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MATHEMATICS

A CLASSIFICATION OF CONTINUOUS TRANSFORMATIONS OF A COMPLEX INTO A SPHERE. I*

By **L. PONTRYAGIN**

(Communicated by I. M. Vinogradow, Member of the Academy, 25. II. 1938)

Two continuous transformations f and g of a topological space R into an n-dimensional sphere S_n** are usually called equivalent if there exists a continuous deformation transforming f into g. Thus all continuous transformations of R into S_n can be divided into classes, each class consisting of equivalent transformations. The problem of determining these classes (for given R and n) is one of the most actual problems in topology. For the case when R is an n-dimensional complex K_n this problem was completely solved by Hopf. For the case when $R=K_r$, $r>n$, it is as yet far from being solved.

In this and some subsequent notes I shall publish my results concerning some particular cases of the problem. In particular, in the present paper I give a complete classification of continuous transformations of the sphere S_{n+1} into S_n (for $n=2, 3, \ldots$).

Let us begin by establishing some general properties of transformations of a topological space into a sphere.

We shall say that two continuous transformations f and g of a space R into a sphere S_n coincide in an open set $V \subset S_n$ if

1) $f^{-1}(V)=g^{-1}(V)=U \subset R$;

2) for every $x \in U$ we have $f(x)=g(x)$.

We can announce now the following three propositions (A, B and C) the proof of which is immediate.

P r o p o s i t i o n A. If two continuous transformations f and g of R into S_n coincide in a non-empty open set $V \subset S_n$, then these transformations are equivalent.

P r o p o s i t i o n B. Let V be a spherical region in S_n, F is a closed subset of the normal space R, V' and F' are the boundaries of resp. V and F (in S_n and R), and f is a continuous transformation of F into $V+V'$ such that $f(F') \subset V'$. Then there exists a continuous transformation g of R into S_n coinciding with f in V.

P r o p o s i t i o n C. Let f be a simplicial transformation of a complex K into the sphere S_n (which we suppose to be decomposed into simp-

* Theorems 2′ and 2″ have been reported on my behalf by S. Lefshetz at the International Mathematical Congress in Oslo, 1936.

** We distinguish between a «sphere» and a «full sphere». A k-dimensional s p h e r e is the boundary of a $(k+1)$-dimensional f u l l s p h e r e.

lexes), V is an open n-dimensional simplex belonging to S_n and $U = f^{-1}(V)$. Then U is homeomorphic to the topological product of V and a certain complex P, i. e. we can correlate every point $z \in U$ with a pair (x, y), where $x \in V$, $y \in P$ so that if z corresponds to (x, y), then $f(z) = f(x, y) = x$. Besides, if K is a manifold, then P is also a manifold.

The three-dimensional sphere S_3 is homeomorphic to a group $\bar{S}_3$ of quaternions. Let H be a monoparametric connected subgroup of $\bar{S}_3$ and consider the topological space $\bar{S}_2$ of all right co-sets of H in $\bar{S}_3$. This space is homeomorphic to S_2. If we denote by $\bar{\varphi}(x)$ $(x \in S_3)$ the co-set containing x, then φ is a continuous transformation of $\bar{S}_3$ into $\bar{S}_2$. The corresponding transformation of S_3 into S_2 we shall denote by φ.

We shall say that a transformation f of a complex K into S_n is homologic to zero, if the image of each n-dimensional cycle Z in K modulo m, where m is an arbitrary natural number, is the vanishing cycle in S_n, i. e. if $f(Z) = 0$.

Lemma. *Let f be a continuous transformation of a complex K into S_2. If f is homologic to zero, there exists a continuous transformation g of K into S_3 such that $f = \varphi g$, i. e. such that for every $x \in K$ we have*

$$f(x) = \varphi[g(x)].$$

The proof of this lemma is based on the proposition C and involves a rather complicated construction in which some special properties of the transformation φ are used.

Hurewicz has proved that two transformations g and h of K into S_3 are equivalent if and only if the transformations φg and φh (of K into S_2) are equivalent.

This theorem of Hurewicz together with the above lemma allow us to reduce the question of the classification of homologic to zero transformations of K into S_2 to the question of the classification of all continuous transformations of K into S_3. And hence we obtain easily the classification of continuous transformations of S_3 into S_2.

