ROOT INVARIANTS IN THE ADAMS SPECTRAL SEQUENCE

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Abstract. Let E be a ring spectrum for which the E-Adams spectral sequence converges. We define a variant of Mahowald's root invariant called the 'filtered root invariant' which takes values in the E_1 term of the E-Adams spectral sequence. The main theorems of this paper concern when these filtered root invariants detect the actual root invariant, and explain a relationship between filtered root invariants and differentials and compositions in the E-Adams spectral sequence. These theorems are compared to some known computations of root invariants at the prime 2. We use the filtered root invariants to compute some low dimensional root invariants of v_1-periodic elements at the primes 2 and 3. We also compute the root invariants of some infinite v_1-periodic families of elements at the prime 3.

Contents

1. Introduction

In his work on metastable homotopy groups [22], Mahowald introduced an invariant that associates to every element α in the stable stems a new element $R(\alpha)$ called the *root invariant* of α. The construction has indeterminacy and so $R(\alpha)$ is

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in general only a coset. The main result of [25] indicates a deep relationship between elements of the stable homotopy groups of spheres which are root invariants and their behavior in the EHP spectral sequence. Mahowald and Ravenel conjecture in [24] that, loosely speaking, the root invariant of a v_n-periodic element is v_{n+1}-periodic. Thus the root invariant is related simultaneously to unstable and chromatic phenomena.

The conjectural relationship between root invariants and the chromatic filtration is based partly on computational evidence. For instance, at the prime 2, we have [25], [12]

$$R(2^i) = \begin{cases} \alpha_{4t} & i = 4t \\ \alpha_{4t+1} & i = 4t+1 \\ \alpha_{4t+1}\alpha_1 & i = 4t+2 \\ \alpha_{4t+1}\alpha_1^2 & i = 4t+3 \end{cases}$$

while at odd primes we have [35], [25]

$$R(p^i) = \alpha_i$$

demonstrating that the root invariant sends v_0-periodic families to v_1-periodic families. For $p \geq 5$ it is also known [35], [25] that

$$\beta_i \in R(\alpha_i)$$

and

$$\beta_{p/2} \in R(\alpha_{p/2}).$$

These computations led the authors of [25] to regard the root invariants as *defining* the n^{th} Greek letter elements as the root invariants of the $(n-1)^{\text{st}}$ Greek letter elements when the relevant Smith-Toda complexes do not exist.

Other evidence of the root invariant raising chromatic filtration is seen in the cohomology of the Steenrod algebra. Mahowald and Shick define a chromatic filtration on $\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ in [26], and Shick proves that an algebraic version of the root invariant increases chromatic filtration in this context in [36].

The Tate spectrum computations of [11], [10], [27], [17], [2], and [16] indicate that the $\mathbb{Z}/p$-Tate spectrum of a v_n-periodic cohomology theory is v_n-torsion. The root invariant is defined using the Tate spectrum of the sphere spectrum, so the results of the papers listed above provide even more evidence that the root invariant of a v_n-periodic element should be v_n-torsion.

The purpose of this paper is to introduce a new variant of the root invariant called the *filtered root invariant*. We apply the filtered root invariant towards the computation of some new root invariants at the prime 3 in low dimensions. We also compute the root invariants of some infinite v_1-families as infinite v_2-families at the prime 3. We will also describe how the theory of filtered root invariants given in this paper works out to give an alternative perspective on known computations of root invariants at the prime 2.

Let X be a finite p-local spectrum, and let E be a ring spectrum for which the E-Adams spectral sequence (E-ASS) converges. Given $\alpha \in \pi_t(X)$, we define its filtered root invariants $R_E^{[k]}(\alpha)$ to be a sequence of cosets of d_r-cycles in the $E_1^{k,*}$ term of the E-ASS converging to $\pi_*(X)$, where r depends on k. These invariants govern the passage between the E-root invariant $R_E(\alpha)$, or the algebraic root invariant $R_{alg}(\alpha)$, and the elements in the E-ASS that detect $R(\alpha)$. The passage of each of these filtered root invariants to the next is governed by differentials and compositions in the E-ASS. Our method is to use algebraic and E-cohomology computations to determine the first of these filtered root invariants and then iteratively deduce the higher ones from differentials and compositions. Under favorable circumstances, the last of these filtered root invariants will detect the root invariant.

In Section 2, we discuss the E-Adams resolution, and various filtered forms of the Tate spectrum which we shall be using to define the the filtered root invariants.

In Section 3, we define the filtered root invariants. We also recall the definitions of the root invariant, the E-root invariant, and the algebraic root invariant. We will sometimes refer to the root invariant as the "homotopy root invariant" to distinguish it from all of the other variants being used.

Our main results are most conveniently stated using the language of Toda brackets introduced in the appendix of [37]. In Section 4, we define a variant of the Toda bracket, the K-Toda bracket, which is taken to be an attaching map in a fixed finite CW-complex K. Some properties of K-Toda brackets are introduced. We also define a version on the E_r term of the E-ASS.

In Section 5, we state the main results which relate the filtered root invariants to root invariants, Adams differentials, compositions, E-root invariants, and algebraic root invariants. If $R_E^{[k]}(\alpha)$ contains a permanent cycle x, then x detects an element of $R(\alpha)$ modulo Adams filtration $k+1$. If $R_E^{[k]}(\alpha)$ does not contain a permanent cycle, there is a formula relating its E-Adams differential to the next filtered root invariant. Thus if one knows a filtered root invariant, one may sometimes deduce the next one from its E-Adams differential. There is a similar result concerning compositions. The zeroth filtered root invariant $R_E^{[0]}(\alpha)$ is the E-root invariant. The first filtered root invariant $R_E^{[1]}(\alpha)$ is the $E \wedge \overline{E}$-root invariant ($\overline{E}$ is the fiber of the unit of E). If $E = H\mathbb{F}_p$, then the first non-trivial filtered root invariant is the algebraic root invariant $R_{alg}(\alpha)$.

We present the proofs of the main theorems of Section 5 in Section 6. The proofs are technical, and it is for this reason that they have been relegated to their own section.

In Section 7 we give examples of these theorems with $E = bo$, and show how the bo resolution computes the root invariants of the elements 2^k at the prime 2. This was the motivating example for this project.

We will need to make some extensive computations in an algebraic Atiyah-Hirzebruch spectral sequence (AAHSS). This is the spectral sequence obtained by applying $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*(-))$ to the cellular filtration of projective space. We introduce this spectral sequence in Section 8. We compute the d_r differentials for small r using formal group methods.

Calculating homotopy root invariants from our filtered root invariants is a delicate business. In an effort to make our low dimensional calculations easier to follow and less ad hoc, we spell out our methodology in the form of Procedure 9.1 which we follow throughout our low dimensional calculations. The description of this procedure is the subject of Section 9.

In Section 10, we compute some BP-filtered root invariants of the elements p^i as well as of the elements $\alpha_{i/j}$. We find that at every odd prime p

$$\pm\alpha_i \in R_{BP}^{[1]}(p^i)$$
$$\pm\beta_{i/j} \in R_{BP}^{[2]}(\alpha_{i/j})$$

These filtered root invariants hold at the prime 2 modulo an indeterminacy which is identified, but not computed, in this paper. The only exceptions are the cases $i = j = 1$ and $i = j = 2$ at the prime 2 (these cases correspond to the existence of the Hopf invariant 1 elements ν and σ). These filtered root invariants are computed by means of manipulation of formulas in BP_*BP arising from p-typical formal groups.

In Section 11 we compute the root invariants

$$R(\beta_1) = \beta_1^p$$

at primes $p > 2$. These root invariants were announced without proof in [25]. This computation is accomplished by applying our theorems to the Toda differential in the Adams-Novikov spectral sequence (ANSS).

Section 12 is devoted to some 2-primary remarks. We explain how the pattern of differentials and hidden extensions in the ANSS reflect the root invariants $R(2^i)$. We explain conjecturally how our theorems should apply to the classical Adams-Spectral sequence (ASS) and the ANSS to compute many low dimensional root invariants at the prime 2.

In Section 13, we apply the methods described to compute some root invariants of the Greek letter elements $\alpha_{i/j}$ that lie within the 100-stem at the prime 3. At the prime 3, β_i is known to be a permanent cycle for $i \equiv 0, 1, 2, 5, 6 \pmod 9$ [4] and is conjectured to exist for $i \equiv 3 \pmod 9$. The element β_3 is a permanent cycle. One might expect from the previous work for $p \geq 5$ that $\beta_i \in R(\alpha_i)$ when β_i exists. Surprisingly, there is at least one instance where β_i exists, yet is not contained in $R(\alpha_i)$. Our low dimensional computations of $R(\alpha_{i/j})$ at $p = 3$ are summarized in Table 1.

TABLE 1. Low dimensional root invariants of $\alpha_{i/j}$ at $p = 3$

Element	Root Invariant
α_1	β_1
α_2	$\pm\beta_1^2\alpha_1$
$\alpha_{3/2}$	$-\beta_{3/2}$
α_3	β_3
α_4	$\pm\beta_1^5$
α_5	β_5
$\alpha_{6/2}$	$\beta_{6/2}$
α_6	$-\beta_6$

All of these root invariants are v_2-periodic in the sense that they are detected in $\pi_*(L_2S^0)$ [34]. This should be compared to the v_1-periodic situation explored in Section 12 at the prime 2, where sometimes α_i exists but does not detect the homotopy root invariant of 2^i, but the elements $R(2^i)$ are nevertheless v_1-periodic. The lesson we learn is that if one believes that the homotopy Greek letter elements should be determined by iterated root invariants (as suggested in [25]) then they will not always agree with the algebraic Greek letter elements, even when the latter are permanent cycles.

These results will be partly generalized to compute $R(\alpha_i)$ for $i \equiv 0, 1, 5 \pmod 9$ at the prime 3 in Section 16. The remainder of this paper is devoted to providing the machinery necessary for this computation.

In the ANSS the β family lies in low Adams-Novikov filtration, but in the ASS this family is in high filtration. For the purposes of infinite chromatic families, it is often useful to take both spectral sequences into account simultaneously. In Section 14 we explain how our framework can be applied to the Mahowald spectral sequence to compute algebraic root invariants. As mentioned earlier, algebraic root invariants are the first non-trivial $H\mathbb{F}_p$ filtered root invariants.

Infinite families of Greek letter elements are constructed as homotopy classes through the use of Smith-Toda complexes. In their computation of $R(\alpha_i)$ for $p \geq 5$, Mahowald and Ravenel introduce modified root invariants [25] which take values in the homotopy groups of certain Smith-Toda complexes. In Section 15 we adapt our results to modified root invariants.

Our modified root invariant methods are applied in Section 16 to make some new computations of the root invariants of some infinite v_1-periodic families at the prime 3. Specifically, we are able to show that

$$(-1)^{i+1}\beta_i \in R(\alpha_i)$$

for $i \equiv 0, 1, 5 \pmod 9$.

This paper represents the author's dissertation work. The author would like to extend his heartfelt gratitude to his adviser, J. Peter May, for his guidance and encouragement, and to Mark Mahowald, for many enlightening conversations regarding the contents of this paper.

Conventions. Throughout this paper we will be working in the stable homotopy category localized at some prime p. We will always denote the quantity $q = 2(p-1)$, as usual. All ordinary homology will be taken with $\mathbb{F}_p$ coefficients. If $p = 2$, let P_N^M denote the stunted projective space with bottom cell in dimension N and top cell in dimension M. Here M and N may be infinite or negative. See [25] for details. If p is odd, then projective space is replaced by $B\Sigma_p$. The complex $B\Sigma_p$ has a stable cell in every positive dimension congruent to 0 or -1 mod q, and we will use the notation P_N^M to indicate the stunted complex with cells in dimensions between N and M. When $M = \infty$, or when $N = -\infty$ the superscript or subscript may be omitted.

Given a spectrum E, the Tate spectrum

$$\Sigma(E \wedge P)_{-\infty} = \Sigma \mathrm{holim}(E \wedge P_{-n})$$

will be denoted tE. To relate this notation to that in [16], we have

$$tE = t(\iota_* E)^{\mathbb{Z}/p}.$$

There is a unit $S^0 \to tE$. For $E = S^0$, this is the inclusion of the 0-cell. If X is a finite complex, then the Segal conjecture for the group $\mathbb{Z}/p$ [3], [9], [32] (also known as Lin's Theorem [20], [21] at $p = 2$ and Gunawardena's Theorem [18] for $p > 2$) implies that the map

$$X = X \wedge S^0 \to X \wedge tS^0 = tX$$

is p-completion (the last equality requires X to be finite).

Suppose A and B are two subsets of a set C. We shall write $A \overset{\cap}{=} B$ to indicate that $A \cap B$ is nonempty. This is useful notation when dealing with operations with indeterminacy. If we are working over a ring R, we shall use the notation $\doteq$ to indicate that two quantities are equal modulo multiplication by a unit in $R^\times$. We shall similarly use the notation $\dot{\in}$ for containment up to multiplication by a unit.

We will denote the regular ideal $(p, v_1, v_2, \ldots, v_{n-1}) \subseteq BP_*$ by I_n.

Finally, we will be using the following abbreviations for spectral sequences.

ASS: The classical Adams spectral sequence.

ANSS: The Adams-Novikov spectral sequence derived from BP.

E**-ASS:** The generalized Adams spectral sequence derived from a ring spectrum E.

AHSS: The Atiyah-Hirzebruch spectral sequence. We will be using the form that computes stable homotopy groups from homology.

AAHSS: The algebraic Atiyah-Hirzebruch spectral sequence, which uses the cellular filtration to compute $\mathrm{Ext}(X)$.

MSS: The Mahowald spectral sequence, which computes $\mathrm{Ext}(X)$ by applying $\mathrm{Ext}(-)$ to an Adams resolution of X.

2. Filtered Tate spectra

Given a ring spectrum E, we will establish some notation for dealing with the E-Adams resolution. We will then mix the Adams filtration with the skeletal filtration in the Tate spectrum tS^0. These filtered Tate spectra will carry the filtered root invariants defined in Section 3. Our treatment of the Adams resolution follows closely that of Bruner in [8, IV.3].

For E a ring spectrum, let $\overline{E}$ be the fiber of the unit, so there is a cofiber sequence

$$\overline{E} \to S^0 \xrightarrow{\eta} E.$$

For X a spectrum, let $W_k(X)$ denote the k-fold smash power $\overline{E}^{(k)} \wedge X$. We shall also use the notation $W_k^l(X)$ to denote the cofiber

$$W_{k+l+1}(X) \to W_k(X) \to W_k^l(X).$$

We may drop the X from the notation when $X = S^0$. Note that with our definitions $W_k^{k-1}(X) \simeq *$. The E-Adams resolution of X now takes the form

$$\begin{tikzcd}
X \arrow[r, equal] & W_0(X) \arrow[d] & W_1(X) \arrow[l] \arrow[d] & W_2(X) \arrow[l] \arrow[d] & W_3(X) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
& W_0^0(X) & W_1^1(X) & W_2^2(X) & W_3^3(X) &
\end{tikzcd}$$

The notation $W_k^l(X)$ is used because the E-ASS for $W_k^l(X)$ is obtained from the E-ASS for X by setting $E_1^{s,t} = 0$ for $s < k$ and $s > l$, and adjusting the differentials accordingly. If the resolution converges to the p-completion (p-localization) of X, then $W_\infty(X) \simeq *$ in the p-complete (p-local) stable homotopy category, and $W_s^\infty(X) \simeq W_s(X)$.

We shall denote $E_r(X)$ for the E_r term of the E-ASS for X. An element of $E_1^{s,t}(X) = \pi_{t-s}(W_s^s(X))$ is a d_r-cycle if and only if it lifts to an element of $\pi_{t-s}(W_s^{s+r})$. Given an element $\alpha \in \pi_n(X)$ we shall let $filt_E(\alpha)$ denote its E-Adams filtration.

In what follows, the reader may find it helpful to assume that the spectra W_s are CW spectra and the maps

$$W_s \to W_{s-1}$$

are the inclusions of subcomplexes. If this is the case, then in what follows the homotopy colimits may simply regarded as unions. As Bruner points out [8, IV.3.1], this assumption represents no loss of generality, since any infinite tower may be replaced with a tower of inclusions of CW-spectra through the use of CW approximation and mapping telescopes. The Tate spectrum $\Sigma^{-1}tS^0$ is bifiltered, as depicted in the following diagram.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
W_0(P^{N+1}) \arrow[u] & W_1(P^{N+1}) \arrow[l] \arrow[u] & W_2(P^{N+1}) \arrow[l] \arrow[u] & \cdots \arrow[l] \\
W_0(P^{N}) \arrow[u] & W_1(P^{N}) \arrow[l] \arrow[u] & W_2(P^{N}) \arrow[l] \arrow[u] & \cdots \arrow[l] \\
W_0(P^{N-1}) \arrow[u] & W_1(P^{N-1}) \arrow[l] \arrow[u] & W_2(P^{N-1}) \arrow[l] \arrow[u] & \cdots \arrow[l] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

In the above diagram, $W_k(P^N)$ is the spectrum $W_k(P^N)_{-\infty} = \operatorname{holim}_M W_k(P^N_{-M})$, where the homotopy inverse limit is taken *after* smashing with W_k.

Given increasing sequences of integers

$$I = \{k_1 < k_2 < \cdots < k_l\}$$
$$J = \{N_1 < N_2 < \cdots < N_l\}$$

we define subsets $S(I,J)$ of $\mathbb{Z} \times \mathbb{Z}$ by

$$S(I,J) = \bigcup_{i=1}^{l} \{(a,b) \,:\, a \geq k_i, b \leq N_i\}.$$

We give the set of all multi-indices (I,J) the structure of a poset by declaring $(I,J) \leq (I',J')$ if and only if $S(I,J) \subseteq S(I',J')$.

Definition (Filtered Tate spectrum)**.** Given sequences

$$I = \{k_1 < k_2 < \cdots < k_l\}$$
$$J = \{N_1 < N_2 < \cdots < N_l\}$$

with $k_i \geq 0$, we define the filtered Tate spectrum (of the sphere) as the homotopy colimit

$$W_I(P^J) = \bigsqcup_i W_{k_i}(P^{N_i}).$$

We allow for the possibility of $N_l = \infty$. More generally, given another pair of sequences $(I',J') \leq (I,J)$, we define spectra

$$W_I^{I'}(P^J_{J'}) = \text{cofiber}\left(W_{I'+1}(P^{J'-1}) \to W_I(P^J)\right)$$

where $I'+1$ (respectively $J'-1$) is the sequence obtained by increasing (decreasing) every element of the sequence by 1.

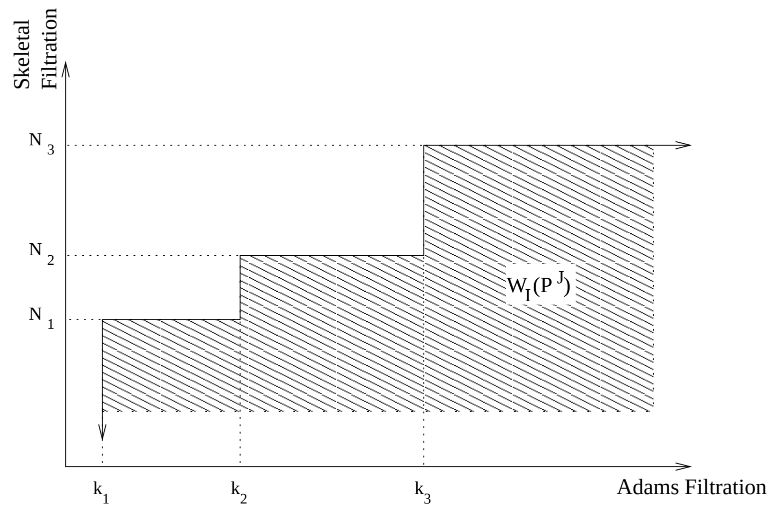

FIGURE 1. The filtered Tate spectrum

Figure 1 displays a diagram of $S(I,J)$ intended to help the reader visualize the filtered Tate spectrum. The entire Tate spectrum is represented by the right half-plane. The shaded region $S(I,J)$ is the portion represented by the filtered Tate spectrum $W_I(P^J)$.

3. Definitions of various forms of the root invariant

In this section we will recall the definitions of the Mahowald root invariant, the E-root invariant, and the algebraic root invariant. We will then define the filtered root invariants.

Definition (*Root invariant*)**.** Let X be a finite complex, and $\alpha \in \pi_t(X)$. The *root invariant* (also called the *Mahowald invariant*) of α is the coset of all dotted arrows making the following diagram commute.

$$\begin{tikzcd}
S^t \arrow[r, dotted] \arrow[d, "\alpha"] & \Sigma^{-N+1}X \arrow[dd] \\
X \arrow[d] & \\
tX \arrow[r] & \Sigma P_{-N} \wedge X
\end{tikzcd}$$

This coset is denoted $R(\alpha)$. Here N is chosen to be minimal such that the composite $S^t \to \Sigma P_{-N} \wedge X$ is non-trivial.

The root invariant is difficult to compute because it involves knowing the homotopy groups of the finite complex X and of $X \wedge P_{-N}$. For this reason, Mahowald and Ravenel [25] introduced a calculable approximation to the root invariant called the E-root invariant, for E a ring spectrum. We find it useful to generalize to arbitrary spectra and to elements of $\pi_*(E)$.

Definition (*E-root invariant*)**.** Let E be a spectrum. Let $x \in \pi_t(E)$. Define the *E-root invariant* of x to be the coset $R_E(\alpha)$ of dotted arrows making the following diagram commute.

$$\begin{tikzcd} S^t \arrow[r, dotted] \arrow[d, "x"] & \Sigma^{-N+1}E \arrow[dd] \\ E \arrow[d] & \\ tE \arrow[r] & \Sigma E \wedge P_{-N} \end{tikzcd}$$

It is quite possible that the composite $S^t \to tE$ is trivial. If this is the case the E-root invariant is said to be *trivial*. Otherwise, in the diagram above we choose N be minimal such that the composite $S^t \to \Sigma E \wedge P_{-N}$ is non-trivial.

If E is a ring spectrum, and $\alpha \in \pi_*(X)$ for X a finite spectrum, let $x = h(\alpha) \in E_*(X)$ be the Hurewicz image of α. We will then refer to the $E \wedge X$-root invariant of $h(\alpha)$ simply as the E-root invariant of α. Therefore, by abuse of notation, we have

$$R_E(\alpha) = R_{E\wedge X}(h(\alpha)).$$

Finally, one may define root invariants in Ext. These are called algebraic root invariants.

Definition (*Algebraic root invariant*)**.** Let α be an element of $\mathrm{Ext}^{s,t}(H_*X)$. We have the following diagram of Ext groups which defines the algebraic root invariant $R_{alg}(\alpha)$.

$$\begin{tikzcd} \mathrm{Ext}^{s,t}(H_*X) \arrow[r, squiggly, "R_{alg}(-)"] \arrow[d, "f"] & \mathrm{Ext}^{s,t+N-1}(H_*X) \arrow[d, "\iota_N"] \\ \mathrm{Ext}^{s,t-1}(H_*P_{-\infty} \wedge X) \arrow[r, "\nu_N"'] & \mathrm{Ext}^{s,t-1}(H_*P_{-N} \wedge X) \end{tikzcd}$$

Here f is induced by the inclusion of the -1-cell of $P_{-\infty}$, ν_N is the projection onto the $-N$-coskeleton, ι_N is inclusion of the $-N$-cell, and N is minimal with respect to the property that $\nu_N \circ f(\alpha)$ is non-zero. Then the algebraic root invariant $R_{alg}(\alpha)$ is defined to be the coset of lifts $\gamma \in \mathrm{Ext}^{s,t+N-1}(H_*X)$ of the element $\nu_N \circ f(\alpha)$.

We wish to extend these definitions to a sequence of filtered root invariants that appear in the E-Adams resolution. Suppose that X is a finite complex and $\alpha \in \pi_t(X)$. We want to lift α over the smallest possible filtered Tate spectrum. To this end, we shall describe a pair of sequences

$$I = \{k_1 < k_2 < \cdots < k_l\}$$
$$J = \{-N_1 < -N_2 < \cdots < -N_l\}$$

associated to α, which we define inductively. Let $k_1 \geq 0$ be maximal such that the composite

$$S^{t-1} \xrightarrow{\alpha} \Sigma^{-1}X \to \Sigma^{-1}tX \to W_0^{k_1-1}(P \wedge X)_{-\infty}$$

is trivial. Next, choose N_1 to be maximal such that the composite

$$S^{t-1} \xrightarrow{\alpha} \Sigma^{-1}X \to \Sigma^{-1}tX \to W_0^{(k_1-1,k_1)}(P_{(-N_1+1,\infty)} \wedge X)$$

is trivial. Inductively, given

$$I' = (k_1, k_2, \dots, k_i)$$
$$J' = (-N_1, -N_2, \dots, -N_i)$$

let k_{i+1} be maximal so that the composite

$$S^{t-1} \xrightarrow{\alpha} \Sigma^{-1}X \to \Sigma^{-1}tX \to W_0^{(I'-1,k_{i+1}-1)}(P_{(J'+1,\infty)} \wedge X)$$

is trivial. If there is no such maximal k_{i+1}, we declare that $k_{i+1} = \infty$ and we are finished. Otherwise, choose N_{i+1} to be maximal such that the composite

$$S^{t-1} \xrightarrow{\alpha} \Sigma^{-1}X \to \Sigma^{-1}tX \to W_0^{(I'-1,k_{i+1}-1,k_{i+1})}(P_{(J'+1,-N_{i+1}+1,\infty)} \wedge X)$$

is trivial, and continue the inductive procedure. We shall refer to the pair (I, J) as the E-bifiltration of α.

Observe that there is an exact sequence

$$\pi_{t-1}(W_I(P^J \wedge X)) \to \pi_t(tX) \to \pi_{t-1}(W^{I-1}(P_{J+1} \wedge X)).$$

Our choice of (I, J) ensures that the image of α in $\pi_{t-1}(W^{I-1}(P_{J+1} \wedge X))$ is trivial. Thus α lifts to an element $f^\alpha \in \pi_{t-1}(W_I(P^J \wedge X))$.

Definition (*Filtered root invariants*)**.** Let X be a finite complex, let E a ring spectrum such that the E-Adams resolution converges, and let α be an element of $\pi_t(X)$ of E-bifiltration (I, J). Given a lift $f^\alpha \in \pi_{t-1}(W_I(P^J \wedge X))$, the k^{th} filtered root invariant is said to be trivial if $k \neq k_i$ for any $k_i \in I$. Otherwise, if $k = k_i$ for some i, we say that the image β of f^α under the collapse map

$$\pi_{t-1}(W_I(P^J \wedge X)) \to \pi_{t-1}(W_{k_i}^{k_i}(\Sigma^{-N_i}X))$$

is an element of the k^{th} *filtered root invariant* of α. The k^{th} filtered root invariant is the coset $R_E^{[k]}(\alpha)$ of $E_1^{k,t+k+N_i-1}(X)$ of all such β as we vary the lift f^α.

Remark 3.1. Let r_i denote the difference $k_{i+1} - k_i$. Then there is a factorization

$$\pi_{t-1}(W_I(P^J \wedge X)) \to \pi_{t-1}(W_{k_i}^{k_{i+1}-1}(\Sigma^{-N_{k_i}}X)) \to \pi_{t-1}(W_{k_i}^{k_i}(\Sigma^{-N_{k_i}}X))$$

Thus any such $\beta \in R_E^{[k]}(\alpha)$ is actually a d_r-cycle for $r < r_i$.

