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ALGEBRAIC TOPOLOGY

$\mathbb{Z}_2$-equivariant James Construction

by

Sławomir RYBICKI

Presented by A. PEŁCZYŃSKI on November 17, 1988

Summary. In this paper we investigate a relation between the James space $\mathbb{J}(X)$ with an action of $\mathbb{Z}_2$ and the $\mathbb{Z}_2$-space $\Omega^V S^V X$ for V the nontrivial one-dimensional representation of the group $\mathbb{Z}_2$. After defining the action on $\mathbb{J}(X)$ we prove that the $\mathbb{Z}_2$-spaces $\mathbb{J}(X)$ and $\Omega^V S^V X$ are weak $\mathbb{Z}_2$-homotopically equivalent. As an application of the result we get a construction of a $\mathbb{Z}_2$-EHP sequence.

Introduction. In this paper by (X, q) we denote arcwise-connected and compact topological $\mathbb{Z}_2$-space X with chosen nondegenerated base point $x_0 \in X$ and a continuous map $q : X \longrightarrow \mathbb{R}_+$ such that

(i) $$q^{-1}(0) = x_0,$$

(ii) $$q(g \cdot x) = q(x) \quad \text{for all } g \in \mathbb{Z}_2 \text{ and } x \in X.$$

In Section 1 of this paper we formulate definitions of spaces $\Gamma^V(X, q)$ and $\Omega^{*V}\Gamma^V(X, q)$ and give some their properties. Moreover, we construct a fibering which fiber is the space $(\Omega^{*V}\Gamma^V(X, q))^{\mathbb{Z}_2}$.

Section 2 contains definition of an action of the group $\mathbb{Z}_2$ on the space $\mathbb{J}(X)$. We also define a continuous $\mathbb{Z}_2$-equivariant map $\lambda : \mathbb{J}(X) \longrightarrow \Omega^{*V}\Gamma^V(X, q)$. In addition we construct a quasi-fibering which fiber is the space $\mathbb{J}(X)^{\mathbb{Z}_2}$.

In Section 3 we formulate and prove the main result of this paper. Namely, we prove that the map λ is a weak $\mathbb{Z}_2$-homotopic equivalence.

As an application, in Section 4 we give a construction of $\mathbb{Z}_2$-EHP sequence for a case not known before.

1. Spaces $\Gamma^V(X, q)$, $S^V X$, $\Omega^{*V}\Gamma^V(X, q)$, $\Omega^V S^V X$ and their properties. Let us define the spaces $S^V X$ and $\Gamma^V(X, q)$ in the following way

$$S^V X = S^V \wedge X \text{ and } \Gamma^V(X, q) = V \times X / \{(v, x) : \|v\| \geqslant q(x)\}.$$

We define the action of the group $\mathbb{Z}_2$ on the spaces $\Gamma^V(X,q)$ and S^VX by the formula $g\cdot|(v,x)| = |(g\cdot v, g\cdot x)|$. The space $\Gamma^V(X,q)$ is called $\mathbb{Z}_2$-Moore suspension on the $\mathbb{Z}_2$-space (X,q). It is easy to see that the above spaces are $\mathbb{Z}_2$-homeomorphic.

Next we define the space $\Omega^{*V}\Gamma^V(X,q)$ as follows

$$\Omega^{*V}\Gamma^V(X,q) = \{(r,f)\in \mathbb{R}_+ \times \overline{F}(V,\Gamma^V(X,q)) : \forall v\in V \|v\| \geqslant r \Rightarrow f(v) = x_0\}.$$

We consider the space $\Omega^{*V}\Gamma^V(X,q)$ as $\mathbb{Z}_2$-space with the action of $\mathbb{Z}_2$ given by the formula $g\cdot(r,f) = (r, g\cdot f)$.