In fact let f be a simplicial transformation of S_3 into S_2. Take any two points a and b of S_2 which are interior to their corresponding simplexes. Then $f^{-1}(a)$ and $f^{-1}(b)$ are cycles in S_3. Hopf has shown that the looping coefficient $v(f)$ of these cycles is an invariant of the class of equivalent transformations. It may be proved that $v(f)$ is the only invariant of this class. In other words we have:

Theorem 1. *Two transormations f and g of S_3 into S_2 are equivalent if and only if $v(f) = v(g)$.*

We proceed now to the case $n \geqslant 3$.

Theorem 2′. *For $n \geqslant 3$ there exist not more than two classes of transformations of S_{n+1} into S_n.*

Proof. Let f be a simplicial transformation of S_{n+1} into S_n, V is an n-dimensional full sphere with the centre a lying in one of the simplexes of S_n, and let $U = f^{-1}(V)$. Then by proposition C, U can be represented as the topological product of V and a manifold P consisting of a finite number of (topological) circles *. Modifying f we can easily make P consist of a single circle. By a second deformation of f we can make the curve (a, P) a differentiable curve (lying in S_{n+1}), (V, y) (for any $y \in P$) a (metric) full sphere normal to (a, P) and the transformation f isometric on (V, y). Let g be another transformation of S_{n+1} into S_n; we can suppose g to be modified in the same manner as f. Besides, any two circles in S_{n+1} being isotopic, we can suppose that for both transformations the (a, P) [and, consequently, the (V, y)] coincide. Further, since the transformations f and g transform every (V, y) into V isometrically, we have $f(x, y) = g[\psi_y(x), y]$, where ψ_y is a rotation of full sphere V. Thus, the comparison of the two transformations f and g has lead us to the family $\{\psi_y\}_{y \in P}$ of rotations of an n-dimensional full sphere where P is a (topological) circle (and ψ_y depends of y in a continuous manner). But since in the manifold of all rotations of an n-dimensional full sphere V $(n \geqslant 3)$ there are only two different homotopic types of circles, it follows that there exist at most two classes of transformations of S_{n+1} into S_n.

* I. e., in our terminology, one-dimensional spheres.

In order to be able to proceed further we need another general proposition.

Proposition D. Any continuous transformation of S_r into S_n can be approximated by an analytic transformation. If two analytic transformations of S_r into S_n are equivalent, there exists an analytic deformation of one into the other.

The proof is based upon the possibility of approximation of a function of several variables by polynomials.

Theorem 2″. *For $n \geqslant 3$ there exist at least two classes of transformations of S_{n+1} into S_n.*

Proof. Let f be an analytic transformation of S_{n+1} into S_n. Then there exists a point $a \in S_n$ such that the set $P = f^{-1}(a)$ consists of a finite number of simple closed analytic curves having no points in common. The number of these curves we denote by β. Now take at a n mutually orthogonal vectors $v_1, \ldots, v_n$ and a point $y \in P$. The transformation f correlates the system $v_1, \ldots, v_n$ to a system of linearly independent vectors $u_1, \ldots, u_n$ orthogonal to P. To these vectors we join a vector u_0 tangent to P at y. Thus, at every point $y \in P$ we have determined a system U_y of $n+1$ linearly independent vectors. Let us now take at each point $z \in S_{n+1}$ a certain system of $n+1$ linearly independent vectors. The manifold of all such systems we denote by M. It may be easily established that the one-dimensional Betti group of M is the cyclic group of order 2. Let Z be an one-dimensional cycle in M not homologous to zero. The set of all systems U_y $(y \in P)$ defines in M a cycle homologous (in M) to the cycle αZ where $\alpha = 0$ or 1. Let $\gamma \equiv \alpha + \beta$ (mod 2). It may be proved that if we deform the transformation f analytically, γ remains constant (mod 2); if β remains constant during the deformation our assertion is evident, because the deformation of U_y is then continuous; the critical moments of the deformation (when β changes its value) require a special (however, rather elementary) consideration.

To prove theorem 2″ it is now sufficient to construct two transformations for which the rests of γ to the modulus 2 have different values 0 and 1. This may be easily done with the aid of proposition B.

In the next note I shall give the classification of continuous transformations of S_{n+2} into S_n.

Mathematical Institute. State University.
Moscow.

