Equivariant Structure on Smash Powers of Commutative Ring Spectra

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Contents

Preface

Introduction

Topological Hochschild homology (THH) of a "ring up to homotopy" has, after its existence and even some calculations had been conjectured by Goodwillie in the 1980's, appeared in several guises. Bökstedt, who gave the first rigorous definition in [Bö], used an ad hoc construction on (simplicial) functors with smash products, since a strictly symmetric monoidal model for the stable homotopy category was not yet available. The construction of such categories via $\mathbb{S}$-modules in [EKMM], symmetric spectra in [HSS] and later via orthogonal spectra in [MMSS] has brought forward various alternative versions that follow Goodwillies original idea of just mimicking algebra more closely.

In particular, THH of a ring spectrum in either of the above categories can be defined as the realization of a simplicial object closely resembling the algebraic Hochschild complex where one uses the symmetric monoidal smash product instead of the algebraic tensor product. These constructions have been carried out and studied by the authors of [EKMM] in terms of $\mathbb{S}$-algebras, by Shipley in the category of symmetric ring spectra in [Sh] and more recently by Kro in the context of orthogonal ring spectra in his thesis [Kr].

For the case of $\mathbb{S}$-algebras in the sense of [EKMM], it is possible to encode the same data via the categorical tensor with the circle group S^1 (cf. [MSV, A]). However, care has to be taken: Already in [EKMM, IX.3.9] the authors remark that, for a commutative $\mathbb{S}$-algebra A, the categorical tensor $A \otimes S^1$, which canonically inherits an S^1-action from the second factor, does not necessarily have the right *equivariant* homotopy type. This is particularly crucial since the S^1-equivariant structure of THH(A), that was already present in Bökstedt's model, is the key ingredient in Bökstedt, Hsiang and Madsen's definition of Topological Cyclic homology (TC).

Recently efforts have been made to generalize the above and study iterated or higher versions of both THH and TC of commutative ring spectra (e.g. [BCD, CDD, Schl, BM]). Especially for these applications it is essential to have a model for THH at hand that generalizes easily, while at the same time displaying the correct equivariant structure. In [BCD] the authors address this by defining a topological analogue of the Loday functor from algebra (cf. [L89]) in the style of Bökstedt's definition. This functor $\Lambda_X(A)$ produces THH when evaluated at $X = S^1$, and the equivariant structure necessary for calculating TC is the one induced by functoriality. The much richer equivariant structure on evaluations of $\Lambda_{-}A$ on tori, leading to Higher Topological Hochschild and Cyclic homology, has been investigated closely in [BCD] and [CDD]. However, the construction of the Loday functor in [BCD] is, due to the setting of simplicial functors, rather complicated and in particular a lot of work has to be done to actually achieve functoriality.

The main goal of this thesis is to define and study the properties of a topological Loday functor in a more accessible framework of spectra, for which we chose the orthogonal spectra of [MMSS]. The main reason for this choice is that we have to deal with the equivariant structure, for which the category of equivariant orthogonal spectra of [MM], together with Kro's exposition, forms a convenient basis.

However, the classical model structures on commutative orthogonal ring spectra are not quite flexible enough for our purposes. To remedy this, we will develop the convenient $\mathbb{S}$-model structures as the main results of the first half of the thesis, both in a non equivariant setting and taking actions of compact Lie groups into account (cf. Theorems 1.3.28,1.3.29,2.3.37 and 2.3.38). This will then allow us to verify that the following rather simple definition is appropriate:

Definition 1 (cf. 3.2.1)**.** The *Loday functor* $\Lambda_{-}(-)$ on $\mathbb{S}$-cofibrant commutative orthogonal ring spectra is the categorical tensor.

In particular, for discrete input, the Loday functor specializes to the fundamental example of the smash powers of commutative ring spectra mentioned in the title (cf. 3.2.5). We completely describe the equivariant structure of such smash powers in terms of their geometric fixed points in Theorem 3.4.26. This result, together with its non discrete analogue Theorem 3.4.48, forms the basis for the verifications that our Loday functor indeed allows the study of higher analogues of THH and TC in the sense of [BCD] and [CDD] in Section 3.6.

A key technical tool used in our construction of the $\mathbb{S}$-model structures is the Assembling Theorem B.2.8, which we formulate and prove in the appendix. It is one of many instances throughout the thesis where we take advantage of so called semi-free objects. Such objects have appeared in the context of symmetric spectra in [S] and [Sh04], but to the authors knowledge a general treatment is given for the first time in this thesis (cf. Subsections 2.2.2 and B.2).

Organization

The thesis is organized as follows. We begin Chapter 1 with a recollection of the definition and elementary categorical properties of orthogonal spectra, before we define the semi-free objects and investigate their properties more closely in Subsection 1.2.2. We continue with a review of the classical model structures from [MMSS], before we construct the convenient $\mathbb{S}$-model structure for orthogonal spectra in Section 1.3. The model structure is extended to commutative orthogonal ring spectra in 1.3.4.

Chapter 2 is laid out parallel to the first, in the equivariant context. We again start with recalling definitions before we spend some time on the semi-free equivariant spectra in Subsection 2.2.2, as they will play crucial roles in most of what follows. We briefly collect some technical results about change of universe and change of group functors as well as the classical model structures for equivariant spectra from [MM]. The construction of the equivariant $\mathbb{S}$-model structure is then given in Section 2.3, and we pass to equivariant commutative ring spectra in 2.3.4.

Chapter 3 begins with a description of the algebraic Loday functor and the categorical tensors for commutative orthogonal ring spectra that form the motivation for our definition of several versions of topological Loday functors in Section 3.2. We then recall definitions and elementary properties for geometric fixed points, before turning to the fixed points of smash powers of different types of cells in 3.3.2, 3.3.3 and 3.3.5. We prove the existence of an equivariant cellular filtration on smash powers of $\mathbb{S}$-cofibrant spectra in 3.4.2, and use it to study geometric fixed points of general smash powers in 3.4.26. Subsection 3.4.3 contains the generalization to non discrete versions of the Loday functor, before we briefly review homotopical properties in 3.5. Together these allow us to both compare our construction to that of [BCD] in subsection 3.5.2, and also study the higher analogues of THH and TC in section 3.6.

The appendix is split into 3 parts. Part A collects some of the category theory we use heavily throughout the thesis, it focuses especially on the enriched theory. Part B deals with model categories, and we in particular restate some of the less commonly used definitions and properties. We collect what we need from unstable homotopy theory in B.1.1, and from the theory of simplicial objects in B.1.2. All of the results in either of these subsections must be well known, but can be hard to find in the literature. In Section B.2 we give the constructions necessary to state and prove the assembling Theorem B.2.8, which underlies the level model structures in Chapters 1 and 2. In Part C we collect results from non stable equivariant homotopy theory which seem to be even harder to find in the literature, in particular if one is interested in formulations in terms of model categories. In Section C.2 we give a brief account of Illman's triangulation theorems, which are crucial for a lot of model theoretic work with compact Lie groups.

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Chapter 1

Orthogonal Spectra

1.1 Introduction

Orthogonal spectra were first defined in [MMSS] in terms of diagram spaces, i.e. functors from certain diagram categories to spaces. Already there, several equivalent definitions were introduced and by now there exist even more descriptions that are commonly used. For the purpose of this overview, let us work with the structurally easiest definition:

Definition 1.1.1. An *orthogonal spectrum* X, consists of a sequence of spaces $\{X_n\}_{n\in\mathbb{N}}$, often called the *levels of* X, together with the following extra structure:

- For each $n \in \mathbb{N}$, an action of the orthogonal group $\mathbf{O}_n$ on the n^{th} space X_n.
- For each $n \in \mathbb{N}$, a *structure map* $\sigma_n\colon X_n \wedge S^1 \to X_{n+1}$, such that for any $n, m \in \mathbb{N}$, the iterated structure map

  $$X_n \wedge S^m \xrightarrow{\cong} X_n \wedge S^1 \wedge \ldots \wedge S^1 \xrightarrow{\sigma_{n+m-1}\circ\ldots\circ(\sigma_n\wedge S^{m-1})} X_{n+1+\ldots+1} \xrightarrow{=} X_{n+m}$$

  is an $\mathbf{O}_n \times \mathbf{O}_m$-equivariant map.

Here, and everywhere throughout this thesis unless we explicitly state otherwise, *space* means a based, compactly generated weak Hausdorff space (e.g. [St]), and maps are assumed to be continuous. Actions of (topological) groups are implied to be basepoint preserving and continuous (cf. A.2.19). The spheres S^n are equipped with an action of $\mathbf{O}_n$ by viewing them as the one-point compactifications $(\mathbb{R}^n)^+$ of the vector spaces $\mathbb{R}^n$, which we always think of as being equipped with the standard basis and inner product. The action of $\mathbf{O}_n \times \mathbf{O}_m$ on X_{n+m} is by inclusion of (n, m)-block matrices into $\mathbf{O}_{n+m}$.

Definition 1.1.2. A *morphism $f : X \to Y$ of orthogonal spectra* is a sequence of maps $\{f_n \colon X_n \to Y_n\}_{n\in\mathbb{N}}$ such that all the diagrams

$$\begin{tikzcd} X_n \wedge S^1 \arrow[r, "f_n\wedge S^1"] \arrow[d, "\sigma_n"'] & Y_n \wedge S^1 \arrow[d, "\sigma_n"] \\ X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1} \end{tikzcd} \tag{1.1.3}$$

commute.
The *category of orthogonal spectra* together with these morphisms will be denoted by $\mathcal{S}p^{\mathcal{O}}$.

Example 1.1.4 (The Sphere Spectrum)**.** The *sphere spectrum* $\mathbb{S}$ is the orthogonal spectrum given in level n by the n-sphere

$$\mathbb{S}_n := S^n,$$

with the action of the $\mathbf{O}_n$ as above by viewing S^n as the one-point compactification of $\mathbb{R}^n$. For the structure maps $\sigma_n \colon S^n \wedge S^1 \to S^{n+1}$ we once and for all chose linear isometries $\mathbb{R}^n \oplus \mathbb{R} \cong \mathbb{R}^{n+1}$ for all natural numbers and again take their one-point compactifications.

Example 1.1.5 (Suspension Spectra)**.** Let A be a space. The *suspension spectrum* $\Sigma^\infty A$ of A is given by

$$(\Sigma^\infty A)_n := A \wedge S^n,$$

i.e. the levelwise smash product of A with $\mathbb{S}$, and the structure maps inherited from $\mathbb{S}$. Note that the commutativity of the diagrams (1.1.3) implies that a morphism $\Sigma^\infty A \to Y$ of spectra is the same as a map $A \to Y_0$ of spaces, i.e. we have a natural isomorphism

$$\mathcal{S}p^{\mathcal{O}}(\Sigma^\infty A, Y) \cong \mathcal{T}(A, Y_0). \tag{1.1.6}$$

The category $\mathcal{S}p^{\mathcal{O}}$ of orthogonal spectra admits rich extra structure. Categorically, it has been shown to be closed symmetric monoidal in [MMSS, 1.7], and hence it permits the study of monoid objects, usually called *orthogonal ring spectra*, as well as algebras and modules over such. The suspension functor above is a specific example of a rather big class of functors linking $\mathcal{S}p^{\mathcal{O}}$ with various categories of spaces. We will review these properties and constructions throughout this section, while also providing definitions of some equivalent categories of orthogonal spectra. Moving towards homotopy theory, the suspension spectrum functor again serves as a nice motivation, as the following definition provides an extension of the well known stable homotopy groups of spaces through postcomposition:

Definition 1.1.7. For an orthogonal spectrum X and $k \in \mathbb{Z}$, define the *homotopy groups* $\pi_k(X)$ as the colimit

$$\pi_k(X) := \operatorname{colim} \pi_{k+n}(X_n),$$

along the maps

$$\pi_{k+n} X_n \xrightarrow{-\wedge S^1} \pi_{k+n+1} X_n \wedge S^1 \xrightarrow{\sigma_*} \pi_{k+n+1} X_{n+1}.$$

A map of spectra is called π_**-isomorphism* if it induces isomorphisms on all such homotopy groups.

This definition does not make use of the orthogonal group actions on the X_i, and indeed in [MMSS, 0.1] the authors show that there is an equivalence between the homotopy categories of orthogonal spectra, obtained by formally inverting the π_*-isomorphisms, and the stable homotopy category $\mathcal{SHC}$ of classical sequential spectra (e.g. [BF]). More precisely the following theorem holds:

Theorem 1.1.8. *[MMSS, 9.2, 10.4, 12.6] The category $\mathcal{S}p^{\mathcal{O}}$ is a cofibrantly generated, stable, proper, monoidal, topological model category with respect to the π_*-isomorphisms, q-cofibrations and q-fibrations. The forgetful functor is the right adjoint of a Quillen equivalence to the classical stable model structure on sequential spectra.*

We will review some results leading up to this theorem which will then be adapted in our construction of the so called $\mathbb{S}$-model structure on orthogonal spectra (1.3.10). This model structure turns out to have similar favourable properties in addition to being "convenient" when lifting model structures to commutative orthogonal ring spectra – both in the literal, and in the technical sense of Theorem 1.3.29.

1.2 Recollections

1.2.1 Definition(s)

Since having different viewpoints available to us will prove advantageous, we are going to recall definitions for several equivalent categories of orthogonal spectra as well as translations between them. A lot of this discussion can be extracted from the above mentioned [MMSS], though we lean our exposition in a slightly different direction.
To begin with, note that the enrichment of the category $\mathcal{T}$ of spaces over itself (A.2.13) also gives $\mathcal{S}p^{\mathcal{O}}$ the structure of a topological category A.2.15:

Definition 1.2.1. The category $\mathcal{S}p^{\mathcal{O}}$ of orthogonal spectra is enriched (cf. A.2) over $(\mathcal{T}, \wedge, S^0)$, by topologizing the set $\mathcal{S}p^{\mathcal{O}}(X, Y)$ as the subspace of the product $\prod_{n\in\mathbb{N}} \mathcal{T}(X_n, Y_n)$ in $\mathcal{T}$, containing those sequences of maps that make the diagrams (1.1.3) commute.

It is also tensored and cotensored (cf. A.2.24) over spaces by using the levelwise smash product and function space on spaces, i.e.

Definition 1.2.2. For a spectrum X and a space A define the *tensor* or *smash product* $A\wedge X$ as the spectrum

$$(A\wedge X)_n := A\wedge X_n,$$

and the structure maps induced from X.
The *cotensor* or *function spectrum* $A \pitchfork X := F(A, X)$ is the spectrum

$$F(A, X)_n := \mathcal{T}(A, X_n),$$

also with the structure maps induced from X.

The second definition of orthogonal spectra we give is in terms of $\mathbb{S}$-modules. Here the term module is to be understood as an object with a right action of a monoid in some monoidal category (cf. A.1.16). To be precise consider the following definition.

Definition 1.2.3. An *orthogonal sequence* or *orthogonal space* X is a sequence $\{X_n\}_{n\in\mathbb{N}}$ of spaces with an action of $\mathbf{O}_n$ on X_n. A *morphism of orthogonal sequences* is a sequence of appropriately equivariant maps. Denote the *category of orthogonal sequences* by $I\mathcal{T}$.

As above, $I\mathcal{T}$ inherits a topological enrichment from $\mathcal{T}$ via the product

$$I\mathcal{T}(X, Y) := \prod_{n\in\mathbb{N}} \mathbf{O}_n\mathcal{T}(X_n, Y_n),$$

where the $\mathbf{O}_n\mathcal{T}$ denote the categories of spaces with $\mathbf{O}_n$-action (cf. A.2.19).
The category $I\mathcal{T}$ is closed symmetric monoidal in the following way:

Definition 1.2.4. Let $X = \{X_n\}$ and $Y = \{Y_n\}$ be orthogonal sequences, their *tensor product* $X \otimes Y$ is the sequence given by

$$(X \otimes Y)_n := \bigvee_{p+q=n} \mathbf{O}_{n+} \wedge_{\mathbf{O}_p \times \mathbf{O}_q} (X_p \wedge Y_q),$$

where the subscript $+$ to $\mathbf{O}_n$ denotes an added disjoint basepoint and each of the the wedge factors is the $\mathbf{O}_n$-space created from the $\mathbf{O}_p \times \mathbf{O}_q$-space by induction along the inclusion of $(p \times q)$-block matrices (cf. C.1.5).

The tensor product is associative by repeatedly using the associativity isomorphisms for the smash product of spaces and the symmetry isomorphisms for the coproduct of spaces. It is also symmetric via the natural isomorphism induced on wedge summands by:

$$\begin{array}{rcl} \mathbf{O}_{n+} \wedge_{\mathbf{O}_p \times \mathbf{O}_q} X_p \wedge Y_q & \to & \mathbf{O}_{n+} \wedge_{\mathbf{O}_q \times \mathbf{O}_p} Y_q \wedge X_p \\ (A, x, y) & \mapsto & (AT_{q,p}, y, x), \end{array}$$

where $T_{q,p}$ is the orthogonal block matrix $\begin{pmatrix} 0 & \mathbb{I}_p \\ \mathbb{I}_q & 0 \end{pmatrix}$ that shuffles the first q past the last p coordinates. A unit object is given by the orthogonal sequence $\{S^0, *, *, \ldots\}$, that has just the basepoint in all levels except zero, where it is S^0.

Definition 1.2.5. Given an orthogonal sequence Y in $I\mathcal{T}$, its *m-fold shift* is the orthogonal sequence $\mathrm{sh}_m Y$ with

$$(\mathrm{sh}_m Y)_n := Y_{m+n}.$$

The $\mathbf{O}_n$-action is via inclusion of lower right block matrices, hence the whole sequence $\mathrm{sh}_m Y$ has an action of $\mathbf{O}_m$, using the levelwise inclusion of upper left block matrices.

This allows us to identify the internal **Hom** functor of $I\mathcal{T}$ as the function sequence $F(X, Y)$, which is defined as

$$F(X, Y)_m := I\mathcal{T}(X, \mathrm{sh}_m Y), \tag{1.2.6}$$

where the $\mathbf{O}_m$-action on $\mathrm{sh}_m Y$ gives the required action on $F(X, Y)_m$.

Remark 1.2.7. In the spirit of Example 1.1.5, we can embed the category of spaces as those sequences that have just a basepoint in all levels except 0. Denote the image of a space A under this functor by A^*. The analogue of the natural isomorphism 1.1.6 then gives

$$I\mathcal{T}(A^*, X) \cong \mathcal{T}(A, X_0), \tag{1.2.8}$$

and we can use this in combination with the closed symmetric monoidal structure on $I\mathcal{T}$ to define tensors and cotensors over $\mathcal{T}$. Just as for spectra, these turn out to be levelwise.

Remark 1.2.9. One can identify the category $I\mathcal{T}$ of orthogonal sequences as above with the $\mathcal{T}$-category $[I, \mathcal{T}]$ of continuous functors $I \to \mathcal{T}$, where I is the category with one object for each natural number, morphism spaces $I(n,n) = \mathbf{O}_{n+}$ and $I(n,m) = *$ for $n \neq m$. Then I is strict monoidal with product $(- + -)$ and symmetric via the continuous natural isomorphism $\tau_{p,q} = T_{p,q} \in \mathbf{O}_{p+q}$. The symmetric monoidal structure on $I\mathcal{T}$ described above is an example of the one constructed in [MMSS, §21].

Remark 1.2.10. This interpretation of $I\mathcal{T}$as an enriched functor category, allows us to reinterpret equation (1.2.8) as a special case of an evaluation adjunction, cf. 1.2.26.

Example 1.2.11 (The Sphere Sequence)**.** Forgetting the structure maps of the sphere spectrum $\mathbb{S}$ yields an orthogonal sequence, which we denote by the same symbol. This sphere sequence is a commutative monoid in $I\mathcal{T}$, where the unit map is the identity in level 0 and the inclusion of the basepoint everywhere else. The multiplication map is given on wedge summands by:

$$\begin{array}{rcl} \mathbf{O}_{n+} \wedge_{\mathbf{O}_p \times \mathbf{O}_q} (\mathbb{R}^p)^+ \wedge (\mathbb{R}^q)^+ & \to & (\mathbb{R}^n)^+ \\ (A, v, w) & \mapsto & (A(v+w)). \end{array}$$

Note that without the addition of the $\mathbb{T}_{p,q}$ in the definition of the symmetry morphism above, the sphere spectrum would not be a *commutative* monoid.

The following lemma is then immediately verified by comparing the explicit Definition 1.2.4 with (1.1.3)

Lemma 1.2.12. *The category $\mathcal{S}p^{\mathcal{O}}$ is equivalent to the category of right $\mathbb{S}$-modules in $I\mathcal{T}$. The equivalence is $\mathcal{T}$-enriched, i.e. through continuous functors.*

This in particular implies, that $\mathcal{S}p^{\mathcal{O}}$ is itself closed symmetric monoidal by A.1.16. Formally we denote the resulting monoidal product and internal **Hom** functor by $\wedge_{\mathbb{S}}$ and $F_{\mathbb{S}}$, but we will usually drop the index $\mathbb{S}$ when the context is clear.

Another alternative definition of orthogonal spectra is as a category of continuous functors itself. The appropriate diagram category is the following:

Definition 1.2.13. Let $\mathbf{O}$ be the topological category with the same objects as I and morphism spaces given by

$$\mathbf{O}(n, n+m) := \mathbf{O}_{n+m+} \wedge_{\mathbf{O}_m} S^m,$$

if $m \geq 0$ and just a basepoint otherwise. Composition is then given by inclusion of upper left block matrices and matrix multiplication on one factor and induced by the isomorphisms $S^m \wedge S^{m'} \cong S^{m+m'}$ on the other.

Remark 1.2.14. The notation is consistent in the sense that the endomorphism spaces $\mathbf{O}(n, n)$ are given by the orthogonal groups $\mathbf{O}_n$. Both ways of writing these have their benefits, which will become more obvious in the next paragraphs, where we discuss coordinate free spectra. For convenience, we will abbreviate $[\mathbf{O}, \mathcal{T}]$ as $\mathbf{O}\mathcal{T}$.

Lemma 1.2.15. *There is an equivalence of categories $\mathcal{S}p^{\mathcal{O}} \leftrightarrows \mathbf{O}\mathcal{T}$, where $\mathbf{O}\mathcal{T}$ is the category of continuous functors $\mathbf{O} \to \mathcal{T}$ and continuous natural transformations between them.*

Proof. This is a special case of Theorem [MMSS, 2.2]. The appropriate functors are given in the following way:
Let X be an orthogonal spectrum in the sense of Definition 1.1.1. Then X defines a continuous functor by setting $X(n) := X_n$ for objects $n \in \mathbf{O}$ and using the map adjoint to the composition

$$X_n \wedge \mathbf{O}_{n+m+} \wedge_{\mathbf{O}_m} S^m \xrightarrow{\sigma} \mathbf{O}_{n+m+} \wedge X_{n+m} \xrightarrow{\text{act.}} X_{n+m}$$

on morphisms.
Both the inverse equivalence and the construction of natural transformations from morphisms of spectra are immediate. □

In the discussion of orthogonal spectra and orthogonal sequences so far, the vector spaces $\mathbb{R}^n$ have played a pivotal role. This can be interpreted as a choice of basis for any given finite dimensional vector space, which is of course non canonical. This choice makes some computations easier, for example when dealing with homotopy groups of spectra (e.g. 1.3.11) or when describing the explicit formulas for the tensor product (1.2.4). However, it is also often helpful to be able to avoid it. This is done via the so called *coordinate free orthogonal spectra*, another category equivalent to $\mathcal{S}p^{\mathcal{O}}$. These could again be studied analogous to 1.1.1, but we chose to present the approach via functor categories:

Definition 1.2.16. Let $\mathcal{I}$ denote the topological category of all *finite dimensional euclidean vector spaces*, and *linear isometries* between them.
Similarly, let $\mathcal{O}$ be the category with the same objects but morphism spaces constructed in the following way:

Construction 1.2.17. For V and W finite dimensional euclidean vector spaces denote by $\mathcal{L}(V,W)$ the space of linear isometric embeddings of V into W, and note that $\mathcal{L}(V,W) = \mathcal{I}(V,W)$ if the dimensions of V and W are equal. Let $\mathcal{E}(V,W)$ be the subbundle of the trivial W-bundle over $\mathcal{L}(V,W)$ given by all points

$$\{(f,w) \in \mathcal{L}(V,W) \times W, w \in f(V)^{\perp}\},$$

i.e. where w is in the orthogonal complement of the image of f in W.
Define $\mathcal{O}(V,W)$ as the Thom space of $\mathcal{E}(V,W)$, obtained by one-point compactification at each fiber and gluing together all the added infinity points. Note that $\mathcal{E}(V,W)$ is empty if $\dim W < \dim V$, hence $\mathcal{O}(V,W)$ is just a basepoint in that case. Define composition in $\mathcal{O}$ as the continuous map:

$$\begin{array}{rcl} \mathcal{O}(W,U) \wedge \mathcal{O}(V,W) & \to & \mathcal{O}(V,U) \\ (g,u),(f,w) & \mapsto & (g \circ f, g(w)+u), \end{array}$$

and the identity maps as $(\mathrm{id}_V, 0)$ in $\mathcal{O}(V,V)$.

Lemma 1.2.18. *For any particular choice of isometric embedding $i : V \to W$, such that $W \cong i(V) \oplus V'$ is an orthogonal direct sum, there is an isomorphism*

$$\mathcal{O}(V,W) \cong \mathbf{O}_{W+} \wedge_{\mathbf{O}_{V'}} S^{V'}.$$

Proof. Let f be an isometric embedding $V \to W$, then there exists a lift g_f,

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "i"'] & W \\ W \arrow[ur, dotted, "\exists g_f"'] & \end{tikzcd},$$

which is a self isometry of W. Define the desired isomorphism via:

$$\begin{array}{rcl} \mathcal{O}(V,W) & \to & \mathbf{O}_{W+} \wedge_{\mathbf{O}_{V'}} S^{V'} \\ (f,w) & \mapsto & (g_f, g_f^{-1} w). \end{array}$$

This is well defined, since the choice of lift only changes the image up to the action of an element of $\mathbf{O}_{V'}$. Note that

$$[g,w] \mapsto (g \circ i, g(w))$$

gives a well defined inverse. $\square$

Remark 1.2.19. In particular, we can consider the category $\mathbf{O}$ as the full subcategory of $\mathcal{O}$, containing the objects $n = \mathbb{R}^n$, equipped with the standard basis, the standard scalar product, and the standard inclusions $\mathbb{R}^n \to \mathbb{R}^{n+1}$. Similarly I is a full subcategory of $\mathcal{I}$. In both cases the subcategories are skeleta. Hence from now on, we will not differentiate the notation for morphism spaces and always use the symbols $\mathcal{O}$ respectively $\mathcal{I}$.

Remark 1.2.20. Note that all non trivial morphisms in $\mathcal{O}$ between vector spaces of the same dimension are isomorphisms. Also, the composition maps

$$\mathcal{O}(V,W)\wedge\mathcal{O}(U,V) \to \mathcal{O}(U,W)$$

factor through the quotient map

$$\mathcal{O}(V,W)\wedge\mathcal{O}(U,V) \to \mathcal{O}(V,W)\wedge_{\mathbf{O}_V}\mathcal{O}(U,V) \longrightarrow \mathcal{O}(U,W),$$

where the quotient is taken along the diagonal action by postcomposition respectively precomposition with the inverse. Here the latter map is an isomorphism if and only if $\dim U = \dim V$ or $\dim V = \dim W$.

Definition 1.2.21. Define the category $\mathcal{OT}$ of *coordinate free orthogonal spectra* as the category $[\mathcal{O},\mathcal{T}]$ of continuous functors $\mathcal{O} \to \mathcal{T}$ and continuous natural transformations between them.

Remark 1.2.22. Since $\mathbf{O}$ is in particular a skeleton of $\mathcal{O}$, we immediately get an equivalence of categories

$$\mathbf{O}\mathcal{T} \leftrightarrows \mathcal{OT}.$$

The right adjoint is the restriction to levels $\mathbb{R}^n$, and the left adjoint is given by the left Kan extension (cf. A.2.3), which in this case specializes to evaluating an orthogonal spectrum X in $\mathbf{O}\mathcal{T}$ at any vector space V:

$$X_V := \mathcal{O}(\mathbb{R}^n, V)\wedge_{\mathbf{O}_n}X_n,$$

where n is the dimension of V. The *generalized structure maps*, i.e. the maps

$$\mathcal{O}(V,W)\wedge X_V \to X_W,$$

are then given by the following compositions:

$$\begin{tikzcd}
\mathcal{O}(V,W)\wedge X_V \arrow[r, "="] & \mathcal{O}(V,W)\wedge\mathcal{O}(\mathbb{R}^n,V)\wedge_{\mathbf{O}_n}X_n \arrow[d] \\
& \mathcal{O}(\mathbb{R}^n,W)\wedge_{\mathbf{O}_n}X_n \arrow[d, "\cong"] \\
& \mathcal{O}(\mathbb{R}^m,W)\wedge_{\mathbf{O}_m}\mathcal{O}(\mathbb{R}^n,\mathbb{R}^m)\wedge_{\mathbf{O}_n}X_n \arrow[d, "(\mathrm{id},\sigma)"] \\
X_W & \mathcal{O}(\mathbb{R}^m,W)\wedge_{\mathbf{O}_m}X_m \arrow[l, "="']
\end{tikzcd} \tag{1.2.23}$$

where again $n = \dim V$, $m = \dim W$, the first vertical map is composition in $\mathcal{O}$, the second one is induced by the isomorphism from Remark 1.2.20, and σ is the structure map of X.

Remark 1.2.24. In particular for a coordinate free orthogonal spectrum X, and for V and W two euclidean vector spaces of the same dimension, there is an $\mathbf{O}_W$-equivariant natural isomorphism

$$X_W \cong \mathcal{O}(V, W) \wedge_{\mathbf{O}_V} X_V.$$

1.2.2 Free and Semi-Free Spectra

In this subsection we generalize the suspension spectra of Example 1.1.5. This will be done in two different ways. The first is via *free spectra* which had already been discussed in [MMSS], and were put to great use in, for example, the construction of (stable) model structures. The second method, via *semi-free spectra* is less well studied. Analogues in the case of symmetric spectra have been studied by Schwede in [S], and also featured prominently in Shipley's constructions in [Sh04].

However, a detailed analysis of a more general, or even just the orthogonal case, has to the author's knowledge not been carried out before. We again chose to elaborate on the approach via functor categories, as it seems to generalize best to the equivariant case, studied in later sections. A more general approach will also be discussed in the Appendix (B.2), where we state some properties for more general categories.

For now, let V be a finite dimensional euclidean vector space and consider the following diagram of $\mathcal{T}$-categories:

$$\begin{tikzcd}
\star \arrow[d] \arrow[dddr] & \\
\mathbf{O}_V \arrow[d] \arrow[ddr] & \\
\mathcal{I} \arrow[d] \arrow[dr] & \\
\mathcal{O} \arrow[r] & \mathcal{T},
\end{tikzcd}$$

where $\star$ is the trivial $\mathcal{T}$-category with one object, say V (cf. A.2.3). It is included into $\mathbf{O}_V$, the $\mathcal{T}$-category with again only one object V, but endomorphism space given by $\mathbf{O}_{V+}$. This one is included into $\mathcal{I}$ as the full subcategory containing the vector space V. As before, $\mathcal{I}$ is included as a subcategory of $\mathcal{O}$. Then an orthogonal spectrum X, viewed as a functor $X\colon \mathcal{O} \to \mathcal{T}$, naturally defines underlying functors out of $\star$, $\mathbf{O}_V$ and $\mathcal{I}$ by restriction, or rather precomposition with the above inclusions. These precompositions therefore define $\mathcal{T}$-functors

$$\begin{tikzcd} \mathcal{O}\mathcal{T} \arrow[r] \arrow[rrrr, bend left, "\mathrm{ev}_V"] \arrow[rr, bend right, "\mathrm{ev}'_V"'] & \mathcal{I}\mathcal{T} \arrow[r] & \mathbf{O}_V\mathcal{T} \arrow[r] & {[\star, \mathcal{T}]} \arrow[r, "\cong"] & \mathcal{T}, \end{tikzcd} \tag{1.2.25}$$

assigning to a spectrum X its underlying orthogonal sequence, the $\mathbf{O}_V$-space X_V and the underlying space X_V, respectively. All of the functors in diagram 1.2.25 have left adjoints. For the case of $V = \{0\}$, these are exactly the functors

$$\begin{tikzcd} \mathcal{O}\mathcal{T} & \mathcal{I}\mathcal{T} \arrow[l, "(-)\otimes\mathbb{S}"'] & \mathbf{O}_0\mathcal{T} \arrow[l, "(-)^*"'] \arrow[ll, bend left, "\Sigma^\infty"] & \mathcal{T}, \arrow[l, "\cong"'] \arrow[lll, bend right, "\Sigma^\infty"'] \end{tikzcd}$$

studied above in 1.1.5 and 1.2.7.
In general we denote the respective left adjoints in the following way:

$$\begin{tikzcd} \mathcal{O}\mathcal{T} & \mathcal{I}\mathcal{T} \arrow[l, "(-)\otimes\mathbb{S}"'] & \mathbf{O}_V\mathcal{T} \arrow[l] \arrow[ll, bend left, "\mathcal{G}_V(-)"] & \mathcal{T} \arrow[l, "(-)\wedge\mathbf{O}_{V+}"'] \arrow[lll, bend right, "\mathcal{F}_V(-)"'] \end{tikzcd} \tag{1.2.26}$$

Formally, the existence of the left adjoints is proved using (topological) left Kan extensions (cf. A.2.3). Since we will make excessive use of these functors however, we will also provide the explicit formulas:

Definition 1.2.27. Let V be a finite dimensional euclidean vector space, let $A \in \mathcal{T}$ be a space. Then the *free orthogonal spectrum* $\mathcal{F}_V A$ is given in a level W by

$$(\mathcal{F}_V A)_W := \mathcal{O}(V, W) \wedge A,$$

with the structure maps induced by composition in $\mathcal{O}$.
This assignment with the obvious extension to maps in $\mathcal{T}$ yields a functor

$$\mathcal{F}_V\colon \mathcal{T} \to \mathcal{O}\mathcal{T}.$$

Remark 1.2.28. Note that for a chosen embedding $V \hookrightarrow V \oplus V' \cong W$, this is isomorphic to

$$(\mathcal{F}_V A)_W \cong \mathbf{O}_{W+} \wedge_{\mathbf{O}_{V'}} A \wedge S^{V'},$$

using Lemma 1.2.18.

The other instance of Σ^∞ above generalizes to the semi-free case in the following way:

Definition 1.2.29. Let V be a finite dimensional euclidean vector space, let $K \in \mathbf{O}_V\mathcal{T}$ be a space with $\mathbf{O}_V$-action. Then the *semi free orthogonal spectrum* $\mathcal{G}_V K$ is defined in a level W via

$$(\mathcal{G}_V K)_W := \mathcal{O}(V, W) \wedge_{\mathbf{O}_V} K,$$

where the action of $\mathbf{O}_V$ on $\mathcal{O}(V, W)$ is by precomposition and diagonal on the smash product. The structure maps are induced by composition in $\mathcal{O}$. As in the free case, this defines a functor

$$\mathcal{G}_V : \mathbf{O}_V\mathcal{T} \to \mathcal{O}\mathcal{T}.$$

Remark 1.2.30. Again using Lemma 1.2.18, an isometric embedding

$$V \hookrightarrow V \oplus V' \cong W$$

yields an isomorphism

$$(\mathcal{G}_V K)_W \cong \mathbf{O}_{W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_{V'}} K \wedge S^{V'}.$$

Remark 1.2.31. As indicated by the diagram 1.2.26 above, the free and semi-free spectrum functors factor through $\mathcal{I}\mathcal{T}$, i.e. we have natural isomorphisms

$$\mathcal{F}_V(-) \cong \mathcal{F}_V^{\mathcal{I}}(-) \otimes \mathbb{S}$$

and

$$\mathcal{G}_V(-) \cong \mathcal{G}_V^{\mathcal{I}}(-) \otimes \mathbb{S},$$

with the *free* respectively *semi-free orthogonal sequence functors* $\mathcal{F}_V^{\mathcal{I}}$ and $\mathcal{G}_V^{\mathcal{I}}$ given by replacing $\mathcal{O}(V, W)$ with $\mathcal{I}(V, W)$ in Definitions 1.2.27 and 1.2.29. Note that the free and semi-free sequences $\mathcal{F}_V^{\mathcal{I}} A$ and $\mathcal{G}_V^{\mathcal{I}} K$ have basepoints in all levels W not isomorphic to V.

Remark 1.2.32. The other factorization indicated in diagram 1.2.26 is also immediately obvious in the spacewise definitions. There is a natural isomorphism

$$\mathcal{F}_V(-) \cong \mathcal{G}_V(\mathbf{O}_{V+} \wedge -).$$

Hence free spectra are in particular semi-free.

Finally, if the reader wants to escape the language of Kan-extensions completely, the following lemma is a necessary exercise.

Lemma 1.2.33. *The two diagrams 1.2.25 and 1.2.26 indeed display adjoint continuous functors, using the above definitions of the free and semi-free spectra. In particular there are natural isomorphisms*

$$\mathcal{OT}(\mathcal{G}_V K, Z) \cong \mathbf{O}_V\mathcal{T}(K, Z_V)$$

and

$$\mathcal{OT}(\mathcal{F}_V L, Z) \cong \mathcal{T}(L, Z_V).$$

Proof. This can actually be checked using only the explicit description of the internal hom object and monoidal product in $\mathcal{IT}$ (1.2.4), respectively $\mathcal{OT}$(1.2.12), the definitions in 1.2.27 and the translation to the coordinate free world in 1.2.22. □

One of the most important properties of (semi-) free spectra, is that it is easy to calculate their smash products with other spectra and in particular with each other. The following proposition makes this precise, and using Remark 1.2.31 also generalizes Lemma [MMSS, 1.8]. It is analogous to Lemma [S, I.4.5-6] in the case of symmetric spectra.

Proposition 1.2.34. *Let V and W be finite dimensional euclidean vector spaces, $K \in \mathbf{O}_V\mathcal{T}$ and $L \in \mathbf{O}_W\mathcal{T}$. Then there is a natural isomorphism*

$$\mathcal{G}_V K \wedge \mathcal{G}_W L \cong \mathcal{G}_{V\oplus W}(\mathbf{O}_{V\oplus W} \wedge_{\mathbf{O}_V\times\mathbf{O}_W} K \wedge L).$$

Proof. We begin by showing the analogue for orthogonal sequences. Let $Y \in \mathcal{IT}$, then there are continuous natural isomorphisms

$$\begin{array}{rcl}
\mathcal{IT}(\mathcal{G}^{\mathcal{I}}_V K \otimes \mathcal{G}^{\mathcal{I}}_W L, Y) & \cong & \mathcal{IT}(\mathcal{G}^{\mathcal{I}}_V K, F(\mathcal{G}^{\mathcal{I}}_W L, Y)) \\
& \cong & \mathbf{O}_V\mathcal{T}(K, F(\mathcal{G}^{\mathcal{I}}_W L, Y)_V) \\
& \cong & \mathbf{O}_V\mathcal{T}(K, \mathcal{IT}(\mathcal{G}^{\mathcal{I}}_W L, \mathrm{sh}_V Y)) \\
& \cong & \mathbf{O}_V\mathcal{T}(K, \mathbf{O}_W\mathcal{T}(L, Y_{V\oplus W})) \\
& \cong & (\mathbf{O}_V \times \mathbf{O}_W)\mathcal{T}(K\wedge L, Y_{V\oplus W}) \\
\mathcal{IT}(\mathcal{G}^{\mathcal{I}}_{V\oplus W}(\mathbf{O}_{V\oplus W_+} \wedge_{\mathbf{O}_V\times\mathbf{O}_W} K\wedge L), Y) & \cong & \mathbf{O}_{V\oplus W}\mathcal{T}(\mathbf{O}_{V\oplus W_+} \wedge_{\mathbf{O}_V\times\mathbf{O}_W} K\wedge L, Y_{V\oplus W}).
\end{array}$$

Hence the proposition holds for sequences, by the Yoneda lemma. Then the result for spectra is implied by the fact that the definition of the smash product of modules gives a natural isomorphism of spectra

$$(X \otimes \mathbb{S}) \wedge_{\mathbb{S}} (Y \otimes \mathbb{S}) \cong (X \otimes Y) \otimes \mathbb{S},$$

together with Remark 1.2.31. □

Before studying more general smash products, we give another useful property for semi-free spectra, which has no immediate analogue in the free case:

Proposition 1.2.35. *Let V and W be euclidean vector spaces of the same finite dimension. Then for $K \in \mathbf{O}_V\mathcal{T}$, there is a natural isomorphism between semi-free spectra*

$$\mathcal{G}_V K \cong \mathcal{G}_W(\mathcal{O}(V,W)\wedge_{\mathbf{O}_V} K).$$

Proof. Note that $\mathcal{O}(V,W)\wedge_{\mathbf{O}_V} K$ is the evaluation $(\mathcal{G}_V K)_W$ by Remark 1.2.24. Again we compare the defining right adjoints. Let Z be any orthogonal spectrum:

$$\begin{aligned}
\mathcal{OT}(\mathcal{G}_V K, Z) &\cong \mathbf{O}_V\mathcal{T}(K, Z_V)\\
&\cong \mathbf{O}_W\mathcal{T}(\mathcal{O}(V,W)\wedge_{\mathbf{O}_V} K, \mathcal{O}(V,W)\wedge_{\mathbf{O}_V} Z_V)\\
&\cong \mathbf{O}_W\mathcal{T}(\mathcal{O}(V,W)\wedge_{\mathbf{O}_V} K, Z_W)\\
&\cong \mathcal{OT}(\mathcal{G}_W\mathcal{O}(V,W)\wedge_{\mathbf{O}_V} K, Z)
\end{aligned}$$

Here the isomorphism in the second line is immediate after choice of basis, but the actual choice has no influence on the map. For the third line one uses the isomorphism given by the generalized structure maps (cf. Remarks 1.2.22, 1.2.24). $\square$

This allows us to state the following proposition in terms of coordinatized orthogonal spectra, saving us the definition of the coordinate free smash product of sequences via the Kan extension.

Proposition 1.2.36. *The smash product of a semi-free spectrum $\mathcal{G}_n K$ with an orthogonal spectrum X is given in level m by*

$$(\mathcal{G}_n K\wedge X)_m \cong \mathbf{O}_m\wedge_{\mathbf{O}_n\times\mathbf{O}_{m-n}} K\wedge X_{m-n},$$

whenever $m \geq n$, and just a basepoint in lower levels. The structure maps are induced from those of X and the $\mathbf{O}_m$-action is by multiplication on the left factor.

Proof. Again we use that $\mathcal{G}_n K\wedge_{\mathbb{S}} X \cong \mathcal{G}_n^{\mathcal{I}} K \otimes X$, where the structure maps, or $\mathbb{S}$-module structure of the right spectrum are induced only from X. We evaluate the tensor product according to Definition 1.2.4 to get the result. $\square$

Corollary 1.2.37. *The smash product of a semi-free spectrum $\mathcal{G}_V K$ with an orthogonal spectrum X is given in level $W \cong V \oplus V'$ by*

$$(\mathcal{G}_V K\wedge X)_W \cong \mathbf{O}_W\wedge_{\mathbf{O}_V\times\mathbf{O}_{V'}} K\wedge X_{V'},$$

and just a basepoint in levels which V does not embed into isometrically.

Another property differentiating semi-free spectra from free ones is the following:

Lemma 1.2.38. *Let G be a (topological) group. Then a (continuous) action of G on a semi-free spectrum $\mathcal{G}_V K$ is equivalent to a (continuous) action on K. Moreover, the levelwise orbit spectrum $[\mathcal{G}_V K]_G$ is isomorphic to $\mathcal{G}_V(K_G)$.*

Proof. An action of G on a spectrum is the same as a map of (topological) monoids $G \to \mathcal{OT}(\mathcal{G}_V K, \mathcal{G}_V K)$. By the defining adjunction for semi-free spectra that is the same as a map $G \to \mathbf{O}_V\mathfrak{T}(K, K)$. To check the statement about orbits, recall that the action of G is through maps of spectra, i.e. in particular through $\mathbf{O}_V$-equivariant maps on K. That is the $\mathbf{O}_V$- and G-actions on K commute, and on levels W we get the isomorphisms

$$[\mathcal{G}_V K_W]_G = [\mathcal{O}(V, W) \wedge_{\mathbf{O}_V} K]_G \cong [\mathcal{O}(V, W) \wedge K]_{\mathbf{O}_V \times G} \cong \mathcal{O}(V, W) \wedge_{\mathbf{O}_V} (K_G).$$

In the light of our later treatment of G-equivariant spectra, one should note that neither V nor W are considered to have actions of G here, i.e. they are both trivial G-representations. □

Remark 1.2.39. Note that semi-free spectra with actions of G are often more naturally indexed on non-trivial representations. However, Proposition 1.2.35 allows us to change the euclidean space used for the indexing as desired, without changing the underlying spectrum. For example consider the spectrum

$$X := \mathcal{G}_V K \wedge \mathcal{G}_V K \cong \mathcal{G}^{\Sigma_2}_{V \oplus V} \left[\mathbf{O}_{V \oplus V} \wedge_{\mathbf{O}_V \times \mathbf{O}_V} K \wedge K\right].$$

There are many different combinations of Σ_2-actions on the smash factors that are possible. The one we are (usually) interested in is the one that comes from the symmetry isomorphism for the smash product. Before applying Lemma 1.2.38 we should switch indexing from the non trivial Σ_2-representation $V \oplus V$ to an isomorphic vector space W on which Σ_2 acts trivially. Then Proposition 1.2.35 gives the isomorphism

$$X \cong \mathcal{G}_W(\mathcal{O}(V \oplus V, W) \wedge_{\mathbf{O}_V \times \mathbf{O}_V} K \wedge K),$$

where the $\mathbf{O}_W$-action commutes with the Σ_2-action, hence we can calculate the orbits as in Lemma 1.2.38. It is then clear that the correct Σ_2-action to consider on

$$\mathbf{O}_{V \oplus V} \wedge_{\mathbf{O}_V \times \mathbf{O}_V} K \wedge K$$

is by permuting factors of $K \wedge K$ and by multiplying with the block permutation matrix from the right, i.e. by precomposition on $\mathbf{O}_{V \oplus V}$.

Later, considering the smash product as an equivariant spectrum, we can get around this change of indexing, and then the action on $\mathbf{O}_{V \oplus V}$ will be by conjugation (cf. 3.3.39).

Remark 1.2.40. Considerations like these are our main reason for using the coordinate free notation as often as possible. The author himself finds it almost impossible to keep track of the correct actions using the language of $\mathcal{S}p^{\mathcal{O}}$ or even $\mathbf{O}\mathcal{T}$.

There are several other categorical constructions on orthogonal spectra that would warrant a longer discussion. We will not present more details here, as the presented material will suffice for our purposes in this thesis. However, the author hopes to get the opportunity to study twisted smash products analogous to Schwede's exposition for the symmetric case in [S, 4.5] in future work. In particular understanding the effect on homotopy groups in relation to an orthogonal version of the monoid $\mathcal{M}$ of injective self-maps of $\mathbb{N}$ (cf. [S, I.6.1]) would simplify technical results like 1.3.11 and 2.3.29 significantly. For a detailed analysis of these one would, for example, require more discussion about the shift functors sh_m of 1.2.4, its spectrum level analogues and various left adjoints associated to these. Another aspect that we have completely neglected here, is that all the functors in diagram 1.2.25 also have right adjoints, i.e. there are cofree and cosemi-free spectra and sequences.

Remark 1.2.41 (Enrichment). So far, all categories discussed in the previous sections have been viewed as $\mathcal{T}$-categories, and this will continue to be our usual viewpoint. However, as closed symmetric monoidal categories, the categories $\mathcal{I}\mathcal{T}$ and $\mathcal{O}\mathcal{T}$ of orthogonal sequences and spectra are naturally enriched over themselves (cf. A.2.13). Additionally, the functor $\mathrm{ev}_0(-)$ from above is strong symmetric monoidal and transports the enrichment of $\mathcal{O}\mathcal{T}$ over itself into the $\mathcal{T}$-enrichment, i.e. the relation

$$F_{\mathbb{S}}(X, Y)_0 \cong \mathcal{O}\mathcal{T}(X, Y)$$

holds for all orthogonal spectra X and Y. In an analogous way, we can use the (semi-)free functor $\mathcal{F}_0 = \mathcal{G}_0 = \Sigma^\infty$ to view $\mathcal{T}$ as enriched over $\mathcal{O}\mathcal{T}$.

1.2.3 Model Structures

The homotopical properties of the category $\mathcal{S}p^{\mathcal{O}}$ of orthogonal spectra have been studied in [MMSS]. The authors construct various model structures, and several of these will play central roles in the further discussions in this thesis. We will recall the necessary definitions and useful properties of these structures in this section, following the exposition in [MMSS, §6-9]

Definition 1.2.42. Let $f : X \to Y$ be a morphism of orthogonal spectra.

(i) f is a *level equivalence* if $f_n : X_n \to Y_n$ is a weak equivalence in $\mathcal{T}$ for all $n \in \mathbb{N}$.

(ii) f is a *level fibration* if $f_n : X_n \to Y_n$ is a Serre fibration, i.e. a fibration in $\mathcal{T}$ for all $n \in \mathbb{N}$.

(iii) f is a *q-cofibration* if f satisfies the left lifting property with respect to all maps that are *level acyclic fibrations*, i.e. maps that are both level equivalences and level fibrations.

Recall the sets I and J that formed the generating cofibrations for the Quillen model structure on $\mathcal{T}$ (B.1.23). Let $\mathcal{F}I$ be the set of all maps $\mathcal{F}_n i$ of orthogonal spectra with $n \in \mathbb{N}$ and $i \in I$, and $\mathcal{F}J$ the analogue construction. Using these definitons, we can state Theorem [MMSS, 6.3]:

Theorem 1.2.43. *The category $\mathcal{S}p^{\mathcal{O}}$ of orthogonal spectra is a compactly generated proper topological model category with respect to the level equivalences, level fibrations, and q-cofibrations. The sets $\mathcal{F}I$ and $\mathcal{F}J$ are the generating q-cofibrations and acyclic q-cofibrations, respectively.*

Just as in the case of spaces, it is often convenient to also keep the h-cofibrations in mind B.1.14. Note that an h-cofibration of orthogonal spectra is a levelwise h-cofibration of spaces. In particular, Corollary B.1.29 implies the following:

Corollary 1.2.44. *If all involved spectra are well based in each level, then the generalized cube Lemma B.1.7 and the generalized cobase change Lemma B.1.5 hold for (levelwise) h-cofibrations and level homotopy equivalences as well as for (levelwise) h-cofibrations and level weak equivalences of spectra.*

Remark 1.2.45. Note that as in B.1.29, we can even relax the assumption of the spectra being well based slightly.

Recall the definition of the *homotopy groups* of an orthogonal spectrum and *π_*-isomorphisms* from 1.1.7. The following is a collection of properties of π_*-isomorphisms that we will use throughout:

Proposition 1.2.46. *[MMSS, 7.4,9.10], paraphrased*

(i) For a based CW complex A, the functor $-\wedge A$ on orthogonal spectra preserves π_*-isomorphisms.

(ii) A morphism of orthogonal spectra is a π_*-isomorphism if and only if its suspension is. The natural map $\eta\colon X \to \Omega\Sigma X$ is a π_*-isomorphism for all orthogonal spectra X.

(iii) The homotopy groups of a wedge of orthogonal spectra are the direct sums of the homotopy groups of the wedge summands, hence a wedge of π_*-isomorphisms is a π_*-isomorphism.

(iv) Cobase changes of maps that are π_*-isomorphisms and levelwise h-cofibrations are π_*-isomorphisms.

