OVERCATEGORIES AND UNDERCATEGORIES OF MODEL CATEGORIES

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If $\mathcal{M}$ is a model category and Z is an object of $\mathcal{M}$, then there are model category structures on the categories $(\mathcal{M}\downarrow Z)$ (the category of objects of $\mathcal{M}$ over Z) and $(Z\downarrow \mathcal{M})$ (the category of objects of $\mathcal{M}$ under Z) under which a map is a cofibration, fibration, or weak equivalence if and only if its image in $\mathcal{M}$ under the forgetful functor is, respectively, a cofibration, fibration, or weak equivalence. It is asserted without proof in [1] that if $\mathcal{M}$ is cofibrantly generated, cellular, or proper, then so is the overcategory $(\mathcal{M}\downarrow Z)$. The purpose of this note is to fill in the proofs of those assertions (see Theorem 1.7) and to state and prove the analogous results for undercategories (see Theorem 2.8).

1. Overcategories

Definition 1.1. If $\mathcal{M}$ is a category and Z is an object of $\mathcal{M}$, then the category $(\mathcal{M}\downarrow Z)$ of *objects of* $\mathcal{M}$ *over* Z is the category in which

- an object is a map $X \to Z$ in $\mathcal{M}$,
- a map from $X \to Z$ to $Y \to Z$ is a map $X \to Y$ in $\mathcal{M}$ such that the triangle

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

  commutes, and
- composition of maps is defined by composition of maps in $\mathcal{M}$.

Definition 1.2. If $\mathcal{M}$ is a category and Z is an object of $\mathcal{M}$, then the *forgetful functor* $\mathrm{G}\colon (\mathcal{M}\downarrow Z) \to \mathcal{M}$ is the functor that takes the object $A \to Z$ of $(\mathcal{M}\downarrow Z)$ to the object A of $\mathcal{M}$ and the map $\begin{tikzcd}[sep=small] A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & Z & \end{tikzcd}$ of $(\mathcal{M}\downarrow Z)$ to the map $A \to B$ of $\mathcal{M}$.

Lemma 1.3. *Let $\mathcal{M}$ be a cocomplete and complete category and let Z be an object of $\mathcal{M}$.*

(1) *The pushout in $(\mathcal{M}\downarrow Z)$ of the diagram*

$$\begin{tikzcd} C \arrow[dr] & A \arrow[l] \arrow[r] \arrow[d] & B \arrow[dl] \\ & Z & \end{tikzcd}$$

is $P \to Z$ where P is the pushout in $\mathcal{M}$ of the diagram

$$C \longleftarrow A \longrightarrow B$$

and the structure map $P \to Z$ is the natural map from the pushout in $\mathcal{M}$.

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(2) *The pullback in* $(\mathcal{M}\downarrow Z)$ *of the diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] & W \arrow[l] \arrow[dl] \\ & Z & \end{tikzcd}$$

is $P \to Z$ *where* P *is the pullback in* $\mathcal{M}$ *of the diagram*

$$X \longrightarrow Y \longleftarrow W$$

and the structure map $P \to Z$ *is the composition* $P \to Y \to Z$.

Proof. The described constructions possess the universal mapping properties that characterize the pushout (or pullback) in $(\mathcal{M}\downarrow Z)$. □

Lemma 1.4. *Let* $\mathcal{M}$ *be a model category and let* Z *be an object of* $\mathcal{M}$. *If* S *is a set of maps in* $\mathcal{M}$ *and* S_Z *is the set of maps in* $(\mathcal{M}\downarrow Z)$ *of the form*

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & Z & \end{tikzcd}$$

in which the map $A \to B$ *is an element of* S, *then a map* $\begin{tikzcd}[sep=tiny] X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$ *in* $(\mathcal{M}\downarrow Z)$ *is a relative* S_Z*-cell complex (see* [1, Definition 10.5.8]*) if and only if the map* $X \to Y$ *in* $\mathcal{M}$ *is a relative* S*-cell complex.*

Proof. This follows from Lemma 1.3. □

Theorem 1.5. *Let* $\mathcal{M}$ *be a cofibrantly generated model category (see* [1, Definition 11.1.2]*) with generating cofibrations* I *and generating trivial cofibration* J, *and let* Z *be an object of* $\mathcal{M}$. *If*

