Homotopy Properties of Thom Complexes
(English translation with the author's comments)

S.P.Novikov[1]

CONTENTS

[1]The author's comments: As it is well-known, calculation of the multiplicative structure of the orientable cobordism ring modulo 2-torsion was announced in the works of J.Milnor (see [18]) and of the present author (see [19]) in 1960. In the same works the ideas of cobordisms were extended. In particular, very important unitary ("complex') cobordism ring was invented and calculated; many results were obtained also by the present author studying special unitary and symplectic cobordisms. Some western topologists (in particular, F.Adams) claimed on the basis of private communication that J.Milnor in fact knew the above mentioned results on the orientable and unitary cobordism rings earlier but nothing was written. F.Hirzebruch announced some Milnors results in the volume of Edinburgh Congress lectures published in 1960. Anyway, no written information about that was available till 1960; nothing was known in the Soviet Union, so the results published in 1960 were obtained completely independently. Let us make some comments concerning the proof. There exist a misunderstanding of that question in the topological literature. Contrary to the Adams claims, the Milnor's work [18] did not contained proof of the theorem describing multiplicative structure of the cobordism ring and its complex analogue. It used the so-called Adams Spectral Sequence only for calculation of the additive structure and proved "no torsion theorem". For the orientable case it was done independently by my friend B.Averbukh [2] using the standard Cartan-Serre technic; it was Averbukh's work that attracted me to this area: I decided to apply here the Adams Spectral Sequence combined with the homological theory of Hopf algebras and coalgebras instead of the standard Cartan-Serre method because my approach worked very well for the multiplicative problems. The present article was presented in 1959/60 as my diploma work at the Algebra Chair in the Moscow State University. In the Introduction (see below) I made mistakable remark that Milnor also calculated the ring structure using Adams Spectral Sequence (exactly as I did myself). However, it was not so: as it was clearly written by Milnor in [18], his plan was completely different; he intended to prove this theorem geometrically in the second part but never wrote it. I cannot understand why F.Adams missed this fundamental fact in his review in the Math Reviews Journal on my Doklady note ([19]). Does it mean that he never looked carefully in these works? As I realized later after personal meeting in Leningrad with Milnor (and Hirzebruch) in 1961 during the last Soviet Math Congress, his plan was to use some specific concrete algebraic varieties in order to construct the additive basis and apply Riemann-Roch Theorem. I described his manifolds in Appendix (they are very useful) but never realized his plan of the proof: my own purely Hopf-algebraic homotopy-theoretical proof was so simple and natural that I believe until now that Milnor lost interest in his geometric proof after seeing my work. I added Appendix in 1961 but forgot to change Introduction written in 1960, so the mistakable remark survived. It is interesting that in 1965 Stong and Hattori published a work dedicated to this subject. They claimed that they found a "first calculation of the complex cobordism ring avoiding the Adams Spectral Sequence" not mentioning exactly where this theorem was proved first. Let me point out that their work was exactly realization of the Milnor's original plan but Stong and Hattori never mentioned that.

Introduction

This paper contains detailed proofs of the author's results published in [19]. Our purpose is to study the inner homology ("cobordism") rings corresponding to the classical Lie groups $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, and $\mathrm{Sp}(n)$ (in what follows, we denote these rings by V_{SO}, V_{U}, V_{SU}, and V_{Sp}) and the realizability of k-dimensional integral cycles in manifolds of dimension $\geq 2k+1$ by smooth orientable submanifolds. As is known, Thom proved [16] that, for any cycle $z_k \in H^k(M^n)$, there exists a number α such that the cycle αz_k is realized by a submanifold, and, for $k \leq 5$, any cycle z_k can be realized (i. e., $\alpha = 1$).

In Section 3.1 of Chapter 3, we prove the following theorem:

Suppose that $n \geq 2k+1$ and the groups $H_i(M^N)$, where $i = k - 2t(p-1) - 1$, have no p-torsion for all $t \geq 1$ and $p \geq 3$. Then any cycle z_k of dimension k is realized by a submanifold.

The proof of this theorem relies on known Thom's constructions and on new results concerning the homotopy structure of complexes constructed by Thom, which have a number of remarkable properties making it possible to essentially reduce many problems of the topology of manifolds to homotopy problems. There are examples showing that the theorem stated above gives a final criterion in terms of homology groups.

In Chapter 2, the algebraic structure of the rings V_{SO}/T, V_{U}, and $VSp \otimes Z_{p^h}$ with $p > 2$ is completely determined; it is also proved that the ring V_{Sp}/T is nonpolynomial (T is the ideal consisting of the elements of finite order; we can assume that all the orders have form z^s)[2]. The results concerning the ring V_{SU} known to this author are given in Appendix 1; we did not include them in the statements of the main theorems. It turned out that the algebraic structure of the ring V_{U} was found somewhat earlier by Milnor [18][3] (also by the method of Adams), who, in addition, specified geometric generators of the rings V_{SO} and V_{U} and gave a final solution to the problem of Pontryagin (Chern) characteristic numbers of smooth (complex analytic, almost complex) manifolds, i. e., stated a necessary and sufficient condition for a set of numbers to be the set of Pontryagin (Chern) numbers of a smooth (almost complex) manifold. No geometric generators

---

[2]The algebra $V_{\mathrm{SO}} \otimes Q$ was descriqbed by Thom [16]. Rokhlin [13] and (independently) Wall [20] determined the structure of 2-torsion in the ring V_{SO} on the basis of the well-known theorem of Rokhlin about the kernel of a homomorphism (see [12]). Averbukh and Milnor (also independently) proved that the ring V_{SO} has no p-torsion for $p > 2$ (see [2, 10, 18]). In [10], a result on the structure of the ring V_{SO}/T is also announced.

[3]In [18], Milnor considered also the ring V_{Spin}, but he obtained no final results about it.

were known before Milnor's work; because the related results of Milnor were very interesting, we included them in Appendix 2 (we had only known (see Section 2.5 of Chapter 2) that, for primes $p \geq 2$, the projective planes $P^{p-1}(\mathbb{C})$ could be taken as generators in dimensions $2p-2$.) The results of this author related to the multiplicative structure of the ring V_{Sp} and to the ring V_{SU} are new.

Chapter 1 contains some geometric and algebraic information about the Thom complexes.

Chapter 2 is concerned with calculating the inner homology rings. It also considers some questions related to these rings (see also Appendix 2).

Chapter 3 deals with various realizations of cycles by submanifolds.

1. Thom Spaces

1.1. ***G*-framed submanifolds. Classes of *L*-equivalent submanifolds.** Consider a compact closed smooth n-manifold M^n endowed with a Riemannian metric. Let G be a subgroup of the group $\mathrm{O}(n-i)$, where $i < n$. Suppose in addition that the manifold M^n is orientable and that the subgroup G of $\mathrm{O}(n-i)$ is connected. We orient somehow the manifold M^n and consider a compact closed manifold W^i smoothly embedded in M^n. The submanifold W^i is also assumed to be oriented. Obviously, in this case, the normal $\mathrm{SO}(n-i)$-bundle ν^{n-i} of W^i in M^n is defined. We consider only those submanifolds of M^n for which the subgroup G of $\mathrm{SO}(n-i)$ is the structural group of the normal bundle ν^{n-i}.

Definition 1. A submanifold W^i of the manifold M^n is said to be G-framed if the normal bundle ν^{n-i} of W^i in M^i is endowed with the structure of a G-bundle, and this structure is fixed.

Now, suppose that N^{n+1} is a compact smooth manifold with boundary M^n and V^{i+1} is its compact smoothly embedded submanifold with boundary $W^i = M^n \cap V^{i+1}$ such that V^{i+1} is orthogonal to the boundary M^n of the manifold N^{n+1}. In this case, we can also consider the normal bundle τ^{n-i} of V^{i+1} in N^{n+1}. Hereafter, we assume all manifolds under consideration to be oriented without mentioning it. So we can treat the bundle τ^{n-i} as an $\mathrm{SO}(n-i)$-bundle of planes $\mathbb{R}^{n-i}$ and define G-framed submanifolds with boundary by analogy with Definition 1 (obviously, their boundaries are closed G-framed submanifold of the boundary M^n of the manifold N^{n+1}).

Following Thom, we introduce a relation of L-equivalence on the set of G-framed closed submanifolds of the closed manifold M^n. Consider the direct product $M^n \times I$ of the manifold by the oriented closed interval $I = [0, 1]$. The manifold with boundary $N^{n+1} = M^n \times N$ is oriented in a natural way. Let W_1^i and W_2^i be G-framed closed submanifolds of M^n. The submanifolds $W_1^i \times 0$ and $W_2^i \times 1$ are also oriented in a natural way in the manifolds $M^n \times 0$ and $M^n \times 1$, respectively, and the oriented submanifold $W_i^1 \times 0 \cup W_2^i \times 1$ of the manifold $M^n \times 0 \cup M^n \times 1$ is G-framed.

Definition 2. G-framed submanifolds W_1^i and W_2^i of the manifold M^n are called L-equivalent if there exists a G-framed submanifold V^{i+1} of the manifold $M^n \times I$ with boundary $W_i^1 \times 0 \cup W_2^i \times 1$.

It is easy to verify that the relation of L-equivalence on G-framed submanifolds is symmetric, reflexive, and transitive; therefore, the set of all G-framed submanifolds of the manifold M^n decomposes into classes of L-equivalent submanifolds. We denote the set of these classes by $V^i(M^n, G)$. Note that each element of the set $V^i(M^n, G)$ determines an integer cycle $z_i \in H_i(M^n)$; thus we have a map

$$\lambda_G \colon V^i(M^n, G) \to H_i(M^n).$$

Definition 3. We call a cycle $z_i \in H_i(M^n)$ *G-realizable* if it belongs to the image of the map λ_G.

Obviously, Definition 3 is equivalent to Thom's definition of G-realizability (see [16]).

1.2. **Thom spaces. The classifying properties of Thom spaces.** In Section 1.1, we fixed a connected subgroup of the group $\mathrm{O}(n-i)$. Now, we assume that this subgroup is closed in $\mathrm{O}(n-i)$. Let B_G be the classifying space of the group G. Without loss of generality, we can assume that B_G is a manifold of sufficiently high dimension. Let $\eta(G)$ be the classifying G-bundle of spheres S^{n-i-1}. We denote the total space of the bundle by E_G and the projection by p_G. The cylinder of the projection is a manifold T_G with boundary E_G. The cylinder T_G can be regarded as the space of the classifying G-bundle of closed balls E^{n-i}. Let us contract the boundary E_G of T_G to a point.

We denote the obtained space by M_G and call it the *Thom space of the subgroup G of the group* $\mathrm{O}(n-i)$. The general theory readily implies that, at $n-i > 1$, the space M_G is simply connected (see, e. g., [16] for $G = \mathrm{SO}(n-i)$). As to the cohomology of M_G, Thom showed that there exists a natural isomorphism $\varphi \colon H^k(B_G) \to H^{k+n-i}(M_G)$.

Let $u_G \in H^{n-i}(M_G)$ be the element equal to $\varphi(1)$. The following theorem is valid.

Thom's Theorem. *An integer cycle $z_i \in H^i(M^n)$ is G-realizable if and only if there exists a map $f \colon M^n \to M_G$ such that the cohomology class $f^*(u_G)$ is Poincaré dual to z_i.*

(If the group G is disconnected, then Thom's theorem holds for cycles modulo 2.)

Thom also related the sets $V^i(M^n, \mathrm{SO}(n-i))$ to the sets of homotopy classes $\pi(M^n, M_G)$ of maps $M^n \to M_G$ [16, Theorem IV.6]. Formally replacing SO with G in the proof of Theorem IV.6, we easily obtain the following lemma.

Lemma 1. *The elements of the set $V^i(M_n, G)$ are in one-to-one correspondence with the elements of the set $\pi(M^n.M_G)$.*

At $i < \left[\dfrac{n}{2}\right]$, both sets naturally turn into Abelian groups. The natural one-to-one correspondence between them established by Thom's theorem is then a group isomorphism. This takes place also in the case of $M^n = S$ (for any i). In what follows, we are only interested in this case.

Let us define a pairing of groups

$$V^{i_1}(S^{n_1}, G_1) \otimes V^{i_2}(S^{n_2}, G_2) \to V^{i_1+i_2}(S^{n_1+n_2}, G_1 \times G_2). \tag{1}$$

The group $G_1 \times G_2$ is assumed to be embedded in the group $\mathrm{SO}(n_1+n_2-i_2-i_2)$. The embedding is determined by the decomposition of the Euclidean space $\mathbb{R}^{n_1+n_2-i_1-i_2}$ in the direct product $\mathbb{R}^{n_1-i_1} \times \mathbb{R}^{n_2-i_2}$, which arises naturally in the case under consideration. To define pairing (1), we choose a representative in each of two arbitrary elements $x_1 \in V^{i_1}(S^{n_1}, G_1)$ and $x_2 \in V^{i_2}(S^{n_2}, G_2)$. Obviously, these representatives are G_1- and G_2-framed submanifolds $W^{i_1} \subset S^{n_1}$ and $W^{i_2} \subset S^{n_2}$, respectively.

The direct product is naturally embedded in $S^{n_1+n_2}$ and $G_1\times G_2$-framed, because the normal bundle of this direct product decomposes in the direct product of the normal bundles of the manifolds W^{i_1} and W^{i_2}. We assume that the group $G_1 \times G_2$ is embedded in the group $\mathrm{SO}(n_1 + n_2 - i_1 - i_2)$ precisely as specified above. The following lemma is valid.

Lemma 2. *There exists a homeomorphism*

$$M_{G_1} \times M_{G_2}/M_{G_1} \vee M_{G_2} \to M_{G_1\times G_2} \tag{2}$$

such that the diagram

$$\begin{array}{ccccc} V^{i_1}(S^{n_1}, G_1) & \oplus & V^{i_2}(S^{n_2}, G_2) & \longrightarrow & V^{i_1+i_2}(S^{n_1+n_2}, G_1 \times G_2) \\ \big\downarrow \wr & & \big\downarrow \wr & & \big\downarrow \wr \\ \pi_{n_1}(M_{G_1}) & \oplus & \pi_{n_2}(M_{G_2}) & \longrightarrow & \pi_{n_1+n_2}(M_{G_1\times G_2}) \end{array} \tag{3}$$

is commutative. (Here the top line is pairing (1) and the bottom line is the pairing of homotopy groups determined by homeomorphism (2).)

Proof. To prove the existence of homeomorphism (2), note that the classifying space $B_{G_1\times B_2}$ of the group $G_1 \times G_2$ decomposes in the direct product $B_{G_1} \times B_{G_2}$. The classifying $G_1 \times G_2$-bundle of planes $\mathbb{R}^{n_1+n_2-i_1-i_2}$ also decomposes in the direct product of the classifying bundles of the G_1- and G_2-bundles, respectively. The classifying sphere G-bundle is obtained from the plane bundle by taking the set of all vectors of length 1 in each fiber. The set of all vectors of length not exceeding 1 gives the classifying bundle of closed balls. Now, taking manifolds of sufficiently large dimensions as B_{G_1} and B_{G_2} and recalling the definition of Thom spaces based on the cylinders T_{G_1} and T_{G_2} of the projections of the classifying sphere bundles, we obtain the natural homeomorphism $T_{G_1\times G_2} = T_{G_1} \times T_{G_2}$.

The cylinders T_G are manifolds with boundaries E_G. In constructing a Thom space, the boundary E_G is contracted to a point. Obviously, the homeomorphism $E_{G_1\times G_2} = T_{G_1} \times E_{G_2} \cup E_{G_1} \times T_{G_2}$ holds. This implies the existence of homeomorphism (2). As to the commutativity of diagram (3), it follows from the geometric meaning of the vertical isomorphisms (see the proof of Theorem IV.4 in [16]). This completes the proof of the lemma. $\square$

Thom mentioned that the space M_e is homeomorphic to the sphere S^{n-i} if the unit group e is treated as a subgroup of the group $\mathrm{O}(n - i)$. On the other hand, for any polyhedron K, the polyhedron $K \times S^{n-i}/K \vee S^{n-i}$ is homeomorphic to the iterated suspension $E^{n-i}K$ over the polyhedron K. Suppose that the group G coincides with one of the classical Lie groups $\mathrm{SO}(k)$, $\mathrm{U}(k)$, $\mathrm{SU}(k)$, and $\mathrm{Sp}(k)$. Obviously, the natural group-subgroup embeddings

$$\begin{array}{lll} \mathrm{SO}(k) \times e \subset \mathrm{SO}(k + 1), & \text{where} & e \in \mathrm{SO}(1), \\ \mathrm{U}(k) \times e \subset \mathrm{U}(k + 1), & \text{where} & e \in \mathrm{U}(1), \\ \mathrm{SU}(k) \times e \subset \mathrm{SU}(k + 1), & \text{where} & e \in \mathrm{SU}(1), \\ \mathrm{Sp}(k) \times e \subset \mathrm{Sp}(k + 1), & \text{where} & e \in \mathrm{Sp}(1), \end{array}$$

determine maps

$$\begin{aligned} &EM_{\mathrm{SO}(k)} \to M_{\mathrm{SO}(k+1)}, && E^2M_{\mathrm{U}(k)} \to M_{\mathrm{U}(k+1)}, \\ &E^2M_{\mathrm{SU}(k)} \to M_{\mathrm{SU}(k+1)}, && E^4M_{\mathrm{Sp}(k)} \to M_{\mathrm{Sp}(k+1)}. \end{aligned} \tag{4}$$

(To construct these maps, it is sufficient to apply Lemma 2. Recall that group-subgroup embeddings $G \subset \overline{G} \subset \mathrm{SO}(k)$ give rise to natural maps $M_G \to M_{\overline{G}}$.)