4. The Toda bracket associated to a complex

In this section we introduce a variant of the Toda bracket. This treatment is essentially a specialization of the treatment of Toda brackets given in the appendix of [37]. Suppose K is a finite CW-spectrum with one bottom dimensional cell and one top dimensional cell. Suspend K accordingly so that it is connective and n-dimensional with one cell in dimension zero and one cell in dimension n. For some other spectrum X we shall define the K*-Toda bracket* to be an operator which, when defined, takes an element of $\pi_t(X)$ to a coset of $\pi_{t+n-1}(X)$. We shall present a dual definition, and show this dual definition is equivalent to our original definition. We will also define a variant on the E_r term of the E-ASS.

In what follows, we let K^j be the j-skeleton of K, and let K_i^j be the quotient K^j/K^{i-1}. We shall omit the top index for the i-coskeleton $K_i = K/K^{i-1}$.

Definition (*K-Toda bracket*)**.** Let

$$f : \Sigma^{-1} K_1 \to S^0$$

be the attaching map of the 1-coskeleton of K to the 0-cell, so that the cofiber of f is K. Let $\nu : K_1 \to S^n$ be the projection onto the top cell. Suppose α is an element of $\pi_t(X)$. We have

$$\pi_t(X) \xleftarrow{\nu_*} \pi_{t+n}(X \wedge K_1) \xrightarrow{f_*} \pi_{t+n-1}(X).$$

We say the K-Toda bracket

$$\langle K \rangle(\alpha) \subseteq \pi_{t+n-1}(X)$$

is *defined* if α is in the image of ν_*. Then the K-Toda bracket is the collection of all $f_*(\gamma) \in \pi_{t+n-1}(X)$ where $\gamma \in \pi_{t+n}(X \wedge K_1)$ is any element satisfying $\nu_*(\gamma) = \alpha$. If $\langle K \rangle(\alpha)$ contains only one element, we will say that it is *strictly defined*.

Remark 4.1. If the $(n-1)$-skeleton K^{n-1} is coreducible, then the attaching map f factors as a composite

$$f : \Sigma^{-1} K_1 \xrightarrow{\nu} S^{n-1} \xrightarrow{\xi} S^0.$$

Then the product $\xi \cdot \alpha$ is in $\langle K \rangle(\alpha)$.

We remark on the relationship to Shipley's treatment of Toda brackets in triangulated categories given in the appendix of [37]. In Shipley's terminology, if K is an m-filtered object in $\{f_1, f_2, \dots, f_{m-1}\}$ with $F_1 K \simeq S^0$ and $F_m K / F_{m-1} K \simeq S^n$, then we have the equality (modulo indeterminacy)

$$\langle K \rangle(\alpha) \overset{\cap}{=} \langle f_1, f_2, \dots f_{m-1}, \alpha \rangle.$$

In particular, if we take K to be the 3-filtered object

$$* \subseteq S^0 \subseteq K^{n-1} \subseteq K$$

with attaching maps

$$\begin{aligned} f_1 :& \Sigma^{-1} K_1^{n-1} \to S^0 \\ f_2 :& S^{n-1} \to K_1^{n-1} \end{aligned}$$

then we have the equality (modulo indeterminacy)

$$\langle K \rangle(\alpha) \overset{\cap}{=} \langle f_1, f_2, \alpha \rangle.$$

The reason we don't have exact equalities is that with the Toda bracket, you can get more indeterminacy by varying the m-filtered object, whereas the m-filtered object is fixed in the case of the K-Toda bracket. However, the K-Toda bracket can also get indeterminacy that is not seen by the Toda bracket if $X \neq S^0$, because there are potentially more choices of lift γ if you smash with X.

In some of our applications the following dual variant will be more natural to work with. Our use of the word 'dual' stems from the use the skeletal instead of coskeletal filtration of K.

Definition (*Dual definition of the K-Toda bracket*)**.** Let

$$g : S^{n-1} \to K^{n-1}$$

be the attaching map of the n-cell of K to the $(n-1)$-skeleton, so that the cofiber of g is K. Let $\iota : S^0 \to K^{n-1}$ be the inclusion of the bottom cell. Suppose α is an element of $\pi_t(X)$. We have

$$\pi_t(X) \xrightarrow{g_*} \pi_{t+n-1}(X \wedge K^{n-1}) \xleftarrow{\iota_*} \pi_{t+n-1}(X)$$

We say the (dual) K-Toda bracket

$$\langle K \rangle(\alpha) \subseteq \pi_{t+n-1}(X)$$

is *defined* if $g_*(\alpha)$ is in the image of ι_*. Then the K-Toda bracket is the collection of all $\gamma \in \pi_{t+n-1}(X)$ where $\iota_*(\gamma) = g_*(\alpha)$.

We reconcile our use of the same notation in both definitions with the following lemma.

Lemma 4.2. The K-Toda bracket and the dual K-Toda bracket are equal.

Proof. Let α, f, g, ι, and ν be the maps given in our two definitions of the K-Toda bracket. For the purposes of this proof, we shall refer to the dual K-Toda bracket as $\langle K \rangle^d(\alpha)$. The map of cofiber sequences makes the following diagram commute.

$$\begin{tikzcd}
S^0 \arrow[r] \arrow[d, "\iota"'] & K \arrow[r] \arrow[d, equal] & K_1 \arrow[r, "f"] \arrow[d, "\nu"'] & S^1 \arrow[d, "\iota"] \\
K^{n-1} \arrow[r] & K \arrow[r] & S^n \arrow[r, "g"'] & \Sigma K^{n-1}
\end{tikzcd}$$

Taking the last square in the above triangle, extending it to a map of cofiber sequences the other way, and smashing with X, we get the following commutative diagram whose columns are exact.

$$\begin{tikzcd}
\pi_{t+n}(X \wedge K_1^{n-1}) \arrow[r, equal] \arrow[d, "\iota'_*"'] & \pi_{t+n}(X \wedge K_1^{n-1}) \arrow[d, "f'_*"] \\
\pi_{t+n}(X \wedge K_1) \arrow[r, "f_*"] \arrow[d, "\nu_*"'] & \pi_{t+n-1}(X) \arrow[d, "\iota_*"] \\
\pi_t(X) \arrow[r, "g_*"'] \arrow[d, "g'_*"'] & \pi_{t+n-1}(X \wedge K^{n-1}) \arrow[d, "\nu'_*"] \\
\pi_{t+n-1}(X \wedge K_1^{n-1}) \arrow[r, equal] & \pi_{t+n-1}(X \wedge K_1^{n-1})
\end{tikzcd}$$

Here ι' is the inclusion and ν' is the projection, and f' and g' are determined from the diagram. The bracket $\langle K \rangle(\alpha)$ is computed by taking the preimage under ν_* and then applying f_*. The bracket $\langle K \rangle^d(\alpha)$ is computed applying g_* and then taking the preimage over ι_*. We see that for any lift $\gamma \in \pi_{t+n}(X \wedge K_1)$ of α, $f_*(\gamma)$ is a lift of $g_*(\alpha)$. It follows that we have $\langle K \rangle(\alpha) \subseteq \langle K \rangle^d(\alpha)$.

For the reverse containment, suppose that $\beta \in \pi_{t+n-1}(X)$ is a lift of $g_*(\alpha)$. Then β is an arbitrary element of $\langle K \rangle^d(\alpha)$. Let $\gamma \in \pi_{t+n}(X \wedge K_1)$ be any lift if α. Such a lift exists since

$$g'_*(\alpha) = \nu'_* g_*(\alpha) = \nu'_* \iota_*(\beta) = 0.$$

We must add a correction term to γ since it is not necessarily the case that $f_*(\gamma) = \beta$. Let δ be the difference $f_*(\gamma) - \beta$. Then $\iota_*(\delta) = 0$, so there is a lift to $\widetilde{\delta} \in$

$\pi_{t+n}(X \wedge K_1^{n-1})$. Then $\gamma' = \gamma - \iota'_*(\widetilde{\delta})$ is another lift of α, and $f_*(\gamma) = \beta$. We have therefore proven that $\langle K \rangle^d(\alpha) \subseteq \langle K \rangle(\alpha)$. □

Remark 4.3. Perhaps a more conceptual way to prove Lemma 4.2 would be to regard the K-Toda bracket as a d_n in the AHSS for K with respect to the coskeletal filtration, and the dual K-Toda bracket as a d_n in the AHSS with respect to the skeletal filtration. There is a comparison of these spectral sequences which is an isomorphism on E_1-terms, hence the spectral sequences are isomorphic, and in particular they have the same differentials d_n. See, for instance, Appendix B of [16].

We need to extend our K-Toda brackets to operations on the E_r term of the E-ASS. Suppose that the attaching map $f : \Sigma^{-1} K_1 \to S^0$ has E-Adams filtration d.

Definition (*K-Toda brackets on $E_r(X)$*)**.** Suppose that α is an element of $E_r^{s,t}(X)$. Choose a lift of α to $\widetilde{\alpha} \in \pi_{t-s}(W_s^{s+r-1}(X))$. Let $\widetilde{f} : \Sigma^{-1}K_1 \to W_d(S^0)$ be a lift of f. Then we have

$$\pi_{t-s}(W_s^{s+r-1}(X)) \xleftarrow{\nu_*} \pi_{t-s+n}(W_s^{s+r-1}(X \wedge K_1)) \xrightarrow{\widetilde{f}_*} \pi_{t+n-1}(W_{s+d}^{s+d+r-1}(X)).$$

If there exists a lift $\widetilde{\alpha}$ which is in the image of ν_*, then we say that the K-Toda bracket is *defined*. Take $\gamma \in \pi_{t-s+n}(W_s^{s+r-1}(X))$ such that $\nu_*(\gamma) = \widetilde{\alpha}$. Then the image of $\widetilde{f}_*(\gamma)$ in $E_r^{s+d,t+d+n-1}(X)$ is in the K-Toda bracket $\langle K \rangle(\alpha)$. We define the K-Toda bracket to be the set of all such images for various choices of $\widetilde{\alpha}$, γ, and $\widetilde{f}$. We shall say that the K-Toda bracket has *E-Adams degree d*.

Remark 4.4. If the K-Toda bracket of α is defined, where $\alpha \in E_\infty^{s,t}(X)$, then if α detects $\overline{\alpha}$, every element of $\langle K \rangle(\alpha)$ detects an element of $\langle K \rangle(\overline{\alpha})$. In particular, if $\langle K \rangle(\alpha)$ contains 0, then there is an element of $\langle K \rangle(\widetilde{\alpha})$ of E-Adams filtration greater than $k + d$.

5. Statement of results

In this section we will state our main results concerning filtered root invariants. The proofs of many of the theorems are given in Section 6. Throughout this section let E be a ring spectrum, let X be a finite complex, and suppose α is an element of $\pi_t(X)$ of E-bifiltration (I, J) with

$$I = (k_1, k_2, \dots, k_l)$$
$$J = (-N_1, -N_2, \dots, -N_l)$$

Theorem 5.1 (*Relationship to homotopy root invariant*)**.** Suppose that $R_E^{[k_i]}(\alpha)$ contains a permanent cycle β. Then there exists an element $\overline{\beta} \in \pi_*(X)$ which β detects such that the following diagram commutes up to elements of E-Adams filtration greater than or equal to k_{i+1}.

$$\begin{tikzcd}
S^t \arrow[r, "\overline{\beta}"'] \arrow[d, "\alpha"'] & \Sigma^{-N_i+1}X \arrow[dd] \\
X \arrow[d] & \\
tX \arrow[r] & \Sigma P_{-N_i} \wedge X
\end{tikzcd}$$

The proof of Theorem 5.1 is deferred to Section 6. If $k_{i+1} = \infty$, then i is equal to l (the maximal index of the bifiltration), and we get the following corollary.

Corollary 5.2. The top filtered root invariant $R_E^{[k_l]}(\alpha)$ is contained in $E_\infty(X)$. Let $\overline{R}_E^{[k_l]}(\alpha)$ be the set of elements of $\pi_*(X)$ detected by $R_E^{[k_l]}(\alpha)$. There are two possibilities:

1. The image of the elements of $\overline{R}_E^{[k_l]}(\alpha)$ in $\pi_*(P_{-N_l} \wedge X)$ under the inclusion of the bottom cell is zero, and the homotopy root invariant lies in a higher stem than the k_l^{th} filtered root invariant.
2. There is an equality (modulo indeterminacy)
$$\overline{R}_E^{[k_l]}(\alpha) \overset{\cap}{=} R(\alpha)$$

Proof. Since $k_l = \infty$, the diagram in Theorem 5.1 commutes modulo elements of infinite Adams filtration. The E-Adams resolution was assumed to converge, so this means the diagram actually commutes. Let $\overline{\beta}$ be the element described in Theorem 5.1. If the image of $\overline{\beta}$ in $\pi_{t-1}(P_{-N_i})$ is zero, then the homotopy root invariant lies in a higher stem than the i^{th} filtered root invariant. Otherwise, $\overline{\beta}$ is an element of $R(\alpha)$. □

Unfortunately, Corollary 5.2 is difficult to invoke in practice. This is because given a filtered root invariant, one usually does not know whether it is the highest one or not. In practice we are in the situation where we have a permanent cycle in a filtered root invariant and we would like to show that the diagram of Theorem 5.1 commutes on the nose. One strategy is to write out the Atiyah-Hirzebruch spectral sequence for $\pi_*(P_{-N_i})$ and try to show there are no elements of higher E-Adams filtration which could be the difference between the image of α and the image of $\overline{\beta}$ in $\pi_*(P_{-N_i})$. This method is outlined in Procedure 9.1.

We now present some theorems which relate filtered root invariants to differentials and compositions in the Adams spectral sequence. Recall that $r_j = k_{j+1} - k_j$.

Theorem 5.3 (*Relationship to Adams differentials*)**.** Suppose that the $P_{-N_i}^{-N_{i+1}}$-Toda bracket has E-Adams degree d and that $d \le r_{i+1}$. Then the following is true.

1. $\langle P_{-N_i}^{-N_{i+1}} \rangle (R_E^{[k_{i+1}]}(\alpha))$ is defined and contains a permanent cycle.
2. $R_E^{[k_i]}(\alpha)$ consists of elements which are d_r cycles for $r < r_i + d$.
3. There is a containment
$$d_{r_i+d} R_E^{[k_i]}(\alpha) \subseteq \langle P_{-N_i}^{-N_{i+1}} \rangle (R_E^{[k_{i+1}]}(\alpha))$$
where both elements are thought of as elements of $E_{r_i+d}^{*,*}(X)$.

The proof of Theorem 5.3 is deferred to Section 6. We intend to use Theorem 5.3 in reverse: given the i^{th} filtered root invariant, we would like to deduce the $(i+1)^{\text{st}}$ filtered root invariant from the presence of an Adams differential. If the differential in Theorem 5.3 is zero, we still may be able to glean some information from the presence of a non-trivial composition.

Theorem 5.4 (*Relationship to compositions*)**.** Suppose that $R_E^{[k_i]}(\alpha)$ is a coset of permanent cycles. Let $\widetilde{R}_E^{[k_i]}(\alpha)$ denote the coset of all lifts of elements of $R_E^{[k_i]}(\alpha)$ to $\pi_{t+N_i-1}(W_{k_i}(X))$. When defined, there are lifts of the Toda brackets $\langle P_{-m}^{-N_i}\rangle(\widetilde{R}_E^{[k_i]}(\alpha))$ to

$$\widetilde{\langle P_{-m}^{-N_i}\rangle}(\widetilde{R}_E^{[k_i]}(\alpha)) \subseteq \pi_{t+m-2}(W_{k_{i+1}}(X)).$$

Let M be the minimal such $m > N_i$ with the property that $\widetilde{\langle P_{-m}^{-N_i}\rangle}(\widetilde{R}_E^{[k_i]}(\alpha))$ contains a non-trivial element. Let d be the E-Adams degree of the Toda bracket $\langle P_{-N_i}^{-N_{i+1}}\rangle(-)$, and suppose that $d \le r_{i+1}$. Then the following is true.

(1) Let

$$\overline{R}_E^{[k_i]}(\alpha) \subseteq \pi_{t+N_i-1}(X)$$

denote the elements which are detected by the permanent cycles of $R_E^{[k_i]}(\alpha)$. Then the Toda bracket

$$\langle P_{-M}^{-N_i}\rangle(\overline{R}_E^{[k_i]}(\alpha)) \subseteq \pi_{t+M-2}(X)$$

is defined.

(2) The Toda bracket $\langle P_{-M}^{-N_{i+1}}\rangle(R_E^{[k_{i+1}]}(\alpha))$ is defined in $E_{r_{i+1}}(X)$, and contains a permanent cycle. We shall denote the collection of all elements which are detected by these permanent cycles by

$$\overline{\langle P_{-M}^{-N_{i+1}}\rangle(R_E^{[k_{i+1}]}(\alpha))} \subseteq \pi_{t+M-2}(X).$$

(3) There is an equality (modulo indeterminacy)

$$\langle P_{-M}^{-N_i}\rangle(\overline{R}_E^{[k_i]}(\alpha)) \overset{\cap}{=} \overline{\langle P_{-M}^{-N_{i+1}}\rangle(R_E^{[k_{i+1}]}(\alpha))}.$$

Remark 5.5. The hypothesis $d \le r_{i+1}$ in Theorems 5.3 and 5.4 is a necessary technical hypothesis to make the proofs work. In practice, d is often equal to 1. Since r_j is always positive, the hypothesis is satisfied in this case.

Finally, we give a partial description of the first filtered root invariant. We begin with a simple observation.

Lemma 5.6. If $filt_E(\alpha) = k$, then $R_E^{[s]}(\alpha)$ is trivial for $s < k$.

Proof. We must show that in the E-bifiltration of α, $k_1 \ge k$. Consider the following diagram.

$$\begin{tikzcd}
\pi_t(X) \arrow[r] \arrow[d] & \pi_t(W_0^{k-1}(X)) \arrow[d] \\
\pi_t(tX) \arrow[r] & \pi_{t-1}(W_0^{k-1}(P_{-\infty}))
\end{tikzcd}$$

Since $filt_E(\alpha) = k$, the image of α in $\pi_t(W_0^{k-1}(X))$ is trivial. Therefore the image of α in $\pi_{t-1}(W_0^{k-1}(P_{-\infty}))$ is trivial. By the maximality of k_1, we have $k_1 \ge k$. $\square$

If $filt_E(\alpha) = 0$, (i.e. when α has a non-trivial Hurewicz image) then we can sometimes identify the first non-trivial filtered root invariant with the E-root invariant.

Proposition 5.7 (*Relationship to the E-root invariant*)**.** The E-root invariant $R_E(\alpha)$ is non-trivial if and only if $R_E^{[0]}(\alpha)$ is non-trivial. If this is the case, regarding $R_E(\alpha)$ as being contained in $E_1^{0,*}(X)$, we have

$$R_E^{[0]}(\alpha) \subseteq R_E(\alpha).$$

Proof. This is immediate from the definitions. □

If $filt_E(\alpha) = 1$, then the E-root invariant is trivial and Lemma 5.6 implies that the zeroth filtered root invariant is trivial. We can however sometimes compute the first filtered root invariant using the $E \wedge \overline{E}$-root invariant.

Proposition 5.8 (*Relationship to the $E\wedge\overline{E}$-root invariant*)**.** Suppose that α has E-Adams filtration 1. Then there exists an element $\widetilde{\alpha} \in \pi_t(E\wedge\overline{E}\wedge X_p^\wedge) = E_1^{1,t+1}(X_p^\wedge)$ which detects α in the E-ASS, and such such that $R_{E\wedge\overline{E}}(\widetilde{\alpha})$ is trivial if and only if $R_E^{[1]}(\alpha)$ is trivial. There is a containment

$$R_E^{[1]}(\alpha) \subseteq R_{E\wedge\overline{E}}(\widetilde{\alpha}).$$

In practice, we will not know what choice of detecting element $\widetilde{\alpha}$ to choose, so the following corollary will prove useful.

Corollary 5.9. Suppose that $filt_E(\alpha) = 1$. If $\widetilde{\alpha} \in \pi_t(E \wedge \overline{E} \wedge X)$ is any element which detects α in the E-ASS, then

$$R_E^{[1]}(\alpha) \subseteq R_{E\wedge\overline{E}}(\widetilde{\alpha}) + A.$$

Here A is the image of the map

$$\pi_t(W_0^{(0,1)}(P_{(-N,-N+1)} \wedge X)) \xrightarrow{\partial} \pi_{t-1}(W_1^1(\Sigma^{-N}X))$$

where ∂ is the boundary homomorphism associated to the cofiber sequence

$$W_1^1(\Sigma^{-N}X) \xrightarrow{\iota} W_0^1(P_{-N} \wedge X) \to W_0^{(0,1)}(P_{(-N,-N+1)} \wedge X)$$

and the $E \wedge \overline{E}$-root invariant of $\widetilde{\alpha}$ is carried by the $-N$-cell of $P_{-\infty}$.

The proof of Proposition 5.8 and Corollary 5.9 is deferred to Section 6. When $E = H\mathbb{F}_p = H$, we can identify the first filtered root invariant with the algebraic root invariant.

Theorem 5.10 (*Relationship to the algebraic root invariant*)**.** If E is the Eilenberg-MacLane spectrum $H\mathbb{F}_p = H$ and α has Adams filtration k, then $k_1 = k$. Furthermore, the filtered root invariant $R_H^{[k]}(\alpha)$ consists of d_1 cycles which detect a coset of non-trivial elements $\overline{R}_H^{[k]}(\alpha) \subseteq E_2^{k,t+k+N_1-1}(X)$, and there exists a choice of $\widetilde{\alpha} \in E_2^{k,t+k}(X)$ which detects α in the ASS such that

$$\overline{R}_H^{[k]}(\alpha) \subseteq R_{alg}(\widetilde{\alpha}).$$

The proof of Theorem 5.10 is deferred to Section 6. We have given a partial scenario as to how the filtered root invariants can be used to calculate root invariants using the E-ASS. One first calculates the zeroth filtered root invariant as an E-root invariant or the first non-trivial root invariant as an algebraic root invariant. Then the idea, while only sometimes correct, is that "if a filtered root invariant does not detect the root invariant, then it either supports a differential or a composition that points to the next filtered root invariant." The last filtered root invariant then has a chance of detecting the homotopy root invariant. We stress that many things can interfere with this actually happening.

6. Proofs of the main theorems

In all of the proofs below, we shall assume that our finite complex X is actually S^0. The general case is no different, but smashing everything with X complicates the notation.

Proof of Theorem 5.1. Let β be an element of $R_E^{[k_i]}(\alpha)$. Then there exists a lift f^α such that β is the image of f^α under the collapse map

$$\pi_{t-1}(W_I(P^J)) \to \pi_{t-1}(W_{k_i}^{k_{i+1}-1}(S^{-N_i})).$$

Consider the following diagram.

$$\begin{tikzcd}
 & S^{t-1} \arrow[d, "\alpha"'] \arrow[ddrrr, "f^\alpha"] \arrow[ddddl, dotted, "\overline{\beta}"'] \arrow[ddddr, dotted, "\widetilde{\beta}"] & & & & \\
 & S^{-1} \arrow[d] & & & & \\
 & P_{-\infty} \arrow[d, "\overline{\nu}"'] & & & W_I(P^J) \arrow[d, "\nu"'] & \\
 & P_{-N_i} & & & W_{k_i}(P_{-N_i}) \arrow[lll] \arrow[dd, "\eta"'] & W_{k_{i+1}}(P_{-N_i}) \arrow[l] \\
S^{-N_i} \arrow[ur, "\overline{\iota}"'] & & W_{k_i}(S^{-N_i}) \arrow[ll] \arrow[urr, "\iota"'] \arrow[dd, "\eta"'] & & & \\
 & & & & W_{k_i}^{k_{i+1}-1}(P_{-N_i}) & \\
 & & W_{k_i}^{k_{i+1}-1}(S^{-N_i}) \arrow[urr, "\iota"'] & & &
\end{tikzcd}$$

In the above diagram, ν is induced from the composite

$$W_I(P^J) \to W_{I'}(P_{-N_i}^{J'}) \to W_{k_i}(P_{-N_i})$$

where $I' = (k_i, \ldots, k_l)$ and $J' = (-N_i, \ldots, -N_l)$. Since $R_E^{[k_i]}(\alpha)$ contains a permanent cycle β, there exists a map $\widetilde{\beta}$ (as above) such that $\eta\widetilde{\beta}$ projects to β, and such that $\iota\eta\widetilde{\beta} = \eta\nu f^\alpha$. If $\delta = \iota\widetilde{\beta} - \nu f^\alpha$, then δ lifts to $\pi_{t-1}(W_{k_{i+1}}(P_{-N_i}))$. Let $\overline{\beta}$ be the map induced by $\widetilde{\beta}$, and denote by $\overline{\delta} \in \pi_{t-1}(P_{-N_i})$ the image of δ. Then $filt_E(\overline{\delta}) \geq k_{i+1}$ and have the following formula.

$$\overline{\iota} \circ \overline{\beta} = \overline{\nu} \circ \alpha + \overline{\delta}$$

This is precisely what we wanted to prove. □

Proof of Theorem 5.3. Fix a lift $f^\alpha \in \pi_{t-1}(W_I(P^J))$ of α. Define a spectrum

$$U = W_{(k_i,k_{i+1},k_{i+2})}(P_{-N_i}^{(-N_i,-N_{i+1},\infty)}).$$

There is a natural map
$$W_I(P^J) \to U$$
and let $\gamma \in \pi_{t-1}(U)$ be the image of f^α under this map.

Since we have assumed that the $P_{-N_i}^{-N_{i+1}}$-Toda bracket has E-Adams degree d, there is a lift of the attaching map
$$f : \Sigma^{-1} P_{-N_i+1}^{-N_{i+1}} \to S^{-N_i}$$
to a map
$$\widetilde{f} : \Sigma^{-1} P_{-N_i+1}^{-N_{i+1}} \to W_d(S^{-N_i})$$
Define a filtered stunted projective space $(P_{-N_i}^{-N_{i+1}})_{[d]}$ by the following cofiber sequence.
$$\Sigma^{-1} P_{-N_i+1}^{-N_{i+1}} \xrightarrow{\widetilde{f}} W_d(S^{-N_i}) \to (P_{-N_i}^{-N_{i+1}})_{[d]}$$
The spectrum U is given by the homotopy pushout
$$\left(W_{k_{i+2}}(P_{-N_i}) \sqcup_{W_{k_{i+2}}(P_{-N_i}^{-N_{i+1}})} W_{k_{i+1}}(P_{-N_i}^{-N_{i+1}}) \right) \sqcup_{W_{k_{i+1}}(S^{-N_i})} W_{k_i}(S^{-N_i}).$$
Since we have assumed that $d \le r_{i+1} = k_{i+2} - k_{i+1}$, the spectrum U admits the equivalent description as
$$\left(W_{k_{i+2}}(P_{-N_i}) \sqcup_{W_{k_{i+2}}(P_{-N_i}^{-N_{i+1}})} W_{k_{i+1}}((P_{-N_i}^{-N_{i+1}})_{[d]}) \right) \sqcup_{W_{k_{i+1}+d}(S^{-N_i})} W_{k_i}(S^{-N_i}).$$
Consider the following commutative diagram.
$$\begin{tikzcd}
& \pi_{t-2}(W_{k_{i+1}+d}^{k_{i+2}+d-1}(S^{-N_i})) & & & \\
& & & \pi_{t-2}(W_{k_{i+1}+d}(S^{-N_i})) \arrow[ull, "\eta"'] \arrow[dr] & \\
& \pi_{t-1}(W_{k_{i+1}}^{k_{i+2}-1}(P_{-N_i+1}^{-N_{i+1}})) \arrow[uu, "\widetilde{f}"] \arrow[dl, "\nu"'] & & & \pi_{t-2}(W_{k_{i+1}}(S^{-N_i})) \\
\pi_{t-1}(W_{k_{i+1}}^{k_{i+2}-1}(S^{-N_{i+1}})) \arrow[uuur, squiggly, bend left, "\langle P_{-N_i}^{-N_i+1} \rangle(-)"] & & & \pi_{t-1}(U) \arrow[lll] \arrow[ull, "g_2"'] \arrow[uu, "\partial_{tot}"] \arrow[r, "g_3"'] & \pi_{t-1}(W_{k_i}^{k_{i+1}-1}(S^{-N_i})) \arrow[u, "\partial"']
\end{tikzcd}$$
Here g_3 is obtained by collapsing out the first and second factors of the homotopy pushout U, and similarly g_2 is obtained by collapsing out the first and third factors. The map ∂_{tot} is the boundary homomorphism of the Meyer-Vietoris sequence, and may be thought of as collapsing out all of the factors of the homotopy pushout. The wavy arrow indicates that the $P_{-N_i}^{-N_{i+1}}$-Toda bracket is taken by taking the inverse image under ν, followed by application of $\widetilde{f}$.