The space $\Omega^{*V}\Gamma^V(X,q)$ is called the *$\mathbb{Z}_2$-space of the Moore loops on the $\mathbb{Z}_2$-Moore suspension of the $\mathbb{Z}_2$-space (X,q).*

LEMMA 1.1. *$\Omega^{*V}\Gamma^V(X,q)$ is homotopically equivalent to Ω^VS^VX.*

We omitt easy proof of this Lemma. Let us introduce a structure of a manoid on the space $\Omega^{*V}\Gamma^V(X,q)$. For (r,f) and $(s,g)\in\Omega^{*V}\Gamma^V(X,q)$ we define the multiplication by $(r,f)*(s,g) = (r+s,h)$, where

$$h(v) = \begin{cases} f(v+s\cdot w) & \text{for } v\in\langle -(r+s)\cdot w, (r-s)\cdot w\rangle \\ g(v-r\cdot w) & \text{for } v\in\langle (r-s)\cdot w, (r+s)\cdot w\rangle \\ x_0 & \text{for } v\notin\langle -(r+s)\cdot w, (r+s)\cdot w\rangle \end{cases}$$

Next we construct a fibering with fiber equal to $(\Omega^{*V}\Gamma^V(X,q))^{\mathbb{Z}_2}$. First we define the $\mathbb{Z}_2$-space of Moore paths

$$P^{*V}(\Gamma^V(X,q)) = \{(r,f)\in\mathbb{R}_+\times\overline{F}(V,\Gamma^V(X,q)) : \exists x,\overline{x}\in\Gamma^V(X,q)\forall v\in V \ \mathrm{or}(v)\cdot\|v\|\geqslant r\Rightarrow f(x) = x \text{ and } \mathrm{or}(v)\cdot\|v\|\leqslant -r\Rightarrow f(x)=\overline{x}\}.$$

The group $\mathbb{Z}_2$ acts on the space $P^{*V}\Gamma^V(X,q)$ by $g\cdot(r,f) = (r,g\cdot f)$. The equation $\overline{p}_1(r,f) = f(r\cdot w)$ defines a continuous map

$$\overline{p}_1 : (P^{*V}\Gamma^V(X,q))^{\mathbb{Z}_2} \longrightarrow \Gamma^V(X,q).$$

Note that $\overline{p}_1^{-1}(x_0) = (\Omega^{*V}\Gamma^V(X,q))^{\mathbb{Z}_2}$.

LEMMA 1.2. *The map $\overline{p}_1 : (P^{*V}\Gamma^V(X,q))^{\mathbb{Z}_2}\longrightarrow\Gamma^V(X,q)$ defined as above is a fibering.*

P r o o f is standard and left to the reader. An easy computation shows that the space $(P^{*V}\Gamma^V(X,q))^{\mathbb{Z}_2}$ has the same homotopy type as the space $X^{\mathbb{Z}_2}$. We define a homeomorphism $e:\Gamma^V(X,q)\longrightarrow\Gamma^V(X,q)$ by $e(|(v,x)|) = |(v,g\cdot x)|$, where g is the nontrivial element of $\mathbb{Z}_2$. The map $p_1 = e\circ\overline{p}_1 : (P^{*V}\Gamma^V(X,q))^{\mathbb{Z}_2}\longrightarrow\Gamma^V(X,q)$ is a fibering.

We see that the exact homotopy sequence for the fibering can be written in the following form

$$\ldots\longrightarrow\pi_n((\Omega^{*V}\Gamma^V(X,q))^{\mathbb{Z}_2})\longrightarrow\pi_n(X^{\mathbb{Z}_2})\longrightarrow$$

$$\longrightarrow\pi_n(\Gamma^V(X,q))\longrightarrow\pi_{n-1}((\Omega^{*V}\Gamma^V(X,q))^{\mathbb{Z}_2})\longrightarrow\ldots$$

2. $\mathbb{Z}_2$-equivariant James construction and its properties. We use the definition of James construction given in [4]. The James construction on the space (X,q) will be denoted by $\mathbb{J}(X)$. The elements of the space $\mathbb{J}(X)$ will be written in the form $x_1\cdot\ldots\cdot x_k$. We define an action of the group $\mathbb{Z}_2$ on the space $\mathbb{J}(X)$ as follows

$$g\cdot(x_1\cdot\ldots\cdot x_k) = \begin{cases} x_1\cdot\ldots\cdot x_k & \text{for } g=0\in\mathbb{Z}_2 \\ (g\cdot x_k)\cdot\ldots\cdot(g\cdot x_1) & \text{for } g=1\in\mathbb{Z}_2 \end{cases}$$