It is easy to show that the maps (4) in stationary dimensions are homotopy equivalences. Indeed, the maps (4) are permutable with the Thom isomorphism $\varphi\colon H^i(B_G) \to H^{k+i}(M_G)$, where $G \subset \mathrm{SO}(k)$. But, as is well known, the maps of the classifying spaces of the groups in (4) (Grassmann manifolds) determine isomorphisms of homology and cohomology groups for small i. The Thom spaces are simply connected; therefore, the maps (4) are homotopy equivalences in stable dimensions.

We denote the groups $V^i(S^n, \mathrm{SO}(n-i))$ with $i < \left[\frac{n}{2}\right]$ by V^i_{SO}, the groups $V^i\Bigl(S^n, \mathrm{U}\Bigl(\frac{n-i}{2}\Bigr)\Bigr)$ with $i < \left[\frac{n}{2}\right]$ such that $n-i$ are even by V^i_{U}, the groups $V^i\Bigl(S^n, \mathrm{SU}\Bigl(\frac{n-i}{2}\Bigr)\Bigr)$ with the same i by V^i_{SU}, and the groups $V^i\Bigl(S^n, \mathrm{Sp}\Bigl(\frac{n-i}{4}\Bigr)\Bigr)$ with $i < \left[\frac{n}{2}\right]$ such that $n-i \equiv 0 \pmod 4$ by V^i_{Sp}. According to the above observation, this notation is valid by virtue of the stabilization of the homotopy groups of Thom spaces.

The pairing (1) directly introduces the structure of graded rings in the direct sums $V_{\mathrm{SO}} = \sum_{i\geq 0} V^i_{\mathrm{SO}}$, $V_{\mathrm{U}} = \sum_{i\geq 0} V^i_{\mathrm{U}}$, $V_{\mathrm{SU}} = \sum_{i\geq 0} V^i_{\mathrm{SU}}$, and $V_{\mathrm{Sp}} = \sum_{i\geq 0} V^i_{\mathrm{Sp}}$. wwwFor these rings, we use the same notations V_{SO}, V_{U}, V_{SU}, and V_{Sp}, respectively.

1.3. **The cohomologies of Thom spaces modulo p for $p > 2$.** Consider the classifying spaces $B_{\mathrm{SO}(2k)}$, $B_{\mathrm{U}(k)}$, $B_{\mathrm{SU}(k)}$, and $B_{\mathrm{Sp}(k)}$. Their cohomology algebras modulo p are well known (see [4]). Namely, $H^*(B_{\mathrm{SO}(2k)}, Z_p)$ is the algebra of polynomials in the Pontryagin classes $p_{4i} \in H^{4i}(B_{\mathrm{SO}(2k)}, Z_p)$, where $0 \leq i < k$, and in the class $w_{2k} \in H^{2i}(B_{\mathrm{SO}(2k)}, Z_p)$. The algebra $H^*(B_{\mathrm{U}(2k)}, Z_p)$ is isomorphic to the algebra of polynomials in generators $c_{2i} \in H^{2i}(B_{\mathrm{U}(k)}, Z_p)$, where $0 \leq i \leq k$; the algebra $H^*(B_{\mathrm{SU}(k)}, Z_p)$ is isomorphic to the algebra of polynomials in generators $c_{2i} \in H^{2i}(B_{\mathrm{SU}(k)}, Z_p)$, where $i \neq 1$; and the algebra $H^*(B_{\mathrm{Sp}(k)}, Z_p)$ is isomorphic to the algebra of polynomials in generators $k_{4i} \in H^{4i}(B_{\mathrm{Sp}(k)}, Z_p)$, where $0 \leq i \leq k$. The generators c_{2i} are Chern classes reduced modulo p, and the generators k_{4i} are symplectic Borel classes (see [5]) reduced modulo p. Thom showed that the algebra $H^*(M_{\mathrm{SO}(2k)}, Z_p)$ is isomorphic to the ideal of the algebra $H^*(B_{\mathrm{SO}(2k)}, Z_p)$ generated by the element w_{2k} (in positive dimensions). Our purpose is to prove similar assertions for the other classical Lie groups. The following lemma is valid.

Lemma 3. *For the classical Lie groups*

$$G = \mathrm{SO}(2k), \quad G = \mathrm{U}(k), \quad G = \mathrm{SU}(k), \quad G = \mathrm{Sp}(k),$$

the homomorphism

$$j^*\colon H^*(M_G, Z_p) \to H^*(B_G, Z_p)$$

generated by the natural embedding $j\colon B_G \subset M_G$ has the following properties:

(a) *j^* is a monomorphism*;

(b) $\operatorname{Im} j^*$ *is equal to the ideal generated by the element* w_{2k} *for the group* $\mathrm{SO}(2k)$*, by the element* c_{2k} *for the groups* $\mathrm{U}(k)$ *and* $\mathrm{SU}(k)$*, and by the element* k_{4k} *for* $\mathrm{Sp}(k)$*.*

Proof. Consider the total space E_G of the classifying G-bundle of spheres S^{2k-1} (if the group coincides with one of the groups $\mathrm{SO}(2k)$, $\mathrm{U}(k)$, and $\mathrm{SU}(k)$) or of spheres S^{4k-1} (if $G = \mathrm{Sp}(k)$).

By the construction of the Thom space M_G (see [2]), the cohomology algebra $H^*(M_G)$ can be identified in positive dimensions with the algebra $H^*(T_G, E_G)$, where T_G is the cylinder of the projection p_G of the classifying sphere bundle. We can write the following exact sequence in cohomology for the pair (T_G, E_G):

$$\dots \longrightarrow H^i(T_G) \xrightarrow{p_G^*} H^i(E_G) \xrightarrow{\delta} H^{i+1}(T_G, E_G) \xrightarrow{j^*} H^{i+1}(T_G) \longrightarrow \dots . \tag{5}$$

The space T_G is homotopy equivalent to the space B_G, and the homomorphisms $H^*(T_G) \to H^*(E_G)$ and $H^*(T_G, E_G) \to H^*(E_G)$ generated by the embeddings coincide with the homomorphisms $p_G^*\colon H^*(B_G) \to H^*(E_G)$ and $j^*\colon H^*(M_G) \to H^*(B_G)$, respectively.

We use the spectral sequence of the classifying sphere G-bundle to study the homomorphism p_G^*. This spectral sequence is well studied in the case under consideration (see [3]). In the spectral sequence of this sphere bundle, the following relations hold:

$$E_2 \approx H^*(S^{2k-1}, Z_p) \otimes H^*(B_G, Z_p)$$

if $G = \mathrm{SO}(2k)$, $G = \mathrm{U}(k)$, or $G = \mathrm{SU}(k)$, and

$$E_2 \approx H^*(S^{4k-1}, Z_p) \otimes H^*(B_G, Z_p)$$

if $G = \mathrm{Sp}(k)$. Let us denote a generator of the group $H^*(S^{2k-1}, Z_p)$ by v^{2k-1} and a generator of the group $H^*(S^{4k-1}, Z_p)$ by v^{4k-1} (we take generating elements being integer generators reduced modulo p). It is well known (see [3]) that $d_{2k}(v^{2k-1} \otimes 1) = 1 \otimes w_{2k}$ for the group $\mathrm{SO}(k)$, $d_{4k}(v^{4k-1} \otimes 1) = 1 \otimes k_{4k}$ for the group $\mathrm{Sp}(k)$, and $d_{2k}(v^{2k-1} \otimes 1) = 1 \otimes c_{2k}$ for the group $\mathrm{U}(k)$ and $\mathrm{SU}(k)$ in the corresponding spectral sequences. Obviously, $E_\infty \approx E_{4k+1}$ for the group $\mathrm{Sp}(k)$ and $E_\infty \approx E_{2k+1}$ for the other Lie groups. Referring to the spectral meaning of the homomorphism p_G^*, we conclude that, in all the cases under consideration, the homomorphism p_G^* is a homomorphism onto the algebra $H^*(E_G, Z_p)$, and its kernel is the required ideal. The homomorphism δ in the exact sequence (5) is trivial. This proves Lemma 3. □

Remark. It is easy to see that the proof of Lemma 3 remains valid at $p = 2$ for all groups except for $\mathrm{SO}(k)$. For this reason, we do not separately examine these cases in the next section.

Now, our main goal is to study the action of the Steenrod powers on the cohomologies of Thom spaces. For this purpose, following [16], we use the Wu generators defined in [4].

Consider symbolic two-dimensional elements $t_1, \dots, t_k$. We impose no relations on these elements. Take the subalgebra of symmetric polynomials in the algebra of polynomials $P(t_1, \dots, t_k)$. We set $c_{2i} = \sum t_1 \circ \dots \circ t_i$ and $p_{4i} = k_{4i} = \sum t_1^2 \circ \dots \circ t_i^2$ for $i < k$, $w_{2k} = c_{2k} = t_1 \circ \dots \circ t_k$, and $k_{4k} = w_{2k}^2$. We can calculate the action of the Steenrod operations on any polynomial by the Cartan formulas. We also set $\beta(t_i) = 0$ $(i = 1, \dots, k)$, where β is the Bockstein homomorphism. Now, to each decomposition ω of a positive integer q in positive integer summands

$q_1, \dots, q_s$ (the decomposition is not ordered) we assign the symmetrized monomial $\sum t_1^{q_1} \circ \dots t_s^{q_s}$. We denote this monomial by v_ω. Similarly, to each decomposition $\overline{\omega}$ of an even positive integer $2q$ in even positive integer summands $2q_1, \dots, 2q_s$ (the decomposition is not ordered) we assign the symmetrized monomial $\sum t_1^{2q_1} \circ \dots \circ t_s^{2q_s} = v_{\overline{\omega}}$. It is known that the algebra of polynomials in generators c_{2i} (treated as symmetric polynomials in the Wu generators) is isomorphic as a module over the Steenrod algebra to the cohomology algebra $H^*(B_{\mathrm{U}(k)}, Z_p)$ (similar isomorphisms hold for the cohomology algebras $H^*(B_{\mathrm{SO}(2k)}, Z_p)$ and $H^*(B_{\mathrm{Sp}(k)}, Z_p)$).

The algebra $H^*(B_{\mathrm{SU}(k)}, Z_p)$ is isomorphic as a module over the Steenrod algebra to the quotient algebra of $H^*(B_{\mathrm{U}(k)}, Z_p)$ modulo the ideal generated by c_2. Relying on Lemma 3 and results of Borel and Serre [4], we shall think of cohomology algebras of Thom spaces as algebras of symmetric polynomials in Wu generators. We say that a decomposition ω (or $\overline{\omega}$) is p-adic if at least one of its terms equals $p^i - 1$. Recall that, in [6], Cartan assigned a certain number, Cartan type, to each cohomology operation (this is the number of occurrences of the Bockstein homomorphisms in the iterated operation).

Lemma 4. *All Steenrod operations of nonzero type in the cohomologies modulo p of the Thom spaces $M_{\mathrm{SO}(2k)}$, $M_{\mathrm{U}(k)}$, $M_{\mathrm{SU}(k)}$ and $M_{\mathrm{Sp}(k)}$ act trivially. For all non-p-adic decompositions $\overline{\omega}$, the Steenrod operations of type zero on the elements w_{2k} and $v_{\overline{\omega}} \circ w_{2k}$ are independent in dimensions smaller than $4k$ and form a Z_p-basis of the algebra $H^*(M_{\mathrm{SO}(2k)}, Z_p)$ in these dimensions. For all non-p-adic decompositions ω, the Steenrod operations of type zero on the elements c_{2k} and $v_\omega \circ c_{2k}$ are independent and form a Z_p-basis of the algebra $H^*(M_{\mathrm{U}(k)}, Z_p)$ in dimensions smaller than $4k$. For all non-p-adic decompositions $\overline{\omega}$, the Steenrod operations of type zero on the elements k_{4k} and $v_{\overline{\omega}} \circ k_{4k}$ are independent and form a Z_p-basis of the algebra $H^*(M_{\mathrm{Sp}(k)}, Z_p)$ in dimensions smaller than $8k$.* (We again remind the reader that $\overline{\omega}$ denote decompositions of even numbers in even summands and ω, of arbitrary integers into integer terms; the polynomials $v_{\overline{\omega}}$ and v_ω are defined above.)

We omit the proof of this lemma; it is a word-for-word repetition of the arguments of Thom (see [11]) and Cartan (see [6]). For the group $\mathrm{SO}(2k)$, the proof is given in [2].

To facilitate formulating the further results, we introduce the graded modules by $H_{\mathrm{SO}}(p)$, $H_{\mathrm{U}}(p)$, $H_{\mathrm{SU}}(p)$, and $H_{\mathrm{Sp}}(p)$ over the Steenrod algebra such that their homogeneous terms are stationary cohomology groups of the corresponding Thom spaces modulo p (here $p = 2$ is allowed). In essence, Lemma 4 is about these modules.

1.4. **Cohomologies of Thom spaces modulo 2.** According to the remark made in Section 3, the same method applies to studying the cohomologies of the Thom spaces $M_{\mathrm{U}(k)}$ and $M_{\mathrm{Sp}(k)}$ modulo 2. Although, the method of [16] does not apply to the groups $\mathrm{SO}(2k)$ and $\mathrm{SU}(k)$, because the cohomologies of the classifying spaces of these groups (as modules over the Steenrod algebra) are not described by subalgebras of the algebra of polynomials in the Wu generators (see Section 1.3).

Consider the iterated Steenrod squares $S^J q$ corresponding to admissible sequences $J = (i_1, \dots, i_s)$ in the sense of Serre (see [14]). For positive integers m and q, we write $J \equiv 0 \pmod q$ if $i_j \equiv 0 \pmod q$ for all j and $J \equiv m \pmod q$ if $i_j \equiv m \pmod q$ for at least one j. We say that *the Steenrod operation $S^J q$ has type m modulo q if $J \equiv m \pmod q$.*

The description of the cohomologies of the classifying spaces $B_{\mathrm{U}(k)}$ and $B_{\mathrm{Sp}(k)}$ given in Section 1.3 applies also to cohomologies modulo 2 (and modulo an arbitrary integer). As in Section 1.3, we introduce a symmetrized monomial v_ω for each decompositions $\omega = (a_1, \dots, a_t)$ of a positive integer in positive integer summands and a symmetrized monomial $v_{\overline{\omega}}$ for each decompositions $\overline{\omega} = (2a_1, \dots, 2a_t)$ of an even positive integer in even positive integer summands. The decompositions are assumed to be unordered.

Lemma 5. *All Steenrod operations of nonzero type modulo* 2 *act trivially in the cohomologies modulo* 2 *of the Thom spaces* $M_{\mathrm{U}(k)}$*,* $M_{\mathrm{SU}(k)}$*, and* $M_{\mathrm{Sp}(k)}$*. The Steenrod operations of nonzero type modulo* 4 *act trivially in the cohomology modulo* 2 *of the Thom space* $M_{\mathrm{Sp}(k)}$*. The Steenrod operations* S^Jq *of type zero modulo* 2 *on all elements* $v_\omega \circ c_{2k}$ *and* c_{2k}*, where the decompositions* ω *contain no terms of the form* $2^t - 2$*, are independent and form a* Z_2*-basis of the algebra* $H^*(M_{\mathrm{U}(k)}, Z_2)$ *in dimensions smaller than* $4k$*. The Steenrod operations* S^Jq *of type zero modulo* 4 *on all elements* $v_{\overline{\omega}} \circ k_{4k}$ *and* k_{4k}*, where the decompositions* $\overline{\omega}$ *contain no terms of the form* $2^t - 4$*, are independent and form a* Z_2*-basis of the algebra* $H^*(M_{\mathrm{Sp}(k)}, Z_2)$ *in dimensions smaller than* $8k$*.*

The proof of this lemma is quite similar to the proof of Lemma 4 and consists in a word-for-word repetition of Thom's argument (see [16, Lemma 11.8, Lemma 11.9, Theorem 11.10]).

How, let us try to study the cohomologies modulo 2 of the Thom spaces $M_{\mathrm{SO}(k)}$ and $M_{\mathrm{SU}(k)}$. As above, we consider only cohomologies of stable dimensions. By analogy with the preceding lemmas, we introduce unordered decompositions $\overline{\omega} = (a_1, \dots, a_t)$ of positive integers $a = \sum a_i$ such that $a \equiv 0 \pmod 4$ in positive terms a_i $(i = 1, \dots, t)$ such that $a_i \equiv 0 \pmod 4$.

