed*, i.e. there is a bifunctor

$$[-,-] : \mathscr{V}_0^{op} \times \mathscr{V}_0 \to \mathscr{V}_0$$

satisfying the Hom-tensor adjunction

1.1. $\mathscr{V}_0(X \Box Y, Z) \cong \mathscr{V}_0(X, [Y, Z])$.

From these data we can deduce natural isomorphisms

1.2. (1) $\mathscr{V}_0(Y, Z) \cong \mathscr{V}_0(\Phi \Box Y, Z]) \cong \mathscr{V}_0(\Phi, [Y, Z])$
(2) $Z \cong [\Phi, Z]$
(3) $[X \Box Y, Z] \cong [X, [Y, Z]]$.

Following Kelly we call a map $f : \Phi \to Z$ in $\mathscr{V}_0$ an *element in* Z. Hence, in accordance with (1.2.1), an element of $[Y, Z]$ is a morphism in $\mathscr{V}_0$.

Definition 1.3. A $\mathscr{V}$*-category* is a category $\mathscr{C}$ whose morphism sets are replaced by objects in $\mathscr{V}$, composition is defined by maps

$$\mathscr{C}(B, C) \Box \mathscr{C}(A, B) \to \mathscr{C}(A, C)$$

in $\mathscr{V}_0$ with identities $j_A : \Phi \to \mathscr{C}(A, A)$, subject to the obvious conditions. $\mathscr{V}$*-functors* and $\mathscr{V}$*-natural transformations* are defined in the obvious way.

Each $\mathscr{V}$-category $\mathscr{C}$ has an *underlying category* $\mathscr{C}_0$ with the same objects as $\mathscr{C}$ and $\mathscr{C}_0(A, B) = \mathscr{V}_0(\Phi, \mathscr{C}(A, B))$. A $\mathscr{V}$-functor $F : \mathscr{B} \to \mathscr{C}$ of $\mathscr{V}$-categories has an *underlying functor* $F_0 : \mathscr{B}_0 \to \mathscr{C}_0$, and the same applies to $\mathscr{V}$-natural transformation. We also say that $\mathscr{C}$ is a $\mathscr{V}$*-enriched* version of $\mathscr{C}_0$.

We have a hom-functor

$$\mathscr{C}_0^{op} \times \mathscr{C}_0 \to \mathscr{V}_0, \quad (A, B) \mapsto \mathscr{C}(A, B).$$

If $f : \Phi \to \mathscr{C}(B, B')$ and $g : \Phi \to \mathscr{C}(A', A)$ are morphisms in $\mathscr{C}_0$ then f_* and g^* are defined by

$$f_* : \mathscr{C}(A, B) \cong \Phi \Box\, \mathscr{C}(A, B) \xrightarrow{f \Box id} \mathscr{C}(B, B') \Box\, \mathscr{C}(A, B) \to \mathscr{C}(A, B')$$

$$g^* : \mathscr{C}(A, B) \cong \mathscr{C}(A, B) \Box \Phi \xrightarrow{id \Box g} \mathscr{C}(A, B) \Box\, \mathscr{C}(A', A) \to \mathscr{C}(A', B).$$

Note that there is a canonical $\mathscr{V}$-enriched version of $\mathscr{V}_0$, also denoted by $\mathscr{V}$: its objects are those of $\mathscr{V}_0$ and $\mathscr{V}(A, B) = [A, B]$.

Definition 1.4. In this paper a $\mathscr{V}$-category $\mathscr{C}$ is called $\mathscr{V}$*-complete* and $\mathscr{V}$*-cocomplete* if each functor $F : \mathscr{J} \to \mathscr{C}_0$, $\mathscr{J}$ a small indexing category, has a limit and a colimit in $\mathscr{C}_0$, and we have natural isomorphisms in $\mathscr{V}_0$

$$\mathscr{C}(X, \lim F) \cong \lim \mathscr{C}(X, F) \qquad \text{and} \qquad \mathscr{C}(\operatorname{colim} F, X) \cong \lim \mathscr{C}(F, X).$$

Definition 1.5. A $\mathscr{V}$-category $\mathscr{C}$ is *tensored* and *cotensored* if there are functors

$$\mathscr{C}_0 \times \mathscr{V}_0 \longrightarrow \mathscr{C}_0, \quad (X, K) \mapsto X \otimes K$$
$$\mathscr{C}_0 \times \mathscr{V}_0^{op} \longrightarrow \mathscr{C}_0, \quad (X, K) \mapsto X^K$$

and natural isomorphisms in $\mathscr{V}_0$

$$\mathscr{C}(X \otimes K, Y) \cong \mathscr{V}(K, \mathscr{C}(X, Y)) \cong \mathscr{C}(X, Y^K).$$

We observe that $\mathscr{V}$ is tensored by $-\Box-$ and cotensored by $[-,-]$. From (1.4) and (1.5) we immediately deduce

1.6. There are natural isomorphisms in $\mathscr{C}_0$

$$X \otimes \Phi \cong X \cong X^\Phi \qquad \text{and} \qquad X \otimes (K \Box L) \cong (X \otimes K) \otimes L.$$

1.7. If $\emptyset$ is initial in $\mathscr{C}_0$ and $*$ terminal, then

$$\emptyset \otimes K \cong \emptyset \qquad \text{and} \qquad *^K \cong *.$$

Definition 1.8. We call a $\mathscr{V}$-functor $F : \mathscr{C} \to \mathscr{B}$ $\mathscr{V}$-left adjoint to the $\mathscr{V}$-functor $U : \mathscr{B} \to \mathscr{C}$ if there is a natural isomorphism

$$\mathscr{B}(F(C), B) \cong \mathscr{C}(C, U(B))$$

in $\mathscr{V}_0$ for all objects $B \in \mathscr{B}$ and $C \in \mathscr{C}$.