(v) *The generalized cobase change and cube lemmas (B.1.5,B.1.7) hold for all orthogonal spectra, (levelwise) h-cofibrations and π_*-isomorphisms.*

(vi) *If X is the colimit of a sequence of h-cofibrations $X_n \to X_{n+1}$, each of which is a π_*-isomorphism, then the map from the initial term X_0 into X is a π_*-isomorphism.*

(vii) *For any morphism $f\colon X \to Y$ of orthogonal spectra, there are natural long exact sequences*

$$\cdots \to \pi_q(Ff) \to \pi_q(X) \to \pi_q(X) \to \pi_{q-1}(Ff) \to \cdots$$

$$\cdots \to \pi_q(X) \to \pi_q(Y) \to \pi_q(Cf) \to \pi_{q-1}(X) \to \cdots ,$$

where Ff and Cf denote the (levelwise) homotopy fiber and cofiber of f. The natural map $Ff \to \Omega Cf$ is a π_*-isomorphism.

In [MMSS, §9] the stable model structure for orthogonal spectra is constructed by adding generating acyclic cofibrations to the level structure from Theorem 1.2.43, so as to include the π_*-isomorphisms as weak equivalences. Alternatively one could think of this as Bousfield localization at the class of π_*-isomorphisms, but the extra control about the generating acyclic cofibrations proves helpful in several places. In particular, since we are going to reuse the specific maps going into the construction, we recall some details:

Definition 1.2.47. Let $\lambda_n : \mathcal{F}_{n+1}S^1 \to \mathcal{F}_n S^0$ be the map adjoint to the inclusion

$$S^1 \to \mathbf{O}_{n+1+}\wedge_{\mathbf{O}_n} S^0 \wedge S^1,$$

sending the sphere to the copy indexed by the identity in $\mathbf{O}_{n+1}$.

Definition 1.2.48. Once again recall the sets of generating cofibrations of spaces I and of generating level acyclic cofibrations $\mathcal{F}J$ from above. Factor all the maps λ_n via the mapping cylinder as $\lambda_n = r_n \circ k_n$ with $k_n : \mathcal{F}_{n+1}S^1 \to M\lambda_n$ a q-cofibration and r_n a deformation retraction. We define K_n to be the set of pushout product maps of the form $k_n \square i$ with $i \in I$. Let K be the union of $\mathcal{F}J$ with the sets K_n for $n \in \mathbb{N}$.

Definition 1.2.49. A *q-fibration* of orthogonal spectra is a map that has the lifting property with respect to K, hence with respect to all maps that are q-cofibrations and π_*-isomorphisms.

The q-fibrations can be characterized more explicitly:

Lemma 1.2.50. *[MMSS, 9.5] A map $p : E \to B$ of orthogonal spectra is a q-fibration, if and only if it is a level fibration and the diagrams*

$$\begin{tikzcd} E_n \arrow[r, "\tilde{\sigma}"] \arrow[d, "p_n"'] & \Omega E_{n+1} \arrow[d, "\Omega p_{n+1}"] \\ B_n \arrow[r, "\tilde{\sigma}"'] & \Omega B_{n+1} \end{tikzcd} \tag{1.2.51}$$

are homotopy pullbacks for every $n \geq 0$.

Definition 1.2.52. An orthogonal spectrum F is an Ω-spectrum, if all the maps $\bar{\sigma} \colon F_n \to \Omega F_{n+1}$ are weak equivalences.

Corollary 1.2.53. *The map $F \to \star$ is is a q-fibration if and only if F is an Ω-spectrum.*

Then the following theorem combines [MMSS, 9.2,12.5 and 12.6] :

Theorem 1.2.54. *The category $\mathcal{S}p^{\mathcal{O}}$ of orthogonal spectra is a compactly generated stable proper topological model category with respect to the π_*-isomorphisms, q-cofibrations and q-fibrations. The sets of generating cofibrations and generating acyclic cofibrations are given by $\mathcal{F}I$ and K, respectively. This model structure satisfies the pushout product axiom and the monoid axiom.*

This model structure is lifted to categories of orthogonal ring spectra and modules and algebras over such in §12 of [MMSS], following the general treatment of such questions from [SS]. To move towards categories of commutative orthogonal ring spectra, however, some more work is needed, and in particular one requires a *positive* variation of the stable model structure, which is constructed in [MMSS, §14]. The generating sets of cofibrations $\mathcal{F}^+I$, $\mathcal{F}^+J$ and K^+ are therefore defined by excluding all maps that require the use of $\mathcal{F}_0$ from their absolute counterparts (cf. 1.2.43 and 1.2.48).

Definition 1.2.55. Let f be a map of orthogonal spectra:

(i) f is a *positive level fibration*, if f_n is a Serre fibration for all $n > 0$.

(ii) f is a *positive level equivalence*, if f_n is a weak equivalence for all $n > 0$.

(iii) f is a *positive q-cofibration* if it is a q-cofibration and f_0 is a homeomorphism.

(iv) f is a *positive q-fibration* if it has the right lifting property with respect to all maps that are positive q-cofibrations and π_*-isomorphisms.

Theorem 1.2.56. *[MMSS, 14.1] The category $\mathcal{S}p^{\mathcal{O}}$ of orthogonal spectra is a compactly generated proper topological model category with respect to the positive level equivalences, positive level fibrations and positive q-cofibrations. The sets $\mathcal{F}^+I$ and $\mathcal{F}^+J$ are the generating sets of cofibrations and acyclic cofibrations, respectively.*

Theorem 1.2.57. *[MMSS, 14.2] The category $\mathcal{S}p^{\mathcal{O}}$ of orthogonal spectra is a compactly generated stable proper topological model category with respect to the π_*-isomorphisms, positive q-fibrations and positive q-cofibrations. The sets $\mathcal{F}^+I$ and K^+ are the generating sets of cofibrations and acyclic cofibrations, respectively. This model structure satisfies the pushout product axiom an the monoid axiom.*

Recall the functor $\mathbb{E}$ that creates free commutative ring spectra, i.e. commutative $\mathbb{S}$-algebras from A.1.19.

Theorem 1.2.58. *[MMSS, 15.1] The category of commutative orthogonal ring spectra is a compactly generated proper topological model category with fibrations and weak equivalences created in the positive stable model structure of $\mathcal{S}p^{\mathcal{O}}$. The sets $\mathbb{E}\mathcal{F}^+I$ and $\mathbb{E}K^+$ are the generating sets of cofibrations and acyclic cofibrations, respectively.*

Theorem 1.2.59. *[MMSS, 15.2] Let R be a commutative orthogonal ring spectrum The category of commutative R-algebras is a compactly generated proper topological model category whose weak equivalences, fibrations and cofibrations created in the category of commutative orthogonal ring spectra.*

1.3 The Convenient Model Structure

In this section, we define a model structure on the category of commutative orthogonal ring spectra which is convenient in the sense that cofibrant objects are already cofibrant as underlying orthogonal spectra. The existence of such a structure has already been hinted at in [Sh04], where an analogue for symmetric spectra is discussed. The differences and, in particular, difficulties that come up when working in the topological world of orthogonal spectra as opposed to the simplicial setup of symmetric spectra, warrant a more detailed discussion.
We begin on space level, where we have to consider some more equivariant homotopy theory before we can get started:

1.3.1 Mixed Model Structures for G-spaces

Let G be a compact Lie group, for example the orthogonal group $\mathbf{O}_V$ of a finite dimensional euclidean vector space V. There are various accounts of the homotopy theory of G-spaces available, we review what we need here in the Appendix C, where we give more detailed references.
Recall the naive and genuine model structures on the category $G\mathfrak{T}$ of G-spaces from C.1.8. For our purposes we want a mix between these two model structures, analogous to [Sh04, 1.3]:

Definition 1.3.1. A map in $G\mathfrak{T}$ is called a *mixed cofibration* if it is a genuine cofibration. It is called a *mixed equivalence* if it is a naive weak equivalence. A *mixed fibration* is a map that has the right lifting property with respect to all mixed cofibrations that are also mixed equivalences.

Theorem 1.3.2. *The mixed fibrations, cofibrations and equivalences give a compactly generated model structure on $G\mathfrak{T}$.*

Proof. Recall the sets I_G and J_G for the generating (acyclic) cofibrations for the genuine model structure from C.1.8. We will have to add additional acyclic cofibrations: Let H be a closed subgroup of G. Consider the projection

$$\pi_H : (G \times_H EH)_+ \to \left({}^{G}\!/\!_{H}\right)_+,$$

which is a naive weak equivalence. Factor π_H via the mapping cylinder $M\pi_H$ as a naively acyclic genuine cofibration j_H followed by a G-deformation retraction r_H. Then we can define the set of generating acyclic mixed cofibrations as:

$$J_{Gm} := J_G \cup \{j_H \square i, H \subset G, i \in I_G\}.$$

We check the conditions from [H, 2.1.19]. Obviously, every map in I_G or J_{Gm} is an h-cofibration. Also, every map in J_{Gm} is in I_G-cof and a weak equivalence,

because the genuine model structure satisfied the pushout product axiom (C.2.7) and Lemma C.1.10(ii). This kind of maps(h-cofibration and naive weak equivalence) is preserved under the cell complex construction. Therefore we immediately get, that $I_G-\mathrm{inj} \subset J_{Gm}-\mathrm{inj}$. Since every map in $I_G-\mathrm{inj}$ is a genuine, thus naive, weak equivalence, the only thing that remains is to show that any map that is both in $J_{Gm}-\mathrm{inj}$ and a naive weak equivalence is already in $I_G-\mathrm{inj}$.
For a map $f : X \to Y$ it is equivalent to be in $J_{Gm}-\mathrm{inj}$ and that f is both a genuine fibration and that it has the right lifting property with respect to all the maps $j_H \square i$ above. By adjointness B.1.12 the latter condition is equivalent to $\pitchfork_\square(j_H, f)$ having the right lifting property with respect to I_G for all subgroups H. But since the genuine model structure is G-topological (cf. C.2.7) this is equivalent to $\pitchfork_\square(j_H, f)$ being a genuine weak equivalence if f already was a genuine fibration. Recall that the defining diagram for $\pitchfork_\square(j_H, f)$ is given by the lower part of the diagram

$$\begin{tikzcd}
\mathcal{T}_G\left({}^{G}\!/\!_{H_+}, X\right) \arrow[d, "\sim"'] \arrow[rr, bend left=15] & & \\
\mathcal{T}_G(M\pi_H, X) \arrow[dr, "\pitchfork_\square(j_H,f)", "\sim"'] & & \mathcal{T}_G\left({}^{G}\!/\!_{H_+}, Y\right) \arrow[d, "\sim"] \\
& P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{T}_G(M\pi_H, Y) \arrow[d, two heads] \\
& \mathcal{T}_G((G \times_H EH)_+, X) \arrow[r] & \mathcal{T}_G((G \times_H EH)_+, Y).
\end{tikzcd}$$

Here the two upper vertical maps are genuine weak equivalences because r_H was a G-homotopy equivalence. The lower right vertical map is a genuine fibration, because j_H is a genuine cofibration.
Now let f be in $J_{Gm}-\mathrm{inj}$, i.e. a naive weak equivalence, such that f^{hH} is a weak equivalence. Passing to G-fixed points in the above diagram then yields:

$$\begin{tikzcd}
X^H \arrow[d, "\sim"'] \arrow[rr, bend left=15] & & \\
G\mathcal{T}(M\pi_H, X) \arrow[dr, "\sim"'] & & Y^H \arrow[d, "\sim"] \\
& P^G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G\mathcal{T}(M\pi_H, Y) \arrow[d, two heads] \\
& X^{hH} \arrow[r, "\sim"] & Y^{hH}.
\end{tikzcd}$$

Hence for all subgroups H of G, the restriction to fixed points f^H is a weak equivalence by the two out of three property and right properness of the standard model structure on spaces. But this exactly says that f is also a genuine equivalence, hence in $I-$inj. $\square$

Remark 1.3.3. Note that this *mixed model structure* is not an instance of the mixing of model structures described in [C, Theorem 2.1]. Applying Cole's method would yield a mixed structure that uses the naive weak equivalences and the genuine fibrations.

1.3.2 Level Structures

We will use semi-free spectra to assemble a model structure on orthogonal spectra that makes use of the mixed model structures in each level. Note that in order to use Theorem B.2.8, we have to restrict to a proper set of levels, i.e. a small full subcategory of $\mathcal{O}$ (cf. B.2.5). The reader might want to think of the explicit skeleton $\mathbf{O}$, but we rather fix some unspecified small subcategory $\mathcal{O}^{\mathcal{V}}$ equivalent to $\mathcal{O}$, in order to get used to some of the notation that will be necessary in the equivariant case (cf. 2.3). Call a vector space V a *level* if it is an object of $\mathcal{O}^{\mathcal{V}}$, and say that a property applies levelwise, if it applies for all V in $\mathcal{O}^{\mathcal{V}}$.

Definition 1.3.4. A morphism $f : X \to Y$ of orthogonal spectra is called a *level mixed fibration*, if for each level V, the map $f_V : X_V \to Y_V$ is a mixed fibration of $\mathbf{O}_V$-spaces. It is called an $\mathbb{S}$-*cofibration* if it has the left lifting property with respect to level mixed fibrations that are also level equivalences.

Proposition 1.3.5. *The* $\mathbb{S}$*-cofibrations, level mixed fibrations and level equivalences give a compactly generated model structure on* $\mathcal{S}p^{\mathcal{O}}$.

Proof. We check the prerequisites of Theorem B.2.21: The mixed model structures on $\mathbf{O}_V$-spaces are cofibrantly generated by construction and satisfy the cofibration hypothesis B.1.31.
As in B.2.6 define

$$\mathcal{G}I := \bigcup_{V\in\mathcal{V}} \mathcal{G}_V I_{\mathbf{O}_V} \qquad\qquad \mathcal{G}J := \bigcup_{V\in\mathcal{V}} \mathcal{G}_V J_{\mathbf{O}_V m}.$$

What is left to check, is that the set $\mathcal{G}J$ consists entirely of level equivalences: For $f\colon X \to Y$ a mixed acyclic cofibration of $\mathbf{O}_V$-spaces and $W = V \oplus V'$ another level, $(\mathcal{G}_V f)_W$ is the map:

$$\mathbf{O}_{W+}\wedge_{\mathbf{O}_V\times\mathbf{O}_{V'}} X\wedge S^{V'} \to \mathbf{O}_{W+}\wedge_{\mathbf{O}_V\times\mathbf{O}_{V'}} Y\wedge S^{V'}.$$

Before factoring out the diagonal action of $\mathbf{O}_V \times \mathbf{O}_{V'}$, this is exactly the map

$$\mathbf{O}_W \wedge f \wedge S^{V'},$$

hence a weak equivalence by C.1.10(ii) and a genuine cofibration by Illman's triangulation theorem C.2.2 together with C.2.7(ii). Since $\mathbf{O}_V \times \mathbf{O}_{V'}$ acts freely on source and target, C.1.10(iv) gives that they are naively cofibrant as $\mathbf{O}_V \times \mathbf{O}_{V'}$-spaces. Finally C.1.10(iii) and Ken Browns Lemma ([H, 1.1.12]) imply that taking the $\mathbf{O}_V \times \mathbf{O}_{V'}$ orbits preserves weak equivalences between naively cofibrant $\mathbf{O}_V \times \mathbf{O}_{V'}$-spaces. □

As with the standard level model structure, there is a positive variation of the $\mathbb{S}$-model structure:

Definition 1.3.6. A morphism $f\colon X \to Y$ of orthogonal spectra is called a *positive mixed fibration*, if for each level V with $\dim(V) \neq 0$, the map $f_V\colon X_V \to Y_V$ is a mixed fibration of $\mathbf{O}_V$-spaces. It is called a *positive* $\mathbb{S}$*-cofibration* if it has the left lifting property with respect to positive mixed fibrations that are also positive level equivalences.

Note that the positive $\mathbb{S}$-cofibrations are exactly the $\mathbb{S}$-cofibrations, that are isomorphisms in level 0.

Proposition 1.3.7. *The positive* $\mathbb{S}$*-cofibrations, positive mixed fibrations and positive level equivalences give a compactly generated model structure on* $\mathcal{S}p^{\mathcal{O}}$*. The sets of generating (acyclic) cofibrations are* $\mathcal{G}^+I$ *and* $\mathcal{G}^+J$*, respectively.*

Proof. Define $\mathcal{G}^+I$ as the subset of $\mathcal{G}I$ using only non-trivial levels V, and analogous for $\mathcal{G}^+J$. The proof of Proposition 1.3.5 holds verbatim, if we replace the mixed model structure on $\mathbf{O}_0$-spaces (or spaces), by the one where every map is a fibration and weak equivalence, so that only isomorphisms are cofibrations. This model structure is cofibrantly generated by an empty set of cofibrations. □

1.3.3 Stable Structures

As in the classical case, we want to localize the level model structures to obtain stable counterparts. Recall the λ-maps, that were already used for this purpose in the classical case 1.2.47. Here we use the coordinate free variation:

Definition 1.3.8. For finite dimensional euclidean vector spaces V and W, define $\lambda_{V,W} : \mathcal{F}_{V\oplus W} S^W \to \mathcal{F}_0 S^0$ to be the map adjoint to the inclusion

$$S^W \to \mathcal{O}(V, V \oplus W) \cong \mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_W} S^W,$$

that corresponds to the inclusion at the identity factor after choosing the embedding $V \to V \oplus W$ indicated by the notation.
Factor the maps $\lambda_{V,W}$ through the mapping cylinder as $r_{V,W} \circ k_{V,W}$. Then denote the set of all $k_{V,W} \Box i$ with V and W in $\mathcal{V}$ and i in $\mathcal{F}I = \mathcal{G}I'$ by K. For the case of the positive structures, form the analogous set K^+ by excluding those $k_{V,W} \Box i$ where V is trivial.

Definition 1.3.9. A morphism of orthogonal spectrum is a *(positive)* $\mathbb{S}$*-fibration*, if it has the right lifting property with respect to all (positive) $\mathbb{S}$-cofibrations that are also π_*-isomorphisms.

Proposition 1.3.10. *The category $\mathcal{S}p^{\mathcal{O}}$ of orthogonal spectra admits a compactly generated proper topological monoidal model structure where the cofibrations are given by the (positive) $\mathbb{S}$-cofibrations, the fibrations are the (positive) $\mathbb{S}$-fibrations and the weak equivalences are given by the π_*-isomorphisms. This* (positive) $\mathbb{S}$-model structure *satisfies the monoid axiom.*

Proof. We again use [H, 2.1.19], treating the absolute (i.e. not positive) case in detail. The positive variation follows similarly, and we point out the differences. The generating cofibrations are given by $\mathbb{S}I := \mathcal{G}I$ whereas the acyclic cofibrations are generated by $\mathbb{S}J := \mathcal{G}J \cup K$. Use the positive variations of the sets for the positive structure to define $\mathbb{S}^+I$ and $\mathbb{S}^+J$. Since every map in K is a q-cofibration it has the left lifting property with respect to level acyclic level fibrations, thus with respect to all level acyclic level genuine fibrations, and is therefore in $\mathbb{S}I{-}\mathrm{cof}$. Thus we already have $\mathbb{S}J{-}\mathrm{cof} \subset \mathbb{S}I{-}\mathrm{cof}$ and $\mathbb{S}I{-}\mathrm{inj} \subset \mathbb{S}J{-}\mathrm{inj}$.

Since every map in $\mathbb{S}J$ is also both an h-cofibration and a π_*-isomorphism, every map in $\mathbb{S}J{-}\mathrm{cell}$ is a π_*-isomorphism (cf. 1.2.46). Every map in $\mathbb{S}I{-}\mathrm{inj}$ is a level equivalence, thus a π_*-isomorphism. Finally, note that any map in $\mathbb{S}J{-}\mathrm{inj}$ is a q-fibration since $\mathcal{F}J \subset \mathcal{G}J$. Therefore, if it is also a π_*-isomorphism, it is an acyclic q-fibration, hence an acyclic level fibration, hence in particular in $\mathcal{G}J{-}\mathrm{inj}$ and a level equivalence. But then it is already in $\mathcal{G}I{-}\mathrm{inj} = \mathbb{S}I{-}\mathrm{inj}$ by the level structure.

For the pushout product axiom one can quickly check, that $\mathbb{S}I \Box \mathbb{S}I \subset \mathbb{S}I$ and that $\mathbb{S}I \Box \mathcal{G}J \subset \mathcal{G}J$ by using that the semi-free spectrum functors and smashing with (orbit-) spaces preserve pushouts, and again using the property C.1.10(ii) as well as 1.2.34. It remains to show, that maps in $\mathbb{S}I \Box K$are π_*-isomorphisms. So let $i\colon A \to B$ in $\mathbb{S}I$ and $k\colon C \to D$ in K. Since the maps in K are q-cofibrations and π_*-isomorphisms, and since the stable model structure on orthogonal spectra satisfies the monoid axiom, the top and bottom maps in the following pushout diagram are π_*-isomorphisms, therefore the cobase change is, since the left vertical map is an h-cofibration. Thus the pushout product is a π_*-isomorphism via the two out of three property.

$$\begin{tikzcd}
A\wedge C \arrow[r] \arrow[d] & A\wedge D \arrow[d] \arrow[ddr, bend left] & \\
B\wedge C \arrow[r] \arrow[drr, bend right] & P \arrow[ul, phantom, "\ulcorner"] \arrow[dr, dotted] & \\
 & & B\wedge D
\end{tikzcd}$$

The pushout product axiom also implies that the model structure is topological by using that the tensor with a space is the same as the smash product with the corresponding suspension spectrum. Since $\mathbb{S}$-cofibrations are h-cofibrations and $\mathbb{S}$-fibrations are level fibrations, properness follows from [MMSS, 9.10]. The unit axiom for monoidal model categories is satisfied in the absolute case because the sphere spectrum $\mathbb{S}$ is cofibrant there. To show it in the positive case, we use Ken Brown's Lemma ([H, 1.1.12]) and the monoid axiom: Since smashing with an arbitrary space sends acyclic cofibrations between cofibrant objects to weak equivalences, it preserves the π_*-isomorphism between the sphere spectrum and its positive hence absolute cofibrant replacement. Finally the monoid axiom itself can be proved as in [MMSS, 12.5], using the following Proposition. □

Proposition 1.3.11. *$\mathbb{S}$-cofibrant spectra are flat, in the sense that for any $\mathbb{S}$-cofibrant spectrum X, the functor $X\wedge -$ preserves π_*-isomorphisms.*

Proof. Because we have to calculate homotopy groups, we will prove this proposition in $\mathbf{O}\mathcal{T}$, i.e. restrict ourselves to those levels of an orthogonal spectrum, that are indexed by the euclidean spaces $\mathbb{R}^n$, abbreviated by the natural number n (cf. 1.2.22).

Since smashing with any spectrum preserves cofiber sequences, and by the long exact sequence for homotopy groups 1.2.46(vi), it suffices to show that if Z is an orthogonal spectrum with $\pi_*(Z) = 0$, then also $\pi_*(X\wedge Z) = 0$. Since smashing with Z preserves the cell complex construction, we can further reduce to the case where X is either the source or the target of one of the generating $\mathbb{S}$-cofibrations, i.e. X is of the form $\mathcal{G}_n\left[\mathbf{O}_n/H_+\wedge S^k_+\right]$ or $\mathcal{G}_n\left[\mathbf{O}_n/H_+\wedge D^k_+\right]$. Since $\mathcal{G}_n\left[\mathbf{O}_n/H_+\wedge K_+\right]$ is equal to $\mathcal{G}_n\left[\mathbf{O}_n/H_+\right]\wedge K_+$ and since we know that smashing with a cofibrant space preserves π_*-isomorphisms it suffices to show that $\mathcal{G}_n\left[\mathbf{O}_n/H_+\right]\wedge Z$ has trivial homotopy groups, if Z did.

Recall that

$$\begin{aligned}
\left(\mathcal{G}_n\left[\mathbf{O}_n/H_+\right]\wedge Z\right)_{n+m} &= \mathbf{O}_{n+m+}\wedge_{\mathbf{O}_n\times\mathbf{O}_m}\left(\mathbf{O}_n/H_+\wedge Z_m\right)\\
&\cong \mathbf{O}_{n+m}/H_+\wedge_{\mathbf{O}_m} Z_m,
\end{aligned}$$

where the structure maps are the composite of the structure map of Z with the (upper left block) inclusion of $\mathbf{O}_{n+m} \to \mathbf{O}_{n+m+1}$ and the projection to the quotient:

$$\mathbf{O}_{n+m}/_{H+} \wedge_{\mathbf{O}_m} Z_m \wedge S^1 \xrightarrow{\mathrm{id}\wedge\sigma} \mathbf{O}_{n+m}/_{H+} \wedge_{\mathbf{O}_m} Z_{m+1} \xrightarrow{p\circ\mathrm{inc}} \mathbf{O}_{n+m+1}/_{H+} \wedge_{\mathbf{O}_{m+1}} Z_{m+1}.$$

The homotopy groups of $\mathcal{G}_n\left[\mathbf{O}_n/_{H+}\right]\wedge Z$ are therefore calculated via the following colimit:

$$\begin{tikzcd}
\dots \arrow[d, "(p\circ\mathrm{inc})_*"] & \\
\pi_*\left(\mathbf{O}_{n+m}/_{H+}\wedge_{\mathbf{O}_m} Z_m\right) \arrow[r, "\sigma_*\circ\Sigma"] & \pi_{*+1}\left(\mathbf{O}_{n+m}/_{H+}\wedge_{\mathbf{O}_m} Z_{m+1}\right) \arrow[d, "(p\circ\mathrm{inc})_*"] & \\
& \pi_{*+1}\left(\mathbf{O}_{n+m+1}/_{H+}\wedge_{\mathbf{O}_{m+1}} Z_{m+1}\right) \arrow[r, "\sigma_*\circ\Sigma"] & \dots
\end{tikzcd}$$

We can add more edges to this diagram without changing the colimit, because cofinal sequential colimits commute:

$$\begin{tikzcd}
\pi_*\left(\mathbf{O}_{n+m}/_{H+}\wedge_{\mathbf{O}_m} Z_m\right) \arrow[r, "\sigma_*\circ\Sigma"] & \pi_{*+1}\left(\mathbf{O}_{n+m}/_{H+}\wedge_{\mathbf{O}_m} Z_{m+1}\right) \arrow[r, "\sigma_*\circ\Sigma"] \arrow[d, "(p\circ\mathrm{inc})_*"] & \pi_{*+2}\left(\mathbf{O}_{n+m}/_{H+}\wedge_{\mathbf{O}_m} Z_{m+2}\right) \arrow[r, "\sigma_*\circ\Sigma"] \arrow[d, "(p\circ\mathrm{inc})_*"] & \dots \\
& \pi_{*+1}\left(\mathbf{O}_{n+m+1}/_{H+}\wedge_{\mathbf{O}_{m+1}} Z_{m+1}\right) \arrow[r, "\sigma_*\circ\Sigma"] & \pi_{*+2}\left(\mathbf{O}_{n+m+1}/_{H+}\wedge_{\mathbf{O}_{m+1}} Z_{m+2}\right) \arrow[r, "\sigma_*\circ\Sigma"] \arrow[d, "(p\circ\mathrm{inc})_*"] & \dots \\
& & \dots &
\end{tikzcd} \tag{1.3.12}$$

Here, taking the colimit along a line calculates the (shifted) homotopy groups of an orthogonal spectrum $\mathbf{O}_{n+m}/_{H+}\wedge_{\mathbf{O}_m} Z$ which is defined in the following way:

$$\left(\mathbf{O}_{n+m}/_{H+}\wedge_{\mathbf{O}_m} Z\right)_k = \mathbf{O}_{n+m}/_{H+}\wedge_{\mathbf{O}_m} Z_{m+k},$$

where the structure maps are inherited from Z. Thus if we show that spectra of this type have trivial homotopy groups, we are done.
We know that $\mathbf{O}_{n+m}/_H$ is a free (right) $\mathbf{O}_m$ space, since H is a subgroup of upper left n-block matrices in $\mathbf{O}_{n+m}$. Therefore it is not only genuinely but also naively cofibrant as an $\mathbf{O}_m$-space, i.e. in the cell complex structure of $\mathbf{O}_{n+m}/_H$ appear only free cells. Both levelwise smash product with spaces and taking levelwise $\mathbf{O}_m$-orbits commute with (levelwise) colimits hence with the cell complex construction. Therefore we can finally reduce to showing that $(I'_{\mathbf{O}_m}\wedge_{\mathbf{O}_m} Z)$-cell complexes are acyclic, where $I'_{\mathbf{O}_m}$ where the generating naive cofibrations on $\mathbf{O}_m$-spaces.

So suppose that $* = X_0 \to X_1 \to \dots \to X_l$ is the cellular filtration of $\mathbf{O}_{n+m}/_{H+}$, and use induction on this filtration. From the cell structure we get the gluing diagrams of spectra:

$$\begin{tikzcd} S^{p-1}_+\wedge \mathbf{O}_{m+}\wedge_{\mathbf{O}_m} Z \arrow[r] \arrow[d] & X_i\wedge_{\mathbf{O}_m} Z \arrow[d] \\ D^p_+\wedge \mathbf{O}_{m+}\wedge_{\mathbf{O}_m} Z \arrow[r] & X_{i+1}\wedge_{\mathbf{O}_m} Z \end{tikzcd} \cong \begin{tikzcd} S^{p-1}_+\wedge Z \arrow[r] \arrow[d] & X_i\wedge_{\mathbf{O}_m} Z \arrow[d] \\ D^p_+\wedge Z \arrow[r] & X_{i+1}\wedge_{\mathbf{O}_m} Z. \end{tikzcd} \tag{1.3.13}$$

Then as Z was acyclic, so are the two spectra in the left columns of the squares. Therefore the top horizontal maps are π_*-isomorphisms, hence so are the bottom maps since the left vertical maps are h-cofibrations. Thus $X_{i+1}\wedge Z$ is acyclic, too. □

Proposition 1.3.14. *The identity functor on the category of orthogonal spectra $\mathcal{S}p^{\mathcal{O}}$ gives a monoidal Quillen equivalence from the usual (positive) stable model structure to the (positive) stable $\mathbb{S}$-model structure.*

Proof. Since any q-cofibration is an $\mathbb{S}$-cofibration, the identity functor preserves cofibrations. Since the weak equivalences are the same in both model structures, it also preserves acyclic cofibrations and detects and preserves weak equivalences. The identity trivially satisfies all other properties of monoidal Quillen functors. □

Note that Proposition 1.3.10 allows us to use Theorem [SS, 4.1] so that for R an orthogonal ring spectrum, we get absolute and positive R-model structures on the categories of R-modules and if R is commutative on (associative) R-algebras.

Theorem 1.3.15. *Let R be an orthogonal ring spectrum.*

(i) The category of left R-modules is a compactly generated proper model category with respect to the π_*-isomorphisms and the underlying (positive) $\mathbb{S}$-fibrations. The sets of generating cofibrations and acyclic cofibrations are $R\wedge\mathbb{S}I$ ($R\wedge\mathbb{S}^+I$) and $R\wedge\mathbb{S}J$ ($R\wedge\mathbb{S}^+J$).

(ii) If R is $\mathbb{S}$-cofibrant, then the forgetful functor from R-modules to orthogonal spectra preserves cofibrations. Hence every cofibrant R-module is cofibrant as an orthogonal spectrum.

(iii) Let R be commutative. The model structures of (i) are monoidal and satisfy the monoid axiom.

(iv) Let R be commutative. The category of R-algebras is a compactly generated right proper model category with respect to the π_*-isomorphisms and the underlying (positive) $\mathbb{S}$-fibrations. The sets of generating cofibrations and acyclic cofibrations are $R\wedge\mathbb{A}\mathbb{S}I$ ($R\wedge\mathbb{A}\mathbb{S}^+I$) and $R\wedge\mathbb{A}\mathbb{S}J$ ($R\wedge\mathbb{A}\mathbb{S}^+J$), where $\mathbb{A}$ is the free associative algebra functor from A.1.19.

(v) Let R be commutative. Every cofibration of R-algebras whose source is (positive) cofibrant as an R-module is also a cofibration of R-modules. In particular, every cofibrant R-algebra is cofibrant as an R-module.

All the above model structures are Quillen equivalent to the classical ones from [*MMSS*, 12.1] *via the identity functor as in 1.3.14.*

Proof. Except for the properness statements, parts (i),(iii) and (iv) (v) are implied by [SS, 4.1]. For part (ii) and the properness use that the necessary (co-)limits are computed in the underlying category of orthogonal spectra. □

Note that a positive analogue to (ii) and (v) in this theorem is not implied by [SS, 4.1], since the sphere spectrum is not positive $\mathbb{S}$-cofibrant. In the next section, we will deal with the lifting of the positive $\mathbb{S}$-model structure to commutative orthogonal ring spectra, and in particular give versions of these statements in Theorem 1.3.29.

1.3.4 Extension to Commutative Ring Spectra

From now on, except when explicitly stated otherwise, we only work with the positive $\mathbb{S}$-model structure. In particular we drop the adjective *positive* when talking about $\mathbb{S}$-cofibrations or $\mathbb{S}$-cofibrant objects. To lift the $\mathbb{S}$-model structure to commutative orthogonal ring spectra, we want to mimic the methods from [MMSS, §15]. The basic idea is to apply Lemma [SS, 2.3] to the free commutative ring spectrum functor $\mathbb{E}$ (cf. A.1.19). In particular we check that $\mathbb{E}\mathbb{S}^+I$ and $\mathbb{E}\mathbb{S}^+J$ both satisfy the cofibration hypothesis (B.1.31) to cover the smallness prerequisites, and that all the maps in $\mathbb{E}\mathbb{S}^+J$-cell are actually weak equivalences.
For the latter, we first consider the following two lemmas, which are the analogues of [MMSS, 15.5] and [EKMM, III 5.1].

Lemma 1.3.16. *Let Y be an orthogonal spectrum and let $X = \mathcal{G}_V\left[K\wedge ({}^{\mathbf{O}_V}\!/_{H})_+\right]$, for K a based CW-complex and V a non trivial euclidean space. Then the quotient map*

$$q\colon \left(E\Sigma_{i+}\wedge_{\Sigma_i}X^{\wedge i}\right)\wedge Y \to X^{\wedge i}_{\Sigma_i}\wedge Y$$

is a level homotopy equivalence.

Proof. We rewrite the target of q in the following way:

$$X^{\wedge i}_{\Sigma_i}\wedge Y = \mathcal{G}_{V^{\oplus i}}\left[\mathbf{O}_{V^{\oplus i}+}\wedge_{\prod_i \mathbf{O}_V}\left(K\wedge {}^{\mathbf{O}_V}\!/_{H+}\right)^{\wedge i}\right]_{\Sigma_i}\wedge Y.$$

Here Σ_i acts on the smash product $\mathbf{O}_{V^{\oplus i}+}\wedge_{\prod_i \mathbf{O}_V}\left(K\wedge {}^{\mathbf{O}_V}\!/_{H+}\right)^{\wedge i}$ diagonally, by permuting the copies of $K\wedge {}^{\mathbf{O}_V}\!/_{H+}$ and by multiplication from the right with block-permutation matrices on $\mathbf{O}_{V^{\oplus i}}$. Letting Σ_i act on $\prod_i \mathbf{O}_V$ by conjugation with the block permutation matrices, we can then rewrite the orbit space in terms of the semi-direct product of $\prod_i \mathbf{O}_V$ and Σ_i:

$$\begin{aligned}\left[\mathbf{O}_{V^{\oplus i}+} \wedge_{\prod_i \mathbf{O}_V} \left(K \wedge \mathbf{O}_V /_{H+}\right)^{\wedge i}\right]_{\Sigma_i} &= \mathbf{O}_{V^{\oplus i}+} \wedge_{\prod_i \mathbf{O}_V \rtimes \Sigma_i} \left(K \wedge \mathbf{O}_V /_{H+}\right)^{\wedge i} \\ &\cong \mathbf{O}_{V^{\oplus i}+} \wedge_{\prod_i H \rtimes \Sigma_i} K^{\wedge i},\end{aligned}$$

using the fact that $\mathbf{O}_V$ acted trivially on K. Now let $W \cong V^{\oplus i} \oplus V'$ be another level. Corollary 1.2.37 implies that

$$\left(X^{\wedge i}_{\Sigma_i} \wedge Y\right)_W \cong \mathbf{O}_{W+} \wedge_{(\prod_i H \rtimes \Sigma_i) \times \mathbf{O}_{V'}} K^{\wedge i} \wedge Y_{V'}.$$

In a similar way we see that

$$\left(E\Sigma_{i+} \wedge_{\Sigma_i} (\mathcal{G}_V K)^{\wedge i} \wedge Y\right)_W \cong (E\Sigma_i \times \mathbf{O}_W)_+ \wedge_{(\prod_i H \rtimes \Sigma_i) \times \mathbf{O}_{V'}} K^i \wedge Y_{V'},$$

where Σ_i acts diagonally on $E\Sigma_i \times \mathbf{O}_W$. The desired quotient map is now induced by $\hat{q} \colon E\Sigma_i \times \mathbf{O}_W \to \mathbf{O}_W$. Since $\mathbf{O}_W$ is a free $(\prod_i \mathbf{O}_V \rtimes \Sigma_i) \times \mathbf{O}_{V'}$ space, the theorems C.2.2 and C.2.3 give, that $\hat{q}$ is weak equivalence between naive $(\prod_i \mathbf{O}_V \rtimes \Sigma_i) \times \mathbf{O}_{V'}$-equivariant cell-complexes, i.e. a $(\prod_i \mathbf{O}_V \rtimes \Sigma_i) \times \mathbf{O}_{V'}$-homotopy equivalence. Equivariant homotopy equivalences are preserved by smashing with any space, and yield homotopy equivalences after passing to orbits with respect to subgroups. □

Lemma 1.3.17. *Let X be a positive $\mathbb{S}$-cofibrant orthogonal spectrum. Then the quotient map*

$$q : E\Sigma_{i+} \wedge_{\Sigma_i} X^{\wedge i} \to X^{\wedge i}_{\Sigma_i}$$

is a π_*-isomorphism.

Proof. We proceed by induction on i and a Σ_i-equivariant cellular filtration of $X^{\wedge i}$. The traditional reference for this in the case of free cells and external smash products is [BMMS, p. 37-38], however, some translation work needs to be done in order to apply their result to our case. We will instead use the filtration introduced by Theorem 3.4.22. For the induction start $i = 1$, the statement is trivially true. Hence let us assume it holds for all $j < i$ and that X is built from A by attaching a single cell $\mathcal{G}_V \left[D^n_+ \wedge \mathbf{O}_V /_{H+}\right]$, where A is itself positively cofibrant. Then Theorem 3.4.22 states that $X^{\wedge i}$ is built from $A^{\wedge i}$ by attaching induced cells of the form

$$\begin{array}{c}\Sigma_i \wedge_{\Sigma_j \times \Sigma_{i-j}} \left(\mathcal{G}_W S^{n-1}_+ \wedge \mathbf{O}_W /_{P+}\right)^{\square j} \wedge A^{\wedge (i-j)} \\ \big\downarrow \\ \Sigma_i \wedge_{\Sigma_j \times \Sigma_{i-j}} \left(\mathcal{G}_W D^n_+ \wedge \mathbf{O}_W /_{P+}\right)^{\wedge j} \wedge A^{\wedge (i-j)}\end{array}$$

(cf. 3.4.24), with cofiber $\Sigma_i \wedge_{\Sigma_j \times \Sigma_{i-j}} \left(\mathcal{G}_W S^n_+ \wedge \mathbf{O}_W/_{P+}\right)^{\wedge j} \wedge A^{\wedge(i-j)}$. Both $E\Sigma_i \wedge_{\Sigma_i} -$ and taking Σ_i-orbits preserve cofiber sequences, hence $\hat{q}$ induces maps of long exact sequences of homotopy groups (cf. 1.2.46). Thus, for the induction step, it suffices to show that the map

$$E\Sigma_i \wedge_{\Sigma_j \times \Sigma_{i-j}} \left(\mathcal{G}_W S^n_+ \wedge \mathbf{O}_W/_{P+}\right)^{\wedge j} \wedge A^{\wedge(i-j)} \to \left[\left(\mathcal{G}_W S^n_+ \wedge \mathbf{O}_W/_{P+}\right)^{\wedge j} \wedge A^{\wedge(i-j)}\right]_{\Sigma_j \times \Sigma_{i-j}}$$

is a π_*-isomorphism. We use that $E\Sigma_i$ is $\Sigma_j \times \Sigma_{i-j}$-equivariantly homotopy equivalent to $E\Sigma_j \times E\Sigma_{i-j}$ to factor this map as:

$$\begin{array}{c}
E\Sigma_j \wedge_{\Sigma_j} \left(\mathcal{G}_W S^n_+ \wedge \mathbf{O}_W/_{P+}\right)^{\wedge j} \wedge E\Sigma_{i-j} \wedge_{\Sigma_{i-j}} A^{\wedge i-j} \\
\Big\downarrow{\scriptstyle \hat{q} \wedge \mathrm{id}} \\
\left[\mathcal{G}_W S^n_+ \wedge \mathbf{O}_W/_{P+}\right]^{\wedge j}_{\Sigma_j} \wedge E\Sigma_{i-j} \wedge_{\Sigma_{i-j}} A^{\wedge i-j} \\
\Big\downarrow{\scriptstyle \mathrm{id} \wedge \hat{q} \wedge \mathrm{id}} \\
\left[\mathcal{G}_W S^n_+ \wedge \mathbf{O}_W/_{P+}\right]^{\wedge j}_{\Sigma_j} \wedge A^{\wedge i-j}_{\Sigma_{i-j}}
\end{array}$$

Here the first map is a level homotopy equivalence by Lemma 1.3.16. The second map is a π_*-isomorphism by the induction hypothesis and Proposition 1.3.11, using that the spectrum $\left[\mathcal{G}_W S^n_+ \wedge \mathbf{O}_W/_{P+}\right]^{\wedge j}_{\Sigma_j}$ is $\mathbb{S}$-cofibrant by the following proposition. $\square$

Proposition 1.3.18. *Let V be a non-trivial vector space and let $X = \mathcal{G}_V[K_+ \wedge \mathbf{O}_V/_{H+}]$ for K either S^n or D^n and H some closed subgroup of $\mathbf{O}_V$. Then the orthogonal spectrum $X^{\wedge i}_{\Sigma_i}$ is $\mathbb{S}$-cofibrant. In particular the inclusion $\mathbb{S} \to \mathbb{E}X$ is an $\mathbb{S}$-cofibration.*

Proof. We assume K is a sphere, the case for a disc is similar. The i-fold smash power of X is the spectrum

$$\left(\mathcal{G}_V[S^n_+ \wedge \mathbf{O}_V/_{H+}]\right)^{\wedge i} \cong \mathcal{G}_{V^{\oplus i}}\left[\left(S^{ni}_+ \wedge \left(\mathbf{O}_{V^{\oplus i}}/_{\prod_i H}\right)_+\right)\right]$$

Since the action on a semi-free spectrum $\mathcal{G}_W L$ is adjoint to an action on L (1.2.38), it suffices to show that

$$M \colon = \left[S^{ni}_+ \wedge \left(\mathbf{O}_{V^{\oplus i}}/_{\prod_i H}\right)_+\right]_{\Sigma_i}$$

is a genuine $\mathbf{O}_{V^{\oplus i}}$-complex. We prove this using Illman's triangulation theorem (C.2.2):

Since $\mathbf{O}_{V^{\oplus i}}$ operates transitively on ${}^{\mathbf{O}_{V^{\oplus i}}}\!\diagup_{\prod_i H}$, the $\mathbf{O}_{V^{\oplus i}}$-orbit space of M is given by $(S_+^{\wedge ni})_{\Sigma_i}$. Then the proof of Illman's triangulation Theorem C.2.4 implies that this space admits a triangulation where the conjugacy class of the stabilizer subgroups $\mathrm{Stab}_s \subset \Sigma_i$ is constant on the open simplices, i.e. the Σ_i-isotropy type is constant on open simplices.
We claim that this triangulation also has the property that the $\mathbf{O}_{V^{\oplus i}}$-isotropy type is constant on open simplices. Let $x = [(s, [A])]$ be an element of M in the preimage of $[s] \in (S_+^{ni})_{\Sigma_i}$. Then an element $B \in \mathbf{O}_{V^{\oplus i}}$ is in Stab_x if and only if $BA = Ah\sigma$ with $h \in \prod_i H$ and $\sigma \in \mathrm{Stab}_s \subset \Sigma_i$. Hence Stab_x is exactly the subgroup $A(\prod_i H \,\mathrm{Stab}_s)A^{-1}$, i.e. the $\mathbf{O}_{V^{\oplus i}}$-isotropy type is constant on the open simplices. Hence C.2.2 implies that M is a genuine $\mathbf{O}_{V^{\oplus i}}$-complex.
Note that in particular the case $K = D^0$, i.e. $D_+^n = D_+^{ni} = S^0$ is allowed here. □

Corollary 1.3.19. *Let X be $\mathcal{G}_V\left[K \wedge ({}^{\mathbf{O}_V}\!\diagup_H)_+\right]$, for K a based CW-complex and V a euclidean space, then the functor $\mathbb{E}X \wedge (-)$ on orthogonal spectra preserves π_*-isomorphisms.*

Proof. Smashing with X preserves π_*-isomorphisms by 1.3.11. Since $E\Sigma_i$ is a free Σ_i-cell complex, so does $E\Sigma_{i+} \wedge_{\Sigma_i} (-)$. Now we can apply Lemma 1.3.16 for each wedge summand in $\mathbb{E}X$. □

Corollary 1.3.20. *The functor $\mathbb{E}$ preserves π_*-isomorphisms between $\mathbb{S}$-cofibrant orthogonal spectra. In particular, each map in $\mathbb{E}J$ is a π_*-isomorphism.*

Proof. Iterated use of the pushout product axiom for the $\mathbb{S}$-model structure 1.3.10 implies that the i-fold smash power of an acyclic cofibration between $\mathbb{S}$-cofibrant spectra is an acyclic cofibration. Both $E\Sigma_{i+} \wedge_{\Sigma_i} (-)$ and taking wedges preserve π_*-isomorphisms hence Ken Brown's Lemma gives the result. □

We will need to calculate realizations of simplicial objects and some other specific colimits in the category of commutative orthogonal ring spectra. Recall Proposition 3.1.13 and the following lemma from [MMSS], that allow us to do so in the underlying category of spectra:

Lemma 1.3.21 ([MMSS, 15.11])**.** *Let $\{R_i \to R_{i+1}\}$ be a sequence of maps of commutative orthogonal ring spectra that are h-cofibrations of orthogonal spectra. Then the underlying orthogonal spectrum of the colimit of the sequence in commutative orthogonal ring spectra is the colimit of the sequence computed in the category of orthogonal spectra.*

The following Proposition is inspired by Lemma 15.9 in [MMSS], it deals with the other part of the cofibration hypothesis for $\mathbb{E}\mathbb{S}^+ I$ and brings us closer to the convenience property 1.3.29.

Proposition 1.3.22. *Let $f\colon X \to Y$ be a wedge of maps in $\mathbb{S}^+I$ and let $\mathbb{E}X \to R$ be a map of commutative orthogonal ring spectra. Then the cobase change $j\colon R \to R\wedge_{\mathbb{E}X}\mathbb{E}Y$ is an underlying h-cofibration of spectra. If smashing with R additionally preserves $\mathbb{S}$-cofibrations, j is even an $\mathbb{S}$-cofibration.*

Proof. As in the proof of [MMSS, 15.9], we identify the inclusion $S^n_+ \to D^{n+1}_+$ of spaces with the realization of the inclusion of 0-simplices of the simplicial space $B_*(S^n_+, S^n_+, S^0)$ (cf. B.1.36). Its q simplices are given by a wedge of $q+1$ copies of S^n_+ with S^0, with degeneracy maps the inclusions of wedge summands and face maps induced from folding maps $S^n_+ \vee S^n_+ \to S^n_+$, respectively the collapse map $S^n_+ \vee S^0 \to S^0$ for the last face in each simplicial level. Note that both the smash product with an $\mathbf{O}_V$-orbit and the semi-free functors $\mathcal{G}_V$ preserve colimits and tensors, hence the simplicial realization. Thus we can express f analogously as the inclusion of 0-simplices of the simplicial orthogonal spectrum $B_*(X, X, T)$, where $X = \bigvee_i \mathcal{G}_{V_i} S^{n_i}_+ \wedge \left(\mathbf{O}_{V_i}/H_i\right)_+$ and $T = \bigvee_i \mathcal{G}_{V_i} S^0 \wedge \left(\mathbf{O}_{V_i}/H_i\right)_+$. Applying $\mathbb{E}$, takes coproducts to smash products, fold maps to multiplication maps, and inclusions of the basepoints to unit maps of commutative orthogonal ring spectra. It also preserves tensors over $\mathcal{U}$, i.e. sends $X\wedge A_+$ to $X \otimes A$, hence it sends the realization of $B_*(X, X, T)$ to the realization of the bar construction $\mathbf{B}_*(\mathbb{E}X, \mathbb{E}X, \mathbb{E}T)$ as defined in B.1.51. Finally, since we can compute geometric realizations in terms of the underlying spectra (3.1.13), and since smashing with R commutes with this realization, we can identify

$$R\wedge_{\mathbb{E}X}\mathbb{E}Y \cong R\wedge_{\mathbb{E}X}\mathbf{B}(\mathbb{E}X, \mathbb{E}X, \mathbb{E}T) \cong \mathbf{B}(R, \mathbb{E}X, \mathbb{E}T). \tag{1.3.23}$$

We look at $\mathbf{B}_*(R, \mathbb{E}X, \mathbb{E}T)$ in more detail: R includes into the 0-simplices $R\wedge\mathbb{E}T$ as a wedge summand, i.e. via an h-cofibration. All the other wedge summands are of the form

$$R\wedge \left(\mathcal{G}_{V_i} S^0 \wedge \mathbf{O}_{V_i}/H_{i+}\right)^{\wedge k}_{\Sigma_k},$$

hence they are $\mathbb{S}$-cofibrant if smashing with R preserves $\mathbb{S}$-cofibrations by 1.3.18. Then in particular $R\wedge\mathbb{E}T$ is $\mathbb{S}$-cofibrant and the inclusion of R is an $\mathbb{S}$-cofibration. The degeneracy maps are given by inclusions

$$R\wedge(\mathbb{E}X)^{\wedge q}\wedge\mathbb{E}T = R\wedge(\mathbb{E}X)^{\wedge r}\wedge\mathbb{S}\wedge(\mathbb{E}X)^{\wedge q-r}\wedge\mathbb{E}T \longrightarrow R\wedge(\mathbb{E}X)^{\wedge q+1}\wedge\mathbb{E}T.$$

Therefore the inclusion of degenerate simplices (B.1.42) is in each level q given by the map

$$R\wedge(\mathbb{S} \to \mathbb{E}X)^{\square q+1}\wedge\mathbb{E}T,$$

which is an h-cofibration because $\mathbb{S} \to \mathbb{E}X$ is an inclusion of a wedge summand. Furthermore 1.3.18 states that $\mathbb{S} \to \mathbb{E}X$ is an $\mathbb{S}$-cofibration. Hence by the pushout product axiom, the inclusion of degenerate simplices is an $\mathbb{S}$-cofibration if smashing with R preserves $\mathbb{S}$-cofibrations. Hence the bar construction is h-proper, and even $\mathbb{S}$-proper for the stronger assumption on R. The result then follows using Proposition B.1.46. □

Remark 1.3.24. Note that particular examples of ring spectra R that preserve positive $\mathbb{S}$-cofibrations under the smash product are all R that are (absolute) $\mathbb{S}$-cofibrant. This includes the very important case of the sphere spectrum $\mathbb{S}$.