Lemma 6. *The algebra* $H^*(M_{\mathrm{SO}(k)}, Z_2)$ *contains a system of elements* $u_{\overline{\omega}} \in H^{k+a}(M_{\mathrm{SO}(k)}, Z_2)$*, where* $a \equiv 0 \pmod 4$ *and* $\overline{\omega} = (a_1, \dots, a_t)$ *is an arbitrary decomposition of* a *in terms* $a_i \equiv 0 \pmod 4$*, and a system of elements* $x_l \in H^{k+i_l}(M_{\mathrm{SO}(k)}, Z_2)$ *such that*

(a) *all Steenrod operations* S^Jq *on the elements* x_l *are independent in dimensions smaller than* $2k$;

(b) *all Steenrod operations* S^Jq *on the elements* $u_{\overline{\omega}}$ *and* $w_k \in H^k(M_{\mathrm{SO}(k)}, Z_2)$ *are independent of the operations on the elements* x_l *if* $J = (i_1, \dots, i_s)$*, where* $i_s > 1$*, and the dimensions of* $S^Jq(u_{\overline{\omega}})$ *and* $S^Jq(w_k)$ *are smaller than* $2k$;

(c) *the elements* $S^Jq(u_{\overline{\omega}})$ *and* $S^Jq(w_k)$ *are zero if* $J = (i_1, \dots, i_s)$*, where* $i_s = 1$;

(d) *all elements* $S^Jq(u_{\overline{\omega}})$*,* $S^Jq(w_k)$*, and* $S^Jq(x_l)$ *form a* Z_2*-basis of the algebra* $H^*(M_{\mathrm{SO}(k)}, Z_2)$ *in dimensions smaller than* $2k$*.*

Before proceeding to prove Lemma 6, note that this lemma makes it easy to study the action of the Steenrod squares in the cohomology modulo 2 of the space $M_{\mathrm{SU}(k)}$. Indeed, recall the description of the algebras $H^*(B_{\mathrm{O}(k)}, Z_2)$ and $H^*(B_{\mathrm{SO}(k)}, Z_2)$ in terms of the one-dimensional Wu generators $y_1, \dots, y_n$. We set $w_i = \sum y_1 \circ \dots \circ y_i$, where $i \le k$. Obviously, $w_k = y_1 \circ \dots \circ y_i$, and all Steenrod operations on the elements w_i are calculated by the Cartan formulas. Wu proved that the algebra $H^*(B_{\mathrm{O}(k)}, Z_2)$ is isomorphic as a module over the Steenrod algebra to the algebra

$P(w_1, \dots, w_k)$ and that the algebra $H^*(B_{\mathrm{SO}(k)}, Z_2)$ is isomorphic as a module over the Steenrod algebra to the quotient algebra of the algebra $P(w_1, \dots, w_k)$ modulo the ideal generated by w_1. An analogy with the description of the algebras $H^*(B_{\mathrm{U}(k)}, Z_2)$ and $H^*(B_{\mathrm{SU}(k)}, Z_2)$ in terms of the two-dimensional Wu generators $t_1, \dots, t_k$ (see Section 1.3) is evident.

In [1], Adams defined an endomorphism

$$h \colon A \to A \tag{6}$$

of the Steenrod algebra $A = A_2$ over the field Z_2 such that $h(S^{2i}q) = S^i q$ and $h(S^{2i+1}q) = 0$.

Consider a dimension-halving isomorphism $\mu \colon P(t_1, \dots, t_k) \to P(y_1, \dots, y_k)$ of graded algebras over the field Z_2. Obviously,

$$\mu(S^J q(x)) = h(S^J q)(\mu(x)), \tag{7}$$

where $x \in P(t_1, \dots, t_k)$. The isomorphism μ induces isomorphisms

$$\mu_1 \colon H^*(B_{\mathrm{U}(k)}, Z_2) \to H^*(B_{\mathrm{O}(k)}, Z_2),$$
$$\mu_2 \colon H^*(B_{\mathrm{SU}(k)}, Z_2) \to H^*(B_{\mathrm{SO}(k)}, Z_2),$$

which also have property (7), and isomorphisms

$$\begin{aligned} \lambda_1 \colon H^*(M_{\mathrm{U}(k)}, Z_2) \to H^*(M_{\mathrm{O}(k)}, Z_2), \\ \lambda_2 \colon H^*(M_{\mathrm{SU}(k)}, Z_2) \to H^*(M_{\mathrm{SO}(k)}, Z_2), \end{aligned} \tag{8}$$

which halve the dimension and have property (7). Obviously, $\lambda_1(\mathrm{U}_{\mathrm{U}(k)}) = \mathrm{U}_{\mathrm{O}(k)}$ and $\lambda_2(\mathrm{U}_{\mathrm{SU}(k)}) = \mathrm{U}_{\mathrm{SO}(k)}$. Thus we obtain the following result.

Corollary 1. *There exists a dimension-halving isomorphism*

$$\lambda_2 \colon H^*(M_{\mathrm{SU}(k)}, Z_2) \to H^*(M_{\mathrm{SO}(k)}, Z_2)$$

such that $\lambda_2(\mathrm{U}_{\mathrm{SU}(k)}) = \mathrm{U}_{\mathrm{SO}(k)}$ *and* $h(S^J q \lambda_2(x)) = \lambda_2(S^J q(x))$ *for all* $x \in H^J(M_{\mathrm{SU}(k)}, Z_2)$ *and* $j < 2k$.

Now, we proceed to prove Lemma 6. Rokhlin's results [12] imply that the kernel of the map $i_* \colon \pi_m(M_{\mathrm{SO}(k)}) \to \pi_m(M_{\mathrm{O}(k)})$ generated by the embedding $i \colon \mathrm{SO}(k) \subset \mathrm{O}(k)$ consists of all elements divisible by 2 for $m < 2k-1$. Let $\pi_m^{(2)}(M_{\mathrm{SO}(k)})$ denote the quotient group of $\pi_m(M_{\mathrm{SO}(k)})$ by the subgroup consisting of all elements of odd order (it is proved in [2] that the groups $\pi_m(M_{\mathrm{SO}(k)})$ contain no elements of odd order, but we do not rely on the results of this paper). The results obtained in [16] imply that we can choose systems of generators $x_i^{(m)}$ in the groups $\pi_m^{(2)}(M_{\mathrm{SO}(k)})$ and $y_j^{(m)}$ in the groups $\pi_m(M_{O(k)})$ such that the map i_* takes the set $\{x_i^{(m)}\}$ to a subset of the set $\{y_j^{(m)}\}$. Thom proved that the space $M_{\mathrm{O}(k)}$ (in stable dimensions) can be assumed homotopy equivalent to a direct product of Eilenberg–MacLane complexes (see [16, 11.6–11.10]). Thus there is a map i_1 of the space $M_{\mathrm{SO}(k)}$ to a direct product Π of Eilenberg–MacLane complexes of type $K(Z_2, n_j)$, whose homotopy groups have generators being in one-to-one correspondence with the $x_i^{(m)}$; moreover, the map i_{1*} takes each element $x_i^{(m)}$ to the corresponding generator of the product. We can assume that the fundamental classes $u_i^{(m)}$ of the factors in this direct product are specified by the equalities $(u_i^{(m)}, i_1 x_i^{(m)}) = 1$ and $(u_i^{(m)}, x_{i'}^{(m')}) = 0$ if $i \neq i'$ or $m \neq m'$.

Let Π_m denote the subproduct of the product Π of Eilenberg–MacLane complexes determined by elements of homotopy groups of dimensions larger than or equal to m. We denote the projection of the map i_1 to Π_m by $i_1^{(m)}$. Consider the well-known Serre fibrations $p_m\colon \widehat{M}_{\mathrm{SO}(k)} \xrightarrow{M_{(m)}} M^{(m)}$ (see [14]), where $\widehat{M}_{\mathrm{SO}(k)}$ denotes a space homotopy equivalent to $M_{\mathrm{SO}(k)}$ and $M^{(m)}$ is the space obtained by pasting all homotopy groups starting with the mth one. The fibers of these fibrations are m-killing spaces for $M_{\mathrm{SO}(k)}$. Let us denote them by $M_{(m)}$. For generators of the groups $\pi_m^{(2)}(M_m)$, we use the same notation $x_i^{(m)}$. The group $H^m(M_{(m)}Z_2)$ is generated by the elements $v_i^{(m)}$ determined by

$$(v_i^{(m)}, x_i^{(m)}) = 0, \quad (v_i^{(m)}, x_{i'}^{(m)}) = 1, \qquad i' \neq i.$$

We treat the space Π_m as a fiber space whose base consists of one point. The map $i_1^{(m)}$ induces a map $\hat{i}_1^{(m)}\colon M_{(m)} \to \Pi_m$.

Obviously, $\hat{i}_1^{(m)*}(u_i^{(m)}) = v_i^{(m)}$. This immediately implies that the transgression for the Serre fibration $p_m\colon \widehat{M}_{\mathrm{SO}(k)} \xrightarrow{M_{(m)}} M^{(m)}$ is trivial at all elements $v_i^{(m)}$, because the map $i_1^{(m)}$ can be regarded as a map of this fiber space to the trivial one specified above. This immediately implies that all factors of the space $M_{\mathrm{SO}(k)}$ in the sense of Postnikov (see [11]) reduced modulo 2 are trivial. Now, the assertion of Lemma 6 follows from Thom's Theorem IV.15 from [16] and the observation that the cohomology groups $H_i(M_{\mathrm{SO}(k)})$ contain no elements of order 4 if $i < 2k - 1$. We mean the cohomology with coefficients in z.

1.5. **Diagonal Homomorphisms.** Let K be an arbitrary polyhedron. By $H^+(K, Z_p)$ we denote its module over the Steenrod algebra $A = A_p$ whose homogeneous terms are cohomology groups of positive dimensions. Let K_1 and K_2 be polyhedra. There is the well-known isomorphism

$$H^+(K_1 \times K_2/K_1 \vee K_2, Z_p) \approx H^+(K_1, Z_p) \otimes H^+(K_2, Z_p).$$

This isomorphism is an isomorphism of A-modules (which makes sense because A is a Hopf algebra). Therefore, the homeomorphism (2) mentioned in Lemma 2 determines the following diagonal homomorphisms generated by the above-mentioned group-subgroup embeddings $\mathrm{SO}(m)\times\mathrm{SO}(n) \subset \mathrm{SO}(m+n)$, $\mathrm{U}(m)\times\mathrm{U}(n) \subset \mathrm{U}(m+n)$, $\mathrm{SU}(m) \times \mathrm{SU}(n) \subset \mathrm{SU}(m+n)$, and $\mathrm{Sp}(m) \times \mathrm{Sp}(n) \subset \mathrm{Sp}(n+m)$:

$$\begin{aligned} H^+(M_{\mathrm{SO}(m+n)}) &\to H^+(M_{\mathrm{SO}(m)}) \quad \otimes H^+(M_{\mathrm{SO}(n)}), \\ H^+(M_{\mathrm{U}(m+n)}) &\to H^+(M_{\mathrm{U}(m)}) \quad \otimes H^+(M_{\mathrm{U}(n)}), \\ H^+(M_{\mathrm{SU}(m+n)}) &\to H^+(M_{\mathrm{SU}(m)}) \quad \otimes H^+(M_{\mathrm{SU}(n)}), \\ H^+(M_{\mathrm{Sp}(m+n)}) &\to H^+(M_{\mathrm{Sp}(m)}) \quad \otimes H^+(M_{\mathrm{Sp}(n)}). \end{aligned} \tag{9}$$

We denote all these homomorphisms by $\Delta_{m,n}$. For the modules $H_{\mathrm{SO}}(p)$, $H_{\mathrm{U}}(p)$, $H_{\mathrm{SU}}(p)$, and $H_{\mathrm{Sp}}(p)$ (see Section 1.3), the homomorphisms $\Delta_{m,n}$ determine the following homomorphisms Δ:

$$\begin{aligned} H_{\mathrm{SO}}(p) &\to H_{\mathrm{SO}}(p) \quad \otimes H_{\mathrm{SO}}(p), \\ H_{\mathrm{U}}(p) &\to H_{\mathrm{U}}(p) \quad \otimes H_{\mathrm{U}}(p), \\ H_{\mathrm{SU}}(p) &\to H_{\mathrm{SU}}(p) \quad \otimes H_{\mathrm{SU}}(p), \\ H_{\mathrm{Sp}}(p) &\to H_{\mathrm{Sp}}(p) \quad \otimes H_{\mathrm{Sp}}(p). \end{aligned} \tag{10}$$

The goal of this section is to calculate homomorphisms (10).

The following lemma is valid.

Lemma 7. *At the generators u_ω and $u_{\overline{\omega}}$ of the modules $H_{\mathrm{SO}}(p)$, $H_{\mathrm{U}}(p)$, and $H_{\mathrm{Sp}}(p)$, the homomorphisms Δ have the forms*

$$\begin{aligned}\Delta(u_\omega) &= \sum_{\substack{(\omega_1,\omega_2)=\omega,\\ \omega_1\neq\omega_2}} [u_{\omega_1}\otimes u_{\omega_2} + u_{\omega_2}\otimes u_{\omega_1}] + \sum_{(\omega_1,\omega_1)=\omega} u_{\omega_1}\otimes u_{\omega_1},\\ \Delta(u_{\overline{\omega}}) &= \sum_{\substack{(\overline{\omega}_1,\overline{\omega}_2)=\overline{\omega},\\ \overline{\omega}_1\neq\overline{\omega}_2}} [u_{\overline{\omega}_1}\otimes u_{\overline{\omega}_2} + u_{\overline{\omega}_2}\otimes u_{\overline{\omega}_1}] + \sum_{(\overline{\omega}_1,\overline{\omega}_1)=\overline{\omega}} u_{\overline{\omega}_1}\otimes u_{\overline{\omega}_1}\end{aligned} \tag{11}$$

for all $p \geq 2$.

Note that, in (10), ω_1 ($\overline{\omega}_1$) is allowed to be the decomposition into the empty set of terms. The generator u_{ω_1} ($u_{\overline{\omega}_1}$) for the empty decomposition ω_1 ($\overline{\omega}_1$) corresponds to the element w_{2k}, c_{2k}, or k_{4k}, as in the preceding sections ($u_{\overline{\omega}}$ denotes the module generator corresponding to the product $v_{\overline{\omega}} \circ w_{2k}$ or $v_{\overline{\omega}} \circ k_{4k}$ from Lemma 4, and u_ω denotes the generator corresponding to the product $v_\omega \circ c_{2k}$ and the generators mentioned in Lemma 6).

Proof. First, consider the modules $H_{\mathrm{SO}}(p)$ with $p > 2$, $H_{\mathrm{U}}(p)$ with $p \geq 2$, and $H_{\mathrm{Sp}}(p)$ with $p \geq 2$. Let us return to the description of the cohomologies of Thom spaces in terms of ideals in the cohomologies of the classifying spaces (see Lemma 3) and the Wu generators. Our immediate goal is to calculate the homomorphisms (9) by using the Whitney formulas for the Pontryagin and Chern classes and for the symplectic Borel classes.

Let m and n be sufficiently large integers, and let $x_1, \dots, x_m, y_1, \dots, y_n$ be the symbolic two-dimensional Wu generators. In the algebra $P(x_1, \dots, x_m, y_1, \dots, y_n)$, consider the elementary symmetric polynomials in the generators $x_1, \dots, x_m, y_1, \dots, y_n$ and $x_1^2, \dots, x_m^2, y_1^2, \dots, y_n^2$. The topological meaning of these polynomials is specified above. In the algebras $P(x_1, \dots, x_m)$ and $P(y_1, \dots, y_n)$ over the field Z_p, we consider similar elementary symmetric polynomials. Note that the homomorphisms $\Delta_{m,n}$ must satisfy the Whitney formulas, and these formulas uniquely determine the homomorphisms (9). Let us formally set

$$\Delta_{m,n}(x_i) = x_i \otimes 1, \quad \Delta_{m,n}(y_j) = 1 \otimes y_j \tag{12}$$

for all $i \leq m$ and $j \leq n$. (We treat the set of elements x_i as the Wu generators for the algebras

$$H^*(B_{\mathrm{SO}(2m)}, Z_p), \quad H^*(B_{\mathrm{U}(m)}, Z_p), \quad \text{and} \quad H^*(B_{\mathrm{Sp}(m)}, Z_p)$$

and the set of elements y_j as the Wu generators for the algebras

$$H^*(B_{\mathrm{SO}(2n)}, Z_p), \quad H^*(B_{\mathrm{U}(n)}, Z_p), \quad \text{and} \quad H^*(B_{\mathrm{Sp}(n)}, Z_p).)$$

Applying (12) to the elementary symmetric polynomials, we observe that (12) implies the fulfillment of the Whitney formulas for all the characteristic classes involved. For this reason, the homomorphisms $\Delta_{m,n}$ defined formally by (11) coincide with the "geometric" homomorphisms $\Delta_{m,n}$ on the symmetric polynomials. Now, let us apply (12) to the polynomials $v_{\overline{\omega}} \circ w_{2k}$, $v_\omega \circ c_{2k}$, $v_{\overline{\omega}} \circ k_{4k}$, w_{2k}, c_{2k}, and k_{4k}. It is easy to see that we immediately obtain the required result. To prove formulas (11) for the module $H_{\mathrm{SO}}(2)$, note that the Pontryagin classes satisfy the

Whitney formulas without taking into account 2-torsion. Let the generators u_ω from Lemma 6 be the polynomials reduced modulo 2 in the Pontryagin classes and the class $w_{2(m+n)}$ which correspond to the expression of the symmetrized monomial

$$\sum x_1^{a_1+1} \circ \cdots \circ x_k^{a_s+1} \circ \ldots x_m \circ y_1 \circ \ldots y_n,$$

where $\omega = (a_1, \ldots, a_s)$ is an arbitrary decomposition of an even number in even summands, in terms of elementary symmetric polynomials in the squared generators x_1^2 and y_j^2 and of the polynomial $w_{2(m+n)} = x_1 \circ \cdots \circ x_m \circ y_1 \circ \cdots \circ y_n$. The same considerations as above imply that formula (9) holds for these elements up to certain elements belonging to the image of the operation Sq^1 (in the integer cohomology without taking into account 2-torsion). This proves the lemma. □

Let us summarize the results obtained in this chapter. In Section 1.2, we associated the sequences of groups $\{G_i = \mathrm{SO}(i)\}$, $\{G_i = \mathrm{U}(i)\}$, $\{G_i = \mathrm{SU}(i)\}$, and $\{G_i = \mathrm{Sp}(i)\}$ with the graded rings V_{SO}, V_{U}, V_{SU}, and V_{Sp}. We shall call these rings the *inner homology rings*. On the other hand, in Sections 1.3–1.5, we associated the same sequences of groups with graded modules over the Steenrod algebra $H_{\mathrm{SO}}(p)$, $H_{\mathrm{U}}(p)$, $H_{\mathrm{SU}}(p)$, and $H_{\mathrm{Sp}}(p)$ for all primes $p \geq 2$. These module were calculated in Sections 1.3–1.4. In Section 1.5, they were associated with the diagonal maps (10).