The space $\mathbb{J}(X)$ with this action of $\mathbb{Z}_2$ is called *$\mathbb{Z}_2$-equivariant James construction on the $\mathbb{Z}_2$-space (X,q).*

Let us define a continuous $\mathbb{Z}_2$-equivariant map $\overline{\lambda}:X\longrightarrow\Omega^{*V}\Gamma^V(X,q)$ by formula $\overline{\lambda}(x) = (q(x),\lambda_x(\ ))$, where $\lambda_x(v) = |(v,x)|$. The map $\overline{\lambda}$ extends to a continuous $\mathbb{Z}_2$-equivariant map $\lambda:\mathbb{J}(X)\longrightarrow\Omega^{*V}\Gamma^V(X,q)$ defined by $\lambda(x_1\cdot\ldots\cdot x_k) = \overline{\lambda}(x_1)\cdot\ldots\cdot\overline{\lambda}(x_k)$.

We shall construct a quasifibering, which fiber is the space $\mathbb{J}(X)^{\mathbb{Z}_2}$. Form a map $f:X\times\mathbb{J}(X)^{\mathbb{Z}_2}\longrightarrow\mathbb{J}(X)^{\mathbb{Z}_2}$ by formula

$$f(x,x_1\cdot\ldots\cdot x_k) = x\cdot x_1\cdot\ldots\cdot x_k\cdot(g\cdot x),$$

where g is the nontrivial element of $\mathbb{Z}_2$. We define the cone $C^V(X,q)$ in the following way

$$C^V(X,q) = \{(v,x)\in V\times X : \|v\|\leqslant q(x)\}/\{(v,x)\in V\times X : \|v\| = q(x) \text{ and } \mathrm{or}(v)\leqslant 0\}.$$

We are given an injection $X\times\mathbb{J}(X)^{\mathbb{Z}_2}\longrightarrow C^V(X,q)\times\mathbb{J}(X)^{\mathbb{Z}_2}$ defined by $(x,y)\longrightarrow(|(q(x)\cdot w,x)|),y)$. Let $E^V = (C^V(X,q)\times\mathbb{J}(X)^{\mathbb{Z}_2})\cup_f\mathbb{J}(X)^{\mathbb{Z}_2}$ be the cylinder of the map f. Now let us define a map $p:E^V\longrightarrow\Gamma^V(X,q)$ as follows

$$p(|(v,x)|,y) = |(v,x)| \text{ and } p(y) = x_0.$$

By the James Lemma in [5] it follows that the map p is a quasifibering. We have obtained the folowing exact sequence of the quasifibering

$$\ldots\longrightarrow\pi_n(\mathbb{J}(X)^{\mathbb{Z}_2})\xrightarrow{i_*}\pi_n(E^V)\xrightarrow{p_*}$$
$$\longrightarrow\pi_n(\Gamma^V(X,q))\xrightarrow{\partial_*}\pi_{n-1}(\mathbb{J}(X^{\mathbb{Z}_2})\longrightarrow\ldots$$

On the space $C^V(X,q)\times\mathbb{J}(X)^{\mathbb{Z}_2}$ we define the following relation

$$(|(q(x)\cdot w,x)|,y)\sim(x_0,x\cdot y\cdot(g\cdot x)),$$

where g is the nontrivial element of $\mathbb{Z}_2$. Note that $E^V = C^V(X,q)\times\mathbb{J}(X)^{\mathbb{Z}_2}/\sim$.

Let us introduce the following filtration E^V_n in the space $E^V : E^V_n = C^V(X,q)\times\mathbb{J}_n(X)/\sim$ for $n = 1,2,3,\ldots$

THEOREM 2.1 (Milnor Theorem). *Let Z and X be topological spaces with filtrations Z_n and Y_n respectively and $f : Y \longrightarrow Z$ be a map preserving the filtrations. If*

a) *for all $n \geqslant 1$ the pairs (Z_{n+1}, Z_n) and (Y_{n+1}, Y_n) are cofiberings,*

b) *for all $n \geqslant 1$ the maps $f_n = f_{|Z_n} : Z_n \longrightarrow Y_n$ are homotopy equivalences,*

then the map $f : Z \longrightarrow Y$ is a homotopy equivalence.