Corollary 1.3.25 ([MMSS, 15.9])**.** *The set $\mathbb{ES}^+I$ of maps of commutative orthogonal ring spectra satisfies the cofibration hypothesis. Since it consists of $\mathbb{ES}I$-cell complexes, so does $\mathbb{ES}^+J$.*

Lemma 1.3.26 ([MMSS, 15.12])**.** *Let $i\colon R \to R'$ be an $\mathbb{S}$-cofibration of commutative orthogonal ring spectra. Then the functor $(-)\wedge_R R'$ on commutative R-algebras preserves π_*-isomorphisms.*

Proof. Assume inductively that i is a cobase change of a wedge of maps in $\mathbb{ES}^+I$. Then as in 1.3.23 we can identify $(-)\wedge_R R'$ with an appropriate $\mathbf{B}(-, \mathbb{E}X, \mathbb{E}T)$. This functor preserves π_*-isomorphisms by B.1.47, since the bar construction is h-proper. □

Finally we get the analogue of [MMSS, 15.4], using the same proof as in the classical case (cf. [MMSS, p. 490]):

Proposition 1.3.27. *Every relative $\mathbb{ES}J$-cell complex is a π_*-isomorphism.*

This once more allows us to use Lemma [SS, 2.3], and we obtain the $\mathbb{S}$-model structure for commutative orthogonal ring spectra:

Theorem 1.3.28. *The underlying positive $\mathbb{S}$-fibrations and π_*-isomorphisms give a compactly generated proper topological model structure on the category of commutative orthogonal ring spectra. The generating (acyclic) cofibrations are given by the sets $\mathbb{ES}^+I$ and $\mathbb{ES}^+J$, respectively.*
Again the identity functor gives a Quillen equivalence to the classical model structure from [MMSS, 15.1].

We call cofibrant objects in this model structure simply $\mathbb{S}$-cofibrant, inspired by the following Theorem, which is implied by the second statement of Proposition 1.3.22, and provides the main motivation for the constructions in this section:

Theorem 1.3.29. *The $\mathbb{S}$-model structure on commutative orthogonal ring spectra is "convenient", i.e. if $\mathbb{E}$ is a commutative orthogonal ring spectrum that is $\mathbb{S}$-cofibrant, it is already (positive) $\mathbb{S}$-cofibrant as an orthogonal spectrum.*

Even slightly more is true:

Theorem 1.3.30. *Let $f: R \to R'$ be a map of commutative orthogonal ring spectra, that is a cofibration in the model structure of Theorem 1.3.28. If the smash product with R preserves $\mathbb{S}$-cofibrations of orthogonal spectra, then f is an underlying $\mathbb{S}$-cofibration.*

Proof. Reduce to the case of a $\mathbb{E}\mathbb{S}^+ I$-cell complex. Induction on the cellular filtration and the stronger second statement of Proposition 1.3.22 give the result. □

Theorem 1.3.31. *For a commutative orthogonal ring spectrum R, the $\mathbb{S}$-model structure induces a compactly generated proper topological model structure on commutative R-algebras. This R-model structure is convenient with respect to the R-model structure on R-modules from Theorem 1.3.15(i).*
The identity functor on commutative R-algebras induces a Quillen equivalence to the classical model structure of [MMSS, 15.2].

Proof. We can use [DS, 3.10]. An analogue of Theorem 1.3.29 is then immediate, since the free commutative R-algebra functor $\mathbb{E}_R$ satisfies $\mathbb{E}_R(-) \cong R \wedge \mathbb{E}(-)$, and thus any cofibration of commutative R-algebras is an underlying R-cofibration. □

Chapter 2

Equivariant Orthogonal Spectra

2.1 Introduction

This chapter mirrors Chapter 1 on orthogonal spectra in an equivariant setting. We once again begin with introducing notation and recalling basic results. We loosely follow the approach from [MM], where the equivariant orthogonal spectra were first defined. Other good references include [Kr] and [S11].

Summing up section 1.2.1, we have described several equivalent categories of orthogonal spectra: $\mathcal{S}p^{\mathcal{O}}$, functor categories $\mathbf{O}\mathcal{T}$ and $\mathcal{O}\mathcal{T}$ as well as $\mathbb{S}$-modules in $I\mathcal{T}$ or $\mathcal{I}\mathcal{T}$. For the equivariant case, there are even more variations, only some of which will be mentioned here. Again, our main focus will be on the functorial approach analogous to 1.2.21, but we start once again with a down to earth version that serves as a good basis for the intuition.

Throughout the whole section, let $G \in \mathcal{T}$ be a topological group. At a later stage we will usually restrict to compact Lie- or discrete groups, but in this categorical part this is not yet necessary.

Definition 2.1.1. A *G-equivariant orthogonal spectrum* is an orthogonal spectrum X, with a continuous action of G, i.e. a morphism $G \to \mathcal{S}p^{\mathcal{O}}(X, X)$ of monoids in in $\mathcal{T}$.

Similar *naive* ways of looking at equivariant spectra have come up before, but have in other contexts been found insufficient in capturing all the desired (homotopy theoretical) information. However, orthogonal spectra have a distinctive advantage in that regard, which is expressed in the following construction:

Construction 2.1.2. Given a G-equivariant orthogonal spectrum X, and an orthogonal representation V of G, we can define the *evaluation of X at V*, as the G-space X_V analogous to 1.2.22 via:

$$X_V := \mathcal{O}(\mathbb{R}^n, V) \wedge_{\mathbf{O}_n} X_n, \tag{2.1.3}$$

where n is the dimension of V. The action of G is then diagonal on the smash product, with $\mathcal{O}(\mathbb{R}^n, V)$ inheriting an action via postcomposition from the action of G on V.

Hence, even though X a priori only contains information at trivial representations, the actions of the orthogonal groups allow us to study X at any finite dimensional orthogonal representation.

Example 2.1.4 (The Sphere Spectrum revisited)**.** Recall the orthogonal spectrum $\mathbb{S}$ from 1.2.11, and equip it with the "trivial" G-action. Then for all n, G acts trivially on the evaluation $\mathbb{S}_{\mathbb{R}^n} = S^n$. However, this equivariant sphere spectrum is far from trivial at more general representations. By 2.1.3 we have

$$\mathbb{S}_V = \mathcal{O}(\mathbb{R}^n, V) \wedge_{\mathbf{O}_n} S^n,$$

for n the dimension of an orthogonal representation V of G. The G action on $\mathbb{S}_V$ is by postcomposition with the action on V on the first factor, which by the definition of the $\mathbf{O}_n$-action on S^n gives a G-equivariant homeomorphism

$$\mathbb{S}_V \cong S^V$$

to the representation sphere S^V, i.e. the one-point compactification of V with the induced G-action.

As this example shows, a lot more equivariant information is hidden in the datum of a G-equivariant spectrum than might be obvious at first glance. The ability to unwrap this information will prove instrumental in various instances, in particular when studying(equivariant) homotopical properties. In the following sections, we will give definitions for several equivalent categories of G-equivariant orthogonal spectra which will shed more light on the above, and, as in the non equivariant case of 1.2.2, allow us to state some of the more technical properties needed throughout.

As in Chapter 1, we will then go on to recall classical results on the stable homotopy theory of G-equivariant orthogonal spectra, before we adapt the methods from Section 1.3, and construct the equivariant $\mathbb{S}$-model structure in 2.3.27. We want to highlight that, contrary to the classical model structures described in [MM] (cf. 2.2.46), the cofibrations in the equivariant $\mathbb{S}$-model structure are independent from the choice of G-universe (2.2.3,2.3.15), which is a consequence of the twisting and untwisting Propositions 2.3.9 and 2.3.6. Both of these appear for the first time in this thesis, and should be seen as analogues of formula (2.1.3) that allow some level of control over fixed point spaces of the levels.

2.2 More Recollections

2.2.1 Definition(s)

One of the first things we should address, is that we did not actually define morphisms of G-equivariant orthogonal spectra in the previous section. The reason for this is that there are actually two equally important different notions of morphisms, forming categories enriched over $G\mathfrak{T}$ or $\mathcal{T}$, respectively.

Definition 2.2.1. Let X and Y be G-equivariant orthogonal spectra. The morphism space $\mathcal{S}p^{\mathcal{O}}(X,Y)$ inherits a G action by conjugation. We define the *morphism G-space* $\mathcal{S}p^{\mathcal{O}}_G(X,Y) \in G\mathfrak{T}$ as $\mathcal{S}p^{\mathcal{O}}_G(X,Y) := \mathcal{S}p^{\mathcal{O}}(X,Y)$, and it is immediate that under the conjugation action, composition is in fact G-equivariant. This gives a *$G\mathfrak{T}$-category of G-equivariant orthogonal spectra* $\mathcal{S}p^{\mathcal{O}}_G$. The G-fixed category of $\mathcal{S}p^{\mathcal{O}}_G$ is denoted by $G\mathcal{S}p^{\mathcal{O}}$, which is a $\mathcal{T}$-category, as usual. The morphisms $G\mathcal{S}p^{\mathcal{O}}(X,Y) = \mathcal{S}p^{\mathcal{O}}_G(X,Y)^G$ exactly correspond to the G-equivariant morphisms of spectra with G-action.

Remark 2.2.2. As mentioned above, there are various different viewpoints on this. For example a map of monoids $G_+ \to \mathcal{S}p^{\mathcal{O}}(X,X)$ in $\mathcal{T}$ is of course the same as a continuous functor from the one object category with morphism space G_+ to $\mathcal{S}p^{\mathcal{O}}$. Hence as in A.2.8 we can form the enriched functor category $[G,\mathcal{S}p^{\mathcal{O}}]$ which turns out to be equivalent to $\mathcal{S}p^{\mathcal{O}}_G$. That is, morphisms in $\mathcal{S}p^{\mathcal{O}}_G$ correspond to natural transformations of functors, whereas morphisms in $G\mathcal{S}p^{\mathcal{O}}$ are only the G-natural transformations.
Similarly observe that the defining adjunctions of the tensor in orthogonal spectra 1.2.2 and the suspension spectrum functor Σ^∞ give natural isomorphisms

$$\mathcal{T}(G_+,\mathcal{S}p^{\mathcal{O}}(X,X)) \cong \mathcal{S}p^{\mathcal{O}}(G_+ \wedge X, X) \cong \mathcal{S}p^{\mathcal{O}}(\Sigma^\infty G_+ \wedge X, X).$$

Hence an action map $\mu\colon G_+ \to \mathcal{S}p^{\mathcal{O}}(X,X)$ is adjoint to some $\bar{\mu}\colon \Sigma^\infty G_+ \wedge X \to X$, and the fact that μ is a map of monoids exactly translates to X being a module over the orthogonal ring spectrum $\mathbb{S}_{[G]} := \Sigma^\infty G_+$ via $\bar{\mu}$. Then morphisms in $\mathcal{S}p^{\mathcal{O}}_G$ correspond to mere spectrum morphisms between $\mathbb{S}_{[G]}$-modules, whereas morphisms in $G\mathcal{S}p^{\mathcal{O}}$ correspond to honest module maps.
Finally in the light of Definition 2.1.1, an action of G on an orthogonal spectrum consists of actions of G on each level $\mathbb{R}^n$, such that the structure maps σ are G-equivariant. Then morphisms in $\mathcal{S}p^{\mathcal{O}}_G$ are morphisms of spectra that are $\mathbf{O}_V$, but not necessarily G-equivariant in each level $\mathbb{R}^n$.

We welcome the reader to pick his favourite out of the above models for his own intuition, however the author has found the approach analogous to 1.2.21 most powerful in dealing with the later theory. Hence following [MM], we construct the equivariant analogue of the category $\mathcal{O}$ in the equivariant context:

Definition 2.2.3. Let $\mathcal{O}_G$ be the topological G category with objects all finite dimensional orthogonal G-representations and morphism G-spaces given by

$$\mathcal{O}_G(V, W) := \mathcal{O}(V, W),$$

where the G-actions are given by the conjugation action through the G-actions on V and W.
To be explicit once, let $(V, \phi\colon G \to \mathbf{O}_V)$ and $(W, \psi\colon G \to \mathbf{O}_W)$ be finite dimensional orthogonal G-representations. Then the space $\mathcal{O}(V, W)$ has elements of the form (f, w) with $f : V \to W$ an isometric embedding and w an element of the orthogonal complement of the image of f in W. The group G acts on $\mathcal{O}(V, W)$ by mapping

$$g, (f, w) \mapsto (\psi(g) \circ f \circ \phi(g^{-1}), \psi(g)w).$$

Remark 2.2.4. Of course we could similarly define $\mathcal{I}_G$ as an analogue to 1.2.3.

Analogous to the non equivariant case, we can then define equivariant spectra as functors:

Definition 2.2.5. A *G-equivariant orthogonal spectrum* X is a continuous G-functor, i.e. a $G\mathfrak{T}$-enriched functor

$$X\colon \mathcal{O}_G \to \mathfrak{T}_G.$$

Denote the $G\mathfrak{T}$-category $[\mathcal{O}_G, \mathfrak{T}_G]$ of such functors and their (not necessarily equivariant) natural transformations by $\mathcal{O}_G\mathcal{T}$. The G-fixed category of only the G-equivariant natural transformations $[\mathcal{O}_G, \mathfrak{T}_G]^G$ will be written as $G\mathcal{O}\mathcal{T}$.

Proposition 2.2.6. *The $G\mathfrak{T}$-categories $\mathcal{S}p_G^{\mathcal{O}}$ and $\mathcal{O}_G\mathcal{T}$ are equivalent, and hence so are their G-fixed categories $G\mathcal{S}p^{\mathcal{O}}$ and $G\mathcal{O}\mathcal{T}$.*

Proof. We give $G\mathfrak{T}$-functors in both directions. Let $X : \mathcal{O}_G \to \mathfrak{T}_G$ be a $G\mathfrak{T}$-functor. Restricting to the trivial representations $\mathbb{R}^n$ in $\mathcal{O}_G$ yields an orthogonal spectrum consisting of G-spaces $X_n \in \mathfrak{T}_G$. We need to check that the levelwise G-actions fit together into an action on the spectrum, i.e. that all structure maps $X_n \wedge S^{m-n} \to X_m$ are G-equivariant, or equivalently that the adjoint

$$\mathcal{O}(\mathbb{R}^n, \mathbb{R}^m) \to \mathfrak{T}_G(X_n, X_m)$$

is. This latter formulation is exactly what it meant for X to be a $G\mathfrak{T}$-functor. On morphisms, an (equivariant) natural transformation $X \to Y$ between $G\mathfrak{T}$-functors by definition consists of (G-equivariant) maps $S^0 \to \mathfrak{T}_G(X_V, Y_V)$ for all $V \in \mathcal{O}_G$, sending the non-basepoint to $\alpha_V : X_V \to Y_V$, such that the diagram

$$\begin{tikzcd} X_V \arrow[r, "\alpha_V"] \arrow[d, "X_f"'] & Y_V \arrow[d, "Y_f"] \\ X_W \arrow[r, "\alpha_W"'] & Y_W \end{tikzcd}$$

is commutative for all $f \in \mathcal{O}_G(V, W)$. Restriction to the trivial representations thus defines a morphism of spectra, which is equivariant if the transformation was.

In the other direction, for X and Y orthogonal spectra with G-action, we already know how to evaluate them at general elements of $\mathcal{O}_G$ by 2.1.2, so we only need to check that the generalized structure maps from 1.2.22 actually yield a $G\mathfrak{T}$-functor, i.e. that the composites

$$\mathcal{O}_G(V, W) \to \mathcal{O}(V, W) \to \mathfrak{T}_G(X_V, Y_V)$$

are G-equivariant. The adjoint of the latter map is made explicit in the diagram 1.2.23, and all involved maps there are G-equivariant, using the conjugation G-action on all the morphism spaces. Similarly one checks that (equivariant) morphisms of spectra yield (equivariant) natural transformations.
Composing these two constructions in one way gives the identity functor on $\mathcal{S}p^{\mathcal{O}}_G$. On the other hand the fact that the natural isomorphisms

$$\begin{aligned} X_W &\cong \mathcal{O}_G(W, W) \wedge_{\mathbf{O}_W} X_W \\ &\cong \mathcal{O}_G(\mathbb{R}^m, W) \wedge_{\mathbf{O}_m} \mathcal{O}_G(W, \mathbb{R}^m) \wedge_{\mathbf{O}_W} X_W \\ &\cong \mathcal{O}_G(\mathbb{R}^m, W) \wedge_{\mathbf{O}_m} X_m \end{aligned}$$

are G-equivariant then shows that the two categories are equivalent. $\square$

Remark 2.2.7. Mixing results from the non equivariant case with the ideas from A.2.27, we see that $\mathcal{S}p^{\mathcal{O}}_G$ is tensored and cotensored over $G\mathfrak{T}$. Tensors and cotensors are calculated levelwise as in the non equivariant case, with the G-action on smash products always being diagonal, and by conjugation on mapping spaces. We get the defining $G\mathfrak{T}$-natural isomorphisms

$$\mathfrak{T}_G(D, \mathcal{S}p^{\mathcal{O}}_G(X, Y)) \cong \mathcal{S}p^{\mathcal{O}}_G(D \wedge X, Y) \cong \mathcal{S}p^{\mathcal{O}}_G(X, F(D, Y))$$

for tensors $\wedge$ and cotensors F. Again analogous to Example A.2.27, taking the G-fixed points of the spaces above yields

$$G\mathfrak{T}(D, \mathcal{S}p^{\mathcal{O}}_G(X, Y)) \cong G\mathcal{S}p^{\mathcal{O}}(D \wedge X, Y) \cong G\mathcal{S}p^{\mathcal{O}}(X, F(D, Y)),$$

hence for D having trivial G-action when considering $D \in \mathcal{T} \subset \mathfrak{T}_G$,

$$G\mathfrak{T}(D, G\mathcal{S}p^{\mathcal{O}}(X, Y)) \cong G\mathcal{S}p^{\mathcal{O}}(D \wedge X, Y) \cong G\mathcal{S}p^{\mathcal{O}}(X, F(D, Y)).$$

That is $G\mathcal{S}p^{\mathcal{O}}$ is tensored and cotensored over $\mathcal{T}$.

Even more can be taken from the non equivariant case, without extra work: Note that given X and Y orthogonal spectra with G-action, their smash product $X \wedge Y$ inherits a diagonal G-action. We can compatibly equip their internal function spectrum with a conjugation G-action. This can be made explicit in terms of the formulas 1.2.4 and 1.2.6, but we will only need the following

Corollary 2.2.8. *The categories $\mathcal{S}p_G^{\mathcal{O}}$ and $G\mathcal{S}p^{\mathcal{O}}$ are closed symmetric monoidal, with smash products and function spaces calculated in $\mathcal{S}p^{\mathcal{O}}$, equipped with the diagonal or conjugation G-action, respectively.*

Remark 2.2.9. Our choice of presentation for the definition of the smash product via $I\mathcal{T}$, respectively the induced tensor on the module category, has the advantage that it is very explicit, and in particular when just dealing with non equivariant spectra it provides a very helpful intuition. However, since we mainly deal with coordinate free versions of (equivariant) spectra, respectively (enriched) functor categories we should also give the equivalent formulation in these terms. The non equivariant case is discussed in detail in [MMSS, §21] based on the more general statement of [D, §3,4]. We give the key steps in the construction:
Let $(\mathcal{D}, \oplus)$ be a skeletally small category enriched over $G\mathcal{T}$ whose underlying category (cf. A.2.12) is symmetric monoidal, and let X and Y be enriched functors $\mathcal{D} \to G\mathcal{T}$. They define an *external smash product*

$$\begin{array}{rrcl} X\bar{\wedge}Y\colon & \mathcal{D} \times \mathcal{D} & \to & \mathcal{T}_G. \\ & (d, d') & \mapsto & X_d \wedge Y_{d'} \end{array}$$

Define the *(internal) smash product* as the GT-enriched left Kan extension

$$\wedge := \mathrm{Lan}_{\oplus}(-\bar{\wedge}-)\colon \mathcal{D} \to \mathcal{T}_G.$$

In particular, by [K, 4.25], we can write the smash product pointwise as the coend

$$(X\wedge Y)_e = \int^{(d,d')\in\mathcal{D}\times\mathcal{D}} \mathcal{D}(d \oplus d', e)\wedge X_d \wedge Y_{d'}.$$

Note that in the cases $\mathcal{D} = \mathcal{O}_G$, and in particular in the non equivariant case for G the trivial group, this definition agrees with the one given above up to canonical natural isomophism. The fact that the definition given here indeed gives a closed symmetric monoidal structure on the underlying category of $[\mathcal{D}, G\mathcal{T}]$ in the more general case can be checked by applying the enriched Kan-extension to the coherence diagrams for $\mathcal{D}$, using the fact that the Kan-extension is natural in all its inputs, together with the fact that $\mathcal{T}_G$ itself was closed symmetric monoidal.
Note that as in [MMSS, 3.3, 23.1-6], for $i\colon \mathcal{D} \to \mathcal{D}'$ a strong symmetric monoidal functor, the functor $\mathbb{U}_i\colon [\mathcal{D}, \mathcal{T}_G] \to [\mathcal{D}', \mathcal{T}_G]$ defined by precomposition is lax symmetric monoidal with respect to the smash product defined above. Its left adjoint, again defined in terms of the enriched left Kan-extension is even *strong* symmetric monoidal.

2.2.2 Free and Semi-Free equivariant Spectra

Identifying $\mathcal{S}p_G^{\mathcal{O}}$ with the enriched functor category $\mathcal{O}_G\mathcal{T}$ allows us to once more study free and semi-free functors. Again the free case is well documented in the literature as it has featured prominently in the constructions of [MM], whereas the semi-free case makes its first appearance in this thesis. Since these are central to our work, we again take some time to be explicit.

Consider for V some finite dimensional G-representation the following diagram of $G\mathfrak{T}$-categories:

$$\begin{tikzcd}
\star \arrow[d] \arrow[ddr] & \\
\mathbf{O}_V \arrow[d] \arrow[dr] & \\
\mathcal{O} \arrow[r] & \mathcal{T},
\end{tikzcd}$$

where $\star$ is the trivial $G\mathfrak{T}$-category with one object, say V and morphism G-space S^0. It is included into $\mathbf{O}_V$, the $G\mathfrak{T}$-category with again only one object V, but endomorphism space given by $\mathbf{O}_{V+}$, with the conjugation G-action. Further include $\mathbf{O}_V$ as a full subcategory of $\mathcal{O}_G$. Then as in the non equivariant case, an orthogonal spectrum X, viewed as a functor $X\colon \mathcal{O}_G \to \mathfrak{T}_G$, naturally defines underlying functors out of $\star$, $\mathbf{O}_V$ and $\mathcal{I}$ by precomposition with the above inclusions.

Once more there is a commuting diagram of functors

$$\begin{tikzcd}
\mathcal{O}_G\mathcal{T} \arrow[r, "\mathrm{ev}_V'"'] \arrow[rrr, bend left, "\mathrm{ev}_V"] & \mathbf{O}_V\mathcal{T} \arrow[r] & {[\star, \mathcal{T}]} \arrow[r, "\cong"] & \mathcal{T},
\end{tikzcd} \tag{2.2.10}$$

with the corresponding diagram of left adjoints given by the enriched left Kan-extensions

$$\begin{tikzcd}
\mathcal{O}\mathcal{T} & \mathbf{O}_V\mathcal{T} \arrow[l, "\mathcal{G}_V^G(-)"] & \mathcal{T}. \arrow[l, "(-)\wedge\mathbf{O}_{V+}"] \arrow[ll, bend right, "\mathcal{F}_V^G(-)"']
\end{tikzcd} \tag{2.2.11}$$

Again we can easily compute the defining Kan-extension explicitly to get the following description of $\mathcal{F}_V^G$:

Definition 2.2.12. Let V be a finite dimensional G-representation, let $A \in \mathfrak{T}_G$ be a G-space. Then the *free orthogonal G-spectrum* $\mathcal{F}_V^G A$ is given in a level W by

$$(\mathcal{F}_V^G A)_W := \mathcal{O}_G(V, W)\wedge A,$$

with the diagonal G-action, and the structure maps induced by composition in $\mathcal{O}_G$. This assignment, with the obvious extension to maps in $\mathcal{T}$, yields a functor

$$\mathcal{F}_V^G\colon \mathfrak{T}_G \to \mathcal{O}_G\mathcal{T}.$$

Since they do not differ on objects, we will denote the restriction to G-fixed categories $\mathcal{F}_V^G\colon G\mathfrak{T} \to G\mathcal{O}\mathcal{T}$ in the same way.

The semi-free case is of course similar, but it is worthwhile to first take a closer look at the source category: By definition, an object of $[\mathbf{O}_V, \mathfrak{T}_G]$ is a $G\mathfrak{T}$-functor $\mathbf{O}_V \to \mathfrak{T}_G$, i.e. a space K with a continuous action of both G and $\mathbf{O}_V$, such that the map

$$\mathbf{O}_V \to \mathfrak{T}_G(K, K)$$

is G-equivariant, i.e. such that the two actions are compatible in the following sense:

$$g(A(g^{-1}k)) = (gAg^{-1})k \quad \text{for all } A \in \mathbf{O}_V,\ g \in G \text{ and } k \in K, \tag{2.2.13}$$

where the left side only uses the actions on K, whereas on the right side, we first act with g on A by conjugation, and then let the result act on k. Replacing k by gk' in 2.2.13, we see that an object of $[\mathbf{O}_V, \mathfrak{T}_G]$ is the same as a space with a continuous action of the semi-direct product $\mathbf{O}_V \rtimes G$. Recall that the multiplication on $\mathbf{O}_V \rtimes G$ is defined as follows:

$$(A, g)(B, h) = (A(gBg^{-1}), gh).$$

Then morphisms in $[\mathbf{O}_V, \mathfrak{T}_G]$ are continuous maps that are $\mathbf{O}_V$-, but not necessarily G-equivariant, so that the morphism spaces again inherit G-actions by conjugation. The G-fixed category of $[\mathbf{O}_V, \mathfrak{T}_G]$ has only those morphisms that are both $\mathbf{O}_V$- and G-equivariant, i.e. it is isomorphic to the category $\mathbf{O}_V \rtimes G\mathcal{T}$ of $\mathbf{O}_V \rtimes G$-spaces. This discussion signifies the importance of the study of semi-direct products for the theory of equivariant orthogonal spectra and we will have to investigate specific properties of spaces with $\mathbf{O}_V \rtimes G$-actions at several places throughout this thesis. Examples include the twisting and untwisting Propositions 2.3.9 and 2.3.6 and Subsection 3.3.3, where we look at fixed points and orbits of $\mathbf{O}_V \rtimes G$-spaces.

Definition 2.2.14. Let V be a finite dimensional G-representation and let $K \in \mathbf{O}_V \rtimes G\mathcal{T}$. Then the *semi free orthogonal spectrum* $\mathcal{G}_V^G K$ is defined in a level W via

$$(\mathcal{G}_V K)_W := \mathcal{O}(V, W) \wedge_{\mathbf{O}_V} K,$$

where the action of $\mathbf{O}_V$ on the smash product is diagonal and on $\mathcal{O}(V, W)$ by precomposition. The G-action is similarly diagonal on the smash product and by conjugation on $\mathcal{O}(V, W)$. The structure maps, and hence the $\mathbf{O}_W$ action, are induced by postcomposition in $\mathcal{O}_G$. As in the free case, this defines a functor

$$\mathcal{G}_V^G\colon [\mathbf{O}_V, \mathfrak{T}_G] \to \mathcal{O}_G\mathcal{T}.$$

Again we make no difference in notation when passing to G-fixed categories and still write $\mathcal{G}_V^G\colon \mathbf{O}_V\rtimes G\mathcal{T} \to G\mathcal{OT}$.

Remark 2.2.15. Note that the $G\mathfrak{T}$-enriched adjunction yields adjunctions on G-fixed categories, i.e the natural isomorphisms

$$\mathcal{O}_G\mathcal{T}(\mathcal{G}_V^G K, Y) \cong [\mathbf{O}_V, \mathfrak{T}_G](K, Y_V)$$

in $G\mathfrak{T}$ yield natural isomorphisms

$$G\mathcal{OT}(\mathcal{G}_V^G K, Y) \cong \mathbf{O}_V\rtimes G\mathcal{T}(K, Y_V),$$

such that the objects $\mathcal{G}_V^G K$ have appropriate universal properties in both settings.

Remark 2.2.16. Note that forgetting the G-actions, the underlying spectrum of a semi-free G-spectrum $\mathcal{G}_V^G K$ is $\mathcal{G}_V K$, i.e. again semi-free, generated by the underlying $\mathbf{O}_V$-space of the $\mathbf{O}_V\rtimes G$-space K. The analogue statement holds for free G-spectra. Compare to 1.2.38.

Therefore, if the target category is clear, we will often drop the superscript from both $\mathcal{F}_V^G$ and $\mathcal{G}_V^G$.
The following results are the analogues of 1.2.34 – 1.2.38, and will similarly prove helpful in several calculations:

Proposition 2.2.17. *Let V and W be finite dimensional G-representations, $K \in \mathbf{O}_V\rtimes G\mathcal{T}$ and $L \in \mathbf{O}_W\rtimes G\mathcal{T}$. Then there is a natural isomorphism*

$$\mathcal{G}_V K\wedge\mathcal{G}_W L \cong \mathcal{G}_{V\oplus W}(\mathbf{O}_{V\oplus W}\wedge_{\mathbf{O}_V\times\mathbf{O}_W} K\wedge L),$$

where the G-action on $V\oplus W$ and $K\wedge L$ is diagonal.

Proof. Since the smash product in $\mathcal{O}_G\mathcal{T}$ is calculated in $\mathcal{OT}$, it suffices to check that the diagonal action on the smash product is the one described in the proposition. □

Proposition 2.2.18. *Let V and W be G-representations of the same finite dimension. Then for $K \in \mathbf{O}_V\rtimes G\mathcal{T}$, there is a natural isomorphism between semi-free spectra*

$$\mathcal{G}_V K \cong \mathcal{G}_W(\mathcal{O}(V,W)\wedge_{\mathbf{O}_V} K).$$

Proof. The proof is exactly as that of 1.2.35, after one checks that the natural isomorphisms used there are all G-equivariant. □

Proposition 2.2.19. *The smash product of a semi-free G-spectrum $\mathcal{G}_V K$ with an orthogonal G-spectrum X is given in level $W \cong V\oplus V'$ by*

$$(\mathcal{G}_V K\wedge X)_W \cong \mathbf{O}_W\wedge_{\mathbf{O}_V\times\mathbf{O}_{V'}} K\wedge X_{V'},$$

and just a basepoint in levels W which V does not embed into isometrically. The G-action on the right space is again diagonal on all three smash factors, through the conjugation action on the first.

Proof. Again this is proven by identifying the diagonal G-actions on smash products in 1.2.37. $\square$

Proposition 2.2.20. *Let Γ be another (topological) group. Then a (continuous) action of Γ on a semi-free G-spectrum $\mathcal{G}_V^G K \in G\mathcal{O}\mathcal{T}$ is equivalent to a (continuous) action of Γ on the $\mathbf{O}_V \rtimes G$-space K, and the levelwise orbit spectrum $\left[\mathcal{G}_V^G K\right]_\Gamma$ is isomorphic to $\mathcal{G}_V(K_\Gamma)$.*

Proof. This again follows exactly as in 1.2.38 because we assumed that the action of Γ is categorical, implying that Γ acts through G-equivariant maps, i.e. the actions of Λ and G commute. $\square$

2.2.3 Universes

Traditionally, choices of universe have played a big role in stable equivariant homotopy theory. Working with orthogonal spectra, however, the universes move to the background, at least for the categorical viewpoint. We recall some of the definitions, so that we can make exact statements about where the choice of universe matters (and where it does not).

Definition 2.2.21. A *G-universe $\mathcal{U}$* is a sum of countably many copies of each orthogonal G-representation in some set of irreducible G-representations, that includes the trivial representation. We say that $\mathcal{U}$ is *complete* if it contains all irreducible representations, and that it is*trivial* if it contains only trivial representations.

Definition 2.2.22. Given a G-universe $\mathcal{U}$, define $\mathcal{O}_G^{\mathcal{U}}$ to be the full subcategory of $\mathcal{O}_G$, containing only those objects V, that are subspaces of $\mathcal{U}$. Then analogous to Definition 2.2.5, let $\mathcal{O}_G^{\mathcal{U}}\mathcal{T}$ be the category of G-functors from $\mathcal{O}_G^{\mathcal{U}}$ to $\mathfrak{T}_G$.

Lemma 2.2.23. *For any choice $\mathcal{U}$ of G-universe, the $G\mathfrak{T}$-category $\mathcal{O}_G^{\mathcal{U}}\mathcal{T}$ is equivalent to $\mathcal{O}_G\mathcal{T}$.*

Proof. The inclusion of categories $\iota\colon \mathcal{O}_G^{\mathcal{U}} \to \mathcal{O}_G$ defines a functor $\mathbb{U}_\iota\colon \mathcal{O}_G\mathcal{T} \to \mathcal{O}_G^{\mathcal{U}}\mathcal{T}$ by precomposition. This functor has a left adjoint we denote by $\mathbb{P}_\iota$ and because ι is the inclusion of a full subcategory the composition $\mathbb{U}_\iota\mathbb{P}_\iota$ is naturally isomorphic to the identity functor. Since $\mathcal{O}_G^{\mathcal{U}}$ in particular contains all trivial representations $\mathbb{R}^n$, the formula for evaluation at a general representation 2.1.2 shows that the functor pair is an equivalence of categories. $\square$

Similarly we could talk about indexing collections instead of universes:

Definition 2.2.24. Let $\mathcal{V}$ be a collection of finite dimensional euclidean G-representations. Call $\mathcal{V}$ *good*, if it contains the trivial one dimensional representation and is closed under direct sums.

Definition 2.2.25. Let $\mathcal{U}$ be a G-universe. An *indexing G-space* in $\mathcal{U}$ is a finite dimensional sub-G-space of $\mathcal{U}$, with the restricted inner product making it an orthogonal G-representation. Define $\mathcal{V}(\mathcal{U})$ to be the good collection of all G-representations that are isomorphic to indexing G-spaces in $\mathcal{U}$.

Similarly one can produce a G-universe out of a given good collection of G-representations. Note that if $\mathcal{U}$ was complete, then $\mathcal{V}(\mathcal{U})$ is the collection of all finite dimensional euclidean G-representation.

Definition 2.2.26. Given a collection $\mathcal{V}$, define $\mathcal{O}_G^{\mathcal{V}}$ to be the full subcategory of $\mathcal{O}_G$, containing only those objects V, that are elements of $\mathcal{V}$. Then analogous to Definition 2.2.5, let $\mathcal{O}_G^{\mathcal{V}}\mathcal{T}$ be the category of G-functors from $\mathcal{O}_G^{\mathcal{V}}$ to $\mathcal{T}_G$.

Again all the categories defined from good collections in this way are equivalent analogous to 2.2.23. Sometimes it is worthwhile to consider similar constructions for collections of G-representations that are not good in the sense of 2.2.24, in particular the following example will come up in our study of geometric fixed points of smash powers:

Example 2.2.27. Let G be a non-trivial finite group, X a finite free G-set (e.g. $X = G$), and let $\mathcal{V}$ be the collection of objects of the full subcategory $\mathcal{O}_G^{\mathrm{reg}}$ of $\mathcal{O}_G$ containing all finite dimensional X-regular G-representations, that is representations of the form $V \otimes \mathbb{R}_{[X]}$ for $V \in \mathcal{O}_G$. Then in particular $\mathcal{O}_G^{\mathrm{reg}}$ is not good because it only contains representations with dimensions a multiple of the order of X. As in 2.2.26, we can form the category of *X-regular orthogonal spectra* $\mathcal{O}_G^{\mathrm{reg}}\mathcal{T}$. Again, the inclusion $i\colon \mathcal{O}_G^{\mathrm{reg}} \to \mathcal{O}_G$ defines a pair of adjoint functors

$$\mathbb{P}_i\colon \mathcal{O}_G^{\mathrm{reg}}\mathcal{T} \leftrightarrows \mathcal{O}_G\mathcal{T}\mathbb{U}_i$$

as above, which still has the property $\mathbb{U}_i\mathbb{P}_i \cong \mathrm{id}$, but is not an equivalence of categories. However, we can as in 2.2.14 define free and semi-free functors for $\mathcal{O}_G^{\mathrm{reg}}\mathcal{T}$. Then the fact that for W in $\mathcal{O}_G^{\mathrm{reg}}$ we can factor the inclusion $\mathbf{O}_W \to \mathcal{O}_G^{\mathrm{reg}}\mathcal{T} \to \mathcal{O}_G$ implies the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{O}\mathcal{T} & \mathcal{O}_G^{\mathrm{reg}}\mathcal{T} \arrow[l, "\mathbb{P}_i"'] & \mathbf{O}_V\mathcal{T} \arrow[l, "\mathcal{G}_W^{\mathrm{reg}}"'] \arrow[ll, bend right, "\mathcal{G}_W(-)"']
\end{tikzcd}$$

of left adjoints to the respective precomposition functors. In particular we get the following natural isomorphism between semi-free functors:

$$\mathcal{G}_W^{\mathrm{reg}} \cong \mathbb{U}_i\mathbb{P}_i\mathcal{G}_W^{\mathrm{reg}} \cong \mathbb{U}_i\mathcal{G}_W. \tag{2.2.28}$$

Note that we only used that $\mathcal{O}_G^{\mathrm{reg}}$ was a full subcategory, so one could easily generalize the formula 2.2.28.

2.2.4 Model structures

Let for the rest of the section G be a compact Lie group. We recall the results on model structures for G-equivariant orthogonal spectra as well as some technical properties and tools from [MM]. The model structures on equivariant orthogonal spectra constructed in [MM, III], depend heavily on a background choice of G-universe that underlies the definition of the equivariant orthogonal spectra [MM, II.2.6]. However, Theorem [MM, V.1.7] validates our viewpoint of not changing the underlying category (as was originally done in [MM, II]). Still, to give exact statements, fix some choice of universe $\mathcal{U}$ and a $G\mathfrak{T}$-skeleton $\operatorname{sk}\mathcal{O}_G^{\mathcal{V}}$ of $\operatorname{sk}\mathcal{O}_G^{\mathcal{U}} \subset \mathcal{O}_G$ throughout this subsection. We call objects of $\operatorname{sk}\mathcal{O}_G^{\mathcal{V}}$ *levels*, and say that some property applies *levelwise*, whenever it holds for all levels V.
Recall the genuine model structure on $G\mathfrak{T}$ from C.1.8, and in particular the sets I_G and J_G of generating cofibrations and acyclic cofibrations, respectively.

Definition 2.2.29. Let $f : X \to Y$ be a morphism of orthogonal G-spectra.

(i) f is a *level equivalence* if $f_V : X_V \to Y_V$ is a genuine equivalence in $G\mathfrak{T}$ for all levels V.

(ii) f is a *level fibration* if $f_V : X_V \to Y_V$ is a genuine fibration in $G\mathfrak{T}$ for all levels V.

(iii) f is a *q-cofibration* if f satisfies the left lifting property with respect to all maps that are both level equivalences and level fibrations.

Let $\mathcal{F}I_G$ be the set of all maps $\mathcal{F}_V i$ with V some level and i in I_G, and $\mathcal{F}J_G$ the analogous construction. Then the following theorem is a combination of [MM, IV.4.6] and [MM, V.1.7]:

Theorem 2.2.30. *The category $G\mathcal{OT}$ of orthogonal G-spectra is a cofibrantly generated proper G-topological model category with respect to the level equivalences, level fibrations and q-cofibrations. The sets $\mathcal{F}I_G$ and $\mathcal{F}J_G$ are the generating q-cofibrations and level acyclic q-cofibrations, respectively.*

Once more, we give a list of homotopical properties we use throughout. Again, orthogonal G-spectra X are called *well based* if the levels X_V are well based G-spaces for all levels V.

Lemma 2.2.31. *If all involved spectra are well based, then the generalized cube lemma and the generalized cobase change lemma hold for levelwise h-cofibrations and levelwise G-homotopy equivalences as well as for levelwise h-cofibrations and level equivalences.*

Remark 2.2.32. As in B.1.29, one can slightly relax the well-based assumption.

Definition 2.2.33. For an orthogonal G-spectrum X, $k \in \mathbb{Z}$ and H a subgroup of G, define the *homotopy groups* $\pi_k^H(X)$ as the colimit

$$\pi_k^H X := \begin{cases} \operatorname{colim}_{V \subset \mathcal{U}} \pi_k(\Omega^V X_V)^H & \text{if } k \geq 0 \\ \operatorname{colim}_{\mathbb{R}^k \subset V \subset \mathcal{U}} \pi_0(\Omega^{V-\mathbb{R}^k} X_V)^H & \text{if } k \leq 0 \end{cases}$$

Remark 2.2.34. cf. [Kr, 3.3.3] Note that this definition depends on the choice of universe $\mathcal{U}$. To be precise, it depends on the choice of $\mathcal{U}$ only up to H-equivalence: Let $\phi\colon V \to W$ be an H-equivariant isometry between levels. Then the vertical maps in the diagram

$$\begin{tikzcd} S^V \arrow[r, "f"] \arrow[d, "\phi_*"'] & X_V \arrow[d, "X_\phi"] \\ S^W \arrow[r, "g"] & X_W \end{tikzcd}$$

are H-equivariant homeomorphisms, and hence there is a homeomorphism between $(\Omega^V X_V)^H$ and $(\Omega^W X_W)^H$, sending a map f to the unique g that makes the diagram commute. The same argument works when one replaces S^V by $S^{V-\mathbb{R}^k}$ in the diagrams, i.e. for negatively indexed homotopy groups.
Similarly, for $\mathcal{U}$ cofinal in another G-universe $\mathcal{U}'$, i.e. if there is a G-equivariant embedding $\mathcal{U} \to \mathcal{U}'$ such that every indexing space in $\mathcal{U}'$ is a G-equivariant subspace of an indexing space in $\mathcal{U}$, a map of G-spectra is a π_*-isomorphism with respect to $\mathcal{U}$, if and only if it is so with respect to $\mathcal{U}'$.

Definition 2.2.35. Let $f : X \to Y$ be a map of orthogonal G-spectra. Then f is an π_*-*isomorphism* if for all (closed) subgroups H of G and $k \in \mathbb{Z}$ the induced map

$$f_*\colon \pi_k^H X \to \pi_k^H Y$$

is an isomorphism.

Again, this definition of course depends on the choice of universe. In cases where the group G in question is ambiguous, we will sometimes introduce a superscript and write π_*^G-*isomorphism*. Similarly, for $\mathscr{H}$ a family of subgroups of G, we write $\pi_*^{\mathscr{H}}$-isomorphism for maps that induce isomorphisms on π_*^H for all $H \in \mathscr{H}$.

Definition 2.2.36. An orthogonal G-spectrum X is an Ω-*spectrum*, if for all levels V and W, the map

$$\bar{\sigma}\colon X_V \to \Omega^W X_{V \oplus W}$$

adjoint to the structure map is a genuine G-equivalence.

Proposition 2.2.37. *[MM, III.9.3] Let $f\colon X \to Y$ be a map of Ω-spectra and let $\mathscr{H}$ be a closed family of subgroups of G. If $\pi_*^H(f)$ is an isomorphism for all $H \in \mathscr{H}$, then $(f_V)^H$ is a weak equivalence for all levels V.*
In particular if f is a π_*-isomorphism, then f is a level equivalence.

The following technical properties of π_*-isomorphisms will be used heavily throughout our work:

Proposition 2.2.38. *[MM, III.3.3, 3.5-3.11, 4.13, paraphrased]*

(i) A level equivalence of orthogonal G-spectra is a π_*-isomorphism.

(ii) For a genuine G-cell complex A, the functor $-\wedge A$ on orthogonal spectra preserves π_*-isomorphisms.

(iii) A morphism f of orthogonal G-spectra is a π_*-isomorphism if and only if any of its equivariant suspensions, i.e. $\Sigma^V f$ for some level V, is. The natural map $\eta\colon X \to \Omega^V\Sigma^V X$ is a π_*-isomorphism for all orthogonal spectra X.

(iv) The homotopy groups of a wedge of orthogonal G-spectra are the direct sums of the homotopy groups of the wedge summands, hence a wedge of π_*-isomorphisms is a π_*-isomorphism.

(v) Cobase changes of maps that are π_*-isomorphisms and levelwise h-cofibrations are π_*-isomorphisms.

(vi) The generalized cobase change and cube lemmas (B.1.5,B.1.7) hold for all orthogonal G-spectra, levelwise h-cofibrations and π_*-isomorphisms.

(vii) If X is the colimit of a sequence of h-cofibrations $X_n \to X_{n+1}$, each of which is a π_*-isomorphism, then the map from the initial term X_0 into X is a π_*-isomorphism.

(viii) For any morphism $f\colon X \to Y$ of orthogonal G-spectra and any $H \subset G$, there are natural long exact sequences

$$\cdots \to \pi_q(Ff) \to \pi_q(X) \to \pi_q(X) \to \pi_{q-1}(Ff) \to \cdots$$

$$\cdots \to \pi_q(X) \to \pi_q(Y) \to \pi_q(Cf) \to \pi_{q-1}(X) \to \cdots ,$$

where Ff and Cf denote the (levelwise) homotopy fiber and cofiber of f. The natural map $Ff \to \Omega Cf$ is a π_*-isomorphism.

All the above statements also hold with π_*-isomorphisms replaced by $\pi_*^{\mathscr{H}}$-isomorphisms for some family $\mathscr{H}$ of (closed) subgroups of G.

Again the level model structures are localized at the π_* isomorphisms to get stable structures, cf. [MM, 3.4]. The following maps play the role analogue to 1.2.47 in the non equivariant case:

Definition 2.2.39. For V and W in $\mathcal{O}_G$, let $\lambda_{V,W} : \mathcal{F}_{V\oplus W} S^W \to \mathcal{F}_V S^0$ be the map adjoint to the inclusion

$$S^W \to \mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} S^0 \wedge S^W,$$

sending the sphere to the copy indexed by the identity in $\mathbf{O}_{V\oplus W}$.

Lemma 2.2.40. *[MM, III.4.5] The maps $\lambda_{V,W}$ are π_*-isomorphisms.*

Definition 2.2.41. Once again recall the sets I_G of generating cofibrations for G-spaces, and $\mathcal{F}J$ of generating level acyclic cofibrations of orthogonal G-spectra from above. Factor all the maps $\lambda_{V,W}$ via the mapping cylinder as $\lambda_{V,W} = r_{V,W} \circ k_{V,W}$, with $r_{V,W}$ a deformation retraction. If source and target of $\lambda_{V,W}$ are q-cofibrant (e.g. if V and W are levels), then $k_{V,W}$ is a q-cofibration. Let K_G be the union of $\mathcal{F}J_G$ and the set of all maps of the form $i \Box k_{V,W}$ with $i \in I_G$ and V and W levels.

Definition 2.2.42. A *q-fibration* of orthogonal G-spectra is a map that has the lifting property with respect to all maps that are q-cofibrations and π_*-isomorphisms.

Again there is a characterization analogous to 1.2.50

Lemma 2.2.43. *[MM, 4.8] A map $p\colon E \to B$ satisfies the right lifting property with respect to K_G if and only if p is a level fibration and the diagrams*

$$\begin{tikzcd} E_V \arrow[r, "\tilde{\sigma}"] \arrow[d, "p_V"'] & \Omega^W E_{V+W} \arrow[d, "\Omega p_{V+W}"] \\ B_V \arrow[r, "\tilde{\sigma}"'] & \Omega^W B_{V+W} \end{tikzcd} \tag{2.2.44}$$

are homotopy pullbacks for all levels V and W, that is the map from E_V to the pullback is a genuine equivalence of G-spaces.

Corollary 2.2.45. *The map $F \to \star$ has the right lifting property with respect to K_G, if and only if F is an Ω-spectrum.*

Then the following theorem combines [MM, III.4.2,III.7.4, 7.5 and IV 6.5]:

Theorem 2.2.46. *The category $G\mathcal{OT}$ is a stable compactly generated proper G-topological model category with respect to the π_*-isomorphisms, q-fibrations and q-cofibrations. The sets $\mathcal{F}I_G$ and K_G are the generating cofibrations and acyclic cofibrations, respectively. This model structure satisfies the pushout product and the monoid axiom.*

An important fact is that this model structure is not just stable, but also *equivariantly stable* in the following sense:

Theorem 2.2.47. *[MM, III.4.14] For any V in $\mathcal{U}$, the functor pair (Σ^V, Ω^V) is a self Quillen equivalence of $G\mathcal{OT}$, with respect to the model structure from 2.2.46.*

This model structure is lifted to categories of orthogonal ring spectra and modules and algebras over such in [MMSS, III.7], following the general treatment of such questions from [SS]. Again when trying to lift this model structure to commutative orthogonal G-ring spectra, we have to consider positive variations of the definitions above. Note that in the equivariant context, *positive* encodes a stricter condition than non equivariantly: The generating sets of cofibrations $\mathcal{F}^+I_G$, $\mathcal{F}^+J_G$ and K_G^+ are therefore defined by excluding all maps that require the use of $\mathcal{F}_V$ with $V^G = 0$ from their absolute counterparts (cf. 2.2.29 and 2.2.41).

Definition 2.2.48. Let f be a map of orthogonal G-spectra:

(i) f is a *positive level fibration*, if f_V is a genuine fibration for all V with $V^G \neq 0$.

(ii) f is a *positive level equivalence*, if f_V is a genuine equivalence for all V with $V^G \neq 0$.

(iii) f is a *positive q-cofibration* if it is a q-cofibration and f_V is a homeomorphism for all V with $V^G = 0$.

(iv) f is a *positive q-fibration* if it has the right lifting property with respect to all maps that are positive q-cofibrations and π_*-isomorphisms.

Theorem 2.2.49. *[MMSS] The category $G\mathcal{S}p^{\mathcal{O}}$ of orthogonal spectra is a compactly generated proper topological model category with respect to the positive level equivalences, positive level fibrations and positive q-cofibrations. The sets $\mathcal{F}^+I_G$ and $\mathcal{F}^+J_G$ are the generating sets of cofibrations and acyclic cofibrations, respectively.*

Theorem 2.2.50. *[MM, 5.3,7.4,7.5] The category $G\mathcal{S}p^{\mathcal{O}}$ of orthogonal G-spectra is a compactly generated stable proper topological model category with respect to the π_*-isomorphisms, positive q-fibrations and positive q-cofibrations. The sets $\mathcal{F}^+I_G$ and K_G^+ are the generating sets of cofibrations and acyclic cofibrations, respectively. This model structure satisfies the pushout product axiom an the monoid axiom.*

Again recall the functor $\mathbb{E}$ from A.1.19.

Theorem 2.2.51. *[MM, 8.1] The category of commutative orthogonal G-ring spectra is a compactly generated proper topological model category with fibrations and weak equivalences are created in the positive stable model structure of $G\mathcal{S}p^{\mathcal{O}}$. The sets $\mathbb{E}\mathcal{F}^+I_G$ and $\mathbb{E}K_G^+$ are the generating sets of cofibrations and acyclic cofibrations, respectively.*

Theorem 2.2.52. *Let R be a commutative orthogonal ring spectrum The category of commutative R-algebras is a compactly generated proper topological model category whose weak equivalences, fibrations and cofibrations created in the model structure 2.2.46 for the category of commutative orthogonal G-ring spectra.*

2.2.5 Change of Groups

Just as in the case of the change of universe, the change of groups in the context of orthogonal G-spectra is severely simplified by the formula for evaluating a spectrum at general representations 2.1.2:

Definition 2.2.53. Let X be an orthogonal G-spectrum, and let V be any H-representation of dimension n for $i\colon H \subset G$ the inclusion of a subgroup. Then the *evaluation of X at V*, is the H-space

$$X_V := \mathcal{O}_H(\mathbb{R}^n, V) \wedge_{\mathbf{O}_n} X_n,$$

where the action of H is diagonal on the smash product, by postcomposition on the first factor and through the inclusion i on X_n (C.1.4).

Hence again, even though a G-spectrum classically only contains information for H-representations that are restrictions of G-representations, an orthogonal G-spectrum already carries all necessary equivariant information. Of course we should assemble the levelwise information into a functor.

Definition 2.2.54. Let $i\colon H \to G$ be the inclusion of a subgroup. Then the assignment $(i^*X)_V := X_V$ for all H-representations V gives the object part of the *restriction functor*

$$i^*\colon G\mathcal{OT} \to H\mathcal{OT}.$$

On morphisms one checks easily that the maps on levels $\mathbb{R}^n$ induce morphisms of H-spectra, which are compatible via the natural isomorphisms 2.1.2.