Our element $\gamma \in \pi_{t-1}(U)$ has compatible images in all of the other groups in the diagram. The image of γ under g_3 projects to an element of $R_E^{[k_i]}(\alpha)$ in E_{r_i}. Following the outside of the diagram from the lower right-hand corner to the top corner counter-clockwise amounts to taking d_{r_i+d} in the E-ASS. Following from the lower right-hand corner to the top clockwise applies the Toda bracket to the k_{i+1}^{st} filtered root invariant. Thus $d_{r_i+d} R_E^{[k_i]}(\alpha)$ and $\langle P_{-N_i}^{-N_{i+1}} \rangle (R_E^{[k_{i+1}]}(\alpha))$ have a common element. $\square$

Proof of Theorem 5.4. Fix a lift $f^\alpha \in \pi_{t-1}(W_I(P^J))$ of α. Let V_m be defined by

$$V_m = W_{(0,k_i,k_{i+1},k_{i+2})}(P_{-m}^{(-N_{i-1},-N_i,-N_{i+1},\infty)})$$

Then V_m is defined by the following homotopy pushout.

$$\begin{gathered} P_{-m}^{-N_{i-1}} \sqcup_{W_{k_i}(P_{-m}^{-N_{i-1}})} W_{k_i}(P_{-m}^{-N_i}) \sqcup_{W_{k_{i+1}}(P_{-m}^{-N_i})} \\ W_{k_{i+1}}(P_{-m}^{-N_{i+1}}) \sqcup_{W_{k_{i+2}}(P_{-m}^{-N_{i+1}})} W_{k_{i+2}}(P_{-m}) \end{gathered}$$

There is a natural map

$$W_I(P^J) \to V_m$$

and we define $\gamma_m \in \pi_{t-1}(V_m)$ to be the image of f^α under this map. We need to lift γ_m a little more. The composite

$$\pi_{t-1}(V_m) \xrightarrow{\partial} \pi_{t-2}(W_{k_{i+1}}(P_{-m}^{-N_i})) \to \pi_{t-2}(W_{k_{i+1}}(S^{-N_i}))$$

sends γ_m to zero, since it carries the E-Adams differential of an element in $R_E^{[k_i]}(\alpha)$, which was hypothesized to be zero. Here ∂ is a Meyer-Vietoris boundary homomorphism. Thus our element γ_m lifts to an element $\widetilde{\gamma}_m \in \pi_{t-1}(\widetilde{V}_m)$ where $\widetilde{V}_m$ is the following spectrum.

$$\begin{gathered} P_{-m}^{-N_{i-1}} \sqcup_{W_{k_i}(P_{-m}^{-N_{i-1}})} W_{k_i}(P_{-m}^{-N_i}) \sqcup_{W_{k_{i+1}}(P_{-m}^{-N_i-1})} \\ W_{k_{i+1}}(P_{-m}^{-N_{i+1}}) \sqcup_{W_{k_{i+2}}(P_{-m}^{-N_{i+1}})} W_{k_{i+2}}(P_{-m}) \end{gathered}$$

The following diagram explains the lifted bracket $\widetilde{\langle P_{-m}^{-N_i} \rangle}(\widetilde{R}_E^{[k_i]}(\alpha))$.

(6.1)
$$\begin{tikzcd} \pi_{t-1}(\widetilde{V}_m) \arrow[r, "\partial"] \arrow[d, "p_2"'] & \pi_{t-2}(W_{k_{i+1}}(P_{-m}^{-N_i-1})) \arrow[d] & \pi_{t-2}(W_{k_{i+1}}(S^{-m})) \arrow[l] \arrow[d] \\ \pi_{t-1}(W_{k_i}(S^{-N_i})) \arrow[r, "g"] \arrow[rr, bend right, squiggly, "\langle P_{-m}^{-N_i-1} \rangle(-)"'] & \pi_{t-2}(W_{k_i}(P_{-m}^{-N_i-1})) & \pi_{t-2}(W_{k_i}(S^{-m})) \arrow[l] \end{tikzcd}$$

Here p_2 is projection onto the second factor of the homotopy pushout $\widetilde{V}_m$, ∂ is a Meyer-Vietoris boundary, and g is the attaching map of the $-N_i$-cell to $P_{-m}^{-N_i-1}$. Assuming (inductively) that for every $N_i < m' < m$,

$$\widetilde{\langle P_{-m'}^{-N_i} \rangle}(\widetilde{R}_E^{[k_i]}(\alpha)) = 0$$

we there is a lift of $\partial(\widetilde{\gamma}_m)$ to $\pi_{t-2}(W_{k_{i+1}}(S^{-m})$. The set of all such lifts is defined to be

$$\widetilde{\langle P_{-m}^{-N_i} \rangle}(\widetilde{R}_E^{[k_i]}(\alpha)) \subseteq \pi_{t-2}(W_{k_{i+1}}(S^{-m}))$$

Diagram 6.1 implies that this set of lifts is indeed a lift of the Toda bracket

$$\langle P_{-m}^{-N_i} \rangle(\widetilde{R}_E^{[k_i]}(\alpha)) \subseteq \pi_{t-2}(W_{k_i}(S^{-m})).$$

Let M be the first m such that $\widetilde{\langle P_{-m}^{-N_i} \rangle}(\widetilde{R}_E^{[k_i]}(\alpha))$ contains a non-trivial element. Let $\widetilde{\gamma} = \widetilde{\gamma}_M$ and let $\widetilde{V} = \widetilde{V}_M$. The image of $\widetilde{\gamma}$ under the composite

$$\pi_{t-1}(\widetilde{V}) \xrightarrow{\partial} \pi_{t-2}(W_{k_{i+1}}(P_{-M}^{-N_i-1})) \to \pi_{t-2}(W_{k_{i+1}}(P_{-M+1}^{-N_i-1}))$$

is zero, and $\widetilde{\gamma}$ lifts even further, to an element $\widetilde{\widetilde{\gamma}} \in \pi_{t-1}(\widetilde{\widetilde{V}})$, where $\widetilde{\widetilde{V}}$ is defined to be the following spectrum.

$$P_{-M}^{-N_{i-1}} \sqcup_{W_{k_i}(P_{-M}^{-N_{i-1}})} W_{k_i}(P_{-M}^{-N_i}) \sqcup_{W_{k_{i+1}}(S^{-M})}$$
$$W_{k_{i+1}}(P_{-M}^{-N_{i+1}}) \sqcup_{W_{k_{i+2}}(P_{-M}^{-N_{i+1}})} W_{k_{i+2}}(P_{-M})$$

Let f be the attaching map

$$f : \Sigma^{-1} P_{-M+1}^{-N_{i+1}} \to S^{-M}.$$

Since $\langle P_{-M}^{-N_{i+1}} \rangle$ has E-Adams degree d, there is a lift

$$\widetilde{f} : \Sigma^{-1} P_{-M+1}^{-N_{i+1}} \to W_d(S^{-M}).$$

Let $(P_{-M}^{-N_{i+1}})_{[d]}$ be the cofiber of $\widetilde{f}$. Our hypothesis that $d \le r_{i+1}$ implies that $\widetilde{\widetilde{V}}$ has the following equivalent description as a homotopy pushout.

$$P_{-M}^{-N_{i-1}} \sqcup_{W_{k_i}(P_{-M}^{-N_{i-1}})} W_{k_i}(P_{-M}^{-N_i}) \sqcup_{W_{k_{i+1}+d}(S^{-M})}$$
$$W_{k_{i+1}}((P_{-M}^{-N_{i+1}})_{[d]}) \sqcup_{W_{k_{i+2}}(P_{-M}^{-N_{i+1}})} W_{k_{i+2}}(P_{-M})$$

Our proof is reduced to chasing the diagram displayed in Figure 2.

$$
\begin{tikzcd}
 & & \pi_{t-2}(S^{-M}) \arrow[dr] & & \pi_{t-1}(S^{-N_i}) \arrow[ll, rightsquigarrow, "\langle P_{-M}^{-N_i}\rangle(-)"'] \arrow[dl, "g"'] \\
 & & & \pi_{t-2}(P_{-M}^{-N_i-1}) & \\
\pi_{t-2}(W^{k_{i+2}+d-1}_{k_{i+1}+d}(S^{-M})) & & \pi_{t-2}(W_{k_{i+1}+d}(S^{-M})) \arrow[ll] \arrow[r] \arrow[uu] & \pi_{t-2}(W_{k_i}(P_{-M}^{-N_i-1})) \arrow[u] & \\
 & \pi_{t-1}(W^{k_{i+2}-1}_{k_{i+1}}(P_{-M+1}^{-N_{i+1}})) \arrow[ul, "\widetilde{f}"'] \arrow[dl] & & & \pi_{t-1}(W_{k_i}(S^{-N_i})) \arrow[ul, "g"'] \arrow[uuu] \arrow[d] \\
\pi_{t-1}(W^{k_{i+2}-1}_{k_{i+1}}(S^{-N_{i+1}})) \arrow[uu, rightsquigarrow, "\langle P_{-M}^{-N_{i+1}}\rangle(-)"] & & \pi_{t-1}(\widetilde{\widetilde{V}}) \arrow[ll] \arrow[rr] \arrow[uu, "\partial"'] \arrow[ul, "p_3"] \arrow[ur] \arrow[urr, "p_2"'] & & \pi_{t-1}(W^{k_{i+1}-1}_{k_i}(S^{-N_i}))
\end{tikzcd}
$$

FIGURE 2. The main diagram for the proof of Theorem 5.4

In Figure 2, $\widetilde{f}$ and g are attaching maps, p_2 is projection onto the second factor of the homotopy pushout $\widetilde{\widetilde{V}}$, p_3 is projection onto the third factor followed by projection onto the $-N_i$-cell, and ∂ is the Meyer-Vietoris boundary. The wavy arrows represent Toda brackets. The element $\widetilde{\widetilde{\gamma}} \in \widetilde{\widetilde{V}}$ maps to compatible elements of every group displayed in Figure 2. The image of $\widetilde{\widetilde{\gamma}}$ in $\pi_{t-1}(W_{k_i}^{k_{i+1}-1}(S^{-N_i}))$ projects to an element of $R_E^{[k_i]}(\alpha)$. The image of $\widetilde{\widetilde{\gamma}}$ in $\pi_{t-1}(W_{k_{i+1}}^{k_{i+2}-1}(S^{-N_{i+1}}))$ projects to an element of $R_E^{[k_{i+1}]}(\alpha)$. Let β be the image of $\widetilde{\widetilde{\gamma}}$ in $\pi_{t-2}(S^{-M})$. Following the outside of Figure 2 from the lower left hand corner clockwise reveals that

$$\beta \in \overline{\langle P_{-M}^{-N_{i+1}} \rangle (R_E^{[k_{i+1}]}(\alpha))}.$$

Following the outside of Figure 2 from the lower right-hand corner counterclockwise gives

$$\beta \in \langle P_{-M}^{-N_i} \rangle (\overline{R}_E^{[k_i]}(\alpha)).$$

Thus β is a common element of the two groups. □

Proof of Proposition 5.8. Suppose that the first filtered root invariant of α is carried by the $-N$-cell of $P_{-\infty}$. If the first filtered root invariant is trivial, then $N = \infty$. Consider the following diagram.

$$\begin{tikzcd}
\pi_t(tS^0) \arrow[rr, bend left, squiggly, "R_E^{[1]}(-)"] & \pi_{t-1}(W_I(P^J)) \arrow[l] \arrow[r] \arrow[dl] \arrow[dd] & \pi_{t-1}(W_1^1(S^{-N})) \arrow[dd, equal] \\
\pi_{t-1}(W_1(P)) \arrow[u] \arrow[d] & & \\
\pi_{t-1}(W_1^1(P)) \arrow[rr, bend right, squiggly, "R_{E\wedge\overline{E}}(-)"'] & \pi_{t-1}(W_1^1(P^{-N})) \arrow[l] \arrow[r] & \pi_{t-1}(W_1^1(S^{-N}))
\end{tikzcd}$$

Let f^α be a lift of the image of α in $\pi_{t-1}(tS^0)$ to $\pi_{t-1}(W_I(P^J))$. Then f^α has compatible images in every group in the diagram. The detecting element $\widetilde{\alpha}$ is the image of f^α in $\pi_{t-1}(W_1^1(P_{-\infty})) = \pi_{t-1}(\Sigma^{-1}E \wedge \overline{E}_p^\wedge)$. □

Proof of Corollary 5.9. First observe that if the first filtered root invariant is carried by the $-M$-cell, then $M \geq N$. Let $\widetilde{\alpha}'$ be a 'preferred' detecting element such as the one described in Proposition 5.8. It suffices to show that if $\beta \in R_{E\wedge\overline{E}}(\widetilde{\alpha})$ and $\beta' \in R_{E\wedge\overline{E}}(\widetilde{\alpha}')$ (or $\beta' = 0$ if $R_{E\wedge\overline{E}}(\widetilde{\alpha}')$ lies in a higher degree than $R_{E\wedge\overline{E}}(\widetilde{\alpha})$), then $\beta - \beta' \in A$. The elements β and β' are the images of elements γ and γ', respectively, under the collapse map

$$\pi_{t-1}(W_1^1(P^{-N})) \to \pi_{t-1}(W_1^1(S^{-N})).$$

where γ and γ' both map to the image of α under the homomorphism

$$\pi_{t-1}(W_1^1(P^{-N})) \xrightarrow{\iota} \pi_{t-1}(W_0^1(P)).$$

Therefore there is an element $\mu \in \pi_t(W_0^{(0,1)}(P_{(-\infty,-N+1)}))$ such that $\gamma-\gamma' = \partial_1(\mu)$, where ∂_1 is the connecting homomorphism of the long exact sequence associated to the cofiber sequence

$$W_1^1(P^{-N}) \to W_0^1(P) \to W_0^{(0,1)}(P_{(-\infty,-N+1)}).$$

A comparison of cofiber sequences shows that there is a commutative diagram

$$\begin{tikzcd}
\pi_t(W_0^{(0,1)}(P_{(-\infty,-N+1)})) \arrow[r, "\partial_1"] \arrow[d, "\nu"'] & \pi_{t-1}(W_1^1(P^{-N})) \arrow[d, "\nu"] \\
\pi_t(W_0^{(0,1)}(P_{(-N,-N+1)})) \arrow[r, "\partial"'] & \pi_{t-1}(W_1^1(S^{-N}))
\end{tikzcd}$$

Thus we see that

$$\beta - \beta' = \nu(\gamma-\gamma') = \nu \circ \partial_1(\mu) = \partial \circ \nu(\mu)$$

so $\beta - \beta'$ is contained in A. $\square$

Proof of Theorem 5.10. We shall refer to the appropriate Ext groups as as the E_2 terms of the ASS. Observe that Lemma 5.6 implies that $k_1 \geq k$. We first will prove that $k_1 = k$. Suppose that $k_1 > k$. Let $\widetilde{\widetilde{\alpha}}$ be a lift of $\widetilde{\alpha}$ to $\pi_t(W_k(S^0))$. Consider the following diagram.

$$\begin{tikzcd}
E_2^{k,t+k-1}(S^{-1}) \arrow[d, "f"'] & \pi_{t-1}(W_k(S^{-1})) \arrow[l] \arrow[d] & \\
E_2^{k,t+k-1}(P_{-\infty}) & \pi_{t-1}(W_k(P)_{-\infty}) \arrow[l] & \pi_{t-1}(W_{k_1}(P)_{-\infty}) \arrow[l]
\end{tikzcd}$$

The element $\widetilde{\widetilde{\alpha}}$ maps to $\widetilde{\alpha} \in E_2^{k,t+k-1}(S^{-1})$. By the definition of k_1, the image of $\widetilde{\widetilde{\alpha}}$ in $\pi_{t-1}(W_k(P)_{-\infty})$ lifts to an element of $\pi_{t-1}(W_{k_1}(P)_{-\infty})$. This implies that $f(\widetilde{\alpha}) = 0$. However, $\widetilde{\alpha}$ is non-zero, and the algebraic Segal conjecture implies that f is an isomorphism. We conclude that $k_1 = k$.

The filtered root invariant $R_H^{[1]}(\alpha)$ is a subset of $E_1^{k,t+N_1+k-1}(S^0)$. All of the attaching maps of $P_{-\infty}$ are of positive Adams filtration, so Theorem 5.3 implies that $R_E^{[1]}(\alpha)$ consists of d_1-cycles. Let $f^\alpha \in \pi_{t-1}(W_I(P^J))$ be a lift of α, and let $\beta \in \pi_{t-1}(W_k^k(S^{-N_1}))$ be the corresponding element in $R_H^{[k]}(\alpha)$. Let $f_1^\alpha \in \pi_{t-1}(W_k(P)_{-\infty})$ be the image of f^α under the natural map

$$W_I(P^J) \to W_k(P)_{-\infty}.$$

Consulting the diagram,

$$\begin{tikzcd}
 & \pi_{t-1}(W_k(P)_{-\infty}) \arrow[d] & \\
E_2^{k,t+k-1}(P_{-N_1}) & \pi_{t-1}(W_k^{k+1}(P_{-N_1})) \arrow[l] \arrow[d] & \pi_{t-1}(W_k^{k+1}(S^{-N_1})) \arrow[l] \arrow[d] \\
 & \pi_{t-1}(W_k^k(P_{-N_1})) & \pi_{t-1}(W_k^k(S^{-N_1})) \arrow[l]
\end{tikzcd}$$

both f_1^α and β have the same image in $\pi_{t-1}(W_k^k(P_{-N_1}))$. Let γ be the image of f_1^α in $\pi_{t-1}(W_k^{k+1}(P_{-N_1}))$. Since β is a d_1-cycle, it lifts to $\widetilde{\beta} \in \pi_{t-1}(W_k^{k+1}(S^{-N_1}))$. Let $\widetilde{\gamma}$ be the image of $\widetilde{\beta}$ in $\pi_{t-1}(W_k^{k+1}(P_{-N_1}))$. It not necessarily the case that $\gamma = \widetilde{\gamma}$, but it is the case that the difference $\gamma - \widetilde{\gamma}$ vanishes in $E_2^{k,t+k-1}(P_{-N_1})$.

Consider the following diagram.

$$\begin{tikzcd}
\pi_t(tS^0) \arrow[rrr, squiggly, "\overline{R}_H^{[k]}(-)"] & & & \pi_{t-1}(W_k^{k+1}(S^{-N_1})) \arrow[dl] \arrow[ddd] \arrow[ddddd, bend left] \\
& \pi_{t-1}(W_I(P^J)) \arrow[ul] \arrow[r] \arrow[dr] \arrow[ddl] & \pi_{t-1}(W_k^k(S^{-N_1})) \arrow[d] & \\
& & \pi_{t-1}(W_k^k(P_{-N_1})) & \\
\pi_{t-1}(W_k(P)_{-\infty}) \arrow[uuu] \arrow[rrr] \arrow[d] & & & \pi_{t-1}(W_k^{k+1}(P_{-N_1})) \arrow[ul] \arrow[d] \\
E_2^{k,t+k-1}(P_{-\infty}) \arrow[rrr, "\nu_{N_1}"'] & & & E_2^{k,t+k-1}(P_{-N_1}) \\
E_2^{k,t+k-1}(S^{-1}) \arrow[u, "\cong", "f"'] \arrow[rrr, squiggly, "R_{alg}(-)"'] & & & E_2^{k,t+k-1}(S^{-N_1}) \arrow[u]
\end{tikzcd}$$

The element $\widetilde{\beta}$ maps to an element of $\overline{R}_H^{[k]}(\alpha)$ in $E_2^{k,t+k-1}(S^{-N_1})$. The map f is an isomorphism, so the image of f_1^α lifts to an element $\widetilde{\alpha} \in E_2^{k,t+k-1}(S^{-1})$. This is the choice of $\widetilde{\alpha}$ which detects α that we appeal to in the statement of Theorem 5.3. Recall that the image of f_1^α in $\pi_{t-1}(W_k^{k+1}(P_{-N_1}))$ was γ while the image of β is $\widetilde{\gamma}$. Since the difference $\gamma - \widetilde{\gamma}$ maps to zero in $E_2^{k,t+k-1}(P_{-N_1})$, we may conclude that the image of $\widetilde{\beta}$ in $E_2^{k,t+k-1}(S^{-N_1})$ is also an element of the algebraic root invariant, *provided the algebraic root invariant does not lie in a higher stem.*

We have proven that either what is claimed in Theorem 5.10 holds, or the algebraic root invariant lives in a larger stem than $R_H^{[1]}(\alpha)$. We shall now show that this cannot happen. Let M be maximal such that the image of $\widetilde{\alpha}$ in $E_2^{k,t+k-1}(S^{-1})$ maps to zero under the composite

$$E_2^{k,t+k-1}(S^{-1}) \xrightarrow{\nu_{M-1} \circ f} E_2^{k,t+k-1}(P_{-M+1}).$$

By the algebraic Segal conjecture, such a finite M exists. We wish to show that $N_1 = M$. So far we know that $N_1 \le M$. In light of the definition of N_1, we simply must show that the α is sent to zero under the composition

$$\pi_{t-1}(S^{-1}) \to \pi_{t-1}(W_0^{(k-1,k)}(P_{(-M+1,\infty)})).$$

To this end consider the following diagram.

$$\begin{tikzcd}
 & \pi_t(W_0^{k-1}(P)_{-\infty}) \arrow[d, "\partial"'] & \pi_t(W_{k-1}^{k-1}(P)_{-\infty}) \arrow[l] \arrow[d, "\nu"] \\
\pi_{t-1}(W_k(P)_{-\infty}) \arrow[r] \arrow[d] & \pi_{t-1}(W_k^k(P_{-M+1})) \arrow[d] & \pi_t(W_{k-1}^{k-1}(P_{-M+1})) \arrow[l, "d_1"] \\
\pi_t(tS^0) \arrow[r] & \pi_{t-1}(W_0^{(k-1,k)}(P_{(-M+1,\infty)})) &
\end{tikzcd}$$

The central vertical column corresponds to part of the long exact sequence for a cofibration. The element $f_1^\alpha \in \pi_{t-1}(W_k(P)_{-\infty})$ maps to the image of α in $\pi_t(tS^0)$. Let g be the image of f_1^α in $\pi_{t-1}(W_k^k(P_{-M+1}))$. By our choice of M, g must vanish in E_2, that is to say, there must be an element $h \in \pi_t(W_{k-1}^{k-1}(P_{-M+1}))$ such that $d_1(h) = g$. The map ν is surjective, because we are dealing with ordinary homology, and the map

$$H_*(P)_{-\infty} \to H_*(P_N)$$

is surjective for every N. Thus h lifts to an element $\widetilde{h} \in \pi_t(W_{k-1}^{k-1}(P)_{-\infty})$. Applying ∂ to the image of $\widetilde{h}$ in $\pi_t(W_0^{k-1}(P)_{-\infty})$, we get g, so g is in the image of ∂. Thus, by exactness of the vertical column, the image of g in $\pi_{t-1}(W_0^{(k-1,k)}(P_{(-M+1,\infty)}))$ is trivial. Thus the image of α in $\pi_t(tS^0)$ maps to zero in $\pi_{t-1}(W_0^{(k-1,k)}(P_{(-M+1,\infty)}))$, which is what we were trying to prove. $\square$

7. *bo* resolutions

The *bo* resolution for the sphere was the motivating example for the theorems in Section 5. The d_1's on the v_1-periodic summand of the 0 and 1-lines of the *bo*-resolution reflect the root invariants of the elements 2^i at the prime 2. We do not rederive these root invariants, but satisfy ourselves in explaining how the theorems of Section 5 play out in this context. Alternatively, one might interpret this section as explaining how to use *bo* to compute the root invariants in the $K(1)$-local category. This section is furthermore meant to explain how the techniques of this paper, when applied to the small resolution of $L_{K(2)}(S^0)$ described in [14], could be used to compute root invariants in the $K(2)$-local category.

Recall that $\Sigma^4 bsp$ is a summand of $bo \wedge \Sigma\overline{bo}$, hence its homotopy is a stable summand of the 1-line of the E_1-term of the *bo*-ASS. The composite of the d_1 of this spectral sequence with the projection onto the summand

$$\pi_t(bo) \xrightarrow{d_1} \pi_{t-1}(bo \wedge \overline{bo}) \to \pi_{t-1}(\Sigma^4 bsp)$$

is the map $\psi^3 - 1$ (up to a unit in $\mathbb{Z}_{(2)}$). The fiber of $\psi^3 - 1$ is the 2-primary J spectrum. Figure 3 shows the v_1-periodic summand of the *bo*-ASS with differentials. In this figure, dots represent copies of $\mathbb{F}_2$ and circles represent copies of $\mathbb{Z}$. The d_1 differentials are (up to a 2-adic unit) multiplication by the power of 2 indicated. What survives are the v_1-periodic elements in π_*^S at $p = 2$.

Our theorems explain the relationship between $R_{bo}(2^k)$ and $R(2^k)$ for all $k > 0$. It is quite straightforward to calculate $R_{bo}(2^k)$ (see [25]). Let η, α, and β be the

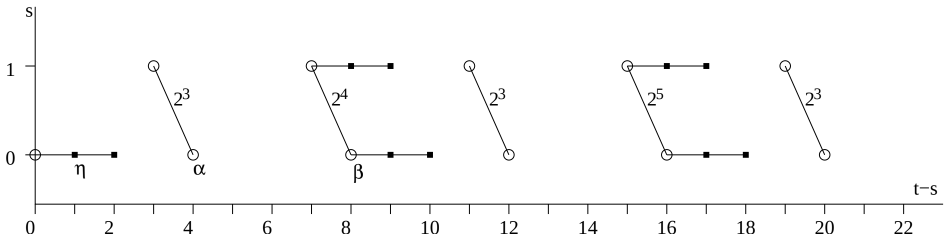

FIGURE 3. The v_1-periodic summand of the bo-ASS.

multiplicative generators of $\pi_*(bo)$ in dimensions 1, 4, and 8, respectively. The we have the following bo-root invariants.

$$R_{bo}(2^k) = \begin{cases} \beta^i & k = 4i \\ \eta\beta^i & k = 4i+1 \\ \eta^2\beta^i & k = 4i+2 \\ \alpha\beta^i & k = 4i+3 \end{cases}$$

These are the filtered root invariants $R^{[0]}_{bo}(2^k)$, by Proposition 5.7. For $k \equiv 1, 2 \pmod 4$, the elements $R^{[0]}_{bo}(2^k)$ are in the Hurewicz image of bo, hence they are permanent cycles. They detect elements in $R(2^k)$.