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Le théorème III est applicable aux intégrales $W_n(x)$, $F_n(x)$, $P_r(x)$, $B_n(x)$, $L_n(x)$, $V_n(x)$ dans le cas $p \geqslant 1$. Une légère complication de raisonnement nous permet d'appliquer le théorème aussi aux intégrales $K_n(x)$ (pour $p \geqslant 1$) et $S_h(x)$ (mais seulement pour $p > 1$, cependant, comme il a été démontré par A. Tulajkoff (6), le résultat reste vrai aussi pour $p = 1$).

Institut de Mathématiques et de Mécanique. Université de Léningrad.

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MATHEMATICS

A CLASSIFICATION OF CONTINUOUS TRANSFORMATIONS OF A COMPLEX INTO A SPHERE. 2*

By L. PONTRYAGIN

(Communicated by I. M. Vinogradow, Member of the Academy, 25. III. 1938)

In this paper we give a classification of transformations of the sphere S_{n+2} into the sphere S_n $(n \geqslant 2)$ and establish some general relations between transformations of S_r into S_n.

Theorem 1. *There exist exactly two classes of transformations of S_4 into S_2.*

Proof. In C. of c. t. 1 it was proved that the number of classes of transformations of S_4 into S_2 is equal to the number of classes of transformations of S_4 into S_3. In the same paper this last number was proved to be 2.

Theorem 2. If $n \geqslant 3$ there exists only one class of transformations of S_{n+2} into S_n.

Proof. According to C. of c. t. 1 we may confine ourselves to analytical transformations of S_{n+2} into S_n. Let f be an analytical transformation of S_{n+2} into S_n and a a point of S_n, such that $f^{-1}(a)$ is a surface in S_{n+2} without singularities. Let further V be a (sufficiently small) spherical neighbourhood of a. Then $U = f^{-1}(V)$ is homeomorphic to the topological product of the full sphere V and a given surface P; moreover, if a point $z \in U$ corresponds (in this homeomorphism) to (x, y) (where $x \in V$ and $y \in P$), then $f(z) = f(x, y) = x$. We can modify f so that it will transform the full sphere (V, y) into the full sphere V isometrically and that (V, y) will be orthogonal to the surface (a, P) for an arbitrary $y \in P$.

We shall first consider the case when P is a sphere. For $n \geqslant 4$ any two differentiable surfaces homeomorphic to a sphere are homotopic in S_{n+2}. If, therefore, f and g are two transformations of the described type, we can suppose that the surface (a, P), and consequently, the full spheres

* The present paper is a continuation of my paper «A classific tion of continuous transformations of a complex into a sphere. 1» [C. R. Acad. Sci. URSS, XIX, № 1—2 (1938)], which will be referred to as C. of c. t. 1.

The convention laid down in C. of c. t. 1 concerning the distinction we make between a k-dimensional «sphere» and a k-dimensional «full sphere» remains valid: under a k-dimensional sphere we understand the boundary of a $(k+1)$-dimensional full sphere.

Theorems 1 and 2 were reported on my behalf by Prof. S. Lefshetz at the International Mathematics Congress in Oslo in 1936.

(V, y) coincide for both transformations. We have thus $(x, y) = g(\psi_y(x), y)$ where ψ_y is a rotation of V depending in a continuous manner on the point $y \in P$. Any image of a two-dimensional sphere being homotopic to zero in the manifold of all rotations of V, we conclude that f and g are equivalent (see C. of c. t. 1, A). The case $n=3$ requires a special consideration (involving some rather complicated arguments), but the result remains true.

It remains to reduce the general case to the case when P is a sphere. It is easily seen that the surface P is always orientable. Let L be a simple closed curve lying in P. A neighbourhood of L can be covered by a monoparametric family $\{L_t\}$, $-1 < t < 1$, of simple closed curves $(L_0 = L)$. Consider in S_{n+2} the family of curves $\{(x, L_t)\}_{x \in V}$. This family is homeomorphic to an $(n+1)$-dimensional full sphere V'. If we now correlate every curve (x, L_t) with the point of V' which corresponds to the curve (x, L_t) in the above homeomorphism, we shall obtain a transformation h' of an open subset of S_{n+2} into V'. Supposing that V' lies in an $(n+1)$-dimensional sphere S_{n+1}, we can extend the transformation h' to a transformation h of S_{n+2} into S_{n+1} (see C. of c. t. 1, B). Two cases are now possible: 1) the transformation h is homotopic to zero; then we shall say that the index of L is zero; 2) h is not homotopic to zero; in this case we shall say that the index of L is unity. It may be proved that homologous curves on P have the same index and that if we add two curves (in the sense of homology) their indices are also added modulo 2. Hence it follows easily that unless P is a system of spheres, there exists on P a curve not homologous to zero, whose index is zero. Let L be such a curve. Denote by P' an differentiable surface in S_{n+2}, homeomorphic to a circle (two-dimensional full sphere), having the curve (a, L) for its boundary and no points in common with the surface (a, P) except the points of (a, L).