Throughout this paper we make the following assumptions

Assumptions 1.9. $\mathscr{V}$ is a self-enriched closed symmetric monoidal category, $\mathscr{V}$-complete and $\mathscr{V}$-cocomplete. $\mathscr{C}$ is a $\mathscr{V}$-category with underlying category $\mathscr{C}_0$. We assume $\mathscr{C}$ to be $\mathscr{V}$-complete and $\mathscr{V}$-cocomplete, tensored and cotensored.

Remark 1.10. (1) If $\mathscr{C} = \mathscr{V}$, then (1.9) is satisfied, if the assumptions for $\mathscr{V}$ hold.
(2) If $\mathscr{V}_0$ has a 0-object then $\mathscr{C}_0$ has a 0-object: Since $\mathscr{V}_0(\Phi, K)$ is not empty for each object K in $\mathscr{V}_0$, the set $\mathscr{C}_0(X, Y) = \mathscr{V}_0(\Phi, \mathscr{C}(X, Y))$ is not empty for each pair of objects X, Y in $\mathscr{C}_0$. Hence the initial and terminal objects in $\mathscr{C}_0$, which exist by our assumptions, are isomorphic.

Example 1.11.

(A) Each of the following categories satisfies the assumptions on $\mathscr{V}$.
  (1) $\mathscr{T}op$, the category of compactly generated spaces in the sense of [18, 5(ii)]. (The examples [18, 5(iii),(iv)] also satisfy the requirements).
  We define $X \Box Y = X \times Y$, and $[X, Y]$ to be the function space in the category.
  (2) $\mathscr{S}\mathscr{S}ets$, the category of simplicial sets with $X \Box Y = X \times Y$, and $[X, Y] = \mathrm{Hom}(X, Y)$, the internal Hom-functor (for details see [13]).
  (3) The based versions $\mathscr{T}op_*$ and $\mathscr{S}\mathscr{S}ets_*$ of (1) and (2). Here we have $X \Box Y = X \wedge Y$, but the same definitions for $[X, Y]$ with the obvious base points.

(B) Take $\mathscr{V}$ to be $\mathscr{T}op$. Let $\mathscr{C}$ be the category of G-spaces in $\mathscr{T}op$, where $G \in \mathscr{T}op$ is a topological group. Then $\mathscr{C}$ satisfies (1.9); $X \otimes K = X \times K$ and $X^K = \mathscr{T}op(K, X)$ with the obvious G-structure.

Additional examples will be given in §4.

2. Fibrations and cofibrations. Throughout this section let $\mathscr{C}$ be a $\mathscr{V}$-enriched category. We assume that $\mathscr{V}$ and $\mathscr{C}$ satisfy the assumptions (1.9). If there is no danger of confusion we denote its underlying category $\mathscr{C}_0$ by the same symbol $\mathscr{C}$. Throughout we also assume that $\mathscr{V}$ has a good cylinder object:

Definition 2.1. A *cylinder object* in $\mathscr{V}$ is an object I equipped with morphisms $i_\varepsilon : \Phi \to I$, $\varepsilon = 0, 1$, and $\pi : I \to \Phi$ such that $\pi \circ i_\varepsilon = id$, $\varepsilon = 0, 1$.

A cylinder object (I, i_0, i_1, π) is called *good* if $(i_0, i_1) : \Phi \sqcup \Phi \to I$ is a strong cofibration in $\mathscr{V}$ (see Definition 2.4 below).

2.2. Recall that a morphism $j : A \to X$ has the *left lifting property* (LLP) for $p : E \to B$ if each commutative square

$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle j}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ X & \longrightarrow & B \end{array}$$

has a diagonal filler $h : X \to E$ making the diagram commute.

Dually, p has the *right lifting property* (RLP) for i if such an h exists.

Definition 2.3. (1) A map $p : E \to B$ in $\mathscr{C}$ is a *fibration* if it has the RLP for all maps $X \otimes i_0 : X \otimes \Phi \to X \otimes I$.
(2) A map $j : A \to X$ in $\mathscr{C}$ is a *cofibration* if it has the LLP for all maps $Z^{i_0} : Z^I \to Z^\Phi = Z$.

Passing to adjoints using (1.5) we see that $j : A \to X$ is a cofibration iff it has the homotopy extension property.

Definition 2.4. Consider morphisms $j : A \to X$, $p : E \to B$, and $i_0 : \Phi \to I$, and a commutative square

$$\begin{array}{ccc} A \otimes I \cup_{A \otimes \Phi} X \otimes \Phi & \longrightarrow & E \\ {\scriptstyle (j \otimes I, X \otimes i_0)}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ X \otimes I & \longrightarrow & B \end{array}$$

where $A \otimes I \cup_{A \otimes \Phi} X \otimes \Phi$ is the pushout of $A \otimes I \overset{A \otimes i_0}{\longleftarrow} A \otimes \Phi \overset{j \otimes \Phi}{\longrightarrow} X \otimes \Phi$. We call $j : A \to X$ a *strong cofibration* if the dotted filler h exists whenever p is a fibration. We call p a *strong fibration* if the dotted filler h exists whenever j is a cofibration.

Notation 2.5. We denote by **scof**, **cof**, **sfib**, and **fib** the classes of (strong) cofibrations and fibrations.