The purpose of the next chapter is to calculate the inner homology rings by the spectral method of Adams.

2. Inner Homology Rings

As mentioned, this chapter is concerned with calculating inner homology rings. The basic theorems of this chapter are stated in Sections 2.4–2.5. The first three sections study extensions of modules over the Steenrod algebra.

2.1. Modules with One Generator[4]. Let $A = \sum_{i\geq 0} A^{(i)}$ be a graded associative algebra over the field Z_p. As usual, we assume that $A^{(0)} = Z_p$ all $A^{(i)}$ are finite-dimensional linear spaces over the initial field. Consider a graded A-module M with one generator u of dimension 0 in which some homogeneous Z_p-basis $\{x_i^{(m)}\}$, where $x_i^{(m)} \in M^{(m)}$, is given. We denote the 1-generated free A-module by the same symbol A (if its generator has dimension 0) and identify it with the algebra A. We denote the generator by 1 and identify it with the identity element of the algebra A. Obviously, a canonical map $\varepsilon\colon A \to M$ of A-modules such that $\varepsilon(1) = u$ is defined. As usual, $\overline{A}$ denotes the ideal of the algebra A consisting of elements of positive dimension.

Suppose that B is a graded subalgebra of A, $B = \sum_{i\geq 0} B^{(i)}$, $B^{(0)} = Z_p$, and $B^{(i)} = B \cap A^{(i)}$. Let M_B denote a 1-generated module equal to $A/A \circ \overline{B}$. As above, we denote its Z_p-basis by $x_i^{(m)}$ and the generator by $u = \varepsilon(1)$, where ε is the natural homomorphism $A \to A/A \circ \overline{B}$. Let $\{y_j^{(k)}\}$ be an arbitrary homogeneous Z_p-basis of the algebra B; in each set $\varepsilon^{-1}(x_i^{(k)})$, we choose and arbitrary element $z_i^{(m)} \in \varepsilon^{-1}(x_i^{(m)})$. The elements $z_i^{(m)}$ are assumed to be homogeneous of the same dimension as $x_i^{(m)}$.

[4]We consider only left A-modules

Definition 4. The subalgebra B of the algebra A is said to be *special* if all products $z_i^{(m)} \circ y_i^{(k)}$ form a homogeneous Z_p-basis of the algebra A and are independent.

As usual, we endow the field Z_p with the trivial structure of an A-module.

Lemma 8. *If the subalgebra B of the algebra A is special, then the following isomorphism holds:*

$$\mathrm{Ext}_A^{s,t}(M_B, Z_p) \approx \mathrm{Ext}_B^{s,t}(Z_p, Z_p). \tag{13}$$

Proof. Let $C_B(Z_p)$ denote the B-free standard complex of the algebra B (see [8]). To prove (13), we construct an A-free acyclic complex $C_A(M_B)$ such that the differential isomorphism

$$\mathrm{Hom}_B^{s,t}(C_B(Z_p), Z_p) \approx \mathrm{Hom}_A^{s,t}(C_A(M_B), Z_p) \tag{14}$$

holds for each pair s, t. Applying the isomorphism (14) to all pairs s, t commuting with the differential, we obtain the isomorphism (13). In constructing the complex $C_A(M_B)$, we shall employ the Z_p-bases of the algebras and of the module M_B specified in Definition 4 of this section; we shall use the same notation.

As $C_A^0(M_B) = \sum_t C_A^{0,t}(M_B)$ we take the free module A, and as $\varepsilon\colon A \to M_B$ we take the map defined at the beginning of this section. Obviously, all elements $y_j^{(k)}$ with $k_q > 0$ are generators of the A-module $\mathrm{Ker}\,\varepsilon \subset C_A^0(M_B)$. Definition 4 directly implies that all A-relations between the generators $y_j^{(k)}$ of the module $\mathrm{Ker}\,\varepsilon$ follow from the commutative relations in the subalgebra B. We shall construct the complex $C_A(M_B) = \Sigma_{s,t} C_A^{s,t}(M_B)$ by induction on s. Suppose that

(a) the complex $C_A(M_B)$ is constructed for all $s \le n$;

(b) generators of the A-module $\mathrm{Ker}\, d_{n-1}\colon C_A^n(M_B) \to C_A^{n-1}(M_B)$ are in one-to-one correspondence with the sequences $(y_{j_1}^{(k_1)}, \dots, y_{j_{n+1}}^{(k_{n+1})})$ of homogeneous elements of positive dimension; we denote these generators by $u(y_{j_1}^{(k_1)}, \dots, y_{j_{n+1}}^{(k_{n+1})})$; for each element $y_i = \sum q_l y_{j_{i,l}}^{(k_{i,l})}$, $u(y_{j_1}^{(k_1)}, \dots, y_i, \dots, y_{j_{n+1}}^{(k_{n+1})})$ denotes the linear combination $\sum_l q_l u(y_{j_1}^{(k_1)}, \dots, y_{j_{i,l}}^{(k_{i,l})}, \dots, y_{j_{n+1}}^{(k_{n+1})})$ of generators of the kernel $\mathrm{Ker}\, d_{n-1}$ (for any $1 \le i \le n+1$);

(c) the generators of the A-module $\mathrm{Ker}\, d_{n-1}$ satisfy the relations

$$\begin{aligned} y \circ u(y_{j_1}^{(k_1)}, \dots, y_{j_{n+1}}^{(k_{n+1})}) &= u(y \circ y_{j_1}^{(k_1)}, \dots, y_{j_{n+1}}^{(k_{n+1})}) \\ &\quad + \sum_{i=1}^{n} (-1)^i u(y, y_{j_1}^{(k_1)}, \dots, y_{j_i}^{(k_i)} \circ y_{j_{i+1}}^{(k_{i+1})}, \dots, y_{j_{n+1}}^{(k_{n+1})}), \end{aligned}$$

where $y \in \overline{B}$;

(d) all relations are linear combinations of the right-hand sides of the relations specified in (c) multiplied on the left by elements of the form $z_i^{(m)} \circ y_j^{(k)}$ (and of the trivial relations);

(e) the dimension of a generator $u(y_{j_1}^{(k_1)}, \dots, y_{j_{i+1}}^{(k_{i+1})})$ is equal to the sum of the dimensions of the elements $y_{j_i}^{(k_i)}$ $(i = 1, \dots, n+1)$.

All these assertions have been proved above for $n = 0$. Now, let us construct the module $C_A^{n+1}(M_B) = \sum_t C_A^{n+1,t}(M_B)$ and the map $d_n\colon C_A^{n+1}(M_B) \to C_A^n(M_B)$. We choose A-generators of the free module $C_A^{n+1}(M_B)$ is such a way that they be in one-to-one correspondence with the elements $u(y_{j_1}^{(k_1)}, \dots, y_{j_{n+1}}^{(k_{n+1})})$. We denote these free generators by $v(y_{j_1}^{(k_1)}, \dots, y_{j_{n+1}}^{(k_{n+1})})$ and define the dimensions of the generators $v(y_{j_1}^{(k_1)}, \dots, y_{j_{n+1}}^{(k_{n+1})})$ and $u(y_{j_1}^{(k_1)}, \dots, y_{j_{n+1}}^{(k_{n+1})})$ to be $\sum k_i$. We set

$$d_{n+1}(v(y_{j_1}^{(k_1)}, \dots, y_{j_{n+1}}^{(k_{n+1})}) = u(y_{j_1}^{(k_1)}, \dots, y_{j_{n+1}}^{(k_{n+1})}).$$

Let us show that the kernel $\operatorname{Ker} d_{n+1}$ satisfies requirements (b)–(e). We set

$$u(y_{j_1}^{(k_1)}, \dots, y_{j_{n+2}}^{(k_{n+2})}) = y_{j_1}^{(k_1)} \circ v(y_{j_2}^{(k_2)}, \dots, y_{j_{n+2}}^{(k_{n+2})}) \\ - \sum_{i=1}^{n+1} (-1)^i v(y_{j_1}^{(k_1)}, \dots, y_{j_i}^{(k_i)} \circ y_{j_{i+1}}^{(k_{i+1})}, \dots, y_{j_{n+2}}^{(k_{n+2})}).$$

Assertions (c) and (e) are easily verified by a direct calculation. To prove (d), we use properties of the bases of the algebra A. Composing an arbitrary relation (i. e., a zero linear combination of elements of the form $z_j^{(m)} \circ y_q^{(l)} \circ u(y_{j_1}^{(k_1)}, \dots, y_{j_{n+2}}^{(k_{n+2})})$), we see that (d) follows readily from properties of the bases and the induction hypotheses.

Obviously, the constructed resolvent $C_A(M_B)$ satisfies (14) by the definition of the standard complex $C_B(Z_p)$ of the algebra B. This completes the proof of the lemma. □

Now, suppose that A is a Hopf algebra and B is a special subalgebra invariant under the diagonal map $\psi\colon A \to A \otimes A$. Then the A-module M_B admits the diagonal map

$$\widetilde{\psi}\colon M_B \to M_B \otimes M_B$$

induced by ψ, which is a homomorphism of A-modules. The homomorphism $\widetilde{\psi}$ can also be regarded as a homomorphism of the A-module M_B to the $A \otimes A$-module $M_B \otimes M_B$ with the property $\widetilde{\psi}(a \circ x) = \psi(a) \circ \widetilde{\psi}(x)$ for $a \in A$ and $x \in M_B$. The homomorphism $\widetilde{\psi}$ endows the direct sum

$$\operatorname{Ext}_A(M_B, Z_p) = \sum_{s,t} \operatorname{Ext}_A^{s,t}(M_B, Z_p)$$

with the structure of a bigraded algebra over the field Z_p.

Lemma 9. *If A is a Hopf algebra and B is its special subalgebra invariant under the diagonal map, then the isomorphism* (13) *is an isomorphism of graded algebras.*

Lemma 9 follows from the commutativity of the diagram

$$\begin{array}{ccc} \operatorname{Ext}_{B\otimes B}(Z_p, Z_p) & \approx & \operatorname{Ext}_{A\otimes A}(M_{B\otimes B}, Z_p)^5 \\ \big\downarrow{\scriptstyle \psi^*} & & \big\downarrow{\scriptstyle \widetilde{\psi}^*} \\ \operatorname{Ext}_B(Z_p, Z_p) & \approx & \operatorname{Ext}_A(M_B, Z_p). \end{array}$$

[5]It remains to note that $B \otimes B$ is special in $A \otimes A$ and the $A \otimes A$-modules $M_B \otimes M_B$ and $M_{B\otimes B}$ are canonically isomorphic.

2.2. **Modules over the Steenrod Algebra. The Case of a Prime $p > 2$.** Adams [1] defined families of elements e'_r, where $r \geq 0$, and $e_{r,k}$, where $r \geq 1$ and $k \geq 0$, in the Steenrod algebra over the field Z_p which have the following very interesting properties:

(a) $e'_r \in A^{(2p^r-1)}$, $e_{r,k} \in A^{(k(2p^r-1))}$, $e_{r,0} = 1$;
(b) $\psi(e'_r) = e'_r \otimes 1 + 1 \otimes e'_r$, $\psi(e_{r,k}) = \sum_{i+j=k} e_{r,i} \otimes e_{r,j}$, where ψ is the Hopf homomorphism of the Steenrod algebra;
(c) if we arbitrarily order the family consisting of the elements e'_r and of the functions $f_r(k) = e_{r,k}$ in a positive integer argument, then the set of monomials being products of elements e'_r and e_{r,k_r} under the substitution of arbitrary arguments k_r (in this ordering) forms a basis of the algebra A;
(d) the elements $e_{r,k}$ have Cartan type zero (see [6]), and the elements e'_r have Cartan type 1;
(e) the elements e'_r anticommute.

In what follows, we rely heavily on the properties of the Adams elements in the Steenrod algebra. Let us return to the modules $H_{\mathrm{SO}}(p)$, $H_{\mathrm{U}}(p)$, and $H_{\mathrm{Sp}}(p)$. Formulas (10) from Section 1.5 of Chapter 1 introduce diagonal homomorphisms in these modules. Thus the bigraded groups

$$\mathrm{Ext}_A(H_{\mathrm{SO}}(p), Z_p) = \sum_{s,t} \mathrm{Ext}_A^{s,t}(H_{\mathrm{SO}}(p), Z_p),$$
$$\mathrm{Ext}_A(H_{\mathrm{U}}(p), Z_p) = \sum_{s,t} \mathrm{Ext}_A^{s,t}(H_{\mathrm{U}}(p), Z_p),$$
$$\mathrm{Ext}_A(H_{\mathrm{Sp}}(p), Z_p) = \sum_{s,t} \mathrm{Ext}_A^{s,t}(H_{\mathrm{Sp}}(p), Z_p)$$

turn naturally into bigraded algebras. Next, recall that, in Section 1.3 of Chapter 1, we introduced decompositions ω and $\overline{\omega}$ of certain numbers in summands of the same form; we call the doubled sum of these summands the *dimension of the decomposition* ω ($\overline{\omega}$) and denote it by $R(\omega)$ ($R(\overline{\omega})$). We also introduced the notion of p-adic decompositions ω and $\overline{\omega}$.

Theorem 1. *The algebras* $\mathrm{Ext}_A(H_{\mathrm{SO}}(p), Z_p)$ *and* $\mathrm{Ext}_A(H_{\mathrm{Sp}}(p), Z_p)$ *are isomorphic, and they are the algebras of polynomials in the generators*

$$\begin{gathered} 1 \in \mathrm{Ext}_A^{0,0}(H_{\mathrm{Sp}}(p), Z_p), \quad z_{4k} \in \mathrm{Ext}_A^{0,4k}(H_{\mathrm{Sp}}(p), Z_p), \qquad 2k \neq p^i - 1, \\ h'_r \in \mathrm{Ext}_A^{1,2p^r-1}(H_{\mathrm{Sp}}(p), Z_p), \qquad r \geq 0. \end{gathered} \tag{15}$$

The algebra $\mathrm{Ext}_A(H_{\mathrm{U}}(p), Z_p)$ *is the algebra of polynomials in the generators*

$$\begin{gathered} 1 \in \mathrm{Ext}_A^{0,0}(H_{\mathrm{U}}(p), Z_p), \quad z_{2k} \in \mathrm{Ext}_A^{0,2k}(H_{\mathrm{U}}(p), Z_p), \qquad k \neq p^i - 1, \\ h'_r \in \mathrm{Ext}_A^{1,2p^r-1}(H_{\mathrm{U}}(p), Z_p), \qquad r \geq 0. \end{gathered} \tag{16}$$

Proof. Let M_β denote the module over the Steenrod algebra with one generator u of dimension 0 where the only nontrivial relation is the identity $\beta(x) = 0$ for all $x \in M_\beta$.

Obviously, the module M_β admits a diagonal map $\Delta\colon M_\beta \to M_\beta \otimes M_\beta$, and the group $\mathrm{Ext}_A(M_\beta, Z_p) = \sum_{s,t} \mathrm{Ext}_A^{s,t}(M_\beta, Z_p)$ is an algebra.

Lemma 10. *The algebra* $\operatorname{Ext}_A(M_\beta, Z_p)$ *is the algebra of polynomials in the generators*

$$1 \in \operatorname{Ext}_A^{0,0}(M_\beta, Z_p), \quad h'_r \in \operatorname{Ext}_A^{0,2p^r-1}(M_\beta, Z_p), \qquad r \geq 0.$$

Proof. To prove the lemma, we use the properties (a)–(e) of the Adams elements and Lemma 9 proved in Section 2.1 of Chapter 2 (see above). As the subalgebra B of A we take the subalgebra generated by the elements e'_r, where $r \geq 0$, and by $e_{r,0} = 1$. We order the Adams elements in such a way that all elements e'_r precede all elements $e_{r,k}$ and take the basis of the Steenrod algebra determined by the property (c) of the Adams elements with this ordering. Obviously, the algebra B satisfies all assumptions of Lemma 9 from Section 2.1 of Chapter 2. The algebra B is the exterior algebra in the generators $e'_r \in B^{(2p^r-1)}$, and therefore, its cohomology algebra is a polynomial algebra. On the other hand, (d) implies that, in this case, $M_B = M_\beta$, which proves the lemma. □

According to Lemmas 4 and 7 from Chapter 1, the modules $H_{\mathrm{SO}}(p)$, $H_{\mathrm{U}}(p)$, and $H_{\mathrm{Sp}}(p)$ are direct sums of modules of the M_β type with the only difference that their generators u_ω and $u_{\overline{\omega}}$, except one, have nonzero dimension equal to $R(\omega)$ or $R(\overline{\omega})$. Let $z_\omega \in \operatorname{Ext}_A^{0,R(\omega)}(H_{\mathrm{U}}(p), Z_p)$, $z_{\overline{\omega}} \in \operatorname{Ext}_A^{0,R(\overline{\omega})}(H_{\mathrm{SO}}(p), Z_p)$, and $z_{\overline{\omega}} \in \operatorname{Ext}_A^{0,R(\overline{\omega})}(H_{\mathrm{Sp}}(p), Z_p)$ be elements of these algebras defined by the equalities

$$\begin{aligned} &(z_\omega, u_\omega) = 1, \qquad &&(z_\omega, u_{\omega_1}) = 0, \qquad &&\omega_1 \neq \omega, \\ &(z_{\overline{\omega}}, u_{\overline{\omega}}) = 1, \qquad &&(z_{\overline{\omega}}, u_{\overline{\omega}_1}) = 0, \qquad &&\overline{\omega}_1 \neq \overline{\omega}. \end{aligned} \tag{17}$$

Lemma 7 from Chapter 1 implies that, in the algebras $\operatorname{Ext}_A(H_{\mathrm{SO}}(p), Z_p)$, $\operatorname{Ext}_A(H_{\mathrm{U}}(p), Z_p)$, and $\operatorname{Ext}_A(H_{\mathrm{Sp}}(p), Z_p)$, the relations

$$z_\omega \circ z_{\omega_1} = z_{(\omega,\omega_1)}, \quad z_{\overline{\omega}} \circ z_{\overline{\omega}_1} = z_{(\overline{\omega},\overline{\omega}_1)}$$

hold for all non-p-adic decompositions ω, ω_1 and $\overline{\omega}$, $\overline{\omega}_1$. Now, it is sufficient to take the elements $z_{\overline{\omega}}$ and z_ω, where the decompositions $\overline{\omega}$ and ω consist of only one term, as the generators z_{4k} and z_{2k}. This completes the proof of the theorem. □

2.3. **Modules over the Steenrod Algebra. The Case of $p = 2$.** Adams studied also bases of the Steenrod algebra modulo 2 in [1]. Namely, he defined a family of elements $e_{r,k} \in A^{(k\cdot 2^r-k)}$ with properties similar to properties (a)–(e) from Section 2.2. In this case, the elements $e_{r,1}$ commute for any r and $e_{r,1}^2 = 0$ (also for any r).