For the proof of the above theorem see for instance [5].

LEMMA 2.1. *The space E^V has the same homotopy type as the space $X^{\mathbb{Z}_2}$.*

Proof. Let g be the nontrivial element of the group $\mathbb{Z}_2$. We define a continuous map $h : E^V \longrightarrow (P^{*V}\Gamma^V(X,q))^{\mathbb{Z}_2}$ as follows

$$h(|(v,x)|, x_1 \cdot \ldots \cdot x_k) = \Big(\sum_{i=1}^{n} \frac{q(x_i)}{2} + q(x) + \mathrm{or}(v) \cdot \|x\|,$$
$$\gamma_1(v,x) * \lambda(x_1 \cdot \ldots \cdot x_k) * \gamma_2(v,x)\Big),$$

where $\gamma_1(v,x) : V \longrightarrow \Gamma^V(X,q)$ is given by

$$\gamma_1(v,x)(t) = \begin{cases} |(-v,x)| & \text{for } \mathrm{or}(t)\cdot\|t\| \leqslant -(q(x) + \mathrm{or}(v)\cdot\|v\|) \\ |((\|t\| - q(x))\cdot w, g\cdot x)| & \text{for } -(q(x)+\mathrm{or}(v)\cdot\|v\|) \leqslant \mathrm{or}(t)\cdot\|t\| \leqslant 0 \\ x_0 & \text{for } 0 \leqslant \mathrm{or}(t)\cdot\|t\| \end{cases}$$

and where $\gamma_2(v,x) : V \longrightarrow \Gamma^V(X,q)$ is defined by

$$\gamma_2(v,x)(t) = \begin{cases} |(v, g\cdot x)| & \text{for } (q(x)+\mathrm{or}(v)\cdot\|v\|) \leqslant \mathrm{or}(t)\cdot\|t\| \\ |((\|t\| - q(x))\cdot w, g\cdot x)| & \text{for } 0 \leqslant \mathrm{or}(t)\cdot\|t\| \leqslant q(x)+\mathrm{or}(v)\cdot\|v\| \\ x_0 & \text{for } \mathrm{or}(t)\cdot\|t\| \leqslant 0 \end{cases}$$

Let $\Pi : \{0\} \times X^{\mathbb{Z}_2} \longrightarrow X^{\mathbb{Z}_2}$ be the projection. We denote the coordinates of the map h as $h(y) = (s(y), t(y))$. We define a map $d : E^V \longrightarrow X^{\mathbb{Z}_2}$ in the following way $d(y) = \Pi(t(y)(0))$. Let us apply Theorem 2.1. to the following case

$$Z = E^V, \quad Y = X^{\mathbb{Z}_2}, \quad Z_n = E^V_n \quad \text{and} \quad f = d.$$

It is easy to see that for $n \geqslant 1$

a) the pairs (Z_{n+1}, Z_n) and (Y_{n+1}, Y_n) are cofiberings,

b) the map $d_n = d_{|Z_n} : Z_n \longrightarrow Y_n$ is a homotopy equivalence.

By Theorem 2.1 it follows that the map d is a homotopy equivalence. □

Using Lemma 2.1 we simplify the exact sequence of the quasifibering $p : E^V \longrightarrow \Gamma^V(X,q)$. We obtain the following exact sequence

$$\ldots \longrightarrow \pi_n(\mathrm{J}(X)^{\mathbb{Z}_2}) \longrightarrow \pi_n(X^{\mathbb{Z}_2}) \longrightarrow$$
$$\longrightarrow \pi_n(\Gamma^V(X,q)) \longrightarrow \pi_{n-1}(\mathrm{J}(X)^{\mathbb{Z}_2}) \longrightarrow \ldots$$

In our notation the Theorem 2.2 has the following form

THEOREM 2.2 (James Theorem). *The map $\lambda : \mathrm{J}(X) \longrightarrow \Omega^{*V}\Gamma^V(X,q)$ is a weak homotopy equivalence.*

The proof of the theorem can be found in [4]. A stronger version of the theorem is due to [9].