Since strong fibrations are defined by a RLP and strong cofibrations by a LLP, we obtain

Lemma 2.6. (1) **cof** *and* **scof** *contain all isomorphisms and are closed under cobase change, retracts (in the category of morphisms and commutative squares), arbitrary sums, and sequential colimits (given a sequence* $X_0 \to X_1 \to X_2 \to \dots$ *in* **cof** *or* **scof** *then* $X_n \to \operatorname{colim}_k X_k$ *is in* **cof** *resp.* **scof** *for all* n*).*

(2) **fib** *and* **sfib** *contain all isomorphisms and are closed under base change, retracts, arbitrary products and sequential limits (defined dually to* (1)*).*
(3) *Strong cofibrations are cofibrations and each* X *in* $\mathscr{C}$ *is strongly cofibrant, i.e.* $\emptyset \to X$ *is a strong cofibration. Strong fibrations are fibrations and each* X *in* $\mathscr{C}$ *is strongly fibrant, i.e.* $X \to *$ *is a strong fibration.*

The main result of this section is

Pairing theorem for cofibrations 2.7. *Suppose that* $\mathscr{V}$ *has a good cylinder object and that we are given* $\mathscr{V}$*-enriched categories* $\mathscr{A}, \mathscr{B}, \mathscr{C}$*, satisfying* (1.9)*,* $\mathscr{V}$*-functors*

$$T : \mathscr{A} \times \mathscr{B} \to \mathscr{C}, \quad U : \mathscr{A}^{op} \times \mathscr{C} \to \mathscr{B}, \quad V : \mathscr{B}^{op} \times \mathscr{C} \to \mathscr{A}$$

and natural isomorphisms in $\mathscr{V}$

$$\mathscr{C}(T(A, B), C) \cong \mathscr{B}(B, U(A, C)) \cong \mathscr{A}(A, V(B, C)).$$

Then: (1) *If* $i : A \to X$ *and* $j : B \to Y$ *are cofibrations in* $\mathscr{A}$ *resp.* $\mathscr{B}$ *and at least one of them is strong, then the morphism*

$$f = (T(i, Y), T(X, j)) : T(A, Y) \cup_{T(A,B)} T(X, B) \to T(X, Y)$$

is a cofibration in $\mathscr{C}$*. If both* i *and* j *are strong,* f *is strong.*
(2) *If* $j : B \to Y$ *is a cofibration in* $\mathscr{B}$ *and* $p : E \to Z$ *a fibration in* $\mathscr{C}$ *and at least one of them is strong, then the morphism*

$$g = (V(j, E), V(Y, p)) : V(Y, E) \to V(B, E) \times_{V(B,Z)} V(Y, Z)$$

is a fibration in $\mathscr{A}$*. If both* j *and* p *are strong,* g *is strong.*

The proof uses the following result

Lemma 2.8. *Let* $U : \mathscr{A} \to \mathscr{B}$ *and* $F : \mathscr{B} \to \mathscr{A}$ *be a pair of* $\mathscr{V}$*-functors between* $\mathscr{V}$*-enriched categories satisfying* (1.9)*. Suppose that* F *is* $\mathscr{V}$*-left adjoint to* U*. Then* F *preserves (strong) cofibrations and* U *preserves (strong) fibrations.*

P r o o f. Note that F preserves colimits and tensors while U preserves limits and cotensors. To show that U preserves fibrations consider

$$\begin{array}{ccc} X \otimes \Phi & \xrightarrow{f} & U(E) \\ {\scriptstyle X \otimes i_0} \downarrow & & \downarrow {\scriptstyle U(p)} \\ X \otimes I & \xrightarrow{g} & U(B) \end{array}$$

with p a fibration. Passage to the adjoint square provides the required filler. The argument for the other statements is similar. $\square$

P r o o f o f t h e p a i r i n g t h e o r e m. Throughout this proof we assume that i is a strong cofibration and p a fibration. We write $T(A, Y) \cup T(X, B)$ for $T(A, Y) \cup_{T(A,B)} T(X, B)$. Given a commutative diagram

$$\text{(D.1)} \qquad \begin{array}{ccc} (T(A, Y) \cup T(X, B)) \otimes I \cup T(X, Y) \otimes \Phi & \longrightarrow & E \\ {\scriptstyle (f \otimes I, T(X,Y) \otimes i_0)} \downarrow & & \downarrow {\scriptstyle p} \\ T(X, Y) \otimes I & \longrightarrow & Z \end{array}$$

we want to produce a filler $h : T(X, Y) \otimes I \to E$. Since $- \otimes I$ has a right adjoint it preserves colimits. Since $T(-, Y)$ has a $\mathscr{V}$-right adjoint it preserves tensors. Hence finding a filler for (D.1) is equivalent to finding a filler for

$$\text{(D.2)}\qquad \begin{array}{ccc} A \otimes I \cup X \otimes \Phi & \longrightarrow & V(Y, E) \\ \big\downarrow & & \big\downarrow g \\ X \otimes I & \longrightarrow & V(B, E) \times_{V(B,Z)} V(Y, Z). \end{array}$$

Since i is strong if suffices to show that g is a fibration in $\mathscr{A}$. This reduces the problem to the case that $A = \emptyset$. Since T preserves colimits in each variable $T(\emptyset, Y) = \emptyset$. Hence (D.1) reduces to

$$\text{(D.3)}\qquad \begin{array}{ccc} T(X, B) \otimes I \cup T(X, Y) \otimes \Phi & \longrightarrow & E \\ \big\downarrow & & \big\downarrow p \\ T(X, Y) \otimes I & \longrightarrow & Z \end{array}$$

which, by the same argument as above, is adjoint to

$$\text{(D.4)}\qquad \begin{array}{ccc} B \otimes I \cup Y \otimes \Phi & \longrightarrow & U(X, E) \\ \big\downarrow & & \big\downarrow U(X,p) \\ Y \otimes I & \longrightarrow & U(X, Z). \end{array}$$

Since $U(X, -)$ has a $\mathscr{V}$-left adjoint it preserves fibrations (2.8). Hence (D.4) admits a filler provided j is strong. So f is a strong cofibration if i and j are strong.