The following theorem is proved by analogy with Theorem 1 from Section 2.2 of Chapter 2.

Theorem 2. *The algebra* $\operatorname{Ext}_A(H_{\mathrm{U}}(2), Z_2)$ *is the algebra of polynomials in the generators*

$$\begin{gathered} 1 \in \operatorname{Ext}_A^{0,0}(H_{\mathrm{U}}(2), Z_2), \quad z_{2k} \in \operatorname{Ext}_A^{0,2k}(H_{\mathrm{U}}(2), Z_2), \qquad k \neq 2^i - 1, \\ h'_r \in \operatorname{Ext}_A^{1,2p^r-1}(H_{\mathrm{U}}(2), Z_2), \qquad r \geq 0. \end{gathered} \tag{18}$$

Proof. The proof of this theorem is similar to the proof of Theorem 1 (in this case, $\beta = Sq^1$). The only difference is that there is no Cartan type in the Steenrod algebra over the field Z_2. As above, M_β denotes the module over the Steenrod algebra with one generator of dimension 0 in which the only relation is the identity $\beta(x) = 0$ for all $x \in M_\beta$. As the subalgebra B of A we take the commutative algebra generated by the elements $e_{r,1}$, where $r = 1$, and by $e_{r,0} = 1$. Obviously, it is special. Let us prove that $M_B = M_\beta$. We use the Adams dividing homomorphism h (see (6) in Section 1.4 of Chapter 1). Obviously, it kills all iterations of type 1 modulo 2 (see Section 1.4 in Chapter 1). Adams proved that

$$\begin{aligned} h(e_{r,2k}) &= e_{r,k}, \\ h(e_{r,2k+1}) &= 0. \end{aligned} \tag{19}$$

These relations imply $h(e_{r,1}) = 0$. Instead of the elements $e_{r,k}$, we can now consider only the elements $e_{r,2^i}$ and construct bases of type (c) (see Section 1.2 of Chapter 1) with the use of only these elements (see [1]). Considering the order under which all elements of the form $e_{r,1}$ precede the elements $e_{r,2^i}$ with $i > 0$, we conclude that the homomorphism h kills only those monomials from the basis that have $e_{r,1}$ on the left. This implies $M_\beta = M_B$ and completes the proof of the theorem. □

Let $\widetilde{H}_{\mathrm{SO}}(2)$ denote the quotient module of $H_{\mathrm{SO}}(2)$ by the A-free part generated by the generator x_i mentioned in Lemma 6 from Section 1.4 of Chapter 1. Lemmas 6 and 7 from Chapter 1 imply the following assertion by analogy with Theorems 1 and 2 of this chapter.

Lemma 11. *The algebra* $\mathrm{Ext}_A(\widetilde{H}_{\mathrm{SO}}(2), Z_2)$ *is the algebra of polynomials in the generators*

$$\begin{gathered} 1 \in \mathrm{Ext}_A^{0,0}, \quad z_{4k} \in \mathrm{Ext}_A^{0,4k}, \qquad k \geq 1, \\ h_0 \in \mathrm{Ext}_A^{1,1}. \end{gathered} \tag{20}$$

The proof of this lemma is similar to the preceding ones. In this case, as the special subalgebra B we must take the subalgebra generated by only one element $e_{1,1} = Sq^1$.

Let $\widetilde{M}_\beta$ denote the module over the Steenrod algebra with one generator u of dimension 0 where the only nontrivial relations are the identity $\beta(x) = 0$ for all $x \in \widetilde{M}_\beta$ and the relation $Sq^2(u) = 0$. Lemma 6 and Corollary 1 imply that the module $H_{\mathrm{SO}}(2)$ is a direct sum of modules of types M_β and $\widetilde{M}_\beta$. The following theorem is valid.

Theorem 3. *The algebra* $\mathrm{Ext}_A(\widetilde{M}_\beta, Z_2)$ *admits the system of generators*

$$\begin{gathered} 1 \in \mathrm{Ext}_A^{0,0}(\widetilde{M}_\beta, Z_2), \quad h_0 \in \mathrm{Ext}_A^{1,1}(\widetilde{M}_\beta, Z_2), \quad h_1 \in \mathrm{Ext}_A^{1,2}(\widetilde{M}_\beta, Z_2), \\ x \in \mathrm{Ext}_A^{3,7}(\widetilde{M}_\beta, Z_2), \quad y \in \mathrm{Ext}_A^{4,12}(\widetilde{M}_\beta, Z_2), \quad h'_r \in \mathrm{Ext}_A^{1,2^r-1}(\widetilde{M}_\beta, Z_2), \quad r \geq 3, \end{gathered} \tag{21}$$

with the relations

$$h_0 h_1 = 0, \quad h_1^3 = 0, \quad x^2 = h_0^2 y, \quad h_1 x = 0; \tag{22}$$

all the other relations are consequences of (22).

Proof. Theorem 3 is proved by the same method of finding a special subalgebra B of the Steenrod algebra A which corresponds to the module $\widetilde{M}_\beta$. As B we take the subalgebra generated by the element $e_{1,2} = Sq^2$ and by all elements of form $e_{r,1}$. The description of the elements $e_{r,k}$ given in [1] implies immediately that $[e_{1,1}; e_{1,2}] = e_{2,1}$ and $[e_{r,1}; e_{1,2}] = 0$ for $r > 1$. (The notation $[a; b]$ is used for $ab + ba$ (commutator).) Let us find the cohomology algebra $H^*(B)$. Obviously, the subalgebra of B generated by the elements $e_{r,1}$ with $r > 1$ is central in B. Let us denote it by C. It is easy to see that the algebra B/C is commutative, and the squares of all its elements are zero (because $e_{1,2}^2 = e_{1,1} \circ e_{2,1}$ in B). The algebra $H^*(C)$ is isomorphic to the algebra of polynomials in the generators $h'_r \in H^{1,2^r-1}(C)$ for all $r \geq 2$. The algebra $H^*(B/C)$ is isomorphic to the algebra of polynomials in the generators $h_0 \in H^{1,1}(B/C)$and $h_1 \in H^{1,2}(B/C)$.

Consider the Serre–Hochschild spectral sequence of the central subalgebra C in the algebra B (see [15]). We know its term E^2; namely, $E_2^{p,q} = H^p(B/C) \otimes H^q(C)$. Simple calculations show that, in the Serre–Hochschild spectral sequence,

$$\begin{gathered} d_2(1 \otimes h'_2) = h_0 h_1 \otimes 1, \quad d_i(1 \otimes h'_r) = 0, \quad r \geq 3, \quad i \geq 2, \\ d_3(1 \otimes {h'_2}^2) = h_1^3 \otimes 1, \quad d_i(1 \otimes {h'_2}^4) = 0, \quad i \geq 2. \end{gathered} \tag{23}$$

Setting $x = h_0 \otimes {h'_2}^2$ and $y = 1 \otimes {h'_2}^4$ and retaining the other notation, we easily obtain the required result. This completes the proof of the theorem. □

Thus, it only remains to study the module $H_{\mathrm{Sp}}(2)$. Let $M_{1,2}$ denote the 1-generated module over the Steenrod algebra with the two identity relations $Sq^1(x) = 0$ and $Sq^2(x) = 0$ for all $x \in M_{1,2}$. We assume the dimension of the generator to be zero. Lemmas 5 and 7 of Chapter 1 show that studying the algebra $\mathrm{Ext}_A(H_{\mathrm{Sp}}(2), Z_2)$ reduces to studying the algebra $\mathrm{Ext}_A(M_{1,2}, Z_2)$. Arguing as above, we can easily show that the algebra $\mathrm{Ext}_{\mathrm{A}}(M_{1,2}, Z_2)$ is isomorphic to $H^*(B)$, where B is the subalgebra of the Steenrod algebra A generated by all elements of form $e_{r,1}$ and $e_{r,2}$. Recalling the description of the elements $e_{r,k}$ given in [1] and performing easy calculations, we see that the elements $e_{r,1}$ and $e_{r,2}$ satisfy the relations

$$\begin{gathered} [e_{r_1,1}; e_{r_2,1}] = 0, \quad e_{r,1}^2, \quad [e_{r,2}; e_{1,1}] = e_{r+1,1}, \\ [e_{r_1,2}; e_{r_2,1}] = 0, \quad r_2 > 1, \quad [e_{r,2}; e_{1,2}] = e_{r+1,1} \circ e_{1,1}, \\ [e_{r_1,2}; e_{r_2,2}] = 0, \quad r_1 > 1, \quad r_2 > 1, \quad e_{1,2}^2 = e_{2,1} \circ e_{1,1}, \\ [e_{r,2}; e_{1,2}] = e_{r+1,1} \circ e_{1,1}, \\ e_{r,2}^2 = 0, \quad r > 1, \end{gathered} \tag{24}$$

and all the other relations are consequences of (24).

Consider the central subalgebra C generated by the elements $e_{r,1}$ with $r \geq 2$ in this algebra. The cohomology algebra $H^*(C)$ is isomorphic to the algebra of polynomials in the generators $h'_r \in H^{1,2^r-1}(C)$ for all $r \geq 2$; this is easily seen from the relations (24). The cohomology algebra $H^*(B/C)$ is isomorphic to the algebra of polynomials in the generators $h_0 \in H^{1,1}(B/C)$ and $h_{r,1} \in H^{1,2^{r+1}-2}(B/C)$ for all $r \geq 1$. It is easy to derive from relations (24) that, in the Serre–Hochschild spectral sequence of the subalgebra C of B, the following relations hold:

$$\begin{aligned} d_2(1 \otimes h'_r) &= h_0 h_{r-1,r} \otimes 1, && r \geq 2, \\ d_3(1 \otimes {h'_r}^2) &= h_{1,1} h_{r-1,r}^2 \otimes 1, && r \geq 2, \\ d_i(1 \otimes {h'_r}^4) &= 0, && i \geq 2, \quad r \geq 2. \end{aligned} \tag{25}$$

We set $x = h_0 \otimes {h'_2}^2$ and $y = 1 \otimes {h'_2}^4$. Obviously, $d_i(x) = 0$ and $d_i(y) = 0$ for all $i \geq 2$. Hence there are elements $x \in \mathrm{Ext}_A^{3,7}(H_{\mathrm{Sp}}(2), Z_2)$ and $y \in \mathrm{Ext}_A^{4,12}(H_{\mathrm{Sp}}(2), Z_2)$ satisfying the relation

$$x^2 = h_0^2 y, \tag{26}$$

where $h_0 \in \mathrm{Ext}_A^{1,1}(H_{\mathrm{Sp}}(2), Z_2)$ (clearly, such an h_0 exists). In addition, (25) obviously implies that, in the algebra $\mathrm{Ext}_A(H_{\mathrm{Sp}}(2), Z_2)$, $h_0^n x \neq 0$ and $h_0^n y \neq 0$ for any n.

2.4. **Inner homology rings.**

Theorem 4. *The quotient ring of V_{SO} modulo 2-torsion is isomorphic to the ring of polynomials in generators u_{4i} of dimension $4i$ for all $i \geq 0$. The ring V_{U} is isomorphic to the ring of polynomials in generators v_{2i} of dimension $2i$ for all $i \geq 0$. The algebras $V_{\mathrm{Sp}} \times Z_p$ are isomorphic to the algebras of polynomials in generators t_{4i} of dimension $4i$ for all $i \geq 0$ at any $p \geq 0$. The ring V_{Sp} has no p-torsion at $p > 2$. The quotient ring of V_{Sp} modulo 2-torsion is not isomorphic to a polynomial ring. There exist elements $x \in V_{\mathrm{Sp}}^4$ and $y \in V_{\mathrm{Sp}}^8$ such that $x^2 - 4y \equiv 0$* (mod 2-*torsion) and they are generators of infinite order in the groups V_{Sp}^4 and V_{Sp}^8.*

Proof. Theorem 4 is proved by the spectral method of Adams; for this reason, we consider it necessary to give a precise statement of the main theorem of [1].

Let K be an arbitrary finite complex. We denote the groups $\pi_{n+i}(E^i K)$, where E is a suspension and i is sufficiently large, by $\pi_n^s(K)$. For the polyhedron $K = K_1 \times K_2/K_1 \vee K_2$, where K_1 and K_2 are finite polyhedra, a pairing of groups $\pi_{n_1}(K_1) \otimes \pi_{n_2}(K_2) \to \pi_{n_1+n_2}(K)$ is defined. The properties of the operation $K_1 \times K_2/K_1 \vee K_2$ imply that this pairing induces a pairing

$$\pi_{n_1}^s(K_1) \otimes \pi_{n_2}^s(K_2) \to \pi_{n_1+n_2}^s(K). \tag{27}$$

On the other hand, it is well known that $H^+(K_1, Z_p) \otimes H^+(K_2, Z_p) \approx H^+(K, Z_p)$. From algebraic considerations, this isomorphism determines a pairing

$$\mathrm{Ext}_A^{s,t}(H^+(K_1, Z_p), Z_p) \otimes \mathrm{Ext}_A^{\bar{s},\bar{t}}(H^+(K_2, Z_p), Z_p) \to \mathrm{Ext}_A^{s+\bar{s},t+\bar{t}}(H^+(K, Z_p), Z_p). \tag{28}$$

Adams' Theorem. *For any polyhedron K, there exists a spectral sequence $\{E_r(K), d_r\}$ such that*

(a) *$E_r(K) \approx \sum_{s,t} E_r^{s,t}$, $d_r \colon E_r^{s,t} \to E_r^{s+r,t+r-1}$, and $E_r^{s,t} = 0$ for $s > t$;*
(b) *$E_2^{s,t} \approx \mathrm{Ext}_A^{s,t}(H^+(K, Z_p), Z_p)$;*
(c) *the group $\sum_{t-s=m} E_\infty^{s,t}$ is adjoint to the quotient group of $\pi_m^s(K)$ modulo the subgroup of the elements whose orders are coprime to p;*
(d) *if $K = K_1 \times K_2/K_1 \vee K_2$, then there are pairings*

$$Q_r \colon E_r^{s,t}(K_1) \otimes E_r^{\bar{s},\bar{t}}(K_2) \to E_r^{s+\bar{s},t+\bar{t}}(K) \tag{28$'$}$$

such that

$$d_r Q_r(x \otimes y) = Q_r(d_r(x) \otimes y) + (-1)^{t-s} Q_r(x \otimes d_r(y)); \tag{28$''$}$$

(e) *the pairing Q_2 coincides (up to signs) with the pairing (28), and the pairing Q_∞ is adjoint to the pairing (27).*

Obviously, Adams' theorem and Lemmas 2 and 7 from Chapter 1 imply the following assertion.

Lemma 12. *There exist spectral sequences of algebras $\{E_r(\mathrm{SO}), d_r\}$, $\{E_r(\mathrm{U}), d_r\}$, $\{E_r(\mathrm{SU}), d_r\}$, and $\{E_r(\mathrm{Sp}), d_r\}$ such that*

(a)

$$\begin{aligned} E_2(\mathrm{SO}) &\approx \mathrm{Ext}_A(H_{\mathrm{SO}}(p), Z_p), \\ E_2(\mathrm{U}) &\approx \mathrm{Ext}_A(H_{\mathrm{U}}(p), Z_p), \\ E_2(\mathrm{SU}) &\approx \mathrm{Ext}_A(H_{\mathrm{SU}}(p), Z_p), \\ E_2(\mathrm{Sp}) &\approx \mathrm{Ext}_A(H_{\mathrm{Sp}}(p), Z_p) \end{aligned}$$

and

(b) *for* $E_r^{(m)} = \sum_{t-s=m} E_r^{s,t}$, *the graded algebras* $E_\infty = \sum_m E_\infty^{(m)}$ *are adjoint to the quotient rings of the rings* V_{SO}, VU, V_{SU}, *and* V_{Sp} *modulo the ideals of elements of orders coprime to* m *for the sequences of groups* $\{\mathrm{SO}(n)\}$, $\{\mathrm{U}(n)\}$, $\{\mathrm{SU}(n)\}$, *and* $\{\mathrm{Sp}(n)\}$.