3. $\mathbb{Z}_2$-equivariant James Theorem.

In this section we formulate and prove the main theorem of the paper.

THEOREM 3.1 ($\mathbb{Z}_2$-equivariant James Theorem). *For every finite $\mathbb{Z}_2$-CW complex K the map*

$$\lambda_* : [K, \mathrm{J}(X)]_{\mathbb{Z}_2} \longrightarrow [K, \Omega^{*V}\Gamma^V(X,q)]_{\mathbb{Z}_2}$$

is an isomorphism.

Proof. An easy computation shows that the map $h : E^V \longrightarrow (P^{*V}\Gamma^V(X,q))^{\mathbb{Z}_2}$ defined in Lemma 2.1 is a homotopy equivalence. It follows immediately that $h_{|\mathrm{J}(X)^{\mathbb{Z}_2}} = \lambda^{\mathbb{Z}_2}$.

We shall prove that the constructed exact sequences of fibering and quasifibering are isomorphic. We have the following commutative diagram

$$\begin{tikzcd} J(X)^{\mathbb{Z}_2} \arrow[r, "\lambda^{\mathbb{Z}_2}"] \arrow[d, "i"] & (\Omega^{*V}\Gamma^V(X,q))^{\mathbb{Z}_2} \arrow[d, "i"] \\ E^V \arrow[r, "h"] \arrow[d, "p"] & (P^{*V}\Gamma^V(X,q))^{\mathbb{Z}_2} \arrow[d, "p_1"] \\ \Gamma^V(X,q) \arrow[r, "I_d"] & \Gamma^V(X,q) \end{tikzcd}$$

By the Five Lemma the map

$$\lambda_*^{\mathbb{Z}_2} : \pi_n(\mathrm{J}(X)^{\mathbb{Z}_2}) \longrightarrow \pi_n((\Omega^{*V}\Gamma^V(X,q))^{\mathbb{Z}_2})$$

is an isomorphism for any $n \in N$. From this it follows that the maps λ_* and $\lambda_*^{\mathbb{Z}_2}$ are isomorphisms. Using the above and Proposition 11.2 in [1] we complete our proof.

4. Application of $\mathbb{Z}_2$-equivariant James Theorem. In this section we use notation introduced in [1], where the exact sequence of $\mathbb{Z}_2$-pair (X,A) was constructed. This sequence is of the following form

$$\ldots \longrightarrow \pi_{p,q}(A) \xrightarrow{i_*} \pi_{p,q}(X) \xrightarrow{j_*} \pi_{p,q}(X,A) \xrightarrow{\partial_*} \pi_{p,q-1}(A) \longrightarrow .$$

Let us fix indexes r and s and write the above exact sequence for the $\mathbb{Z}_2$-pair $(\mathbb{J}(\Sigma^{r,s}), \Sigma^{r,s})$

$$\ldots \longrightarrow \pi_{p,q}(\Sigma^{r,s}) \xrightarrow{i_*} \pi_{p,q}(\mathbb{J}(\Sigma^{r,s})) \xrightarrow{j_*}$$
$$\longrightarrow \pi_{p,q}(\mathbb{J}(\Sigma^{r,s}), \Sigma^{r,s}) \xrightarrow{\partial_*} \pi_{p,q-1}(\Sigma^{r,s}) \longrightarrow \ldots$$

The spaces $\Omega^{*V}\Gamma^{V}(X,q)$ and $\Omega^{1,0}\Sigma^{1,0}$ are $\mathbb{Z}_2$-homotopically equivalent. Denote by γ a homotopy equivalence between these spaces. By the properties of the exact sequence of pair (X,A) and Theorem 3.1 we deduce that the above sequence is equivalent to the sequence

$$(*)\qquad \ldots \longrightarrow \pi_{p,q}(\Sigma^{r,s}) \xrightarrow{\Sigma^{1,0}} \pi_{p+1,q}(\Sigma^{r+1,s}) \xrightarrow{H_*}$$
$$\longrightarrow \pi_{p,q}(\mathbb{J}(\Sigma^{r,s}), \Sigma^{r,s}) \xrightarrow{\partial_*} \pi_{p,q-1}(\Sigma^{r,s}) \longrightarrow \ldots$$