If we only want to show that f is a cofibration we go through the same argument with $Z = *$. Then $V(B, Z) = *$ and $U(X, Z) = *$, because the functors $V(B, -)$ and $U(X, -)$ preserve limits. But then (D.4) has the required filler if j is just a cofibration.

The proof of part (2) is similar using the equivalence between filling (D.1) and (D.2) in the opposite direction. □

We apply the pairing theorem to the functors

$$\begin{array}{ll} T : \mathscr{C} \times \mathscr{V} \to \mathscr{C}, & (X, K) \mapsto X \otimes K \\ U : \mathscr{C}^{op} \times \mathscr{C} \to \mathscr{V}, & (X, Y) \mapsto \mathscr{C}(X, Y) \\ V : \mathscr{V}^{op} \times \mathscr{C} \to \mathscr{C}, & (K, X) \mapsto X^K. \end{array}$$

Corollary 2.9. *Let $i : K \to L$ be a strong cofibration in $\mathscr{V}$.*

(1) *If $j : A \to X$ is a (strong) cofibration in $\mathscr{C}$, so is*

$$f = (j \otimes L, X \otimes i) : A \otimes L \cup X \otimes K \to X \otimes L.$$

(2) *If $p : E \to B$ is a (strong) fibration in $\mathscr{C}$, so is*

$$g = (E^i, p^L) : E^L \to E^K \times_{B^K} B^L.$$

Since $(i_0, i_1) : \Phi \sqcup \Phi \to I$ is a strong cofibration and $X \otimes (\Phi \sqcup \Phi) \cong X \sqcup X$ and $X^{\Phi \sqcup \Phi} \cong X \times X$ we have the following important special case of Corollary 2.9:

Corollary 2.10. *Given a (strong) cofibration* $j : A \to X$ *and a (strong) fibration* $p : E \to B$ *in* $\mathscr{C}$ *then*

$$(X \otimes i_0,\, j \otimes I,\, X \otimes i_1) : X \cup A \otimes I \cup X \to X \otimes I$$

is a (strong) cofibration and

$$\left(p^I, E^{i_0}, E^{i_1}\right) : E^I \to B^I \times_{B\times B} E \times E$$

a (strong) fibration.

Applying the pairing theorem to the functors

$$\begin{aligned} T &: \mathscr{V} \times \mathscr{C} \to \mathscr{C}, \quad (K, X) \mapsto X \otimes K \\ U &: \mathscr{V}^{op} \times \mathscr{C} \to \mathscr{C},\ (K, X) \mapsto X^K \\ V &: \mathscr{C}^{op} \times \mathscr{C} \to \mathscr{V},\ (X, Y) \mapsto \mathscr{C}(X, Y) \end{aligned}$$

gives

Corollary 2.11. *Given a cofibration* $j : A \to X$ *and a fibration* $p : E \to B$ *in* $\mathscr{C}$ *and at least one of them is strong, then*

$$(j^*, p_*) : \mathscr{C}(X, E) \to \mathscr{C}(A, E) \times_{\mathscr{C}(A,B)} \mathscr{C}(X, B)$$

is a fibration in $\mathscr{V}$*. If* j *and* p *are strong then* (j^*, p_*) *is strong.*

3. Model type structures. For relative homological algebra one would like to have model category structures in the sense of Quillen [13] on our categories with genuine homotopy equivalences defined via the cylinder functor as the weak equivalences. In important cases, such as spectra, those structures are not known to exist. Fortunately one can do with less: fibration or cofibration structures suffice (e.g. see [1]). Our cofibrations and fibrations give rise to such structures.

Since I is a good cylinder object we have (e.g. see [1, I. 3.5]).

Lemma 3.1. *Homotopy is an equivalence relation on* $\mathscr{C}(X, Y)$*.*

Recall the notations **cof** etc. from (2.5). Let **eq** denote the class of homotopy equivalences in $\mathscr{C}$. Our results imply (for definitions see [1])

Proposition 3.2. *If* I *is a good cylinder object then* $(\mathscr{C}, \mathbf{cof}, \mathbf{sfib}, -\otimes I, (-)^I, \emptyset, *)$ *and* $(\mathscr{C}, \mathbf{scof}, \mathbf{fib}, -\otimes I, (-)^I, \emptyset, *)$ *are* IP*-categories. In particular,* $(\mathscr{C}, \mathbf{eq}, \mathbf{cof})$ *and* $(\mathscr{C}, \mathbf{eq}, \mathbf{scof})$ *are cofibration categories with all objects cofibrant and fibrant, and* $(\mathscr{C}, \mathbf{eq}, \mathbf{fib})$ *and* $(\mathscr{C}, \mathbf{eq}, \mathbf{sfib})$ *are fibration categories with all objects cofibrant and fibrant* (2.6.3).