This construction enables us to modify f so as to make the genus of P smaller by one. Repeating this process we shall come at last to a transformation for which the corresponding surface P consists of a system of spheres; and a system of spheres may be easily reduced to a single sphere.

The theorem is thus proved.

It seems to be usefull to introduce the operations of addition of classes of equivalent transformations of S_r into S_n, thus transforming the set of all such classes into a group.

Definition. Let S_r' and S_r'' be two oriented r-dimensional spheres, and f' and f'' their transformations into the n-dimensional sphere S_n. We shall say that the transformations f' and f'' are equivalent, if there exists a homeomorphism h of S_r'' on S_r' which preserves the orientation and for which the transformation $f'h$ and f'' of S_r'' into S_n are equivalent. Thus, all transformations of oriented r-dimensional spheres into S_n are divided into classes of equivalent transformations.

We shall define now the sum of two classes as follows. Let A and B be two classes and let $f \in A$ and $g \in B$. Denote by S_r' the sphere transformed by f and by S_r'' that transformed by g. Let further V' and V'' be two full spheres resp. in S_r' and S_r''; since f and g are defined only up to equivalent transformations, we may suppose that $f(V') = g(V'') = a$ where a is a point of S_n. Now if we join the oriented manifolds $S_r' - V'$ and $S_r'' - V''$ by identifying their boundaries so that the orientations are preserved, we obtain an oriented sphere S_r'''; the transformations f and g of S_r' and S_r'' define a continuous transformation h of S_r''' into S_n.

It may be proved that the class C, to which h belongs, depends neither on the choice of f and g within the classes A and B, nor on the way of identification of the boundaries of $S_r' - V'$ and $S_r'' - V''$ (provided that the condition about the preservation of orientations is fulfilled), but only on the classes A and B. This class C we shall call the sum of A and B and write $C = A + B$. The operation of addition thus defined possesses all the properties of a group operation. The zero of this group is the class of transformations homotopic to zero. If A is a class of transformations and $f \in A$ is a transformation of S_r' into S_n, then the same transformation f applied to the sphere S_r'', differing from S_r' by orientation, determines the class $-A$ opposite to A.

The group thus obtained is isomorphic to one of the homotopy groups constructed by Hurewicz.

The problem of classification of continuous transformations of S_{n+k} into S_n may be now more precisely formulated as the problem of determination of the group of transformations of S_{n+k} into S_n. This group we shall denote by P_k^n.

From the results already ortained it follows that P_1^2 is a free cyclic group; P_1^n for $n \geqslant 3$ is a cyclic group of the second order, as also is P_2^2, while for $n \geqslant 3$ the group P_2^n consists of a single element.

We have no reasons to suppose that P_k^n is always a cyclic group. But another conjecture suggested by the above results is true. We have:

Theorem 3. *For $n \geqslant k+2$ P_k^n is isomorphic to $P_k^{k+2} = P_k$.*

The proof (which is not very complicated) depends upon the propositions D and A of C. of c. t. 1.

In view of this theorem the problem arises to determine the group P_k. It seems that this problem is closely related to the study of homotopic properties of the group of orthogonal matrices.

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Pontryagin, L.

A classification of continuous transformations of a complex into a sphere. I, II. (English)