For $k \equiv 3, 4 \pmod 4$, the elements $R_{bo}(2^k)$ are not in the bo-Hurewicz image. Therefore, the elements $R^{[0]}_{bo}(2^k)$ support differentials. We have

$$d_1 R^{[0]}_{bo}(2^k) = 2 \cdot \widetilde{\alpha}_k$$

where $\widetilde{\alpha}_k$ survives to a v_1-periodic element of order 2 in dimension $4k-1$. In the bo-bifiltration of 2^k, the $-N_1$ cell carries the zeroth filtered root invariant, where N_1 is given by

$$N_1 = \begin{cases} 8i+5 & k = 4i+3 \\ 8i+1 & k = 4i. \end{cases}$$

The first cell to attach to the $-N_1$ cell in P_{-N_1} is the $-N_1+1$ cell, and the attaching map is the degree 2 map. Therefore, we may deduce from Theorem 5.3 that $\widetilde{\alpha}_k$ is an element of $R^{[1]}_{bo}(2^k)$. The element $\widetilde{\alpha}_k$ detects the homotopy root invariant $R(2^k)$.

8. The algebraic Atiyah-Hirzebruch spectral sequence

As discussed after the statement of Corollary 5.2, it is often the case that one may know $R^{[k]}_E(\alpha)$ contains a permanent cycle in the E-Adams spectral sequence, but nothing else. One would want to conclude that this permanent cycle detects the homotopy root invariant $R(\alpha)$, but Theorem 5.1 says this is only true modulo obstructions in higher E-Adams filtration. However, it is sometimes the case that brute force computations in the E_2-term of the E-Adams spectral sequence for P_N will yield enough data to eliminate such obstructions. This section is concerned with computations of the E_2-term $E_2(P_N)$. Specifically, we shall assume that $E = BP$ and describe the algebraic Atiyah-Hirzebruch spectral sequence (AAHSS), which computes $\mathrm{Ext}(BP_*P_N)$ from $\mathrm{Ext}(BP_*)$. In Section 9, we describe a procedure

(Procedure 9.1) that explains how a limited knowledge of differentials in the AAHSS can be exploited to compute homotopy root invariants from filtered root invariants.

The AAHSS is the spectral sequence obtained by applying Ext to the skeletal filtration of a complex. The AAHSS has appeared in various forms in the literature. In [22], Mahowald uses ordinary cohomology, and uses this spectral sequence to compute Ext of various stunted projective spaces. In [35], Sadofsky uses the BP version that is the subject of this section. There is some overlap with this section and the computations in [35].

We will first describe the AAHSS for computing the Adams-Novikov E_2-term $E_2(P_N)$, where $N \equiv -1 \pmod q$. Let F be the universal p-typical formal group law associated to BP, and write the p-series of F as

$$[p]_F(x) = \sum_{i \geq 0}^{F} v_i x^{p^i} = \sum_j c_j x^{(p-1)j+1}$$

where v_i are the Araki generators. Then we can say the following about the coefficients c_i.

$$\begin{aligned}
c_0 &= p \\
c_1 &= v_1 \\
c_j &\equiv 0 \pmod p \text{ for } 1 < j < p+1 \\
c_{p+1} &\equiv v_2 \pmod p
\end{aligned}$$

There are short exact sequences (compare with [35, 2.3])

$$0 \to BP_*(S^{kq-1}) \xrightarrow{\phi} BP_*(P_N^{kq-1}) \to BP_*(P_N^{kq}) \to 0 \tag{8.1}$$

$$0 \to BP_*(P_N^{(k-1)q}) \to BP_*(P_N^{kq-1}) \xrightarrow{\nu} BP_*(S^{kq-1}) \to 0 \tag{8.2}$$

and these give rise to long exact sequences upon applying Ext. They have the following boundary homomorphisms.

$$\begin{aligned}
\delta_1 &: \mathrm{Ext}^{s,t}(BP_* P_N^{kq}) \to \mathrm{Ext}^{s+1,t}(BP_* S^{kq-1}) \\
\delta_2 &: \mathrm{Ext}^{s,t}(BP_* S^{kq-1}) \to \mathrm{Ext}^{s+1,t}(BP_* P_N^{(k-1)q})
\end{aligned}$$

For $l \leq k$, $BP_*(P_{lq-1}^{kq})$ is generated as a BP_*-module by elements e_{jq-1} in dimension $jq-1$. The map ϕ in the short exact sequence 8.1 is given by

$$\phi(\iota_{kq-1}) \mapsto \sum_i c_i e_{q(k-i)-1}.$$

Splicing these sequences together creates an exact couple, and the resulting spectral sequence is the AAHSS. We shall index it just as one indexes the stable EHP spectral sequence [33, 1.5]. In fact, if $N = q-1$ then this is precisely the BP-algebraic stable EHP spectral sequence. Thus we have a spectral sequence

$$E_1^{k,n,s} \Rightarrow \mathrm{Ext}^{s,s+k}(BP_* P_N)$$

where the E_1 term is described below.

$$\begin{aligned}
E_1^{k,2m,s} &= \mathrm{Ext}^{s,s+k}(BP_* S^{mq-1}) \\
E_1^{k,2m+1,s} &= \mathrm{Ext}^{s+1,s+k}(BP_* S^{mq-1})
\end{aligned}$$

The indexing works out so that

$$d_r : E_r^{k,n,s} \to E_r^{k-1,n-r,s+1}.$$

If we wish to compute $\mathrm{Ext}(BP_*P_N)$ for $N = lq$, simply truncate the AAHSS for $\mathrm{Ext}(BP_*P_{N-1})$ by setting $E_1^{k,2l,s} = 0$ for all k and s.

We shall refer to an element in the E_1-term of the AAHSS by its name in $\mathrm{Ext}(BP_*)$ and the cell that is borne on. Thus if $\gamma \in \mathrm{Ext}(BP_*)$ is in $E_1^{k,2m,s}$, then we shall refer to it as $\gamma[mq-1]$. Likewise, if γ is in $E_1^{k,2m+1,s}$, we shall refer to it as $\gamma[mq]$.

In order to implement Procedure 9.1 we shall need to know how to explicitly compute differentials in the filtered spectral sequence. It is useful to use the diagram below. *In this diagram and the remainder of this section we will drop BP_* from our* Ext *notation for compactness.*

$$\begin{tikzcd}
 & \mathrm{Ext}^{s+1,s+k}(P_N^{lq-1}) \arrow[r, "\nu_*"'] \arrow[dd] & \mathrm{Ext}^{s+1,s+k}(S^{lq-1}) \arrow[d, equal] \\
 & & E_1^{k-1,2l,s+1} \\
 & \mathrm{Ext}^{s+1,s+k}(P_N^{lq}) \arrow[r, "\delta_1"'] \arrow[d] & \mathrm{Ext}^{s+2,s+k}(S^{lq-1}) \arrow[d, equal] \\
 & \vdots \arrow[dd] & E_1^{k-1,2l+1,s+1} \\
E_1^{k,2m,s} \arrow[d, equal] & & \\
\mathrm{Ext}^{s,s+k}(S^{mq-1}) \arrow[r, "\delta_2"'] & \mathrm{Ext}^{s+1,s+k}(P_N^{(m-1)q}) \arrow[dd] & \\
E_1^{k,2m+1,s} \arrow[d, equal] & & \\
\mathrm{Ext}^{s+1,s+k}(S^{mq-1}) \arrow[r, "\phi_*"'] & \mathrm{Ext}^{s+1,s+k}(P_N^{mq-1}) &
\end{tikzcd}$$

Suppose $\gamma[n]$ is an element of E_1, where $n = mq - \epsilon$, $\epsilon = 0, 1$. Let γ_1 be the image of γ in $\mathrm{Ext}^{s+1,s+k}(P_N^{mq-1})$ or $\mathrm{Ext}^{s+1,s+k}(P_N^{(m-1)q})$, depending on the value of ϵ. Then lift γ_1 as far as possible up the tower in the center of the above diagram. Suppose that γ_1 lifts to $\gamma_2 \in \mathrm{Ext}^{s+1,s+k}(P_N^{lq-\epsilon_1})$, where $\epsilon_1 = 0, 1$. Then if γ_3 is the image of γ_2 under ν_* or δ_1 (depending on the value of ϵ_1), there is an AAHSS differential

$$d_r(\gamma[n]) = \gamma_3[n']$$

where $n' = lq - \epsilon$.

The above description makes the following proposition clear. In the statement of this proposition, and in what follows, when we say that differentials are computed modulo I_n, we mean is that we are considering their images in $\mathrm{Ext}(BP_*(-)/I_n)$.

Proposition 8.1. If r is odd, then we have

$$d_r(\alpha[mq-1]) = 0.$$

Modulo I_n, we have the differential

$$d_{2p^n-1}(\alpha[mq]) = v_n \cdot \alpha[mq - 2p^n + 1].$$

Proof. Thinking of $\alpha[mq-1]$ as an element of $\mathrm{Ext}^{s,s+k}(S^{mq-1})$, suppose γ is a lift of $\delta_2(\alpha[mq-1])$ to $\mathrm{Ext}^{s+1,s+k}(P_N^{lq})$. Then $\delta_1\gamma = 0$ since $d^2 = 0$ in the cobar complex.

For the second part, take $\alpha \in \mathrm{Ext}^{s+1,s+k}(S^{mq-1})$. Clearly, it is the case that

$$\phi_*(\alpha) \underset{(\mathrm{mod}\ I_n)}{\equiv} v_n\alpha[mq - 2p^n + 1] + \cdots$$

lifts to $\mathrm{Ext}^{s+1,s+k}(P^{mq-2p^n+1})$, and the image of this lift under the map ν_* is

$$v_n \cdot \alpha \in \mathrm{Ext}^{s+1,s+k}(S^{mq-2p^n+1}).$$

$\square$

In order to compute δ_1 and δ_2, we must know something of the BP_*BP comodule structure of $BP_*(P_N)$. It suffices to understand the BP_*BP comodule structure of $BP_*(B\mathbb{Z}/p)$, since P_0 is a stable summand of $B\mathbb{Z}/p_+$. The BP_*BP comodule structure of $BP_*(P_N)$ may then be deduced from James periodicity. In [35], it is proven that there is a cofiber sequence

$$\mathbb{C}P^\infty \to (\mathbb{C}P^\infty)^{\xi^p} \to \Sigma B\mathbb{Z}/p$$

(where ξ is the canonical complex line bundle on $\mathbb{C}P^\infty$) which induces the short exact sequence below.

$$0 \to BP_*\mathbb{C}P^\infty \to BP_*(\mathbb{C}P^\infty)^{\xi^p} \to BP_*\Sigma B\mathbb{Z}/p \to 0$$

Since $BP_*(\mathbb{C}P^\infty)^{\xi^p}$ surjects onto $BP_*\Sigma B\mathbb{Z}/p$, it suffices to understand the BP_*BP-comodule structure of the former.

Recall that $BP^*(\mathbb{C}P^\infty) = BP^*[[x]]$ where x is the Euler class of ξ. Therefore, the image of the inclusion

$$BP^*(\mathbb{C}P^\infty)^{\xi^p} \to BP^*(\mathbb{C}P^\infty)$$

is the ideal

$$[p]_F(x) \cdot BP^*[[x]] \subseteq BP^*(\mathbb{C}P^\infty).$$

We shall identify $BP^*(\mathbb{C}P^\infty)^{\xi^p}$ with this ideal.

Let y_{2k} be the generator in $BP_{2k}((\mathbb{C}P^\infty)^{\xi^p})$ dual to $[p]_F(x) \cdot x^{k-1}$. Let $f(x)$ be the formal power series over BP_*BP whose inverse is given by

$$f^{-1}(x) = \sum_{i \geq o}^{F} t_i x^{p^i}.$$

The power series $f(x)$ is the universal strict isomorphism of p-typical formal group laws.

Proposition 8.2. The BP_*BP coaction on y_{2k} is given below.

$$\psi(y_{2k}) = \sum_{i+j=k} \left(\frac{f([p]_F(x))}{[p]_F(x)} (f(x))^{j-1} \right)_{k-1} \otimes y_{2j} \tag{8.3}$$

Here the subscript $k-1$ indicates the coefficient of x^{k-1}.

Proof. We digress for a moment on cooperations. For a spectrum X, the left unit $\eta_L : BP \to BP \wedge BP$ induces the BP_*BP coaction

$$\psi : BP_*(X) \xrightarrow{(\eta_L)_*} (BP \wedge BP)_*(X) \cong BP_*BP \otimes_{BP_*} BP_*(X).$$

It is sometimes convenient to consider the dual coaction on cohomology

$$\psi^* : BP^*(X) \xrightarrow{(\eta_L)_*} (BP \wedge BP)^*(X).$$

In Theorem 11.3 of Part II of [1], Adams describes the MU_*MU coaction on $MU_*(\mathbb{C}P^\infty)$. When translated into BP-theory, the coaction formula reads

$$\psi(e_{2k}) = \sum_{i+j=k} \left((f(x))^j\right)_k \otimes e_{2j}.$$

Here, e_{2k} is the generator of $BP_{2k}(\mathbb{C}P^\infty)$ dual to $x^k \in BP^{2k}(\mathbb{C}P^\infty)$. Upon dualization, we get the following formula for the dual coaction on an element $h(x)$ in $BP^*(\mathbb{C}P^\infty)$.

$$\psi^*(h(x)) = f_*h(f(x))$$

The polynomial $f_*h(x)$ is the polynomial obtained from applying the right unit to the coefficients of $h(x)$. We deduce the dual coaction on an element $[p]_F(x) \cdot h(x)$ in $BP^*((\mathbb{C}P^\infty)^{\xi^p})$, regarded as the ideal $([p]_F(x))$ contained in $BP^*[[x]]$.

$$\begin{aligned}
\psi^*([p]_F(x) \cdot h(x)) &= f_*[p]_F(f(x)) \cdot f_*h(f(x)) \\
&= [p]_{f_*F}(f(x)) \cdot f_*h(f(x)) \\
&= f([p]_F(x)) \cdot f_*h(f(x)) \\
&= [p]_F(x) \cdot \frac{f([p]_F(x))}{[p]_F(x)} \cdot f_*h(f(x))
\end{aligned}$$

Here, f_*F is the pushforward of the formal group law F under the map f. Letting $h(x) = x^{j-1}$ and dualizing, we have the desired formula for the coaction on $BP^*((\mathbb{C}P^\infty)^{\xi^p})$. $\square$

We wish to use the above coaction formula to compute differentials in the filtered spectral sequence. We begin with $p \geq 3$. The first few terms of the relevant power series are listed below. Everything in what follows is written in terms of Hazewinkel generators.

$$[p]_F(x) = px + (1 - p^{p-1})v_1 x^p + \mathcal{O}(x^p + 1) \tag{8.4}$$

$$f(x) = x - t_1 x^p + (pt_1^2 + v_1 t_1)x^{2p-1} + \mathcal{O}(x^{2p}) \tag{8.5}$$

$$\begin{aligned}\frac{f([p]_F(x))}{[p]_F(x)} &= 1 - p^{p-1}t_1 x^{p-1} + \\
&\quad p^{p-2}(p^{p+1}t_1^2 + (1 - p - p^{p-1} + 2p)v_1 t_1)x^{2p-2} + \mathcal{O}(x^{2p-1})\end{aligned} \tag{8.6}$$

We may compute these series further if we work modulo the ideal $I_2 = (p, v_1)$.

$$[p]_F(x) \equiv v_2 x^{p^2} + \mathcal{O}(x^{p^2} + 1) \pmod{I_2} \tag{8.7}$$

$$f(x) \equiv x - t_1 x^p + \mathcal{O}(x^{p^2}) \pmod{I_2} \tag{8.8}$$

$$\frac{f([p]_F(x))}{[p]_F(x)} \equiv 1 + \mathcal{O}(x^{p^2-1}) \pmod{I_2} \tag{8.9}$$

These formulas give rise to the following proposition.

Proposition 8.3. In the filtered spectral sequence, the following formulas for differentials hold up to multiplication by a unit in $\mathbb{Z}_{(p)}$.

$$d_2(\alpha[kq-1]) \doteq \begin{cases} \alpha_1 \cdot \alpha[(k-1)q-1] & k \not\equiv 0 \pmod p \\ 0 & k \equiv 0 \pmod p \end{cases}$$

$$d_2(\alpha[kq]) \underset{(\mathrm{mod}\ I_1)}{\dot{\equiv}} \begin{cases} \alpha_1 \cdot \alpha[(k-1)q] & k \not\equiv 1 \pmod p \\ 0 & k \equiv 1 \pmod p \end{cases}$$

$$d_q(\alpha[kq-1]) \underset{(\mathrm{mod}\ I_2)}{\dot{\in}} \begin{cases} \langle \alpha, \underbrace{\alpha_1, \dots, \alpha_1}_{p-1} \rangle [(k-p+1)q-1] & k \equiv -1 \pmod p \\ 0 & k \not\equiv -1 \pmod p \end{cases}$$

$$d_q(\alpha[kq]) \underset{(\mathrm{mod}\ I_2)}{\dot{\in}} \begin{cases} \langle \alpha, \underbrace{\alpha_1, \dots, \alpha_1}_{p-1} \rangle [(k-p+1)q] & k \equiv 0 \pmod p \\ 0 & k \not\equiv 0 \pmod p \end{cases}$$

Proof. We shall begin by computing $d_2(\alpha[kq-1])$. In light of the coaction equation 8.3, we compute from 8.5 and 8.6

$$\frac{f([p]_F(x))}{[p]_F(x)} \left(f(x)\right)^r = x^r - (r + p^{p-1}) t_1 x^{r+p-1} + \mathcal{O}(x^{r+p}).$$

We may conclude that the coaction formula on the generator e_{kq-1} of $BP_*(P_0^\infty)$ is given by

$$\psi(e_{kq-1}) = 1 \otimes e_{kq-1} + ((k-1)(p-1)-1) t_1 \otimes e_{(k-1)q-1} + \cdots \tag{8.10}$$

Now t_1 represents α_1 in the cobar complex, and $p \cdot \alpha_1 = 0$.

We now compute d_2.

$$\delta_2(\alpha[kq-1]) = ((k-1)(p-1)-1)\alpha | t_1 [(k-1)q-1] + \cdots$$

and it follows immediately that

$$\begin{aligned} d_2(\alpha[kq-1]) &= ((k-1)(p-1)-1)\alpha_1 \cdot \alpha[(k-1)q-1] \\ &\equiv -k\alpha_1 \cdot \alpha[(k-1)q-1] \pmod p \end{aligned}$$

Of course, since $p \cdot \alpha_1 = 0$, one needs only to compute this differential modulo p.

Now we shall deal with $d_2(\alpha[kq])$. We shall work modulo I_1.

$$\phi_* \alpha[kq] \equiv v_1 \cdot \alpha v_1 [(k-1)q-1] + \cdots$$

Therefore $\phi_* \alpha[kq]$ lifts to $\mathrm{Ext}(P_0^{(k-1)q})$. So

$$d_2(\alpha[kq]) = \delta_1(v_1 \cdot \alpha[(k-1)q-1] + \cdots).$$

The boundary homomorphism δ_1 is computed below on the level of cobar complexes.

$$\begin{tikzcd} C^*(P_0^{(k-1)q-1}) \arrow[r, "\iota_*"] \arrow[dr, "d"] & C^*(P^{(k-1)q}) \\ C^*(S^{(k-1)q-1}) \arrow[r, "\phi_*"] & C^*(P_0^{(k-1)q-1}) \end{tikzcd}$$

$$\begin{tikzcd} v_1 \cdot \alpha[(k-1)q-1] + \cdots \arrow[r, mapsto] & v_1 \cdot \alpha[(k-1)q-1] + \cdots \\ -(k-1)\alpha|t_1[(k-1)q-1] \arrow[r, mapsto] & -(k-1)v_1 \cdot \alpha|t_1[(k-2)q-1] + \cdots \end{tikzcd}$$

Here, $C^*(-)$ represents the cobar complex. We conclude that

$$\delta_1(v_1 \cdot \alpha[(k-1)q-1] + \cdots) \equiv (1-k)\alpha|t_1[(k-1)q-1] \pmod{I_1}$$

The d_q's are proved similarly. The relevant information to be gathered from Equations 8.8 and 8.9 is

$$\begin{aligned} \frac{f([p]_F(x))}{[p]_F(x)}(f(x))^r \underset{(\mathrm{mod}\ I_2)}{\equiv} & x^r + \binom{r}{1}(-t_1)x^{r+(p-1)} + \cdots \\ & + \binom{r}{p-1}(-t_1)^{p-1}x^{r+(p-1)^2} + \mathcal{O}(x^{r+(p-1)^2+1}) \end{aligned}$$

The Massey product

$$\langle \alpha, \underbrace{\alpha_1, \ldots, \alpha_1}_{p-1} \rangle$$

corresponds (up to multiplication by a unit) to

$$\alpha|t_1^{p-1} + \cdots$$

in the cobar complex. We have that

$$d_q(\alpha[kq-1]) \underset{(\mathrm{mod}\ I_2)}{\in} \binom{r}{p-1}\langle \alpha, \underbrace{\alpha_1, \ldots, \alpha_1}_{p-1} \rangle[(k-p+1)q-1].$$

where $r = (k-p+1)(p-1)-1$. Consider the coefficient

$$\binom{r}{p-1} = \frac{r \cdot (r-1) \cdots (r-p+2)}{(p-1)!}.$$

Since $p \not| \, p-1$, in order for p to not divide this binomial coefficient, it must be the case that $r \equiv -1 \pmod{p}$. In terms of k, this translates to $k \equiv -1 \pmod{p}$.

Just like in the calculation of $d_2\alpha[kq]$, the map δ_1 introduces an offset in coefficients which results in the claimed formula for $d_q\alpha[kq]$. $\square$

It is interesting to note that these differentials, combined with the image of J differentials, are enough to compute the AAHSS completely through a certain range at odd primes.

In $\mathrm{Ext}(BP_*)$, many elements live in an α_1-β_1 tower, that is, a copy of $P[\beta_1] \otimes E(\alpha_1)$. The following general observation says that elements of the E_1-term of the

AAHSS which lie within an α_1-β_1 tower typically either support differentials are the target of differentials.

Proposition 8.4. Let $\gamma \in \mathrm{Ext}(BP_*)$. Suppose furthermore that there exist elements λ and μ in the E_2 term of the Adams-Novikov spectral sequence such that either

$$\alpha_1 \lambda = \gamma$$
$$\langle \gamma, \underbrace{\alpha_1, \ldots, \alpha_1}_{p-1} \rangle = \mu$$

or

$$\langle \lambda, \underbrace{\alpha_1, \ldots, \alpha_1}_{p-1} \rangle = \gamma$$
$$\alpha_1 \gamma = \mu.$$

(Here we insist that the Massey products have no indeterminacy.) Assume furthermore that p and v_1 do not divide any of λ, μ, or γ. Then, in the AAHSS, $\gamma[n]$ is either killed by a differential, or supports a non-trivial differential.

Proof. The condition on p and v_1 not dividing any of the elements λ, μ, or γ is just to ensure that we may use the formulas in Proposition 8.3. Let us suppose that γ satisfies the first set of conditions.

Case 1: $n \equiv -1 \pmod q$: Write $n = mq - 1$.
- Subcase (a): $m \equiv -1 \pmod p$: Then $d_q \gamma[mq-1] = \mu[(m-p+1)q-1]$ or $d_i\gamma[mq-1] \neq 0$ for some $i < q$.
- Subcase (b): $m \not\equiv -1 \pmod p$: Then $d_2\lambda[(m+1)q-1] = \gamma[mq-1]$.

Case 2: $n \equiv 0 \pmod q$: Write $n = mq$.
- Subcase (a): $m \equiv 0 \pmod p$: Then either $d_q\gamma[mq] = \mu[(m-p+1)q]$ or we have $d_i\gamma[mq] \neq 0$ for some $i < q$.
- Subcase (b): $m \not\equiv 0 \pmod p$: Then $d_2\lambda[(m+1)q] = \gamma[mq]$.

Suppose now that γ satisfies the second set of conditions.

Case 1: $n \equiv -1 \pmod q$: Write $n = mq - 1$.
- Subcase (a): $m \equiv 0 \pmod p$: Then $d_q\lambda[(m+p-1)q-1] = \gamma[mq-1]$.
- Subcase (b): $m \not\equiv 0 \pmod p$: Then $d_2\gamma[mq-1] = \mu[(m-1)q-1]$.

Case 2: $n \equiv 0 \pmod q$: Write $n = mq$.
- Subcase (a): $m \equiv 1 \pmod p$: Then $d_q\lambda[(m+p-1)q] = \gamma[mq]$.
- Subcase (b): $m \not\equiv 1 \pmod p$: Then $d_2\gamma[mq] = \mu[(m-1)q]$ or $d_1\gamma[mq] \neq 0$. (This is the case of $p \cdot \gamma \neq 0$.)

□

It is also useful to record all of the differentials supported by the v_1-periodic elements of $\mathrm{Ext}(BP_*)$ for $p > 2$ in the AAHSS. These results are used in executing Step 4 of Procedure 9.1.

Proposition 8.5. In the AAHSS, the differentials supported by v_1-periodic elements whose targets are v_1-periodic elements are given by

$$d_1(\alpha_{i/j}[kq]) = \alpha_{i/(j-1)}[kq-1] \text{ for } j > 1$$
$$d_{2j+1}(\alpha_i[kq]) \doteq \tilde{\alpha}_{i+j}[(k-j)q-1] \text{ where } \nu_p(k+i) = j-1.$$

(Here $\tilde{\alpha}_l$ is $\alpha_{l/m}$ for $m = \nu_p(l) + 1$. It is the additive generator of imJ in the $lq - 1$ stem.)

Proof. These differentials detect the corresponding differentials in the AHSS which converge to the stable homotopy of P_N. These differentials were computed for P_0 by Thompson in [38] and are summarized in [33]. ☐

There are still v_1-periodic elements of $\mathrm{Ext}(BP_*)$ which are eligible to support differentials in the filtered spectral sequence (or subsequently in the Adams-Novikov spectral sequence for P_N) whose targets are elements which are not v_1-periodic. These elements are described below.

Corollary 8.6. The only elements in imJ in the AAHSS for $P_{-\infty}$ which are neither targets or sources of the differentials described in Proposition 8.5 are

$$\tilde{\alpha}_i[kq-1]$$

where $p^i | (k+i)$.

We obtain the following consequence which will become quite relevant when we execute Step 4 of Procedure 9.1.

Proposition 8.7. None of the elements $\beta_{i/j}[kq]$ are the target of differentials in the AAHSS or subsequently in the Adams-Novikov spectral sequence for $\pi_*(P_N)$.

Proof. The only elements which could kill $\beta_{i/j}[kq]$ in the filtered spectral sequence are of the form $\alpha_n[mq]$. But these cannot by Corollary 8.6. There are no elements in the E_2 term of the Adams-Novikov spectral sequence which can kill $\beta_{i/j}[kq]$. Such elements would have to lie in Adams-Novikov filtration 0. ☐

For $p = 2$, the series are tabulated below.

$$[2]_F(x) = 2x - v_1x^2 + 2v_1^2x^3 - (7v_2 + 8v_1^3)x^4 + \mathcal{O}(x^5) \tag{8.11}$$

$$f(x) = x - t_1x^2 + (2t_1^2 + v_1t_1)x^3 + (-5t_1^3 - t_2 - 5v_1t_1^2 - v_1^2t_1)x^4 + \mathcal{O}(x^5) \tag{8.12}$$

$$\frac{f([2]_F(x))}{[2]_F(x)} = 1 - 2t_1x + (8t_1^2 + 5t_1v_1)x^2 + (-40t_1^3 - 8t_2 - 48v_1t_1^2 - 14v_1^2t_1)x^3 + \mathcal{O}(x^4) \tag{8.13}$$

We obtain the following differentials at $p = 2$.