(1) I_Z *is the set of maps in* $(\mathcal{M}\downarrow Z)$ *of the form*

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & Z & \end{tikzcd} \tag{1.6}$$

in which the map $A \to B$ *is an element of* I *and*

(2) J_Z *is the set of maps in* $(\mathcal{M}\downarrow Z)$ *of the form* (1.6) *in which the map* $A \to B$ *is an element of* J,

then the standard model category structure on $(\mathcal{M}\downarrow Z)$ *(in which a map* $\begin{tikzcd}[sep=tiny] X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$ *is a cofibration, fibration, or weak equivalence in* $(\mathcal{M}\downarrow Z)$ *if and only if the map* $X \to Y$ *is, respectively, a cofibration, fibration, or weak equivalence in* $\mathcal{M}$*) is cofibrantly generated, with generating cofibrations* I_Z *and generating trivial cofibrations* J_Z.

Proof. We will show that the set I_Z permits the small object argument and that a map is a trivial fibration if and only if it has the right lifting property with respect to I_Z; the proof of the analogous statement for J_Z is similar.

Lemma 1.3 implies that the forgetful functor $\mathrm{G}\colon (\mathcal{M}\downarrow Z) \to \mathcal{M}$ (see Definition 1.2) takes a relative I_Z-cell complex in $(\mathcal{M}\downarrow Z)$ to a relative I-cell complex in $\mathcal{M}$, and so the set I_Z permits the small object argument.

Since every element of I_Z is a cofibration in $(\mathcal{M}\downarrow Z)$, every trivial fibration in $(\mathcal{M}\downarrow Z)$ has the right lifting property with respect to every element of I_Z. To show that every map with the right lifting property with respect to I_Z is a trivial fibration, it is sufficient to show that every cofibration is a retract of a relative I_Z-cell complex (see [1, Proposition 10.3.2]). Let $\begin{smallmatrix} X \longrightarrow Y \\ \searrow\ \swarrow \\ Z \end{smallmatrix}$ be a cofibration in $(\mathcal{M}\downarrow Z)$; then the map $X \to Y$ is a cofibration in $\mathcal{M}$, and we can factor it as $X \to W \to Y$ in $\mathcal{M}$ where $X \to W$ is a relative I-cell complex and $W \to Y$ is a trivial fibration. Since $\begin{smallmatrix} W \longrightarrow Y \\ \searrow\ \swarrow \\ Z \end{smallmatrix}$ is a trivial fibration in $(Z\downarrow\mathcal{M})$, the retract argument ([1, Proposition 7.2.2]) now implies that $\begin{smallmatrix} X \longrightarrow Y \\ \searrow\ \swarrow \\ Z \end{smallmatrix}$ is a retract of $\begin{smallmatrix} X \longrightarrow W \\ \searrow\ \swarrow \\ Z \end{smallmatrix}$, and Lemma 1.4 implies that $\begin{smallmatrix} X \longrightarrow W \\ \searrow\ \swarrow \\ Z \end{smallmatrix}$ is a relative I_Z-cell complex. □

Theorem 1.7. *Let $\mathcal{M}$ be a model category and let Z be an object of $\mathcal{M}$.*

(1) *If $\mathcal{M}$ is cofibrantly generated, then so is $(\mathcal{M}\downarrow Z)$.*
(2) *If $\mathcal{M}$ is cellular, then so is $(\mathcal{M}\downarrow Z)$.*
(3) *If $\mathcal{M}$ is left proper, right proper, or proper, then so is $(\mathcal{M}\downarrow Z)$.*

Proof. Part 1 follows from Theorem 1.5, part 2 follows from Theorem 1.5 and Lemma 1.4, and part 3 follows from Lemma 1.3. □

2. Undercategories

Definition 2.1. If $\mathcal{M}$ is a category and Z is an object of $\mathcal{M}$, then the category $(Z\downarrow\mathcal{M})$ of *objects of* $\mathcal{M}$ *under* Z is the category in which

- an object is a map $Z \to X$ in $\mathcal{M}$,
- a map from $Z \to X$ to $Z \to Y$ is a map $X \to Y$ in $\mathcal{M}$ such that the triangle

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

  commutes, and
- composition of maps is defined by composition of maps in $\mathcal{M}$.