Assertion (a) of Adams' theorem implies $d_r(E_r^{(m)}) \subset E_r^{(m-1)}$. Theorems 1 and 2 and Lemma 11 from Chapter 2 imply that, in the case under consideration, the groups $E_2^{s,t}(\mathrm{SO})$, $E_2^{s,t}(\mathrm{U})$, and $E_2^{s,t}(\mathrm{Sp})$ are zero if $t-s=1$ (possibly except the groups $E_2^{s,t}(\mathrm{Sp})$ and $E_2^{s,t}(\mathrm{SO})$ at $p=2$). Therefore, in the Adams spectral sequences specified in Lemma 12 from this section, all differentials are trivial. This observation and Lemma 11 from Chapter 2 give the isomorphisms

$$\begin{aligned} E_\infty(\mathrm{SO}) &\approx \mathrm{Ext}_A(H_{\mathrm{SO}}(p), Z_p), && p \geq 2, \\ E_\infty(\mathrm{U}) &\approx \mathrm{Ext}_A(H_{\mathrm{U}}(p), Z_p), && p \geq 2, \\ E_\infty(\mathrm{Sp}) &\approx \mathrm{Ext}_A(H_{\mathrm{Sp}}(p), Z_p), && p > 2. \end{aligned} \tag{29}$$

It is shown above that these algebra are polynomial (see Sections 2.2–2.3 in Chapter 2). Consider the elements $h_0 \in \mathrm{Ext}_A^{1,1}$ for all the algebras under examination. It is well known that multiplication by such an element is adjoint to multiplication by p in homotopy groups (see [1]). A comparison of the obtained results for all prime p gives all assertions of the theorem except the last one. Relations (25) and (26) imply that the elements $x \in \mathrm{Ext}_A^{3,7}(H_{\mathrm{Sp}}(2), Z_p)$ and $y \in \mathrm{Ext}_A^{4,12}(H_{\mathrm{Sp}}(2), Z_p)$ are cycles for all differentials in the spectral sequence of Adams and that, in the ring V_{Sp}, we have $x^2 - 4y \equiv 0$ (mod 2-torsion). There is no p-torsion in the rings V_{SO}, V_{U}, and V_{Sp}, because in the corresponding algebras $E_\infty(\mathrm{SO})$, $E_\infty(\mathrm{U})$, and $E_\infty(\mathrm{Sp})$, relations of the form $h_0^n z = 0$ can hold for no n and z.

This concludes the proof of Theorem 4. □

2.5. **Characteristic Numbers and the Image of the Hurewicz Homomorphism in Thom Spaces.** Consider the Thom space M_G corresponding to a subgroup G of the group $\mathrm{SO}(n)$. Let W^i be a compact closed smooth oriented manifold smoothly embedded in the sphere S^{n+i}. Obviously, the $\mathrm{SO}(n)$-bundle of planes $\mathbb{R}^n$ normal to the manifold W^i in the sphere S^{n+1} is defined. We denote it by ν^n and assume that it is endowed with the structure of a G-bundle, as in Section 1.1 of Chapter 1. Let p be the classifying map of this bundle to a plane G-bundle over B_G, and let $x \in H^i(B_G, Z_p)$ be an arbitrary cohomology class. We call the inner product $(p^*x, [W^i])$ the *characteristic number of the manifold* W^i *corresponding to the element* x ($[W^i]$ denotes the fundamental cycle of the manifold W^i with chosen orientation). We denote this number by $x[W^i]$ or by $x[\nu^i]$. Recall Thom's isomorphism $\varphi\colon H^i(B_G) \to H^{n+i}(M_G)$ and Thom's construction of the map $f(\nu^n, W^i)\colon S^{n+i} \to M_G$ for the manifold W^i embedded in the sphere S^{n+1} as specified above. The following lemma is valid.

Lemma 13. *If $[S^{n+1}]$ is the fundamental cycle of the sphere oriented consistently with a submanifold W^i G-framed in the sphere S^{n+1}, then*

$$(f(\nu^n, W^i)_*[S^{n+i}], \varphi(x)) = x[W^i] \tag{30}$$

for any $x \in H^i(B_G, Z)$.

Proof. Consider the closed tubular ε-neighborhood $\widetilde{T}(W^i)$ of the G-framed submanifold W^i of the sphere S^{n+i} for a sufficiently small ε. We assume that the map $f(\nu^i, W^i)\colon S^{n+i} \to M_G$ is t-regular in the tubular neighborhood $\widetilde{T}(W^i)$ (see [16]). The Thom isomorphisms $\varphi\colon H^i(W^i) \to H^{n+l}(\widetilde{T}(W^i), \partial\widetilde{T}(W^i))$ arise for all $l \geq 0$.

Let $m_0 \in M_G$ be the point of the Thom space obtained by contracting the boundary E_G of the cylinder T_G to a point, and let E^{n+i}_δ be the complement in the sphere to a very small ball neighborhood of radius δ of an interior point in $\widetilde{T}(W^i)$. Obviously, an embedding of pairs

$$j\colon (\widetilde{T}(W^i), \partial\widetilde{T}(W^i)) \subset (S^{n+i}, E^{n+i}_\delta)$$

is defined. The map of pairs $f(\nu^n, W^i)\colon (S^{n+i}, x_0) \to (M_G, m_0)$ is completely equivalent to the map of pairs

$$\tilde{f}(\nu^n, W^i)\colon (S^{n+i}, E^{n+i}_\delta) \to (M_G, m_0), \qquad x_0 \in E^{n+i}_\delta.$$

Let us denote the composition $\tilde{f}(\nu^n, W^i) \circ f$ by g. By the definition of a t-regular map $f(\nu^n, W^i)$, it induces a map of pairs $f_1(\nu^n, W^i)\colon (\widetilde{T}(W^i), \partial\widetilde{T}(W^i)) \to (M_G, m_0)$ commuting with φ. Let μ be the fundamental cycle of the manifold $\widetilde{T}(W^i)$ modulo the boundary. Since the maps under consideration are regular, we have $(f_1(\nu^n, W^i)_*(\mu), \varphi(x)) = (\varphi^{-1}(\mu), p^*(x)) = x[W^i]$, where $p\colon W^i \to B_G$ is the map induced by $f_1(\nu^n, W^i)$ on the subspace W^i. It is also obvious that

$$(g^*(\mu), \varphi(x)) = (\tilde{f}(\nu^n, W^i)_* \circ j_*(\mu), \varphi(x)) = (\tilde{f}(\nu^n, W^i)_*[S^{n+i}], \varphi(x)).$$

It only remains to note that $f(\nu^n, W^i)_*[S^{n+i}] = g_*(\mu)$, because $j_*(\mu) = [S^{n+i}]$ and $\tilde{f}(\nu^n, W^i)_*[S^{n+i}] = f(\nu^n, W^i)_*[S^{n+i}]$.

This concludes the proof of the lemma. □

Let us return to the case where the group G is one of the classical Lie groups. As in Chapter 2, we shell describe the cohomology of the Thom spaces and of the spaces $B_{\mathrm{SO}(2k)}$, $B_{\mathrm{U}(k)}$, and $B_{\mathrm{Sp}(k)}$ in terms of the two-dimensional Wu generators $t_1, \ldots, t_k$. According to Milnor's lectures on characteristic classes (see [9]), $j^*(p_{4i}) = \sum_{m+l=2i} c_{2m} \circ c_{2l}$, where $j\colon \mathrm{U}(k) \to \mathrm{SO}(2k)$ is the natural group-subgroup embedding. Therefore, the quotient ring of $H^*(B_{\mathrm{SO}(2k)})$ modulo 2-torsion can be thought of as the ring of polynomials in the generators $\sum_{m+l=2i} c_{2m} \circ c_{2l} = \sum t_1^2 \circ \cdots \circ t_i^2$ being elementary symmetric polynomials in the squared Wu generators and the polynomial $w_{2k} = t_1 \circ \cdots \circ t_k$ (this is obvious for the rings $H^*(B_{\mathrm{U}(k)})$ and $H^*(B_{\mathrm{Sp}(k)})$, because these rings have no torsion). Let ω and $\overline{\omega}$ be the same decompositions as in Chapters 1 and 2, and let v_ω and $v_{\overline{\omega}}$ be the symmetrized monomials corresponding to decompositions ω and $\overline{\omega}$ (see Section 1.3 in Chapter 1). As above, we use $R(\omega)$ and $R(\overline{\omega})$ to denote the dimensions of the elements v_ω and $v_{\overline{\omega}}$ in the rings of symmetric polynomials in the generators $t_1, \ldots, t_k$. Obviously, v_ω belongs to $H^{R(\omega)}(B_{\mathrm{U}(k)})$ and $v_{\overline{\omega}}$ belongs to $H^{R(\overline{\omega})}(B_{\mathrm{SO}(2k)})$ or to $H^{R(\overline{\omega})}(B_{\mathrm{Sp}(2k)})$ (more precisely, to the quotient group of $H^{R(\overline{\omega})}(B_{\mathrm{SO}(2k)})$ modulo 2-torsion). We refer to the characteristic numbers of the accordingly framed manifolds that correspond to the elements v_ω and $v_{\overline{\omega}}$ as the ω- ($\overline{\omega}$-) numbers of these manifolds[6]. We are especially interested in the cases where $\omega = (k)$ and $R(\omega) = 2k$ and where $\overline{\omega} = (2k)$ and $R(\overline{\omega}) = 4k$. Lemma 13 allows us to consider the characteristic numbers of elements of the rings V_{SO}, V_{U}, V_{SU}, and V_{Sp}.

Theorem 5. *The $\overline{(2k)}$-number of a $4k$-dimensional polynomial generator of the quotient ring of V_{SO} modulo 2-torsion is equal to p if $2k = p^i - 1$, where $p > 2$, and to 1 if $2k \neq p^i - 1$ for all $p > 2$; the (k)-number of a $2k$-dimensional polynomial generator of the ring V_{U} is equal to p if $k = p^i - 1$, where $p \geq 2$, and to 1 if $k \neq p^i - 1$ for all $p \geq 2$. The minimal nonzero $\overline{(2k)}$-number of a $4k$-dimensional symplectically framed manifold is equal to $2^s p$ if $k = p^i - 1$, where $p > 2$, and to 2^s if $k \neq p^i - 1$ for all $p > 2$, where $s \geq 0$.* (By a minimal number, we mean a number with minimum absolute value.)

Proof. We shall prove this theorem by using homotopy arguments relying on equality (30). We give the proof only for the ring V_{U}, because the proofs for the rings V_{SO} and V_{Sp} are quite similar.

Consider the module $H_U(p)$. As mentioned (see Sections 2.2–2.3 of Chapter 2), the module $H_U(p)$ with an arbitrary $p \geq 2$ is a direct sum of modules M_β^ω of the M_β type with generators u_ω for all non-p-adic decompositions ω (and with a generator u of dimension 0). In the proof of Theorem 2 of Chapter 2, it was mentioned that the module M_B corresponds to a special subalgebra B in the Steenrod algebra A generated by all the elements e'_r and by 1 (see Section 2.2 of Chapter 2) for $p > 2$ and by all the elements $e_{r,1}$ and by 1 (see Section 2.3 of Chapter 2) for $p = 2$, i.e., $M_\beta = M_B$. Consider the resolvent $C_A(M_B)$ constructed in proving the isomorphism (14). Let $C_A(H_{\mathrm{U}}(p))$ denote the direct sum $\sum_\omega C_A(M_\beta^\omega)$, where the resolvents $C_A(M_\beta^\omega)$ are similar to the resolvent $C_A(M_\beta)$ with the only difference that the dimensions of all elements are shifted by $R(\omega)$; the resolvents $C_A(M_\beta^\omega)$ coincide with minimal resolvents of the special subalgebra; the sum is over all non-p-adic decompositions ω.

In what follows, we study only the maps

$$\begin{aligned} \varepsilon\colon C_A^0(H_{\mathrm{U}}(p)) &\to H_{\mathrm{U}}(p), \\ d_0\colon C_A^1(H_{\mathrm{U}}(p)) &\to C_A^0(H_{\mathrm{U}}(p)). \end{aligned}$$

Take the Thom space $M_{\mathrm{U}(k)}$ for a sufficiently large k. It is aspherical in dimensions smaller than $2k$. Following [1], consider a realization $Y = \{Y_{-1} \supset Y_0 \subset \dots Y_n\}$ of the free acyclic resolvent $C_A(H_{\mathrm{U}}(p))$ (see [11, Chapter II]). We assume that the realization Y is polyhedral and the number n is sufficiently large. By the definition of realizations of resolvents, the space Y_{-1} is homotopy equivalent to the space $M_{\mathrm{U}(k)}$ (k is large), the A-modules $H^*(Y_{i-1}, Y_i; Z_p)$ are isomorphic (up to a sufficiently large dimension) to the A-modules $C_A^i(H_{\mathrm{U}}(p))$ for $i \geq 0$, the maps $\delta_i^*\colon H^*(Y_{i-1}, Y_i; Z_p) \to H^*(Y_{i-2}, Y_{i-1}; Z_p)$ coincide with the maps $d_{i-1}\colon C_A^i(H_{\mathrm{U}}(p)) \to C_A^{i-1}(H_{\mathrm{U}}(p))$ for all $i \geq 1$, and the map $\delta_0^*\colon H^*(Y_{-1}, Y_0; Z_p) \to H^*(Y_{-1}; Z_p)$ coincides with the map $\varepsilon\colon C_A^0(H_{\mathrm{U}}(p)) \to H_{\mathrm{U}}(p)$. (We imply here that the cohomology is over the field Z_p.) Obviously, in the case under consideration, we have $\pi_{2k+t}(Y_{i-1}, Y_i) \approx \mathrm{Hom}_A^t(C_A^i(H_{\mathrm{U}}(p), Z_p)$ for all $t < 4k - 1$ and $i \geq 0$. For the pair (Y_{-1}, Y_0), consider

[6] The properties of the ω- ($\overline{\omega}$-) numbers are given in Milnor's lectures [9], where these numbers are denoted by S_ω ($S_{\overline{\omega}}$).

the cohomology exact sequences

$$\ldots \longrightarrow H^q(Y_{-1}, Y_0; Z_p) \xrightarrow{\delta_0^*} H^q(Y_{-1}; Z_p) \xrightarrow{j^*} H^q(Y_0; Z_p) \longrightarrow \ldots\,, \tag{31$'$}$$

$$\ldots \longrightarrow H^q(Y_{-1}, Y_0; Z) \xrightarrow{\bar\delta_0^*} H^q(Y_{-1}; Z) \xrightarrow{\bar j^*} H^q(Y_0; Z) \longrightarrow \ldots\,. \tag{31$''$}$$

The homomorphism δ_0^* in the sequence (31$'$) is an epimorphism if $q < 4k-1$; therefore, the homomorphism j^* is trivial. But the groups $H^q(Y_{-1}; Z)$ have no torsion, and the homomorphism $\bar\delta_0^*$ in the sequence (31$''$) is also trivial, because all groups $H^q(Y_{-1}, Y_0; Z)$ are finite and they are direct sums of the groups Z_{p_i}. Since the homomorphism j^* in (31$'$) is trivial, the image $\operatorname{Im} \bar j^*$ is divisible by p in the group $H^q(Y_0; Z)$. The inclusion $H^q(Y_0; Z)/\operatorname{Im} \bar j^* \subset H^{q+1}(Y_{-1}, Y_0; Z)$ implies that the image $\operatorname{Im} \bar j^*$ is not divisible by numbers of form ap, where $|a| > 1$, because the groups $H^t(Y_{-1}, Y_0; Z)$ with $t < 4k-1$ are direct sums of the groups Z_{p_i} (see [7]). This implies that the image of $\bar j_*\colon H_q(Y_0; Z) \to H_q(Y_{-1}; Z)$ consists of all elements having the form px, where $x \in H_q(Y_{-1}; Z)$. Now, consider the elements $\tilde z_{2l} \in \operatorname{Hom}_A^{2l}(C_A^0(H_{\mathrm{U}}(p), Z_p)$ which determine the elements (16) or (18) in the groups $\operatorname{Ext}_A^{0,2l}(H_{\mathrm{U}}(p), Z_p)$ at $l \neq p^i - 1$. By the definition of the realization of the resolvent, we can assume that $\tilde z_{2l} \in \pi_{2l+2k}(Y_{-1}, Y_0)$. Moreover, since the differentials in the Adams spectral sequence are trivial, we can assume that the elements $\tilde z_{2l} \in \pi_{2l+2k}(Y_{-1}, Y_0)$ belong to the image $\delta_{0*}(\pi_{2l+2k}(Y_{-1}))$. Let $H\colon \pi_i(K) \to H_i(K, Z)$ be the Hurewicz homomorphism. Take an element $\tilde{\tilde z}_{2l} \in \pi_{2l+2k}(Y_{-1})$ for which the inner product $(H\tilde{\tilde z}_{2l}, v_l \circ c_{2k})$ has minimal absolute value. Clearly, $\delta_{0*}\tilde{\tilde z}_{2l} = \lambda \tilde z_{2l}$, and λ is coprime to p, because the map δ_{0*} is an epimorphism of homotopy groups and, by the construction of the resolvent, the cycle $H\tilde z_{2l}$ is a δ_{0*}-image of a cycle x_{2l} such that $(x_{2l}, v_{(l)} \circ c_{2k}) \neq 0$. This implies that the inner product $(H\tilde{\tilde z}_{2l}, v_{(l)} \circ c_{2k})$ is coprime to p if $l \neq p^i - 1$. Comparing the obtained results for different p, we conclude that the inner product $(H\tilde{\tilde z}_{2l}, v_{(l)} \circ c_{2k})$ is equal to ± 1 if $l \neq p^i - 1$ for any $p \geq 2$ and it is equal to $\pm p^s$ if $l = p^i - 1$. The number s is not known so far. It remains to find it. For this purpose, consider the cohomology exact sequences for the pair (Y_0, Y_1):

$$\ldots \longrightarrow H^q(Y_0, Y_1; Z_p) \xrightarrow{i^*} H^q(Y_0; Z_p) \longrightarrow H^q(Y_1; Z_p) \longrightarrow \ldots\,, \tag{32$'$}$$

$$\ldots \longrightarrow H^q(Y_0, Y_1; Z) \xrightarrow{\bar i^*} H^q(Y_0; Z) \longrightarrow H^q(Y_1; Z) \longrightarrow \ldots\,. \tag{32$''$}$$

Recall that we have calculated the module $H^*(Y_0; Z_p) \approx \operatorname{Ker} \varepsilon$ in a similar case in Section 2.1 of Chapter 2 in the proof of Lemma 8, which applies to the case under consideration. The homomorphism i^* in the sequence (32$'$) is an epimorphism, as well as δ_0^* in (31$'$). It readily follows from considerations similar to the argument used above that there exists an element $\tilde{\tilde z}_{2l} \in \pi_{2k+2l}(Y_0)$ for which the inner product $(H\tilde{\tilde z}_{2l}, y_{2l})$ is coprime to p, where $y_{2l} \in H^{2k+2l}(Y_0)$ is an element such that $pj^*(c_{2k} \circ v_{(l)}) - y_{2l} = \sum \lambda_i j^*(v_{\omega_i} \circ c_{2k})$, where $\omega_i \neq (l)$ (j^* is the homomorphism $H^*(Y_{-1} \to H^*(Y_0))$.