Remark 2.2.55. Note that for V an H-representation which is the restriction of a G-representation, i.e. $V \cong i^*W$, we have H-equivariant homeomorphisms

$$(i^*X)_V \cong i^*(X_W).$$

Remark 2.2.56. Considering the different enrichments, we can consider $\mathcal{O}_G\mathcal{T}$ as an $H\mathfrak{T}$-enriched category by transporting the $G\mathfrak{T}$-enrichment along the space level restriction functor, which is strong symmetric monoidal (cf. A.2.9). Doing so, the functor i^* becomes an $H\mathcal{T}$-functor $\mathcal{O}_G\mathcal{T} \to \mathcal{O}_H\mathcal{T}$, whose H-fixed point functor is the one in Definition 2.2.54.

The functor i^* has both a left and a right adjoint, the latter of which will not be used in our work, so we omit a formal definition.

Definition 2.2.57. Let Y be an orthogonal H-spectrum. Then the *induced G-spectrum $G_+ \wedge_H Y$* is the orthogonal G-spectrum given by

$$(G_+ \wedge_H Y)_V := G_+ \wedge_H Y_{i^*V},$$

with the (G equivariant) structure maps given as the composite

$$\begin{aligned} \mathcal{O}_G(V,W) \wedge (G_+ \wedge_H Y_{i^*V}) &\cong G_+ \wedge_H (i^* \mathcal{O}_G(V,W) \wedge Y_{i^*V}) \\ &\cong G_+ \wedge_H (\mathcal{O}_H(i^*V, i^*W) \wedge Y_{i^*V}) \\ &\to G_+ \wedge_H Y_{i^*W}, \end{aligned}$$

where the first first isomorphism is an instance of C.1.6, and the last map is the structure map of the H-spectrum Y.

Remark 2.2.58. Again, viewing $G\mathcal{OT}$ as an $H\mathfrak{T}$-category, this even gives an enriched functor.

The interactions of these change of group functors with the model structures from the previous section are well documented in [MM, V.2]. We will only repeat the ones that we explicitly use throughout. Let G be a compact Lie group and H a closed subgroup.

Lemma 2.2.59. *[MM, V.2.2] The restriction functor $i^* \colon G\mathcal{OT} \to H\mathcal{OT}$ "preserves π_*-isomorphisms".*

This looks innocent, but of course we should make the statement precise in terms of the involved choice of universes. So to be specific let $\mathcal{U}_G$ be a G-universe such that $i^*\mathcal{U}_G$ is cofinal in an H-universe $\mathcal{U}_H$, that is, any finite dimensional subspace V of $\mathcal{U}_H$ can be H-equivariantly embedded into a subspace of $i^*\mathcal{U}_G$.

Lemma 2.2.60. *The restriction functor i^* sends π_*^G-isomorphisms defined in terms of the universe $\mathcal{U}_G$ to π_*^H-isomorphisms defined in terms of the universe $\mathcal{U}_H$.*

Here we implicitly used Remark 2.2.34. Under the same assumption the following statement about the left adjoint is immediate:

Lemma 2.2.61. *The induction functor $G_+ \wedge_H (-)$ maps levelwise H-homotopy equivalences to levelwise G-homotopy equivalences.*

Similarly, if we additionally assume that $\mathcal{U}_H$ is even H-isomorphic to $i^*\mathcal{U}_G$, we get the following lemma:

Lemma 2.2.62. *The induction functor $G_+ \wedge_H (-)$ "preserves (acyclic) q-cofibrations", i.e. sends (acyclic) q-cofibrations defined in terms of $\mathcal{U}_H$ to acyclic q-cofibrations defined in terms of $\mathcal{U}_G$.*

2.3 Equivariant Convenient Model Structures

In this section we give analogues to the convenient model structure from Theorem 1.3.28 in the case of equivariant orthogonal (commutative ring) spectra. The non equivariant case will serve as a blueprint for the constructions. Let again for the whole section G denote a compact Lie group.

2.3.1 More Mixed Model Structures

Recall the evaluation G-functors from G-spectra to $\mathbf{O}_V \rtimes G\mathcal{T}$, and their left adjoints the semi free G-spectrum functors from Definition 2.2.14. Analogous to 1.3.5 we want to construct a model structure on $G\mathcal{S}p^{\mathcal{O}}$ by lifting a family of model structures on the various categories $\mathbf{O}_V \rtimes G\mathcal{T}$ along the functors $\mathcal{G}_V$.

For each orthogonal G-representation V, we therefore consider the genuine model structure on $\mathbf{O}_V \rtimes G\mathcal{T}$ as in C.1.8. Recall that the generating (acyclic) cofibrations were given by the sets

$$I_{\mathbf{O}_V \rtimes G\mathcal{T}} = \left\{ i\colon (S^{n-1} \times {}^{\mathbf{O}_V \rtimes G}\!/_{H})_+ \to (D^n \times {}^{\mathbf{O}_V \rtimes G}\!/_{H})_+,\ n \geq 0,\ H \text{ closed} \right\}$$

$$J_{\mathbf{O}_V \rtimes G\mathcal{T}} = \left\{ i_0\colon (D^n \times {}^{\mathbf{O}_V \rtimes G}\!/_{H})_+ \to (D^n \times [0,1] \times {}^{\mathbf{O}_V \rtimes G}\!/_{H})_+,\ n \geq 0,\ H \text{ closed} \right\}.$$

We abbreviate the notation by writing $I_V := I_{\mathbf{O}_V \rtimes G\mathcal{T}}$ and $J_V := J_{\mathbf{O}_V \rtimes G\mathcal{T}}$ instead. Also recall that for any closed family $\mathscr{F}$ of subgroups of $\mathbf{O}_V \rtimes G$ (cf. C.1.1), the $\mathscr{F}$-equivalences are defined as those maps that are weak equivalences on H fixed points, for all H in $\mathscr{F}$ (C.1.12). Similar to the mixed model structure we constructed on general G-spaces, we want to construct model structures that are a mix between the genuine and a more naive structure on $\mathbf{O}_V \rtimes G$-spaces. Compared to the structure above, we in the end want to relax the requirements on weak equivalences to exactly allow for maps that are naive $\mathbf{O}_V$- but genuine G-equivalences. The precise statement is the following theorem:

Theorem 2.3.1. *Let $\mathscr{F}$ be a closed family of closed subgroups of $\mathbf{O}_V \rtimes G$. The category $\mathbf{O}_V \rtimes G\mathcal{T}$ has a compactly generated model category structure with cofibrations the genuine cofibrations and weak equivalences the $\mathscr{F}$-equivalences.*

Proof. Recall the universal $\mathscr{F}$-space $E\mathscr{F}$ from C.1.18. Let H be a subgroup of $\mathbf{O}_V \rtimes G$ that is not in $\mathscr{F}$. Then as in 1.3.2 consider the projection

$$\pi_H\colon (\mathbf{O}_V \rtimes G \times_H E\mathscr{F})_+ \to ({}^{\mathbf{O}_V \rtimes G}\!/_{H})_+,$$

which is a $\mathscr{F}$-equivalence (C.1.19). Factor $\pi_H = r_H \circ j_H$ via the mapping cylinder $M\pi_H$ as an $\mathscr{F}$-acyclic genuine cofibration j_H, followed by an $\mathbf{O}_V \rtimes G$-deformation retraction r_H. Then define the new set of generating acyclic mixed cofibrations as

$$J_{V\,m} := J_V \cup \{j_H \square i, i \in I_V, H \notin \mathscr{F}\}.$$

The proof is then the same as the one of Theorem 1.3.2, if we substitute naive equivalence by $\mathscr{F}$-equivalences, and use that the $\mathscr{F}$-model structure is G-topological (C.1.13), instead of the fact that naive weak equivalences induce equivalences on homotopy fixed points. The new reference for closure properties of $\mathscr{F}$-equivalences is C.1.16. As it is the main point of complication, we show again that any map that is both in $J_{V\,m}-$inj and a naive weak equivalence is already in I_G-inj: For a map $f : X \to Y$ it is equivalent to be in $J_{V\,m}-$inj and that f is both a genuine fibration and that it has the right lifting property with respect to all the maps $j_H \square i$ from above. By adjointness B.1.12 the latter condition is equivalent to $\pitchfork_\square(j_H, f)$ having the right lifting property with respect to I_G for H not in $\mathscr{F}$. But since the genuine model structure is $\mathbf{O}_V\rtimes G$-topological (cf. C.2.7) this is equivalent to $\pitchfork_\square(j_H, f)$ being a genuine weak equivalence if f already was a genuine fibration. Again the defining diagram for $\pitchfork_\square(j_H, f)$ is given by the lower part of the diagram

$$\begin{tikzcd}
\mathcal{T}_{\mathbf{O}_V\rtimes G}\left({}^{\mathbf{O}_V\rtimes G}/_{H_+}, X\right) \arrow[d, "\sim"'] \arrow[rr, bend left] & & \\
\mathcal{T}_{\mathbf{O}_V\rtimes G}(M\pi_H, X) \arrow[dr, "\pitchfork_\square(j_H,f)", "\sim"'] & & \mathcal{T}_{\mathbf{O}_V\rtimes G}\left({}^{\mathbf{O}_V\rtimes G}/_{H_+}, Y\right) \arrow[d, "\sim"] \\
& P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{T}_{\mathbf{O}_V\rtimes G}(M\pi_H, Y) \arrow[d] \\
& \mathcal{T}_{\mathbf{O}_V\rtimes G}((\mathbf{O}_V\rtimes G\times_H E\mathscr{F})_+, X) \arrow[r] & \mathcal{T}_{\mathbf{O}_V\rtimes G}((\mathbf{O}_V\rtimes G\times_H E\mathscr{F})_+, Y).
\end{tikzcd}$$

In the situation we are interested in f is in $J_{V\,m}-$inj and a $\mathscr{F}$-equivalence, such that also the lower horizontal map is a $\mathbf{O}_V\rtimes G$-equivalence since $\mathbf{O}_V\rtimes G \times_H E\mathscr{F}$ is a $\mathscr{F}$-complex. Passing to $\mathbf{O}_V\rtimes G$-fixed points in the above diagram then yields that the maps f^H are also weak equivalences for all H not in $\mathscr{F}$, that is f is already a genuine $\mathbf{O}_V\rtimes G$ weak equivalence, hence in I_V-inj. $\square$

Note the following fact from the proof:

Corollary 2.3.2. *A map $f\colon E \to B$ is a fibration in the model structure of 2.3.1 if it is a genuine $\mathbf{O}_V\rtimes G$-fibration, and the maps $\pitchfork_\square\,(j_H, f)$ are genuine weak equivalences of G-spaces for all $P \notin \mathscr{F}$.*

Remark 2.3.3. Several variations on the family $\mathscr{F}$ will be important for us. Of most immediate use will be the family of subgroups which generate $\mathbf{O}_V$-free orbits:

$$\begin{aligned}
\mathscr{F}_{\text{ree}} &:= \{P \subset \mathbf{O}_V\rtimes G,\ \text{s.t. } {}^{\mathbf{O}_V\rtimes G}/_{P} \text{ is } \mathbf{O}_V\text{-free}\} \\
&= \{P \subset \mathbf{O}_V\rtimes G,\ \text{s.t. } P \not\ni (A, e) \text{ for } A \neq \mathrm{id}_V\} \\
&= \{P \subset \mathbf{O}_V\rtimes G,\ \text{s.t. } \mathrm{pr}_2\colon P \to G \text{ is injective}\}.
\end{aligned}$$

We usually do not indicate the specific G-representation V in the notation for the families; where confusion is possible, we add the superscript $\mathscr{F}_{\text{ree}}^V$.

2.3.2 Level Structures

In this section, we aim to use Theorem B.2.8 to assemble these various mixed model structures to a mixed level model structure on G equivariant orthogonal spectra. We will, however, begin with some more details about the cofibrations, weak equivalences and fibrations that will form the assembled structure.

Definition 2.3.4. A morphism of orthogonal G-spectra is an $\mathbb{S}$-cofibration, if it has the left lifting property with respect to those morphisms that are levelwise equivalences and fibrations in the model structures of Theorem 2.3.1, with respect to the families $\mathscr{F}_{\text{ree}}$.

It is important to notice that levelwise $\mathscr{F}_{\text{ree}}$ equivalences are not the same as level equivalences in the sense of 2.2.29. The following definition and subsequent corollaries will make the difference more obvious:

Definition 2.3.5. A morphism $f : X \to Y$ is a *strong level equivalence (fibration)* if for all (closed) subgroups $i\colon H \hookrightarrow G$ the restriction (i^*f) to H-spectra has the property that for *all* H-representations W, the map $(i^*f)_W$ is an H-equivalence (fibration).
An orthogonal G-spectrum X is a *strong Ω-spectrum* if for all (closed) subgroups $i\colon H \hookrightarrow G$ the restriction (i^*f) to H-spectra has the property that for *all* H-representations V and W, the adjoint structure map $\bar{\sigma}\colon X_V \to \Omega^W X_{V\oplus W}$ is an H-equivalence.

In particular the definitions of strong level equivalences, fibrations and Ω-spectra are independent of the choice of universe. Comparisons to these strong types of maps will be very useful when proving the model category axioms in Theorems 2.3.13 and 2.3.27. The core idea lies in the following two propositions, which should be seen as analogues of the formulas for evaluating a spectrum at any G-representation, now also including fixed points. For more information on the semi-direct product see also Subsection 3.3.3, which gives a good intuition of why one needs to take care when passing to fixed points.

Proposition 2.3.6 (Untwisting). *Let X be an orthogonal G-spectrum and let $(V, \phi\colon G \to \mathbf{O}_V)$ a G-representation. Then for any subgroup $P \in \mathscr{F}_{\text{ree}}^V$, there is a subgroup $i\colon H \hookrightarrow G$, a group homomorphism $\psi\colon H \to P$ and an H-representation V' of the same dimension as V, such that there is an isomorphism*

$$\psi^*(X_V) \cong (i^*X)_{V'}$$

of H-spaces. In particular $X_V^P \cong (i^*X)_{V'}^H$. This isomorphism is natural in X.

Proof. Since the $\mathbf{O}_V$-action on ${\mathbf{O}_V \rtimes G}/_{P}$ is free, the group homomorphism

$$P \xrightarrow{\text{inc}} \mathbf{O}_V \rtimes G \xrightarrow{\text{pr}_2} G$$

is injective. We denote the image of P in G by H. There is a splitting

$$\psi : H \xrightarrow{\cong} P \to \mathbf{O}_V \rtimes G$$

of the projection to G, and we denote $\psi(h) := (O_h, h) \in P \subset \mathbf{O}_V \rtimes G$. In particular because ψ is a group homomorphism we get the following relations on the elements O_h of $\mathbf{O}_V$:

$$O_{gh} = O_g \psi(g) O_h \psi(g)^{-1}.$$

The same relations then guarantee, that the map $\varphi : H \to \mathbf{O}_V$, $h \mapsto O_h \phi(h)$ is a group homomorphism. In particular we get a (new) H representation (V, φ), which we denote by V'. Now consider the isomorphisms from 2.1.2:

$$X_V \cong \mathcal{O}(V, V) \wedge_{\mathbf{O}_V} X_V \tag{2.3.7}$$

$$(i^* X)_{V'} \cong \mathcal{O}(V, V') \wedge_{\mathbf{O}_V} X_V.$$

We need to check H-equivariance, so let us take a closer look at the actions: Let $[(f, x)]$ be an element in $\psi^*(\mathcal{O}(V, V) \wedge_{\mathbf{O}_V} X_V)$, then $h \in H$ acts through ψ, i.e. by sending $[(f, x)]$ to $[(O_h \phi(h) f \phi(h)^{-1}, hx)]$. Similarly, considering $[(f, x)]$ as an element of $\mathcal{O}(V, V') \wedge_{\mathbf{O}_V} X_V$, the action of h takes it to $[(\varphi(h) f \phi(h)^{-1}, hx)]$. Hence by construction of φ, the "identity map" sending $[(f, x)]$ to $[(f, x)]$ is an H-homeomorphism from $\psi^* X_V$ to $(i^* X)_{V'}$ under the identifications 2.3.7. Naturality in X is immediate from the construction. □

Corollary 2.3.8. *Strong level equivalences are levelwise $\mathscr{F}_{\text{ree}}$-equivalences and strong level fibrations are levelwise $\mathscr{F}_{\text{ree}}$-fibrations.*

Proposition 2.3.9 (Twisting)**.** *Let X be an orthogonal G-spectrum, $(W, \varphi \colon G \to \mathbf{O}_W)$ an n-dimensional G-representation and $i \colon H \hookrightarrow G$ a subgroup. For any n-dimensional H-representation $(V, \phi \colon H \to \mathbf{O}_V$ and any subgroup $K \subset H$, there is a (continuous) homomorphism $\psi \colon K \cong L \subset \mathbf{O}_W \rtimes G$, such that L is in $\mathscr{F}_{\text{ree}}^W$ and there is an isomorphism*

$$(i^* X)_V \cong \psi^*(X_W)$$

of K-spaces. In particular $X_W^L \cong (i^* X)_V^K$. This isomorphism is natural in X.

Proof. We chose bases for V and W in order to identify the orthogonal groups $\mathbf{O}_V$ and $\mathbf{O}_W$. Different choices of bases will in the end yield potentially different subgroups L, but we are only concerned with the existence. It is enough to show the statement for $K = H$, since we can restrict X to a K-spectrum first. Again consider

$$\begin{aligned} (i^*X)_V &\cong \mathcal{O}(W,V)\wedge_{\mathbf{O}_W} X_W \\ X_W &\cong \mathcal{O}(W,W)\wedge_{\mathbf{O}_W} X_W. \end{aligned} \tag{2.3.10}$$

Similar to the untwisting case, we look at the H-actions: For $[(f,x)]$ in $(i^*X)_V$, an element $h \in H$ sends it to $[(\phi(h)f\varphi(h)^{-1}, hx)]$. On elements $[(f,x)]$ of X_W, an element $(A,g) \in \mathbf{O}_W \rtimes G$ operates by sending it to $[(A\varphi(g)f\varphi(g)^{-1}, gx)]$. Hence defining $\psi : H \to \mathbf{O}_W \rtimes G$ as

$$h \mapsto (\phi(h)\varphi(h)^{-1}, h)$$

yields the desired H-isomorphism $(i^*X)_V \cong \psi^*(X_n)$. Naturality in X is again immediate from the construction. □

Corollary 2.3.11. *For all $n \geq 0$ choose a G-representation V_n of dimension n. Morphisms that are $\mathscr{F}_{\text{ree}}$-equivalences (fibrations) in all levels V_n, are strong level equivalences (fibrations).*

Corollary 2.3.12. *In particular, strong level equivalences and levelwise $\mathscr{F}_{\text{ree}}$-equivalences are the same.*

Theorem 2.3.13. *The $\mathbb{S}$-cofibrations and strong level equivalences give a compactly generated G-topological model category structure on the category $G\mathcal{S}p^{\mathcal{O}}$ of G-equivariant orthogonal spectra.*

Proof. We use the Assembling Theorem B.2.8. Proposition B.2.10 shows that the resulting structure is G-topological. The sets of generating cofibrations are

$$\mathcal{G}I := \left\{ \bigcup_{V \in \mathcal{O}_G} \mathcal{G}_V I_V \right\} \quad \text{and} \quad \mathcal{G}J := \left\{ \bigcup_{V \in \mathcal{O}_G} \mathcal{G}_V J_{V\,m} \right\}.$$

To apply Theorem B.2.8 we need to check that the maps in $\mathcal{G}J$ are actually levelwise $\mathscr{F}_{\text{ree}}$-equivalences. So let $\mathcal{G}f : \mathcal{G}_V X \to \mathcal{G}_V Y$ a generating acyclic cofibration, i.e. $f \colon X \to Y$ is a generating acyclic mixed cofibration, in particular a genuine cofibration between genuine cofibrant $\mathbf{O}_V \rtimes G$-spaces, that is $\mathscr{F}_{\text{ree}}$-acyclic. We prove that $\mathcal{G}_V f$ is a strong level equivalence. In particular, let first $H = G$, and W any G-representation. Then f_W is the map

$$f_W \colon \mathcal{O}_G(V,W)\wedge_{\mathbf{O}_V} X_V \to \mathcal{O}(V,W)\wedge_{\mathbf{O}_V} Y_V.$$

Before passing to the $\mathbf{O}_V$-orbits, this is the map $\mathcal{O}(V,W)\wedge f_V$, which is a $\mathscr{F}_{\text{ree}}$-equivalence between $\mathscr{F}_{\text{ree}}$-cell complexes, by Lemma C.2.7(i) and (iii). Therefore

B.1.3 implies that it is even a $\mathbf{O}_V \rtimes G$-homotopy equivalence, and hence a G-equivalence after passing to $\mathbf{O}_V$-orbits.
Let now $i\colon H \hookrightarrow G$ be an inclusion of a closed subgroup and j the corresponding inclusion $\mathbf{O}_V \rtimes H \hookrightarrow \mathbf{O}_V \rtimes G$. Then 2.1.2 shows that for any $\mathbf{O}_V \rtimes G$-space K, the H-spectra $i^*(\mathcal{G}_V^G K)$ and $\mathcal{G}_{i^*V}^H j^*K$ are isomorphic. Since $j^*X \to j^*Y$ is again a $\mathscr{F}_{\mathrm{ree}}^{i^*V}$-equivalence between genuinely cofibrant $\mathbf{O}_V \rtimes H$-spaces, the same argument as in the above case of $H = G$ shows that f is even a strong level equivalence. □

Again note the following immediate consequence of the description of the fibrations in the Assembling Theorem B.2.8, and Corollary 2.3.2:

Corollary 2.3.14. *Any fibration in the model structure of Theorem 2.3.13 is a genuine $\mathbf{O}_V \rtimes G$-fibration in each level, in particular a genuine G-fibration in each level, and a strong level fibration by 2.3.9.*

Finally we record the following property:

Lemma 2.3.15. *The definition of $\mathbb{S}$-cofibrations is independent from the choice of universe.*

Proof. By the proof of 2.3.13, the q-cofibrations are exactly retracts of $\mathcal{G}I$-cell complexes. For any n chose a G-representation V_n of dimension n (for example the trivial one). We show that any $\mathcal{G}I$-cell complex is already a $\bigcup \mathcal{G}_{V_n} I_{V_n}$-complex. Indeed, let $\dim(W) = n$ and let

$$\mathcal{G}_W i : \mathcal{G}_W(S^{k-1} \times {}^{\mathbf{O}_W \rtimes G}\!/\!_P)_+ \to \mathcal{G}_W(D^k \times {}^{\mathbf{O}_V \rtimes G}\!/\!_P)_+$$

be in $\mathcal{G}I$. By 2.2.18, it suffices to show that

$$\mathcal{O}_G(W, V_n) \wedge_{\mathbf{O}_W} {}^{\mathbf{O}_W \rtimes G}\!/\!_{P_+}$$

is a genuine $\mathbf{O}_{V_n} \rtimes G$-complex. Before passing to the orbits, this is the space $\mathcal{O}_G(W, V_n) \wedge (\mathbf{O}_W \rtimes G)_+$, which is a genuine $P^{\mathrm{op}} \times ((\mathbf{O}_{V_n} \times \mathbf{O}_W) \rtimes G)$-complex by C.2.4, hence the result follows. □

Corollary 2.3.16. *The model structure in Theorem 2.3.13 is independent from the choice of universe.*

Again we can consider positive variations of this model structure analogous to 1.3.7, keep in mind, that in the equivariant setting a level V is called positive, if the dimension of the fixed point space V^G is positive.
Similar to [MM, IV.6], one can restrict attention to a specific closed family $\mathscr{H}$ of subgroups of G. To do so, we *only* have to adapt the definition of the families $\mathscr{F}_{\mathrm{ree}}$ in 2.3.3, and accordingly the definition of appropriately strong level equivalences.

Definition 2.3.17. Let $\mathscr{H}$ be a closed family of subgroups of G. For a representation V of G define the closed family $\mathscr{F}_{\text{ree}}^{\mathscr{H}}$ of subgroups of $\mathbf{O}_V \rtimes G$ as the intersection of $\mathscr{F}_{\text{ree}} \cap \mathscr{F}_{\mathscr{H}}$, where $\mathscr{F}_{\mathscr{H}}$ contains exactly those subgroups whose image under the projection to G is in $\mathscr{H}$.

Definition 2.3.18. A morphism $f : X \to Y$ is an *$\mathscr{H}$-strong level equivalence* if for all subgroups $H \in \mathscr{H}$ the restriction $(i^* f)$ to H-spectra has the property that for *all* H-representations W, the map $(i^* f)_W$ is an H-equivalence.

Then the assembling theorem B.2.8 implies the following analogue to 2.3.13

Theorem 2.3.19. *The $\mathbb{S}$-cofibrations and $\mathscr{H}$-strong level equivalences give a compactly generated G-topological model structure on the category $G\mathcal{S}p^{\mathcal{O}}$ of equivariant orthogonal spectra.*

Proof. The only thing we need to check, is that $\mathscr{H}$-strong level equivalences are levelwise $\mathscr{F}_{\text{ree}}^{\mathscr{H}}$-equivalences, and that the proposed generating acyclic cofibrations coming from the assembling theorem are indeed $\mathscr{H}$-strong level equivalences. Both are proved in the exact same way as above, together with the fact that the Twisting and Untwisting propositions 2.3.9 and 2.3.6 respect the property that the projection of a subgroup to the second factor is in $\mathscr{H}$. $\square$

Again this model structure is independent of the choice of universe.

2.3.3 Stable Structures

We construct stable versions of the $\mathbb{S}$-model structures analogous to [MM, III.4]. Unless explicitly stated otherwise, all π_*-isomorphisms are with respect to a complete universe. Just as in the case of the usual stable model structure on orthogonal G-spectra, we have to add more generating acyclic cofibrations.

Definition 2.3.20. For all closed subgroups $iH \hookrightarrow G$ and H-representations V and W, define

$$\lambda_{V,W}^H := \mathcal{F}_{V \oplus W}^H S^W \to \mathcal{F}_V^H S^0.$$

We induce up to G-spectra, setting $\lambda_{V,W}^G := G_+ \wedge_H \lambda_{V,W}^H$. Then as in the classical case, factor $\lambda_{V,W}^G$ via the mapping cylinder as $\lambda_{V,W}^G = r_{V,W} \circ k_{V,W}$ and let $\mathbb{S}J$ be the union of $\mathcal{G}J$ and the set of all maps of the form $i \square k_{V,W}$ with $i \in \mathcal{G}_0 I_0 = \mathcal{F}_0 I_0$. For the positive case, exclude those $k_{V,W}$ where V does not contain a trivial H-representation of dimension at least 1.

Lemma 2.3.21. *For all H-representations V and W as above, the map $k_{V,W}$ is an $\mathbb{S}$-cofibration and a π_*-isomorphism.*

Proof. We begin by showing that source and target of $\lambda^G_{V,W}$ are $\mathbb{S}I_G$-cofibrant, which implies the first part of the statement since the level structure is (G-) topological. Since source and target of $\lambda^H_{V,W}$ are $\mathbb{S}I_H$-cofibrant, 2.3.15 implies that it suffices to show that the inducing up functor $G_+\wedge_H(-)$ sends morphisms in $\mathcal{G}^H_{\mathbb{R}^n}I_{\mathbb{R}^n}$ to $\mathbb{S}I$-cell complexes. By comparison of the respective right adjoints, we get natural isomorphisms

$$G_+\wedge_H\mathcal{G}^H_{\mathbb{R}^n}K \cong \mathcal{G}_{\mathbb{R}^n}(G_+\wedge_H K) \cong \mathcal{G}_{\mathbb{R}^n}(\mathbf{O}_{\mathbb{R}^n}\rtimes G\wedge_{\mathbf{O}_{\mathbb{R}^n}\rtimes H}K).$$

Since the space level restriction functor preserves genuine fibrations, its left adjoint preserves genuine cell complexes.
For the second part, using a cofibrant approximation functor $(-)^c$ for the classical level model structure on $H\mathcal{OT}$ with respect to the H-universe containing the restricted G-representations. We get a diagram of H-spectra

$$\begin{tikzcd}
(\mathcal{F}^H_{V\oplus W}S^W)^c \arrow[r] \arrow[d] & (\mathcal{F}^H_V S^0)^c \arrow[d] \\
\mathcal{F}^H_{V\oplus W}S^W \arrow[r, "\lambda^H_{V,W}"] & \mathcal{F}^H_V S^0,
\end{tikzcd}$$

in which the top horizontal map is a π_*-isomorphism between q-cofibrant H-spectra, and the vertical maps are levelwise genuine H-equivalences between levelwise genuine $\mathbf{O}_V\rtimes H$-complexes, hence H-homotopy equivalences. Applying the inducing up functor to this diagram yields that $\lambda^G_{V,W}$ is a π_*-isomorphism by 2.2.61 and 2.2.62. $\square$

As usual for this kind of discussion, we need a characterization of the right lifting property with respect to $\mathbb{S}J$:

Proposition 2.3.22. *A morphism $p : E \to B$ of orthogonal G-spectra has the right lifting property with respect to $\mathbb{S}J$ ($\mathbb{S}^+J$), if and only if it is a fibration in the (positive) level structure of Theorem 2.3.13, and the diagrams*

$$\begin{tikzcd}
(i^*E)_V \arrow[r, "\tilde{\sigma}"] \arrow[d, "(i^*p)_V"'] & \Omega(i^*E)_{V\oplus W} \arrow[d, "\Omega^W(i^*p)_{V\oplus W}"] \\
(i^*B)_V \arrow[r, "\tilde{\sigma}"'] & \Omega^W(i^*B)_{V\oplus W}
\end{tikzcd} \qquad (2.3.23)$$

are homotopy pullbacks for all closed subgroups $H \subset G$ and H-representations V and W (which do contain trivial sub-H-representations, i.e. are positive).

Proof. Since $\mathbb{S}J$ contains $\mathcal{G}J$, any map that has the right lifting property with respect to $\mathbb{S}J$ is a fibration in the level $\mathbb{S}$-model structure. Additionally by the adjunctions A.2.32, it has the lifting property with respect to all the maps $i\Box k_{V,W}$, if and only if the maps $\mathcal{O}_G\mathcal{T}(k^*_{V,W}, p_*)$ are genuine acyclic fibrations in $G\mathfrak{T}$. Since $k_{V,W}$ is an $\mathbb{S}$-cofibration, $\mathcal{O}_G\mathcal{T}(k^*_{V,W}, p_*)$ is always a genuine fibration because the level structure is G-topological. Since $r_{V,W}$ was a levelwise G-homotopy equivalence, $\mathcal{O}_G\mathcal{T}(k^*_{V,W}, p_*)$ is a genuine weak equivalence if and only if $\mathcal{O}_G\mathcal{T}((\lambda^G_{V,W})^*, p_*)$ is. Finally use the defining adjunctions to identify the diagram in the proposition. □

Corollary 2.3.24. *If the morphism $E \to \star$ has the right lifting property with respect to $\mathbb{S}J$ ($\mathbb{S}^+J$), then E is a strong (positive) Ω-spectrum, in particular for all closed subgroups $i : H \hookrightarrow G$, i^*E is a (positive) H-Ω-spectrum.*

Proposition 2.3.25. *A π_*-isomorphism between strong (positive) Ω-spectra is a strong (positive) level equivalence.*

Proof. Since for all subgroups $i \colon H \hookrightarrow G$ the functor i^* preserves π_*-isomorphisms by 2.2.60, this is a direct consequence of 2.2.37. □

Definition 2.3.26. A morphism of orthogonal G-spectra is an $\mathbb{S}$-*fibration* if it has the right lifting property with respect to all morphisms that are $\mathbb{S}$-cofibrations and π_*-isomorphisms.

Theorem 2.3.27. *The category $G\mathcal{S}p^{\mathcal{O}}$ of orthogonal G-spectra is a proper monoidal model category with respect to the π_*-isomorphisms, (positive) $\mathbb{S}$-fibrations and (positive) $\mathbb{S}$-cofibrations. This* (positive) $\mathbb{S}$-model structure *satisfies the monoid axiom.*

Proof. The proof is similar to the classical case of [MM, III.4.2] and the non equivariant case of 1.3.10. The generating cofibrations and acyclic cofibrations are given by $\mathbb{S}I := \mathcal{G}I$ and $\mathbb{S}J$, respectively $\mathbb{S}^+I$ and $\mathbb{S}^+J$ for the positive case. All the maps in $\mathbb{S}J$ are in $\mathbb{S}I-$cof and π_*-isomorphisms, in particular also h-cofibrations, so that every map in $\mathbb{S}J-$cell is a π_*-isomorphism. We have to check, that every map $p \in \mathbb{S}J-$inj that is a π_*-isomorphism is already in $\mathbb{S}I-$inj. By the level model structure and Theorem 2.3.8 it suffices to show that p is a strong level equivalence. Since p has the right lifting property with respect to $\mathbb{S}I$ it is in particular a levelwise $\mathscr{F}_{\mathrm{ree}}$ fibration, i.e. a strong level fibration by Corollary 2.3.11. Thus it suffices to prove that the (levelwise) fiber F

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ \star \arrow[r] & B \end{tikzcd}$$

is strong level acyclic, by the long exact sequences of homotopy groups for all (strong) levels. But $F \to \star$ is again a π_*-isomorphism (2.2.38(viii)) and inherits the right lifting property with respect to $\mathbb{S}J$. Hence F is a strong Ω-spectrum and therefore strong level acyclic by 2.3.25.

The new reference for properness is [MM, III.4.13]. The pushout product axiom follows from the fact that π_*-isomorphisms are preserved both under the smash product with h-cofibrant spaces and under cobase change along h-cofibrations, together with the following proposition. The monoid axiom is similar to the non equivariant case implied by Proposition 2.3.29. $\square$

Proposition 2.3.28. $\mathbb{S}I \square \mathbb{S}I \subset \mathbb{S}I-$cell.

Proof. Let

$$\mathcal{G}_V\left(\mathbf{O}_V \rtimes G/_{H_{1+}} \wedge \left[S^{n-1} \to D^n\right]\right) \text{ and } \mathcal{G}_W\left(\mathbf{O}_W \rtimes G/_{H_{2+}} \wedge \left[S^{m-1} \to D^m\right]\right)$$

be maps in $\mathcal{G}I$. Then their pushout product is isomorphic to:

$$\mathcal{G}_{V\oplus W}\left(\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} \mathbf{O}_V \rtimes G/_{H_{1+}} \wedge \mathbf{O}_W \rtimes G/_{H_{2+}} \wedge \left[S^{n+m-1} \to D^{n+m}\right]\right).$$

Since as a left adjoint, smashing with a space preserves colimits, it suffices to show that

$$\mathbf{O}_{V\oplus W+} \wedge_{\mathbf{O}_V \times \mathbf{O}_W} \mathbf{O}_V \rtimes G/_{H_{1+}} \wedge \mathbf{O}_W \rtimes G/_{H_{2+}}$$

is a genuine $\mathbf{O}_{V\oplus W} \rtimes G$-cell complex. As a $\mathbf{O}_{V\oplus W} \rtimes G$-space, this is isomorphic to

$$\mathbf{O}_{V\oplus W} \rtimes (G \times G)/_{H_1 \times H_{2+}},$$

where $H_1 \times H_2$ acts on $\mathbf{O}_{V\oplus W} \rtimes (G \times G)$ from the right via:

$$[(A, g, g').(J_1, j_1, J_2, j_2)] := A\begin{pmatrix} gJ_1g^{-1} & 0 \\ 0 & g'J_2g'^{-1} \end{pmatrix}, gj_1, g'j_2$$

So the manifold $\mathbf{O}_{V\oplus W} \rtimes (G \times G)_+$ has an action by $\mathbf{O}_{V\oplus W} \rtimes G \times (H_1 \times H_2)^{\mathrm{op}}$ and is hence triangulable as an $\mathbf{O}_{V\oplus W} \rtimes G \times (H_1 \times H_2)^{\mathrm{op}}$-complex by [Ill83, 7.1]. Since taking orbits preserves colimits, so is

$$\mathbf{O}_{V\oplus W} \rtimes (G \times G)/_{H_1 \times H_{2+}}.$$

Finally, since $\mathbf{O}_{V\oplus W} \rtimes G \times (H_1 \times H_2)^{\mathrm{op}}$ orbit types are triangulable as $\mathbf{O}_{V\oplus W} \rtimes G$-complexes by C.2.5, it is genuinely cofibrant as desired.

$\square$

Proposition 2.3.29. *$\mathbb{S}$-cofibrant equivariant spectra are flat, in the sense that for any $\mathbb{S}$-cofibrant G-spectrum X, the functor $X \wedge -$ preserves equivariant π_*-isomorphisms.*

Proof. Let Z be a G-spectrum with trivial equivariant stable homotopy groups. Using the same arguments as in the proof of the non equivariant case (cf. Proposition 1.3.11), we can reduce to showing that $\mathcal{G}_V{}^{\mathbf{O}_V \rtimes G}\!/\!_{H_+} \wedge Z$ is still acyclic. Using Theorem [MM, 3.6] to replace Z by its shift by an arbitrary G-representation $W = V \oplus V'$ and again following the non equivariant case yields that we can reduce to showing that the levelwise smash product

$$\left(\mathbf{O}_{V\oplus V'}{}_+ \wedge_{\mathbf{O}_V} {}^{\mathbf{O}_V \rtimes G}\!\big/\!_{H_+}\right) \wedge_{\mathbf{O}_{V'}} Z$$

is still acyclic. The left space is a $\mathbf{O}_{V'}{}^{\mathrm{op}} \rtimes G$-complex by C.2.4, and like in the non equivariant case, the right action by $\mathbf{O}_{V'}$ is free. Then again a cell induction, this time using Theorem [MM, III.3.11], gives the desired result. □

Once again we can restrict attention to a closed family of subgroups of G, all the results above hold with the π_*-isomorphisms replaced by $\pi_*^{\mathscr{H}}$-isomorphisms so that we get:

Theorem 2.3.30. *The category $G\mathcal{S}p^{\mathcal{O}}$ of orthogonal G-spectra is a proper monoidal model category with respect to the $\pi_*^{\mathscr{H}}$-isomorphisms, (positive) $\mathbb{S}^{\mathscr{H}}$-fibrations and (positive) $\mathbb{S}$-cofibrations. This* (positive) $\mathbb{S}^{\mathscr{H}}$-model structure *satisfies the monoid axiom.*

Proposition 2.3.31. *The identity functor on the category of equivariant orthogonal spectra gives a monoidal Quillen equivalence from the classical (positive) stable model structure to the (positive) stable $\mathbb{S}^{\mathscr{H}}$-model structure.*

Proof. Since any q-cofibration is an $\mathbb{S}$-cofibration, the identity functor preserves cofibrations. Since the weak equivalences are the same in both model structures, it also preserves acyclic cofibrations and detects and preserves weak equivalences. The identity trivially satisfies all other properties of monoidal Quillen functors. □

Remark 2.3.32. The same proof implies a similar statement for the (positive) $\mathbb{S}^{\mathscr{H}}$-model structure and the "classical" (positive) stable $\mathscr{H}$-model structure on $G\mathcal{OT}$ from [MM, IV.6.5].

Analogous to 1.3.15 we use Theorem [SS, 4.1] to lift the (positive) $\mathbb{S}$-model structures to categories of modules and algebras:

Theorem 2.3.33. *Let R be an orthogonal ring G-spectrum.*

(i) The category of left R-modules is a compactly generated proper model category with respect to the π_*-isomorphisms and the underlying (positive) $\mathbb{S}$-fibrations. The sets of generating cofibrations and acyclic cofibrations are $R\wedge\mathbb{S}I$ ($R\wedge\mathbb{S}^+I$), and $R\wedge\mathbb{S}^+J$ ($R\wedge\mathbb{S}^+J$).

(ii) If R is $\mathbb{S}$-cofibrant, then the forgetful functor from R-modules to orthogonal spectra preserves cofibrations. Hence every cofibrant R-module is cofibrant as an orthogonal spectrum.

(iii) Let R be commutative. The model structures of (i) are monoidal and satisfy the monoid axiom.

(iv) Let R be commutative. The category of R-algebras is a compactly generated right proper model category with respect to the π_*-isomorphisms and the underlying (positive) $\mathbb{S}$-fibrations. The sets of generating cofibrations and acyclic cofibrations are $R \wedge \mathbb{A}\mathbb{S}I$ ($R \wedge \mathbb{A}\mathbb{S}^+ I$) and $R \wedge \mathbb{A}\mathbb{S}J$ ($R \wedge \mathbb{A}\mathbb{S}^+ J$).

(v) Let R be commutative. Every cofibration of R-algebras whose source is (positive) cofibrant as an R-module is also a cofibration of R-modules. In particular, every cofibrant R-algebra is cofibrant as an R-module.

All the above model structures are Quillen equivalent to the classical ones from [MM, III.7.6] via the identity functor as in 2.3.31, and admit variations for closed families $\mathscr{H}$ of subgroups of G.

As in the classical case, we will deal with the lift to commutative algebras in a separate section.

2.3.4 Extension to Commutative Ring Spectra

As usual, we only work with the positive $\mathbb{S}$- or positive $\mathbb{S}^{\mathscr{H}}$-model structure in this section, to keep notation simpler, we will not indicate the choice of closed family in the notation. Generally we will be rather brief, as most of the discussion is analogous to section 1.3.4, and the discussion in the classical case from [MM, III.8]. Once again we start with the technical lemmas analogous to 1.3.16 and 1.3.17:

Lemma 2.3.34. *Let Y be an orthogonal G-spectrum and let X be $\mathcal{G}_V\left[(\mathbf{O}_V \rtimes G/H)_+ \wedge K\right]$, for K a based CW-complex and $V^G \neq 0$. Then the quotient map*

$$q\colon \left(E\Sigma_{i+} \wedge_{\Sigma_i} X^{\wedge i}\right) \wedge Y \to X^{\wedge i}_{\Sigma_i} \wedge Y$$

is an eventual level G-equivalence, hence an equivariant π_*-isomorphism.

Proof. We prove that q is a G-equivalence in all levels $W \cong_G V^{\oplus i} \oplus V'$. There the target of q_W is given by:

$$\left[\mathbf{O}_{W+} \wedge_{\prod \mathbf{O}_V \times \mathbf{O}_{V'}} \left(\mathbf{O}_V \rtimes G/H_+\right)^{\wedge i} \wedge K^{\wedge i} \wedge Y_{V'}\right]_{\Sigma_i}.$$

Before taking Σ_i orbits this is isomorphic as an $\mathbf{O}_W \rtimes G$-space to

$$\left({}^{\mathbf{O}_W \rtimes G^{\times i}} \diagup_{\prod H} \right)_+ \wedge_{\mathbf{O}_{V'}} K^{\wedge i} \wedge Y V'$$

in a similar way as in the proof of the pushout product axiom in Theorem 2.3.28. Here, the group Σ_i acts diagonally on the first three factors, via multiplying with block permutation matrices induced by permuting V-summands from the right on the first, and by permuting (smash) factors on the second and third. Under this identification, we can rewrite the orbit space as

$$\mathbf{O}_W \rtimes G^{\times i}{}_+ \wedge_{\left[(\prod H^{\mathrm{op}} \rtimes \Sigma_i) \times \mathbf{O}_{V'}\right]} K^{\wedge i} \wedge Y_{V'}.$$

Doing the same for its source, we see that the map q_W comes from taking the $\Lambda := (\prod H^{\mathrm{op}} \rtimes \Sigma_i) \times \mathbf{O}_{V'}$-orbits of the map of $\Gamma := G \ltimes \left[(\prod H^{\mathrm{op}} \rtimes \Sigma_i) \times \mathbf{O}_{V'}\right]$-spaces

$$\hat{q} \colon E\Sigma_{i+} \wedge \mathbf{O}_W \rtimes G^{\times i} \wedge K^{\wedge i} \wedge Y_{V'} \to \mathbf{O}_W \rtimes G^{\times i} \wedge K^{\wedge i} \wedge Y_{V'},$$

that collapses $E\Sigma_i$. Note that $E\Sigma_i$ is Γ-homotopy equivalent to $E\mathscr{F}$, where $\mathscr{F}$ is the family of subgroups that intersect trivially with Σ_i. Since $\mathbf{O}_W \rtimes G^{\times i}$ is an $\mathscr{F}$-complex, $\hat{q}$ is a Γ-homotopy equivalence, hence $q_W = {}^{\hat{q}}\diagup_{\Lambda}$ is a G-homotopy equivalence as desired. □

Lemma 2.3.35. *Let X a positive $\mathbb{S}$-cofibrant G-spectrum. Then the quotient map*

$$q : E\Sigma_{i+} \wedge_{\Sigma_i} X^{\wedge i} \to X^{\wedge i}_{\Sigma_i}$$

is a π_*-isomorphism.

Proof. The proof is exactly as the one of Lemma 1.3.17 when one replaces $\mathbf{O}_W$ by $\mathbf{O}_W \rtimes G$. The new reference for the very last step is the following proposition. □

Proposition 2.3.36. *Let V contain a positive dimensional trivial G-representation and let $X = \mathcal{G}_V[K_+ \wedge {}^{\mathbf{O}_V \rtimes G}\diagup_{H+}]$ for K either S^n or D^n and H some closed subgroup of $\mathbf{O}_V \rtimes G$. Then the orthogonal spectrum $X^{\wedge i}_{\Sigma_i}$ is $\mathbb{S}$-cofibrant. In particular the inclusion $\mathbb{S} \to \mathbb{E}X$ is an $\mathbb{S}$-cofibration.*

Proof. We make the case of K a disc explicit this time, the argument for spheres is similar. The i-fold smash power of X is the spectrum

$$\left(\mathcal{G}_V[D^{ni}_+ \wedge {}^{\mathbf{O}_V \rtimes G}\diagup_{H}]\right)^{\wedge i} \cong \mathcal{G}_{V^{\oplus i}} \left[\left(D^{ni}_+ \wedge \left({}^{\mathbf{O}_{V^{\oplus i}} \rtimes G^{\times i}} \diagup_{\prod_i H} \right)_+ \right) \right].$$

Here the actions on $\mathbf{O}_{V^{\oplus i}} \rtimes G^{\times i}$ are again as in the proof of Proposition 2.3.28, with an additional action of Σ_i from the right on $\mathbf{O}_{V^{\oplus i}}$ and permuting factors of G. Hence it suffices to show that

$$\left[D^{ni}_+ \wedge \left(\mathbf{O}_V \rtimes G^{\times i} \Big/ \prod_i H \right)_+ \right]_{\Sigma_i}$$

is a $\mathbf{O}_{V^{\oplus i}} \rtimes G$-complex. Again this follows from C.2.5,C.2.2 and C.2.6, using that $\mathbf{O}_{V^{\oplus i}} \rtimes G^{\times i}$ is a manifold with an action of

$$\Gamma := (\mathbf{O}_{V^{\oplus i}} \rtimes G) \times (\prod_i H^{\mathrm{op}} \rtimes \Sigma_i),$$

and that D^{ni} is a Σ_i- and hence also a Γ-complex. □

From here on, the arguments from the non equivariant case apply verbatim. We omit listing all the intermediate statements in their equivariant formulations, and only give the final results:

Theorem 2.3.37. *The underlying positive $\mathbb{S}$-fibrations and π_*-isomorphisms give a compactly generated proper topological model structure on the category of commutative orthogonal G-ring spectra. The generating (acyclic) cofibrations are given by the sets $\mathbb{ES}^+ I$ and $\mathbb{ES}^+ J$, respectively.*
Again the identity functor gives a Quillen equivalence to the classical model structure from [MM, III.8.1].

As usual, we call cofibrant objects in this model structure simply $\mathbb{S}$-cofibrant, inspired by the following Theorem:

Theorem 2.3.38. *The $\mathbb{S}$-model structure on commutative orthogonal ring G-spectra is "convenient", i.e. if $\mathbb{E}$ is a commutative orthogonal G-ring spectrum that is $\mathbb{S}$-cofibrant, it is already (positive) $\mathbb{S}$-cofibrant as a G-equivariant orthogonal spectrum.*

As in the non equivariant case, even slightly more is true:

Theorem 2.3.39. *Let $f\colon R \to R'$ be a map of commutative orthogonal G-ring spectra, that is a cofibration in the model structure of Theorem 2.3.37. If the smash product with R preserves $\mathbb{S}$-cofibrations of equivariant orthogonal spectra, then f is an underlying $\mathbb{S}$-cofibration.*

Finally we can again change the underlying base ring:

Theorem 2.3.40. *For a commutative orthogonal G-ring spectrum R, the $\mathbb{S}$-model structure induces a compactly generated proper topological model structure on commutative R-algebras. This R-model structure is convenient with respect to the R-model structure on R-modules from Theorem 2.3.33(i).*
The identity functor on commutative R-algebras induces a Quillen equivalence to the classical model structure one would get by applying [DS, 3.10] to the structure from [MM, III.8.1].

Chapter 3

Smash Powers and the Loday Functor

3.1 Introduction

3.1.1 Hochschild Homology

Hochschild Homology, since it was defined by Hochschild for the case of algebras over fields in 1944 ([HH]), vas adapted to various more general contexts and has proven a valuable tool both in algebra, topology and geometry. We will be very brief in recalling some basics, not touching on most of the rich theory that follows. A very readable and thorough introduction can for example be found in Loday's book on Cyclic Homology [L], where in particular everything that follows here can be extracted from. We shall focus on the commutative setting, and will always assume that the coefficients are the commutative algebra itself. Let for the whole section R denote a commutative unital ground ring, and denote the tensor product over R simply by $\otimes$.

Definition 3.1.1. Let A be a commutative R-algebra. *Hochschild Homology of A* is the homology $HH(A)$ of the chain complex

$$\begin{tikzcd} \vdots \arrow[d] \\ A \otimes A \otimes A \otimes A \arrow[d] \\ A \otimes A \otimes A \arrow[d] \\ A \otimes A \arrow[d] \\ A \end{tikzcd}$$

with differential

$$b = \sum_{i=0}^{n} (-1)^i b_i,$$

where for $i < n$ the b_i are given by

$$b_i(a_0, \dots, a_n) := (a_0, \dots, a_i.a_{i+1}, \dots, a_n),$$

and the final b_n by

$$b_n(a_0, \dots, a_n) := (a_n.a_0, a_1 \dots, a_{n-1}).$$

Loday realized that this definition could be seen as a special case of a functorial construction on R-algebras (cf. [L89, 4.2]):

Definition 3.1.2. Let $\mathcal{F}$in be the category of finite sets and $R_{\mathcal{CA}\mathrm{lg}}$ the category of commutative R-algebras. Define the *algebraic Loday Functor*

$$\Lambda_{(-)}(-) \colon \mathcal{F}\mathrm{in} \times R_{\mathcal{CA}\mathrm{lg}} \to R_{\mathcal{CA}\mathrm{lg}},$$

on objects as the iterated tensor product

$$\Lambda_X(A) \quad := \quad \underbrace{A \otimes \ldots \otimes A}_{|X| - times}.$$

On morphisms $(f, \phi) : (X, A) \to (Y, B)$ by

$$(f, \phi)(a_x)_{x \in X} := \left(\prod_{f(x)=y} \phi(a_x) \right)_{y \in Y},$$

where the product over an empty indexing set is to be understood as the unit in R.

Remark 3.1.3. Note that we really use that A is a *commutative* R-algebra when defining the functor from $\mathcal{F}$in. If we would restrict to the category of finite sets and isomorphisms, the same formulas would give a functor with input mere R-modules. Extending to non surjective maps requires a unit map of some sort, and non injective but at least monotonous maps between ordered sets would only require an associative multiplication.

It is this formula that we emulate in the topological setting of commutative orthogonal ring spectra, where we construct a continuous analogue to the Loday functor. One application of this topological Loday functor, is a convenient definition of topological Hochschild homology in the same spirit as the following definition:
The algebraic Loday functor gives an explicit example of functors for which we can define Hochschild homology:

Definition 3.1.4. Let $F\colon \mathcal{F}\mathrm{in} \to R_{\mathcal{A}\mathrm{lg}}$ be a functor. Define its Hochschild homology $HH(F)$ as the homology of the simplicial algebra

$$\Delta^{\mathrm{op}} \xrightarrow{S^1} \mathcal{F}\mathrm{in} \xrightarrow{F} R_{\mathcal{A}\mathrm{lg}}.$$

Then immediately, inspection of the defining chain complex yields that Hochschild homology of commutative algebras is the same as Hochschild homology of the Loday functor:

$$HH(A) = HH(\Lambda_{(-)}A). \tag{3.1.5}$$

3.1.2 Tensors in Commutative Orthogonal Ring Spectra

Before we move on to a more thorough discussion of the topological Loday functor from the Introduction (Definition 1), we have to state some results about the tensor in the category of commutative orthogonal ring spectra (cf. A.2.24). The results and methods are the orthogonal spectrum analogues of those in [EKMM, VII.3].