If α_* is the isomorphism of the groups $\pi_{p,q}(\Omega^{1,0}\Sigma^{1,0}\Sigma^{r,s})$ and $\pi_{p+1,q}(\Sigma^{r+1,s})$, then $\Sigma^{1,0} = \alpha_* \circ \gamma_* \circ \lambda_* \circ i_*$ and $H_* = j_* \circ \lambda_*^{-1} \circ \gamma_*^{-1} \circ \alpha_*^{-1}$. It can be verified that the groups $\pi_{p,q}(\mathbb{J}(\Sigma^{r,s}), \Sigma^{r,s})$ and $\pi_{p,q}(\mathbb{J}(\Sigma^{r,s} \wedge \Sigma^{r,s}))$ are isomorphic for $p+q < 3\cdot(r+s)-2$ and $q < r+s-1$. Denote by σ_* this isomorphism.

We are given an action of $\mathbb{Z}_2$ on the space $\Sigma^{r,s} \wedge \Sigma^{r,s}$ by

$$(**)\qquad g\cdot(x\wedge y) = \begin{cases} (g\cdot y)\wedge(g\cdot x) & \text{for } g = 1 \in \mathbb{Z}_2 \\ x \wedge y & \text{for } g = 0 \in \mathbb{Z}_2 \end{cases}$$

PROPOSITION 4.1. *Assume that $p+q < 3\cdot(r+s)-2$ and $q < r+s-1$. The simpliest form of the exact sequence for the pair $(\mathbb{J}(\Sigma^{r,s}), \Sigma^{r,s})$ is the following*

$$\ldots \longrightarrow \pi_{p,q}(\Sigma^{r,s}) \xrightarrow{\Sigma^{1,0}} \pi_{p+1,q}(\Sigma^{r+1,s}) \xrightarrow{\eta_*\circ\sigma_*\circ H_*}$$
$$\longrightarrow \pi_{p+1,q}(\Sigma^{r+s+1,r+s}) \xrightarrow{\partial_*\circ\sigma_*^{-1}\circ\eta_*^{-1}} \pi_{p,q-1}(\Sigma^{r,s}) \longrightarrow \ldots$$

P r o o f. Using the isomorphism σ_* we simplify the sequence $(*)$

$$\ldots \longrightarrow \pi_{p,q}(\Sigma^{r,s}) \xrightarrow{\Sigma^{1,0}} \pi_{p+1,q}(\Sigma^{r+1,s}) \xrightarrow{\sigma_*\circ H_*}$$
$$\longrightarrow \pi_{p,q}(\mathbb{J}(\Sigma^{r,s} \wedge \Sigma^{r,s})) \xrightarrow{\partial_*\circ\sigma_*^{-1}} \pi_{p,q-1}(\Sigma^{r,s}) \longrightarrow \ldots$$

It can be proved that the space $\Sigma^{r,s} \wedge \Sigma^{r,s}$ with action $(**)$ of $\mathbb{Z}_2$ is $\mathbb{Z}_2$-homeomorphic to the space $\Sigma^{r+s,r+s}$.

Applying the above remark and Theorem 3.1 we conclude that the group $\pi_{p,q}(\mathbb{J}(\Sigma^{r,s} \cap \Sigma^{r,s}))$ is isomorphic to $\pi_{p+1,q}(\Sigma^{r+s+1,r+s})$. Denote the isomorphism by η_*.

Finally we get

$$\ldots \longrightarrow \pi_{p,q}(\Sigma^{r,s}) \xrightarrow{\Sigma^{1,0}} \pi_{p+1,q}(\Sigma^{r+1,s}) \xrightarrow{\eta_*\circ\sigma_*\circ H_*}$$
$$\longrightarrow \pi_{p+1,q}(\Sigma^{r+s+1,r+s}) \xrightarrow{\partial_*\circ\sigma_*^{-1}\circ\eta_*^{-1}} \pi_{p,q-1}(\Sigma^{r,s}) \longrightarrow \ldots$$

□

NOTATIONS.