Remark 3.3. This result has a number of consequences:

(1) The gluing and cogluing lemmas hold [1, II. 1.2].
(2) There are factorizations of $f : X \to Y$ into a strong cofibration followed by a homotopy equivalence (given by the mapping cylinder construction) and into a homotopy equivalence followed by a strong fibration (given by the mapping path space construction). In particular, $\pi : X \otimes I \to X$ is a homotopy equivalence [1, I.3.12 and I.3.13].
(3) If $f : X \to Y$ is a fibration and a homotopy equivalence then f is a fiberwise deformation retraction. If f is a cofibration and a homotopy equivalence then f admits a strong deformation retraction [1, II.1.12].

Proposition 3.4. *The IP-categories of Proposition* 3.2 *satisfy the Continuity Axiom* [1, p. 182], *i.e. given a commutative ladder*

$$\begin{array}{ccccccc} A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \\ B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots \end{array}$$

such that each f_i and g_i is a cofibration and each h_i a homotopy equivalence then $\operatorname{colim} h_i : \operatorname{colim} A_i \to \operatorname{colim} B_i$ *is a homotopy equivalence. Suppose h_0, each f_i and g_i and each induced map*

$$q_i : B_i \cup_{A_i} A_{i+1} \longrightarrow B_{i+1}$$

is a (strong) cofibration, then $\operatorname{colim} h_i$ *is a (strong) cofibration.*

The dual statements hold, too. (We leave the formal proofs as an exercise.)

Using these results the proof of the following characterization of (strong) cofibrations and fibrations is fairly standard.

Proposition 3.5. *In the notation of* (2.2) *and* (2.5)

(1) $j \in \mathbf{cof} \Leftrightarrow$ *j has the LLP for all* $p \in \mathbf{sfib} \cap \mathbf{eq}$,
(2) $p \in \mathbf{sfib} \Leftrightarrow$ *p has the RLP for all* $j \in \mathbf{cof} \cap \mathbf{eq}$,
(3) $j \in \mathbf{cof} \cap \mathbf{eq} \Leftrightarrow$ *j has the LLP for all* $p \in \mathbf{sfib}$,
(4) $j \in \mathbf{scof} \cap \mathbf{eq} \Leftrightarrow$ *j has the LLP for all* $p \in \mathbf{fib}$,
(5) $p \in \mathbf{fib} \Leftrightarrow$ *p has the RLP for all* $j \in \mathbf{scof} \cap \mathbf{eq}$,
(6) $j \in \mathbf{scof} \Leftrightarrow$ *j has the LLP for all* $p \in \mathbf{fib} \cap \mathbf{eq}$,
(7) $p \in \mathbf{fib} \cap \mathbf{eq} \Leftrightarrow$ *p has the RLP for all* $j \in \mathbf{scof}$,
(8) $p \in \mathbf{sfib} \cap \mathbf{eq} \Leftrightarrow$ *p has the RLP for all* $j \in \mathbf{cof}$.

Proof. Each of the eight implications from left to right is one of the four cases of the following observation: *The diagram in* (2.2) *has a filler provided that j is a cofibration, p is a fibration, (at least) one of the two strong and one of them a homotopy equivalence.*

Below we give details for the case that p is strong and a homotopy equivalence. The other three cases are similar.

The implications from right to left are proved by suitably specializing the "test map" j or p respectively. E.g. in (1) we take $p = Z^{i_0} : Z^I \to Z$, which is a strong fibration by (2.9) and a homotopy equivalence by (3.3).

Let $j : A \to X$ be a cofibration and $p : E \to B$ be a strong fibration and a homotopy equivalence. By (3.3) there is a section $s : B \to E$ of p and a fiber homotopy equivalence

$$s \circ p \simeq_B id_E.$$

Take a map $(f, g) : j \to p$

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ {\scriptstyle j}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ X & \xrightarrow{g} & B \end{array} \quad .$$

The morphism $k = s \circ g : X \to E$ makes the lower triangle commutative and the upper commute by a homotopy

$$H : A \otimes I \to E, \quad H : k \circ j \simeq f$$

with $p \circ H = p \circ f \circ (A \otimes \pi)$. Since p is a strong fibration, the diagram

$$\begin{array}{ccc} X \otimes \Phi \cup A \otimes I & \xrightarrow{(k,H)} & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ X \otimes I & \xrightarrow[g \circ (X \otimes \pi)]{} & B \end{array}$$

admits a lift $K : X \otimes I \to E$. The morphisms $K \circ i_1 : X \to E$ is the required filler h. □

Addendum to the Pairing theorem 3.6. In the notation of the pairing theorem we have:

(1) if i and j are cofibrations, one of them strong and one of them a homotopy equivalence, then f is a homotopy equivalence.
(2) if j is a cofibration, p a fibration, one of them strong and one of them a homotopy equivalence, then g is a homotopy equivalence.

Proof. We consider commutative diagrams

(D.5)
$$\begin{array}{ccc} T(A, Y) \cup T(X, B) & \longrightarrow & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ T(X, Y) & \longrightarrow & Z \end{array}$$

and their adjoints

(D.6)
$$\begin{array}{ccc} A & \longrightarrow & V(Y, E) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle g} \\ X & \longrightarrow & V(B, E) \times_{V(B,Z)} V(Y, Z) \end{array}$$

Filling the first is equivalent to filling the second.