Proposition 8.8. In the filtered spectral sequence, the following formulas for differentials hold.

$$d_2(\alpha[k]) = \begin{cases} \alpha_1 \cdot \alpha[2(k-1)-1] & k \equiv 0,1 \pmod 4 \\ 0 & k \equiv 2,3 \pmod 4 \end{cases}$$

$$d_4(\alpha[k]) \underset{(\mathrm{mod}\ I_2)}{\equiv} \begin{cases} \alpha_{2/2} \cdot \alpha[2(k-2)-1] & k \equiv 0,1,2,3 \pmod 8 \\ 0 & k \equiv 4,5,6,7 \pmod 8 \end{cases}$$

9. Procedure for low dimensional calculations of root invariants

In this section, we let $E = BP$, and concentrate on BP-filtered root invariants. The following procedure is used in later sections to compute homotopy root invariants from filtered root invariants using Theorem 5.1. It only has a chance to work through a finite range, and is very crude. We state it mainly to codify everything that must be checked in general to see that a filtered root invariant detects a homotopy root invariant.

Procedure 9.1. Suppose we are in the situation where we know β is an element of $R^{[k]}_{BP}(\alpha)$, and we know that β is a permanent cycle. Let ι_{-N} be the inclusion of the $-N$ cell of P_{-N}, where $-N = -N_i$ is the index of the bifiltration of α that corresponds to the cell that bears the k^{th} filtered root invariant. Let $\nu_{-N}\alpha$ be the image of α in $\pi_*(P_{-N})$. Then Theorem 5.1 tells us that $\nu_{-N}\alpha = \iota_{-N}\beta$ modulo elements in higher Adams-Novikov filtration.

Step 1: Make a list

$$\gamma_i[-n_i]$$

of additive generators in the E_1 term of the AAHSS for $\mathrm{Ext}(P_{-N})$, where γ_i lies in Adams-Novikov filtration s_i and stem k_i, satisfying the following.

(1) The homological degree s_i satisfies

$$s_i > \begin{cases} k & n_i \equiv 1 \pmod q \\ k+1 & n_i \equiv 0 \pmod q \end{cases}$$

(2) The stem k_i is greater than $p \cdot |\alpha|$, and less than $|\beta|$.

(3) We have $n_i = k_i - |\alpha| + 1$, and $n_i \equiv 0, 1 \pmod q$.

These are precisely the conditions required for $\gamma_i[n_i]$ to be a candidate to survive in the AAHSS to an element in the same stem, but higher Adams-Novikov filtration, as $\iota_{-N}\beta$. Condition (3) is a consequence of P_{-N} having cells in dimensions congruent to $0, -1$ modulo q. We apply the first inequality in (2) because Jones's theorem [29] will preclude the possibility of these surviving to detect the difference of $\iota_{-N}\overline{\beta}$ and $\nu_{-N}\alpha$. Thus we have

$$\nu_{-N}\alpha = \beta[-N] + \sum_i a_i \gamma_i[-n_i] + (\text{ terms born on cells } < -(p-1)|\alpha| + 1).$$

Stem 2.: Attempt to determine which $\gamma_i[n_i]$ are killed in the filtration spectral sequence, or subsequently, in the Adams-Novikov spectral sequence for computing $\pi_*(P_{-N})$. This will limit the possibilities for what can detect the difference of $\iota_{-N}\overline{\beta}$ and $\nu_{-N}\alpha$.

Step 3: First eliminate those γ_i which are not permanent cycles in the ANSS. These cannot be root invariants. Then, attempt to show that every remaining non-trivial linear combination of the elements $\gamma_i[n_i]$ which are not killed in the AAHSS actually supports a non-trivial differential in the AHSS for computing $\pi_*(P_{-\infty})$. Suppose this is the case. Then, if $\iota_{-N}\overline{\beta} - \nu_{-N}\alpha$ is non-trivial, and is born on cells above the $-N$-cell, it will project non-trivially to $\pi_*(P_{-M})$ for $M < N$, and will represent the image of the homotopy root invariant. The homotopy root invariant is a permanent cycle in the AHSS for $\pi_*(P_{-\infty})$. Thus we may assume that the difference $\iota_{-N}\overline{\beta} - \nu_{-N}\alpha$ is actually born on cells above the $-(p-1)|\alpha| + 1$ cell. If this difference was non-trivial, we would violate Jones's theorem [29]. The difference is therefore trivial, and $\iota_{-N}\overline{\beta} = \nu_{-N}\alpha$.

Step 4: We will have shown that $\overline{\beta} \in R(\alpha)$ if we know that $\iota_{-N}\overline{\beta} \neq 0$. It would suffice to show that $\beta[-N]$ is not killed in the AAHSS, and also survives to a non-zero element in the ANSS for computing $\pi_*(P_{-N})$. Form a list of all of the elements

$$\eta_i[n_i]$$

in the E_1-term of the AAHSS where η_i is in Adams-Novikov filtration s_i and stem k_i such that:

(1) $s_i \leq \begin{cases} k & -N \equiv -1 \pmod q \\ k-1 & -N \equiv 0 \pmod q \end{cases}$

(2) $k_i \leq |\beta|$

(3) $n_i = -N + |\alpha| - k_i + 1 \equiv 0, -1 \pmod q$

These are precisely the elements which have a chance of killing $\beta[-N]$ in the AAHSS, or subsequently in the ANSS. Next, show these elements actually do not kill $\beta[-N]$.

Remark 9.2. By comparing the definition of differentials in the AAHSS with the differentials in the AHSS, it is clear that if γ detects $\overline{\gamma}$ in the ANSS for computing π_*^S, then if $\gamma[n]$ supports a non-trivial AAHSS differential

$$d_r(\gamma[n]) = \eta[m]$$

where η is a permanent cycle in the ANSS, then either $\overline{\gamma}$ supports a non-trivial AHSS d_i for $i < r$, or there is an AHSS differential

$$d_r(\overline{\gamma}[n]) = \overline{\eta}[m]. \tag{9.1}$$

where η detects $\overline{\eta}$. However, the differential given in Equation 9.1 may be trivial in the AHSS if $\overline{\eta}$ is the target of a shorter differential.

Since in Step 3 of Procedure 9.1 we are only concerned with whether the elements $\overline{\gamma}_i[n_i]$ support non-trivial differentials in the AHSS, it suffices to show that the elements $\gamma_i[n_i]$ support non-trivial differentials in the AAHSS. One must then make sure that the targets of the AAHSS differentials are not the targets of shorter differentials in the AHSS.

10. BP-filtered root invariants of some Greek letter elements

In this section we compute the first two BP-filtered root invariants of some chromatic families. If $\alpha = \alpha^{(n)}_{i/j_1,\dots,j_k}$ is the n^{th} algebraic Greek letter element of the ANSS which survives to a non-trivial element of π_*^S, then one might expect that it should be the case that

$$\alpha^{(n+1)}_{i/j_1,\dots,j_k} \mathrel{\dot{\in}} R^{[n+1]}_{BP}(\alpha^{(n)}_{i/j_1,\dots,j_k}) \tag{10.1}$$

The purpose of this section is to show that Equation 10.1 holds for $n = 0$ at all primes and for $n = 1$ at odd primes. Modulo an indeterminacy group which we do not compute, we also show that the $n = 1$ case of Equation 10.1 is true at the prime 2. Throughout this section, for a ring spectrum E, we shall let $\widetilde{E} \simeq \Sigma\overline{E}$ denote the cofiber of the unit. Also, whereas in Section 8 we used Hazewinkel generators, *in this section we always use Araki generators.* This is because the p-series is more naturally expressed in the Araki generators.

For appropriate i and j, the elements which generate the 1-line of the ANSS are $\alpha_{i/j} \in BP_*BP$ and are given by

$$\alpha_{i/j} = \frac{\eta_R(v_1)^i - v_1^i}{p^j}.$$

Proposition 10.1. The first two filtered root invariants of p^i are given by

$$(-v_1)^i \in R^{[0]}_{BP}(p^i)$$
$$(-1)^i \cdot \alpha_i \in R^{[1]}_{BP}(p^i)$$

Define, for appropriate i and j,

$$\widetilde{\beta}_{i/j} = \frac{(v_2 + v_1 t_1^p - v_1^p t_1)^i - v_2^i}{v_1^j} \in BP_*BP.$$

Observe that the image of $\widetilde{\beta}_{i/j}$ in BP_*BP/I_1 coincides with

$$\frac{\eta_R(v_2)^i - v_2^i}{v_1^j}.$$

It follows from the definition of the algebraic Greek letter elements that, in the cobar complex, $d_1(\widetilde{\beta}_{i/j}) = p \cdot \beta_{i/j}$ unless $p = 2$ and $i = j = 1$, in which case $\widetilde{\beta}_1 = \alpha_{2/2}$, which is a permanent cycle that detects ν.

In order to compute the first filtered root invariants of the elements $\alpha_{i/j}$, we first compute the $BP \wedge \widetilde{BP}$-root invariants, and then invoke Corollary 5.9. Note that $\widetilde{BP} = \Sigma \overline{BP}$, so the $BP \wedge \widetilde{BP}$-root invariant is just the suspension of the $BP \wedge \overline{BP}$-root invariant. The following result holds for any prime.

Proposition 10.2. The $BP \wedge \widetilde{BP}$-root invariant of $\alpha_{i/j}$ is given by

$$R_{BP \wedge \widetilde{BP}}(\alpha_{i/j}) = (-1)^{i-j} \widetilde{\beta}_{i/j} + p\widetilde{\beta}_{i/j} BP_* \widetilde{BP}$$

Corollary 10.3. Suppose the Greek letter element $\alpha_{i/j}$ exists and is a permanent cycle in the ANSS. Let $\overline{\alpha}_{i/j} \in \pi_*^S$ be the element that $\alpha_{i/j}$ detects. Then the zeroth filtered root invariant of $\overline{\alpha}_{i/j}$ is trivial. If p is odd, the first filtered root invariant of $\overline{\alpha}_{i/j}$ is given by

$$(-1)^{i-j} \widetilde{\beta}_{i/j} + c \cdot \widetilde{\alpha}_{i(p+1)+j} \in R^{[1]}_{BP}(\overline{\alpha}_{i/j})$$

where c is some constant. The second filtered root invariant is given by

$$(-1)^{i-j} \beta_{i/j} \in R^{[2]}_{BP}(\overline{\alpha}_{i/j}).$$

Deducing the first filtered root invariant in Corollary 10.3 from Proposition 10.2 amounts to computing the indeterminacy group A described in Corollary 5.9. The second filtered root invariant then follows from Theorem 5.3. We describe a spectral sequence which computes A (10.8). We fully compute A for odd primes p and describe some of the 2-primary aspects of this computation in Remark 10.6.

We have shown that $\alpha_k \mathrel{\dot{\in}} R^{[1]}_{BP}(p^k)$. It is shown in [25], [35] that these elements survive in the Adams-Novikov spectral sequence to elements of $R(p^k)$. Similarly, we have shown that $\beta_k \mathrel{\dot{\in}} R^{[2]}_{BP}(\overline{\alpha}_k)$. Again, in [25], [35] it is shown that these elements survive to elements of $R(\overline{\alpha}_k)$ for $p \geq 5$. Sadofsky goes further in [35] to show that $\beta_{p/2} \in R(\overline{\alpha}_{p/2})$ for $p \geq 5$.

In general, we have shown that if $p^{k-1}|s$, then $\beta_{s/k} \mathrel{\dot{\in}} R_{BP}^{[2]}(\alpha_{s/k})$. According to the summary presented in [33, 5.5], the elements $\beta_{s/k}$ exist and are permanent cycles in the ANSS for $p \geq 5$ for these values of k and s. However, without any additional information, we can only deduce the conclusions of Theorem 5.1. In Sections 12 and 13 we indicate what these filtered root invariant calculations mean for the computation of homotopy root invariants in low dimensions.

In proving Propositions 10.1 and 10.2, we shall need to make use of the following well known computation (see Lemma 2.1 of [2]).

Lemma 10.4. Let E be a complex oriented spectrum whose associated p-series $[p]_E(x)$ is not a zero divisor in $E^*[[x]]$. Then the coefficient ring of the Tate spectrum is given by

$$\pi_*(tE) = (E_*[[x]]/([p]_E(x)))[x^{-1}] = E_*((x))/([p]_E(x))$$

where the degree of x is -2. Furthermore, the inclusions of the projective spectra are described as the inclusion of the fractional ideal

$$x^N E_*[[x]]/([p]_E(x)/x) = E_*(\Sigma(B\mathbb{Z}/p)^{-2N}) \to \pi_*(tE) = E_*((x))/([p]_E(x)).$$

According to Appendix B of [16], the skeletal and coskeletal filtrations of $(E \wedge P)_{-\infty}$ give rise to the same notions of the E-root invariant. It follows that E-root invariants are quite easy to compute for complex orientable spectra E provided one has some knowledge of the p-series of E. The method is outlined in the following corollary.

Corollary 10.5. Suppose that α is an element of E_*. Viewing $\overline{\alpha}$ as the image of the constant power series α in $E_*((x))/([p]_E(x))$, suppose that n is maximal so that

$$\overline{\alpha} = a_n x^n + \mathcal{O}(x^{n+1})$$

with $a_n \in E_*$ nonzero. Then a_n is an element of $R_E(\alpha)$.

Proof of Proposition 10.1. We first compute $R_{BP}(p^i)$. The p-series $[p]_F(x)$ of the universal p-typical formal group F is given by

$$[p](x) = px +_F v_1 x^p +_F v_2 x^{p^2} +_F \cdots$$

where v_i are the Araki generators of BP_*. In tBP_*, we have the relation

$$p = -v_1 x^{p-1} + \mathcal{O}(x^p)$$

which gives, upon taking the i^{th} power,

$$p^i = (-v_1)^i x^{(p-1)i} + \mathcal{O}(x^{(p-1)i+1}).$$

Any other expression of p^i in terms of $x^{(p-1)i}$ and higher order terms will have a leading coefficient that differs by an element of the ideal $pBP_* \subset BP_*$. Using Corollary 10.5 and Proposition 5.7, we may conclude that

$$(-v_1)^i + pBP_* = R_{BP}(p^i) \supseteq R_{BP}^{[0]}(p^i).$$

We calculate the Adams-Novikov d_1 on this coset as

$$d_1(v_1^i) = \eta_R(v_1^i) - v_1^i.$$

The algebraic Greek letters α_i are defined by

$$\alpha_i = \frac{\eta_R(v_1^i) - v_1^i}{p}$$

so we are in the situation where

$$d_1(R^{[0]}_{BP}(p^i)) \subseteq p \cdot (-1)^i \alpha_i + p^2 \alpha_i BP_* BP.$$

By Theorem 5.3, we may conclude that the first filtered root invariant $R^{[1]}_{BP}(p^i)$ is contained in the coset $(-1)^i \alpha_i + p\alpha_i BP_* BP$. □

Proof of Proposition 10.2. The proof consists of two parts. In Part 1 we prove the proposition with v_1 inverted, and in Part 2 we prove that the $v_1^{-1}BP \wedge \widetilde{BP}$-root invariant can be lifted to compute the $BP \wedge \widetilde{BP}$-root invariant.

Part 1: computing the $v_1^{-1}BP \wedge \widetilde{BP}$-root invariant

We may calculate $tBP \wedge \widetilde{BP}_*$ from tBP_*BP using the split short exact sequence

$$0 \to tBP_* \to tBP \wedge BP_* \to tBP \wedge \widetilde{BP}_* \to 0.$$

Applying Lemma 10.4 to the left and right complex orientations of BP_*BP, we get

$$\begin{aligned} tBP \wedge BP_* = BP \wedge BP_*((x_L))/([p]_L(x_L)) \\ BP \wedge BP_*((x_R))/([p]_R(x_R)). \end{aligned}$$

Here, the p-series $[p]_L(x)$, $[p]_R(x)$, and the coordinate x_R are given by the formulas

$$\begin{aligned} [p]_L(x) &= px +_{F_L} v_1 x^p +_{F_L} v_2 x^{p^2} +_{F_L} \cdots \\ [p]_R(x) &= px +_{F_R} \eta_R(v_1) x^p +_{F_R} \eta_R(v_2) x^{p^2} +_{F_R} \cdots \\ x_L &= x_R +_{F_L} t_1 x_R^p +_{F_L} t_2 x_R^{p^2} +_{F_L} \cdots . \end{aligned}$$

The formal group laws F_L and F_R are the p-typical formal group laws over BP_*BP induced by left and right units η_L and η_R, respectively. Consider the following computation in $tBP \wedge \widetilde{BP}_*$.

$$\begin{aligned} px_R &= [-1]_{F_R}\left(\eta_R(v_1) x_R^p +_{F_R} \eta_R(v_2) x_R^{p^2} +_{F_R} \cdots\right) \\ &= -\eta_R(v_1) x_R^p - \eta_R(v_2) x_R^{p^2} + \mathcal{O}(x_R^{p^2+1}) \end{aligned} \tag{10.2}$$

The last equality for $p > 2$ follows from the fact that for any p-typical formal group law G where $p > 2$, the -1-series is given by

$$[-1]_G(x) = -x.$$

For $p = 2$ this is not true, but it turns out that the last equality of Equation 10.2 still holds up to the power indicated in $tBP_*\widetilde{BP}$. The two expressions we want to be equal are written out explicitly below.

$$\begin{aligned} &[-1]_{F_R}\left(\eta_R(v_1)x_R^2 +_{F_R} \eta_R(v_2) x_R^4 +_{F_R} \cdots\right) \\ &\quad = (2t_1 - v_1)x_R^2 + (14t_2 - 4t_1^3 - v_1 t_1^2 - 3v_1^2 t_1 - v_2 + v_1^3)x_R^4 + \mathcal{O}(x_R^5) \end{aligned} \tag{10.3}$$

$$\begin{aligned} &-\eta_R(v_1)x_R^2 - \eta_R(v_2)x_R^4 + \mathcal{O}(x_R^5) \\ &\quad = (2t_1 - v_1)x_R^2 + (14t_2 + 4t_1^3 - 13v_1 t_1^2 + 3v_1^2 t_1 - v_2)x_R^4 + \mathcal{O}(x_R^5) \end{aligned} \tag{10.4}$$

The coefficient of x_R^2 is the same in Equations 10.3 and 10.4. Since we are working in $tBP_*\widetilde{BP}$, we are working modulo the 2-series, and thus we only need the coefficients of x_R^4 to be equivalent modulo 2. However, the coefficient of x_R^4 in Equation 10.3

has an extra v_1^3, but since we are working in the reduced setting of $tBP_*\widetilde{BP}$, we have $v_1 x_L^4 = 0$. We may switch to x_R since $x_L^4 = x_R^4 + \mathcal{O}(x_R^5)$.

Returning to our manipulations of the p-series, upon dividing Equation 10.2 by x_R, we get

$$p = -\eta_R(v_1)x_R^{p-1} - \eta_R(v_2)x_R^{p^2-1} + \mathcal{O}(x_R^{p^2})$$

which implies

$$\eta_R(v_1) = -\eta_R(v_2)x_R^{p^2-p} + \mathcal{O}(x_R^{p^2-p+1})$$

since the image of p is zero in $BP_*\widetilde{BP}$. Taking the i^{th} power and exploiting the fact that the image of v_1^i in $BP_*\widetilde{BP}$ is also zero, we may write

$$p^j \alpha_{i/j} = \eta_R(v_1)^i - v_1^i = (-\eta_R(v_2))^i x_R^{i(p^2-p)} + \mathcal{O}(x_R^{i(p^2-p)+1}). \tag{10.5}$$

Since we are modding out by the p-series in x_L, we have the relation

$$\begin{aligned} px_L &= [-1]_{F_L}\left(v_1 x_L^p +_{F_L} v_2 x_L^{p^2} +_{F_L} \cdots\right) \\ &= -v_1 x_L^p + \mathcal{O}(x_L^{p^2}) \end{aligned}$$

or, dividing by x_L and taking the j^{th} power,

$$p^j = (-v_1)^j x_L^{j(p-1)} + \mathcal{O}(x_L^{j(p-1)+p^2-p}) \tag{10.6}$$

Upon combining Equation 10.6 with Equation 10.5, and using the fact that $x_R^k = x_L^k + \mathcal{O}(x_R^{k+1})$, our original expression for $\alpha_{i/j}$ becomes the following.

$$\begin{aligned} &(-v_1)^j \alpha_{i/j} + \alpha_{i/j}\mathcal{O}(x_R^{p^2-p}) \\ &\qquad = (-1)^i(\eta_R(v_2)^i - v_2^i)x_R^{i(p^2-p)-j(p-1)} + \mathcal{O}(x_R^{i(p^2-p)-j(p-1)+1}) \end{aligned}$$

Explicit formulas (see, for instance, [33, 4.3.21]) reveal that

$$\eta_R(v_2) = v_2 + v_1 t_1^p - v_1^p t_1 + py$$

for some $y \in BP_*BP$. Using this formula and Equation 10.6 to write elements which are divisible by p in terms of higher order elements, we have

$$\begin{aligned} &(-v_1)^j \alpha_{i/j} + \alpha_{i/j}\mathcal{O}(x_R^{p^2-p}) \\ &\qquad = (-1)^i v_1^j \tilde{\beta}_{i/j} x_R^{i(p^2-p)-j(p-1)} + \mathcal{O}(x_R^{i(p^2-p)-j(p-1)+1}) \end{aligned} \tag{10.7}$$

In order to solve for $\alpha_{i/j}$, we would like to divide by v_1^j. Therefore, we shall finish our algebraic manipulations in $tv_1^{-1}BP \wedge \widetilde{BP}_*$. We will then show that we can pull back our results to results in $tBP \wedge \widetilde{BP}_*$. Taking the image of Equation 10.7 in $tv_1^{-1}BP \wedge BP_*$ and dividing by $(-v_1)^j$, we get the expression

$$\alpha_{i/j} = (-1)^{i-j}\widetilde{\beta}_{i/j} x_R^{i(p^2-p)-j(p-1)} + \alpha_{i/j}\mathcal{O}(x_R^{p^2-p}) + \mathcal{O}(x_R^{i(p^2-p)-j(p-1)+1})$$

By successively substituting the left hand side of this expression into the right hand side, we obtain the expression below.

$$\alpha_{i/j} = (-1)^{i-j}\tilde{\beta}_{i/j} x_R^{i(p^2-p)-j(p-1)} + \text{higher order terms}.$$

We may conclude that $R_{v_1^{-1}BP \wedge \widetilde{BP}}(\alpha_{i/j}) = \widetilde{\beta}_{i/j}$, or that the $v_1^{-1}BP \wedge \widetilde{BP}$-root invariant lives in a higher stem.

Part 2: lifting to the $BP \wedge \widetilde{BP}$-root invariant

We claim that the localization map

$$BP \wedge \widetilde{BP}_*(P_{-N}) \to v_1^{-1} BP \wedge \widetilde{BP}_*(P_{-N})$$

is an inclusion. In order to see this, chase the following diagram.

$$\begin{tikzcd}
BP_*(P_{-N}) \arrow[r] \arrow[d, hook] & BP \wedge BP_*(P_{-N}) \arrow[r] \arrow[d, hook] & BP \wedge \widetilde{BP}_*(P_{-N}) \arrow[d] \\
v_1^{-1} BP_*(P_{-N}) \arrow[r] & v_1^{-1} BP \wedge BP_*(P_{-N}) \arrow[r] & v_1^{-1} BP \wedge \widetilde{BP}_*(P_{-N})
\end{tikzcd}$$

The relevant observations are that the other two localization maps in the diagram are inclusions (since there is no v_1 torsion in these groups), and the top and bottom sequences are compatibly split cofiber sequences.

We wish to deduce something about the $BP \wedge \widetilde{BP}$ root invariant from the computation of the $v_1^{-1} BP \wedge \widetilde{BP}$ root invariant. We refer to the following diagram.

$$\begin{tikzcd}
v_1^{-1} BP \wedge \widetilde{BP}_* \arrow[rrr, squiggly, "R_{v_1^{-1} BP \wedge \widetilde{BP}}(-)"] \arrow[ddd] & & & v_1^{-1} BP \wedge \widetilde{BP}_*(S^{-N+1}) \arrow[ddd, "\iota"] \\
& BP \wedge \widetilde{BP}_* \arrow[r, squiggly, "R_{BP \wedge \widetilde{BP}}(-)"] \arrow[ul] \arrow[d] & BP \wedge \widetilde{BP}_*(S^{-N+1}) \arrow[ur] \arrow[d, "\iota"] & \\
& tBP \wedge \widetilde{BP}_* \arrow[r, "\nu"'] \arrow[dl] & BP \wedge \widetilde{BP}_*(\Sigma P_{-N}) \arrow[dr] & \\
tv_1^{-1} BP \wedge \widetilde{BP}_* \arrow[rrr, "\nu"'] & & & v_1^{-1} BP \wedge \widetilde{BP}_*(\Sigma P_{-N})
\end{tikzcd}$$

Here N equals $2i(p^2 - p) - 2j(p-1) + 1$. We have shown that $\iota((-1)^{i-j} \widetilde{\beta}_{i/j}) = \nu(\alpha_{i/j})$ after v_1 is inverted. Because the v_1-localization map is an inclusion, we may conclude that $\iota((-1)^{i-j} \widetilde{\beta}_{i/j}) = \nu(\alpha_{i/j})$ in $BP \wedge \widetilde{BP}_*(\Sigma P_{-N})$. Therefore, we have computed the $BP_* \widetilde{BP}$-root invariant

$$R_{BP \wedge \widetilde{BP}}(\alpha_{i/j}) = (-1)^{i-j} \widetilde{\beta}_{i/j} + p\widetilde{\beta}_{i/j} BP_* \widetilde{BP}$$

or it lies in a larger stem. The latter cannot be the case, however, as $\iota(\widetilde{\beta}_{i/j})$ is non-zero. $\square$

We wish to apply Corollary 5.9 to Proposition 10.2 and conclude that

$$R_{BP}^{[1]}(\overline{\alpha}_{i/j}) \subseteq (-1)^{i-j} \widetilde{\beta}_{i/j} + p\widetilde{\beta}_{i/j} BP_* \widetilde{BP} + A.$$

Here A is the image of the boundary homomorphism

$$\pi_{iq-1}(W_0^{(0,1)}(P_{(-(ip-j)q-1, -(ip-j)q)})) \xrightarrow{\partial} \pi_{iq-2}(W_1^1(S^{-(ip-j)q-1})).$$

Let N be $(ip-j)q + 1$.

The group $\pi_*(W_0^{(0,1)}(P_{(-N,-N+1)}))$ may be computed from the AHSS associated to the filtration

$$W_0^0(S^{-N}) \subset W_0^{(0,1)}(P_{(-N,-N+1)}^{-N+1}) \subset W_0^{(0,1)}(P_{(-N,-N+1)}^{-N+2}) \subset \cdots .$$

The resulting spectral sequence takes the form

$$E_1^{k,l} = \begin{cases} 0 & l \not\equiv 0,-1 \pmod q \\ \pi_k(W_0^1(S^l)) & l \equiv 0,-1 \pmod q,\ l > -N \\ \pi_k(W_0^0(S^{-N})) & l = -N \end{cases} \tag{10.8}$$

and converges to $\pi_k(W_0^{(0,1)}(P_{(-N,-N+1)}))$. The groups $\pi_*(W_0^1(S^l))$ in the E_1-term of the spectral sequence (10.8) may be computed by taking the 0 and 1-lines of the E_1-term of the ANSS for the sphere, and taking the cohomology with respect to the d_1 from the 0-line to the 1-line. The differentials in spectral sequence (10.8) are a restriction of the differentials in the AAHSS.

The image of the map ∂ is generated by the image of Adams-Novikov d_1's supported on the 0-line, the subgroup $pW_1^1(S^{-N})$, and the images of the higher differentials in the AAHSS whose sources are permanent cycles in spectral sequence (10.8) and whose targets are elements in Adams-Novikov filtration 1 that are carried by the $-N$-cell.

We remark that for $0 \neq x \in \pi_*(W_1^1(S^0))$ which is *not* a permanent cycle, we have a non-trivial differential

$$d_1(x[kq]) = px[kq-1].$$

The reason the differential must be non-trivial is that there is no torsion in the E_1-term of the ANSS, and if px were the target of a d_1 in the ANSS, then x would have to be a d_1-cycle. Thus, if we are looking for longer differentials in spectral sequence (10.8) supported by kq-cells, we may restrict our search to those which are actually d_1-cycles.