We have already seen, that the category $\mathcal{S}p^{\mathcal{O}}$ is tensored and cotensored over $\mathcal{T}$ in Remark 1.2.2. That the analogue is true for the associated categories of modules, algebras and commutative algebras, i.e. for orthogonal ring spectra, commutative ring spectra as well as modules and algebras over such, is an entirely categorical argument:

Theorem 3.1.6. *[EKMM, VII 2.10],[MMSS, 5.1] Let $\mathcal{C}$ be a topologically complete and cocomplete category and let $\mathbb{M} : \mathcal{C} \to \mathcal{C}$ be a continuous monad that preserves reflexive coequalizers. Then the category $\mathcal{C}[\mathbb{M}]$ of $\mathbb{M}$-algebras is topologically bicomplete with limits and cotensors created in $\mathcal{C}$.*

This is used in two different settings. For categories of R-modules, for R an orthogonal ring spectrum, the defining monad is just the smash product with R, which as a left adjoint satisfies the prerequisite of the theorem. In particular since (enriched) left adjoints preserve (enriched) colimits, we get the slightly stronger

Corollary 3.1.7. *Let R be an orthogonal ring spectrum, then the category of R-modules is topologically bicomplete with limits, colimits, tensors and cotensors calculated in the category $\mathcal{S}p^{\mathcal{O}}$.*

The other situation we want to apply Theorem 3.1.6 to, is the case of algebras and, even more important, commutative algebras. One particular point where we have to be careful is that in order to make correct sense of the forgetful functor being topologically enriched, one should not use the category $\mathcal{T}$ for the enrichment. This stems from the fact, that the trivial map between underlying spectra of two ring spectra is almost never a map of ring spectra. Hence whenever we speak of a topological enrichment where ring spectra are involved, we will mean an enrichment over the category $\mathcal{U}$ of unbased spaces, and transport all the $\mathcal{T}$ enrichments via the forgetful lax monoidal $\mathcal{U}$-functor $\mathcal{T} \to \mathcal{U}$. (cf. A.2.15). Within this interpretation, Theorem 3.1.6 still holds, and the following result allows us to apply it:

Theorem 3.1.8. *[EKMM, II.7.2],[MMSS, 5.2] Let $\mathcal{C}$ be a cocomplete closed symmetric monoidal category. Then the monads that define monoids and commutative monoids in $\mathcal{C}$ preserve reflexive coequalizers.*

Corollary 3.1.9. *Let R be a commutative orthogonal ring spectrum (e.g. $R = \mathbb{S}$), then the category of (commutative) R-algebras is topologically bicomplete with limits and cotensors created in R-modules.*

We will not review the proofs of these, but instead focus on the the part of the theory that will help us compute tensors and cotensors. Obviously the case of cotensors is simply:

Corollary 3.1.10. *Let R be a commutative orthogonal ring spectrum (e.g. $R = \mathbb{S}$), then in categories of R-modules and (commutative) R-algebras limits and cotensors are created in orthogonal spectra.*

For the case of tensors, the result that will be most useful to us is the analogue to [EKMM, VII.3.4]. In the following let always R denote a commutative orthogonal ring spectrum as above, and let $\mathcal{C}$ denote either one of the categories of R-modules, R-algebras or commutative R-algebras.

Theorem 3.1.11. *Let A be an object of $\mathcal{C}$ and X_* a simplicial space. There is a natural isomorphism*

$$A \otimes_{\mathcal{C}} |X_*| \cong |A \otimes_{\mathcal{C}} X_*|$$

in $\mathcal{C}$. Here the realization on the right side is to be understood in the category of orthogonal spectra.

This is a consequence of the following two results, which are the orthogonal spectrum versions of [EKMM, VII.3.2, VII.3.3]:

Proposition 3.1.12. *Let A be an object of $\mathcal{C}$ and X_* a simplicial space. There is a natural isomorphism*

$$A \otimes_{\mathcal{C}} |X_*| \cong |A \otimes_{\mathcal{C}} X_*|_{\mathcal{C}},$$

where the realization on the right side is in $\mathcal{C}$.

Proof. As in [EKMM, VII.3.2], this is done by comparing the right adjoints of the realization functors (cf. B.1.38) and using the defining adjunctions of tensors and cotensors (cf. A.2.24). Let B be some object of $\mathcal{C}$, then we have natural isomorphisms

$$\begin{aligned}
\mathcal{C}(A \otimes_{\mathcal{C}} |X_*|, B) &\cong \mathcal{U}(|X_*|, \mathcal{C}(A, B)) \\
&\cong s\mathcal{U}(X_*, \Delta \pitchfork \mathcal{C}(A, B)) \\
&\cong s\mathcal{U}(X_*, \mathcal{C}(A, \Delta \pitchfork_{\mathcal{C}} B)) \\
&\cong s\mathcal{C}((A \otimes_{\mathcal{C}} X)_*, \Delta \pitchfork_{\mathcal{C}} B) \\
&\cong \mathcal{C}(|(A \otimes_{\mathcal{C}} X)_*|_{\mathcal{C}}, B),
\end{aligned}$$

so the Yoneda-lemma gives the result. □

Proposition 3.1.13. *Let A_* be a simplicial object of $\mathcal{C}$, then we have a natural isomorphism*

$$|A_*|_{\mathcal{C}} \cong |A_*|.$$

Proof. We should first describe how the realization $|A_*| = |A_*|_{\mathcal{S}p^{\mathcal{O}}}$ is an object of $\mathcal{C}$ again. We treat the case of commutative orthogonal ring spectra over $R = \mathbb{S}$, all the others are similar. For the unit morphism $\mathbb{S} \to |A_*|$ view $\mathbb{S}$ as $|\mathbb{S}_*|$, the realization of the constant simplicial spectrum, and use that since the simplicial structure maps of A_* are ring maps, the collection of unit maps of the A_q gives a map of simplicial ring spectra $\mathbb{S}_* \to A_*$ which induces a map on realizations. For the multiplication, first recall that the geometric realization is defined as a coend (B.1.37). Since coends and tensors in orthogonal spectra are defined levelwise, we have a natural isomorphism $|A_*|_V \cong |(A_*)_V|$ in each level V. Recall the coend definition of the smash product of orthogonal spectra and the fact that for simplicial spaces the realization commutes with both the smash product and the inducing up functor. The Fubini theorem for coends ([McL, IX.8]) then implies a natural isomorphism of spectra

$$|A_*| \wedge |A_*| \cong |A_* \wedge A_*|.$$

Here the latter smash product is calculated separately in each simplicial level. Hence the multiplication maps of the A_q induce multiplication maps on the realization. It is tedious, but not too hard to verify the associativity, unitality, commutativity and coherence conditions, and we omit more details here.
Going back to the proof, we continue as in case of (commutative) $\mathbb{S}$-algebras in the EKMM setting. For (commutative) orthogonal ring spectra A and B and a space X we claim that a morphism $A \otimes_{\mathcal{C}} X \to B$ of ring spectra determines and is determined by a morphism of spectra

$$A \otimes X = A \wedge X_+ \to B,$$

such that for all points $x \in X$ the map

$$A \cong A \wedge S^0 \xrightarrow{A \wedge i_x} A \wedge X_+ \longrightarrow B$$

is a map of ring spectra. To see this take a look at the defining adjunctions

$$\mathcal{C}(A \otimes_{\mathcal{C}} X, B) \cong \mathcal{U}(X, \mathcal{C}(A, B)) \longrightarrow \mathcal{U}(X, \mathcal{S}p^{\mathcal{O}}(A, B)) \cong \mathcal{S}p^{\mathcal{O}}(A \wedge X_+, B),$$

where the middle arrow is induced by the faithful (!) forgetful functor $\mathcal{C} \to \mathcal{S}p^{\mathcal{O}}$. Note that this is the point where the enrichment over $\mathcal{U}$ instead of $\mathcal{T}$ proves handy, since otherwise we would have to be very careful with trivial maps here.
Given a simplicial object A_* in $\mathcal{C}$ a morphism $\hat{g} \in \mathcal{C}(|A_*|, B)$ is completely determined by a morphism of spectra $\hat{f} \in \mathcal{S}p^{\mathcal{O}}(|A_*|, B)$, which by B.1.38 is adjoint to a morphism of simplicial spectra $f \in s\mathcal{S}p^{\mathcal{O}}(A_*, \Delta \pitchfork_{\mathcal{S}p^{\mathcal{O}}} B)$. For each simplicial level q, the morphism f_q is adjoint to a morphism of spectra $\hat{f}_q \colon A_q \otimes_{\mathcal{S}p^{\mathcal{O}}} \Delta^q \to B$ that is pointwise a morphism of ring spectra. As we saw above, these exactly correspond to algebra morphisms $\hat{h}_q \colon A_q \otimes_{\mathcal{C}} \Delta^q \to B$, whose adjoints fit together into a map of simplicial ring spectra $h \in s\mathcal{C}(A, \Delta \pitchfork_{\mathcal{C}} B)$. Altogether we defined a natural isomorphism

$$\mathcal{C}(|A_*|, B) \cong s\mathcal{C}(A_*, \Delta \pitchfork_{\mathcal{C}} B),$$

i.e. we showed that $|A_*|$ and $|A_*|_{\mathcal{C}}$ have the same right adjoint. This proves the proposition. □

The immense usefulness of Theorem 3.1.11 stems from the fact, that for discrete spaces X, tensors $A \otimes_{\mathcal{C}} X$ are easily computable in any topologically enriched category:

Proposition 3.1.14. *Let $\mathcal{D}$ be a category which is enriched and tensored over either $\mathcal{T}$ or $\mathcal{U}$. Let A be an object of $\mathcal{D}$ and X a discrete space in $\mathcal{U}$. Then there is a continuous natural isomorphism*

$$A \otimes X \cong \coprod_X A.$$

Proof. We use the defining universal properties of tensors and coproducts. Let B be some object of $\mathcal{D}$, we have natural isomorphisms

$$\begin{aligned} \mathcal{D}(A \otimes X, B) &\cong \mathcal{T}(X, \mathcal{D}(A, B)) \\ &\cong \mathcal{T}(\coprod_{x \in X} \{x, *\}, \mathcal{D}(A, B)) \\ &\cong \prod_X \mathcal{D}(A, B) \\ &\cong \mathcal{D}(\coprod_X A, B). \end{aligned}$$

Hence the topological Yoneda lemma gives the desired continuous natural isomorphism. In the case that the enrichment is only over $\mathcal{U}$, the adjunctions are completely analogous. □

In the case we are most interested in, using Lemma A.1.18, we get

Corollary 3.1.15. *Let A be a commutative orthogonal ring spectrum, X a discrete space, then there is a (continuous) natural isomorphism*

$$A \otimes X \cong \bigwedge_X A,$$

between the tensor of A with X and the X-fold smash power of A, i.e. the X-fold smash product of A with itself.

This is the main point of motivation for the translation from the algebraic case which we present in the next subsection.

3.2 The Loday Functor

Let R be a commutative orthogonal ring spectrum, i.e. a commutative $\mathbb{S}$-algebra in $\mathcal{S}p^{\mathcal{O}}$. Let A be a commutative R-algebra, and let X be a space. The aim of this section is to give a functorial model for the X-fold derived smash product over R of A with itself. For the case where both A is connective and R is the sphere spectrum itself, a similar construction has been carried out in the world of Γ-spaces in [BCD, Section 4]. The construction we give is much simpler than the one presented in op. cit., so it will be crucial to study its properties in detail, so as to make sure, that we indeed capture all the desired homotopy theoretic information. In particular the equivariant homotopy type, when X is equipped with an action of some (compact Lie-) group G, will require some care.

Let throughout the whole section, R denote a commutative orthogonal ring spectrum, for example the sphere spectrum $R = \mathbb{S}$. Denote by $R_{\mathcal{C}\mathcal{A}\mathrm{lg}}$ the category of commutative R-algebras. The fundamental definition of this chapter is the analogue of Definition 1 in the Introduction:

Definition 3.2.1. The *Loday functor* $\Lambda^R_{(-)}(-)\colon \mathcal{U} \times R_{\mathcal{C}\mathcal{A}\mathrm{lg}} \to R_{\mathcal{C}\mathcal{A}\mathrm{lg}}$ is the continuous bifunctor

$$\Lambda^R_X(A) := A \otimes_{R_{\mathcal{C}\mathcal{A}\mathrm{lg}}} X.$$

The functoriality and continuity in both variables is inherent in the definition of the categorical tensor (cf. A.2.24). We will generally leave out the superscript and just write $\Lambda_X(A)$ if $R = \mathbb{S}$. As in the algebraic case we can extend the realm of definition in the second variable, by restricting the first (cf. Remark 3.1.3). We will describe these extensions in more detail, as they will help us with actually evaluating the Loday functor. Let $R_{\mathcal{M}\mathrm{od}}$ be the category of modules over R and $\mathcal{F}\mathrm{in}_I$ the category of finite sets and isomorphisms between them.

Definition 3.2.2. Define the Loday functor $\mathcal{F}\mathrm{in}_I \times R_{\mathcal{M}\mathrm{od}} \to R_{\mathcal{M}\mathrm{od}}$ on objects via

$$\Lambda^R_X(A) \;\; := \;\; \underbrace{A\wedge_R \ldots \wedge_R A}_{|X|\text{-fold}}.$$

Functoriality in A is then immediate from the functoriality of $\wedge_R$, whereas functoriality in X makes use of the shuffle permutation induced by the symmetry isomorphism for $\wedge_R$.

This can be extended further: Let $\mathcal{S}\mathrm{et}_{\mathrm{in}}$ be the category of Sets and injections and let $R_{u\mathcal{M}\mathrm{od}}$ be the category of unital R-modules, or R-modules under R, i.e. modules M with a distinguished map of R-modules $\iota_M \colon R \to M$.

Definition 3.2.3. Define the Loday functor $\mathcal{S}\mathrm{et}_{\mathrm{in}} \times R_{u\mathcal{M}\mathrm{od}} \to R_{u\mathcal{M}\mathrm{od}}$ on objects via

$$\Lambda_X^R(A) := \operatorname*{colim}_{F\subset X, F\in\mathcal{F}\mathrm{in}} \Lambda_F^R(A),$$

and on inclusions $i\colon I \to J$ of finite sets as follows:

$$\Lambda_I^R(A) = \bigwedge_I A \cong \bigwedge_I A \wedge_R \bigwedge_{J\setminus I} R \xrightarrow{\mathrm{id}\wedge \bigwedge_{J\setminus I}\iota_A} \bigwedge_J A = \Lambda_J^R(A)$$

To allow for non injective maps, we have to move to the category of (unital) associative R-algebras $R_{\mathcal{A}\mathrm{lg}}$. Let also $\mathcal{S}\mathrm{et}_{\mathrm{ord}}$ be the category of ordered sets.

Definition 3.2.4. Define the Loday functor $\mathcal{S}\mathrm{et}_{\mathrm{ord}} \times R_{\mathcal{A}\mathrm{lg}} \to R_{\mathcal{A}\mathrm{lg}}$ on objects as in 3.2.2. The functoriality along non injective order preserving maps $f : X \to Y$ then uses the multiplication map of the algebra on smash factors whose index is mapped to the same element:

$$f_* := \quad \begin{tikzcd} \Lambda_X^R A \arrow[r] \arrow[d, "\cong"'] & \Lambda_Y^R A, \\ \bigwedge_{y\in Y} \Lambda_{f^{-1}(y)}^R A \arrow[ur, "\bigwedge_{y\in Y}\mu_A"'] & \end{tikzcd}$$

where the multiplication out of an empty smash product is to be understood as the unit map, and the smash factors have to be kept in the order induced from X, respectively Y.

In the light of Proposition 3.1.14, we see that by embedding the various categories of sets above as discrete spaces in $\mathcal{U}$ and by forgetting parts of the structures of commutative R-algebras, all of the definitions above agree, whenever several of them make sense. For the case of infinite discrete X we need to assume that the unit map of A is an h-cofibration, so that the colimit in Definition 3.2.2 is the same when calculated in the underlying category of spectra. In particular the final version, analogous to Definition 3.1.2 in the algebraic case is the following:

Definition 3.2.5. The discrete Loday functor $\mathcal{S}\mathrm{et} \times R_{\mathcal{C}\mathcal{A}\mathrm{lg}} \to R_{\mathcal{C}\mathcal{A}\mathrm{lg}}$ is defined on objects via

$$\Lambda_X^R(A) := \bigwedge_X A = \coprod_X A,$$

where the coproduct is in $R_{\mathcal{C}\mathcal{A}\mathrm{lg}}$. On morphisms $f : X \to Y$ of sets, define $\Lambda_f^R(A)$ via

$$f_* := \quad \begin{tikzcd} \Lambda_X^R A \arrow[r] \arrow[d, "\cong"'] & \Lambda_Y^R A, \\ \bigwedge_{y\in Y} \Lambda_{f^{-1}(y)}^R A \arrow[ur, "\bigwedge_{y\in Y}\mu_A"'] & \end{tikzcd}$$

where the first map in the composite reshuffles the smash factors using the symmetry isomorphism. The choice of shuffle then does not influence the multiplication in the second map, since A was strictly commutative.

Finally note that Proposition 3.1.11 implies the following lemma:

Lemma 3.2.6. *For X the realization $|Y_*|$ of a simplicial space Y_* and A a commutative R-algebra, there is a natural isomorphism*

$$\Lambda_X^R(A) \cong |\Lambda_{Y_*}^R(A)|,$$

where the realization on the right is in orthogonal spectra.

Remark 3.2.7. For a morphism $\varphi : R \to S$ of commutative orthogonal ring spectra, we get adjoint pairs of functors between the categories of R-, respectively S-modules, -algebras and commutative S-algebras. All the left adjoints are given by *induction*, i.e. using $S \wedge_R (-)$, which is strong monoidal and preserves tensors, hence commutes with all versions of the Loday functor from above. The respective right adjoint functors do in general not exhibit similar properties.

Note that since all group actions are through isomorphisms, a (continuous) action of a (topological) group G on X induces (continuous) actions on the targets of the Loday functor by precomposition as follows

$$G \longrightarrow \mathcal{A}(X, X) \xrightarrow{\Lambda_-^R A} \mathcal{C}(\Lambda_X^R A, \Lambda_X^R A) \, ,$$

where ($\mathcal{A}$, $\mathcal{C}$) is any of the pairs of categories for which we defined the Loday functor above and $A \in \mathcal{C}$. In this light, we can for each of the above definitions and any (topological) group G consider equivariant analogues

$$\Lambda_{(-)}(-) \colon [G, \mathcal{A}] \times \mathcal{C} \to [G, \mathcal{C}].$$

As already discussed in the Introduction, it is crucial to investigate the equivariant properties of these Loday functors, to make sure that they are usable for our applications. The next sections will be devoted to this topic.

3.3 Fixed Points of Cells

We are interested in equivariant structures that actions of a compact Lie group G on the space X that we use as input for the Loday functor induce on the output. As usual in equivariant homotopy theory, this implies studying the associated fixed point spectra. In our case, there is another version of fixed points which is much more amenable to computation. We recall the definitions and some of the relations between the two versions from [MM, V.4]. Since we want to work with the $\mathbb{S}$-model structure (1.3.28) for commutative orthogonal ring spectra we will have to extend the classical results to a bigger class of (cofibrant) spectra throughout the following sections.

3.3.1 Fixed Point Spectra

For a closed normal subgroup N of G, consider the short exact sequence E of compact Lie-groups:

$$E: \quad 1 \to N \to G \xrightarrow{\epsilon} J \to 1,$$

where ϵ denotes the projection on N-orbits.
Similar to the case of equivariant spaces C.1, where N-fixed points of G-spaces inherit J-actions, we want to consider fixed point functors $G\mathcal{S}p^{\mathcal{O}} \to J\mathcal{S}p^{\mathcal{O}}$, or rather $G\mathcal{O}\mathcal{T} \to J\mathcal{O}\mathcal{T}$.

Remark 3.3.1. Note that for subgroups H that are not necessarily normal, we can first restrict the G-actions to the normalizer N_H of H in G before taking fixed points, to get a functor

$$G\mathcal{S}p^{\mathcal{O}} \to N_H\mathcal{S}p^{\mathcal{O}} \to W_H\mathcal{S}p^{\mathcal{O}},$$

to spectra with actions of the Weyl group $W_H := {}^{N_H}\!/_{H}$.

We give two different notions of N-fixed point spectra in $J\mathcal{O}\mathcal{T}$ for a given G-spectrum, both constructions make use of an intermediate category:

Definition 3.3.2. In the above situation denote by $\mathcal{O}_E$ the J-category with objects the same as $\mathcal{O}_G$, but where the morphisms are given by

$$\mathcal{O}_E(V, W) := \mathcal{O}_G(V, W)^N,$$

i.e. the N-fixed points of the morphism space in $\mathcal{O}_G$, which are exactly the N-equivariant morphisms.
Define two *categories of E-spectra*: The $J\mathcal{T}$-category of $J\mathcal{T}$-functors $[\mathcal{O}_E, \mathcal{T}_J]$ denoted as $\mathcal{O}_E\mathcal{T}$, and its J-fixed $\mathcal{T}$-category $E\mathcal{O}\mathcal{T}$ i.e. continuous J-functors $\mathcal{O}_E \to \mathcal{T}_J$ and (J-) natural transformations (cf. A.2.8).

As for all our previous examples of such enriched functor categories, we can consider appropriate evaluation functors and their left adjoints, the free and semi-free E-spectra given by enriched left Kan extensions (cf. 2.2.14). One can define a closed symmetric monoidal structure on $\mathcal{O}_E\mathcal{T}$, but the approach analogous to 1.2.4 is not particularly enlightening in this case. Thus we will from now on use the definition of the smash product in terms of the (enriched) left Kan extension as in Remark 2.2.9.

Remark 3.3.3. *Careful:* The category $\mathcal{O}_E\mathcal{T}$ depends on the choice of G-universe! An E-spectrum $X \in \mathcal{O}_E\mathcal{T}$ is *not* completely determined by its evaluations at trivial representations if N is not trivial. In particular a formula computing the evaluation at V from the evaluation at V' analogous to (2.1.2) requires an N-equivariant isometry between V and V'. We will unless otherwise stated use a complete G-universe, i.e consider $\mathcal{O}_E$ as the N-fixed category of $\mathcal{O}_G$. For some of the more technical properties of the fixed point functors, however, we need to be able to change this point of view (cf. 3.3.52). We will then indicate the universe via the superscript $\mathcal{O}_E^{\mathcal{U}}$.

The first step in the definition of the fixed point functors is the following:

Definition 3.3.4. The *levelwise N-fixed point functor*

$$\mathrm{Fix}^N \colon G\mathcal{O}\mathcal{T} \to E\mathcal{O}\mathcal{T}$$

assigns to a G-spectrum X the E-spectrum $\mathrm{Fix}^N X$, where $[\mathrm{Fix}^N X]_V$ is given by X_V^N for every finite dimensional G-representation V. This indeed gives an E-spectrum, since for V and W G-representations, the structure maps

$$X_V^N \wedge \mathcal{O}_E(V, W) \to X_W^N$$

are restrictions of the structure maps for X to N-fixed points.

Remark 3.3.5. If one wants to view this functor as enriched, i.e. between $\mathcal{O}_G\mathcal{T}$ and $\mathcal{O}_E\mathcal{T}$ one needs to make them enriched over the same category first. In particular one could take the N-fixed category of $\mathcal{O}_G\mathcal{T}$, from where the above assignment immediately gives a $J\mathcal{T}$-functor. Since all model category theoretic considerations happen in the G-fixed categories, we will not make use of this point of view however, so Definition 3.3.4 is good enough for our purposes.

Remark 3.3.6. The functor Fix^N is lax symmetric monoidal. Indeed, by the universal properties of the left Kan-extension that defines the smash products (cf. 2.2.9) there is a natural isomorphism

$$\mathcal{O}_G\mathcal{T}(X \wedge Y, X \wedge Y) \;\;\cong\;\; [\mathcal{O}_G \times \mathcal{O}_G, \mathfrak{T}_G](X \bar{\wedge} Y, (X \wedge Y) \circ \oplus).$$

Which in particular says that for any level $V \oplus W$, the identity map of $X \wedge Y$ determines morphisms

$$X_V^N \wedge Y_W^N \quad \to \quad (X \wedge Y)_{V \oplus W}^N$$

on N-fixed points. The universal property of the smash product in $\mathcal{O}_E\mathcal{T}$ assembles these into a morphism

$$\mathrm{Fix}^N X \wedge \mathrm{Fix}^N Y \to \mathrm{Fix}^N(X \wedge Y).$$

Note that the unit object for the smash product in $\mathcal{O}_E\mathcal{T}$ is $\mathcal{F}_0^E S^0 \cong \mathrm{Fix}^N \mathbb{S}$.

To define the fixed point functors, we need to close the gap between $E\mathcal{S}p^{\mathcal{O}}$ and $J\mathcal{S}p^{\mathcal{O}}$. Since these are both defined as categories of J-functors to $\mathcal{T}_J$, functors between their source categories will do the trick. In particular consider the following:

Definition 3.3.7. Let $\nu\colon \mathcal{O}_J \to \mathcal{O}_E$ be given on objects by

$$\begin{array}{rrcl} \nu\colon & \mathcal{O}_J & \to & \mathcal{O}_E \\ & W & \mapsto & \epsilon^* W \end{array}$$

and on morphisms by

$$\begin{array}{rcl} \mathcal{O}_J(V, V') & \to & \mathcal{O}_E(\epsilon^* V, \epsilon^* V'), \\ (f, s) & \mapsto & (\epsilon^* f, \epsilon^* s) = (f, s) \end{array}$$

where $f\colon V \to V'$ is an isometric embedding and s an element in the orthogonal complement of $f(V)$.

Definition 3.3.8. Let $\phi\colon \mathcal{O}_E \to \mathcal{O}_J$ be given on objects by

$$\begin{array}{rrcl} \phi & : \mathcal{O}_E & \to & \mathcal{O}_J \\ & V & \mapsto & V^N \end{array}$$

and on morphisms by

$$\begin{array}{rcl} \mathcal{O}_E(W, W') & \to & \mathcal{O}_J(W^N, W'^N). \\ (g, t) & \mapsto & (g^N, t) \end{array}$$

Since $\mathcal{O}_E(W, W') = \mathcal{O}_G(W, W')^N$, the isometry g as above indeed maps the N-fixed points of W into those of W'. Similarly t is indeed in W'^N.

Both of these functors give "pullback" functors on $\mathcal{O}_E$- or $\mathcal{O}_J$-spectra by precomposition:

$$\begin{array}{rcl} \mathbb{U}_\nu \colon \mathcal{O}_E\mathcal{T} & \to & \mathcal{O}_J\mathcal{T} \\ Y & \mapsto & Y \circ \nu \\ & & [\mathbb{U}_\nu Y]_W = Y_{\epsilon^* W} \end{array}$$

$$\begin{array}{rcl} \mathbb{U}_\phi \colon \mathcal{O}_J\mathcal{T} & \to & \mathcal{O}_E\mathcal{T} \\ X & \mapsto & X \circ \phi \\ & & [\mathbb{U}_\phi X]_V = X_{V^N} \end{array}$$

Remark 3.3.9. Note that $\phi \circ \nu = \mathrm{id}$, and hence $\mathbb{U}_\nu \circ \mathbb{U}_\phi = \mathrm{id}$.

Since $\mathcal{T}_J$ is enriched and bicomplete over itself, enriched left Kan extensions exist and provide left adjoints to both of these functors (cf. A.2.34). We denote them by $\mathbb{P}_\nu$ and $\mathbb{P}_\phi$, respectively. Similarly there exist right adjoints to $\mathbb{U}_\nu$ and $\mathbb{U}_\phi$, but we will not make use of these.

We can finally define the fixed point functors:

Definition 3.3.10. Define the *categorical N-fixed point functor*

$$(-)^N \colon G\mathcal{S}p^{\mathcal{O}} \to J\mathcal{S}p^{\mathcal{O}}$$

as the composition $(-)^N := \mathbb{U}_\nu \circ \mathrm{Fix}^N$, and define the *geometric N-fixed point functor*

$$\Phi^N \colon G\mathcal{S}p^{\mathcal{O}} \to J\mathcal{S}p^{\mathcal{O}}$$

as the composition $\Phi^N := \mathbb{P}_\phi \circ \mathrm{Fix}^N$.

Remark 3.3.11. The name *categorical* is validated by the fact that the categorical fixed point functor is indeed the same as taking the categorical limit of the functor $N \to G \to \mathcal{S}p^{\mathcal{O}}$ that defines the N-action on an orthogonal G-spectrum (cf. 2.1.1), as is easily checked on trivial representations. In particular, the categorical fixed point functor does not depend on the choice of universe. Note that since the categorical fixed point functor is lax monoidal, we can view the category $G\mathcal{O}\mathcal{T}$ as enriched over $\mathcal{O}\mathcal{T}$, with the G-fixed point spectrum of the internal function spectrum as morphism object. From this viewpoint the fixed-point functor is then also the *enriched* right adjoint of the functor ϵ^*, that equips a J-spectrum with the trivial G-action. Since we rarely work with the spectral enrichment this viewpoint is not of particular importance for us, except for the helpful Lemma 3.3.45

Remark 3.3.12. Note that this definition of the geometric fixed point $\Phi^N X$ for a G-spectrum X is the one introduced by Mandell and May in [MM, V.4.3]. However, this does not generally agree with the classical definition in terms of the categorical fixed points $(X\wedge\tilde{E}\mathscr{F})^N$, see e.g. [LMS, II.9]. Up to weak equivalence the two concepts agree for sufficiently cofibrant G-spectra X. We will go into more detail in subsection 3.5, where we in particular losen the cofibrancy condition far enough to apply to equivariant spectra in the image of the Loday functor.

Remark 3.3.9 gives a comparison map between the two fixed point functors:

Definition 3.3.13. Let $\gamma\colon X^N \to \Phi^N X$ be the natural map of J-spectra

$$\begin{tikzcd} \mathbb{U}_\nu \mathrm{Fix}^N X \arrow[r, "\mathbb{U}_\nu(\eta)"] \arrow[rrr, bend left, "\gamma"] & \mathbb{U}_\nu \mathbb{U}_\phi \mathbb{P}_\phi \mathrm{Fix}^N X \arrow[r, "="] & \mathbb{P}_\phi \mathrm{Fix}^N X \arrow[r, "="] & \Phi^N X, \end{tikzcd}$$

induced by the adjunction unit $\eta\colon \mathrm{id} \to \mathbb{U}_\phi\mathbb{P}_\phi$.

The difference between $(-)^N$ and Φ^N is usually significant. We will recall more of the technicalities from [MM, V.4] in 3.5, before we give an exact statement in Proposition 3.5.15, after we loosen the cofibrancy conditions from the classical case in 3.5.11.

Remark 3.3.14. Note that this emphasizes the dependence of $\mathcal{EOT}$ and Φ^N on the choice of universe, since in particular for a G-universe $\mathcal{U}$ containing only trivial N-representations, it is easily seen that the functor ϕ is an equivalence and hence the functor $\Phi^N_{\mathcal{U}}$ is naturally isomorphic to $(-)^N$ via the natural map γ from above.

Before going into more detail though, we will list properties of the fixed point functors and study their interaction with free spectra, which forms the basis for computations on fixed points of smash powers.

Proposition 3.3.15. *[MM, III.1.6] The geometric fixed point functor Φ^N preserves coproducts, pushouts along h-cofibrations, sequential colimits along h-cofibra-tions and tensors with spaces.*

Proof. The functor Fix^N preserves these by C.1.2. Since $\mathbb{P}_\phi$ is a continuous left adjoint, the result follows. □

Proposition 3.3.16. *[MM, IV.4.5] For any finite dimensional G-representation V and any G-space K, there is a natural isomorphism of J-spectra*

$$\mathcal{F}^J_{V^N} K^N \cong \Phi^N \mathcal{F}^G_V K.$$

Proof. Recall that $\mathcal{F}^G_V K$ is given in level W by $\mathcal{O}_G(V,W)\wedge K$ where G acts diagonally on the smash product, by conjugation on the morphism space and on K via its G-space structure. Thus $(\mathcal{F}^G_V K)^N_W = \mathcal{O}_G(V,W)^N \wedge K^N = \mathcal{O}_E(V,W)\wedge K^N$, and hence

$$\mathrm{Fix}^N \mathcal{F}^G_V K = \mathcal{F}^E_V K^N. \tag{3.3.17}$$

The next step is showing that $\mathbb{P}_\phi \mathcal{F}^E_V L$ is isomorphic to $\mathcal{F}^J_{V^N} L$, so for any J-spectrum Z take a look at the morphism space:

$$\begin{array}{rcll} J\mathcal{OT}(\mathbb{P}_\phi \mathcal{F}^E_V L, Z) & \cong & E\mathcal{OT}(\mathcal{F}^E_V L, \mathbb{U}_\phi Z) & \\ & \cong & \mathcal{T}_J(L, (\mathbb{U}_\phi Z)_V) & \\ & = & \mathcal{T}_J(L, Z_{V^N}) & \cong \quad J\mathcal{OT}(\mathcal{F}^J_{V^N} L, Z) \end{array}$$

Hence the Yoneda lemma gives the desired natural isomorphism. Summing up, the functors $\mathcal{F}^E_V$ and $\mathcal{F}^J_{V^N}$ have the same right adjoint to $\mathcal{T}_J$, and are therefore naturally isomorphic. □

Corollary 3.3.18. *cf. [MM, IV.4.5] The geometric fixed point functor preserves q-cofibrations and acyclic q-cofibrations.*

Proposition 3.3.19. *[MM, IV.4.7] The functor Φ^N is lax monoidal and for q-cofibrant orthogonal G-spectra X and Y the natural (equivariant) morphism of J-spectra*

$$\alpha\colon \Phi^N X \wedge \Phi^N Y \to \Phi^N(X \wedge Y),$$

is an isomorphism.

Both the proof of the acyclic part of 3.3.18 and the second half of 3.3.19 rely on a good understanding of a cellular filtration of the □-product of cellular maps, respectively the smash product of cellular objects. It seems hard to actually find these explicitly spelled out in the literature, but since we are going to have to work with such filtrations in more detail later, we will give some more details on the proofs at the end of Subsection 3.4.1. First, we will investigate the geometric fixed points of the spectra appearing as smash powers of generating cofibrations, and some more specialised cells.

Of course, motivated by the discrete Loday functor from 3.2.5, the examples of G-spectra where we are most interested in calculating geometric fixed points, are smash powers of orthogonal spectra. We begin with studying the fixed points of free and semi-free spectra before we move to more general spectra 3.3.5, and general cofibrant spectra in Section 3.4.

3.3.2 Free Cells

Let us begin by investigating what happens in the case of free spectra. For X a finite discrete set with an action of a discrete group G, the discrete Loday functor Λ_X sends an orthogonal spectrum A to its X-fold smash power $A^{\wedge X}$. As in Remark 1.2.39, the action of G on X induces an action on $\Lambda_X(A)$ by permuting the smash factors using the symmetry isomorphism for the smash product in $\mathcal{OT}$.

Lemma 3.3.20. *For X a finite discrete G-set, $K \in \mathcal{T}$ a space and $A = \mathcal{F}_V K$ a free orthogonal spectrum, there is a natural isomorphism of G-spectra*

$$\Lambda_X(A) = (\mathcal{F}_V K)^{\wedge X} \cong \mathcal{F}^G_{V^{\oplus X}} K^{\wedge X},$$

where G acts on the vector space $V^{\oplus X}$ by permuting summands and on the spectrum $(\mathcal{F}_V K)^{\wedge X}$ and the space $K^{\wedge X}$ by permuting smash factors.

Proof. Recall the symmetry isomorphism for orthogonal spectra induced by 1.2.4. Evaluated at a trivial representation $\mathbb{R}^n$, (1.2.34) gives the isomorphism

$$(\mathcal{F}_V K)^{\wedge X}_{\mathbb{R}^n} \cong \mathcal{O}(V^{\oplus X}, \mathbb{R}^n) \wedge K^{\wedge X},$$

where the permutation of smash factors translates to permuting smash factors of $K^{\wedge X}$ and precomposing with the appropriate block-permutation matrices on $\mathcal{O}(V^{\oplus X}, \mathbb{R}^n)$. Thus the description of $\mathcal{F}^G_{V^{\oplus X}}$ from 2.2.12 gives the desired result. □

In combination with 3.3.16 we get:

Proposition 3.3.21. *For X a finite discrete G-set, N a normal subgroup of G, $K \in \mathcal{T}$ a space and $A = \mathcal{F}_V K$ a free orthogonal spectrum, there is a natural isomorphism of J-spectra*

$$\Phi^N \Lambda_X(A) \cong \mathcal{F}^J_{V^{\oplus X_N}} K^{\wedge X_N} \cong \Lambda_{X_N}(A),$$

where X_N is the orbit J-space of X factoring out the N-action.

Proof. The only thing left to do is that the N-fixed point spaces of $V^{\oplus X}$ and $K^{\wedge X}$ are indeed J-isomorphic to $V^{\oplus X_N}$ respectively $K^{\wedge X_N}$. Specifying the actions on elements makes this immediate. □

Remark 3.3.22. This proposition serves as a good starting point for calculating geometric fixed points of smash powers of general q-cofibrant orthogonal spectra. This has for example been studied in the special case of G a cyclic group by Kro in [Kr, 3.10.1] and Hill, Hopkins and Ravenel in [HRR, B.96]. However, since we are ultimately interested in evaluating the Loday functor on commutative ring spectra, we would rather not have to apply q-cofibrant approximation. In particular since the classical stable model structure does not satisfy the convenience property of 1.3.29, a q-cofibrant replacement of an orthogonal ring spectrum is in general only E_∞ instead of strictly commutative. To remedy this, we first study smash powers of semi-free spectra and later generalize to $\mathbb{S}$-cofibrant spectra.

3.3.3 Semi-Free Cells

Orbits and Fixed Points for semi-direct Products

Throughout our work, actions of groups that are semi-direct products appear in various places. In particular every level A_V of an orthogonal G-spectrum A is equipped with an action of the semi-direct product $\mathbf{O}_V \rtimes G$. We will recall a few elementary properties, before investigating the more complicated interactions of the orbit and fixed point functors that play a role in computing the fixed points of smash powers in Proposition 3.3.37. To begin with we restate the definition, in order to fix some notation:

Definition 3.3.23. Let $(G, \cdot, e) \in \mathcal{T}$ be a topological group acting on another group $(O, ., E) \in \mathcal{T}$ through automorphisms, i.e. via a group homomorphism $\phi\colon G \to \mathrm{Aut}(O)$. For $g \in G$ denote $\phi_g := \phi(g)$. Then the *semi-direct product* $O \rtimes G$ is the product space $O \times G$ equipped with the multiplication defined by

$$\begin{array}{rcl} (O \rtimes G) \times (O \rtimes G) & \to & (O \rtimes G) \\ (A, g), (B, h) & \mapsto & (A.\phi_g(B), g \cdot h) \end{array}$$

It is usually far to cumbersome to explicitly write the signs "·" and "." for the of G and O, so we often omit them. The following properties are elementary:

Lemma 3.3.24. *(i) Mapping $A \in O$ to $(A, e) \in O \rtimes G$ embeds O as a (closed) normal subgroup.*

(ii) Mapping $g \in G$ to $(E, g) \in O \rtimes G$ embeds G as a closed subgroup.

(iii) For $A \in O$ and $g \in G$, the following elements of the semi-direct product are equal:

$$(\phi_g(A), e) = (E, g)(A, e)(E, g)^{-1}$$

(iv) G is normal in $O \rtimes G$ if and only if ϕ is trivial, i.e. the semi-direct product is actually the direct product.

(v) The projection $\mathrm{pr}_2\colon O \rtimes G \to G$ to the second factor is a group homomorphism.

Motivated by the first three points in the lemma, we will often drop ϕ from the notation as well, and instead use $gAg^{-1} = \phi_g(A)$.

Lemma 3.3.25. *Let $Z \in \mathcal{T}$ be a space with an action of the semi-direct product $O \rtimes G$, and denote for $z \in Z$ its stabilizer by $\mathrm{Stab}_z \subset O \rtimes G$. Then the action of O on Z is free (away from the basepoint), if and only if for any point $z \in Z$ (except the basepoint), the projection to the second factor $\mathrm{pr}_2\colon \mathrm{Stab}_z \to G$ is injective.*

Proof. Note that, for $z \in Z$, its stabilizer under the restricted action is

$$\mathrm{Stab}_z^O = \mathrm{Stab}_z \cap (O \rtimes \{e\}).$$

Recall that an action is free (away from the basepoint) if and only if the stabilizers for all elements (except the basepoint) are trivial. □

For any space Z with an action of the semi-direct product, its orbit space Z_O inherits an action of G since O was normal in $O \rtimes G$. We want to investigate in how far taking such O-orbits commutes with taking fixed points with respect to subgroups of G.

Proposition 3.3.26. *Let G be a group acting on another group O by group homomorphisms. Let Z be a space with compatible actions of both O and G, i.e. Z a $O \rtimes G$-space. Further assume that O acts freely on Z (away from the basepoint). Then the canonical map from the quotient of the fixed points into the fixed points of the quotient*

$$Z^G / O^G \to \left[Z / O \right]^G \tag{3.3.27}$$

is injective.

Proof. Assume $[z_1]$ and $[z_2]$ in Z^G / O^G map to the same element in the target, i.e. $z_1 = Az_2$, for some A in O. Then for any g in G we have

$$Az_2 = z_1 = gz_1 = gAz_2 = (gAg^{-1})gz_2 = (gAg^{-1})z_2.$$

But since the O action on Z was free, this implies that $A = gAg^{-1}$ for all $g \in G$, i.e. $A \in O^G$ and therefore $[z_1] = [z_2]$. □

Surjectivity can not be guaranteed in this generality, as the following example illustrates:

Example 3.3.28. Let $Z = S(\mathbb{C})_+$ the unit circle in $\mathbb{C}$, with some disjoint basepoint. Let $O = \mathbb{Z}_4$ and $G = Z_2$ such that the action of the non trivial element in G maps an element of O to its inverse. In particular $O \rtimes G$ is the Dihedral group D_4. Then O acts freely on $S(\mathbb{C})$ through rotations by 90 degrees, G acts by complex conjugation, and one checks that the actions compatibly fit together into an action of D_4. Take a closer look at source and target of the map (3.3.27):
Note that O^G is the subgroup of self inverse elements $\mathbb{Z}_2 \subset \mathbb{Z}_4$, i.e. generated by the rotation by 180 degrees. The G-fixed point space $(S(\mathbb{C}))^G$ has 2 points which are in the same O^G-orbit, i.e. the source of (3.3.27) contains only one point. On the other hand, taking orbits first, we see that Z / O is isomorphic to $S(\mathbb{C})$, with the action of G again given by complex conjugation, i.e. target of (3.3.27) consists of 2 points, such that the map can not be surjective.

The intuition behind the failure in surjectivity is that there are "diagonal" copies of G in $O \rtimes G$, and points with isotropy type of such a diagonal copy contribute to the target of (3.3.27) but not to the source. Motivated by this, we will give a formal sufficient condition for the surjectivity of (3.3.27). First, consider the simple case of only one $O \rtimes G$-orbit, and take a closer look at the target space:

Lemma 3.3.29. *Let in the setup of Proposition 3.3.26 the space Z consist of single O-free $O \rtimes G$-orbit, i.e. $Z = {}^{O \rtimes G}\!/_{P}$ for some (closed) subgroup P of $O \rtimes G$. Then $({}^{Z}\!/_{O})^G$ contains at most one element, and is non empty if and only if the projection $\mathrm{pr}_2 : P \to G$ to the second factor is an isomorphism.*

Proof. The projection is injective by 3.3.25. Hence ${}^{Z}\!/_{O} \cong {}^{G}\!/_{\mathrm{pr}_2 P}$ as G-spaces, and the latter has exactly one G-fixed point if and only if $\mathrm{pr}_2(P) = G$, and is empty otherwise. □

On the other hand, the following elementary fact gives a characterization of the source:

Lemma 3.3.30. *Let $H \in \mathcal{T}$ be a group with subgroups P and G. Then the space $\left({}^{H}\!/_{P}\right)^G$ is non empty, if and only if G is subconjugate to G.*
In particular the fixed point space is a quotient of the subspace of all those elements $h \in H$, that conjugate G into P.

Proof. Let hP be a point in the orbit space. Then hP is G-fixed, if and only if for all $g \in G$ we have $ghP = hP$, or equivalently $h^{-1}gh \in P$. □

Proposition 3.3.31. *If Z is a genuinely cofibrant $O \rtimes G$-space, such that for all orbit types P appearing in the cell-decomposition of Z, the projection to the second factor $\mathrm{pr}_2(P) \cong G$ is an isomorphism, if and only if P is subconjugate to $\{E\} \rtimes G \subset O \rtimes G$, then the map* (3.3.27) *is an isomorphism.*

Proof. Note that both taking fixed points and taking orbits preserves the cell-complex construction by C.1.2. Hence the natural map (3.3.27) induces a natural isomorphism of cell diagrams, hence an isomorphism on the transfinite composition. □

Corollary 3.3.32. *If $O \rtimes G$ has the property, that subgroups P are isomorphic to G along the projection to the second factor if and only if they are subconjugate to G, then for any genuinely cofibrant $O \rtimes G$-space the natural map* (3.3.27) *is an isomorphism.*

Example 3.3.33. Note that 3.3.28 gives a non-example of this. In particular, the semi-direct product D_4 has a subgroup P of order 2 generated by $(2,1) \in Z_4 \rtimes \mathbb{Z}_2$ which acts on $S(\mathbb{C})$ via the reflection with respect to the imaginary axis. Note that since this element is in the center of D_4 the subgroup it generates can not be conjugate to G. In particular, since $S(\mathbb{C})$ has two P-fixed points, an equivariant cell decomposition will have to use a cell of type ${O \rtimes G}/{P_+} \wedge S_0$.

The main example we want to apply Corollary 3.3.32 to is the following:

Example 3.3.34. Let G be a discrete group, X a discrete G-space and let V be a finite dimensional G-representation. Let in the above notation $O = \prod_X \mathbf{O}_V$ with G acting by permuting the factors. Then the conditions of Corollary 3.3.32 are satisfied by the semi-direct product $\prod_X \mathbf{O}_V \rtimes G$:

Indeed, consider a subgroup P with $\mathrm{pr}_2(P) \cong G$ and let

$$\begin{array}{rccl} \psi\colon & G & \to & P \subset O \rtimes G \\ & g & \mapsto & (\{A_x^g\}_{x\in X}, g) \end{array}$$

be the inverse of pr_2. Looking at the multiplication in $\mathbf{O} \rtimes G$, the fact that ψ is a group homomorphism, i.e. $\psi(gh) = (\{A_x^{gh}\}_{x\in X}, gh)$ gives formulas

$$A_x^g A_{g^{-1}x}^h = A_x^{gh} \qquad \forall g,h \in G,\ x \in X \tag{3.3.35}$$

Now chose a system of representatives R for the G-orbits in X. Let $B = \{B_x\} \in O$ be the element given by

$$B_x = B_{hr} := A_r^{h^{-1}},$$

where $x = hr$ is the unique presentation of x with $h \in G$ and $r \in R$. Combining the formulas above gives us that for any $(\{A_x^g\}, g) \in P$, we have:

$$\begin{aligned} (\{B_x\}, e)(\{A_x^g\}, g)(\{B_x\}, e)^{-1} &= (\{A_r^{h^{-1}} A_{hr}^g\}, g)(\{B_x\}, e)^{-1} \\ &= (\{A_r^{h^{-1}g}\}, g)(\{A_x\}, e)^{-1} \\ &= (\{A_r^{h^{-1}g}\}\{(A_{g^{-1}hr})^{-1}\}, g) \\ (E, g) &= (\{A_r^{h^{-1}g}(A_r^{h^{-1}g})^{-1}\}, g) \end{aligned}$$

Hence P is subconjugate to $\{e\} \rtimes G$ via $B \in O$ and therefore taking O-orbits commutes with taking G-fixed points.

Remark 3.3.36. Note that for non-discrete groups G and spaces X, a similar statement still holds under slightly stronger assumptions: Let O be the group $\mathcal{T}(X, \mathbf{O}_V)$ where G acts by precomposition with the action on X. Assume there exists a continuous splitting of the projection $X \to X_N$. Then again for any continuous splitting ψ of $O \rtimes G \overset{\mathrm{pr}_2}{\to} G$, the image $\psi(G)$ is subconjugate to $\{\mathrm{const}_e\} \rtimes G$. The proof is exactly as above with the slightly more complicated function notation.

Proposition 3.3.37. *Let G be a discrete group, X a discrete G-space and let V be a finite dimensional G-representation. Let in the above notation O be the group $\prod_X \mathbf{O}_V$ with G acting by permuting the factors, then for all genuinely cofibrant $O \rtimes G$-spaces Z, taking O-orbits commutes with taking H-fixed points for all subgroups $H \subset G$, in the sense that the canonical map*

$$Z^H \diagup O^H \to \left[Z \diagup O\right]^H,$$

is an isomorphism.

Proof. Note that for subgroups $H \subset G$, any free G-set is a free H-set, and any genuinely cofibrant $\prod_X \mathbf{O}_V \rtimes G$-space is also genuinely cofibrant as a $\prod_X \mathbf{O}_V \rtimes H$-space by C.2.5. We can then apply Corollary 3.3.32 with the help of Example 3.3.34 for all choices of H. □

Fixed Points of Semi-Free Cells

The result for semi-free spectra analogous to Proposition 3.3.21, is somewhat more involved. In particular there is no general analogue of formula (3.3.17), as the following counterexample shows:

Example 3.3.38. Let G be the group $\mathbb{Z}_2$ with two elements, and $\mathbb{R}^2$ the trivial G-representation. Let $S(2)$ be the unit sphere in $\mathbb{R}^2$ also with the trivial $\mathbb{Z}_2$-action. Consider the semi-free $\mathbb{Z}_2$-spectrum $\mathcal{G}_{\mathbb{R}^2} S(2)$. By 2.2.18, there is a natural G-equivariant isomorphism

$$\mathcal{G}_{R^2} S(2) \cong \mathcal{G}_{\mathbb{C}} \left[\mathcal{O}_{\mathbb{Z}_2}(\mathbb{R}^2, \mathbb{C}) \wedge_{\mathbf{O}_2} S(2)\right] \cong \mathcal{G}_{\mathbb{C}} S(\mathbb{C}),$$

where we consider $\mathbb{C}$ as a $\mathbb{Z}_2$-representation via the complex conjugation, and $S(\mathbb{C})$ is the unit sphere in $\mathbb{C}$ which inherits a $\mathbb{Z}_2$-action. Note that $S(\mathbb{C})$ is an $\mathbf{O}_{\mathbb{C}} \rtimes \mathbb{Z}_2$-space.

If a formula analogous to 3.3.21 would hold for general semi-free spectra, i.e. if $\Phi^N \mathcal{G}_V K \cong \mathcal{G}_{V^N} K^N$, the above would yield a contradiction since

$$\mathcal{G}_{R^2} S^1 \ncong \mathcal{G}_{\mathbb{R}} S^0.$$

Note that the methods of the proof in the free case break already in the first step, since in particular

$$\left(\mathcal{O}_{\mathbb{Z}_2}(\mathbb{R}^2, \mathbb{C}) \wedge_{\mathbf{O}_2} S(2)\right)^{\mathbb{Z}_2} \ncong \mathcal{O}_{\mathbb{Z}_2}(\mathbb{R}^2, \mathbb{C}) \wedge_{\mathbf{O}_2{}^{\mathbb{Z}_2}} S(2)^{\mathbb{Z}_2} \cong S(\mathbb{C})^{\mathbb{Z}_2},$$

i.e. taking the $\mathbb{Z}_2$-fixed points does not commute with taking the $\mathbf{O}_2$-orbits for the $\mathbf{O}_2 \rtimes \mathbb{Z}_2$-space $\mathcal{O}(\mathbb{R}^2, \mathbb{C}) \wedge S(2)$.