To prove (1) let p be some strong fibration. By (3.5(3)) it suffices to show that (D.6) and hence (D.5) has a filler. By the pairing theorem g is a fibration. Interchanging i and j if necessary we assume that i is a homotopy equivalence. If the cofibration i is strong, then (D.6) has a filler by (3.5(4) or (5)). Otherwise the cofibration j is strong, hence so is the fibration g by the pairing theorem, and the required filler exists by (3.5(2) or (3)).

To prove (2) let i be some strong cofibration. By (3.5(7)) it suffices to show that (D.5) and hence (D.6) has a filler. By the pairing theorem f is a cofibration, which is strong if j is. Suppose first that p is a homotopy equivalence. Then (D.5) has a filler by (3.5(6) or (7)) if j is a strong cofibration, and by (3.5(1) or (8)) if p is a strong fibration. Now suppose that j is a homotopy equivalence. Then so is f by part (1). Hence (D.5) has a filler by (3.5(2) or (3)) if p is a strong fibration. Otherwise j is a strong cofibration and so is f. Hence the filler exists by (3.5(4) or (5)). □

4. Examples.

4.1. $\mathscr{T}op$: Recall from (1.11) that $\mathscr{T}op$ is a self-enriched category satisfying Assumption (1.9). The unit interval I is a good cylinder object. Cofibrations and fibrations are the usual ones. Strong cofibrations are exactly the closed cofibrations. This follows from (3.5(6)) and the $\mathscr{T}op$-version of [17, Prop. 1]. The proof of [17, Prop. 1] (where the underlying category is that of all topological spaces) also works in $\mathscr{T}op$. We do not know of a similar characterization of strong fibrations but using the $\mathscr{T}op$-version of [16, Cor. 5] it is not hard to show

Proposition 4.1.1. *Any fibration* $p : E \to B$ *with* B *weakly Hausdorff is strong.*

Our pairing theorem provides a minor extension of [16, Thm. 10].

Proposition 4.1.2. *Let* $i : A \subset X$ *be a cofibration and* $p : E \to B$ *a fibration. If* A *is closed or* B *weakly Hausdorff the natural map* $E^X \longrightarrow E^A \times_{B^A} B^X$ *is a fibration.*

4.2. $\mathscr{T}op_*$: The category $\mathscr{T}op_*$ of based compactly generated spaces is self-enriched and satisfies Assumption 1.9. By (3.5(6)) and the $\mathscr{T}op_*$-version of [17, Prop. 1(b)] each strong cofibration in $\mathscr{T}op_*$ is closed. Conversely, if $j : A \to X$ is a closed cofibration in $\mathscr{T}op_*$ of well-pointed spaces (i.e. the inclusion of the base point is a closed cofibration) then j is a strong cofibration in $\mathscr{T}op_*$: by the $\mathscr{T}op$-version of [17, Prop. 9] the map j is also a closed cofibration in $\mathscr{T}op$. Since any fibration in $\mathscr{T}op_*$ is also a fibration in $\mathscr{T}op$ the required filler for (2.4) exists in $\mathscr{T}op$. Since we started with a diagram in $\mathscr{T}op_*$ this filler factors through the reduced cylinder.

The pairing theorem gives a minor extension of [17, Prop. 12].

Proposition 4.2.1. *Let* $A \subset X$ *be a closed cofibration in* $\mathscr{T}op_*$ *with* A *and* X *well-pointed. Then* $A \wedge Y \cup X \wedge B \longrightarrow X \wedge Y$ *is a cofibration in* $\mathscr{T}op_*$ *for any cofibration* $B \subset Y$ *in* $\mathscr{T}op_*$.

4.3. $\mathscr{H}\mathscr{T}op$ and $\mathscr{H}\mathscr{T}op_*$, the category of compactly generated weak Hausdorff spaces and its based version: Details about these categories can be found in [10] and [12]. Since every fibration in $\mathscr{H}\mathscr{T}op$ is strong by (4.1.1), every cofibration is strong by (3.5). A slight modification of [5, (1.17)] shows that cofibrations are closed. For $\mathscr{H}\mathscr{T}op_*$ our arguments of Example 4.2 imply that all fibrations are strong and that cofibrations between well-pointed objects are closed and strong.

4.4. Spectra. We assume the reader to be familiar with the definitions of module and ring spectra of [7].

Let $\mathscr{S}p$ denote the category of spectra in the sense of [11] and $\mathscr{L}$ the linear isometry operad. We denote the categories of $\mathscr{L}$-spectra and S-module spectra by $\mathscr{L}\mathscr{S}p$ and ${}_S\mathscr{M}$. Let A and B be S-algebra spectra and ${}_A\mathscr{M}_B$ the category of left A-right B-module spectra. This category as well as $\mathscr{S}p$ and $\mathscr{L}\mathscr{S}p$ is $\mathscr{H}\mathscr{T}op_*$-enriched satisfying (1.9).

Let R be a commutative S-algebra. Let ${}_R\mathscr{C}\mathscr{A}lg$ and ${}_R\mathscr{A}lg$ denote the categories of (commutative) R-algebras. These categories are $\mathscr{H}\mathscr{T}op$-enriched and satisfy (1.9).