We now use spectral sequence (10.8) to compute the indeterminacy group A for odd primes p.

Proof of Corollary 10.3. Since $filt_{BP}(\overline{\alpha}_{i/j}) = 1$, Lemma 5.6 implies the filtered root invariant $R_{BP}^{[0]}(\overline{\alpha}_{i/j})$ is trivial. In Proposition 10.2, we found the $BP \wedge \widetilde{BP}$-root invariants $R_{BP\wedge\widetilde{BP}}(\alpha_{i/j})$. We will now deduce the first filtered root invariant $R_{BP}^{[1]}(\overline{\alpha}_{i/j})$ through the application of Corollary 5.8. We just need to compute A using spectral sequence (10.8).

The indeterminacy group A is generated by the images of Adams-Novikov d_1's, the subgroup $pW_1^1(S^{-N})$, and the higher AAHSS differentials. We just need to compute the latter. The only elements of spectral sequence (10.8) that could contribute to A are those of the form

$$\alpha_{k/l}[(i-k)q], \qquad k \leq ip-j.$$

Proposition 8.5 tells us that only one can contribute to δ and that contribution is given by

$$\partial(\alpha_{i(p+1)-j-l}[-(ip-j-l)q]) = \widetilde{\alpha}_{i(p+1)-j}[-(ip-j)q-1]$$

where $l = \nu_p(i)+1$. Thus A is also spanned by the element $\widetilde{\alpha}_{-i(p+1)+j}$. We conclude that

$$(-1)^{i-j}\widetilde{\beta}_{i/j} + c \cdot \widetilde{\alpha}_{-i(p+1)+j} \in R_{BP}^{[1]}(\overline{\alpha}_{i/j})$$

where c is some constant.

To prove the second part of the proposition we appeal to Theorem 5.3. There is an Adams-Novikov differential $d_1(\widetilde{\beta}_{i/j}) = p \cdot \beta_{i/j}$. The filtered root invariant $R^{[1]}_{BP}(\alpha_{i/j})$ is carried by the $-N_1$-cell, where $N_1 = 2(i(p^2-p) - j(p-1)) + 1 = (ip-j)q + 1$. The first cell to attach nontrivially to this cell is the $-(ip-j)q$-cell, and this is by the degree p map. We may conclude that

$$(-1)^{i-j}\beta_{i/j} \in R^{[2]}_{BP}(\overline{\alpha}_{i/j}).$$

□

Remark 10.6. Computing the group A at the prime 2 requires a more careful analysis. The AAHSS differentials don't follow immediately from the J-spectrum AHSS differentials computed in [23], because the varying Adams-Novikov filtrations of the v_1-periodic elements. For instance, it turns out that

$$\alpha_{4/4} = \widetilde{\beta}_{2/2} + x_7 \in R^{[1]}_{BP}(\overline{\alpha}_{2/2})$$

where

$$x_7 \equiv v_2 t_1 + v_1(t_2 + t_1^3) \pmod 2.$$

If $i = j = 1$, then $\widetilde{\beta}_1$ is a permanent cycle which represents the element $\alpha_{2/2}$. If $i = j = 2$, then $\widetilde{\beta}_{2/2} + x_7$ is a permanent cycle which represents the element $\alpha_{4/4}$. Low dimensional calculations seem to indicate that in all other circumstances we have

$$\beta_{i/j} + c \cdot \widetilde{\alpha}_{3i-j+1}\alpha_1 \in R^{[2]}_{BP}(\overline{\alpha}_{i/j})$$

where c is some constant which may or may not be zero and $\widetilde{\alpha}_k$ is equal to $\alpha_{i/j}$ with j maximal. The anomalous cases with $i = j = 1, 2$ correspond to the existence of the 'extra' Hopf invariant 1 elements ν and σ, which are detected in the ANSS by $\alpha_{2/2}$ and $\alpha_{4/4}$, respectively. These filtered root invariants are thus consistent with the homotopy root invariant, which takes each Hopf invariant 1 element to the next one, if it exists.

11. Computation of $R(\beta_1)$ at odd primes

In this section we will compute the root invariant of β_1 at odd primes. This result was stated, but not proved, in [25]. We do this by first computing the BP-filtered root invariants, and then by executing Step 4 of Procedure 9.1.

Proposition 11.1. For $p > 2$, the top filtered root invariant of β_1 is given by $R^{[2p]}_{BP}(\beta_1) \doteq \beta_1^p$.

Sketch of Proof. We first wish to show that $\beta_{p/p} \in R^{[2]}_{BP}(\beta_1)$. In [33], it is shown that modulo the ideal I_1, the representatives for β_1 and $\beta_{p/p}$ in the cobar complex are given by

$$\beta_1 \equiv -\frac{1}{p}\sum_{0<i<p}\binom{p}{i} t_1^i | t_1^{p-i} \pmod{I_1}$$

$$\beta_{p/p} \equiv -\frac{1}{p}\sum_{0<i<p^2}\binom{p^2}{i} t_1^i | t_1^{p^2-i} \pmod{I_1}$$

Now, $\nu_p\binom{p^2}{i} = 1$ if and only if $p|i$. Therefore, the expression for $\beta_{p/p}$ modulo I_1 may be simplified.

$$\beta_{p/p} \equiv -\frac{1}{p} \sum_{0<i<p} \binom{p^2}{ip} t_1^{ip} | t_1^{p(p-i)} \pmod{I_1}$$

Since $\binom{p}{i} \equiv \binom{p^2}{pi} \pmod{p}$ for $0 < i < p$, we see that in $\mathrm{Ext}_{BP_*BP/I_1}(BP_*)$, the element $\beta_{p/p}$ is obtained from β_1 by application of the 0^{th} algebraic Steenrod operation. By computing the reduction map

$$\mathrm{Ext}_{BP_*BP}(BP_*) \to \mathrm{Ext}_{BP_*BP/I_1}(BP_*/I_1)$$

we may conclude that

$$P^0(\beta_1) \equiv \beta_{p/p} \pmod{p}$$

in $\mathrm{Ext}_{BP_*BP}(BP_*)$. By an algebraic analog of Jones's Kahn-Priddy theorem, this coincides with the BP-algebraic root invariant, which corresponds to the first non-trivial filtered root invariant. Thus we have the filtered root invariant

$$\beta_{p/p} \in R_{BP}^{[2]}(\beta_1).$$

In the ANSS, the element $\beta_{p/p}$ supports the Toda differential.

$$d_{q+1}(\beta_{p/p}) \doteq \beta_1^p \alpha_1$$

The $-(p^2-p-1)q-1$ cell attaches to the $-(p^2-p)q-1$ cell of $P_{-\infty}$ with attaching map α_1. Therefore, Theorem 5.3 tells us that

$$\beta_1^p \mathrel{\dot{\in}} R_{BP}^{[2p]}(\beta_1).$$

There is no room for indeterminacy in the ANSS. $\square$

Corollary 11.2. For $p > 2$, the homotopy root invariant of β_1 is given by

$$R(\beta_1) \doteq \beta_1^p.$$

Proof. The element β_1^p lies on the Adams-Novikov vanishing line, so it must be the top filtered root invariant. Therefore, we may apply Corollary 5.2 to see that either the image of the element β_1^p under the inclusion of the $(-(p^2-p-1)q-1)$-cell of $P_{-(p^2-p-1)q-1}$ is null, or β_1^p actually detects the homotopy root invariant. We must therefore show that the element $\beta_1^p[-(p^2-p-1)q-1]$ in the AHSS for $P_{-(p^2-p-1)q-1}$ is not the target of a differential. We will actually show that $\beta_1^p[-(p^2-p-1)q-1]$ is not the target of a differential in the AAHSS, and that there are no possible sources of differentials in the ANSS for $P_{-(p^2-p-1)q-1}$ with target $\beta_1^p[-(p^2-p-1)q-1]$.

According to the low dimensional computations of the ANSS at odd primes given in [33, Ch. 4], the only elements in the E_1-term of the AAHSS which can kill $\beta_1^p[-(p^2-p-1)q-1]$ in either the AAHSS or the ANSS are the elements

$$\begin{aligned} \beta_k[-(k-1)(p+1)q] & \qquad 1 \le k < p \\ \alpha_{k/l}[-(k-p)q-1] & \qquad 1 \le k < p^2, 1 \le l \le \nu_p(k)+1 \end{aligned}$$

as well as elements in α_1-β_1 towers, i.e. those that satisfy the hypotheses of Proposition 8.4. These latter elements cannot kill $\beta_1^p[-(p^2-p-1)q-1]$ in the AAHSS and cannot survive to kill anything in the ANSS by Proposition 8.4. Some care must be taken at $p=3$, but in this low dimensional range there are no deviations from this pattern.

By Proposition 8.3, the elements $\beta_k[-(k-1)(p+1)q]$ support non-trivial AAHSS differentials

$$d_2(\beta_k[-(k-1)(p+1)q]) \doteq \alpha_1 \cdot \beta_k[-k(p+1)q]).$$

According to Proposition 8.5, for $l < \nu_p(k)+1$, we have differentials in the AAHSS

$$d_1(\alpha_{k/l+1}[-(k-p)q]) \doteq \alpha_{k/l}[-(k-p)q-1]$$

whereas for $l = \nu_p(k)+1$ and $k \geq 3$ we have

$$d_5(\alpha_{k-2}[-(k-p-2)q]) \doteq \alpha_{k/l}[-(k-p)q-1].$$

For $k = 2$ we have [38]

$$d_4(1[pq-1]) = \alpha_2[(p-2)q-1].$$

Finally, Proposition 8.3 implies that

$$d_q(\alpha_1[(p-1)q-1]) \doteq \beta_1[-1].$$

There are no elements left to kill $\beta_1^p[-(p^2-p-1)q-1]$. $\square$

12. Low dimensional computations of root invariants at $p = 2$

In this section we will explain how the techniques of this paper play out at the prime 2. We will first explain how the root invariants of the elements 2^i are formed from the perspective of the ANSS. This analysis provides an explanation for the pattern of differentials and hidden extensions in the v_1-periodic part of the ANSS. Assuming that the predictions of Remark 10.6 are correct, we analyze the root invariants of some v_1-periodic elements in the ANSS. We compare this analysis to what one sees in the ASS, beginning with the algebraic root invariant calculations of Bruner [6]. Throughout this section the reader should refer to Figure 4, which depicts the ANSS chart for the sphere at the prime 2 through the 29-stem.

We begin with the root invariants $R(2^k)$. We have already computed in Proposition 10.1 that

$$\alpha_k \in R_{BP}^{[1]}(2^k).$$

Our description of $R(2^k)$ depends on the value of k modulo 4. If $\alpha_{i/j}$ is a permanent cycle in the ANSS, we will use the notation $\overline{\alpha}_{i/j}$ to denote an element of π_*^S that it detects.

- **$k \equiv 1 \pmod 4$:** The element α_k is a permanent cycle which detects the homotopy root invariant $R(2^k)$.
- **$k \equiv 2 \pmod 4$:** While α_k is a permanent cycle, it does not detect an element of $R(2^k)$. This is an instance where Theorem 5.4 comes into play. The $-2k$-cell attaches to the $-2k-1$-cell of P_{-2k-1}^{-2k} by the degree 2 map, and the $-2k+1$-cell attaches to the $-2k-1$-cell in P_{-2k-1}^{-2k+1} by the map α_1. There is a hidden extension $\overline{\alpha}_1 \cdot \overline{\alpha}_{k-1}\overline{\alpha}_1 = 2 \cdot \overline{\alpha}_k$, so we may use Theorem 5.4 to deduce that

  $$\alpha_{k-1}\alpha_1 \in R_{BP}^{[2]}(2^k).$$

  The element $\alpha_{k-1}\alpha_1$ detects the homotopy root invariant $R(2^k)$.
- **$k \equiv 3 \pmod 4$:** The first filtered root invariant α_k is not a permanent cycle, so we turn to Theorem 5.3. There is an Adams-Novikov differential

  $$d_3(\alpha_k) = \alpha_1 \cdot \alpha_{k-2}\alpha_1^2.$$

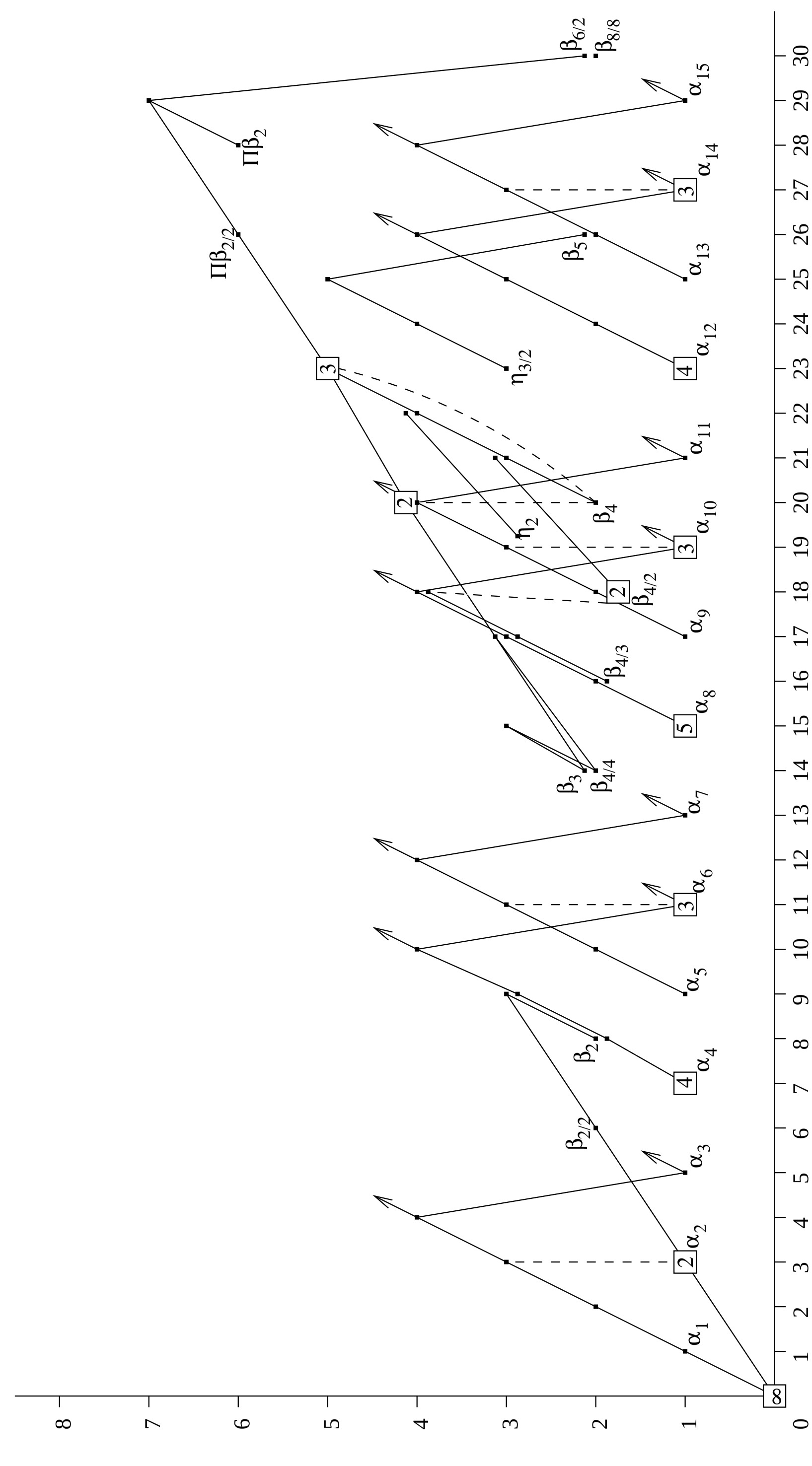

FIGURE 4. The Adams-Novikov spectral sequence at $p = 2$.

The $-2k$-cell attaches to the $-2k+2$-cell in P_{-2k}^{-2k+2} with attaching map α_1. We conclude that

$$\alpha_{k-2}\alpha_1^2 \in R_{BP}^{[3]}(2^k).$$

The element $\alpha_{k-2}\alpha_1^2$ detects an element of the homotopy root invariant $R(2^k)$.

$k \equiv 4 \pmod 4$: The element α_k is a permanent cycle in the ANSS, and detects the element of order 2 in the image of J in the $2k-1$-stem. This is the root invariant $R(2^k)$.

We will now describe the situation for the root invariants of some of the low dimensional v_1-periodic elements. Everything which follows is conjectural, because it relies on the predictions of Remark 10.6 and also the AHSS calculations required by Procedure 9.1. In what follows, for an element α in the ASS or ANSS, we use the notation $\overline{\alpha}$ to denote an element that it detects in homotopy. We denote the Eilenberg-Maclane spectrum $H\mathbb{F}_2$ by H. We describe what happens first in the ASS, and then the ANSS. All of the algebraic root invariants used were taken from [6]. These algebraic root invariants coincide with the first non-trivial filtered root invariants by Theorem 5.10. The computations are summarized below.

$R(\overline{\alpha}_1)$ — ASS: The element $\overline{\alpha}_1 = \eta$ is detected by h_1. We have $h_2 \in R_{alg}(h_1)$. The element h_2 detects $\nu \in R(\eta)$.

ANSS: We have $\alpha_{2/2} \in R_{BP}^{[1]}(\overline{\alpha}_1)$. The element $\alpha_{2/2}$ also detects $\nu \in R(\eta)$.

$R(\overline{\alpha}_1^2)$ — ASS: The algebraic root invariant is given by $h_2^2 \in R_{alg}(h_1^2)$, and h_2^2 detects the homotopy root invariant $\nu^2 \in R(\eta^2)$.

$R(\overline{\alpha}_1^3)$ — ASS: We have $h_2^3 \in R_{alg}(h_1^3)$, which detects the homotopy root invariant $\nu^3 \in R(\eta^3)$.

$R(\overline{\alpha}_{2/2})$ — ASS: The element $\overline{\alpha}_{2/2}$ is detected by h_2. The algebraic root invariant is given by $h_3 \in R_{alg}(h_2)$. The element h_3 detects the homotopy root invariant $\sigma \in R(\nu)$.

ANSS: As indicated in Remark 10.6, the first filtered root invariant is given by $\alpha_{4/4} \in R_{BP}^{[1]}(\alpha_{2/2})$, and $\alpha_{4/4}$ detects σ.

$R(\overline{\alpha}_2)$ — ASS: The element $\overline{\alpha}_2$ is detected by h_0h_2, and $h_1h_3 \in R_{alg}(h_0h_2)$, which gives $\eta\sigma \in R(2\nu)$.

ANSS: The second filtered root invariant is given by $\beta_2 + c\alpha_{4/4}\alpha_1 \in R_{BP}^{[2]}(\overline{\alpha}_2)$, where $c \in \mathbb{Z}/2$. Now the elements β_2 and $\alpha_{4/4}\alpha_1$ correspond to the elements c_0 and h_3h_1 in the ASS, respectively. The ANSS perspective may be reconciled with the ASS if $c = 1$, since the Adams filtration of c_0 is greater than the Adams filtration of h_3h_1. The ASS will only detect the element of lowest filtration in the sum $c_0 + h_3h_1$.

$R(\overline{\alpha}_{4/4})$ ASS: The element $\overline{\alpha}_{4/4}$ is detected by h_3 in the ASS. It is known [25] that $\sigma^2 \in R(\sigma)$. This may be seen, for instance, using the fact that $h_4 \in R_{alg}(h_3)$, which coincides with the first filtered root invariant $R_H^{[1]}(h_3)$ by Theorem 5.10. The Hopf invariant 1 differential $d_2(h_4) = h_0 h_3^2$ allows one apply Theorem 5.3 and get $h_3^2 \in R_H^{[2]}(h_3)$. The element h_3^2 detects $\sigma^2 \in R(\sigma)$.

ANSS: We have the second filtered root invariant $\beta_{4/4} \in R_{BP}^{[2]}(\overline{\alpha}_{4/4})$, and $\beta_{4/4}$ detects σ^2.

$R(\overline{\alpha}_{4/3})$ ASS: The element $\overline{\alpha}_{4/3}$ is detected by $h_0 h_3$ in the ASS. The algebraic root invariant is $h_1 h_4 \in R_{alg}(h_0 h_3)$, which presumably detects the homotopy root invariant $\eta_4 \in R(2\sigma)$.

ANSS: The second filtered root invariant is $\beta_{4/3} + c\alpha_{8/5}\alpha_1 \in R_{BP}^{[2]}(\overline{\alpha}_{4/3})$, for $c \in \mathbb{Z}/2$. The element $\beta_{4/3}$ detects η_4. The ImJ differentials given in [23] seem to indicate that $c = 0$.

$R(\overline{\alpha}_{4/2})$ ASS: The element $\overline{\alpha}_{4/2}$ is detected by $h_0^2 h_3$. The algebraic root invariant is $h_1^2 h_4 \in R_{alg}(h_0^2 h_3)$, and this presumably detects the homotopy root invariant $\eta\eta_4 \in R(4\sigma)$.

ANSS: The second filtered root invariant is given by $\beta_{4/2} + c\alpha_9\alpha_1 \in R_{BP}^{[2]}(\overline{\alpha}_{4/2})$, where $c \in \mathbb{Z}/2$. The element $\beta_{4/2} + c\alpha_9\alpha_1$ is a permanent cycle but does not survive to the homotopy root invariant. We appeal to Theorem 5.4 to find a higher filtered root invariant. In P_{-13}^{-12} the -12-cell attaches to the -13-cell with degree 2 attaching map, and the -11-cell attaches to the -13-cell in P_{-13}^{-11} by η. There is a hidden extension in the ANSS given by $\overline{\alpha}_1 \cdot \overline{\alpha}_1\overline{\beta}_{4/3} = 2 \cdot (\overline{\beta}_{4/2} + c\overline{\alpha}_9\overline{\alpha}_1)$, which indicates that we have a higher filtered root invariant $\alpha_1\beta_{4/3} \in R_{BP}^{[3]}(\overline{\alpha}_{4/2})$. The element $\alpha_1\beta_{4/3}$ detects $\eta\eta_4$.

$R(\overline{\alpha}_4)$ ASS: The element $\overline{\alpha}_4$ is detected by $h_0^3 h_3$, with algebraic root invariant $h_1^3 h_4 \in R_{alg}(h_0^3 h_3)$. Presumably this detects the homotopy root invariant $\eta^2\eta_4 \in R(8\sigma)$.

ANSS: The second filtered root invariant is given by $\beta_4 \in R^{[2]}_{BP}(\overline{\alpha}_4)$. The element β_4 corresponds to the element g in the ASS. The element β_4 is a permanent cycle, so we use Theorem 5.4 to look for a higher root invariant. In P^{-12}_{-15} the -14-cell attaches to the -15-cell by the degree 2 map, and there is a Toda bracket $\langle P^{-12}_{-15}\rangle(-) \overset{\cap}{=} \langle 2, \alpha_1, -\rangle$. There is a hidden extension in the ANSS given by $2\overline{\beta}_4 = (\overline{\beta}_{4/4}\overline{\beta}_{2/2})/2$, and in the ANSS there is a Toda bracket $(\beta_{4/4}\beta_{2/2})/2 \in \langle 2, \alpha_1, \beta_{4/3}\alpha_1^2\rangle$. We conclude using Theorem 5.4 that we have the higher filtered root invariant $\beta_{4/3}\alpha_1^2 \in R^{[4]}_{BP}(\overline{\alpha}_4)$. The element $\beta_{4/3}\alpha_1^2$ detects $\eta^2\eta_4$.

$R(\overline{\alpha}_{4/4}\overline{\alpha}_1)$ ASS: The element $\overline{\alpha}_{4/4}\overline{\alpha}_1$ is detected by h_3h_1 with algebraic root invariant $h_4h_2 \in R_{alg}(h_3h_1)$. This seems to indicate that $\nu^* \in R(\eta\sigma)$.

ANSS: The root invariant of $\overline{\alpha}_{4/4}$ is given by $\overline{\beta}_{4/4}$, so one might initially suspect that $\overline{\alpha}_{2/2}\overline{\beta}_{4/4}$ would detect the homotopy root invariant $R(\overline{\alpha}_{4/4}\overline{\alpha}_1)$. However, the -6-cell attaches to the -10-cell with attaching map ν. Thus in the AAHSS for P_{-10}, there is a differential $d_4(\beta_{4/4}[-6]) = \alpha_{2/2}\cdot\beta_{4/4}[-10]$. Looking at the attaching map structure of P^{-6}_{-11}, this differential actually pushes the root invariant to $\beta_{4/2,2} \in \langle \alpha_{2/2}, 2, \beta_{4/4}\rangle$, and this element detects ν^*.

$R(\overline{\alpha}_{4/4}\overline{\alpha}_1^2)$ ASS: The element $\overline{\alpha}_{4/4}\overline{\alpha}_1^2$ is detected by $h_3h_1^2$, with algebraic root invariant $h_4h_2^2 \in R_{alg}(h_3h_1^2)$. It seems likely that this detects the homotopy root invariant $\nu^*\nu \in R(\eta^2\sigma)$.

$R(\overline{\alpha}_5)$ ASS: The element $\overline{\alpha}_5$ is detected by Ph_1, with algebraic root invariant $h_2g \in R_{alg}(Ph_1)$. Presumably this detects the homotopy root invariant $\nu\overline{\kappa} \in R(\overline{\alpha}_5)$.

ANSS: The second filtered root invariant is given by $\beta_5 + c\alpha_{13}\alpha_1 \in R^{[2]}_{BP}(\overline{\alpha}_5)$, where $c \in \mathbb{Z}/2$. Theorem 5.3 applies to the differential $d_3(\beta_5) = \alpha_1^2\eta_{3/2}$ to give the higher filtered root invariant $\alpha_1\eta_{3/2} \in R^{[4]}_{BP}(\overline{\alpha}_5)$. The element $\alpha_1\eta_{3/2}$ corresponds to the element $h_4c_0h_1$ in the ASS. Using the May spectral sequence, we see that there is a Massey product $h_2^2g \in \langle h_0, h_1, h_4c_0h_1\rangle$. The element h_2^2g is a permanent cycle which corresponds to the element $\Pi\beta_{2/2}$ in the ANSS. We conclude that in the ANSS there is a hidden Toda bracket $\overline{\alpha}_{2/2}\cdot\overline{\beta}_4\overline{\alpha}_1^3/8 = \overline{\Pi\beta_{2/2}} \in \langle 2, \overline{\alpha}_1, \overline{\alpha}_1\overline{\eta}_{3/2}\rangle$. In P^{-15}_{-19}, the -16 cell attaches to the -19-cell with the Toda bracket $\langle 2, \overline{\alpha}_1, -\rangle$ and the -15-cell attaches to the -19-cell with attaching map $\overline{\alpha}_{2/2}$. We conclude, using Theorem 5.4, that we have another higher filtered root invariant $\beta_4\alpha_1^3/8 \in R^{[5]}_{BP}(\overline{\alpha}_5)$. The element $\beta_4\alpha_1^3/8$ detects $\nu\overline{\kappa}$.

$R(\overline{\alpha}_5\overline{\alpha}_1)$ ASS: The element $\overline{\alpha}_5\overline{\alpha}_1$ is detected by Ph_1^2, with algebraic root invariant $r \in R_{alg}(Ph_1^2)$. Theorem 5.10 implies that we have the filtered root invariant $r \in R_H^{[6]}(\overline{\alpha}_5\overline{\alpha}_1)$. The element r supports the Adams differential $d_3(r) = h_1 d_0^2$. Thus, we may use Theorem 5.3 to deduce that there is a higher filtered root invariant $d_0^2 \in R_H^{[8]}(\overline{\alpha}_5\overline{\alpha}_1)$. Presumably this detects the homotopy root invariant $\kappa^2 \in R(\overline{\alpha}_5\overline{\alpha}_1)$.