Proposition 2.2. *If $\mathcal{M}$ is a cocomplete category and Z is an object of $\mathcal{M}$, then the forgetful functor $\mathrm{U}\colon (Z\downarrow\mathcal{M}) \to \mathcal{M}$ that takes the object $Z \to Y$ to Y is right adjoint to the functor $\mathrm{F}\colon \mathcal{M} \to (Z\downarrow\mathcal{M})$ that takes the object X of $\mathcal{M}$ to $Z \to Z \amalg X$ (where that structure map is the natural injection into the coproduct).*

Proof. If X is an object of $\mathcal{M}$ and $Z \to Y$ is an object of $(Z\downarrow\mathcal{M})$, then the universal mapping property of the coproduct implies that a map $\begin{smallmatrix} Z \\ \swarrow\ \searrow \\ Z\amalg X \longrightarrow Y \end{smallmatrix}$ in $(Z\downarrow\mathcal{M})$ is entirely determined by the choice of a map $X \to Y$ in $\mathcal{M}$. □

Lemma 2.3. *Let $\mathcal{M}$ be a cocomplete and complete category and let Z be an object of $\mathcal{M}$.*

(1) *The pushout in* $(Z \downarrow \mathcal{M})$ *of the diagram*

$$\begin{tikzcd} & Z \arrow[dl] \arrow[d] \arrow[dr] & \\ C & A \arrow[l] \arrow[r] & B \end{tikzcd}$$

is $Z \to P$ *where* P *is the pushout in* $\mathcal{M}$ *of the diagram*

$$C \longleftarrow A \longrightarrow B$$

and the structure map $Z \to P$ *is the composition* $Z \to A \to P$.

(2) *The pullback in* $(Z \downarrow \mathcal{M})$ *of the diagram*

$$\begin{tikzcd} & Z \arrow[dl] \arrow[d] \arrow[dr] & \\ X \arrow[r] & Y & W \arrow[l] \end{tikzcd}$$

is $Z \to P$ *where* P *is the pullback in* $\mathcal{M}$ *of the diagram*

$$X \longrightarrow Y \longleftarrow W$$

and the structure map $Z \to P$ *is the natural map to the pullback in* $\mathcal{M}$.

Proof. The described constructions possess the universal mapping properties that characterize the pushout (or pullback) in $(Z \downarrow \mathcal{M})$. □

Proposition 2.4. *Let* $\mathcal{M}$ *be a cocomplete category, let* Z *be an object of* $\mathcal{M}$, *and let* $\mathrm{F}\colon \mathcal{M} \rightleftarrows (Z \downarrow \mathcal{M}) :\mathrm{U}$ *be the adjoint pair of Proposition* 2.2. *If* $f\colon A \to B$ *is a map in* $\mathcal{M}$ *and*

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr, "i_X"] & \\ Z \amalg A \arrow[rr, "i_X \amalg g"'] & & X \end{tikzcd}$$

is a map in $(Z \downarrow \mathcal{M})$, *then the pushout in* $(Z \downarrow \mathcal{M})$ *of the diagram*

$$\begin{tikzcd} & Z \arrow[dl, "i_X"'] \arrow[d] \arrow[dr] & \\ X & Z \amalg A \arrow[l, "i_X \amalg g"] \arrow[r, "1_Z \amalg f"'] & Z \amalg B \end{tikzcd}$$

is $Z \to P$ *where* P *is the pushout in* $\mathcal{M}$ *of the diagram*

$$X \xleftarrow{\ g\ } A \xrightarrow{\ f\ } B$$

and the structure map $Z \to P$ *is the composition* $Z \xrightarrow{i_X} X \to P$.

Proof. The described construction possesses the universal mapping property required of the pushout in $(Z \downarrow \mathcal{M})$. □

Proposition 2.5. *Let* $\mathcal{M}$ *be a cocomplete category, let* Z *be an object of* $\mathcal{M}$, *and let* $\mathrm{F}\colon \mathcal{M} \rightleftarrows (Z \downarrow \mathcal{M}) :\mathrm{U}$ *be the adjoint pair of Proposition* 2.2. *If* S *is a set of maps in* $\mathcal{M}$, *then a relative* $\mathrm{F}S$*-cell complex (see* [1, Definition 10.5.8]*)* $\begin{tikzcd}[sep=small] & Z \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$ *in*

$(Z\downarrow \mathcal{M})$ is a relative S-cell complex $X \to Y$ in $\mathcal{M}$ with structure maps defined by composition with the structure map of $Z \to X$.