These categories all have closed model category structures. The cofibrations and fibrations of these structures are called q-cofibrations and q-fibrations. The weak equivalences are maps which are weak equivalences of underlying spectra. The *q-fibrations* are the Serre fibrations for $\mathscr{S}p$ and $\mathscr{L}\mathscr{S}p$, and the morphisms $f : X \to Y$ for which $F(id, f) : F_{\mathscr{L}}(S, X) \to F_{\mathscr{L}}(S, Y)$ is a Serre fibration of spectra in all other cases. For details see [7]. From (2.9) we can deduce

Proposition 4.4.1. *In all these categories the q-cofibrations are strong cofibrations.*

For homological algebra arguments it is often important to know the behavior of fibrations and cofibrations under certain forgetful functors. For a commutative S-algebra R we consider the following forgetful functors

$$\begin{array}{llllllllll} {}_R\mathscr{C}\mathscr{A}lg & & & & & & & & & \\ & \searrow^{U_5} & & & & & & & & \\ {}_R\mathscr{A}lg & \xrightarrow{U_4} & {}_R\mathscr{M} & \xrightarrow{U_3} & {}_S\mathscr{M} & \xrightarrow{U_2} & \mathscr{L}\mathscr{S}p & \xrightarrow{U_1} & \mathscr{S}p \end{array}$$

Proposition 4.4.2.

(1) U_1 *preserves cofibrations and (strong) fibrations. Its left adjoint* $\mathbb{L}$ *and the composite* $U_1 \circ \mathbb{L}$ *preserve (strong) cofibrations.*
(2) U_2 *preserves (strong) cofibrations. Its right adjoint* $S \wedge_{\mathscr{L}} -$ *preserves (strong) cofibrations and (strong) fibrations.*
(3) U_3 *preserves (strong) cofibrations and (strong) fibrations.*
(4) U_4 *and* U_5 *map q-cofibrations to strong cofibrations.*

Proof. Since $U_1 \circ \mathbb{L}$ has a right adjoint [7, I.4], the functor U_1 preserves limits and colimits [2, 4.3.2] Hence it preserves pushouts and tensors and therefore cofibrations. The other statements of (1) to (3) follow from (2.8), because $S \wedge_{\mathscr{L}} -$ has $F_{\mathscr{L}}(S, -)$ as right adjoint, and U_3 has $R \wedge_S -$ as left and $F_S(R, -)$ as right adjoint. Statement (4) follows from the proof of [7, VII. 3.9]. □

Given an S-algebra C and a map of S-algebras $f : B \to A$ we have an obvious forgetful functor

$$U : {}_A\mathscr{M}_C \to {}_B\mathscr{M}_C.$$

U has a left adjoint $A \wedge_B -$ and a right adjoint $F_B(A, -)$. Hence we have

Proposition 4.4.3. *The forgetful functor* $U : {}_A\mathscr{M}_C \to {}_B\mathscr{M}_C$ *preserves (strong) cofibrations and (strong) fibrations.*

The smash-product

$${}_A\mathscr{M}_B \times {}_B\mathscr{M}_C \to {}_A\mathscr{M}_C, \quad (M, N) \mapsto M \wedge_B N$$

satisfies the requirements of (2.7). We obtain the following pairing results, which simplify arguments in [7].

Proposition 4.4.4. *Suppose we are given cofibrations* $i : K \to M$ *in* ${}_A\mathscr{M}_B$ *and* $j : L \to N$ *in* ${}_B\mathscr{M}_C$, *and a fibration* $p : Y \to Z$ *in* ${}_A\mathscr{M}_C$, *then:*

$$f = (i \wedge_B N, M \wedge_B j) : K \wedge_B N \cup M \wedge_B L \to M \wedge_B N$$

is a cofibration in ${}_A\mathscr{M}_C$ if i or j is strong. If, in addition, one is a homotopy equivalence, then f is a homotopy equivalence. If both i and j are strong then f is strong.

$$g = (F_C(j, Y), F_C(N, p)) : F_C(N, Y) \to F_C(L, Y) \times_{F_C(L,Z)} F_C(N, Z)$$

is a fibration in ${}_A\mathscr{M}_B$ if j or p is strong. If, in addition, one is a homotopy equivalence, then g is a homotopy equivalence. If both j and p are strong then g is strong.

Proposition 4.4.5. *Let $i : K \to L$ be a cofibration in $\mathscr{HT}op_*$ of well-pointed spaces. Then*
(1) *If $j : M \to N$ is a (strong) cofibration in ${}_A\mathscr{M}_B$, so is*

$$f = (j \wedge L, N \wedge i) : M \wedge L \cup N \wedge K \to N \wedge L.$$

If, in addition, j or i is a homotopy equivalence, so is f.
(2) *If $p : M \to N$ is a (strong) fibration in ${}_A\mathscr{M}_B$, so is*

$$g = (M^i, p^L) : M^L \to M^K \times_{N^K} N^L.$$

If, in addition, p or i is a homotopy equivalence, so is g.

4.5. Diagrams in $\mathscr{T}$op. Let $\mathscr{A}$ be topologically enriched small indexing category. The category $\mathscr{T}op^{\mathscr{A}}$ of continuous $\mathscr{A}$-diagrams is canonically $\mathscr{T}op$-enriched and satisfies (1.9). For $D \in \mathscr{T}op^{\mathscr{A}}$ and $K \in \mathscr{T}op$ define $(D \otimes K)(A) = D(A) \times K$ and $D^K(A) = \mathscr{T}op(K, A)$. The unit interval is a good cylinder object.

4.6. Complexes. Let R be a commutative ring and $\mathscr{C}plx_R$ the category of (unbounded) chain complexes of R-modules. The tensor product of chain complexes and the *Hom*-complex make $\mathscr{C}plx_R$ a self-enriched category satisfying (1.9). The cellular chain complex of the unit interval is a good cylinder object I [6].