$R(\overline{\alpha}_5\overline{\alpha}_1^2)$ ASS: The element $\overline{\alpha}_5\overline{\alpha}_1^2$ is detected by Ph_1^3, with algebraic root invariant $h_1 q \in R_{alg}(Ph_1^3)$. It seems likely that this element detects the homotopy root invariant.

$R(\overline{\alpha}_{6/2})$ ASS: This element presents a very interesting story: it is an instance where the ASS seems to give us nothing yet the ANSS tells us the homotopy root invariant. The element $\overline{\alpha}_{6/2}$ corresponds to the element Ph_2 in the ASS, with algebraic root invariant $h_0^3 h_4^2 \in R_{alg}(Ph_2)$. The element $h_0^3 h_4^2$ is killed by a differential in the ASS, and it is not clear what a candidate for the root invariant should be.

ANSS: The second filtered root invariant is given by $\beta_{6/2} \in R_{BP}^{[2]}(\overline{\alpha}_{6/2})$. There is an Adams-Novikov differential $d_5(\beta_{6/2}) = \alpha_{2/2} \cdot \Pi\beta_{2/2}$. Theorem 5.3 indicates that we have the higher filtered root invariant $\Pi\beta_{2/2} \in R_{BP}^{[6]}(\alpha_{6/2})$. It appears that this detects the homotopy root invariant $\nu^2\overline{\kappa} \in R(\overline{\alpha}_{6/2})$.

$R(\overline{\alpha}_6)$ ASS: The element $\overline{\alpha}_6$ is detected by Ph_2h_0, with algebraic root invariant $q \in R_{alg}(Ph_2h_0)$. It seems likely that this detects the homotopy root invariant.

13. Low dimensional computations of root invariants at $p = 3$

The aim of this section is to use knowledge of the ANSS for π_*^S in the first 100 stems to compute the homotopy root invariants of some low dimensional Greek letter elements $\alpha_{i/j}$ at $p = 3$. These results are summarized in the following proposition.

Proposition 13.1. We have the following root invariants at $p = 3$.

$$\begin{aligned} R(\alpha_1) &= \beta_1 \\ R(\alpha_2) &\doteq \beta_1^2\alpha_1 \\ R(\alpha_{3/2}) &= -\beta_{3/2} \\ R(\alpha_3) &= \beta_3 \\ R(\alpha_4) &\doteq \beta_1^5 \\ R(\alpha_5) &= \beta_5 \\ R(\alpha_{6/2}) &= \beta_{6/2} \\ R(\alpha_6) &= -\beta_6 \end{aligned}$$

It is interesting to note that although β_2 exists, it fails to be the root invariant of α_2. The element β_4 does not exist, so it cannot be the root invariant of α_4. In

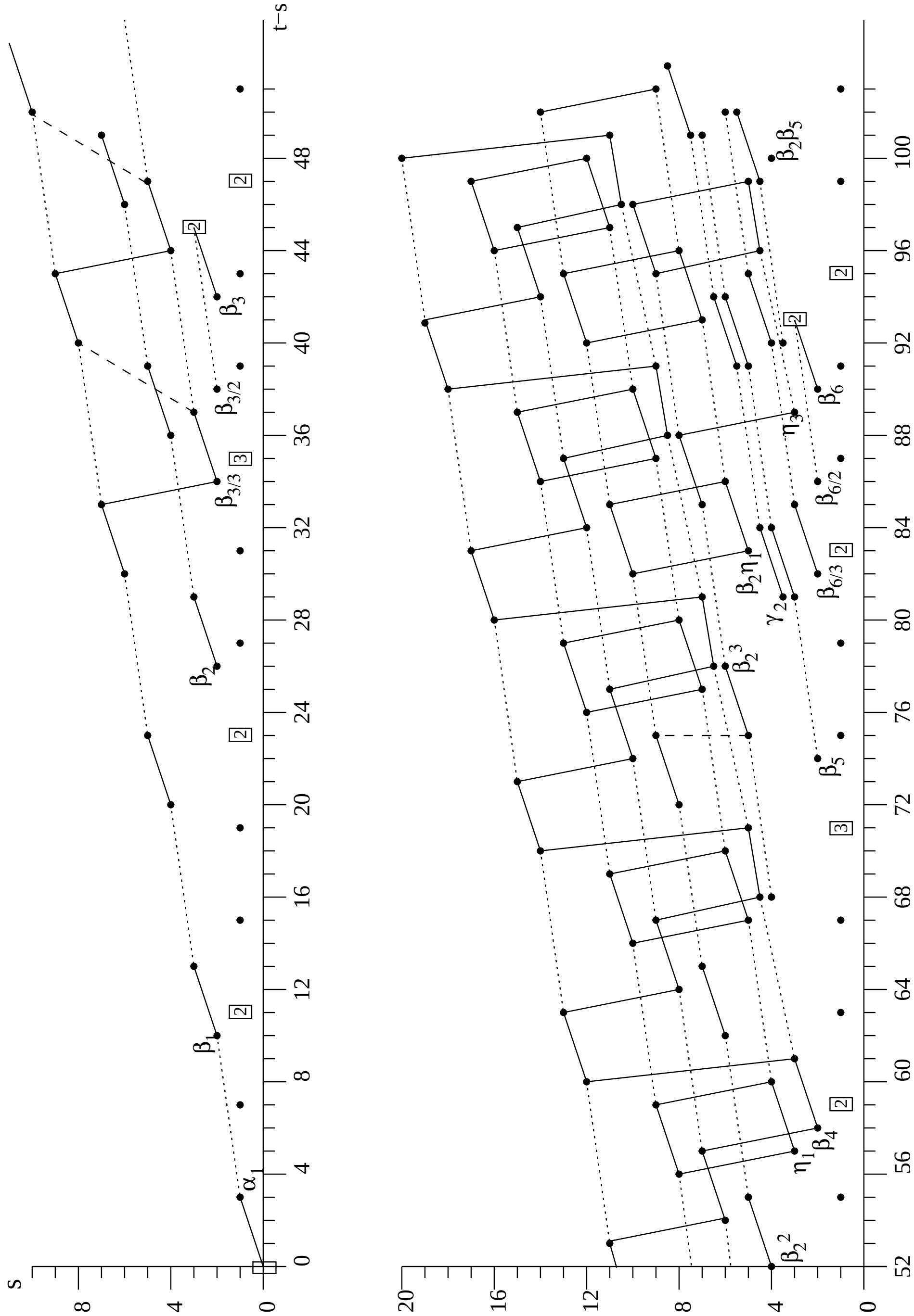

FIGURE 5. The Adams-Novikov spectral sequence at $p = 3$

Section 16 we will prove that $\beta_i \mathrel{\dot\in} R(\alpha_i)$ for $i \equiv 0, 1, 5 \pmod 9$. The remainder of this section is devoted to proving Proposition 13.1.

Figure 5 shows the Adams-Novikov E_2 term. These charts were created from the computations in [33]. Solid lines represent multiplication by α_1 and dotted lines represent the Massey product $\langle -, \alpha_1, \alpha_1 \rangle$. Dashed lines represent hidden extensions.

In the Section 10, we proved in Corollary 10.3 that we have filtered root invariants

$$(-1)^{i-j}\beta_{i/j} \in R^{[2]}(\alpha_{i/j}).$$

We supplement those results with two higher filtered root invariants particular to $p = 3$.

Proposition 13.2. We have the following higher filtered root invariant of α_2.

$$\beta_1^2\alpha_1 \mathrel{\dot{\in}} R^{[5]}_{BP}(\alpha_2)$$

Proof. We know by Corollary 10.3 that

$$-\beta_2 \in R^{[2]}_{BP}(\alpha_2).$$

The element β_2 is a permanent cycle in the ANSS, which detects an element $\overline{\beta}_2 \in \pi_*^S$. We shall apply Theorem 5.4 to determine the higher filtered root invariant from the following hidden Toda bracket

$$\beta_1^3 \doteq \langle \alpha_1, 3, \overline{\beta}_2 \rangle$$

(see, for example, [33]). Considering the attaching map structure of $P_{-\infty}$ we have the following equalities of Toda brackets (the first is an equality of homotopy Toda brackets whereas the second is an equality of Toda brackets in the ANSS).

$$\langle P^{-20}_{-25} \rangle(\overline{\beta}_2) \doteq \langle \alpha_1, 3, \overline{\beta}_2 \rangle \doteq \beta_1^3$$
$$\langle P^{-17}_{-25} \rangle(\beta_1^2\alpha_1) \doteq \langle \alpha_1, \alpha_1, \beta_1^2\alpha_1 \rangle \doteq \beta_1^3$$

We conclude from Theorem 5.4 that

$$\beta_1^2\alpha_1 \mathrel{\dot{\in}} R^{[5]}_{BP}(\alpha_2).$$

□

Proposition 13.3. We have the following higher filtered root invariants of α_4.

$$\beta_{3/3}\beta_1^2 \mathrel{\dot{\in}} R^{[6]}_{BP}(\alpha_4) \tag{13.1}$$

$$\beta_1^5 \mathrel{\dot{\in}} R^{[10]}_{BP}(\alpha_4) \tag{13.2}$$

Proof. We know from Proposition 10.3 that

$$-\beta_4 \in R^{[2]}_{BP}(\alpha_4)$$

but β_4 is not a permanent cycle. There are Adams-Novikov differentials

$$d_5(\beta_4) \doteq \beta_{3/3}\beta_1^2\alpha_1$$
$$d_5(\beta_{3/3}\beta_1^2) \doteq \beta_1^5\alpha_1.$$

The higher filtered root invariants of α_4 are derived from two applications of Theorem 5.3. The relevant Toda brackets are computed below.

$$\langle P^{-36}_{-40} \rangle(\beta_{3/3}\beta_1^2) \doteq \alpha_1 \cdot \beta_{3/3}\beta_1^2$$
$$\langle P^{-40}_{-44} \rangle(\beta_1^5) \doteq \alpha_1 \cdot \beta_1^5$$

□

We now apply Procedure 9.1 to our filtered root invariants. The results of executing Steps 1–3 in Procedure 9.1 are summarized in Table 2. The meanings of the contents of the columns are described below.

Element: contains the element α which we wish to calculate the root invariant of.

$\mathbf{R^{[k]}}$**:** indicates the k^{th} BP-filtered root invariant β of α. We would like β to detect an element of $R(\overline{\alpha})$.

Cell: indicates the cell of $P_{-\infty}$ that carries the filtered root invariant.

$\{\gamma_{\mathbf{i}}[\mathbf{n_i}]\}$**:** contains the list of elements of the ANSS which could survive to the difference between the filtered root invariant and the homotopy root invariant. This is the list described in Step 1 of Procedure 9.1.

Diffs: indicates the differentials in the AAHSS which kill the elements $\gamma_i[n_i]$, as described in Step 2 of Procedure 9.1, or the differentials supported by the $\gamma_i[n_i]$, as described in Step 3 of Procedure 9.1. If the element γ_i supports a non-trivial differential in the ANSS, we shall place a "(1)" to indicate that it is not a permanent cycle, and so cannot be a homotopy root invariant. If the $\gamma_i[n_i]$ satisfies the conditions of Proposition 8.4, then Steps 2 and 3 of Procedure 9.1 are automatically satisfied, as explained in Remark 9.2. We shall indicate this with a "(2)".

TABLE 2. Steps 1–3 of Procedure 9.1 at $p = 3$

Element	$R^{[k]}$	Cell	$\{\gamma_i[n_i]\}$	Diffs
α_1	β_1	$-2q$	—	—
α_2	$\beta_1^2\alpha_1$	$-4q-1$	—	—
$\alpha_{3/2}$	$\beta_{3/2}$	$-7q$	—	—
α_3	β_3	$-8q$	$\beta_2\beta_1\alpha_1[-7q-1]$	(2)
α_4	β_1^5	$-9q$	—	—
α_5	β_5	$-14q$	$\beta_4\beta_1\alpha_1[-13q-1]$	(1)
			$\eta_1\beta_1[-12q-1]$	(1)
			$\eta_1\beta_1\alpha_1[-13q]$	(1)
			$\beta_2^2\beta_1[-11q]$	(2)
			$\beta_{3/3}\beta_1^3\alpha_1[-12q-1]$	(1)
			$\beta_2\beta_1^3\alpha_1[-10q-1]$	(2)
			$\beta_2\beta_1^4[-12q]$	(2)
			$\beta_1^6\alpha_1[-11q-1]$	(2)
			$\beta_1^7[-13q]$	(2)
$\alpha_{6/2}$	$\beta_{6/2}$	$-16q$	$\beta_2\eta_1[-15q-1]$	(1)
			$\beta_2^3/\alpha_1[-13q-1]$	(2)
			$\beta_4\beta_1\alpha_1[-12q-1]$	(1)
			$\beta_2^3[-14q]$	(2)
			$\eta_1\beta_1\alpha_1[-12q]$	(1)
			$\beta_2^2\beta_1^2\alpha_1[-13q-1]$	(2)
			$\beta_2^2\beta_1^3[-15q]$	(2)
			$\beta_{3/3}\beta_1^4[-13q]$	(2)
			$\beta_2\beta_1^5\alpha_1[-14q-1]$	(2)
			$\beta_1^7[-12q]$	(2)

Element	$R^{[k]}$	Cell	$\{\gamma_i[n_i]\}$	Diffs
			$\beta_1^8\alpha_1[-15q-1]$	(2)
α_6	β_6	$-17q$	$\eta_1\beta_2[-15q-1]$	(1)
			$\beta_2^3/\alpha_1[-13q-1]$	(2)
			$\beta_4\beta_1\alpha_1[-12q-1]$	(1)
			$\eta_1\beta_2\alpha_1[-16q]$	(1)
			$\beta_2^3[-14q]$	(2)
			$\eta_1\beta_1\alpha_1[-12q]$	(1)
			$\eta_1\beta_1^3[-16q-1]$	(1)
			$\beta_2^2\beta_1^2\alpha_1$	(2)
			$\beta_2^2\beta_1^3[-15q]$	(2)
			$\beta_{3/3}\beta_1^4[-13q]$	(1)
			$\beta_{3/3}\beta_1^5\alpha_1[-16q-1]$	(2)
			$\beta_2\beta_1^5\alpha_1[-14q-1]$	(2)
			$\beta_2\beta_1^6[-16q]$	(2)
			$\beta_1^8\alpha_1[-15q-1]$	(2)

Some care must be taken in the case of α_3. If $\langle\alpha_1, 3, \beta_3\rangle$ is nonzero (i.e. contains $\pm\beta_2\beta_1^2$), then the AAHSS differential

$$d_4(\beta_2\beta_1\alpha_1[-7q-1]) \doteq \beta_2\beta_1^2[-9q-1]$$

may be trivial in the AHSS with an intervening differential supported by $\beta_3[-8q]$ (see Remark 9.2). This does not happen, though, as the Toda bracket $\langle\alpha_1, 3, \beta_3\rangle$ is actually zero. It is easily seen to have zero indeterminacy. If the Toda bracket were non-trivial, it would have to contain $\pm\beta_2\beta_1^2$, and thus $\alpha_1 \cdot \langle\alpha_1, 3, \beta_3\rangle$ would be nonzero. This cannot be the case, since we have

$$\alpha_1 \cdot \langle\alpha_1, 3, \beta_3\rangle = -\langle\alpha_1, \alpha_1, 3\rangle \cdot \beta_3 = \alpha_2 \cdot \beta_3 = 0.$$

Now we must complete Step 4 of Procedure 9.1. We must determine that the filtered root invariants survive to non-trivial elements in the AHSS, and subsequently in the ANSS. For this we must determine which elements in the AAHSS (or subsequently in the ANSS) can support differentials which kill the filtered root invariants.

Proposition 8.7 implies that for all of the $\alpha_{i/j}$ above whose filtered root invariants were $\beta_{i/j}$, the image of $\beta_{i/j}$ in the appropriate stunted projective space is non-trivial. Thus we conclude that

$$\beta_{i/j} \mathrel{\dot{\in}} R(\alpha_{i/j})$$

for $i \leq 6$ and $i \neq 2, 4$.

Step 4 of Procedure 9.1 is completed for α_2 and α_4 in the following lemmas. In both of these lemmas, we denote by $\overline{x}$ the element in homotopy detected by the ANSS element x. There is no ambiguity arising from higher Adams-Novikov filtration for the elements that we will be considering.

Lemma 13.4. The element $\overline{\beta_1^2\alpha_1}[-4q-1]$ is not the target of a differential in the AHSS for P_{-4q-1}.

Proof. Differentials which could kill $\overline{\beta_1^2\alpha_1}[-4q-1]$ must have their source in the 7-stem. Proposition 8.5 demonstrates that in the the targets of any AHSS differentials supported by ImJ elements in the 7-stem are also ImJ elements. In the range we are considering, there is no room for shorter differentials. The only element left which could kill $\overline{\beta_1^2\alpha_1}[-4q-1]$ is $\overline{\beta_1^2}[-3q-1]$. However, the complex P_{-4q-1}^{-3q-1} is reducible, so there can be no such differential. $\square$

Lemma 13.5. The element $\overline{\beta_1^5}[-9q]$ is not the target of a differential in the AHSS for P_{-9q}.

Proof. Exactly as in the proof of Lemma 13.4, the differentials supported by ImJ elements in the 15-stem can only hit other ImJ elements. The only elements left which can support differentials are given below.

$$\overline{\beta_1^2}[-q-1] \qquad \overline{\beta_1^2\alpha_1}[-2q] \qquad \overline{\beta_2\beta_1}[-5q-1]$$
$$\overline{\beta_1^4}[-6q-1] \qquad \overline{\beta_{3/3}\beta_1\alpha_1}[-8q]$$

All but $\overline{\beta_{3/3}\beta_1\alpha_1}[-8q]$ are the target or source of the AHSS d_2 differentials displayed below.

$$\overline{\beta_1^2}[-q-1] \longmapsto \overline{\beta_1^2\alpha_1}[-2q-1]$$
$$\overline{\beta_1^2}[-q] \longmapsto \overline{\beta_1^2\alpha_1}[-2q]$$
$$\overline{\beta_2\beta_1}[-5q-1] \longmapsto \overline{\beta_2\beta_1\alpha_1}[-6q-1]$$
$$\overline{\beta_{3/3}\alpha_1}[-5q-1] \longmapsto \overline{\beta_1^4}[-6q-1]$$

The last of these is the result of a hidden α_1-extension in the ANSS. The remaining element $\overline{\beta_{3/3}\beta_1\alpha_1}[-8q]$ cannot support a differential whose target is $\overline{\beta_1^5}[-9q]$, because P_{-9q}^{-8q} is coreducible. $\square$

14. Algebraic filtered root invariants

In this section we will describe the Mahowald spectral sequence (MSS), which is a spectral sequence that computes $\mathrm{Ext}(H_*X)$ by applying $\mathrm{Ext}(H_*-)$ to an E-Adams resolution of X. This spectral sequence is described in [28]. We will briefly recall its construction for the reader's convenience. We will then define algebraic E-filtered root invariants, and indicate how some of the results of Section 5 carry over to the algebraic setting to compute algebraic root invariants from the algebraic filtered root invariants in the MSS.

Let E be a ring spectrum and suppose that H is an E-ring spectrum in the sense that there is a map of ring spectra $E \to H$. Suppose furthermore that H_*H is a flat H_* module. Let X be a (finite) spectrum. By applying H_* to the cofiber sequences that make up the E-Adams resolution, we obtain short exact sequences

$$0 \to H_*(W_s(X)) \to H_*(W_s^s(X)) \to H_*(\Sigma W_{s+1}(X)) \to 0 \tag{14.1}$$

which are split by the E-action map

$$H \wedge W_s^s(X) = H \wedge E \wedge \overline{E}^{(s)} \wedge X \to H \wedge \overline{E}^{(s)} \wedge X = H \wedge W_s(X).$$

Thus the short exact sequences 14.1 give rise to long exact sequences when we apply $\mathrm{Ext}_{H_*H}(-)$. Therefore, upon applying Ext to the E-Adams resolution, we get a spectral sequence

$$E_1^{s,t,k}(H_*X) = \mathrm{Ext}^{s-k,t}(H_*W_k^k(X)) \Rightarrow \mathrm{Ext}^{s,t}(H_*X)$$

called the Mahowald spectral sequence (MSS). For the remainder of this section we shall assume $H = H\mathbb{F}_p$.

Many of the notions that we defined for the homotopy root invariant carry over to the algebraic context. We may define Tate comodules (over the dual Steenrod algebra)

$$t(H_*X) = \varprojlim_N (H_*(X \wedge \Sigma P_{-N})) = (tH \wedge X)_*.$$

Here, it is essential that the limit is taken *after* taking homology. We may use this to define algebraic E-root invariants.

Definition (*Algebraic E-root invariant*)**.** Let α be an element of $\mathrm{Ext}^{s,t}(H_*X)$. We have the following diagram of Ext groups which defines the algebraic E-root invariant $R_{E,alg}(\alpha)$.

$$\begin{tikzcd} \mathrm{Ext}^{s,t}(H_*X) \arrow[r, squiggly, "R_{E,alg}(-)"] \arrow[d, "f"'] & \mathrm{Ext}^{s,t}(H_*E \wedge \Sigma^{-N+1}X) \arrow[d, "\iota_N"'] \\ \mathrm{Ext}^{s,t}(t(H_*(E \wedge X))) \arrow[r, "\nu_N"'] & \mathrm{Ext}^{s,t}(H_*(E \wedge \Sigma P_{-N} \wedge X)) \end{tikzcd}$$

Here f is induced by the inclusion of the 0-cell of tS^0, ν_N is the projection onto the $-N$-coskeleton, ι_N is inclusion of the $-N$-cell, and N is minimal with respect to the property that $\nu_N \circ f(\alpha)$ is non-zero. Then the algebraic E-root invariant $R_{E,alg}(\alpha)$ is defined to be the coset of lifts $\gamma \in \mathrm{Ext}^{s,t+N-1}(H_*E \wedge X)$ of the element $\nu_N \circ f(\alpha)$. It could be the the case that $f(\alpha) = 0$, in which case we say that the algebraic E-root invariant is trivial.

For simplicity, we now restrict our attention to the case $E = BP$, and we work at an odd prime p. Let $\overline{\Lambda}$ be the periodic lambda algebra [15]. The cohomology of $\overline{\Lambda}$ is $\mathrm{Ext}(H_*)$. The MSS has a concrete description in terms of $\overline{\Lambda}$. Define a decreasing filtration on $\overline{\Lambda}$ by

$$\overline{\Lambda} = F_0\overline{\Lambda} \subset F_1\overline{\Lambda} \subset F_2\overline{\Lambda} \subset \cdots$$

where $F_k\overline{\Lambda}$ is the subcomplex of $\overline{\Lambda}$ generated by monomials containing k or more λ_i's. The spectral sequence associated to the filtered complex $\{F_k\overline{\Lambda}\}$ is isomorphic to the MSS *starting with the E_2-term.* The spectral sequence of the filtered complex would agree with the MSS on the level of E_1-terms if one were to apply $\mathrm{Ext}(H_*-)$ to the correct BP-resolution (which differs from the canonical BP-resolution). Therefore, we shall also refer to the spectral sequence associated to the filtered complex $\{F_k\overline{\Lambda}\}$ as a MSS.

In analogy with the spaces $W_s(X)$, define subcomplexes of $\overline{\Lambda} \otimes H_*(X)$ (the complex which computes $\mathrm{Ext}(H_*X)$) by

$$W_k(H_*X) = F_k\overline{\Lambda} \otimes H_*(X).$$

We define quotients $W_k^l(H_*X)$ by

$$W_k^l(H_*X) = W_k(H_*X)/W_{l+1}(H_*X).$$

The cohomology $H^*(W_k^l(H_*X))$ is computed by restricting the MSS by setting the $E_1^{*,*,i}$-term equal to zero for $i < k$ or $i > l$. We define complexes $W_k^l(H_*P^N)$ to be the inverse limit

$$W_k^l(H_*P^N) = \varprojlim_M W_k^l(H_*P^N_{-M}).$$

For sequences

$$I = \{k_1 < k_2 < \cdots < k_l\}$$
$$J = \{N_1 < N_2 < \cdots < N_l\}$$

we can define filtered Tate complexes

$$W_I(H_*P^J \wedge X) = \sum_i W_{k_i}(H_*P^{N_i} \wedge X).$$

Given another pair of sequences $(I', J') \leq (I, J)$, we define complexes

$$W_I^{I'}(H_*P_{J'}^J \wedge X) = W_I(H_*P^J \wedge X)/W_{I'+1}(H_*P^{J'-1} \wedge X)$$

where $I'+1$ (respectively $J'-1$) is the sequence obtained by increasing (decreasing) every element of the sequence by 1.

We shall now define algebraic filtered root invariants in analogy with the definition of filtered root invariants given in Section 3. Let α be an element of $\mathrm{Ext}^{s,t}(H_*X)$. We shall describe a pair of sequences

$$I = \{k_1 < k_2 < \cdots < k_l\}$$
$$J = \{-N_1 < -N_2 < \cdots < -N_l\}$$

associated to α, which we define inductively. Let $k_1 \geq 0$ be maximal such that the image of α under the composite

$$\mathrm{Ext}^{s,t-1}(H_*\Sigma^{-1}X) \to H^{s,t-1}(W_0^{k_1-1}(H_*P \wedge X)_{-\infty})$$

is trivial. Next, choose N_1 to be maximal such that the image of α under the composite

$$\mathrm{Ext}^{s,t-1}(H_*\Sigma^{-1}X) \to H^{s,t-1}(W_0^{(k_1-1,k_1)}(H_*P_{(-N_1+1,\infty)} \wedge X))$$

is trivial. Inductively, given

$$I' = (k_1, k_2, \ldots, k_i)$$
$$J' = (-N_1, -N_2, \ldots, -N_i)$$

let k_{i+1} be maximal so that the image of α under the composite

$$\mathrm{Ext}^{s,t-1}(H_*\Sigma^{-1}X) \to H^{s,t-1}(W_0^{(I'-1,k_{i+1}-1)}(H_*P_{(J'+1,\infty)} \wedge X))$$

is trivial. If there is no such maximal k_{i+1}, we declare that $k_{i+1} = \infty$ and we are finished. Otherwise, choose N_{i+1} to be maximal such that the composite

$$\mathrm{Ext}^{s,t-1}(H_*\Sigma^{-1}X) \to H^{s,t-1}(W_0^{(I'-1,k_{i+1}-1,k_{i+1})}(H_*P_{(J'+1,-N_{i+1}+1,\infty)} \wedge X))$$

is trivial, and continue the inductive procedure. We shall refer to the pair (I, J) as the BP-bifiltration of α.

Observe that there is an exact sequence

$$H^{s,t-1}(W_I(H_*P^J \wedge X)) \to \mathrm{Ext}^{s,t-1}(t(H_*\Sigma^{-1}X)) \to H^{s,t-1}(W^{I-1}(H_*P_{J+1} \wedge X)).$$

Our choice of (I, J) ensures that the image of α in $H^{s,t-1}(W^{I-1}(H_*P_{J+1} \wedge X))$ is trivial. Thus α lifts to an element $f^\alpha \in H^{s,t-1}(W_I(H_*P^J \wedge X))$.