Comparing this result with the preceding one, we obtain the required assertion. To this end, it suffices to apply Lemma 13 about the relation between the found inner products with characteristic numbers. To complete the proof of the theorem, it remains to note that the (l)-number of an element of the ring V_{U} is equal to 0 if this element decomposes into a linear combination of products of elements of smaller dimensions. Therefore, the (l)-number of a polynomial generator of dimension $2l$ is minimal, and any element $x \in V_{\mathrm{U}}^{2l}$ whose (l)-number has minimum absolute value can be taken as a polynomial generator[7]. The $\overline{(2l)}$-numbers in the rings V_{SO} and V_{Sp} have similar properties. This concludes the proof of the theorem. □

As is known (see [9]), in the ring V_{SO}, the set of complex projective planes $P^{2k}(\mathbb{C})$ forms a polynomial subring (to be more precise, the $P^{2k}(\mathbb{C})$ with natural normal framing can be taken as representatives of elements x_{4k} forming a polynomial subring in the ring V_{SO} such that the quotient group $V_{\mathrm{SO}}/P(x_4, x_8, \dots)$ consists of elements of finite order). We call the coefficient of a polynomial $4k$-dimensional generator in the decomposition of an element $x \in V_{\mathrm{SO}}^{4k}$ in the generators the *multiplicity of the element* x. The absolute value of the multiplicity does not depend on the choice of polynomial generators in the ring V_{SO} (in its quotient ring modulo 2-torsion). By the *multiplicity of the representative-manifolds* of an element x, we mean the multiplicity of x. Theorem 5 has the following corollary.

Corollary 2. *The multiplicity of the complex projective planes $P^{2k}(\mathbb{C})$ in the ring V_{SO} is equal to $2k+1$ if $2k+1 \neq p^i$ for any $p > 2$ and to $\frac{2k+1}{p}$ if $2k+1 = p^i$, where $p > 2$.*

Each complex analytic manifold is embedded in a real affine even-dimensional space. The embedding endows it with a normal complex framing inverse to the tangent bundle. This allows us to consider the multiplicity of a complex analytic manifold in the ring V_{U}.

Corollary 3.[8] *In the ring V_{U}, the multiplicities of the projective planes $P^k(\mathbb{C})$ are equal to $k+1$ if $k+1 \neq p^i$ for any $p \geq 2$ and to $\frac{k+1}{p}$ if $k+1 = p^i$, where $p \geq 2$.*

To prove Corollaries 2 and 3, it is sufficient to note that the $\overline{(2k)}$-numbers of the $P^{2k}(\mathbb{C})$ in V_{SO} and the (k)-numbers of the $P^k(\mathbb{C})$ in V_{U} are equal to $2k+1$ and $k+1$, respectively, which implies the required assertions. (Obviously, the $\overline{(2k)}$- and (k)-numbers are, respectively, some polynomials in the Pontryagin and Chern classes of the normal bundles inverse to tangent bundles. In the case under consideration, these polynomials are easily calculated by using symmetric polynomials in the Wu generators. For the tangent classes of Pontryagin (Chern), this is done in [19]. The $\overline{(2k)}$-number (the (k)-number) of a normal bundle equals the negative $\overline{(2k)}$-number (the (k)-number) of the tangent bundle. This can be easily derived from the Whitney formula written in terms of the $\overline{\omega}$- and ω-numbers.)

3. Realization of cycles

3.1. **The Possibility of G-Realization of Cycles.** Let M^n be a compact closed smooth oriented manifold.

---

[7]This readily follows from the following Whitney formulas for classes (of Pontryagin, of Chern, and symplectic):

$$\omega(\xi \otimes \eta) = \sum_{\substack{(\omega_1,\omega_2)=\omega,\\ \omega_1\neq\omega_2}} [\omega_1(\xi)\omega_2(\eta) + \omega_2(\xi)\omega_1(\eta)] + \sum_{(\omega_1,\omega_1)=\omega} \omega_1(\xi)\omega_1(\eta). \tag{32}$$

[8]Milnor specified manifolds $H_{r,t} \subset P^r(\mathbb{C}) \times P^t(\mathbb{C})$, where $r > 1$ and $t > 1$, of dimension $2k = 2(r+t-1)$ such that $(k)[H_{r,t}] = -\binom{r+t}{r}$. These manifolds are algebraic (see [17]).

Definition 5. The dimension i of the manifold M^n is said to be *p-regular*, where p is a prime, if $2i < n$ and all groups $H_{i-2q(p-1)-1}(M^n, Z)$ have no p-torsion for $q \geq 1$.

Theorem 6. *If the dimension i of a manifold M^n is p_s-regular for some (finite or infinite) set $\{p_s\}$ of odd primes, then, for each integer cycle $z_i \in H_i(M^n, Z)$, there exists an odd number α such that it is coprime to all the p_s and the cycle αz_i can be realized by a submanifold. If a cycle $z_i \in H_i(M^n, Z)$ is realized by a submanifold, the dimension i is 2-regular, and $n - i \equiv 0 \pmod 2$, then the cycle z_i is $\mathrm{U}\binom{n-i}{2}$-realizable. If a cycle $z_i \in H_i(M^n, Z)$ is realized by a submanifold, $2i < n$, and $n - i \equiv 0 \pmod 4$, then the cycle $2^t z_i$ is $\mathrm{Sp}\binom{n-i}{4}$-realizable provided that t is sufficiently large.*

Proof. We shall prove the theorem by the method of Thom, relying on the homotopy structure of the spaces $M_{\mathrm{SO}(n-i)}$, $M_{\mathrm{U}\left(\frac{n-i}{2}\right)}$, and $M_{\mathrm{Sp}\left(\frac{n-i}{4}\right)}$ studied in the preceding chapters.

We start with proving the first assertion of the theorem. Consider the cohomology class $z^{n-i} \in H^{n-i}(M^n, Z)$ dual to the cycle z_i. Let us construct a map $q\colon M^n \to M_{\mathrm{SO}(n-i)}$ such that $q^* U_{\mathrm{SO}(n-i)} = \alpha z^{n-i}$, where α is an odd number coprime to all the p_s. Lemma 6 from Chapter 1 implies that, for $k = 2(n-i)-2$, the k-th Postnikov complex $M^{(k)}$ (see [11]) of the space $M_{\mathrm{SO}(n-i)}$ is homotopy equivalent to the direct product of some space $\widetilde{M}_{\mathrm{SO}(n-i)}$ and Eilenberg–MacLane complexes of type $K(Z_2, l)$. Moreover, by Theorem 4 from Chapter 2, the factors $\widetilde{M}_{\mathrm{SO}(n-i)}$ in this product can be chosen in such a way that all the groups $\pi_i(\widetilde{M}_{\mathrm{SO}(n-i)})$ be free Abelian and differ from zero only at $t \equiv 0 \pmod 4$. We denote the Postnikov complexes of the space $\widetilde{M}_{\mathrm{SO}(n-i)}$ by $\widetilde{M}^{(q)}$. Obviously, $\pi_t(\widetilde{M}^{(q)}) = 0$ if $t < n - i$ or $t > q$. Let us denote the spaces of type $K(\pi_q(\widetilde{M}_{\mathrm{SO}(n-i)}), q)$ by K_q. The results of [6] imply that the groups $H^{q+t}(Z, q; Z)$ with $t \leq q - 1$ are finite, and they are direct sums of groups of the form Z_p, where $p \geq 2$.

These considerations and the natural fibrations $\eta_q\colon \widetilde{M}^{(q)} \to \widetilde{M}^{(q-1)}$ with fibers K readily imply that the groups $H^{q+t}(\widetilde{M}^{(q)}, Z)$ are finite if $0 < t < 2(n-i)$ and that the Postnikov factors $\Phi_q \in H^{q+2}(\widetilde{M}^{(q)}, \pi_{q+1}(\widetilde{M}_{\mathrm{SO}(n-i)}))$ are cohomology classes of finite order with coefficients in a free Abelian group. Let us denote the order of the factor Φ_q by λ_q. Lemma 5 from Chapter 1 implies that λ_q is odd. We denote the fundamental cohomology class of the complex K_q by u_q. Now, let us construct a family of maps $q_q\colon M^n \to \widetilde{M}^{(q)}$ such that $\eta_q(g_q) = g_{q-1}$ and $q^*_{n-i}(u_{n-i}) = \alpha z^{n-i}$. Recall that, under the assumptions made above, the sets of homotopy classes of the maps $\pi(M^n, \widetilde{M}^{(q)})$ form Abelian groups, and for any pair of elements $h_1, h_2 \in \pi(M^n, \widetilde{M}^{(q)})$, the induced homomorphisms of cohomology groups satisfy the equality $(h_1 + h_2)^* = h_1^* + h_2^*$. Let $\widetilde{H}^{q,t} \subset H^{q+t}(\widetilde{M}^{(q)}, Z)$ be the subgroup of the group $H^{q+t}(\widetilde{M}^{(q)}, Z)$ consisting of the elements of finite order coprime to all the p_s and of the elements of order coprime to λ_q. Suppose given a map $f_q\colon M^n \to \widetilde{M}^{(q)}$ such that $f_q^*(\Phi_q) = 0$. We denote the homotopy class of a map f by $\{f\}$.

Lemma 14. *There exist a map* $f_{q+1}\colon M^n \to \widetilde{M}^{(q+1)}$ *and an odd number* α_q *coprime to all the* p_s *such that* $\{\eta_{q+1} f_{q+1}\} = \alpha_q\{f_q\}$ *and* $f^*_{q+1}(\widetilde{H}^{q+1,t}) \subset \operatorname{Im} f^*_q$ *if* $t+q+1 \le q+n-i$.

Proof. Consider a spectral sequence of the fibration η_{q+1} with coefficients in the group $\pi_{q+1}(\widetilde{M}_{\mathrm{SO}(n-i)})$ which reduces to an exact sequence in small dimensions. As usual, we denote the transgression in the fibration by τ. Obviously, $\tau(u_{q+1}) = \Phi$ and $\tau(\lambda_q u_{q+1}) = 0$. It is also clear that $\tau(x) = 0$ if x is an element of finite order coprime to λ_q whose dimension is smaller than $n-i-q$. Each element $\tilde{x} \in H^*(\widetilde{M}^{(q+1)}, \pi_{q+1}(\widetilde{M}_{\mathrm{SO}(n-i)}))$ adjoint to x is a stable primary cohomology operation (up to an element $y \in \eta^*_{q+1}(H^*(\widetilde{M}^{(q)})))$ on the element $\tilde{u}_{q+1}$ adjoint to $\lambda_q u_{q+1}$. It is well known that, for any map $\tilde{f}_{q+1}\colon M^n \to \widetilde{M}^{(q+1)}$ such that $\eta_{q+1}\tilde{f}_{q+1} = f_q$ and any element $z \in H^{q+1}(M^n, \pi_{q+1}(\widetilde{M}_{\mathrm{SO}(n-i)}))$, there exists a map $\tilde{f}'_{q+1}\colon M^n \to \widetilde{M}^{(q+1)}$ for which ${\tilde{f}'}^*_{q+1}(\tilde{u}_{q+1}) - \tilde{f}^*_{q+1}(\tilde{u}_{q+1}) = \lambda_q z$. Therefore, we can find a map $\tilde{f}_{q+1}\colon M^n \to \widetilde{M}^{(q+1)}$ such that ${\tilde{f}'}^*_{q+1}(\tilde{x}) \subset f^*_q(H^*(\widetilde{M}^{(q)}))$ and $\eta_{q+1}\tilde{f}_{q+1} = f_q$, where $\tilde{x}$ is the element adjoint to $x \in H^t(K_{q+1}; \pi_{q+1}(\widetilde{M}_{\mathrm{SO}(n-i)}))$ with $t < n-i+q$. (The order of x is assumed to be coprime to λ_q.)

Let α_q denote the number of elements in the quotient group

$$\left[\tilde{f}^*_{q+1}\Bigl(\sum_{t=q+2}^{n-1} \widetilde{H}^{q+1,t}\Bigr)\right] \Big/ \left[\operatorname{Im} f^*_q \cap \tilde{f}^*_{q+1}\Bigl(\sum_{t=q+2}^{n-1} \widetilde{H}^{q+1,t}\Bigr)\right].$$

Since λ_q is odd, the construction of the map $\tilde{f}^*_{q+1}$ implies that α_q is odd and coprime to all the p_s. Setting $\{f_{q+1}\} = \alpha_q\{\tilde{f}_{q+1}\}$, we obtain the required assertion. This completes the proof of the lemma. □

Now, let us construct a family of maps $f_q\colon M^n \to \widetilde{M}^{(q)}$ such that $f^*_{n-i}(u_{n-i}) = z^{n-i}$ and $\{\eta_{q+1} f_{q+1}\} = \alpha_q$, where the α_q satisfy the assumptions of Lemma 14. We argue by induction on q.

Note that $\lambda_{n-i-1} = 1$. This obviously implies that $\operatorname{Im} \alpha_{n-i-1} f^*_{n-i} = 0$ in dimensions larger than $n-i$, because the groups $H^{n-i+2q(p-1)+1}(M^n)$ nave no p_s-torsion for $q \ge 1$[9]. Now, suppose that the maps f_j are constructed for all $j \le m$ and $f^*_j(H^t(\widetilde{M}^{(j)}, Z)) = 0$ for $j < t < 2(n-i)$. Without loss of generality, we can assume that $m - n + i \equiv 3 \pmod 4$. We divide the numbers p_s into two classes: the first consists of the numbers for which λ_q is coprime to p_s and the second of all remaining numbers. It is easy to see that, for the numbers p_s from the first class, the map $f_{m+1}\colon M^n \to \widetilde{M}^{(m+1)}$ also satisfies the induction hypotheses, i. e., $f^*_{m+1}(\tilde{x}) = 0$ if $\tilde{x} \in H^t(\widetilde{M}^{(m+1)}, Z)$ for $t > m+1$ and the order of $\tilde{x}$ is divided by p_s (this is readily implied by Lemma 14, the conditions of the theorem, and the structure of the groups $H^t(K_{m+1}, Z)$). Consider the case where the number p_s belongs to the second class. In this case, λ_m is divided by p_s. The factor $\Phi_m \in H^{m+2}(\widetilde{M}^{(m)}, \pi_{m+1}(\widetilde{M}_{\mathrm{SO}(n-i)}))$ can be treated as the partial operation $\Phi(\eta^*_m \circ \cdots \circ \eta^*_{n-i+1}(u_{n-i}))$ on the element $\eta^*_m \circ \cdots \circ \eta^*_{n-i+1}(u_{n-i})$. Let us decompose the element Φ_m as $\Phi_m = \Phi^{(1)}_m + \Phi^{(2)}_m$, where the order of $\Phi^{(2)}_m$ is coprime to p_s and the order of $\Phi^{(1)}_m$ is a number of form

[9] As generators of the Steenrod algebra of stable primary cohomology operations $\theta^i \in H^{n+j}(Z, n; Z)$ with $j \le n+1$, elements of dimensions $2q(p-1)+1$ with $q \ge 1$ can be taken.

p_s^i. Both elements $\Phi_m^{(1)}$ and $\Phi_m^{(2)}$ can be treated as partial operations on the same element $\eta_m^* \circ \cdots \circ \eta_{n-i+1}^*(u_{n-i})$ defined on the same kernels.

The following lemma is valid.

Lemma 15. *Let Φ be a partial stable cohomology operation on $\eta_m^* \circ \cdots \circ \eta_{n-i+1}^*(u_{n-i})$ such that it increases dimension by $m-n+i+2$ and its domain and range are subgroups of cohomology groups with coefficients in free Abelian groups. If the cohomology operation $p^l\Phi$ is trivial for some odd prime p, $\eta_{m+1}^*\Phi(\eta_m^* \circ \cdots \circ \eta_{n-i+1}^*(u_{n-i})) = 0$, and $m-n+i \not\equiv -1 \pmod{2p-2}$, then the operation Φ is also trivial.*

The assertion of the lemma readily follows from the homotopy structure of the Thom spaces studied in Chapter 2.