Proposition 4.6.1. *A chain map $p : X \to Y$ is a fibration iff each $p_n : X_n \to Y_n$ has an R-linear section.*

P r o o f. Passing to the adjoint situation we have to show that p is a fibration iff the commutative square

$$\begin{array}{ccc} R & \xrightarrow{\;f\;} & \mathrm{Hom}_R(A, X) \\ {\scriptstyle i_0}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p_*} \\ I & \xrightarrow{\;g\;} & \mathrm{Hom}_R(A, Y) \end{array}$$

has a filler h for each chain complex A. Here R is considered as chain complex concentrated in dimension 0 and $i_0 : R \to I_0 = R \oplus R$ maps r to $(0, -r)$. The proposition is a consequence of the following observation: Given f, the chain maps g correspond bijectively to the maps

$$g_1 : R \longrightarrow (\mathrm{Hom}_R(A, Y))_1 = \prod_{n \in Z} \mathrm{Hom}_R(A_n, Y_{n+1})$$

and the same holds for h. □

As a consequence we obtain (see also [4, Example 3.3])

Proposition 4.6.2. *$\mathscr{C}plx_R$ with chain equivalences, fibrations, and strong cofibrations is a symmetric monoidal proper closed model category under tensor product, in which all objects are cofibrant and fibrant (for definitions see* [8]*).*

P r o o f. Since $\mathscr{C}plx_R$ is an IP-category by (3.2), we only have to verify the factorization axiom to obtain a closed model structure [1, I.4a.1]. Each chain map $f : X \to Y$ factors as $f = q \circ j : X \to Z_f \to Y$, where Z_f is the mapping cylinder, j is a strong cofibration, and q is a chain equivalence, admitting a section [1, I.3.12]. Hence q is a fibration. Since $\mathscr{C}plx_R$ is a P-category, there is a factorization $f = p \circ i$ into a fibration p and a chain equivalence i. We factor i into a strong cofibration j followed by a fibration and chain equivalence q. Since j is a chain equivalence $f = (p \circ q) \circ j$ is the required second factorization of f.

By the pairing theorem the model structure is monoidal. □

4.7. $\mathscr{C}$at. The usual product of categories and the functor categories make the category $\mathscr{C}at$ of small categories a self-enriched category satisfying (1.9). The category $\mathscr{I}s$ consisting of exactly one non-trivial isomorphism

$$0 \underset{f^{-1}}{\overset{f}{\rightleftarrows}} 1$$

is a good cylinder object. The resulting homotopy theory is the theory of equivalent categories.

Proposition 4.7.1. *Any inclusion $j : \mathscr{A} \subset \mathscr{B}$ is a strong cofibration.*

P r o o f. We have to find a filler K for each commutative square

$$\begin{array}{ccc}
\mathscr{A} \times \mathscr{I}s \cup_{\mathscr{A}\times 0} \mathscr{B} \times 0 & \xrightarrow{(F,K_0)} & \mathscr{X} \\
{\scriptstyle (j\times\mathscr{I}s,\mathscr{B}\times i_0)}\downarrow & \overset{K}{\nearrow} & \downarrow{\scriptstyle p} \\
\mathscr{B} \times \mathscr{I}s & \xrightarrow{L} & \mathscr{Y}
\end{array}$$

with p a fibration. F determines and is determined by two functors $F_0,\ F_1 : \mathscr{A} \to \mathscr{X}$ and a natural isomorphism $\alpha : F_0 \cong F_1$. We have to construct a functor $K_1 : \mathscr{B} \to \mathscr{X}$ and a natural isomorphism $\beta : K_0 \cong K_1$, which define the filler K. For $A \in \mathscr{A}$ we have to take $K_1(A) = F_1(A)$ and $\beta(A) = \alpha(A)$. Since p is a fibration, there is a functor $K_B : \{B\} \times \mathscr{I}s \to \mathscr{X}$ such that $K_B(B, 0) = K_0(B)$ and $p \circ K_B = L|\{B\} \times \mathscr{I}s$ for each $B \in \mathscr{B}$. For $B \notin \mathscr{A}$ we define $K_1(B) = K_B(B, 1)$ and $\beta(B) = K_B(B, f)$. For any morphism $g : B_1 \to B_2$ in $\mathscr{B}$ we define $K_1(g) = \beta(B_2) \circ K_0(g) \circ \beta(B_1)^{-1}$. This gives us the required filler K. □

Proposition 4.7.2. *$\mathscr{C}at$ with equivalences of categories, fibrations, and strong cofibrations is a symmetric monoidal proper closed model category under product, in which all objects are cofibrant and fibrant.*

P r o o f. The proof is dual to the one of (4.6.2). Any functor $F : \mathscr{A} \to \mathscr{B}$ factors into $F = P \circ S : \mathscr{A} \to \mathscr{P}_F \to \mathscr{B}$, where $\mathscr{P}_F$ is the mapping path space, S is an equivalence, which admits a retraction, and P is a fibration. By (4.7.1) S is a strong cofibration. Now proceed like in the proof of (4.6.2). □

4.8. Let R be a commutative ring. The category $\mathscr{S}\mathscr{M}od_R$ of simplicial R-moduls is self-enriched satisfying (1.9). Dimensionwise tensor product defines the tensor and the internal *Hom*-functor the cotensor. $R(\Delta^1)$ is a good cylinder object. By arguments similar to (4.6) and (4.7) we obtain

Proposition 4.8.1. *$\mathscr{SM}od_R$ with simplicial homotopy equivalences, fibrations, and strong cofibrations is a symmetric monoidal proper closed model category under dimensionwise tensor product, in which all objects are cofibrant and fibrant.*

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