Definition (*Algebraic filtered root invariants*)**.** Let X be a finite complex and let α be an element of $\mathrm{Ext}^{s,t}(H_*X)$ of BP-bifiltration (I, J). Given a lift $f^\alpha \in H^{s,t-1}(W_I(H_*P^J \wedge X))$, the k^{th} algebraic filtered root invariant is said to be trivial if $k \neq k_i$ for any $k_i \in I$. Otherwise, if $k = k_i$ for some i, we say that the image β of f^α under the quotient map

$$H^{s,t-1}(W_I(H_*P^J \wedge X)) \to H^{s,t-1}(W_{k_i}^{k_i}(H_*\Sigma^{-N_i}X))$$

is an element of the k^{th} *algebraic filtered root invariant* of α. The k^{th} algebraic filtered root invariant is the coset $R^{[k]}_{E,alg}(\alpha)$ of the MSS E_1-term $E_1^{s,t+N_i-1,k}(H_*X)$ of all such β as we vary the lift f^α.

We wish to indicate how our filtered root invariant theorems carry over to the algebraic context. In order to do this we must produce algebraic versions of K-Toda brackets. Suppose that M is a finite A_*-comodule concentrated in degrees 0 through n with a single $\mathbb{F}_p$ generator in degrees 0 and n. In what follows, we let M^j be the sub-comodule of M consisting of elements of degree less than or equal to j, and let M_i^j be the quotient M^j/M^{i-1}. We shall omit the top index for the quotient $M_i = M/M^{i-1}$.

Definition (*Algebraic M-Toda bracket*)**.** Let

$$f : \mathrm{Ext}^{s,t}(M_1) \to \mathrm{Ext}^{s+1,t}(\mathbb{F}_p)$$

be the connecting homomorphism associated to the short exact sequence

$$0 \to \mathbb{F}_p \to M \to M_1 \to 0.$$

Let $\nu : M_1 \to \Sigma^n\mathbb{F}_p$ be the projection onto the top generator. Suppose α is an element of $\mathrm{Ext}^{s,t}(H_*X)$. We have

$$\mathrm{Ext}^{s,t}(H_*X) \xleftarrow{\nu_*} \mathrm{Ext}^{s,t+n}(H_*X \otimes M_1) \xrightarrow{f} \mathrm{Ext}^{s+1,t+n}(H_*X).$$

We say the algebraic M-Toda bracket

$$\langle M\rangle(\alpha) \subseteq \mathrm{Ext}^{s+1,t+n}(H_*X)$$

is *defined* if α is in the image of ν_*. Then the algebraic M-Toda bracket is the collection of all $f(\gamma) \in \mathrm{Ext}^{s+1,t+n}(H_*X)$ where $\gamma \in \mathrm{Ext}^{s,t+n}(H_*X \otimes M_1)$ is any element satisfying $\nu_*(\gamma) = \alpha$.

We leave it to the reader to construct the dually defined algebraic M-Toda bracket analogous to the dually defined K-Toda bracket, and MSS version of the algebraic M-Toda bracket analogous to the E-ASS version of the K-Toda bracket. The definitions of these Toda brackets given in Section 4 carry over verbatim to the algebraic context.

With these definitions, our filtered root invariant results given in Section 5 have analogous algebraic statements. The first algebraic filtered root invariant is the algebraic BP-root invariant. If the k^{th} algebraic filtered root invariant is a permanent cycle in the MSS, then it is the algebraic root invariant modulo BP-filtration greater than k. One may deduce higher algebraic filtered root invariants from differentials and compositions in the MSS. In other words, there are algebraic versions of Theorem 5.1, Corollary 5.2, Theorem 5.3, Theorem 5.4, Proposition 5.7, Proposition 5.8, and Corollary 5.9, where one makes the following replacements:

- Replace π_* with Ext.
- Replace root invariants with algebraic root invariants.

- Replace E-root invariants with algebraic E-root invariants.
- Replace filtered root invariants with algebraic filtered root invariants.
- Replace K-Toda brackets with algebraic $H_*(K)$-Toda brackets.
- Specialize from E to BP.
- Replace the E-ASS with the MSS.

15. Modified filtered root invariants

In this section we will recall the modified root invariant that Mahowald and Ravenel define in [25]. We will then explain how to define modified versions of filtered root invariants, and how to adapt our theorems to compute modified root invariants. We also discuss algebraic modified filtered root invariants, which is the tool we will be using in Section 16.

Let X be a finite complex. We shall begin be recalling the definition of the modified root invariant $R'(\alpha)$ for an element $\alpha \in \pi_t(X)$. Let $V(0)$ be the mod p Moore spectrum. We define modified versions of the stunted Tate spectra P_M^N which will have the same underlying space, but which we shall view as being built out of $V(0)$ instead of the sphere spectrum. To this end, define spectra

$$P'^N = \begin{cases} P^{N+1} & \text{if } N \text{ is of the form } kq-1 \\ P^N & \text{otherwise.} \end{cases}$$

Likewise, for $M \leq N$, define quotient spectra

$$P'^N_M = P'^N / P'^{M-1}.$$

Thus the spectrum P'^N_M has a $V(0)$-cell in every dimension from M to N congruent to -1 modulo q. There are cofiber sequences

$$P'^{(l-1)q-1}_{kq-1} \to P'^{lq-1}_{kq-1} \to \Sigma^{lq-1} V(0).$$

Definition (*Modified root invariant*)**.** Let X be a finite complex, and suppose we are given $\alpha \in \pi_t(X)$. The *modified root invariant* of α is the coset of all dotted arrows making the following diagram commute.

$$\begin{tikzcd} S^t \arrow[r, dotted] \arrow[d, "\alpha"] & \Sigma^{-N+1} V(0) \wedge X \arrow[dd] \\ X \arrow[d] & \\ tX \arrow[r] & \Sigma P'_{-N} \wedge X \end{tikzcd}$$

This coset is denoted $R'(\alpha)$. Here N is chosen to be minimal such that the composite $S^t \to \Sigma P'_{-N} \wedge X$ is non-trivial.

Thus the definition of the modified root invariant differs from the definition of the root invariant in that it takes values in $\pi_*(V(0) \wedge X)$ instead of $\pi_*(X)$, and that P has been replaced with P'. By making these replacements elsewhere, we may produce modified versions of our other definitions, as summarized below. In the case of the algebraic filtered root invariants, we choose to use the quotient $\overline{\Lambda}_{(0)} = \overline{\Lambda}/(v_0)$ described in [15]. The cohomology of $\overline{\Lambda}_{(0)}$ is given by

$$H^*(\overline{\Lambda}_{(0)}) = \mathrm{Ext}(H_* V(0)).$$

One has modified versions of the subcomplexes $W_k(H_*X)$ given by
$$W'_k(H_*X) = Im(W_k(H_*X) \hookrightarrow \overline{\Lambda} \otimes H_*(X) \to \overline{\Lambda}_{(0)} \otimes H_*(X)).$$
By modifying our definitions, we produce:

- Modified E-root invariants
$$R'_E : \pi_*(X) \rightsquigarrow \pi_*(E \wedge V(0) \wedge X)$$
- Modified algebraic root invariants
$$R'_{alg} : \mathrm{Ext}(H_*X) \rightsquigarrow \mathrm{Ext}(H_*V(0) \wedge X)$$
- Modified filtered Tate spectra
$$W_I(P'^J)$$
- Modified filtered root invariants
$$R_E^{[k]'} : \pi_*(X) \rightsquigarrow \pi_*(W_k^k(V(0) \wedge X))$$
- Modified algebraic E-root invariants
$$R'_{E,alg} : \mathrm{Ext}(H_*X) \rightsquigarrow \mathrm{Ext}(H_*E \wedge V(0) \wedge X)$$
- Modified algebraic filtered root invariants
$$R_{E,alg}^{[k]'} : \mathrm{Ext}(H_*X) \rightsquigarrow H^*({W'}_k^k(H_*X))$$

We would like to reformulate the results of Section 5 in terms of modified root invariants. We need a modified version of the Toda brackets which appear in the statements of the main theorems. Suppose that K is a finite complex built out of $V(0)$ with a single bottom $V(0)$-cell in dimension 0 and a single top $V(0)$-cell in dimension n. Let K^j be the j^{th} $V(0)$-skeleton of K, and let K_i^j be the quotient K^j/K^{i-1}. We shall omit the top index for the i^{th} $V(0)$-coskeleton $K_i = K/K^{i-1}$.

Definition (*Modified K-Toda bracket*)**.** Let
$$f : \Sigma^{-1}K_1 \to V(0)$$
be the attaching map of the first $V(0)$-coskeleton of K to the zeroth $V(0)$-cell, so that the cofiber of f is K. Let $\nu : K_1 \to \Sigma^n V(0)$ be the projection onto the top $V(0)$-cell. Suppose α is an element of $\pi_t(X \wedge V(0))$. We have
$$\pi_t(X \wedge V(0)) \xleftarrow{\nu_*} \pi_{t+n}(X \wedge K_1) \xrightarrow{f_*} \pi_{t+n-1}(X \wedge V(0)).$$
We say the modified K-Toda bracket
$$\langle K\rangle'(\alpha) \subseteq \pi_{t+n-1}(X \wedge V(0))$$
is *defined* if α is in the image of ν_*. Then the modified K-Toda bracket is the collection of all $f_*(\gamma) \in \pi_{t+n-1}(X \wedge V(0))$ where $\gamma \in \pi_{t+n}(X \wedge K_1)$ is any element satisfying $\nu_*(\gamma) = \alpha$.

Similarly we may provide a modified version of the dually defined and E-ASS K-Toda brackets given in Section 4. Furthermore, for an A_*-comodule M which is cofree over the coalgebra $E[\tau_0]$, it is straightforward to define the modified algebraic M-Toda bracket by varying the definition of the M-Toda bracket given in Section 14.

Modified versions of all of the results in Section 5 hold, with proofs that go through with only superficial changes. Specifically, one must make the following adjustments.

- Replace P with P'
- Replace root invariants with modified root invariants.
- Replace E-root invariants with modified E-root invariants.
- Replace filtered root invariants with modified filtered root invariants.
- Replace K-Toda brackets with modified K-Toda brackets.
- Replace the E-ASS for X with the E-ASS for $X \wedge V(0)$.

Furthermore, there are algebraic modified versions of Theorems 5.1, 5.3 and 5.4, Propositions 5.7 and 5.8, and Corollaries 5.2 and 5.9, where one makes the following replacements:

- Replace P with P'.
- Replace π_* with Ext.
- Replace root invariants with modified algebraic root invariants.
- Replace E-root invariants with modified algebraic E-root invariants.
- Replace filtered root invariants with modified algebraic filtered root invariants.
- Replace K-Toda brackets with modified algebraic $H_*(K)$-Toda brackets.
- Specialize from E to BP.
- Replace the E-ASS for X with the MSS for $H_*X \wedge V(0)$

We now modify our definitions even further, to give variations of the second modified filtered root invariant $R''(\alpha)$ given in [25]. In what follows we shall always be working at a prime $p \geq 3$.

It is observed in [25] that there is a splitting

$$P_{kq-1}^{(l+1)q} \wedge V(0) \simeq P''^{lq-1}_{kq-1} \vee \Sigma^{kq} V(0) \vee \Sigma^{(l+1)q-1} V(0)$$

In the notation of [25], we have

$$P''^{lq-1}_{kq-1} = \overline{P}^{lq}_{kq-1}$$

The spectra P''^{lq-1}_{kq-1} are built out of the Smith-Toda complex $V(1)$. There is a $V(1)$-cell every in dimension from $kq-1$ to $lq-1$ congruent to $-1 \mod q$. The decomposition of $P''_{-\infty}$ into $V(1)$-cells is described on the level of cohomology in the following lemma, which is a straightforward computation.

Lemma 15.1. Let e_n^* be the generator of $H^*(P)_{-\infty}$ in dimension n, where $n \equiv 0, -1 \pmod q$. The cohomology of the Moore spectrum is given by

$$H^*(V(0)) = E[Q_0]$$

as a module over the Steenrod algebra. The cohomology $H^*(P'')_{-\infty} = H^*(P \wedge V(0))_{-\infty}$ is free over the subalgebra $E[Q_0, Q_1]$ of the Steenrod algebra on the generators $e^*_{kq-1} \otimes 1$. The action $E[Q_0, Q_1]$ on the free $E[Q_0, Q_1]$-submodule generated by $e^*_{kq-1} \otimes 1$ is then given by the following formulas.

$$\begin{aligned}
Q_0(e^*_{kq-1} \otimes 1) &= e^*_{kq} \otimes 1 - e^*_{kq-1} \otimes Q_0 \\
Q_1(e^*_{kq-1} \otimes 1) &= e^*_{(k+1)q} \otimes 1 \\
Q_1(e^*_{kq} \otimes 1 - e^*_{kq-1} \otimes Q_0) &= -e^*_{(k+1)q} \otimes Q_0 \\
Q_0(e^*_{(k+1)q} \otimes 1) &= e^*_{(k+1)q} \otimes Q_0
\end{aligned}$$

It is convenient to allow arbitrary subscripts and superscripts, so we define for integers $M \leq N$

$$P''^{N}_{M} = P''^{lq-1}_{kq-1}$$

where k (respectively l) is minimal (maximal) such that $M \le kq-1 \le N$ ($M \le lq-1 \le N$). If there is no such k and l, then we have ${P''}_M^N = *$.

The following lemma is Lemma 3.7(e) of [25].

Lemma 15.2. If k and l are congruent to $0 \pmod p$, then $H^*({P''}_{kq-1}^{lq-1})$ is free over the subalgebra $A(1)$ of the Steenrod algebra.

Since there is no difference between the spectra $P''_{-\infty}$ and $P_{-\infty} \wedge V(0)$, and since $V(0)$ is p-complete, there is an analog of Lin's theorem [25, Lemma 3.7(b)].

Lemma 15.3. The map

$$\Sigma^{-1} V(0) \to P''_{-\infty}$$

is an equivalence.

In light of Lemma 15.3, Mahowald and Ravenel [25] define a second modified root invariant.

Definition (*Second modified root invariant*)**.** Let X be a finite complex, and suppose we are given $\alpha \in \pi_t(X \wedge V(0))$. The *second modified root invariant* of α is the coset of all dotted arrows making the following diagram commute.

$$\begin{tikzcd}
S^t \arrow[r, dotted] \arrow[d, "\alpha"] & \Sigma^{-N+1} V(1) \wedge X \arrow[dd] \\
X \wedge V(0) \arrow[d] & \\
tX \wedge V(0) \arrow[r] & \Sigma P''_{-N} \wedge X
\end{tikzcd}$$

This coset is denoted $R''(\alpha)$. Here N is chosen to be minimal such that the composite $S^t \to \Sigma P''_{-N} \wedge X$ is non-trivial.

The definition of the second modified root invariant may be obtained from the definition of the (first) modified root invariant by smashing X with $V(0)$, replacing P' with P'', and replacing $V(0)$ with $V(1)$. Similarly, one may obtain second modified versions of all of the other definitions we have been working with, as summarized below. The second modified lambda complex $W''_k(H_*X)$ is the appropriate subcomplex of $\overline{\Lambda}_{(1)} \otimes H_*(X)$, where $\overline{\Lambda}_{(1)} = \overline{\Lambda}/(v_0, v_1)$.

- Second modified E-root invariants
$$R''_E : \pi_*(X \wedge V(0)) \rightsquigarrow \pi_*(E \wedge V(1) \wedge X)$$
- Second modified algebraic root invariants
$$R''_{alg} : \mathrm{Ext}(H_*X \wedge V(0)) \rightsquigarrow \mathrm{Ext}(H_*V(1) \wedge X)$$
- Second modified filtered Tate spectra
$$W_I({P''}^J)$$
- Second modified filtered root invariants
$$R_E^{[k]''} : \pi_*(X \wedge V(0)) \rightsquigarrow \pi_*(W_k^k(V(1) \wedge X))$$
- Second modified algebraic E-root invariants
$$R''_{E,alg} : \mathrm{Ext}(H_*X \wedge V(0)) \rightsquigarrow \mathrm{Ext}(H_*E \wedge V(1) \wedge X)$$

- Second modified algebraic filtered root invariants

$$R^{[k]''}_{E,alg} : \mathrm{Ext}(H_*X \wedge V(0)) \rightsquigarrow H^*({W''}^k_k(H_*X))$$

If K is a finite complex built from $V(1)$ with single bottom and top dimensional $V(1)$-cells, then we may define the second modified K-Toda bracket

$$\langle K \rangle'' : \pi_*(V(1) \wedge X) \rightsquigarrow \pi_*(V(1) \wedge X).$$

Likewise, if M is an A_*-comodule which is cofree over $E[\tau_0, \tau_1]$, one can define a second modified algebraic M-Toda bracket on $\mathrm{Ext}(H_*V(1) \wedge X)$. Just as outlined for the case of modified root invariants in the first half of this section, second modified and second modified algebraic versions of the results of Section 5 hold.

16. Computation of some infinite families of root invariants at $p = 3$

In this section we will extend the computation

$$(-v_2)^i \in R''(v_1^i)$$

of [25] for $p \geq 5$ to the prime 3 for $i \equiv 0, 1, 5 \pmod 9$. We will use our modified root invariant computations to deduce that, at the prime 3, the root invariant of the element $\alpha_i \in \pi_*^S$ is given by the element β_i for $i \equiv 0, 1, 5 \pmod 9$.

Throughout this section we work at the prime 3. Low dimensional computations indicate that there is a map

$$v_2 : S^{16} \to V(1).$$

Oka [31] demonstrates that there is a map

$$v_2^5 : S^{80} \to V(1).$$

Composing these maps with iterates of the map

$$v_2^9 : \Sigma^{144} V(1) \to V(1)$$

given in [4], we have maps

$$v_2^i : S^{16i} \to V(1)$$

for $i \cong 0, 1, 5 \pmod 9$. The computation of $\pi_*(L_2 V(1))$ given in [13] indicates that these are the only i for which v_2^i can exist.

Theorem 16.1. At the prime 3, for $i \equiv 0, 1, 5 \pmod 9$, the second modified root invariant of $v_1^i \in \pi_*(V(0))$ is given by

$$(-v_2)^i \in R''(v_1^i).$$

We shall prove Theorem 16.1 with a sequence of lemmas.

Lemma 16.2. For all i, we have the second modified BP-root invariant

$$(-v_2)^i \in R''_{BP}(v_1^i).$$

Proof. Modulo I_1, the 3-series of the formal group associated to BP is given by

$$[3](x) \equiv v_1 x^3 + v_2 x^9 + \cdots \pmod{I_1}.$$

Thus in $tBP \wedge V(0)_*$ (see Lemma 10.4), we have

$$v_1 = -v_2 x^6 + \cdots$$

and upon taking the i^{th} power, we have

$$(v_1)^i = (-v_2)^i x^{6i} + \cdots$$

Using the second modified version of Corollary 10.5, we may deduce the result. $\square$

Lemma 16.3. For all i, we have the second modified algebraic BP-root invariant

$$(-v_2)^i \in R''_{BP,alg}(v_1^i).$$

Here, v_1 and v_2 are viewed as elements in the cohomology of the periodic lambda algebra.

Proof. Apply $\mathrm{Ext}(H_* -)$ to the diagram which realizes the second modified BP-root invariant given by Lemma 16.2. □

In order to eliminate obstructions in higher Adams filtration, we prove the following lemma. This lemma is essentially contained in the proof of Lemma 3.10 of [25].

Lemma 16.4. We have the following Ext calculation.

$$\mathrm{Ext}^{s,iq-1+s}(H_*P''_{-3iq-1}) = \begin{cases} 0 & s > i \\ \mathbb{F}_p\{v_2^i\} & s = i \end{cases}$$

Proof. In Lemma 15.2, we saw that $H^*(P''_{-3iq-1})$ is free over the subalgebra $A(1)$ of the Steenrod algebra on generators in dimensions congruent to $-1 \pmod{3q}$. Therefore $\mathrm{Ext}_{A_*}(H_*P''_{-3iq-1})$ is built out of $\mathrm{Ext}_{A_*}(\Sigma^{3kq-1}A(1)_*)$ for $k \geq -i$. Now the May spectral sequence E_2-term for $\mathrm{Ext}_{A_*}(A(1)_*)$ is given by

$$E_2 = E[h_{i,j} : i \geq 1, j \geq 0, i+j \geq 2] \otimes P[b_{i,j} : i \geq 1, j \geq 0, i+j \geq 2] \otimes P[v_i : i \geq 2].$$

Analyzing the degrees that these elements live in establishes that the only elements in $\mathrm{Ext}_{A_*}(A(1)_*)$ which lie on or above the line of slope $1/|v_2|$ (in the $(t-s,s)$-plane) are those of the form $h_{1,1}^{\epsilon_1}h_{2,0}^{\epsilon_2}v_2^k$ for $\epsilon_j = 0, 1$. Therefore, one easily deduces that there are no elements which lie above the line of slope $1/|v_2|$ passing through the point $(0, -3iq-1)$ which lie in the dimensions we are considering, and the only element which lies in the $iq-1$ stem with Adams filtration equal to i is v_2^i. □

Lemma 16.5. For all i, we have the second modified algebraic root invariant

$$(-v_2)^i \in R''_{alg}(v_1^i)$$

Proof. Applying the second modified algebraic version of Proposition 5.7, we may deduce that the zeroth second modified algebraic filtered root invariant is given by

$$(-v_2^i) \in R^{[0]\ ''}_{BP,alg}(v_1^i).$$

The element v_2^i is a permanent cycle in the MSS for $\mathrm{Ext}(H_*V(1))$, so we may use the second modified algebraic version of Theorem 5.1 to deduce that the difference of the images of $(-v_2)^i$ and v_1^i in

$$\mathrm{Ext}^{i,iq-1+i}(H_*P''_{-3iq-1})$$

is of BP-filtration greater than 0. Lemma 16.4 says that there are no non-zero elements of this Ext group which could represent the difference of these images, so we actually have

$$(-v_2)^i \in R''_{alg}(v_1^i).$$

□

We shall need a slightly different result for $i \equiv 5 \pmod 9$.

Lemma 16.6. Modulo indeterminacy, the second modified algebraic root invariant of v_1^{9t+5} is also given by

$$(-v_2)^{9t+5} \pm v_2^{9t} v_3 g_0 b_0 \in R''_{alg}(v_1^{9t+5})$$

Proof. By Lemma 16.4, the element $v_2^{9t} v_3 g_0 b_0$ is in the kernel of the map

$$\mathrm{Ext}^{i,iq-1+i}(H_* \Sigma^{-3iq-1} V(1)) \to \mathrm{Ext}^{i,iq-1+i}(H_* P''_{-3iq-1})$$

where $i = 9t + 5$. Therefore it is in the indeterminacy of the second modified algebraic root invariant. □

Proof of Theorem 16.1. The element v_2^i is a permanent cycle in the ASS for $V(1)$ for $i = 0, 1$. The existence of the self map v_2^9 of the complex $V(1)$ implies that v_2^i is a permanent cycle in the ASS for $i \equiv 0, 1 \pmod 9$. However, in the ASS for $V(1)$, the element v_2^5 is *not* a permanent cycle; the element

$$v_2^5 \pm v_3 g_0 b_0$$

is the permanent cycle that detects the Oka element $v_2^5 \in \pi_{80}(V(1))$ [4, Remark 8.2]. Thus the element $v_2^{9t+5} \in \pi_*(V(1))$ is detected in the ASS by the permanent cycle $v_2^{9t+5} \pm v_2^{9t} v_3 g_0 b_0$. For uniformity of notation, let $\widetilde{v_2^i}$ denote the ASS element that detects $v_2^i \in \pi_*(V(1))$ for $i \equiv 0, 1, 5 \pmod 9$.

Lemmas 16.5 and 16.6 imply that we have second modified algebraic root invariants

$$(-1)^i \widetilde{v_2^i} \in R''_{alg}(v_1^i)$$

where $i \equiv 0, 1, 5 \pmod 9$. By the second modified version of Theorem 5.10, we have second modified filtered root invariants given by

$$(-1)^i \widetilde{v_2^i} \in R_H^{[i]''}(v_1^i).$$

The elements $\widetilde{v_2^i}$ are permanent cycles in the ASS, so by the second modified version of Theorem 5.1, we may conclude that the difference between the images of $(-v_2)^i$ and v_1^i in $\pi_{iq-1}(P''_{-3iq-1})$ has Adams filtration greater than i. According to Lemma 16.4, there are no such elements, so the images of $(-v_2)^i$ and v_1^i in $\pi_{iq-1}(P''_{-3iq-1})$ are actually equal, and we have

$$(-v_2)^i \in R''(v_1^i).$$

□

We will now use our second modified root invariant computations to deduce the following.

Theorem 16.7. We have root invariants

$$(-1)^{i+1} \beta_i \in R(\alpha_i)$$

for $i \equiv 0, 1, 5 \pmod 9$ at $p = 3$.

Proof. Let ν denote the map which is given by projection onto the top cell of $V(0)$. Since P''_{jq-1} is a summand of $P_{jq-1} \wedge V(0)$, we have an induced map

$$\nu' : P''_{jq-1} \to \Sigma P_{(j+1)q}.$$

We have the following diagram for $i \equiv 0, 1, 5 \pmod 9$.

$$\begin{tikzcd}
S^{qi-1} \arrow[rrr, "(-v_2)^i"'] \arrow[d, "\alpha_i"'] \arrow[dr, "v_1^i"] \arrow[rrrr, bend left, "(-1)^i\beta_i"] & & & \Sigma^{-3iq-1}V(1) \arrow[r, "\nu''"'] \arrow[dd] & S^{(-3i+1)q+1} \arrow[ddd, "\iota"] \\
S^0 \arrow[ddd] & \Sigma^{-1}V(0) \arrow[l, "\nu"] \arrow[d] \arrow[r, phantom, "(*)"] & {} & {} \arrow[r, phantom, "(-1)"] & {} \\
& P''_{-\infty} \arrow[rr] \arrow[dl, "\nu"'] & & P''_{-3iq-1} \arrow[dr, "\nu'"] & \\
\Sigma P_{-\infty} \arrow[rrrr] & & & & \Sigma P_{(-3i+1)q}
\end{tikzcd}$$

The commutivity of the inner portion $(*)$ is our modified root invariant computations of v_1^i. The portion of the diagram marked with a "(-1)" commutes with the introduction of a minus sign. This is due to the fact that the elements β_i are given by $\nu''_*(v_2^i)$ where

$$\nu'' : V(1) \to S^6$$

is the projection onto the cell corresponding to the element Q_1Q_0 in the cohomology group

$$H^*(V(1)) = E[Q_0, Q_1].$$

According to the computations of Lemma 15.1, this is the negative of the cell in P''_{-i3q-1} which corresponds to the cohomology element

$$e^*_{(-3i+1)q} \otimes Q_0 = Q_0Q_1(e^*_{-3iq-1} \otimes 1).$$

Following the outer portion of the diagram, and introducing the minus sign, we see that we have

$$(-1)^{i+1}\beta_i \in R(\alpha_i)$$

unless $\iota_*(\beta_i) = 0$. That is not possible by Proposition 8.7. $\square$

Remark 16.8. If one believes that the root invariant takes v_1^9 multiplication to v_2^9 multiplication at $p = 3$, analogous to the results of Sadofsky [35] and Johnson [12], then it is seems likely that $\beta_i \mathrel{\dot{\in}} R(\alpha_i)$ for $i \equiv 0, 1, 3, 5, 6 \pmod 9$. The present methods will not extend because v_2^i is not a permanent cycle in the ANSS for $V(1)$ for $i \not\equiv 0, 1, 5 \pmod 9$. In fact, at the time of writing we still do not know that the elements β_{9t+3} exist. The elements $\beta_{9t+1}\beta_1\alpha_1$ and $\beta_{9t+1}\beta_1^4$ do exist, and are non-trivial by calculations of Shimomura and Wang [34]. We conjecture that we have root invariants

$$\beta_{9t+1}\beta_1\alpha_1 \mathrel{\dot{\in}} R(\alpha_{9t+2})$$

$$\beta_{9t+1}\beta_1^4 \mathrel{\dot{\in}} R(\alpha_{9t+4})$$

The case $t = 0$ was already handled in Section 13.

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