V — a one-dimensional nontrivial representation of the group $\mathbb{Z}_2$,

$w \in V$ — a versor of the representation V,

$\mathrm{or}(v) = \mathrm{sgn}(k)$, where $v = k\cdot w$,

$S^V = V \cup \{\infty\}$ — one-point compactification of the representation V with a chosen base point $x_0 = \infty$,

$\overline{F}(A,B)$ — the space of all continuous maps from a topological space A into a topological space B, considered with compact open topology,

$F(A,B)$ — the space of all these continuous maps from the topological space A into the topological space B, which preserve base points, considered with compact-open topology,

G — is any group,

$[Z,T]_G$ — the set of G-homotopy classes of G-equivariant maps from a G-space Z to a G-space T preserving base points,

X^G — the set of all fixed points of the action of the group G on the G-space X,

$f^H : X^H \longrightarrow Y^H$ — the restriction of the G-equivariant map $f : X \longrightarrow Y$ to the set of fixed points of the action of the subgroup H of the group G,

$\Omega^V S^V X = F(S^V, S^V \wedge X)$.

Given $\mathbb{Z}_2$-spaces A and B we define the following action of the group $\mathbb{Z}_2$ on the space $\overline{F}(A,B)$: $(g\cdot f)(a) = g\cdot f(g\cdot a)$, where $f \in \overline{F}(A,B)$, $g \in \mathbb{Z}_2$ and $a \in A$.

We introduce the action of the group $\mathbb{Z}_2$ on the space $F(A,B)$ in analogous way.

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ALGEBRAIC TOPOLOGY

Skein Modules of 3-Manifolds

by

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Presented by A. BIALYNICKI-BIRULA on January 25, 1989

Summary. It is natural to try to place the new polynomial invariants of links in algebraic topology (e.g. to try to interpret them using homology or homotopy groups). However, one can think that these new polynomial invariants are byproducts of a new more delicate algebraic invariant of 3-manifolds which measures the obstruction to isotopy of links (which are homotopic). We propose such an algebraic invariant based on skein theory introduced by Conway (1969) and developed by Giller (1982) as well as Lickorish and Millett (1987).

Let M be an oriented 3-manifold and R a commutative ring with 1. For $r_0, \ldots, r_{k-1} \in R$ we define the kth skein module $\mathcal{S}_k(M;R)\ (r_0, \ldots, r_{k-1})$ as follows:

Let $\mathcal{L}(M)$ be the set of all ambient isotopy classes of oriented links in M. Let $\mathcal{M}(\mathcal{L}, R)$ be a free R-module generated by $\mathcal{L}(M)$ and $S_{\mathcal{L}(M)}(r_0, \ldots, r_{k-1})$ the submodule generated by linear skein expressions $r_0L_0 + r_1L_1 + \ldots + r_{k-1}L_{k-1}$, where $L_0, L_1, \ldots, L_{k-1}$ are classes of links identical except the parts shown in Fig. 1.

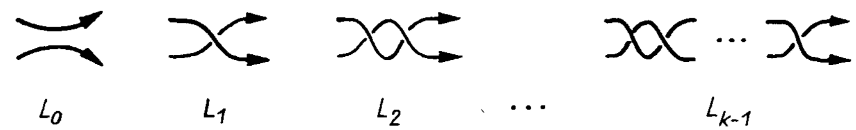

Fig. 1

DEFINITION 1 . The R-module

$$\mathcal{S}_k(M;R)(r_0, \ldots, r_{k-1}) = \mathcal{M}(\mathcal{L}, R)/S_{\mathcal{L}(M)}(r_0, \ldots, r_{k-1})$$

is called *the kth skein module of M.*

EXAMPLE 2.

(a) $\mathcal{S}_k(M;R)(0, \ldots, 0) = \mathcal{M}(\mathcal{L}, R)$,