An analogous result to Proposition 3.3.20, however, still holds if we restrict ourselves to certain "regular" semi free spectra.
Let us again start by investigating the X-fold smash powers of (semi-)free spectra, where X still denotes a finite set with an action of a discrete group G.

Lemma 3.3.39. *For X a finite discrete G-set, V a euclidean vector space, $K \in \mathbf{O}_V\mathcal{T}$ and $A = \mathcal{G}_V K$ a semi-free orthogonal spectrum, there is a natural isomorphism of G-spectra*

$$\Lambda_X(A) = (\mathcal{G}_V K)^{\wedge X} \cong \mathcal{G}^G_{V^{\oplus X}}\left[\mathbf{O}_{V^{\oplus X}} \wedge_{\prod\limits_X \mathbf{O}_V} K^{\wedge X}\right], \tag{3.3.40}$$

where G acts on the vector space $V^{\oplus X}$ by permuting summands and on the spectrum $(\mathcal{G}_V K)^{\wedge X}$, the space $K^{\wedge X}$ and the group $\prod\limits_X \mathbf{O}_V$ by permuting (smash) factors.

Proof. The product $\Pi_X \mathbf{O}_V$ is a G-equivariant subgroup of $\mathbf{O}_{V^{\oplus X}}$ via the inclusion as block-diagonal matrices. Note that $K^{\wedge X}$ is a $\Pi_X \mathbf{O}_V \rtimes G$ space with the action defined by

$$\begin{array}{rcl} \prod\limits_X \mathbf{O}_V \rtimes G_+ \wedge K^{\wedge X} & \to & K^{\wedge X}. \\ (\{A_x\}_{x\in X}, g),\ \{k_x\}_{x\in X} & \mapsto & \{A_x k_{g^{-1}x}\}_{x\in X} \end{array}$$

so that the spectrum on the right side of (3.3.40) is indeed well defined. As above, the formula (2.2.17) for the smash product of semi-free spectra gives the result after identifying the action induced by the symmetry isomorphism. $\square$

So for the case of smash powers of semi-free spectra, we only need to calculate geometric fixed points of very specific semi-free G-spectra. The following result will be fundamental for our further work:

Theorem 3.3.41. *Let A be the source or the target of a generating $\mathbb{S}$-cofibration and X a finite free G-set. Then there is a canonical isomorphism of J-spectra*

$$\Lambda_{X_N}(A) \cong \Phi^N(\Lambda_X(A)).$$

We split the proof into two parts, dealing with Fix^N and $\mathbb{P}_\phi$ separately. The first part relies heavily on our findings about the interactions between orbits and fixed points from Subsection 3.3.3. In particular Proposition 3.3.37 allows us to prove the following:

Proposition 3.3.42. *Let $A = \mathcal{G}_V K$ be the source or the target of a generating $\mathbb{S}$-cofibration and X a finite free G-set. Then there is a canonical isomorphism of E-spectra*

$$\mathrm{Fix}^N([\mathcal{G}_V K]^{\wedge X}) \cong \mathcal{G}^E_{V^{\oplus X}}(\mathcal{O}_E(V^{\oplus X}, V^{\oplus X}) \wedge_{\prod_{X_N} \mathbf{O}_V} K^{\wedge X_N}).$$

Proof. We show the isomorphism in each level W for W a G-representation. The spectrum on the right side of the equation is given in that level by the $\mathcal{O}_E(W) \rtimes J$-space

$$\mathcal{O}_E(V^{\oplus X}, W) \wedge_{\prod_{X_N} \mathbf{O}_V} K^{\wedge X_N},$$

where the group J acts diagonally, by conjugation on the morphism space and via the permutation of factors on the smash product. Since the space level fixed point functor is monoidal (cf. C.1.2), we can identify $K^{\wedge X_N}$ with the diagonal copy of itself in $K^{\wedge X}$ and similar for $\prod_{X_n} \mathbf{O}_V$. Also, recall that $\mathrm{Fix}^N(\mathcal{G}_V K)^{\wedge X}$ is given in level W by

$$(\mathcal{O}_G(V^{\oplus X}, W) \wedge_{\prod_X \mathbf{O}_V} K^{\wedge X})^N,$$

where G acted diagonally, by conjugation on the morphism space and by permutation of the factors on the smash power.
Thus what we need to show is that the canonical inclusion of the coequalizer of the fixed points into the fixed points of the coequalizer

$$\mathcal{O}_G(V^{\oplus X}, W)^N \wedge_{\prod_X \mathbf{O}_V{}^N} (K^{\wedge X})^N \to (\mathcal{O}_G(V^{\oplus X}, W) \wedge_{\prod_X \mathbf{O}_V} K^{\wedge X})^N, \tag{3.3.43}$$

is indeed an isomorphism. This is exactly Proposition 3.3.37, if we can check the cofibrancy condition posed there. We shall make use of our assumption on A: Since $K = \mathbf{O}_V/P_+ \wedge L$, with L either a sphere or a disc, the map (3.3.43) is isomorphic to

$$\mathcal{O}_G(V^{\oplus X}, W)^N \wedge_{\prod_X P^N} (L^{\wedge X})^N \to (\mathcal{O}_G(V^{\oplus X}, W) \wedge_{\prod_X P} L^{\wedge X})^N.$$

Then $L^{\wedge X}$ is again a sphere or disc, hence a manifold, with smooth action of G and trivial action of $\prod_X P$, and therefore a genuine $\prod_X P \rtimes G$-complex by C.2.4. Since the genuine model structure is monoidal C.2.7, the smash product $\mathcal{O}_G(V^{\oplus X}, W) \wedge L^{\wedge X}$ is still cofibrant, hence Proposition 3.3.37 proves that the map (3.3.43) is an isomorphism. □

The second part of the proof of Theorem 3.3.41 is then given by the following lemma:

Lemma 3.3.44. *For K an $\mathbf{O}_V$-space, X a finite discrete G-set, there is a canonical isomorphism of J-spectra*

$$\mathbb{P}_\phi \mathcal{G}^E_{V^{\oplus X}}(\mathcal{O}_E(V^{\oplus X}, V^{\oplus X}) \wedge_{\prod_{X_N} \mathbf{O}_V} K^{\wedge X_N}) \cong (\mathcal{G}_V K)^{\wedge X_N}.$$

Proof. This follows directly from the defining adjunctions of the involved functors, similar to the free case in 3.3.20. Let Z be any J-spectrum, then maps from either of the J-spectra in the proposition to Z are represented by maps of $(\Pi_{X_N} \mathbf{O}_V \rtimes J)$-spaces

$$K^{\wedge X_N} \to (\mathbb{U}_\phi Z)_{V^{\oplus X_N}}.$$

Hence the Yoneda lemma gives the desired isomorphism. □

This finishes the proof of Theorem 3.3.21. As mentioned before, it will form the basis for the identification of the geometric fixed points of smash powers of general $\mathbb{S}$-cofibrant spectra in 3.4.26. We need two more crucial inputs, though. The first is Kro's observation on the interaction of the geometric fixed point functor with induced spectra (cf. 2.2.57) from [Kr, 3.8.10]. Since we will in particular need analogous results for the functors restricting to H-spectra for H a subgroup of G (cf. 2.2.54) to lift our results into to the case of G a compact Lie group, we will go into more details in the next section. The second part is an equivariant cellular filtration of the smash power via "regular" cells, and we will give the necessary constructions in 3.4.1.

3.3.4 Fixed Points and Change of Groups

For the whole section let G be a compact Lie-group, and H a closed subgroup of G with inclusion map $i\colon H \to G$. Recall the definitions of the restriction functor

$$i^*\colon G\mathcal{OT} \to H\mathcal{OT},$$

from 2.2.54 and its left adjoint induction functor

$$G_+\wedge_H(-)\colon H\mathcal{OT} \to G\mathcal{OT}$$

from 2.2.57. The following lemma will be helpful later, it is the spectrum level analogue of C.1.6, and makes use of the spectral enrichment mentioned in Remark 3.3.11.

Lemma 3.3.45. *For an orthogonal H-spectrum X and an orthogonal G-spectrum Y, there is a natural isomorphism:*

$$(G_+\wedge_H X)\wedge Y \cong G_+\wedge_H(i^*X\wedge Y).$$

Proof. Note that we intentionally use the category $G\mathcal{S}p^{\mathcal{O}}$, since the spectral enrichment there makes following the actions much easier. Let Z be any orthogonal G-spectrum, there are natural isomorphisms

$$\begin{aligned} G\mathcal{S}p^{\mathcal{O}}((G_+\wedge_H X)\wedge Y, Z) &\cong G\mathcal{S}p^{\mathcal{O}}(G_+\wedge_H X, F_G(Y,Z)) \\ &\cong H\mathcal{S}p^{\mathcal{O}}(X, i^*F_G(Y,Z)) \\ &\cong H\mathcal{S}p^{\mathcal{O}}(X, F_H(i^*Y, i^*Z) \\ &\cong H\mathcal{S}p^{\mathcal{O}}(X\wedge i^*Y, i^*Z) \\ &\cong G\mathcal{S}p^{\mathcal{O}}(G_+\wedge_H(X\wedge i^*Y), Z), \end{aligned}$$

so the Yoneda lemma gives the desired result. □

We recollect the material from [Kr, 3.8.2], both for completeness and to adapt notation to our conventions. Afterwards we expand the results to the restriction functor.
At first we need to give Kro's Definition of the change of sequence functors for the categories $\mathcal{O}_E\mathcal{T}$ (cf. [Kr, 3.8.7]). The main point of difficulty stems from the fact that, contrary to the case of G-spectra, the change of universe does not necessarily give an equivalence of categories, as mentioned in 3.3.3.

Let $j\colon E \to E_0$ be the morphism of short exact sequences of compact Lie groups:

$$\begin{tikzcd} E_0\colon \arrow[d, "j"'] & 1 \arrow[r] & N \arrow[r] \arrow[d] & H \arrow[r] \arrow[d, "i"'] & J_0 \arrow[r] \arrow[d, "i_1"'] & 1\,, \\ E\colon & 1 \arrow[r] & N \arrow[r] & G \arrow[r] & J \arrow[r] & 1 \end{tikzcd} \tag{3.3.46}$$

where N is normal in G and hence in H, and J and J_0 denote the respective quotients.

Definition 3.3.47. Let $\mathcal{V} \subset \mathcal{V}'$ be two good collections of G-representations. Then the inclusion $\mathcal{O}_G^{\mathcal{V}} \to \mathcal{O}_G^{\mathcal{V}'}$ induces an inclusion of N-fixed categories and hence a forgetful functor $\mathbb{U} : \mathcal{O}_E^{\mathcal{V}}\mathcal{T} \to \mathcal{O}_E^{\mathcal{V}'}\mathcal{T}$. It has a left adjoint $\mathbb{P} : \mathcal{O}_E^{\mathcal{V}'}\mathcal{T} \to \mathcal{O}_E^{\mathcal{V}}\mathcal{T}$. These give the *change of universe functors* for $\mathcal{O}_E$ spaces.

Now for $\mathcal{V}$ a good collection of G-representations, let $i^*\mathcal{V}$ be the induced collection of H-representations i^*V that appear as restrictions of some $V \in \mathcal{V}$.

Definition 3.3.48. The *change of sequence functor* $j^* : \mathcal{O}_E^{\mathcal{V}}\mathcal{T} \to \mathcal{O}_{E_0}^{i^*\mathcal{V}}\mathcal{T}$ is given by sending an $\mathcal{O}_E^{\mathcal{V}}$-space X to the $\mathcal{O}_{E_0}^{i^*\mathcal{V}}$-space given by

$$(j^*X)_{i^*V} := i_1^* X_V.$$

The structure maps are transported directly, using $\mathcal{O}_{E_0}(i^*V, i^*W) \cong i_1^*\mathcal{O}_E(V, W)$.

Again this functor has both a left and a right adjoint. We will only make use of the left adjoint:

Definition 3.3.49. Let Y be a $\mathcal{O}_{E_0}^{\iota^*\mathcal{V}}$-space, then the *induced $\mathcal{O}_E^{\mathcal{V}}$-space* $J_+{\wedge_{J_0}}Y$ is given by

$$(J_+{\wedge_{J_0}}Y)_V := J_+{\wedge_{J_0}}Y_{\iota^*V}$$

The (J-equivariant) structure maps are given as the composite

$$\begin{aligned} \mathcal{O}_E^{\mathcal{V}}(V,W)\wedge(J_+{\wedge_{J_0}}Y_{\iota^*V}) &\cong J_+{\wedge_{J_0}}(\iota_1^*\mathcal{O}_E^{\mathcal{V}}(V,W)\wedge Y_{\iota^*V}) \\ &\cong J_+{\wedge_{J_0}}(\mathcal{O}_{E_0}^{\iota^*\mathcal{V}}(\iota^*V,\iota^*W)\wedge Y_{\iota^*V}) \\ &\to J_+{\wedge_{J_0}}Y_{\iota^*W}. \end{aligned}$$

Note that for N the trivial group, $\mathcal{O}_E$ spaces are just orthogonal G-spectra, and the change of sequence functors specialize to the change of groups functors from 2.2.5.

Induced Spectra and (Geometric) Fixed Points

We give a recollection of [Kr, 3.8.3] before we prove further results in the same spirit that will be needed in particular when dealing with infinite groups. Let as in (3.3.46) E and E_0 be exact sequences of compact Lie groups. The following condition is an important prerequisite already in Kro's exposition, and will be important for us as well:

Condition 1. Let G be a compact Lie group and H a closed subgroup. Let W be an orthogonal H-representation. Then there exists an orthogonal G-representation V and an isometric embedding $W \to V$ that induces an isomorphism on H-fixed point spaces, i.e. $V^H = W^H$.

Remark 3.3.50. To the authors knowledge it is not known if this condition is satisfied in general. As Kro already states in [Kr, 3.8.11] it suffices to look at the irreducible subrepresentations of V one at a time, and extending the trivial representation is trivial. The condition is certainly satisfied when the index of H in G is finite since for V irreducible and non trivial, both V^H and $(G \times_H V)^H$ are the zero vector space, hence the induced representation (which is still finite dimensional) extends V as desired. We prove that Condition 1 is satisfied for all finite subgroups of tori in 3.4.45. Since the extension problem is transitive, this covers a big class of configurations for H and G. Note that the case where H is finite suffices for all applications of the theory we give, so a proof of Condition 1 for H a maximal torus of a compact Lie group G would immediately be very fruitful. (cf. 3.4.50).

Proposition 3.3.51. *[Kr, 3.8.10] Suppose Condition 1 is satisfied for the normal subgroup N of G. Then for orthogonal H-spectra X, there is a natural isomorphism of J-spectra*

$$J_+ \wedge_{J_0} (\Phi^N X) \cong \Phi^N (G_+ \wedge_H X).$$

In particular Kro proves that the following diagram of functors commutes up to an equivalence of categories

$$\begin{tikzcd}
\mathcal{O}_H^{\mathcal{W}}\mathcal{T} \arrow[r, "\mathrm{Fix}^N"] \arrow[d, "\mathbb{U}"'] & \mathcal{O}_{E_0}^{\mathcal{W}}\mathcal{T} \arrow[r, "\mathbb{P}_{\varphi_0}^{\mathcal{W}}"] \arrow[d, "\mathbb{U}"] & \mathcal{O}_{J_0}\mathcal{T} \arrow[d, "\mathbb{U}"] \\
\mathcal{O}_H^{i^*\mathcal{V}}\mathcal{T} \arrow[r, "\mathrm{Fix}^N"] \arrow[d, "G_+\wedge_H -"'] & \mathcal{O}_{E_0}^{i^*\mathcal{V}}\mathcal{T} \arrow[r, "\mathbb{P}_{\varphi_0}^{i^*\mathcal{V}}"] \arrow[d, "J_+\wedge_{J_0} -"] & \mathcal{O}_{J_0}\mathcal{T} \arrow[d, "J_+\wedge_{J_0} -"] \\
\mathcal{O}_G^{\mathcal{V}}\mathcal{T} \arrow[r, "\mathrm{Fix}^N"] & \mathcal{O}_E^{\mathcal{V}}\mathcal{T} \arrow[r, "\mathbb{P}_{\varphi}^{\mathcal{V}}"] & \mathcal{O}_J\mathcal{T},
\end{tikzcd} \tag{3.3.52}$$

where $\mathcal{W}$ is the good collection associated to a complete H-universe (cf. 2.2.25), $\mathcal{V}$ similar for a complete G-universe. As usual $i^*\mathcal{V}$ denotes the restricted H-collection.

All the functors $\mathbb{U}$ are induced by the precomposition with some H-equivariant inclusion of collections $i^*\mathcal{V} \to \mathcal{W}$, with the center one being the change of sequence functor 3.3.48, and the outer ones being instances of the equivalence of categories from 2.2.23. Note that we omit the J- and J_0-universes from the notation, since the change of universe on the right hand side of the diagram is always an equivalence of categories. The commutativity is proven piecewise, using the following lemmas:

Lemma 3.3.53. *[Kr, 3.8.12] Suppose Condition 1 is satisfied for the normal subgroup N of G. Then for $\mathcal{O}_E^{\mathcal{W}}$-spaces Y, there is a natural isomorphism of J_0-spectra*

$$\mathbb{P}_{\phi_0}^{\iota^*\mathcal{V}}\mathbb{U}Y \cong \mathbb{U}\mathbb{P}_{\phi_0}^{\mathcal{W}}Y.$$

In the proof of this lemma, Condition 1 is used to identify two specific coends. We will not repeat the proofs.

Lemma 3.3.54. *[Kr, 3.8.13] Let X be an orthogonal H-spectrum. There is a natural isomorphism of $\mathcal{O}_E^{\mathcal{V}}$ spaces*

$$J_+\wedge_{J_0}(\mathbb{U}\mathrm{Fix}^N X) \cong \mathrm{Fix}^N(G_+\wedge_H \mathbb{U}X).$$

Lemma 3.3.55. *[Kr, 2.8.14] The following diagram of left adjoints commutes:*

$$\begin{tikzcd}
\mathcal{O}_{E_0}^{\iota^*\mathcal{V}}\mathcal{T} \arrow[r, "\mathbb{P}_{\phi_0}^{\iota^*\mathcal{V}}"] \arrow[d, "J_+\wedge_{J_0}-"'] & \mathcal{O}_{J_0}\mathcal{T} \arrow[d, "J_+\wedge_{J_0}-"] \\
\mathcal{O}_E^{\mathcal{V}}\mathcal{T} \arrow[r, "\mathbb{P}_\phi"] & \mathcal{O}_J\mathcal{T}
\end{tikzcd}$$

Lemma 3.3.56. *[Kr, 3.8.14] The following diagram of right adjoints commutes:*

$$\begin{tikzcd}
\mathcal{O}_{E_0}^{\iota^*\mathcal{V}}\mathcal{T} & \mathcal{O}_{J_0}\mathcal{T} \arrow[l, "\mathbb{U}_{\phi_0}^{\iota^*\mathcal{V}}"'] \\
\mathcal{O}_E^{\mathcal{V}}\mathcal{T} \arrow[u, "j^*"] & \mathcal{O}_J\mathcal{T} \arrow[l, "\mathbb{U}_\phi"'] \arrow[u, "\iota_1^*"']
\end{tikzcd}$$

Restricted spectra and (Geometric) Fixed Points

Let as in (3.3.46) E and E_0 be exact sequences of groups. The following proposition is similar in spirit to Kro's 3.3.51 from above, we will use it when moving from finite to compact Lie groups 3.4.50.

Proposition 3.3.57. *For orthogonal G-spectra Y, taking categorical N-fixed points commutes with the restriction to H-spectra, i.e. there is a natural isomorphism of J_0-spectra*

$$i_0^*(Y^N) \cong (i^*Y)^N.$$

If additionally $N \subset G$ satisfies Condition 1, then the same holds for geometric N-fixed points, i.e. there is a natural isomorphism of J_0-spectra

$$i_0^*\Phi^N Y \cong \Phi^N(i^*Y).$$

Proof. For the first statement it suffices to compare the evaluations of $i_0^*(Y^N)$ and $i_0^*(Y^N)$ at a trivial J_0-representation V. Since for G-spaces, taking fixed points commutes with the restriction (cf. C.1) they both evaluate to $(Y_V)^H$.
The second statement needs more work. Let $\mathcal{V}$ and $\mathcal{W}$ be good collections associated to complete G-, respectively H-universes as in 3.3.52. Then the desired result is the commutativity of the outer square in the following diagram:

$$\begin{tikzcd}
\mathcal{O}_H^{\mathcal{W}}\mathcal{T} \arrow[r, "\mathrm{Fix}^N"] \arrow[d, shift left, "\mathbb{U}"] & \mathcal{O}_{E_0}^{\mathcal{W}}\mathcal{T} \arrow[r, "\mathbb{P}^{\mathcal{W}}_{\varphi_0}"] \arrow[d, shift left, "\mathbb{U}"] & \mathcal{O}_{J_0}\mathcal{T} \arrow[d, shift right, "\mathbb{U}"'] \\
\mathcal{O}_H^{i^*\mathcal{V}}\mathcal{T} \arrow[r, "\mathrm{Fix}^N"] \arrow[u, shift left, "\mathbb{P}"] & \mathcal{O}_{E_0}^{i^*\mathcal{V}}\mathcal{T} \arrow[r, "\mathbb{P}^{i^*\mathcal{V}}_{\varphi_0}"] \arrow[u, shift left, "\mathbb{P}"] & \mathcal{O}_{J_0}\mathcal{T} \arrow[u, shift right, "\mathbb{P}"'] \\
\mathcal{O}_G^{\mathcal{V}}\mathcal{T} \arrow[r, "\mathrm{Fix}^N"] \arrow[u, "i^*"] & \mathcal{O}_E^{\mathcal{V}}\mathcal{T} \arrow[r, "\mathbb{P}^{\mathcal{V}}_{\varphi}"] \arrow[u, "i_0^*"] & \mathcal{O}_J\mathcal{T} \arrow[u, "i_0^*"]
\end{tikzcd} \tag{3.3.58}$$

That the lower left square commutes can again be checked on evaluations. Commutativity of the top rectangle follows from a diagram chase: For the top right square evaluating at levels shows immediately that the corresponding diagram of right adjoints commutes. Note that since Condition 1 is satisfied, Lemma 3.3.53 gives that the two ways from the top center to the top right around the top right square differ only by an equivalence of categories. In particular $\mathbb{P}^{\mathcal{W}}_{\varphi_0}$ is naturally isomorphic to the factorization $\mathbb{P} \circ \mathbb{P}^{i^*\mathcal{V}}_{\varphi_0} \circ \mathbb{U}$. In the top left square note that $\mathbb{P} \circ \mathrm{Fix}^N \cong \mathbb{P} \circ \mathrm{Fix}^N \circ \mathbb{U}\mathbb{P}$. Since $\mathbb{U}$ and Fix^N commute, this implies $\mathbb{P} \circ \mathrm{Fix}^N \cong \mathbb{P}\mathbb{U} \circ \mathrm{Fix}^H \circ \mathbb{P}$. Postcomposing the latter with $\mathbb{P}^{\mathcal{W}}_{\varphi_0} \cong \mathbb{P} \circ \mathbb{P}^{i^*\mathcal{V}}_{\varphi_0} \circ \mathbb{U}$ yields $\mathbb{P} \circ \mathbb{P}^{i^*\mathcal{V}}_{\varphi_0} \circ \mathbb{U}\mathbb{P}\mathbb{U} \circ \mathrm{Fix}^N \circ \mathbb{P}$ which is equal to $\mathbb{P} \circ \mathbb{P}^{i^*\mathcal{V}}_{\varphi_0} \circ \mathbb{U} \circ \mathrm{Fix}^N \circ \mathbb{P}$ because $\mathcal{O}_{E_0}^{i^*\mathcal{V}}$ is a full subcategory of $\mathcal{O}_{E_0}^{\mathcal{W}}$. This exactly says that the top rectangle commutes.
The only thing left to prove is that the lower right square commutes. We again look at evaluations at trivial representations $\mathbb{R}^n$. For X in $\mathcal{O}_E^{\mathcal{V}}$, the lower way around the square evaluates as

$$i_0^*\left(\int^{V\in\mathcal{V}} \mathcal{O}_J(V^N, \mathbb{R}^n) \wedge X_V\right) = \int^{V\in\mathcal{V}} i_0^*\mathcal{O}_J(V^N, \mathbb{R}^n) \wedge i_0^* X_V.$$

The upper way yields

$$\int^{i^*V \in i^*\mathcal{V}} \mathcal{O}_{J_0}((i^*V)^N, R^n) \wedge i_0^* X_V.$$

Again we have $(i_0^* V)^N = i_0^*(V^N)$, hence

$$i_0^* \mathcal{O}_J(V^N, \mathbb{R}^n) = \mathcal{O}_{J_0}(i_0^*(V^N), \mathbb{R}^n) = \mathcal{O}_{J_0}((i_0^* V)^N, \mathbb{R}^n),$$

and these identifications are natural with respect to morphisms in $\mathcal{O}_J$. Thus the two coends are naturally isomorphic. Alternatively we could have used that i^* also has a right adjoint. □

3.3.5 Induced Regular Cells

We go back to finite discrete groups G for this subsection. As we have seen in Example 3.3.38, the class of semi-free G-spectra is too big to fully control the geometric fixed point functor. Our studies of the smash powers of semi-free non equivariant spectra has given us a specific example of a class where such control is possible. Now we will define classes of *regular spectra*, and *induced regular spectra* which the smash powers are examples of.

Definition 3.3.59. Let V be a finite dimensional euclidean G-representation and let K be an $\mathbf{O}_V \rtimes G$-space. Then K is called *regular* if there is a G-equivariant subgroup $Q \cong \Pi_X(P)$ of $\mathbf{O}_V$, such that X is a free G-set and K represents a $Q \rtimes G$-space, i.e. there is a G-equivariant isomorphism

$$K \cong \mathbf{O}_{V+} \wedge_Q L \cong \mathbf{O}_{V+} \wedge_{\prod\limits_X P} L.$$

A semi-free G-spectrum $\mathcal{G}_V K$ is called *regular* if K is.

Remark 3.3.60. Note that free spectra are regular, since we can chose Q and P as the trivial groups.

Note in particular that since $Q \subset \mathbf{O}_V$ is G-equivariant, there are inclusions of N-fixed subgroups $Q^N \subset (\mathbf{O}_V)^N$ for all subgroups N of G.

Remark 3.3.61. Since all the involved functors preserve colimits and since inducing up preserves genuine cofibrations by C.1.17, the regular semi-free spectrum $\mathcal{G}_V K \cong \mathcal{G}_V(\mathbf{O}_{V+} \wedge_Q L)$ is $\mathbb{S}$-cofibrant if L is genuinely $Q \rtimes G$-cofibrant.

Example 3.3.62. The most important example of a regular semi-free G-spectrum is the smash power

$$(\mathcal{G}_V K)^{\wedge X} \cong \mathcal{G}_{V^{\oplus X}}[\mathbf{O}_{V^{\oplus X}} \wedge_{\prod_X \mathbf{O}_V} K^{\wedge X}],$$

where X is a finite free G-set.

The following proposition is proved completely analogous to 3.3.41, using that Proposition 3.3.37 is slightly more general than was needed there.

Proposition 3.3.63. *Let N be a normal subgroup of G with quotient group denoted by J. Let V be a finite dimensional euclidean G-representation and let $\mathcal{G}_V^G K$ be a regular semi-free G-spectrum. Then there is a natural isomorphism of J-spectra*

$$\mathcal{G}^J_{V^N} K^N \cong \Phi^N(\mathcal{G}^G_V K).$$

Combining this result with Proposition 3.3.51 for induced spectra motivates the following definition.

Definition 3.3.64. A semi-free G-spectrum A is called *induced regular* if there is an isomorphism of G-spectra

$$A \cong G_+ \wedge_H B,$$

for H a subgroup of G and B a regular semi-free H-spectrum.

Note that by Remark 3.3.50 Condition 1 is always satisfied in the case of finite groups, so we get the following characterizations of geometric fixed points for induced regular spectra which is analogous to Kro's Lemma 3.10.8:

Theorem 3.3.65. *Let G be a finite group and H and N subgroups with N normal. Then for an induced regular semi-free G-spectrum $G_+ \wedge_H \mathcal{G}_V K$ its geometric fixed points are given by the natural isomorphism:*

$$\Phi^N(G_+ \wedge_H [\mathcal{G}_V K]) \cong \begin{cases} {}^{G}\!/_{N+} \wedge_{H/N} [\mathcal{G}_{V^N} K^N] & \text{if } N \subset H \\ \ast & \text{otherwise.} \end{cases}$$

Proof. Only the second part needs additional comment. Note that already for H-spaces K we have $(G_+ \wedge_H A)^N \cong \star$ if N is not contained in H, hence the functor Fix^N is already trivial on the induced spectrum (cf. [Kr, 3.10.8]). □

Note that since the $\mathbb{S}$-cofibrations are independent from the choice of universe by 2.3.15, an argument similar to the one in proof of 2.3.21 gives, that the inducing up functor $G_+ \wedge_H (-)$ preserves $\mathbb{S}$-cofibrations. Hence as above, the induced regular semi-free spectrum $G_+ \wedge_H \mathcal{G}_V^H K$ is $\mathbb{S}$-cofibrant if K is genuinely $Q \rtimes H$-cofibrant. Similarly one gets the following

Lemma 3.3.66. *For V an H-representation, X a finite free H set, $Q = \Pi_X P$ an H-equivariant subgroup of $\mathbf{O}_V$ and i a genuine cofibration of $Q \rtimes H$-spaces, the map*

$$f := G_+ \wedge_H \mathcal{G}_V^H [\mathbf{O}_{V+} \wedge_Q i]$$

of G-spectra is an $\mathbb{S}$-cofibration. Denote the class of $\mathbb{S}$-cofibrations of this form by $\mathbb{S}^G_{\mathrm{reg}}$.

Remark 3.3.67. For $\mathbb{S}^G_{\mathrm{reg}}$-cell complexes, Theorem 3.3.65 allows us to easily compute the geometric fixed points via a cell induction. This could be used to define a class of cofibrations very much in the spirit of Kro's *induced cells* ([Kr, 3.4.4]) and *orbit cells* ([Kr, 3.4.6]), but more general than both. Since we are not going to construct model structures or even replacement functors for any of these classes, we will not go into this generality. Instead we will focus on the type of cells that will appear in the cell structure for the smash powers.

Finally we give a name for the cells that we will use in the equivariant filtration theorem:

Definition 3.3.68. An $\mathbb{S}$-cofibration f of orthogonal G-spectra is an *induced regular cell* if it arises from a generating $\mathbb{S}$-cofibration i of orthogonal spectra via

$$f = G_+ \wedge_H i^{\square H},$$

for H a normal subgroup of G, and $\square$ the pushout product construction from A.2.28. We denote the *class of all induced regular cells* by $\mathrm{Ind}^{\mathrm{reg}}$.

The significance of the pushout product will become more obvious in the next section, when we give details on equivariant filtrations (cf. in particular Lemma 3.4.6). The following remark, however, is immediate:

Remark 3.3.69. Both source and target of an induced regular cell are induced regular in the sense of 3.3.64. For the target this is obvious, since for i the map $\mathcal{G}_V\left(\mathbf{O}_V/P_+ \wedge [S^{n-1 \to D^n}]\right)$, the formula 1.2.34 together with the fact that inducing up preserves colimits yield that $i^{\square H}$ is isomorphic to the map

$$\imath^{\square H} \cong \mathcal{G}_{V \oplus H}\left(\mathbf{O}_{V \oplus H}/\Pi_H P_+ \wedge [S^{|H|n-1} \to D^{|H|n}]\right),$$

where $|H|$ is the order of H. In particular it is represented by the inclusion of the boundary sphere of the $\Pi_H P \rtimes G$ space $D^{|H|n}$, where $\Pi_H P$ acts trivially, and H acts by permuting coordinates blockwise.

3.4 Fixed Points of Smash Powers

3.4.1 Cellular Filtrations

We construct the cellular structures that form the technical heart of this chapter. We generalize Kro's approach from [Kr, 2.2], correcting some minor mistakes along the way. In particular we drop the assumption that all λ-sequences are $\mathbb{N}$-sequences in order to be able to attach cells one at a time, and work in general categories. This allows us to apply the theory in a lot of different contexts, cf. 1.3.17, 2.3.35, 3.3.18 and 3.3.19, but also for a potential extension of our results to multiplicative norm constructions (cf. Remark 3.4.44).

Pushouts and Pushout Products

Recall the following property of pushouts which is independent of the category we work in:

Lemma 3.4.1. *Let $\mathcal{C}$ be a category and consider the following diagram in $\mathcal{C}$:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & P \arrow[r] & Q \end{tikzcd} \tag{3.4.2}$$

(i) If both the left and the right subsquare of the diagram are pushout diagrams, then so is the outer rectangle.

(ii) If both the left subsquare and the outer rectangle are pushout diagrams, then so is the right subsquare.

Proof. One either checks the universal properties directly, or uses cofinal subcategories of the diagram categories underlying 3.4.2. □

Lemma 3.4.3. *Let $\mathcal{C}$ have all pushouts. Consider a commutative cube in $\mathcal{C}$, where either the top and bottom faces or the left and right faces are pushouts:*

$$\begin{tikzcd} & A_0 \arrow[rr] \arrow[dl] \arrow[dd] & & X_0 \arrow[dl] \arrow[dd] \\ Y_0 \arrow[rr] \arrow[dd] & & P_0 \arrow[dd] & \\ & A_1 \arrow[rr] \arrow[dl] & & X_1 \arrow[dl] \\ Y_1 \arrow[rr] & & P_1 & \end{tikzcd} . \tag{3.4.4}$$

Then extending the cube by taking the pushouts of the front- and back face, the induced square is again pushout:

$$\begin{tikzcd}
& A_0 \arrow[rr] \arrow[dl] \arrow[dd] & & X_0 \arrow[dl] \arrow[dd] \\
Y_0 \arrow[rr] \arrow[dd] & & P_0 \arrow[dd] \arrow[r, phantom, "\ulcorner"] & \\
& A_1 \arrow[rr, dashed] \arrow[dl] & & Q_b \arrow[dl] \arrow[dr] \\
Y_1 \arrow[rr] \arrow[ur, phantom, "\ulcorner"] & & Q_f \arrow[dr] & & X_1 \arrow[dl] \\
& & & P_1
\end{tikzcd} \quad . \tag{3.4.5}$$

Proof. Assume the top and bottom faces of 3.4.4 are pushout. The composition of the top- with the front face in 3.4.5 is pushout by 3.4.1 (*i*), hence also the bottom face by (*ii*). Thus, again using 3.4.1(*ii*), so is the extended square. Analogously if the composition of the left and the front face are pushout, the right face is, hence the extended square. □

From now on we work in a closed symmetric monoidal category, so that we can talk about the pushout product construction of A.2.28.

Lemma 3.4.6. *Let* $(\mathcal{C}, \wedge)$ *be closed symmetric monoidal and have all pushouts. Then given two pushout squares*

$$\begin{tikzcd}
A_b \arrow[r, "g_b"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_b \arrow[d] \\
A_f \arrow[r, "g_f"'] & X_f
\end{tikzcd} \qquad \qquad
\begin{tikzcd}
B_b \arrow[r, "h_b"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y_b \arrow[d] \\
B_f \arrow[r, "h_f"'] & Y_f
\end{tikzcd} ,$$

their row-wise pushout product is also a pushout square:

$$\begin{tikzcd}
Q_b \arrow[r, "g_b \square h_b"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_b \wedge Y_b \arrow[d] \\
Q_f \arrow[r, "g_f \square h_f"'] & X_f \wedge Y_f
\end{tikzcd} .$$

Proof. We want to use Lemma 3.4.3, the relevant cube is given by:

$$\begin{tikzcd}
& A_b\wedge B_b \arrow[rr] \arrow[dl] \arrow[dd] & & A_b\wedge Y_b \arrow[dl] \arrow[dd] \\
A_f\wedge B_f \arrow[rr] \arrow[dd] & & A_f\wedge Y_f \arrow[dd] \\
& X_b\wedge B_b \arrow[rr] \arrow[dl] & & Q_b \arrow[dl] \arrow[dr] \arrow[ul, phantom, "\ulcorner", very near start] \\
X_f\wedge B_f \arrow[rr] & & Q_f \arrow[dr] \arrow[ul, phantom, "\ulcorner", very near start] & & X_b\wedge Y_b \arrow[dl] \\
& & & X_f\wedge Y_f
\end{tikzcd} \tag{3.4.7}$$

Here we cannot use the above lemma directly, because the top face and the bottom rectangle are in general not pushouts. However, we can split diagram 3.4.7 in two pieces, factoring all the maps from the back to the front face:

$$\begin{tikzcd}
& A_b\wedge B_b \arrow[rr] \arrow[dl] \arrow[dd] & & A_b\wedge Y_b \arrow[dl] \arrow[dd] \\
A_b\wedge B_f \arrow[rr] \arrow[dd] & & A_b\wedge Y_f \arrow[dd] \\
& X_b\wedge B_b \arrow[rr] \arrow[dl] & & Q_b \arrow[dl] \arrow[dr] \arrow[ul, phantom, "\ulcorner", very near start] \\
X_b\wedge B_f \arrow[rr] & & Q_m \arrow[dr] \arrow[ul, phantom, "\ulcorner", very near start] & & X_b\wedge Y_b \arrow[dl] \\
& & & X_b\wedge Y_f
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_b\wedge B_f \arrow[rr] \arrow[dl] \arrow[dd] & & A_b\wedge Y_f \arrow[dl] \arrow[dd] \\
A_f\wedge B_f \arrow[rr] \arrow[dd] & & A_f\wedge Y_f \arrow[dd] \\
& X_b\wedge B_f \arrow[rr] \arrow[dl] & & Q_m \arrow[dl] \arrow[dr] \arrow[ul, phantom, "\ulcorner", very near start] \\
X_f\wedge B_f \arrow[rr] & & Q_f \arrow[dr] \arrow[ul, phantom, "\ulcorner", very near start] & & X_b\wedge Y_f \arrow[dl] \\
& & & X_f\wedge Y_f
\end{tikzcd}$$

Since smashing with an object is a left adjoint it preserves pushouts, hence the two extended cubes are both examples of 3.4.5. Therefore Lemma 3.4.3 gives that both of the two extended squares are pushout, and Lemma 3.4.1(i) completes the proof. □

Relative Cellular Maps

We will use Lemma 3.4.6 to recognize a relative cellular structure on the □-product of relative cellular maps. Recall the following definition (e.g. [H, 2.1.9]):

Definition 3.4.8. Let $\mathcal{C}$ be a category with pushouts, and I a class of morphisms of $\mathcal{C}$. Then a morphism $f\colon A \to X$ in $\mathcal{C}$ is *relative I-cellular*, if it is a transfinite composition of pushouts of coproducts of elements of I.

Remark 3.4.9. Let $f\colon A \to X$ be a relative I-cellular map, and let $A = X_0 \to X_1 \to \dots$ be a λ-sequence that exhibits this structure, i.e. λ an ordinal and for any $\alpha \le \lambda$ we have pushout diagrams

$$\begin{tikzcd}
S_\alpha \arrow[r, "\sigma_\alpha"] \arrow[d, "i_\alpha"'] & \operatorname*{colim}_{\beta<\alpha} X_\alpha \arrow[d, "f_\alpha"] \\
D_\alpha \arrow[r] & X_\alpha, \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where i_α is a coproduct $\coprod_{c \in C_\alpha} i_c$, with all the maps i_c in I, and C_α empty whenever α is a limit ordinal. Then the union of the C_α is partially ordered, with $i_c \in C_\alpha$ smaller than $i_d \in C_\beta$ if and only if $\alpha < \beta$. In this situation, we say that $\bigcup_{\alpha \leq \lambda} C_\alpha$ indexes the attached cells of f in the λ-sequence.

The following lemma helps with keeping the “length” of the transfinite composition in check when the domains of the morphisms in I are sufficiently small:

Lemma 3.4.10. *Let $f \colon A \to X$ be an I-cell complex and assume that the domains of the maps in I are κ-small. Then there is a κ-sequence of maps exhibiting f as relative I-cellular.*

Proof. Assume that f is the transfinite composition of a λ-sequence $\{X_\alpha\}_{\alpha \leq \lambda}$ that exhibits the a cellular structure, i.e. for $\alpha < \lambda$ there are pushout diagrams

$$\begin{tikzcd} S_\alpha \arrow[r, "\sigma_\alpha"] \arrow[d, "i_\alpha"'] \arrow[dr, phantom, "\ulcorner"] & \operatorname*{colim}_{\beta<\alpha} X_\alpha \arrow[d, "f_\alpha"] \\ D_\alpha \arrow[r] & X_\alpha, \end{tikzcd}$$

such that i_α is the identity of the initial object for α a limit ordinal, and a coproduct of maps in I otherwise. For $\gamma \leq \kappa$, define sets $C_\gamma^<$ and C_γ as well as commutative diagrams

$$\begin{tikzcd} \operatorname*{colim}_{\delta<\gamma} X_\delta \arrow[r] \arrow[d] & \operatorname*{colim}_{\delta<\gamma} Y_\delta \arrow[d] \arrow[dr] & \\ X_\gamma \arrow[r] \arrow[rr, bend right] & Y_\gamma, \arrow[r] & X \end{tikzcd} \tag{3.4.11}$$

by transfinite induction: Let $C_0 := 0$ and $Y_0 := X_0 = A$. Continuing, for μ a limit ordinal let C_μ be empty.Otherwise define the set

$$C_\gamma^< := \{\alpha \leq \lambda, \sigma_\alpha \text{ factors through } Y_{\gamma-1}\}.$$

Further let $C_\gamma := C_\gamma^< \setminus \bigcup_{\delta<\gamma} C_\delta^<$. Finally define Y_γ as the pushout

$$\begin{tikzcd} \coprod_{\alpha \in C_\gamma} S_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \operatorname*{colim}_{\delta<\gamma} Y_\delta \arrow[d] \\ \coprod_{\alpha \in C_\gamma} D_\alpha \arrow[r] & Y_\gamma \end{tikzcd}$$

Define a map $Y_\gamma \to X$ on the attached cells $S_\alpha \to D_\alpha$ by going through the X_α. Note that $\sigma_\gamma \colon S_\gamma \to X$ factors through $\operatorname{colim}_{\delta<\gamma} X_\delta$, hence we get a map $X_\gamma \to Y_\gamma$

which fits into the diagram 3.4.11. Finally note that since S_α is κ-small, all attaching maps σ_α for $\alpha \le \lambda$ factor through some X_γ, hence through Y_γ. the union $\bigcup_{\gamma\le\kappa} C_\gamma$ contains all $\alpha \le \lambda$. Therefore there are canonical maps in both directions between the colimits

$$\operatorname*{colim}_{\alpha\le\lambda} X_\alpha \cong \operatorname*{colim}_{\gamma\le\kappa} Y_\gamma,$$

which are isomorphisms by cofinality. □

Remark 3.4.12. Note that attaching cells via coproducts, gives a partial order on the set of cells. Every such partially ordered set can be linearly ordered as in [H, 2.1.11], which corresponds to giving a λ-sequence in which the cells are attached one at a time. Lemma 3.4.10 gives us a much more convenient way to revert this process, than simply forgetting the extra information. Returning to a closed symmetric monoidal category $(\mathcal{C}, \wedge)$, observe that taking coproducts interacts distributive with the smash product, hence also with the $\square$-product. We therefore allow ourselves to switch freely between attaching cells one at a time or in bigger groups via the coproduct.

The main theorem we aim for in this subsection is the following:

Theorem 3.4.13. *Let $(\mathcal{C}, \wedge)$ be closed symmetric monoidal with all small colimits. Let I and J be sets of morphisms in $\mathcal{C}$ and let $f\colon A \to X$ and $g\colon B \to Y$ be relative I- and J-cellular, respectively. Then their pushout product $f\square g$ is relative $(I\square J)$-cellular.*
In particular, if λ and μ are partially ordered indexing sets for cells of f and g, respectively, then $\lambda \times \mu$ is a partially ordered indexing set for cells of $f\square g$.

Proof. We assume without loss of generality (cf. 3.4.12) that λ and μ are ordinals linearly indexing the cells of f and g, respectively. That is for each $\alpha \le \lambda$ we have a pushout diagram

$$\begin{tikzcd} S_\alpha \arrow[r, "i_\alpha\in I"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & D_\alpha \arrow[d] \\ \operatorname*{colim}_{\gamma<\alpha} X_\gamma \arrow[r, "f_\alpha"'] & X_\alpha, \end{tikzcd}$$

such that the λ-sequence $A = X_0 \to X_\lambda = X$ is the map f, and and analogous for g. Chose the product partial order on $\lambda \times \mu$, i.e. $(\gamma, \delta) < (\alpha, \beta)$ if and only if $\gamma < \alpha$ and $\delta < \beta$. Let $E\colon \lambda \times \mu \to \mathcal{C}$ be the sequence defined by the pushout diagrams

$$\begin{tikzcd} A\wedge B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_\alpha\wedge B \arrow[d] \\ A\wedge Y_\beta \arrow[r] & E_{\alpha,\beta}, \end{tikzcd}$$

and note that $E_{\lambda,\mu}$ is the source of $f\Box g$. We claim that the desired filtration is then given by $\{F_{\alpha,\beta}\}$, the (pointwise) pushout of $\lambda \times \mu$-sequences in the diagram:

$$\begin{tikzcd} E_{\alpha,\beta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_\alpha \wedge Y_\beta \arrow[d] \\ E_{\lambda,\mu} \arrow[r] & F_{\alpha,\beta}, \end{tikzcd} \tag{3.4.14}$$

where $E_{\lambda,\mu}$ is the constant sequence. To prove the claim, note that the transformation $E \to X_{(-)}\wedge Y_{(-)}$ of sequences factors through the sequence P, given pointwise as the pushout

$$\begin{tikzcd} \operatorname*{colim}_{\gamma<\alpha,\delta<\beta} X_\gamma \wedge Y_\delta \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \operatorname*{colim}_{\delta<\beta} X_\alpha \wedge Y_\delta \arrow[d] & \\ \operatorname*{colim}_{\gamma<\alpha} X_\gamma \wedge Y_\beta \arrow[r] & P_{(\alpha,\beta)} \arrow[r, "f_\alpha \Box g_\beta"] & X_\alpha \wedge Y_\beta. \end{tikzcd} \tag{3.4.15}$$

That is, $P_{\alpha,\beta}$ is the source of the map $f_\alpha \Box g_\beta$. We apply the cobase change as in 3.4.14 to this factorization to get the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & P \arrow[r, "f_{(-)}\Box g_{(-)}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X_{(-)}\wedge Y_{(-)} \arrow[d] \\ E_{\lambda,\mu} \arrow[r] & P \amalg_E E_{\lambda,\mu} \arrow[r] & F. \end{tikzcd} \tag{3.4.16}$$

Now comparing the colimits pointwise, cofinality lets us identify $P_{\alpha,\beta} \amalg_E E_{\lambda,\mu}$ as the following pushout:

$$\begin{tikzcd} \operatorname*{colim}_{\gamma<\alpha,\delta<\beta} F_{\gamma,\delta} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \operatorname*{colim}_{\delta<\beta} F_{\alpha,\delta} \arrow[d] & \\ \operatorname*{colim}_{\gamma<\alpha} F_{\gamma,\beta} \arrow[r] & P_{\alpha,\beta} \amalg_E E_{\lambda,\mu} \arrow[r] & F_{\alpha,\beta}. \end{tikzcd}$$

In particular we have

$$P \amalg_E E_{\lambda,\mu} \cong \operatorname*{colim}_{(\gamma,\delta)<(\alpha,\beta)} F_{\alpha,\beta},$$

and the map

$$\operatorname*{colim}_{(\gamma,\delta)<(\alpha,\beta)} F_{\gamma,\delta} \to F_{\alpha,beta}$$

is a cobase change of $f_\alpha \Box g_\beta$. Therefore to show that F is indeed a filtration by $I\Box J$-cells, it suffices by 3.4.16 to show that $f_\alpha \Box g_\beta$ is the attaching of a $I\Box J$-cell. This is a consequence of Lemma 3.4.6, which implies that there are pushout diagrams

$$\begin{tikzcd} S_{(\alpha,\beta)} \arrow[r, "i_\alpha \Box j_\beta \in I\Box J"] \arrow[d] & D_{(\alpha,\beta)} \arrow[d] \\ P_{(\alpha,\beta)} \arrow[r, "f_\alpha \Box g_\beta"] \arrow[ur, phantom, "\ulcorner", very near end] & X_\alpha \wedge Y_\beta. \end{tikzcd}$$

Note that as in Remark 3.4.12 we can extend the partial order on $\lambda \times \mu$ to a linear one, finishing the proof. $\square$

Remark 3.4.17. In a lot of cases of interest, for example $\mathcal{C} = \mathcal{T}$, with I and J the sets of generating (acyclic) cofibrations, we will actually have that $I\Box J \subset J(-\text{cell})$, such that the above proposition also gives $f\Box g$ the structure of a relative J-cellular map.

Remark 3.4.18. In categories where we can think of the maps in I, J and $I\Box J$ as inclusions, and of the filtered colimits as unions of subobjects, the intuition behind the filtration in the theorem simplifies significantly. In particular the cellular maps then give a λ-sequence of inclusions of subobjects

$$A \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X,$$

and similar for $B \hookrightarrow Y$. Note that $f\Box g$ is the inclusion

$$f\Box g\colon X\wedge B \cup_{A\wedge B} A\wedge Y = X\wedge B \cup A\wedge Y \hookrightarrow X\wedge Y,$$

and the filtration given by the theorem is through objects

$$F_{\alpha,\beta} = X\wedge B \cup X_\alpha \wedge Y_\beta \cup A\wedge Y.$$

Corollary 3.4.19. *The monoidal product $X\wedge Y$ of an I-cellular X object with a J-cellular object Y is $I\Box J$-cellular.*

Corollary 3.4.20. *In the situation of Theorem 3.4.13, the map $f\wedge g$ is relative $(I\Box J)\cup K$-cellular, where K is the set of maps $I\wedge B \cup A\wedge J$. In particular, if A and B are themselves I-, respectively J-cellular, $f\wedge g$ is even $I\Box J$-cellular.*

Proof. Use the theorem on the maps $\star \to A \to X$ and $\star \to B \to Y$ which are $I \cup \{\star \to A\}$-, respectively $J \cup \{\star \to B\}$-cellular. Note that the indexing of the filtrations is shifted, and the new filtration factors through $F_{1,1} = A\wedge B$. All the later cells are then of type $(I\Box J)\cup K$. $\square$

Corollary 3.4.21. *Since the $\square$-product is associative, Theorem 3.4.13 immediately gives specific filtrations for iterated $\square$-products of maps. The indexing set for the cells of the iterated $\square$ is always given by the product of the indexing sets with some (linear) order that is compatible with the product partial order.*

We now give the proofs of Corollary 3.3.18 and Proposition 3.3.19:

Proof of 3.3.18. Recall the generating cofibrations and acyclic cofibrations from 2.2.30 and 2.2.41. By 3.3.16 and C.1.17, Φ^N sends $\mathcal{F}I_G$ cells to $\mathcal{F}I_J$ cells. Since it also preserves the mapping cylinder construction and hence sends the maps $k_{V,W}$ in the definition of the generating acyclic cofibrations in $G\mathcal{OT}$ to their counterparts in $J\mathcal{OT}$. Note that $k_{V,W}\square i$ is mapped to $k_{V^N,W^N}\square i^N$, which is K_J-cellular by Theorem 3.4.13. By 3.3.15, Φ^N preserves the cell complex construction. $\square$

Proof of 3.3.19. (cf. [MM, 4.7]) The natural map α is J-equivariant by definition. It is an isomorphism for X and Y free G-spectra by 3.3.16. This implies that α induces a bijection of sets along which we identify $\Phi^N(\mathcal{F}^G I\square\mathcal{F}^G I)$ and $(\Phi^N\mathcal{F}^G I)\square(\Phi^N\mathcal{F}^G I)$. Abbreviate this set as $\hat{I}$. Let now X and Y be $\mathcal{F}^G I$-cellular, and chose specific cellular filtrations with indexing sets C respectively D for the attached cells. Then by Proposition 3.3.15, $\Phi^N X$ is $\Phi^N(\mathcal{F}^G I)$-cellular with the cells still indexed by the same set C, and similar for $\Phi^N Y$. Theorem 3.4.13 then gives explicit filtrations of $\Phi^N X\wedge\Phi^N Y$ and $(\Phi^N X\wedge Y)$ as $\hat{I}$-cellular objects with the same indexing set $C\times D$, and α exactly transports one filtration diagram into the other. Since retracts are preserved by any functor, this proves the proposition. $\square$

We will give more applications of Theorem 3.4.13 later, when we study filtrations of smash powers (cf. 3.4.22).