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Beide Noten behandeln die Aufgabe, die Abbildungsklassen der Sphäre S^{n+k} in die Sphäre S^n aufzuzählen, oder die – etwas präzisere – Aufgabe, die Homotopiegruppen $\pi_{n+k}(S^n)$ im Sinne von *Hurewicz* (Proc. Akad. Wet. Amsterdam 38 (1935), 112-119; F. d. M. 61_{I}, 618) zu bestimmen. Für beliebiges $k > 0$ wird, ohne Beweis, der Satz ausgesprochen: (A) Für $n > k + 2$ ist $\pi_{n+k}(S^n)$ isomorph mit der Gruppe $\pi_{2k+2}(S^{k+2}) = P_k$. Die Vermutung wird geäußert, daß P_k in engem Zusammenhang mit der i-ten Homotopiegruppe der Mannigfaltigkeit der n-dimensionalen Drehungsgruppe stehe. Dies wird für $k = 1$ und für $k = 2$ bestätigt: Die Tatsachen, daß die Fundamentalgruppe der n-dimensionalen Drehungsgruppe für $n \geqq 3$ zyklisch von der Ordnung 2 und daß die zweiten Homotopiegruppen der Drehungsgruppen die Nullgruppen sind, werden benutzt, um zu zeigen: (B) $\pi_{n+1}(S^n)$ ist für $n \geqq 3$ zyklisch von der Ordnung 2; (C) $\pi_{n+2}(S^n)$ ist für $n \geqq 3$ die Nullgruppe. Hierzu gelten noch die Ergänzungen: (B′) $\pi_3(S^2)$ ist frei zyklisch; (C′) $\pi_4(S^2)$ ist zyklisch von der Ordnung 2; ferner ist $\pi_m(S^1)$ bekanntlich für alle $m > 1$ die Nullgruppe. Die Beweise der Sätze (B), (C) und ihrer Ergänzungen werden kurz skizziert; sie beruhen außer auf den erwähnten Homotopie-Eigenschaften der Drehungsgruppen auf der Tatsache, daß jede stetige Abbildung der S^m in die S^n durch analytische Abbildungen approximiert werden kann. Die Sätze, abgesehen von (A), sind bereits 1936 dem Internationalen Mathematikerkongreß in Oslo mitgeteilt worden. Die Sätze (A), (B), (C′) sind 1937 von *Freudenthal* (Über die Klassen der Sphärenabbildungen I, Compositio math., Groningen, 5 (1937), 299-314; F. d. M. 63_{II}) auf ganz anderem Wege ausführlich bewiesen worden; Satz (B′) stammt von *Hurewicz* (a. a. O.).

Hopf, H.; Prof. (Zürich)

in S. We define the terms "tangent line" and "tangent plane". Our principal results may be stated as follows: if K is a point continuum in S and if P is a point of K, then the set of all lines tangent to K at P is a line continuum and the set of all planes tangent to K at P is a plane continuum.

SUR LES TRANSFORMATIONS DES SPHÈRES EN SPHÈRES

Par L. S. Pontrjagin, Moscou.

L'auteur étudie les classes de transformations univoques d'une S_{n+k} en une S_n et obtient ce théorème définitif pour $k=1, 2$.

Théorème. Soit $P(n, k)$ le nombre de classes de transformations univoques de S_{n+k} en S_n. On a alors

$$P(n, 1) = \begin{cases} 1 & \text{pour } n=1 \\ \infty & \text{pour } n=2 \\ 2 & \text{pour } n>2 . \end{cases}$$

$$P(n, 2) = \begin{cases} 1 & \text{pour } n \neq 2 \\ 2 & \text{pour } n=2 . \end{cases}$$

ON HOMOLOGIES IN GENERAL SPACES

By B. Kaufmann, Cambridge.

Let F be an arbitrary r-dimensional set in R^n, and let $Z^p \sim 0$ be an arbitrary homology in F. We assume $Z^p = z_1^p, z_2^p, \cdots, z_k^p, \cdots$ to be a true cycle in F with a variable modulus m_k.[1] By B we denote an arbitrary carrier of Z^p, i. e. a closed subset of F containing all vertices of the cycles z_k^p for all $k=1, 2, \cdots$.

We say a subset A of F *destroys* the homology $Z^p \sim 0$ in F if Z^p is totally[1] $\not\sim 0$ in any compact subset of $F-A$. We can assume A to be a closed subset of F outside a carrier B of Z^p such that Z^p is totally $\not\sim 0$ in B. Then we can obtain the following theorems, which were conjectured by P. Alexandroff[1] (dimensionstheoretischer Verschlingungssatz):

Theorem H_1. There exists always an at most $(r-h-1)$-dimensional subset $F^{(r-h-1)}$ of $F (0 \leqq h \leqq r-1)$ which destroys the homology $Z^h \sim 0$ in F, i. e. Z^h is totally $\not\sim 0$ in $F - F^{(r-h-1)}$.

[1] See P. Alexandroff, "Dimensionstheorie", Math. Annalen, 106 (1932), 161—238.