Lemma 15 implies that the partial cohomology operation $\Phi_m^{(1)}$ is trivial. Now, we can construct a map f_{m+1} such that $\{\eta_{m+1}f_{m+1}\} = \{f_m\} \cdot \alpha_m$, where α_m satisfies the assumptions of Lemma 14, and the image of f_{m+1}^* is trivial in dimensions larger than $m+1$. For this purpose, it is sufficient to apply Lemmas 14 and 15 to all primes p_s from the second class. This gives us the family of maps f_q. It determines a family of maps $\tilde{\tilde{f}}_q\colon M^n \to M_{\mathrm{SO}(n-i)}^{(q)}$ such that $\{\tilde{\tilde{\eta}}_{q+1}\tilde{\tilde{f}}_{q+1}\} = \alpha_q\{\tilde{\tilde{f}}_q\}$ and $\tilde{\tilde{f}}_{n-i}(u_{n-i}) = z^{n-i}$, where $M_{\mathrm{SO}(n-i)}^{(q)}$ is the Postnikov complex of the space $M_{\mathrm{SO}(n-i)}$ and $\tilde{\tilde{\eta}}\colon M_{\mathrm{SO}(n-i)}^{(q+1)} \to M_{\mathrm{SO}(n-i)}^{(q)}$ is the natural projection.

We set $\{g_q\} = \alpha_{n-1} \circ \cdots \circ \alpha_q\{\tilde{\tilde{f}}_q\}$. Clearly, $\{\tilde{\tilde{\eta}}_{q+1}g_{q+1}\} = \{g_q\}$ and $g_{n-i}^*(u_{n-i}) = \alpha_{n-i} \circ \cdots \circ \alpha_{n-i+1} \circ u_{n-i}$. We have constructed a family of maps g_q with the required properties. This proves the first assertion of the theorem. The remaining assertions are proved similarly. $\square$

It follows from the proof of Theorem 6 and the structure of the groups $H^t(B_{\mathrm{U}(m)}, Z)$ that, for any cocycle $z^{2i} \in H^{2i}(B_{\mathrm{U}(m)}, Z)$, there exists a map $g\colon \widetilde{B}_{\mathrm{U}(m)}^{(4i-1)} \to M_{\mathrm{U}(i)}$ of the $(4i-1)$-dimensional skeleton such that $g^*(u_{\mathrm{U}(i)}) = z^{2i}$. This implies the following assertion.

Corollary 4. *The homology class dual to an arbitrary polynomial $P(c_2, c_4, \dots)$ in the Chern classes of any $\mathrm{U}(m)$-bundle over the manifold M^n admits a $\mathrm{U}(i)$-realization if the dimension of the polynomial is equal to $2i$, where $2i > \left[\dfrac{n+1}{2}\right]$.*

Corollary 5. *A cycle $z_i \in H_i(M^n)$ with $i < \left[\dfrac{n}{2}\right]$ can be realized as a submanifold if $2^k z_i = 0$.*

Appendix 1
On the Structure of the Ring V_{SU}

As is known, in all domains of coefficients, the cohomology algebras $H^*(B_{\mathrm{SU}(k)})$ can be described by symmetric polynomials in the Wu generators $t_1, \dots, t_k$ with taking into account the relation $t_1 + \cdots + t_k = 0$, as well as the algebras $H^*(M_{\mathrm{SU}})$.

Suppose that $\omega = (a_1, \dots, a_s)$, $v_\omega = \sum t_1^{a_1} \circ \cdots \circ t_s^{a_s}$, and

$$u_\omega = v_\omega \circ c_{2k} = \sum t_1^{a_1+1} \circ \cdots \circ t_s^{a_s+1} \circ t_{s+1} \circ \cdots \circ t_k,$$

as in Chapter 1. Suppose also that $\sum a_i < k$.

Definition. A decomposition $\omega = (a_1, \dots, a_s)$ is called *p-admissible* if the number of subscripts i such that $a_i = p^l$ divides p for any $l \geq 0$. (Note that, for a field of characteristic zero, this means that $a_i \neq 1$ for $i = 1, \dots, s$.)

Lemma 16. *Each module $H_{\mathrm{SU}}(p)$ with $p > 2$ is isomorphic to a direct sum $\sum_\omega M_\beta^\omega$ of M_β-type modules M_β^ω with generators u_ω corresponding to p-admissible non-p-adic decompositions $\omega = (a_1, \dots, a_s)$. The dimension of the generator u_ω is equal to $2\left(\sum a_i\right)$.*

The module $H_{\mathrm{SU}}(p)$ admits a diagonal map

$$\Delta\colon H_{\mathrm{SU}}(p) \to H_{\mathrm{SU}}(p) \otimes H_{\mathrm{SU}}(p),$$

which has the same form on the generators u_ω as the diagonal map considered in Section 1.5 of Chapter 1 (with the p-admissibility of decompositions taken into account). In addition, the relations (9) hold modulo decomposable elements rather than absolutely. Thus the following lemma is valid.

Lemma 17. *The algebra $\mathrm{Ext}_A(H_{\mathrm{SU}(p)}, Z_p)$ is isomorphic to the algebra of polynomials in the generators*

$$1 \in \mathrm{Ext}_A^{0,0}(H_{\mathrm{SU}}(p), Z_p), \quad h'_r \in \mathrm{Ext}_A^{1,2p^r-1}(H_{\mathrm{SU}}(p), Z_p), \quad r \geq 0,$$
$$z_{(k)} \in \mathrm{Ext}_A^{0,2k}(H_{\mathrm{SU}}(p), Z_p), \quad k \neq p^r, \quad p^r - 1, \quad r \geq 0,$$
$$z_{(\omega_r)} \in \mathrm{Ext}_A^{0,2p^{(r+1)}}(H_{\mathrm{SU}(p)}, Z_p), \quad r \geq 0, \quad \omega_r = \frac{1}{p^{l_r}}(p^{r+1})^{10}.$$

Using an argument similar to that of Section 2.4 of Chapter 2, we can deduce the following theorem from these lemmas.

Theorem 7. *The ring $V_{\mathrm{SU}} \otimes Z_{p^h}$ is isomorphic to the ring of polynomials in the generators v_{2i} $(i = 0, 2, 3, 4, \dots)$ for all $p > 2$ and $h \neq 0$. The ring V_{SU} has no p-torsion for $p > 2$.*

Using the same method as in the proof of Theorem 5 from Chapter 2 and taking into account the p-admissibility of decompositions, we can deduce the following theorem from Lemma 13 from Chapter 2.

Theorem 8. *A sequence of* SU*-framed manifolds $M^4, M^6, M^8, \dots$ is a system of polynomial generators of the ring $V_{\mathrm{SU}} \otimes Z_{p^h}$ if and only if the following conditions on the Chern ω-numbers of the* SU*-framings hold:*

$$(k)[M^{2k}] \not\equiv 0 \pmod{p}, \qquad k \neq p^i, p^i - 1,$$
$$\frac{1}{p}(k)[M^{2k}] \not\equiv 0 \pmod{p}, \qquad k = p^i - 1,$$
$$\frac{1}{p^{l_s}}(p^{s+1})[M^{2p^{s+1}}] \not\equiv 0 \pmod{p}, \qquad s \geq 0, \quad l_s \geq 1.$$

(Note that $(p^{s+1})[M^{2p^{s+1}}] \equiv 0 \pmod{p}$, because $c_2 = (1) = 0$.)

Now, consider the case of $p = 2$.

According to Corollary 1 from Chapter 1, we have $H_{\mathrm{SU}}(2) = \sum_i M_\beta^{(i)} + \sum_\omega \widetilde{M}_\beta^\omega$, where the modules $\widetilde{M}_\beta^\omega$ are quotient modules of M_β^ω-type modules by the relation

---

[10] Since $\sum t_i = 0$, we have $\sum t_i^{p^{r+1}} = p^{l_r} \sum \lambda_{r,i} \circ u_{\omega_{r,i}}$, where l_r is maximal, $\omega_{r,1} = (p^r, \dots, p^r)$, and $\lambda_{r,1} \neq 0$.

$Sq^2(u_\omega) = 0$. The dimension of the generator u_ω, where ω is an arbitrary decomposition of the number $8a$ into terms $(8a_1, \ldots, 8a_s)$ with $a_i > 0$, is equal to $8a$. The dimensions of the generators of the modules $M_\beta^{(i)}$ are even. We set $N_\beta = \sum M_\beta^{(i)}$ and $\widetilde{N}_\beta = \sum_\omega \widetilde{M}_\beta^\omega$. Obviously, we have

$$\mathrm{Ext}_A^{s,t}(H_{\mathrm{SU}}(2), Z_2) \approx \mathrm{Ext}_A^{s,t}(N_\beta, Z_2) + \mathrm{Ext}_A^{s,t}(\widetilde{N}_\beta, Z_2).$$

The algebras $\mathrm{Ext}_A(M_\beta, Z_2)$ and $\mathrm{Ext}_A(\widetilde{M}_\beta, Z_2)$ are calculated in Chapter 2 (see also Theorem 3 for the algebra $\mathrm{Ext}_A(\widetilde{M}_\beta, Z_2)$). As above,

$$h_0 \in \mathrm{Ext}_A^{1,1}(H_{\mathrm{SU}}(2), Z_2), \quad h_1 \in \mathrm{Ext}_A^{1,2}(H_{\mathrm{SU}}(2), Z_2)$$

denote known elements satisfying the relations $h_0 h_1 = 0$, $h_1 \neq 0$, and $h_1^3 = 0$ (see Theorem 3).

The results of Section 2.3 of Chapter 2 readily imply the following assertion.

Lemma 18. *If $h_0^k x = 0$, where $k > 0$ and $x \in \mathrm{Ext}_A^{s,t}(H_{\mathrm{SU}}(2), Z_2)$, then $x = h_1 y$. If $x \in \mathrm{Ext}_A^{s,t}(\widetilde{N}_\beta, Z_2)$, then $h_1^2 x = 0$ implies $h_1 x = 0$ for $t - s = 2k$. If $t - s = 2k + 1$, then x always equals $h_1 y_1$. If $x \in \mathrm{Ext}_A^{s,t}(N_\beta, Z_2)$, then $h_1 x = h_1 y$, where $y \in \mathrm{Ext}_A^{s,t}(\widetilde{N}_\beta, Z_2)$, and $t - s = 2k$ whenever $x \neq 0$.*

Now, consider the Adams spectral sequence specified in Section 2.4 of Chapter 2 (see also [1]).

Lemma 18 and the multiplicative properties of the Adams spectral sequence imply the following theorem.

Theorem 9. *If $x \in \mathrm{Ext}_A^{s,t}(N_\beta, Z_2)$, then $d_i(x) = d_i(y)$, where $y \in \mathrm{Ext}_A^{s,t}(\widetilde{N}_\beta, Z_2)$, for all $i \geq 2$. The elements $h_0 \in \mathrm{Ext}_A^{1,1}(H_{\mathrm{SU}}(2), Z_2)$ and $h_1 \in \mathrm{Ext}_A^{1,2}(H_{\mathrm{SU}}(2), Z_2)$ are cycles for all differentials. If $h_0 x \neq 0$ for $x \in E_r^{s,t}(H_{\mathrm{SU}}(2), Z_2)$, then $x \neq d_r(y)$ for any $y \in E_r^{s-r,t-r+1}(H_{\mathrm{SU}}(2), Z_2)$ $(r \geq 2)$. If $t - s = 2k + 1$ and $x \in E_r^{s,t}$, then $x = h_1 y$ and $h_1 x \neq 0$ for all $r \geq 2$. If $x \in E_2^{s,t}$, $t - s = 2k$, and $x = h_1^2 y$, then $d_i(x) = 0$ for $i \geq 2$.*

Since multiplication by the element h_0 in E_∞ is adjoint to multiplication by 2 in the ring V_{SU} and the element $h_1 \in E_\infty^{1,2}$ determines an element $\bar{h}_1$ of V_{SU} such that $2\bar{h}_1 = 0$, $\bar{h}_1^2 \neq 0$, and $\bar{h}_1^3 = 0$, Theorem 9 implies the following result.

Corollary 6. *The groups V_{SU}^{2k+1} have no elements of order 4 for all $k \geq 2$. Moreover, if $x \in V_{\mathrm{SU}}^{2k+1}$, then $2x = 0$ and $x = \bar{h}_1 y$, where the element $y \in V_{\mathrm{SU}}^{2k}$ can be assumed to have infinite order, and $\bar{h}_1^2 y = \bar{h}_1 x \neq 0$ whenever $x \neq 0$.*

Theorem 3 from Chapter 2 shows that, in the algebras $E_2 = E_2(\mathrm{SU}) = \mathrm{Ext}_A(H_{\mathrm{SU}}(2), Z_2)$ and $E_\infty = E_\infty(\mathrm{SU})$, the relation $h_0 x = 0$ always implies $h_0 x = 0$. This gives rise to the question: Can the groups V_{SU}^{2k} contain elements of order 4?

Appendix 2
The Milnor Generators of the Rings V_{SO} and V_{U}

Consider an algebraic submanifold $H_{r,t} \subset P^r(\mathbb{C}) \times P^t(\mathbb{C})$ realizing the cycle $P^{r-1}(\mathbb{C}) \times P^t(\mathbb{C}) + P^r(\mathbb{C}) \times P^{t-1}(\mathbb{C})$ without singularities. It is easy to show that

$$(r + t - 1)[H_{r,t}] = -\binom{r+t}{r}.$$

It is also known that $(r+t-1)[P^{r+t-1}(\mathbb{C})] = +(r+t)$. Note that the greatest common divisor of the numbers $\left\{\binom{k}{i}\right\}$ $(i = 1, \dots, k-1)$ is equal to 1 if $k \neq p^l$ for any prime $p \geq 2$ and to 2 if $k = p^l$. Thus, taking a linear combination of the manifolds $H_{r,t}$ and $P^{r+t-1}(\mathbb{C})$ with $r+t = \text{const}$, we can obtain a manifold Σ^{r+t-1} such that

$$(r+t-1)[\Sigma^{r+t-1}] = \begin{cases} 1, & r+t \neq p^l, \\ p, & r+t = p^l. \end{cases}$$

According to Theorem 5 from Chapter 2 (see also [17]), the sequence of manifolds

$$\Sigma^1, \Sigma^2, \dots, \Sigma^k, \dots$$

is a system of polynomial generators of the ring V_{U}. Now, consider the natural ring homomorphism $V_{\mathrm{U}} \to V_{\mathrm{SO}}/T$, where V_{SO}/T is the quotient ring of V_{SO} modulo 2-torsion. It is easy to show that the composition

$$V_{\mathrm{U}} \to V_{\mathrm{SO}} \to V_{\mathrm{SO}}/T$$

is an epimorphism. Therefore, the manifolds Σ^{2k} generate the ring V_{SO}/T. The characteristic numbers of the manifolds Σ^k are easy to calculate, and the question of what set of numbers can be the set of Pontryagin numbers for a smooth manifold is completely solved by Milnor (as well as similar questions concerning the Chern numbers of algebraic, complex analytic, almost complex, and U-framed manifolds).

References

[1] J. F. Adams, On the structure and applications of the Steenrod algebra, *Comm. Math. Helv.*, **32**, No. 3 (1958), 180–214.

[2] B. G. Averbukh, The algebraic structure of inner homology groups, *Dokl. Akad. Nauk SSSR*, **125**, No. 1 (1959), 11–14.

[3] A. Borel, Sur la cohomologie des éspaces fibres principaux et des éspaces homogenes de groupes de Lie compacts, *Ann. of Math.*, **57** (1953), 115–207.

[4] A. Borel and J. P. Serre, Groupes de Lie et puissances réduites de Steenrod, *Amer. Math. J.*, **75** (1953), 409–448.

[5] A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces. 1, *Amer. Math. J.*, *80* (1958), 459–538.

[6] H. Cartan, Sur l'itération des opérations de Steenrod, *Comm. Math. Helv.*, **29**, No. 1 (1955), 40–58.

[7] H. Cartan, *Seminaire*, 1954/55.

[8] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Univ. Press, 1956, 1–390.

[9] J. Milnor, *Lectures on Characteristic Classes*, Princeton Univ., 1958 (mimeographed).

[10] J. Milnor, On the cobordism ring, *Notice of Amer. Math. Soc.*, **5** (1958), 457.

[11] M. M. Postnikov, A study on the homotopy theory of continuous maps, *Trudy Mat. Inst. im. V. A. Steklova*, **XXXXVI** (1955).

[12] V. A. Rokhlin, Inner homology, *Dokl. Akad. Nauk SSSR*, **89**, No. 5 (1953), 789–792.

[13] V. A. Rokhlin, Inner homology, *Dokl. Akad. Nauk SSSR*, **119**, No. 5 (1958), 876–879.

[14] J. P. Serre, Cohomologie modulo 2 des compléxes d'Eilenberg–MacLane, *Comm. Math. Helv.*, **27**, No. 3 (1953), 198–232.

[15] J. P. Serre and G. Hochschild, Cohomologies of group extensions, *Trans. Amer. Math. Soc.*, **74** (1953), 110–134.

[16] R. Thom, Quelques propriétés globales de variétés différentiables, *Comm. Math. Helv.*, **28** (1954), 17–86.

[17] F. Hirzebruch, Die complexe Mannigfaltigkeiten, *Proc. Intern. Congress in Edinburg*, 1960.

[18] J. Milnor, On the cobordism ring and a complex analogue. I, *Amer. Math. J.*, **82**, No. 3 (1960), 505–521.

[19] S. P. Novikov, On some problems of the topology of manifolds related to the theory of Thom spaces, *Dokl. Akad. Nauk SSSR*, **132**, No. 5 (1960), 1031–1034.

[20] G. T. G. Wall, Determination of the cobordism ring, *Ann. of Math.*, **72**, No. 2 (1960), 292–311.

Moscow State University