3.4.2 Equivariant Cellular Filtrations

We finally give the equivariant cellular structure for smash powers. Let L be an $\mathbb{S}$-cofibrant orthogonal spectrum. For G a finite group and X a finite G-set, we will give a filtration of the map

$$(\star\to L)^{\square X}\cong(\star\to L^{\wedge X})=\Lambda_X(L),$$

by $\mathbb{S}$-cofibrations of G-spectra using Theorem 3.4.13. It will follow from the construction that, when X is G-free, the attaching maps are all $\mathbb{S}$-cofibrations between induced regular $\mathbb{S}$-cofibrant G-spectra. This allows us to calculate the geometric fixed points of $\Lambda_X(A)$ along the lines of 3.3.41. Our methods are inspired by [Kr, 3.10.1], where a similar filtration is given for the case that $X=G=C_q$ a finite cyclic group.

Theorem 3.4.22. *Let G be a finite group and let X be a finite G-set. Then for L an $\mathbb{S}I$-cellular orthogonal spectrum, its X-fold smash power $L^{\wedge X}$ is an $\mathbb{S}I_G$-cellular G-spectrum.*
In particular let λ be an ordinal indexing the cells of L, then the X-fold product $\lambda^{\times X}$ has a G action and its set $\sigma = (\lambda^{\times X})_G$ of G-orbits indexes the $\mathbb{S}I_G$-cells of $L^{\wedge X}$, i.e.:

For every $[\alpha] \in \sigma$ there is a pushout diagram of G-spectra:

$$\begin{tikzcd}
G_+\wedge_H \left[\mathcal{G}_V^H(S^{n-1} \times \mathbf{O}_V/P)_+\right] \arrow[r] \arrow[d] & G_+\wedge_H \left[\mathcal{G}_V(D^n \times \mathbf{O}_V/P)_+\right] \arrow[d] \\
\bigcup_{[\beta]<[\alpha]} (L^{\wedge X})_{[\beta]} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & (L^{\wedge X})_{[\alpha]},
\end{tikzcd}$$

where the map in the top row is induced up from a generating $\mathbb{S}$-cofibration of H-spectra, for H the stabilizer subgroup of $[\alpha]$ in G. These diagrams define for each $[\alpha]$ in σ a subspectrum $(L^{\wedge X})_{[\alpha]}$ of $L^{\wedge X}$, such the union

$$\bigcup_{[\alpha]\in D} (L^{\wedge X})_{[\alpha]} = L^{\wedge X}.$$

Proof. Let $\star = L_0 \to L_1 \to \ldots \to L$ be the λ-sequence exhibiting L as $\mathbb{S}I$-cellular, i.e. the maps are cobase changes of generating $\mathbb{S}$-cofibrations. As in Remark 3.4.18, we think of the L_μ as subspectra of L, since the $\mathbb{S}$-cofibrations are in particular levelwise inclusions. Recall from Theorem 3.4.13 that as an orthogonal spectrum $L^{\wedge X}$ has a cellular filtration indexed by $\lambda^{\times X}$ equipped with the product partial order, i.e.

$$\beta = \{\beta_x\}_{x\in X} < \alpha = \{\alpha_x\}_{x\in X} \Leftrightarrow \beta_x < \alpha_x \quad \forall x \in X.$$

This implies that there is a $\lambda^{\times X}$-diagram of orthogonal spectra $L_\alpha^{\wedge X}$ such that

$$\operatorname*{colim}_{\alpha\in\lambda^{\times X}} L_\alpha^{\wedge X} \cong L^{\wedge X}.$$

Note that since the $\mathbb{S}$-model structure is monoidal, the maps $L_\beta^{\wedge X} \to L_\alpha^{\wedge X}$ for $\beta < \alpha$ are $\mathbb{S}$-cofibrations as well, so we similarly treat the $L_\alpha^{\wedge X}$ as subspectra $\bigwedge_{x\in X} L_{\alpha_x}$ of $L^{\wedge X}$, and their filtered colimits as unions.
Now let $\alpha = \{\alpha_x\}_{x\in X}$ represent an orbit $[\alpha] \in D$, we define

$$(L^{\wedge X})_{[\alpha]} := \operatorname*{colim}_{g\in G} L_{g\alpha}^{\wedge X} = \bigcup_{g\in G} L_{g\alpha}^{\wedge X} = \bigcup_{g\in G} \bigwedge_{x\in X} L_{\alpha_{g^{-1}x}},$$

which is obviously independent of the choice of representative of $[\alpha]$, and invariant under the G-action. Note that the $L_{[\alpha]}$ are partially ordered by inclusion, compatibly with

$$[\beta] < [\alpha] \quad \Leftrightarrow \quad g\beta < \alpha \qquad \forall g \in G.$$

We investigate the attaching maps closer. For each α_x there is a pushout diagram $\mathbf{D}_{\alpha_\mathbf{x}}$ of spectra

$$\mathbf{D}_{\alpha_\mathbf{x}} := \begin{tikzcd} \mathcal{G}_{V_x}(S^{n_x-1} \times \mathbf{O}_{V_x}/P_x)_+ \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{G}_{V_x}(D^{n_x} \times \mathbf{O}_{V_x}/P_x)_+ \arrow[d] \\ \bigcup_{\beta<\alpha_x} L_\beta \arrow[r] & L_{\alpha_x}. \end{tikzcd}$$

with the top map a generating $\mathbb{S}$-cofibration. By 3.4.6 we can take the row-wise $\square$-product of all the diagrams $\mathbf{D}_{\alpha_\mathbf{x}}$ for $x \in X$ to get a pushout diagram

$$\mathbf{D}_\alpha := \begin{tikzcd} \mathcal{G}_{V'}(S^{n'-1} \times \mathbf{O}_{V'}/P')_+ \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{G}_{V'}(D^{n'} \times \mathbf{O}_{V'}/P')_+ \arrow[d] \\ \bigcup_{\beta<\alpha} \bigwedge_{x\in X} L_{\beta_x} \arrow[r] & \bigwedge_{x\in X} L_{\alpha_x}. \end{tikzcd}$$

Here $V' = \bigoplus_{x\in X} V_x$, $n' = \sum_{x\in X} n_x$ and $P' = \prod_{x\in X} P_x$. Note that each of the corners in the diagram $\mathbf{D}_\alpha$ has an action of the stabilizer subgroup Stab_α of α in $\lambda^{\times X}$ induced by permuting smash factors, and the maps are all Stab_α-equivariant.
Now G acts on all these diagrams $\{\mathbf{D}_\alpha\}_{\alpha\in\lambda^{\times \mathbf{X}}}$ by permutation, and we can form the union over all the diagrams in the G-orbit of one particular diagram $\mathbf{D}_\alpha$ to get the final diagram $\mathbf{D}_{[\alpha]}$:

$$\mathbf{D}_{[\alpha]} : = \begin{tikzcd} G_+ \wedge_{\mathrm{Stab}_\alpha} \left[\mathcal{G}_{V'}(S^{n'-1} \times \mathbf{O}_{V'}/P')_+\right] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & G_+ \wedge_{\mathrm{Stab}_\alpha} \left[\mathcal{G}_{V'}(D^{n'} \times \mathbf{O}_{V'}/P')_+\right] \arrow[d] \\ \bigcup_{[\beta]<[\alpha]} (L^{\wedge X})_{[\beta]} \arrow[r] & (L^{\wedge X})_{[\alpha]} \end{tikzcd}.$$

By Remark 3.3.66, the induced map in the top row is an $\mathbb{S}$-cofibration of G-spectra as desired. Note that if α_x is a limit ordinal for any $x \in X$, the top row in $D_{[\alpha]}$ becomes trivial, hence the lower map is an isomorphism. $\square$

Looking closer at the cellular filtration given by Theorem 3.4.13 and Corollary 3.4.20, we can immediately get a relative version, admitting the same proof:

Proposition 3.4.23. *Let in the situation of Theorem 3.4.22 K be a sub-$\mathbb{S}I$-cell complex of L, then $K^{\wedge X}$ is an equivariant sub-$\mathbb{S}I_G$-cell complex of $L^{\wedge X}$, and for $i\colon K \to L$ the inclusion, so is the source of $i^{\square X}$.*

Proof. We can find a cellular filtration as in the proof of Theorem 3.4.22, with $K = L_\mu$ for some $\mu \in \lambda$. Then in the notation from above,

$$K^{\wedge X} = L^{\wedge X}_{[\hat{\mu}]},$$

where $\hat{\mu}$ is the element $\{\mu\}_{x\in X}$ of $\lambda^{\times X}$, and $i^{\square X}$ is the map

$$L^{\wedge X}_{[\delta]} \to L^{\wedge X},$$

where $[\delta]$ is the colimit in the partially ordered set σ over all those elements $[\alpha]$ such that $\alpha_x \le \mu$ for at least one $x \in X$. $\square$

Remark 3.4.24. Note that if L is built from K by attaching a single cell along a generating $\mathbb{S}$-cofibration i, then we can think of the map $\star \to K$ as another single cell as in Corollary 3.4.20, i.e. think of λ as having 2 elements, one representing the map $\star \to K$, and one representing i. For G a symmetric group Σ_n acting on a set X with n elements, σ also has n elements, and the cells one needs to attach to $K^{\wedge X}$ to get $L^{\wedge X}$ are all of the form

$$\Sigma_n \wedge_{\Sigma_m \times \Sigma_{n-m}} i^{\square m} \wedge K^{\wedge n-m}.$$

In the case where the action of G on X is free, we can further identify the cells in the filtration given by Theorem 3.4.22:

Proposition 3.4.25. *Let in the situation of Theorem 3.4.22 the G-set X be free, then the filtration is by induced regular cells, i.e. all the pushout diagrams $D_{[\alpha]}$ are of the form*

$$\mathbf{D}_{[\alpha]}\colon = \begin{array}{ccc} G_+\wedge_H \left[\mathcal{G}_V(S^{n-1}\times \mathbf{O}_V/P)_+\right]^{\square H} & \longrightarrow & G_+\wedge_H \left[\mathcal{G}_V(D^n\times \mathbf{O}_V/P)_+\right]^{\wedge H} \\ \downarrow & & \downarrow \\ \bigcup_{[\beta]<[\alpha]} (L^{\wedge X})_{[\beta]} & \longrightarrow & (L^{\wedge X})_{[\alpha]} \end{array},$$

where the top map is induced up from the H-fold $\square$-product of a generating $\mathbb{S}$-cofibration of orthogonal spectra.

Proof. We continue using the notation from the proof of Theorem 3.4.22. Recall that the top map in the attaching diagram D_α above was given by the iterated $\Box$-product of all the maps i_{α_X} for $x \in X$, with the Stab_α-action permuting $\Box$-factors that are identical. Since G acts freely on X, so does Stab_α. Therefore choosing any system of representatives R for the Stab_α-orbits of X, we get Stab_α-equivariant isomorphisms

$$\begin{tikzcd}
\mathcal{G}_{V'}(S^{n'-1} \times \mathbf{O}_{V'}/P')_+ \arrow[r] \arrow[d, "\cong"'] & \mathcal{G}_{V'}(D^{n'} \times \mathbf{O}_{V'}/P')_+ \arrow[d, "\cong"'] \\
{[\mathcal{G}_V(S^{n-1} \times \mathbf{O}_V/P)_+]^{\Box \mathrm{Stab}_\alpha}} \arrow[r] & {[\mathcal{G}_V(D^n \times \mathbf{O}_V/P)_+]^{\wedge \mathrm{Stab}_\alpha}},
\end{tikzcd}$$

where $V\colon = \bigoplus_{r\in R} V_r$, $n\colon = \sum_{r\in R} n_r$ and $P\colon = \prod_{r\in R} P_r$. $\Box$

Together with Theorem 3.3.65, we can now calculate the geometric fixed points of smash powers of $\mathbb{S}$-cofibrant spectra. Let as above

$$1 \to N \to G \xrightarrow{\epsilon} J \to 1,$$

be a short exact sequence of finite discrete groups, and X a finite free G-set.

Theorem 3.4.26. *For $\mathbb{S}$-cofibrant orthogonal spectra L and finite free G-set, there is a natural isomorphism of J-spectra*

$$\Lambda_{X_N} L \cong \Phi^N(\Lambda_X L).$$

Proof. This is again quite similar in spirit to the analogue theorem [Kr, 3.10.7]. It suffices to look at $\mathbb{S}I$-cellular L, since retracts are preserved by any functor. We keep the notation from the proof of Theorem 3.4.22 as far as possible, and let σ' denote the J-orbits of $\lambda^{\times X_N}$. The projection $\epsilon G \to J$ induces a "diagonal" map

$$\begin{array}{rccc} \epsilon^*\colon & \lambda^{\times X_N} & \to & \lambda^{\times X}, \\ & \{\kappa_{[x]}\}_{[x]\in X_n} & \mapsto & \{\kappa_{[x]}\}_{x\in X} \end{array}$$

which descends to orbits to give a map $\epsilon^*\colon \sigma' \to \sigma$. Note that in the cell structure for $\mathrm{L}^{\wedge X}$, cells that are not indexed by $\epsilon^*[\alpha]$ do not contribute to the geometric fixed points: By induction over the cellular filtration, assume that for all $[\beta] < [\alpha]$ in σ we have

$$\Phi^N(\bigcup_{[\delta]\le[\beta]} L^{\wedge X}_{[\delta]}) \cong \Phi^N(\bigcup_{[\epsilon^*\kappa]\le\beta} L^{\wedge X}_{[\epsilon^*\kappa]}),$$

then the same is true for β replaced by α: In the case $\alpha \in \epsilon^*\sigma'$ there is nothing to do, otherwise note that for $[\alpha] \notin \epsilon^*\sigma'$, the group N is not contained in Stab_α, hence in the attaching diagram $D_{[\alpha]}$, the top row has trivial geometric fixed points by 3.3.65. Since taking geometric fixed points commutes with the cell-complex construction, we therefore get colimit diagrams for $\Phi^N L^{\wedge X}$ and $L^{\wedge X_N}$ of exactly the same shape, with attaching diagrams indexed by $\epsilon^* \sigma' \cong \sigma'$. Then again by induction on the cellular filtration we show that the J-spectra $L_{[\kappa]}^{\wedge X_N}$ and $\Phi^N(L_{\epsilon^*[\kappa]}^{\wedge X})$ are isomorphic. Herefore compare the attaching diagrams:

$$\begin{tikzcd}
\bigcup_{[\gamma]<[\kappa]} \Phi^N (L^{\wedge X})_{[\epsilon^* \gamma]} \arrow[d, "\cong"'] & \Phi^N \left(G_+ \wedge_H [\mathcal{G}_V (S^{n-1} \times \mathbf{O}_{V/P})_+]^{\square H} \right) \arrow[l] \arrow[r] \arrow[d, "\cong"'] & \Phi^N \left(G_+ \wedge_H [\mathcal{G}_V (D^n \times \mathbf{O}_{V/P})_+]^{\wedge H} \right) \arrow[d, "\cong"'] \\
\bigcup_{[\gamma]<[\kappa]} (L^{\wedge X_N})_{[\gamma]} & J_+ \wedge_{J_0} [\mathcal{G}_V (S^{n-1} \times \mathbf{O}_{V/P})_+]^{\square J_0} \arrow[l] \arrow[r] & J_+ \wedge_{J_0} [\mathcal{G}_V (D^n \times \mathbf{O}_{V/P})_+]^{\wedge J_0}
\end{tikzcd}$$

where the vertical maps are isomorphisms by 3.3.65, hence induce an isomorphism on pushouts. Naturality is after the discussion so far only obvious for cellular morphisms. For all morphisms between $\mathbb{S}$-cofibrant orthogonal spectra and for all isomorphisms of finite free G-sets naturality follows from the construction 3.4.28, cf. Remark 3.4.32. $\square$

3.4.3 Fixed Points and the Loday Functor

The following proposition gives the important naturality property, which we will need when generalizing the above result to infinite and non-discrete spaces. It is inspired by Kro's proposed "diagonal map" $L^q \to \Phi^{C_r} L^{rq}$ in the case of finite cyclic groups. The definition given in [Kr, 3.10.4], however, seems to mix up the left and right adjoints involved. The present author is doubtful that a strict map can be defined in the way described there, even in the cyclic case, if L is not at least $\mathbb{S}$-cofibrant. We will instead define a natural zig-zag of maps, where the arrow in the wrong direction becomes an isomorphism for $\mathbb{S}$-cofibrant input.
Before we give the construction, recall the full subcategories of $\mathcal{O}_G^{\text{reg}} \subset \mathcal{O}_G$ and $\mathcal{O}_J^{\text{reg}} \subset \mathcal{O}_J$ associated to the sets X and X_N from Example 2.2.27. Let as usual $\mathcal{O}_E^{\text{reg}}$ be the N-fixed category of $\mathcal{O}_G^{\text{reg}}$, and note that the following diagram of functors commutes:

$$\begin{tikzcd}
\mathcal{O}_E \arrow[d, "\phi"'] & \mathcal{O}_E^{\text{reg}} \arrow[l, "i"'] \arrow[d, "\phi^{\text{reg}}"] \\
\mathcal{O}_J & \mathcal{O}_J^{\text{reg}} \arrow[l, "j"]
\end{tikzcd}$$

Where ϕ^{reg} is the restriction of ϕ, sending a regular representation $V^{\oplus X}$ to $(V^{\oplus X})^N$. Note that the latter is indeed regular and we usually identify $(V^{\oplus X})^N \cong V^{\oplus X_N}$. The diagram then implies the following natural isomorphisms for the restriction functors and their left adjoints:

$$\mathbb{P}_j \mathbb{P}_\phi^{\text{reg}} \cong \mathbb{P}_\phi \mathbb{P}_i \qquad \mathbb{U}_\phi^{\text{reg}} \mathbb{U}_j \cong \mathbb{U}_i \mathbb{U}_\phi. \tag{3.4.27}$$

Proposition 3.4.28. *For G a finite group and X a finite free G-space, and L any orthogonal spectrum there is a natural* diagonal zig-zag *of J-spectra*

$$\begin{tikzcd} L^{\wedge X_N} & \mathbb{P}_j\mathbb{U}_j L^{\wedge X_N} \arrow[l, "\varepsilon_j"'] \arrow[r] & \mathbb{P}_\phi\mathbb{P}_i\mathbb{U}_i\mathrm{Fix}^N(L^{\wedge X}) \arrow[r, "\mathbb{P}_\phi(\varepsilon_i)"] & \Phi^N(L^{\wedge X}) . \arrow[from=1-1, to=1-4, bend right, dotted, "\Delta(X,L)"'] \end{tikzcd}$$

For $\mathbb{S}$-cofibrant spectra L, the first map is an isomorphism such that the pointed composite $\Delta(X,L)$ exists and is the natural isomorphism from 3.4.26, which we therefore call the diagonal isomorphism*.*

Proof. The maps ε_j and ε_i are the counits of the adjoint pair $(\mathbb{P}_j, \mathbb{U}_j)$, respectively $(\mathbb{P}_i, \mathbb{U}_i)$ and are hence natural. The second map requires more work. First note that by (3.4.27) its target is naturally isomorphic to $\mathbb{P}_j\mathbb{P}_\phi^{\mathrm{reg}}\mathbb{U}_i\mathrm{Fix}^N(L^{\wedge X})$, so it suffices to define a natural map of regular J-spectra

$$\mathbb{U}_j L^{\wedge X_N} \to \mathbb{P}_\phi^{\mathrm{reg}}\mathbb{U}_i\mathrm{Fix}^N(L^{\wedge X}),$$

to which we then apply $\mathbb{P}_j$. We define the map levelwise, so let $U = V^{\oplus X_N}$ be a regular J-representation. By the coend definition of the smash products $L^{\wedge X}$ and $L^{\wedge X_N}$, as well as the evaluation of $\mathbb{P}_\phi^{\mathrm{reg}}$, it suffices to give morphisms

$$\bigwedge_{\bigoplus V_{[x]}=U} L_{V_{[x]}} \longrightarrow \left[\int^{W\in\mathcal{O}_E^{\mathrm{reg}}} \mathcal{O}_J^{\mathrm{reg}}(W^N, U^N) \wedge \int^{\bigoplus W_x \cong W} \mathbf{O}_W \wedge_{\prod_X \mathbf{O}_{W_x}} \bigwedge_{x\in X} L_{W_x} \right]^N . \tag{3.4.29}$$

But since U is regular, $W = V^{\oplus X}$ gives a preferred point in the first coend, and for each partition $\bigoplus V_x \cong U$ the choice $W_x = V_{[x]}$ gives a preferred point in the second coend, so that we can map $\bigwedge_{[x]\in X_N} L_{[x]}$ to the copy of $\bigwedge_{x\in X} L_[x]$ indexed by the identities of W^N and W, via the diagonal map

$$\begin{array}{ccc} \bigwedge_{[x]\in X_N} L_{[x]} & \longrightarrow & \bigwedge_{x\in X} L_{[x]}. \\ \{l_{[x]} \in L_{[x]}\}_{[x]\in X_N} & \mapsto & \{l_{[x]} \in L_{[x]}\})_{x\in X} \end{array}$$

Note that the map is obviously J-equivariant and maps into the coend of the fixed points, hence into the fixed points of the coend. It is compatible with the structure maps of L and natural with respect to all maps $L \to K$ of orthogonal spectra.
To see that the instance of ε_j is an isomorphism for $\mathbb{S}$-cofibrant L, note that it is an isomorphism for any semi-free J-spectrum $\mathcal{G}_K^V$ with V regular by (2.2.28). By Theorem 3.4.22, a cell induction then gives the result. Similarly it suffices to show

that for semi-free spectra the zig-zag yields the isomorphism from 3.3.41. For R a euclidean vector space and $L = \mathcal{G}_R K$, the maps (3.4.29) are represented by the map of $\prod_{X_N} \mathbf{O}_R$-spaces out of $K^{\wedge X_N}$ determined on the level $U = R^{\wedge X_N}$. Since this map is adjoint to the unit map

$$\eta_\phi \colon \mathcal{G}_U^E \left(\mathbf{O}_{R^{\wedge X}} \wedge_{\prod_X \mathbf{O}_R} K^{\wedge X}\right)^N \to \mathbb{U}_\phi \mathbb{P}_\phi \mathcal{G}_U^E \left(\mathbf{O}_{R^{\wedge X}} \wedge_{\prod_X \mathbf{O}_R} K^{\wedge X}\right)^N,$$

the result is implied by the Yoneda lemma and our description of the isomorphism (3.3.41). □

Corollary 3.4.30. *Let L and L' be $\mathbb{S}$-cofibrant orthogonal spectra, then the natural map α from 3.3.19 is an isomorphism*

$$\Phi^N(\Lambda_X L) \wedge \Phi^N(\Lambda_X L') \cong \Phi^N(\Lambda_X L \wedge \Lambda_X L').$$

Proof. Since $\Lambda_X L \wedge \Lambda_X L' \cong \Lambda_X(L \wedge L')$ via a G-equivariant shuffle permutation the map α is up to isomorphism the map

$$L^{\wedge X_N} \wedge L'^{\wedge X_N} \cong (L \wedge L')^{\wedge X_N}.$$

□

Corollary 3.4.31. *Let L be an $\mathbb{S}$-cofibrant orthogonal spectrum and let B be an* $\mathrm{Ind}^{\mathrm{reg}}$*-cellular G-spectrum, then the natural map α from 3.3.65 is an isomorphism*

$$\Phi^N(\Lambda_X L) \wedge \Phi^N(B) \cong \Phi^N(\Lambda_X L \wedge B).$$

Proof. First assume B is itself of the form $B \cong G_+ \wedge_H (L')^{\wedge H}$. If N is not contained in H, both source and target of α are trivial. Otherwise we have

$$L^{\wedge X} \wedge B \cong G_+ \wedge_H (L^{\wedge X} \wedge L'^{\wedge H})$$

by 3.3.45. Hence Theorem 3.3.65 gives that α is J-isomorphic to a map

$$L^{X_N} \wedge {}^{G}\!/_{N+} \wedge_{H/N} L'^{\wedge N} \cong {}^{G}\!/_{N+} \wedge_{H/N} (L^{X_N} \wedge L'^{N}),$$

which is another instance of 3.3.45 The general result follows by a cell induction over the cells of B. □

Remark 3.4.32. Since the Loday functor for mere spectra is in the X-variable only defined with respect to finite sets and isomorphisms between them, the best thing one can hope for is that the diagonal zig-zag of 3.4.28 is natural in the X variable with respect to isomorphisms of finite free G-sets. Then indeed, comparing the appropriate shuffle permutations through which G and J act, naturality in X is immediate for semi-free spectra, and can be followed through the coends in the proof of 3.4.28 with little more effort.

Corollary 3.4.33. *The diagonal isomorphism respects decompositions of X into G-orbits, i.e. for X a finite free G-set and L an $\mathbb{S}$-cofibrant orthogonal spectrum, we get a commutative diagram of natural isomorphisms*

$$\begin{tikzcd}
(X^{\wedge G/N})^{\wedge X_G} \arrow[d, "\Delta(G,L)^{\wedge X_G}"'] & L^{\wedge X_N} \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[dd, "\Delta(X,L)"] & (L^{\wedge X_G})^{\wedge G/N} \arrow[dd, "\Delta(G,L^{\wedge X_G})"] \\
(\Phi^N L^{\wedge G})^{\wedge X_G} \arrow[d, "\alpha"'] & & \\
\Phi^N((L^{\wedge G})^{\wedge X_G}) & \Phi^N L^{\wedge X} \arrow[l, "\cong"'] \arrow[r, "\cong"] & \Phi^N((L^{\wedge X_G})^{\wedge G})
\end{tikzcd}$$

with the lower left vertical map an iterated version of the isomorphism α from 3.3.19.

Proof. Note that for L $\mathbb{S}$-cofibrant, a map out of $L^{\wedge X_N}$ is completely determined by its values on X_N-regular levels by the universal properties of the semi-free spectra appearing in the cell decomposition of Theorem 3.4.22. On these levels the right rectangle commutes by definition of the diagonal zig-zag. For the left rectangle, comparing the universal property defining α from 3.3.6 to the definition of the middle map in the diagonal zig-zag 3.4.28 gives the result. □

Varying the Input Categories

In 3.2, we have defined several versions of the Loday functor, with the input categories varying between finite discrete sets together with general orthogonal spectra, to general spaces and commutative orthogonal ring spectra. In the previous discussion we have restricted the first input further, to finite free G-sets, to study the equivariant structure induced on the output. Inspired by Remark 3.4.32, we will from now view the diagonal isomorphism as a natural transformation of functors

$$\Delta(\cdot,-)\colon \quad \Lambda_{(\cdot)_N}(-) \;\longrightarrow\; \Phi^N(\Lambda_{(\cdot)}(-)),$$

and study in how far we can vary the input categories.

We begin with checking naturality of the diagonal isomorphism with respect to injections of finite free G-sets. As usual, we have to adapt the cofibrancy condition.

Definition 3.4.34. An orthogonal spectrum L is $\mathbb{S}$-*cofibrant under* $\mathbb{S}$ if it is equipped with a designated $\mathbb{S}$-cofibration $\mathbb{S} \to L$.

Example 3.4.35. Every $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum is $\mathbb{S}$-cofibrant under $\mathbb{S}$ via its unit map by 1.3.22, which in particular says that the unit maps for the commutative ring spectra appearing in the generating $\mathbb{S}$-cofibrations is a *positive* $\mathbb{S}$-cofibration.

Lemma 3.4.36. *The diagonal map is natural with respect to finite free G-sets and equivariant inclusions, and spectra $\mathbb{S}$-cofibrant under $\mathbb{S}$.*

Proof. Let L be cofibrant under $\mathbb{S}$, and let $X \to Y$ be an equivariant inclusion of finite free G-sets. There is an $\mathbb{S}I$-cellular structure for L such that the designated map $\mathbb{S} \to L$ is an inclusion of a subcomplex. Then $L^{\wedge X} \to L^{\wedge Y}$ is an inclusion of an equivariant subcomplex as in 3.4.23. Similarly $L^{\wedge X_N}$ is an equivariant subcomplex of $L^{\wedge Y_N}$ and it follows as from the proof of 3.4.26 that the diagonal isomorphism respects the inclusion of subcomplexes. $\square$

Moving towards infinite free G-sets X, the Loday functor is defined in 3.2.3 as via the colimit along inclusions of finite free G-subsets of X.

$$\Lambda_X(L) := \operatorname*{colim}_{F\subset X \text{ finite}} \Lambda_F L.$$

Note that by the proof of the previous lemma, this is a filtered colimit along h-cofibrations, so that it is preserved by Φ^N.

Lemma 3.4.37. *The diagonal isomorphism exists and is natural with respect to free G-sets and equivariant inclusions, and spectra $\mathbb{S}$-cofibrant under $\mathbb{S}$.*

Proof. We begin with the existence. Let L be cofibrant under $\mathbb{S}$, and let X be a free G-sets. The finite subsets of X_N are orbits of finite subsets of X hence there is a natural map $\Lambda_{X_N} L \to \Phi^N(\Lambda_X L)$ which is the colimit of isomorphisms hence an isomorphism itself. Naturality follows since equivariant inclusions induce inclusions of indexing categories for the colimits. $\square$

As an alternative proof, we can use Corollary 3.4.33: Let $f\colon X \to Y = X \cup Z$ be an equivariant inclusion of free G-sets. Then f respects the orbit decomposition, i.e.

$$X \cong_G \bigcup_{X_G} G, \qquad Y \cong_G \bigcup_{X_G} G \cup \bigcup_{Z_G} G,$$

and f corresponds to the obvious inclusion. Hence $\Lambda_f L$ is isomorphic to the map

$$\Lambda_G(\Lambda_{X_G} L) \cong \Lambda_G(\Lambda_{X_G} \wedge \Lambda_{Z_G}\mathbb{S}) \to \Lambda_{Y_G}(\Lambda_{Y_G} L),$$

i.e. it is the smash power of a map of $\mathbb{S}$-cofibrant spectra, so 3.4.28 gives the result. In particular, we even get that the map $\Lambda_{X_G} L \to \Lambda_{Y_G} L$ is a (non equivariant) $\mathbb{S}$-cofibration, thus the induced map of smash powers is an $\mathbb{S}$-cofibration of G-spaces by 3.3.66. To see this, filter Y_G through finite sets Y_i and let $X_i = f^{-1}Y_i$, $Z_i = Y_i \backslash f(X_i)$. As in 3.4.23, each of the maps $L^{\wedge X_i} \cong L^{\wedge X_i} \wedge \mathbb{S}^{\wedge Z_i} \to L^{\wedge Y_i}$ is an $\mathbb{S}$-cofibration, hence so is their colimit.
Finally we move towards $\mathbb{S}$-cofibrant commutative orthogonal ring spectra, where

we want to work with the definition of the Loday functor in terms of the categorical tensor with spaces, i.e. for X a space and A an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum, $\Lambda_X A = A \otimes X$, where the tensor is in the category of commutative orthogonal ring spectra (cf. 3.1.2). As discussed in 3.1.14, the tensor specializes to the smash power for discrete inputs X, so all the results from above still apply. Note that we can now extend the naturality results to not necessarily injective maps:

Lemma 3.4.38. *The diagonal isomorphism from 3.4.37 is natural with respect to free G-sets and equivariant maps, and $\mathbb{S}$-cofibrant commutative orthogonal ring spectra.*

Proof. Let $f\colon X \to Y$ be an equivariant map between free G-sets. Similar to the above discussion, we filter X and Y by finite free G-sets X_i and Y_i and consider f as a colimit of maps $f_i\colon X_i \to Y_i$, where the transformations $\lambda_{f_i} A \to f_j$ for $i \leq j$ are along $\mathbb{S}$-cofibrations. Thus it suffices to check naturality for not necessarily injective equivariant maps between finite free G-sets. Here we can once again use the splitting into orbits, and the fact that the diagonal map is natural with respect to all morphisms between $\mathbb{S}$-cofibrant spectra. As in Corollary 3.4.33, we see that for $X \to Y$ an equivariant morphism of free G-sets, the map on smash powers $\Lambda_X A \to \Lambda_Y A$ is the G-fold smash power of the map $\Lambda_{X_G} A \to \Lambda_{Y_G} A$, hence the result follows. □

We can finally move on towards non discrete spaces. We begin with spaces that are geometric realizations of simplicial sets, since there we have Proposition 3.1.11, which makes computing the Loday functor much easier and in particular allows the following extension of the diagonal isomorphism:

Proposition 3.4.39. *For free G-simplicial sets X_* and equivariant maps between them, and $\mathbb{S}$-cofibrant commutative orthogonal ring spectra A, the diagonal map exists and is a natural isomorphism*

$$\Lambda_{|(X_*)|_N} A \cong \Phi^N(\Lambda_{|X_*|} A).$$

Proof. By 3.1.11, for a free G-simplicial set X_*, and an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum A, the tensor $A \otimes |X_*|$ is naturally isomorphic to the realization of the simplicial orthogonal spectrum

$$q \mapsto (A \otimes X_q) \cong A^{\wedge X_q} \cong \Lambda_{X_q} A.$$

By B.1.44, the geometric realization of this spectrum is the colimit along the skeleton filtration, which is along levelwise h-cofibrations since the simplicial spectrum is h-good (cf. B.1.49, B.1.50 and B.1.46): Every simplicial degeneracy map s_i is an injection of free G-sets, hence as in the comment before Lemma 3.4.37, it induces an

$\mathbb{S}$- hence h-cofibration under the Loday functor. In particular, taking the geometric fixed points commutes with the geometric realization, since Fix^N does. Therefore the diagonal maps $\Delta(X_q, A)$ for each simplicial level induce an isomorphism on realizations. It is natural since maps of free G-simplicial sets are levelwise maps of free G-sets. □

Remark 3.4.40. For X the realization of a simplicial set, the diagonal map constructed in 3.4.39, does not depend on the simplicial model. Given two simplicial sets X_* and Y_* such that $|X_*| \cong |Y_*|$, there is a zig-zag of maps of simplicial sets between them, that realizes to the isomorphism, e.g. via the singular complex of $|Y_*|$. Thus we can use the naturality for simplicial maps.

We can continue this to work towards general cofibrant free G-spaces X, in particular since the generating naive cofibrations are given by the inclusions $G \wedge (S^{n-1} \to D^n)$, which are the realizations of simplicial maps $G \wedge (\partial\Delta^n \to \Delta^n)$, we can write every cofibrant G-space as a colimit of pushouts of maps between spaces that are realizations of free G-simplicial sets. Note that tensoring with A of course preserves this colimit, but also maps it to a colimit along $\mathbb{S}$-cofibrations since the $\mathbb{S}$-model structure on commutative orthogonal ring spectra satisfies the pushout product axiom. In particular particular we can take geometric fixed points before going to the colimit, hence use the lemma for the simplicial case and a cell induction to get the following:

Lemma 3.4.41. *For naively cofibrant G-spaces X and equivariant maps between them, and $\mathbb{S}$-cofibrant commutative orthogonal ring spectra A, the diagonal isomorphism exists and is natural with respect to morphisms $X \to Y$ that are realizations of simplicial maps.*

Remark 3.4.42. Note that every equivariant map between naively cofibrant G-spaces is homotopy equivalent to the realization of a simplicial map via the unit of the adjunction between spaces and simplicial maps and the general Whitehead theorem B.1.3 for the naive model structure. Since the Loday functor is continuous in both variables, this implies that the diagonal isomorphism of 3.4.41 is natural with respect to all continuous equivariant maps *up to homotopy equivalence*. The homotopy equivalence can be seen as one of orthogonal spectra which at each time is a morphism of ring spectra (cf. proof of 3.1.13).

This concludes our study of the case of a fixed finite group, since we have reached the other end of the generality of the definition of the Loday functor from 3.2. So far we have not touched upon functoriality of the Loday functor or naturality of the diagonal map for *changing the group*, so let now $\phi\colon H \to G$ be a homomorphism of topological groups. As usual we can look at the restriction functor $\phi^*\colon G\mathfrak{T} \to H\mathcal{T}$, and it is immediate, that it commutes with the Loday functor in the sense that for orthogonal ring spectra A, there is a natural isomorphism of commutative H-ring spectra

$$\Lambda_{\phi^* X} A \cong \phi^* \Lambda_X A,$$

where on the right side ϕ^* is the restriction functor on commutative ring spectra, analogous to 2.2.5. Note, that since ϕ^* does not send free G-sets to free H-sets if ϕ is not injective, we have in general no control over the diagonal map. Therefore we will only consider the case where ϕ is the inclusion $i\colon H \to G$ of a subgroup. For $N \subset H$, this leads to the following:

Proposition 3.4.43. *The restriction functor preserves the diagonal map.*

Proof. Recall from Remark 3.3.50 that Condition 1 is always satisfied in the case of finite groups. Hence as in the proof of Proposition 3.3.57 the right half of (3.3.58) commutes, i.e. the restriction commutes with $\mathbb{P}_\phi$. Therefore all the functors in the definition of the diagonal zig-zag commute with the restriction, and so does the whole zig-zag. The restriction functor preserves the colimit along the inclusions of finite subsets as well as the geometric realization and the cell complex construction, which we used to extend the diagonal map from the case of spectra above. □

Remark 3.4.44. The relation of the Loday functor with the induction of an H-set X along the inclusion $i\colon H \to G$ is more subtle. Intuitively the Loday functor itself should be viewed as the $\wedge$-induction of a spectrum with action of the trivial group to a spectrum with G-action. If one wants to start with an H-spectrum instead, this generalizes the study of multiplicative norm constructions, where X is assumed to be a discrete subgroup H of G. These norm functors have been famously put to use in the recent proof of the Kervaire-invariant problem by Hill, Hopkins and Ravenel. An introduction can be found in [HRR, A.3,4], or [S11, 8,9], both of which only became available very shortly before this thesis was finished. The author learned about the interplay from Stefan Schwede, during a visit in Bonn in November 2010, where he presented the results of this thesis. As the study of multiplicative norms of course has to address some of the same questions we discussed here, we point out some similarities and differences to [HRR]. Due to the fact that the works are independent, the notation and viewpoint are rather different. First of all one should note the difference in model structures. We worked with the $\mathbb{S}$-model structure instead of the classical q-model structure on commutative orthogonal ring spectra, in order to get around the q-cofibrant replacement (cf. 3.3.22). [HRR] address this problem by proving that the symmetric powers appearing in the generating q-cofibrations are "very flat" (cf. [HRR, B.13,63]), which allows them to construct a natural weak equivalence calculating geometric fixed points. Their method has the advantage that it is more easily applicable to the general multiplicative norm case they aim to study. Our method on the other hand allows us to recognize the diagonal map as a natural *isomorphism*, strengthening their statement. Note that the “Slice Cells” discussed in [HRR, 4.1] are special cases of generating $\mathbb{S}I_G$-cofibrations in our language, and from this viewpoint the filtration given in [HRR, A.4.3] and our Theorem 3.4.22 achieve similar goals – an equivariant filtration of the smash power – with different methods. Finally note that the change of the indexing of the smash power away from a non discrete set we worked for in this section, is only addressed in the side note [HRR, A.35]. Since all groups discussed there are finite, this is not a major point in [HRR], but as we are going to move towards tori and more general compact Lie groups now, the details become important.

Infinite Groups

We now leave the realm of finite groups and move back to the case of compact Lie groups that is the main focus of our results. In particular to deal with topological Hochschild homology and (higher) Cyclic homology, we are interested in the case where G is a torus. The first thing we should address is that Condition 1 is actually satisfied in these cases:

Lemma 3.4.45. *Condition 1 is satisfied when G is the n-torus and H is the kernel of an isogeny of G, in particular for $G \cong S^1$ and H a finite subgroup.*

Proof. Let $G = S^1 \times \ldots \times S^1 = \mathbb{T}^n$. Observe that the group structure on $\mathbb{T}^n$ is inherited from $(\mathbb{R}^n, +)$ via the isomorphisms

$$\begin{array}{rcl} \mathbb{R}/\mathbb{Z} & \cong & S^1 \\ a+b & \mapsto & [a][b] \\ k \cdot a & \mapsto & [a]^k \quad \text{for} \quad k \in \mathbb{Z} \end{array}$$

Given an isogeny α of $\mathbb{T}^n$, its kernel L is a finite subgroup. Note that there is a matrix $A = (a_{i,j}) \in M_n(\mathbb{Z}) \cap Gl_n(\mathbb{Q})$ such that $\alpha : \mathbb{T}^n \to T^n$ is induced by $(A\cdot) : \mathbb{R}^n \to \mathbb{R}^n$. Under this correspondence, the kernel of α is isomorphic to the projection of $A^{-1}\mathbb{Z}^n \subset \mathbb{Q}^n \subset \mathbb{R}^n$ to $\mathbb{R}^n/\mathbb{Z}^n$. Note that $A^{-1}\mathbb{Z}^n$ is finitely generated by the columns $c_1, \ldots c_n$ of A^{-1}. The orders ρ_i of these generators in $\mathbb{R}^n/\mathbb{Z}^n$ are given by the least common multiple of the denominators of entries $c_{i,j}$ of the respective columns c_i. Let W be an H-representation via $\phi : H \to \mathbf{O}_W$. We can restrict ourselves to irreducible representations W, hence assume that $\dim W \leq 2$. (The orthogonal matrices $\phi(l_i)$ commute and hence can be brought to normal form simultaneously). If $W = \mathbb{R}$ is the trivial L-representation, we can define $V := W$ as the trivial G-representation and are done. If W is one dimensional, define $V := \mathbb{C} \cong_{\mathbb{R}} \mathbb{R}^2$ with the standard metric and let $W \to V$ be the embedding as the real line. We prolong ϕ to $\mathbf{O}_V$, by mapping the non-trivial element of $\mathbf{O}_W$ to the rotation of order 2. For W of dimension 2, note that no element of H can be mapped to an element with negative determinant of $\mathbf{O}_W$ – if it does it has two different eigenspaces which have to be preserved by all other elements of $\phi(H)$, hence W could not have been irreducible. The generators of H have to map to elements of $\mathbf{O}_\mathbb{C}$ of order that divides their own. In particular $\phi(c_i)$ acts on $\mathbb{C}$ by multiplication with a root of unity $\zeta_{\rho_i}^{k_i}$, such that $\phi(c_i)^x$ is well defined for $x \in \mathbb{R}$. Define an action of $\mathbb{T}^n$ on $V = \mathbb{C}$ by:

$$\begin{array}{ccc} \mathbb{R}^n/\mathbb{Z}^n \times V \xrightarrow{A\cdot\times V} \mathbb{R}^n/A\mathbb{Z}^n \times V & \xrightarrow{\mu} & V \\ [(x_1,\ldots,x_n)], z & \longmapsto & \prod_i \phi(c_i)^{x_i}\cdot z \end{array}$$

The definition of the lower map obviously gives an action of $(\mathbb{R}^n, +)$ on V, since the $\phi(c_i)$ commute. To see that it descends to an action of $\mathbb{T}^n$, we have to show that it is trivial on $\mathbb{Z}^n$. Let e_j be one of the standard base vectors of $\mathbb{R}^n$. Then multiplication with A takes e_j to the coordinate vector $(a_{1,j},\ldots,a_{n,j})$. We need to check that $\prod_i \phi(c_i)^{a_{i,j}} = 1$ in $\mathbb{C}$:

In $H \subset \mathbb{R}^n/\mathbb{Z}^n$ we can form the $\mathbb{Z}$-linear combination $\Sigma_i a_{i,j} c_i$. Since the c_i were the columns of A^{-1} this gives back the standard basis vector e_j which is congruent to 0 modulo $\mathbb{Z}^n$, i.e. the $\mathbb{Z}$-linear combination $\Sigma_i a_{i,j} l_i$ is equal to the unit in L. Since ϕ is a group homomorphism this implies that $\prod_i \phi(l_i)^{a_{i,j}} = 1$ in $\mathbf{O}_\mathbb{C}$, i.e. $\prod_i \phi(c_i)^{a_{i,j}} = 1$ in $\mathbb{C}$ as desired. This action extends the action of W by construction, since a generator c_i of H maps to e_i under the multiplication with A, hence acts via $\phi(c_i)$ on V. The fixed points of V under the $\mathbb{T}^n$ action are trivial, since no $\phi(c_i)$ that is not trivial fixes a point except the origin. □

We will need some more properties of tori, we begin with the one dimensional case.

Lemma 3.4.46. *Let C_n be a finite (cyclic) subgroup of order n in S^1. There is a simplicial C_n-set $(S_n)_*$ such that $|(S_n)_*| \cong S^1$.*

Proof. This is easily done by hand, or by applying edgewise subdivision to the standard simplicial model of S^1. □

Lemma 3.4.47. *Let H be the kernel of an isogeny α of the torus $\mathbb{T}^n$. There is a simplicial H-set T_* such that $|T_*| \cong \mathbb{T}^n$.*

Proof. We combine the methods of the two lemmas above. For the intuition that underlies the following, it is best to think of $\mathbb{R}^n$ as the n-fold product of the infinite simplicial complex $\mathbb{R}$, which has vertices lying on the points in $\mathbb{Z}$, and edges between them. We identify the action of H on this complex, and then produce a finer complex where the action is simplicial. As above associate to α an integer matrix $A \in M_n(\mathbb{Z}) \cap Gl_n(\mathbb{Q})$. In the notation above, the action of an element H on an element $[r] \in {}^{R^N}\!/_{Z^N}$ corresponds to adding a linear combinations of columns of A^{-1} to r. All the columns of A^{-1} are in $\det(A) \cdot \mathbb{Z}^n$ by Cramer's rule. Hence the difference of r to its image under the action of any $h \in H$ is in $\det(A) \cdot \mathbb{Z}^N$ as well. Thus let T_* be the n-fold product of simplicial sets

$$T_* = S_* \times \ldots \times S_*,$$

where S_* is a $C_{\det A}$-equivariant simplicial model of S^1. The action of H on the realization corresponds to a simplicial action on the resulting simplicial complex, hence T_* is the desired H-simplicial set. □

Finally we can state the result about the diagonal map for our main case of interest:

Theorem 3.4.48. *Let $G = \mathbb{T}^n$ and N the kernel of an isogeny (such that Condition 1 is satisfied). For an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum A and a naively cofibrant G-space X, there is an isomorphism of ${}^{G}\!/_{N}$-spectra*

$$\Lambda_{X_N} A \to \Phi^N \Lambda_X.$$

The isomorphism is natural, and restricts to the diagonal isomorphism under the restriction of the G-action to any finite subgroup of ${}^{G}\!/_{N}$.

Proof. We begin with constructing the map and check equivariance afterwards. Restricting to an arbitrary finite subgroup K of G that contains H, Proposition 3.4.39 gives a diagonal map

$$\Lambda_{X_N} A \to \Phi^N \Lambda_X N$$

of ${}^{K}\!/_{N}$-spectra. For K_1 and K_2 two such subgroups, the two maps they define restrict to the same map of ${}^{K_1 \cap K_2}\!/_{N}$-spectra, since the diagonal maps are preserved under restriction (3.4.43). In particular all of these maps restrict to the same underlying map of spectra. Note that for normal subgroups $N \subset G$ an element $[g]$ in the quotient group ${}^{G}\!/_{N}$ has finite order, if and only if the subgroup generated by g intersects H non trivially, and in particular if and only if the subgroup generated by $\{g\} \cup N$ contains N as a subgroup of finite index. This implies that we constructed a map between spectra with J-action, which is equivariant with respect to the action of all points in J that have finite order. Since ${}^{G}\!/_{H}$ is isomorphic to G, we know that the points of finite order are exactly the rational points in ${}^{G}\!/_{H} \cong \mathbb{T}^n$. Since the rational points are dense in $\mathbb{T}^N$, and the actions on J-spectra are continuous, the map is indeed J-equivariant. □

Since every element of a compact Lie group lies in a maximal torus, and in particular contains points of finite order in every one of its neighborhoods, this argument can be used in more general settings, as soon as one has control over Condition 1 and a property analogous to 3.4.47:

Condition 2. Let G be a topological group and H a subgroup. Assume there exists a simplicial H-set G_*, such that there is a H-equivariant isomorphism

$$|G_*| \cong G$$

.

Remark 3.4.49. As we have seen above, the tori $\mathbb{T}^n$ satisfy Condition 2 with respect to all kernels of isogenies. To the knowledge of the author, the completely general case is unknown. Note that Illman's triangulation theorem C.2.4 constructs an H-equivariant triangulation of G, but since the usual methods that produce simplicial sets from simplicial complexes fail for Illmans equivariant simplices ([Ill83, §3]), this is not enough.

Theorem 3.4.50. *Let G be a compact Lie group, and N a normal subgroup. Assume that G satisfies Conditions 1 and 2 with respect to all subgroups K containing N, such that N has finite index in K. For an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum A and a naively cofibrant G-space X, there is an isomorphism of ${}^{G}\!/_{N}$-spectra*

$$\Lambda_{N} A \to \Phi^N \Lambda_X N.$$

The isomorphism is natural, and restricts to the diagonal map under the restriction of the action to any finite subgroup of ${}^{G}\!/_{N}$.

Before we end this section, let us briefly say something about the change of base rings. Recall that we defined R-model structures for the categories of R-modules for R a commutative orthogonal ring spectrum. In particular, the generating R-cofibrations were given by smashing R with the generating $\mathbb{S}$-cofibrations. Recall that the category of R-modules is symmetric monoidal with respect to the monoidal product $\wedge_R$, defined via the coequalizer diagram A.1.17, and we defined the R-Loday functor using this product in 3.2. We would like to state a result analogous to Theorem 3.4.26, but there is an obstruction: There is a priori no guarantee that for an arbitrary, or even for an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum the geometric fixed points of R equipped with the trivial G-action, are isomorphic to R itself. Note that the sphere spectrum of course has this property by 3.3.16, since $\mathbb{S} \cong \mathcal{F}^0_{S^0}$. Excluding this case, we still get the following

Theorem 3.4.51. *Let G be a compact Lie group and N a normal subgroup. Assume that G satisfies Conditions 1 and 2 with respect to all subgroups K containing N,* *such that N has finite index in K. Let R be an $\mathbb{S}$-cofibrant orthogonal ring spectrum such that $\Phi^N R \cong R$. There is a natural isomorphism*

$$\Lambda^R_{X_N} A \cong \Phi^N(\Lambda^R_X A),$$

if A is an R-cofibrant commutative R-algebra and X a naively cofibrant G-space, or if A is an $\mathbb{R}$-cofibrant R-module and X is a finite free G-set.

Proof. For the spectrum case, since L is R-cofibrant, it is isomorphic to $R \wedge K$ with K an $\mathbb{S}I$-cellular spectrum. Applying The smash power $L^{\wedge_R X}$ is then isomorphic to $R \wedge K^{\wedge X}$. Here G acts trivially on R and in the usual way on the smash power of K. Then by 3.3.60 and 3.4.31 the result follows. For the algebra case we can follow the discussion for $R = \mathbb{S}$ above. $\square$

3.5 Homotopical Properties

We finally turn to investigating the homotopy theoretical properties of the Loday functor. For one, this this will allow us to establish the comparison result to the BCD model. On the other hand it is very good to know, that the $\mathbb{S}$-model structures and in particular the induced regular cells are sufficiently well behaved to allow for the standard tools of equivariant stable homotopy theory to apply, without having to resort to q-cofibrant replacements.

3.5.1 Homotopy Groups

The main point we want to establish is the characterization of geometric fixed points for smash powers of 3.5.13. But before we can give precise statements, we have to once again recall definitions from [MM, V.4], the first one being homotopy groups for $\mathcal{O}_E$-spectra, geometric homotopy groups, and several homomorphisms between the different homotopy groups associated to a G-spectrum. Let as usual

$$E: \qquad 1 \longrightarrow N \longrightarrow G \xrightarrow{\epsilon} J \longrightarrow 1$$

be a sequence of compact Lie groups.