Proof. This follows from Proposition 2.4. □

Proposition 2.6. *Let $\mathcal{M}$ be a cocomplete category, let Z be an object of $\mathcal{M}$, and let $\mathrm{F}\colon \mathcal{M} \rightleftarrows (Z\downarrow \mathcal{M}) :\mathrm{U}$ be the adjoint pair of Proposition 2.2. If S is a set of maps in $\mathcal{M}$ that permits the small object argument (see [1, Definition 10.5.15]), then $\mathrm{F}S$ is a set of maps in $(Z\downarrow \mathcal{M})$ that permits the small object argument.*

Proof. This follows from Proposition 2.5 and the adjointness of the functors F and U. □

Theorem 2.7. *Let $\mathcal{M}$ be a cofibrantly generated model category (see [1, Definition 11.1.2]) with generating cofibrations I and generating trivial cofibrations J. If Z is an object of $\mathcal{M}$ and $\mathrm{F}\colon \mathcal{M} \rightleftarrows (Z\downarrow \mathcal{M}) :\mathrm{U}$ is the adjoint pair of Proposition 2.2, then the standard model category structure on $(Z\downarrow \mathcal{M})$ (in which a map* $\begin{smallmatrix} & Z & \\ \swarrow & & \searrow \\ X & \longrightarrow & Y\end{smallmatrix}$ *is a cofibration, fibration, or weak equivalence in $(Z\downarrow \mathcal{M})$ if and only if the map $X \to Y$ is, respectively, a cofibration, fibration, or weak equivalence in $\mathcal{M}$) is cofibrantly generated, with generating cofibrations*

$$\mathrm{F}I = \left\{ \begin{smallmatrix} & Z & \\ \swarrow & & \searrow \\ Z\amalg X & \longrightarrow & Z\amalg Y\end{smallmatrix} = \mathrm{F}(A \to B) \;\Big|\; (A\to B) \in I \right\}$$

and generating trivial cofibrations

$$\mathrm{F}J = \left\{ \begin{smallmatrix} & Z & \\ \swarrow & & \searrow \\ Z\amalg X & \longrightarrow & Z\amalg Y\end{smallmatrix} = \mathrm{F}(A \to B) \;\Big|\; (A\to B) \in J \right\} .$$

Proof. We will use [1, Theorem 11.3.2] to show that there is a cofibrantly generated model category structure on $(Z\downarrow \mathcal{M})$ with generating cofibrations $\mathrm{F}I$ and generating trivial cofibration $\mathrm{F}J$, after which we will show that this coincides with the standard model category structure on $(Z\downarrow \mathcal{M})$.

To apply [1, Theorem 11.3.2], we must show that

(1) both of the sets $\mathrm{F}I$ and $\mathrm{F}J$ permit the small object argument, and
(2) U takes relative $\mathrm{F}J$-cell complexes in $(Z\downarrow \mathcal{M})$ to weak equivalences in $\mathcal{M}$.

The first condition follows from Proposition 2.6, and the second condition follows from Proposition 2.5, since a relative J-cell complex is a trivial cofibration in $\mathcal{M}$.

Thus, $\mathrm{F}I$ and $\mathrm{F}J$ are the generating cofibrations and generating trivial cofibrations of some model category structure on $(Z\downarrow \mathcal{M})$. To see that this is the standard one, we must show that a map in $(Z\downarrow \mathcal{M})$ is a cofibration, fibration, or weak equivalence if and only if its image under U is, respectively, a cofibration, fibration, or weak equivalence in $\mathcal{M}$. For the weak equivalences, this follows from [1, Theorem 11.3.2]. Since the fibrations of $(Z\downarrow \mathcal{M})$ are the maps with the right lifting property with respect to every element of $\mathrm{F}J$, the adjointness of F and U implies that these are exactly the maps whose images under U have the right lifting property with respect to J, i.e., exactly the maps whose images under U are fibrations in $\mathcal{M}$. Finally, since the fibrations and the weak equivalences of a model category structure determine the cofibrations, the two model category structures on $(Z\downarrow \mathcal{M})$ must have the same cofibrations as well. □

Theorem 2.8. *Let $\mathcal{M}$ be a model category and let Z be an object of $\mathcal{M}$.*

(1) *If* $\mathcal{M}$ *is cofibrantly generated, then so is* $(Z \downarrow \mathcal{M})$.
(2) *If* $\mathcal{M}$ *is cellular, then so is* $(Z \downarrow \mathcal{M})$.
(3) *If* $\mathcal{M}$ *is left proper, right proper, or proper, then so is* $(Z \downarrow \mathcal{M})$.

Proof. Part 1 follows from Theorem 2.7, part 2 follows from Theorem 2.7 and Proposition 2.5, and part 3 follows from Lemma 2.3. □

References

1. P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.

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