Definition 3.5.1. [MM, 4.8] Let Y a $\mathcal{O}_E$-spectrum and X an orthogonal G-spectrum. Let $H/N = K \subset J$, with H a subgroup of G containing N.

(i) Define the *homotopy groups of* Y via

$$\pi_q^K Y := \begin{cases} \operatorname{colim}_{V \in \mathcal{O}_E} \pi_q(\Omega^{V^N} Y_V)^K & \text{if } q \geq 0 \\ \operatorname{colim}_{\mathbb{R}^q \subset V \in \mathcal{O}_E} \pi_0(\Omega^{V^N - \mathbb{R}^q} Y_V)^K & \text{if } q \leq 0 \end{cases}$$

(ii) Define a natural homomorphism

$$\zeta\colon \pi_*^K(\mathbb{U}_\nu Y) \to \pi_*^K(Y),$$

by restricting the defining colimit system of $\pi_*^K Y$ to only those V such that $V = \nu^* W$, i.e. $V = V^N$.

(iii) Define the *geometric homotopy groups of* X as

$$\rho_q^K(X) := \pi_q^K(\mathrm{Fix}^N X).$$

(iv) Define a natural homomorphism

$$\psi\colon \pi_*^K(X^N) \to \pi_*^H(X),$$

by restricting the defining colimit system of $\pi_*^H(X)$ to N-fixed V as above, using that for these

$$(\Omega^V X_V^N)^K \cong (\Omega^V X_V)^H.$$

(v) Define a natural homomorphism

$$\omega\colon \pi_*^H(X) \to \rho_*^K(X),$$

by sending an element of $\pi_q^H(X)$ represented by an H-equivariant map $f\colon S^q \wedge S^V \to X_V$ to the element of $\rho_q^K(X)$ represented by the K-equivariant map $f^N\colon S^q \wedge S^{V^N} \to X_V^N$ for $q \geq 0$ and similar for $q \leq 0$.

(vi) A morphism of $\mathcal{O}_E$-spectra is a π_*-*isomorphism* if it induces isomorphisms on all homotopy groups π_*^K, a morphism of orthogonal G-spectra is a ρ_*-*isomorphism* if it induces isomorphisms on all geometric homotopy groups ρ_*^K.

Note that ψ is a natural isomorphism if X is a G-Ω-spectrum, in particular we can use the fixed point spectra of a fibrant approximation of X to calculate the equivariant homotopy groups of X. This implies the following characterization

Lemma 3.5.2. *Let $\mathscr{H}$ be a closed family of normal subgroups of G. A morphism f of G-Ω-spectra is an $\pi_*^{\mathscr{H}}$-isomorphism if and only if $\mathrm{Fix}^H f$ is a non equivariant π_*-isomorphism for all $H \in \mathscr{H}$.*

We continue to follow through the arguments in [MM, V.4]:

Lemma 3.5.3. *[MM, V.4.9] The natural homomorphisms in Definition 3.5.1 are related via*

$$\zeta = \omega \circ \psi.$$

Lemma 3.5.4. *[MM, V.4.10] For orthogonal J-spectra Z, the homomorphism*

$$\zeta\colon \pi_*^K(Z) = \pi_*^K(\mathbb{U}_\nu \mathbb{U}_\phi Z) \to \pi_*^K(\mathbb{U}_\phi Z)$$

is an isomorphism.

This allows the following definition:

Definition 3.5.5. For orthogonal G-spectra X, define a natural homomorphism

$$\eta_*\colon \rho_*^K(X) \xrightarrow{=} \pi_*^K \mathrm{Fix}^N X \xrightarrow{\eta_\phi} \pi_*^K(\mathbb{U}_\phi \mathbb{P}_\phi \mathrm{Fix}^N X) \xrightarrow{\zeta^{-1}} \pi_*^K(\mathbb{P}_\phi \mathrm{Fix}^N X) \xrightarrow{=} \pi_*^K(\Phi^N X),$$

where η_ϕ is the unit of the adjoint pair $(\mathbb{P}_\phi, \mathbb{U}_\phi)$ from 3.3.8.

Recall the map γ from 3.3.13.

Lemma 3.5.6. *[MM, V.4.11] Let $K = {}^{H}\!/_{N}$, with $N \subset H$. For orthogonal Ω-G-spectra X, the map $\gamma_*\colon \pi_*^K(X^N) \cong \pi_*^K(\Phi^N X)$ is the composite*

$$\pi_*^K(X^N) \xrightarrow{\cong} \pi_*^H(X) \xrightarrow{\omega} \rho_*^K(X) \xrightarrow{\eta^*} \pi_*^K(\Phi^N X).$$

The following lemma is the first point where we have to adapt the argument to fit our more general cells:

Proposition 3.5.7. *cf. [MM, V.4.12] The map $\eta_*\colon \rho_*^K(X) \to \pi_*^K(\Phi^N X)$ is an isomorphism for $(\mathrm{Ind}^{\mathrm{reg}} \cup \mathcal{F}I_G)$-cellular orthogonal G-spectra X.*

Proof. As in the classical case, because Φ^N preserves cofiber sequences, wedges, and colimits of sequences of h-cofibrations, it suffices to check that η_* is an isomorphism on all objects of the form

$$X := G_+ \wedge_P (\mathcal{G}_V(L)^{\wedge P}),$$

with L a genuine $\mathbf{O}_V \rtimes P$-cell complex. Note that if N is not a subgroup of P, then $\mathrm{Fix}^N X$ is trivial, hence there is nothing to prove. Otherwise, as in the proof of Proposition 3.3.41 we get

$$\mathrm{Fix}^N X = J_+ \ \wedge_{J_1} \ \mathcal{G}^E_{V^{\oplus P}} \left[\mathbf{O}_{V^{\oplus P}} \ \wedge_{\prod\limits_{J_1} \mathbf{O}_V} \ L^{\wedge J_1} \right],$$

where $J_1 = {}^{G}\!/_{P}$. Writing down the defining colimit systems, we see that η_* is the map

$$\operatorname*{colim}_{W \in \epsilon^* \mathcal{O}_J} \operatorname*{colim}_{U^N = W} \pi_q \left(\Omega^W \left[J_+ \wedge_{J_1} \mathcal{O}_E(V^{\oplus P}, U) \wedge_{\prod\limits_{J_1} \mathbf{O}_V} L^{\wedge J_1} \right] \right)^K$$

$$\longrightarrow \operatorname*{colim}_{W \in \epsilon^* \mathcal{O}_J} \pi_q \left(\Omega^W \left[J_+ \wedge_{J_1} \mathcal{O}_J(V^{\oplus J_1}, W) \wedge_{\prod\limits_{J_1} \mathbf{O}_V} L^{\wedge J_1} \right] \right)^K.$$

Hence it suffices to prove that

$$p\colon \operatorname*{hocolim}_{U^N = W} \mathcal{O}_E(V^{\oplus P}, U) \to \mathcal{O}_J(V^{\oplus J_1}, W)$$

is a $\Pi_{J_1} \rtimes J_1$-homotopy equivalence, where the map is induced by the restriction to the N-fixed space $V^{\oplus J_1} \subset V^{\oplus P}$ (cf. 3.3.8).
From the definition of $\mathcal{O}_E$ (3.3.2), recall that $\mathcal{O}_E(V^{\oplus P}, U) = \mathcal{O}_G(V^{\oplus P}, U)^N$. Note that any N-equivariant isometry has to preserve fixed spaces and orthogonal complements, hence for $W \oplus U' \cong U$ and $\oplus^{\oplus J_1} V' \cong V^{\oplus P}$ orthogonal decompositions, there is an isomorphism

$$\mathcal{O}_E(V^{\oplus P}, U) \cong \mathcal{O}_J(V^{\oplus J_1}, W) \times \mathcal{L}(V', U')^N,$$

with $\mathcal{L}(V', U')$ the space of linear isometric embeddings from 1.2.17, and the map p corresponds is induced from the projections to the first factor. It therefore suffices to prove that $\mathcal{L}(V', \operatorname{colim} U')^N \to *$ is a $\Pi_{J_1}\mathbf{O}_V \rtimes J_1$-homotopy equivalence. Since V' was the orthogonal complement of $V^{\oplus J_1}$, the $\Pi_{J_1}\mathbf{O}_V$ action is trivial, so [LMS, II.1.5] gives the desired result. □

We continue following [MM, V.4]: Recall the definition of the universal $\mathscr{F}$-space $E\mathscr{F}$ for a closed family of subgroups of G from C.1.18.

Definition 3.5.8. For N a normal subgroup, let $\mathscr{F} = \mathscr{F}[N]$ be the family of subgroups of G that do not contain N. Let $E\mathscr{F}$ be the universal $\mathscr{F}$-space, and let $\tilde{E}\mathscr{F}$ be the cofiber of the quotient map $E\mathscr{F}_+ \to S^0$ that collapses $E\mathscr{F}$ to the non basepoint. For orthogonal G-spectra X, the map $\lambda\colon S^0 \to \tilde{E}\mathscr{F}$ induces a natural map $\lambda\colon X \to X \wedge \tilde{E}\mathscr{F}$.

Note that $(\tilde{E}\mathscr{F})^H = S^0$ if H contains N, and $(\tilde{E}\mathscr{F})^H$ is contractible otherwise.

Lemma 3.5.9. *[MM, V.4.15] For orthogonal G-spectra X, the map*

$$\Phi^N\lambda\colon \Phi^N X \to \Phi^N(X \wedge \tilde{E}\mathscr{F})$$

is a natural isomorphism of orthogonal J-spectra.

The following lemma is another point where we need to be careful about the type of cofibrant objects:

Lemma 3.5.10. *cf. [MM, V.4.16] Let $K = {}^{H}\!/\!_{N}$, with $N \subset H$. For $\mathrm{Ind}^{\mathrm{reg}} \cup \mathcal{F}I_G$-cellular orthogonal G-spectra X, the map*

$$\omega\colon \pi_*^H(X \wedge \tilde{E}\mathscr{F}) \to \rho_*^K(X \wedge \tilde{E}\mathscr{F})$$

is an isomorphism.

Proof. Note that Proposition [LMS, 9.3], which is essential for the proof given in [MM, 4.16] also holds for the weaker assumption of genuine G-cell complexes instead of G-CW-complexes. In particular there are bijections

$$[A, B \wedge \tilde{E}\mathscr{F}]_G \cong [A^N, B \wedge \tilde{E}\mathscr{F}]_G \cong [A^N, B]_G$$

between sets of G-homotopy classes for A any representation sphere and B a level of an induced regular spectrum, which is a genuine G-cell complex by 3.3.66 and C.2.5. Hence as in the classical case, the map

$$\omega\colon \operatorname*{colim}_V \pi_q \left[\Omega^V(X_V \wedge \tilde{E}\mathscr{F})\right]^H \to \operatorname*{colim}_V \pi_q \left[\Omega^{V^N}(X_V \wedge \tilde{E}\mathscr{F})^N\right]^K$$

is a colimit of isomorphisms. □

Finally we can state the main purpose of this excursion, with the classical proof applying verbatim, using our modified results above:

Proposition 3.5.11. *[MM, V.4.17] For* $\mathrm{Ind}^{\mathrm{reg}}$*-cellular orthogonal* G*-spectra* X*, the diagram*

$$R(X \wedge \tilde{E}\mathscr{F})^N \xrightarrow{\gamma} \Phi^N R(X \wedge \tilde{E}\mathscr{F}) \xleftarrow{\Phi^N(\xi\lambda)} \Phi^N X \tag{3.5.12}$$

displays a pair of natural π_**-isomorphisms of orthogonal* J*-spectra, where* R *is a fibrant replacement functor from the classical stable model structure.*

Remark 3.5.13. The significance of this proposition stems from the fact, that there are alternative definitions for the geometric fixed points of a G-spectrum. Classically, one would take the leftmost J-spectrum in the zigzag (3.5.12) as the definition. The proposition then tells us that the homotopy type of the geometric fixed points of an $\mathrm{Ind}^{\mathrm{reg}}$-cellular G-spectrum calculated in terms of Definition 3.3.10 is “correct”, even without first applying a q-cofibrant replacement functor. Note that in the spirit of Remark 3.3.67, we could use the same proofs to extend 3.5.7 and 3.5.10 and hence Proposition 3.5.11 and Proposition 3.5.15 below to $\mathbb{S}_{\mathrm{reg}}$-cellular spectra. However the added generality makes the notation in the proofs even more convoluted, and we only need the weaker result.

The importance of this remark stems from the fact that we want to be able to use the “fundamental cofibration sequence”. It is the following homotopy-cofiber sequence of (non equivariant) orthogonal spectra:

$$[R(X \wedge E\mathscr{F}_+)]^N \longrightarrow [R(X)]^N \longrightarrow [R(X \wedge \tilde{E}\mathscr{F})]^N\ , \tag{3.5.14}$$

which arises from the defining cofiber sequence of $\tilde{E}\mathscr{F}$ by smashing with X, fibrant replacement and passing to categorical fixed points. We saw above, that the homotopy groups of the right spectrum are closely related to the homotopy groups of the geometric fixed points of X. Together with Lemma 3.5.2 from above this implies the following statement:

Proposition 3.5.15. *Let* $\mathscr{H}$ *be a closed family of normal subgroups of* G *and let* X *and* Y *be* $\mathrm{Ind}^{\mathrm{reg}} \cup \mathcal{F}I_G$*-cellular. Then for a morphism* $f\colon X \to Y$*, the following are equivalent:*

(i) The map f *is a* $\pi_*^{\mathscr{H}}$*-isomorphism.*

(ii) For all $H \in \mathscr{H}$ *the map* $\Phi^H f$ *is a (non equivariant)* π_**-isomorphism.*

Proof. Note that since $\mathrm{Ind}^{\mathrm{reg}}$ consists of $\mathbb{S}$-cofibrations, X and Y are levelwise genuine G- hence N-complexes. We compare the maps induced on the homotopy cofiber sequences for $N \in \mathscr{H}$:

$$\begin{tikzcd}
{[R(X \wedge E\mathscr{F}_+)]^N} \arrow[r] \arrow[d] & {[R(X)]^N} \arrow[r] \arrow[d] & {[R(X \wedge \tilde{E}\mathscr{F})]^N} \arrow[d] \\
{[R(Y \wedge E\mathscr{F}_+)]^N} \arrow[r] & {[R(Y)]^N} \arrow[r] & {[R(Y \wedge \tilde{E}\mathscr{F})]^N}
\end{tikzcd} \tag{3.5.16}$$

We use induction on the size of the family $\mathscr{H}$, which is possible since G is compact (cf. [tD, 1.25.15]). For the trivial family, the result is true. We claim first, that both (i) and (ii) imply that the left vertical map is a π_*-isomorphism, for (i) this is trivial by 2.2.37. For (ii) we use the induction hypothesis which implies that f is an $\mathscr{H} \cap \mathscr{F}$-equivalence. Since $\mathscr{F}$ is the family of subgroups not containing N, $\mathscr{H} \cap \mathscr{F}$ is the family $\mathscr{F}_N$ of all proper subgroups of N. Note that as an N-space, $E\mathscr{F}$ is a universal $\mathscr{F}_N$-space. Thus $R(X \wedge E\mathscr{F}_+) \to R(Y \wedge E\mathscr{F}_+)$ is a levelwise $\mathscr{F}_N$ equivalence between genuine $\mathscr{F}_N$-complexes, thus an N-homotopy equivalence and the claim follows. Finally Lemma 3.5.2 and Proposition 3.5.11 finish the proof, since they show that property (i) is equivalent to the second vertical map in (3.5.16) being a π_*-isomorphism for all N, and (ii) being equivalent to the third vertical map in (3.5.16) being a π_*-isomorphism. □

This has the following immediate consequences:

Corollary 3.5.17. *Let $G = \mathbb{T}^n$ be the n-torus and let $\mathscr{H}$ be the family of kernels of isogenies. Let X be a free cofibrant $\mathbb{T}^n$-space. Then the Loday functor $\Lambda_X(-)$ sends π_*-isomorphisms between $\mathbb{S}$-cofibrant commutative orthogonal ring spectra to $\pi_*^{\mathscr{H}}$-equivalences of commutative orthogonal $\mathbb{T}^n$-spectra.*

Proof. Since the $\mathbb{S}$-model structure on commutative orthogonal ring spectra is topological, we know that the tensor with a cofibrant space non equivariantly preserves π_*-isomorphisms between cofibrant commutative $\mathbb{S}$-cofibrations. By Theorem 3.4.48, it therefore also induces non equivariant π_*-isomorphisms in geometric fixed points with respect to all subgroups $H \in \mathscr{H}$, so Proposition 3.5.15 gives the result. □

Similarly for more general compact Lie groups Theorem 3.4.50 gives the following analogue:

Corollary 3.5.18. *Let G be a compact Lie group and let $\mathscr{H}$ be a closed family of normal subgroups that is closed under extensions of finite index, such that G satisfies Conditions 1 and 2 with respect to all $H \in \mathscr{H}$. Let X be a free cofibrant G-space. Then the Loday functor $\Lambda_X(-)$ sends π_*-isomorphisms between $\mathbb{S}$-cofibrant commutative orthogonal ring spectra to $\pi_*^{\mathscr{H}}$-equivalences of commutative orthogonal G-spectra.*

Remark 3.5.19. Analogous results hold for the cases of mere $\mathbb{S}$-cofibrant spectra and finite free G-sets X as well as spectra $\mathbb{S}$-cofibrant under $\mathbb{S}$ and infinite free G-sets X (cf. Subsection 3.4.3).

3.5.2 Comparison to the [BCD]-model

We compare the Loday functor of Section 3.2 to the model constructed in Section 4 of [BCD]. Since op. cit. is written in the language of Γ-spaces, respectively simplicial functors, we cannot do so directly, but instead have to use comparison theorems such as the ones given in [MMSS, §0] and [SS03, §7]. As mentioned in the introduction to [SS03, §7] the corresponding comparisons of categories of commutative monoids do not extend over the whole range of the comparison. Developing such comparison results beyond the scope of this thesis, hence we restrict our comparison to the underlying spectra, respectively the equivariant structure. Since the weak equivalences considered between commutative monoids are in both contexts created in the underlying category, this seems satisfactory.
We begin with the non equivariant discussion. Recall the definition of a strong monoidal Quillen equivalence from [SS03, 3.6]. The discussion in the introduction of [MMSS] and [SS03, 7.1] state that there is a diagram of strong monoidal Quillen equivalences

$$\begin{tikzcd} \mathcal{SF} \arrow[r, bend left, "\mathbb{PT}"] & \mathcal{WT} \arrow[l, bend left] \arrow[r, bend right, "\mathbb{U}"'] & \mathcal{OT}, \arrow[l, bend right, "\mathbb{P}"'] \end{tikzcd} \tag{3.5.20}$$

where $\mathcal{SF}$ is the category of simplicial functors and $\mathcal{WT}$ is the category of continuous functors from finite CW-complexes to spaces, respectively, and the model structure on $\mathcal{OT}$ is the classical stable one from 1.2.54. The functor $\mathbb{T}$ is the postcomposition with the simplicial complex, and the instances of $\mathbb{P}$ and FU are prolongation and restriction functors analogous to 2.2.23. We displayed the strong monoidal left adjoints as the top arrow.
Since these Quillen equivalences are not composable as such, we compare the two constructions of the Loday functor in $\mathcal{WT}$. However, since it would take us too far to recall the whole construction from [BCD], we will immediatel reduce our comparison to smash powers with the help of the following lemma. We denote the [BCD]-Loday functor by $\hat{\Lambda}$.

Lemma 3.5.21. *[BCD, 4.4.4] If A is cofibrant and T is a finite set, then there is a chain of stable equivalences between $\hat{\Lambda}_T A$ and the T-fold smash product $\bigwedge_T A$.*

In particular the functor $\hat{\Lambda}_T$ models the T-fold derived smash product of A with itself. Since the Loday functor we defined for orthogonal spectra has the analogous property, and since by the (3.5.20) the homotopy categories of $\mathcal{SF}$ and $\mathcal{OT}$ are monoidally equivalent, this could already be seen as a successful comparison. We can say a little bit more: Recall that the identity functor on $\mathcal{OT}$ gave strong monoidal Quillen equivalences between the absolute, positive and $\mathbb{S}$-model structures:

$$\begin{tikzcd} \mathcal{OT} \arrow[r, bend right, "\mathrm{id}"'] & \mathcal{OT}_+ \arrow[l, bend right, "\mathrm{id}"'] \arrow[r, bend left, "\mathrm{id}"] & \mathcal{OT}_{\mathbb{S}}. \arrow[l, bend left, "\mathrm{id}"] \end{tikzcd} \tag{3.5.22}$$

Since the identity functors are in particular strong monoidal, the standard methods for (monoidal) Quillen pairs (and [H, 1.3.13, (b)]) give:

Lemma 3.5.23. *For A a cofibrant simplicial functor and X a finite set, there is a chain of natural stable equivalences in $\mathcal{WT}$ connecting*

$$\mathbb{PT}(A^{\wedge X}) \simeq \mathbb{P}((\ \textit{cof.}\ \mathbb{U}\ \textit{fib.}\ \mathbb{PT}A)^{\wedge X}),$$

where fib. denotes a functorial fibrant replacement in $\mathcal{WT}$, and cof. denotes a functorial $\mathbb{S}$-cofibrant replacement in $\mathcal{OT}$.

The comparison of the equivariant properties is similarly obstructed by the fact that there seem to be no usable comparison results on model category level between equivariant orthogonal spectra and equivariant simplicial functors. We therefore only compare on equivariant homotopy categories, making use of our results on the equivariant structure of $\Lambda_X A$ surrounding Proposition 3.5.15 and the analogous result [BCD, 5.2.5]:

Lemma 3.5.24. *Let G be a finite abelian group, X a free G-simplicial set and A a commutative monoid in simplicial functors. The homotopy fiber of the map*

$$[\hat{\Lambda}_X A]^G \to \operatorname*{holim}_{0\neq H\subset G} \left[\hat{\Lambda}_{X_H} A\right]^{G/H} \tag{3.5.25}$$

induced by the restriction maps is connected by a chain of natural maps that are stable equivalences to the homotopy orbit spectrum $[\hat{\Lambda}_X A]_{hG}$.

There are two important translations to be made here. The first is identifying the target of (3.5.25) with the geometric fixed points (defined in terms of $\tilde{E}\mathscr{F}$, cf. Remark 3.5.13) as in [HM, 2.1]. The second is the identification of the homotopy fiber $[\Lambda_X A\wedge E\mathscr{F}_+]^H$ of 3.5.14 with the homotopy orbits $[\Lambda_X A\wedge E\mathscr{F}_+]_H$ in the homotopy category via the Adams isomorphism as in [MM, VI.4.6]. Then the two (co-)fiber sequences in homotopy category exactly say, that $\Lambda_X A$ and $\hat{\Lambda}_X A$ have the same equivariant structure, i.e. the same homotopy type on all fixed points with respect to (finite) subgroups.

3.6 THH and TC

We finish the discussion of the Loday functor by identifying the structure necessary to define higher topological Hochschild homology and higher topological cyclic homology. We follow Kro's approach from [Kr, §5] for the one dimensional theory, and use it as motivation and guideline for our treatment of the higher analogues in subsection 3.6.2.

3.6.1 Classical Structure

Topological Hochschild Homology

We finally put the Loday functor we constructed to use in defining topological Hochschild Homology for commutative orthogonal ring spectra. We do not try to give more of an overview over the existing theory in other settings than was already attempted in the introduction, and instead redirect the interested reader to [Sh] for the case of symmetric spectra, [MSV] for $\mathbb{S}$-algebras in the sense of [EKMM] or [Mad] and [DGM] for a general overview over the classical approach and applications towards TC and K-theory. As we already mentioned in the introduction, we will use the following simple definition:

Definition 3.6.1. Let A be an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum. Define the *topological Hochschild homology spectrum* $\mathrm{THH}(L)$ to be the commutative orthogonal S^1-spectrum

$$\mathrm{THH}(L) := \Lambda_{S^1} A \cong A \otimes S^1.$$

Of course a similar definition is possible for non-cofibrant commutative ring spectra, but for the definition to have homotopical meaning, the assumption is necessary. As usual, we can always precompose with a cofibrant replacement functor for the $\mathbb{S}$-model structure. We want to allow ourselves, however, to not cofibrantly replace again if we start with a cofibrant spectrum though, so we do not put the $\mathbb{S}$-cofibrant replacement in the definition. We note once more, that contrary to the classical model structure, the cofibrant replacement takes place in the category of commutative orthogonal ring spectra, so that the tensor definition still makes sense.
Note that Theorem 3.1.11, together with the standard simplicial model for S^1, implies the following lemma:

Lemma 3.6.2. *For a commutative orthogonal ring spectrum A, the topological Hochschild homology spectrum* $\mathrm{THH}(A)$ *is isomorphic to the geometric realization of the simplicial commutative ring spectrum given by* $\mathrm{THH}(A)_q := A^{\wedge q+1}$*, with the* *simplicial structure maps given by*

$$d_i = \begin{cases} \mathrm{id}^{\wedge i} \wedge \mu \wedge \mathrm{id}^{q-i-1} & \text{for} \quad 0 \leq i < q \\ (\mu \wedge \mathrm{id}^{\wedge q-1}) \circ T_{q,1} & \text{for} \quad i = q \end{cases}$$

$$s_i = \mathrm{id}^{\wedge i+1} \wedge \eta \wedge \mathrm{id}^{q-i}$$

where μ and η are the multiplication respectively unit map of A and $T_{q,1}$ is the action of the shuffle permutation mapping

$$A^{\wedge q+1} = A^{\wedge q} \wedge A \xrightarrow{T_{q,1}} A \wedge A^{\wedge q} = A^{\wedge q+1}.$$

Recall that for A an orthogonal ring spectrum whose unit $\mathbb{S} \to A$ is a q-cofibration, Kro defines an orthogonal spectrum $\overline{\mathrm{THH}}(A)$ in [Kr, 5.2.1] via a simplicial spectrum analogous to the Lemma. Hence in particular, by 3.1.13 we get that Definition 3.6.1 yields a commutative orthogonal ring spectrum $\mathrm{THH}(A)$ whose underlying spectrum is isomorphic to $\overline{\mathrm{THH}}(A)$ in Kro's sense, so we immediately stop distinguishing between the notation.
For a ring spectrum A whose unit $\mathbb{S} \to A$ is only a closed inclusion of spectra, Kro further defines a functor Γ in [Kr, 2.2.13], such that $\Gamma(A) \to A$ is a map of ring spectra which is an underlying level fibration of orthogonal spectra. Denote the cofibrant replacement functor in the $\mathbb{S}$-model structure for commutative orthogonal ring spectra by $\mathcal{E}$. The following lemma shows that the homotopy type of the spectrum $\mathrm{THH}(A)$ does not depend on which of the two cofibrant replacements we chose:

Lemma 3.6.3. *For a commutative orthogonal ring spectrum A whose unit $\mathbb{S} \to A$ is a an underlying closed inclusion of spectra, there is a π_*-isomorphism*

$$\mathrm{THH}(\Gamma(A)) \to \mathrm{THH}(\mathcal{E}(A)).$$

Proof. Since acyclic $\mathbb{S}$-fibrations of commutative orthogonal ring spectra are in particular acyclic q-fibrations of underlying spectra, the lifting property in the q-model structure gives a π_*-isomorphism $f\colon \Gamma(A) \to \mathcal{E}(A)$. We use the simplical spectrum from Lemma 3.6.2 to calculate THH. Both of the resulting simplicial spectra are h-proper by the same argument as in the proof of in 1.3.22. Since both $\Gamma(A)$ and $\mathcal{E}(A)$ are $\mathbb{S}$-cofibrant as spectra, f induces a π_*-isomorphism $f^{\wedge q}$ in each simplicial level by the pushout product axiom for the $\mathbb{S}$-model structure. Hence Proposition B.1.47 implies, that the induced map on realizations is a π_*-isomorphism. □

Of course we also need to know that the equivariant homotopy type agrees, which will be an immediate consequence of Lemma 3.6.10 below.

Definition 3.6.4. Let $G = S^1$ be the circle and let $\mathscr{H}$ be the closed family of finite (cyclic) subgroups. An morphism of orthogonal S^1-spectra is called a *cyclotomic* π_*-isomorphicm if it is a $\pi_*^{\mathscr{H}}$-isomorphism, i.e. if it induces isomorphisms on π_*^C for all subgroups $C \in \mathscr{H}$.

Definition 3.6.5. For $C \in \mathscr{H}$, let ρ_C be the isomorphism

$$\rho_C \colon S^1 \cong {}^{S^1}\!/_{C}.$$

Denote as usual by ρ_C^* the restriction functor from ${}^{S^1}\!/_{C}$-equivariant orthogonal spectra to S^1-equivariant orthogonal spectra.

Then the following result extends Kro's Theorem 5.2.5:

Theorem 3.6.6. *Let A be an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum then there are isomorphisms of S^1-equivariant commutative orthogonal ring spectra*

$$\rho_C^* \Phi^C \operatorname{THH}(A) \cong \operatorname{THH}(A).$$

For $f \colon A \to B$ a π_*-isomorphism of $\mathbb{S}$-cofibrant commutative orthogonal ring spectra,

$$\operatorname{THH}(f) \colon \operatorname{THH}(A) \to \operatorname{THH}(B)$$

is a cyclotomic π_*-isomorphism of S^1-equivariant commutative orthogonal ring spectra.

Proof. The first part is immediate from Theorem 3.4.48 and the fact that $\rho^* \Lambda_X A \cong \Lambda_{\rho^* X} A$ as in the discussion before 3.4.43. The second part is the one-dimensional case of Corollary 3.5.17 □

This exhibits $\operatorname{THH}(A)$ as an especially strong example of a *cyclotomic spectrum*. We are going to give the explicit definition after the next construction:

Construction 3.6.7. Let $G = S^1$, and let C, D and E be finite subgroups such that $\rho_C(D) = {}^{E}\!/_{C}$. In particular we have that for an S^1 space X there is an isomorphism of fixed points

$$(X^C)^{E/C} \cong X^E, \quad \text{and hence} \quad (\rho_C^*(X^C))^D \cong X^E.$$

The same formulas hold for categorical fixed points of an S^1-spectrum L.
In other setups of equivariant stable homotopy theory and cyclotomic specta, one can sometimes also identify

$$\rho_D^* \Phi^D \rho_C^* \Phi^C L = \rho_E^* \Phi^E L, \tag{3.6.8}$$

see for example [HM, Definition 2.2]. *However:* care needs to be taken when adapting this to the orthogonal case, since both the classical Definition of the geometric fixed point functor Φ is different from the one we used here (c.f Remark 3.5.13 and [HM, p.32]), and there are "spectrification" functors hidden in the classical notation. In our setting, the equal sign in (3.6.8) is certainly not warranted. However, by Proposition 3.3.57 there is a natural isomorphism

$$\rho_D^*\Phi^D\rho_C^*\Phi^C L \to \rho_D^*\rho_E^{D*}\Phi^{E/C}\Phi^C L,$$

where $\rho_E^D\colon S^1/D \to S^1/E$ is the isomorphism $\rho_D^{-1}\circ\rho_E$.
Writing down the defining coends one easily constructs a natural map

$$\Phi^E L \to \Phi^{E/C}\Phi^C L,$$

using that a coend of fixed points maps into the fixed points of the coend. For induced regular semi-free spectra (3.3.64) this map is an isomorphism via the identifications of Theorem 3.3.65, hence the same is true for $\mathbb{S}_{\mathrm{reg}}$-cellular (3.3.67) and in particular q-cofibrant or $\mathrm{Ind}^{\mathrm{reg}}$-cellular spectra by a cell induction argument.

This allows us to formulate the following definition:

Definition 3.6.9. (cf. [Kr, 5.1.3]) An orthogonal S^1-spectrum L is cyclotomic, if for all $C \in \mathscr{H}$ there are π_*-isomorphisms

$$r_C\colon \rho_C^*\Phi^C L \to L,$$

such that the diagrams

$$\begin{tikzcd}
\rho_D^*\Phi^D\rho_C^*\Phi^C L \arrow[d, "\rho_D^*\Phi^D r_C"'] & \rho_E^*\Phi^E L \arrow[l] \arrow[d, "r_E"] \\
\rho_D^*\Phi^D L \arrow[r, "r_D"] & L
\end{tikzcd}$$

commute for all subgroups C, D and E in $\mathscr{H}$, such that $\rho_C(D) = E/C$.
A *map of cyclotomic spectra* is a morphism of orthogonal S^1-spectra which commutes with the cyclotomic structure maps r_C for all $C \in \mathscr{H}$.

Note that by the naturality of the diagonal map (3.4.39), the functor $\mathrm{THH}(-) \cong \Lambda_{S^1}(-)$ not only produces cyclotomic spectra, but also maps of cyclotomic spectra from morphisms between $\mathbb{S}$-cofibrant commutative ring spectra.
The following is a generalization of [Kr, 5.1.5]

Lemma 3.6.10. *Let L and L' be $\mathbb{S}_{\mathrm{reg}}$-cellular S^1-spectra (e.g. q-cofibrant or* $\mathrm{Ind}^{\mathrm{reg}}$*-cellular). A map $f\colon L \to L'$ of cyclotomic spectra is a cyclotomic π_*-isomorphism if and only if it is a non-equivariant π_*-isomorphism.*

Proof. The proof is immediate from Proposition 3.5.15 and the two out of three property for cyclotomic π_*-isomorphisms. $\square$

Note that this looks weaker than [Kr, 5.1.3] on first glance, but Kro suppresses the cofibrancy hypothesis, and in particular only provides proofs for the q-cofibrant case. As a corollary we get that the cyclotomic homotopy type of the underlying spectrum of our version of THH for commutative ring spectra agrees with the one Kro constructs:

Corollary 3.6.11. *The non equivariant π_*-isomorphism* $\mathrm{THH}(\Gamma(A)) \to \mathrm{THH}(C(A))$ *in Lemma 3.6.3 is even a cyclotomic π_*-isomorphism of cyclotomic spectra.*

Topological Cyclic Homology

We finally discuss TC. As in the previous paragraph, we do not try to give an overview over the existing theory or even recall results, but rather show how to adapt the definitions to the setting of orthogonal spectra in order to motivate the approach to the higher theory. Better places to read about the classical constructions are for example [BHM], [HM] and again [DGM] and [Mad]. We will again stay close to Kro's exposition from [Kr, 5.1].

Definition 3.6.12. (cf. [HM, 4.1]) Let $\mathbb{I}$ be the category with objects the natural numbers $\{1, 2, 3, \ldots\}$. The morphisms of $\mathbb{I}$ are generated by the Restrictions $R_r\colon rm \to m$ and the Frobenii $F^r\colon mr \to m$, subject to the following set of relations:

$$\begin{aligned} R_1 = F^1 &= \mathrm{id}_m \\ R_r R_s &= R_{rs} \\ F^r F^s &= F^{sr} \\ R_r F^s &= F^s R_r \end{aligned} \tag{3.6.13}$$

Note that we were careful about the ordering of products mr versus rm in $\mathbb{N}$. This is of course not of consequence here, but will become important when passing to higher dimensional analogues.

Construction 3.6.14. A cyclotomic spectrum L defines a functor $\mathbb{I} \to \mathcal{OT}$ by mapping $n \in \mathbb{I}$ to the categorical fixed point spectrum L^{C_n}, where C_n is the cyclic subgroup with n elements of S^1. The actions of the Frobenius maps are then given by the inclusions of fixed points

$$F^r\colon L^{C_{mr}} \cong \left(L^{C_m}\right)^{C_r} \to L^{C_m},$$

whereas the Restriction maps make use of the map γ from 3.3.13:

$$R_r\colon L^{C_{rm}} \cong (L^{C_r})^{C_m} \xrightarrow{\gamma^{C_m}} (\Phi^{C_r}L)^{C_m} \xrightarrow{r_{C_r}} L^{C_m}.$$

We suppressed several instances of maps ρ from the notation to keep the formulas readable. The appropriate relations can then be checked using the definition of cyclotomicity, but since we will spend more time on these later (3.6.27), we omit the details for now.

Note that since the model structures on $\mathcal{OT}$ we discuss are topological in the sense of B.1.10, it is in particular simplicial via the Quillen equivalence between spaces and simplicial sets. Hence there is a concrete model in $\mathcal{OT}$ for the homotopy limit (e.g. [Hir, 18.1.8]) in the following definition:

Definition 3.6.15. Let L be a cyclotomic spectrum, the *topological cyclic homology spectrum* $\mathrm{TC}(L)$ is the orthogonal spectrum

$$\mathrm{TC}(L) := \operatorname*{holim}_{n\in I} T^{C_n}.$$

For a commutative orthogonal ring spectrum A, abbreviate $\mathrm{TC}(\mathrm{THH}(A))$ as $\mathrm{TC}(A)$, and call $\mathrm{TC}(A)$ the *topological Cyclic homology spectrum* of A.

Remark 3.6.16. Note that talking about cyclotomic commutative ring spectra here does not gain a lot of benefits, even though our construction of $\mathrm{THH}(A)$ gives the structure for free: Since the homotopy limit involves some objectwise fibrant replacement, we can only hope for $\mathrm{TC}(L)$ to have the correct homotopy type if L is at least an Ω-spectrum. Since we cannot guarantee, that the *fibrant* replacement functor in the $\mathbb{S}$-model structure preserves the Φ^N, we have to use a q-fibrant replacement functor, which in general destroys the strict commutativity. The q-fibrant replacement functor Q associated to the stable model structure on S^1-spectra from 2.2.46 in particular also preserves cyclotomicity by [Kr, 5.1.10] so it seems most natural to use it here. However, the functoriality of the Loday functor $\Lambda_X(A)$ will allow us to identify the higher analogue of the cyclotomic structure much easier than in the classical setup, so it is still worthwhile to use it even when dealing with TC and not just THH.

3.6.2 Higher Structure

In this final section, we identify the higher structure on $\Lambda_G(A)$, that is used when defining higher topological Cyclic homology or Covering Homology as in [CDD], respectively [BCD]. We fix some compact Lie group G that satisfies Conditions 1 and 2 with respect to a closed family $\mathscr{H}$ of kernels of isogenies. The main example we have in mind is $G = \mathbb{T}^n$ the n-dimensional torus, with the family $\mathscr{H}$ of all kernels of isogenies, since this is gives the higher analogue of topological Cyclic homology. Since the structure we study below depends on the choice of $\mathscr{H}$, or rather the choice of the isogenies, i.e. self covers of G, this theory is usually titled Covering Homology (cf. [CDD]).

Definition 3.6.17. Let A be an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum. Define the *higher topological Hochschild homology spectrum* $\mathrm{THH}^n(L)$ to be the commutative orthogonal S^1-spectrum

$$\mathrm{THH}^n(L) := \Lambda_{\mathbb{T}^n} A \cong A \otimes \mathbb{T}^n.$$

Note that by the defining adjunctions of the categorical tensor, $\mathrm{THH}^n(A)$ is isomorphic to the n-fold iterated application of THH to A, and in particular one could study the S^1-equivariant structure induced by each of these iterations separately, however the Loday functor allows us to also investigate much more intricate diagonal phenomena since we have the whole $\mathbb{T}^n$-equivariant structure available. We begin with a proposed definition of the higher analogue of cyclotomicity mentioned in [CDD, 2.1]:

Definition 3.6.18. Let G and $\mathscr{H}$ be as above. A morphism of orthogonal G-spectra is called a *cyclotomic π_*-isomorphism* if it is a $\pi_*^{\mathscr{H}}$-isomorphism.

Definition 3.6.19. Let $\alpha\colon G \to G$ be an isogeny with kernel denoted by L_α, such that there is an isomorphism of groups

$$\rho_\alpha\colon G \cong {}^{G}\!/_{L_\alpha}.$$

Denote the inverse of ρ_α by ϕ_α (cf. [CDD, 2.2]).

Remark 3.6.20. Note that the notation is coherent with Definition 3.6.5, where the isogeny α of S^1 associated to a cyclic subgroup of order n is of course the map induced by raising a complex number z to the n^{th}-power z^n. Since in the one-dimensional case such (orientation preserving) isogenies and their kernels are in one to one correspondence, there is no loss of information in the indexing.

Construction 3.6.21. For α and β isogenies we can form their composite $\alpha \circ \beta = \alpha\beta$ and their kernels $L_\beta \subset L_{\alpha\beta}$ satisfy

$$\rho_\beta(L_\alpha) = {}^{L_{\alpha\beta}}\!/_{L_\beta}.$$

Hence as in 3.6.7 there is a natural map

$$\Phi^{L_{\alpha\beta}} A \to \Phi^{{}^{L_{\alpha\beta}}\!/_{L_\beta}} \Phi^{L_\beta} A, \tag{3.6.22}$$

which using Proposition 3.3.57 induces a natural map

$$\rho_{\alpha\beta}^*\Phi^{L_{\alpha\beta}}A \to \rho_\alpha^*\Phi^{L_\alpha}\rho_\beta^*\Phi^{L_\beta}A,$$

and both of these become isomorphisms for $\mathbb{S}_{\mathrm{reg}}$-cellular G-spectra A.

Definition 3.6.23. A cyclotomic orthogonal G-spectrum is an orthogonal G-spectrum A, together with cyclotomic π_*-isomorphisms

$$r_\alpha \colon \rho_\alpha^*\Phi^{L_\alpha}A \to A,$$

for all isogenies α whose kernel L_α is in $\mathscr{H}$, such that the diagrams

$$\begin{tikzcd}
\rho_\alpha^*\Phi^{L_\alpha}\rho_\beta^*\Phi^{L_\beta}A \arrow[d, "\rho_\alpha^*\Phi^\alpha r_\beta"'] & \rho_{\alpha\beta}^*\Phi^{L_{\alpha\beta}}A \arrow[l] \arrow[d, "r"] \\
\rho_\alpha^*\Phi^\alpha A \arrow[r, "r_\alpha"] & A
\end{tikzcd}$$

commute for all isogenies α and β with L_α, L_β and $L_{\alpha\beta}$ in $\mathscr{H}$.
A *map of cyclotomic spectra* is a morphism of orthogonal S^1-spectra which commutes with the cyclotomic structure maps r_α.

One can chose to fix a collection $\mathcal{I}$ of isogenies instead of the family $\mathscr{H}$ of kernels and get a similar definition. For example to get a complete analogue of the one dimensional case from Defintion 3.6.9, one should restrict to orientation preserving isogenies of S^1. The analogue of Theorem 3.6.6 uses our results 3.5.18 and 3.4.50 on the equivariant structure of the Loday functor for more general compact Lie groups:

Proposition 3.6.24. *Let A be an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum then the underlying G-spectrum of $\Lambda_G(A)$ is cyclotomic and $\Lambda_G(-)$ sends π_*-isomorphisms to cyclotomic maps that are cyclotomic π_*-isomorphisms.*

We want to identify the Restriction and Frobenius maps that are used to define the covering homology and in particular higher topological Cyclic homology (cf. [CDD, 2.2]). Since the indexing via isogenies and, in particular, the maps ρ_α complicate the notation significantly, we start by defining relative versions, postponing the coordinate change ensued by changing back to G-spectra to Definition 3.6.28:

Definition 3.6.25. Let $N \subset H \subset G$ be a sequence of subgroups and denote by $\rho_H^N \colon {G}/{N} \to {G}/{H}$ the associated projection. For A an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum, the ${G}/{H}$-spectrum $(\Lambda_X A)^H$ becomes a ${G}/{N}$-spectrum via the restriction along ρ_H^N. The *Frobenius map* F_N^H is the morphism of ${G}/{N}$-spectra

$$F_N^H \colon (\Lambda_X A)^H \to (\Lambda_X A)^N$$

given by the inclusion of fixed points on on each level.

Note that F_N^H defines a natural transformation $(-)^H \to (-)^N$.

Definition 3.6.26. Let $N \subset H \subset G$ be a sequence of subgroups and let A be an $\mathbb{S}$-cofibrant orthogonal spectrum as above. The *Restriction map* R_N^H is the natural map of ${}^{G}\!/\!_{H}$-spectra

$$R_N^H \colon (\Lambda_X A)^H \cong \left((\Lambda_X A)^N\right)^{H/N} \xrightarrow{\gamma^{H/N}} \left(\Phi^N \Lambda_X A\right)^{H/N} \cong \left(\Lambda_{X_N} A\right)^{H/N}$$

The following is the first example of a higher (relative) version of the relations (3.6.13):

Proposition 3.6.27. *The following diagram is commutative for all $\mathbb{S}$-cofibrant orthogonal spectra A and configurations of subgroups $N \subset H \subset J \subset G$:*

$$\begin{tikzcd}
(\Lambda_X A)^J \arrow[r, "R_N^J"] \arrow[d, "F_H^J"'] & \left(\Lambda_{X_N} A\right)^{J/N} \arrow[d, "F_{H/N}^{J/N}"] \\
(\Lambda_X A)^H \arrow[r, "R_N^H"'] & \left(\Lambda_{X_N} A\right)^{H/N}
\end{tikzcd}$$

Proof. We split the diagram into the following:

$$\begin{tikzcd}
(\Lambda_X A)^J \arrow[r, "\cong"] \arrow[d, "F_H^J"'] & \left((\Lambda_X A)^N\right)^{J/N} \arrow[r, "\gamma^{J/N}"] \arrow[d, "F_{H/N}^{J/N}"] & \left(\Phi^N \Lambda_X A\right)^{J/N} \arrow[r, "\cong"] \arrow[d, "F_{H/N}^{J/N}"] & \left(\Lambda_{X_N} A\right)^{J/N} \arrow[d, "F_{H/N}^{J/N}"] \\
(\Lambda_X A)^H \arrow[r, "\cong"'] & \left((\Lambda_X A)^N\right)^{H/N} \arrow[r, "\gamma^{H/N}"'] & \left(\Phi^N \Lambda_X A\right)^{H/N} \arrow[r, "\cong"'] & \left(\Lambda_{X_N} A\right)^{H/N}
\end{tikzcd}$$

The two right squares are commutative by the naturality of the Frobenius $F_{H/N}^{J/N}$. Commutativity of the left square can be checked levelwise: Let B be any J spectrum, and V a representation of ${}^{J/N}\!/\!_{H/N} \cong {}^{J}\!/\!_{H}$, then the diagram of inclusions of fixed points commutes:

$$\begin{tikzcd}
(B_V)^J \arrow[r, "\cong"] \arrow[d] & \left(B_V^N\right)^{J/N} \arrow[d] \\
(B_V)^H \arrow[r, "\cong"] & \left(B_V^N\right)^{H/N}
\end{tikzcd}$$

∎

Switching back to the isogeny notation, yields the following:

Definition 3.6.28. Let α and β be isogenies of G as above, and view $(\Lambda_G)^{L_\alpha}$ as a G-spectrum via ρ_α. Then define the *Frobenius maps* F^α as the natural morphism of G-spectra

$$F^\alpha := \rho_\beta^* \circ F_{L_\beta}^{L_{\alpha\beta}} \circ \phi_{\alpha\beta}^* \colon (\Lambda_G A)^{L_{\alpha\beta}} \to (\Lambda_G A)^{L_\beta},$$

and the *Restriction maps* R_β as the natural morphism of G-spectra

$$R_\beta := \rho_\alpha^* \circ (\rho_\beta^*)^{L_\alpha} \circ R_{L_\beta}^{L_{\alpha\beta}} \circ \phi_{\alpha\beta}^* \colon (\Lambda_G A)^{L_{\alpha\beta}} \to (\Lambda_{G/L_\beta} A)^{L_\alpha} \cong (\Lambda_G A)^{L_\alpha},$$

Corollary 3.6.29. *The Restriction and Frobenius maps satisfy the following relations:*

$$\begin{aligned} F^\alpha &= \mathrm{id} \quad \textit{for } \alpha \textit{ invertible}, \\ F^\beta F^\alpha &= F^{\alpha\beta}, \\ R_\beta R_\alpha &= R_{\beta\alpha}, \\ R_\beta F^\alpha &= F^\alpha R_\beta. \end{aligned}$$

Proof. For the first one, use that $\rho_H^N = \rho_H \circ \phi_N$. The second one is immediate in the relative version and the third one uses the isomorphism (3.6.22). The last relation follows from (3.6.27). Note that even though γ is the identity map between the categorical and geometric $\{e\}$-fixed points, R_α is not usually trivial for α invertible because of the coordinate changes ρ^* involved. □

For completeness we very briefly repeat the higher analogues of Definitions 3.6.12 and 3.6.15, for more details, see [CDD, 2.3]:
Denote by $\mathcal{C}$ be the category with one object and morphisms the isogenies $\alpha \in \mathcal{I}$, respectively with $L_\alpha \in \mathscr{H}$.

Definition 3.6.30. Let $\mathcal{A}r_{\mathcal{C}}$ be the twisted arrow category of $\mathcal{C}$, i.e. the category with the isogenies as objects and morphisms $\alpha \to \beta$ given by diagrams

$$\begin{tikzcd} \star \arrow[r, "\gamma"] \arrow[d, "\alpha"'] & \star \arrow[d, "\beta"] \\ \star & \star \arrow[l, "\delta"] \end{tikzcd}$$

with composition given by horizontal concatenation of diagrams. Note that every such morphism is represented by the equation $\alpha = \delta \circ \beta \circ \gamma$ and factors as

$$\begin{tikzcd} \star \arrow[r, "\mathrm{id}"] \arrow[d, "\alpha"'] & \star \arrow[r, "\gamma"] \arrow[d, "\beta\gamma"] & \star \arrow[d, "\beta"] \\ \star & \star \arrow[l, "\delta"] & \star \arrow[l, "\mathrm{id}"] \end{tikzcd}$$

.

Construction 3.6.31. For A an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum, we define a functor from $\mathcal{A}r_{\mathcal{C}} \to G\mathcal{OT}$, by sending the isogeny α to the categorical fixed point spectrum $(\Lambda_G A)^{L_\alpha}$, viewed as a G-spectrum via $\rho_a^* lpha$. Morphisms $\delta\beta = \delta \circ \beta \circ \mathrm{id}$ are sent to the Frobenii F^δ and morphisms $\beta\gamma = \mathrm{id} \circ \beta \circ \gamma$ to the Restrictions R_γ. Functoriality is a consequence of the relations from Corollary 3.6.29.

Definition 3.6.32. Let A be an $\mathbb{S}$-cofibrant commutative orthogonal ring spectrum the *covering homology spectrum* $\mathrm{TC}^{\mathcal{C}}(A)$ associated to $\mathcal{C}$ is the homotopy limit

$$\mathrm{TC}^{\mathcal{C}}(A) := \operatorname*{holim}_{\alpha \in \mathcal{A}r_{\mathcal{C}}} (\Lambda_G A)^{L_\alpha}.$$

As in the the one dimensional case, this becomes homotopically meaningful, only after applying a fibrant replacement to $\Lambda_G(A)$ (cf. Remark 3.6.16).
There is even more structure on $\Lambda_G A$. In [CDD, 2.5,3.2], the authors define Verschiebung maps for the general case and higher differentials for the special case of the p-adic n-torus. Both the definitions and verifications of relations analogous to 3.6.29 rely heavily on a good understanding of the stable equivariant transfer:

Definition 3.6.33. [CDD, 2.4] Let G and $\mathcal{I}$ be as above. For α and β isogenies in $\mathcal{I}$, chose a finite dimensional orthogonal G-representation W and an open G-embedding

$$i \colon W \times G/L_\beta \to W \times G/L_{\alpha\beta}$$

over the projection $\rho_{\alpha\beta} \circ \phi_\beta \colon G/L_\beta \to G/L_{\alpha\beta}$. Applying the Thom construction to this embedding yields a G-equivariant map

$$tr_\alpha = tr_\beta^{\alpha\beta} \colon S^W \wedge (G/L_{\alpha\beta})_+ \to S^W \wedge (G/L_\beta)_+$$

called the transfer, which does not depend on the choice of W or the embedding i up to stable equivariant homotopy.

Proofs for both the existence and the properties of the transfer maps are spread throughout the literature, the author has found the exposition in the original paper [KP] and its follow-up [KP78] very helpful, since in particular the existence is treated nicely there, which is usually omitted in later accounts. The exposition in [LMS, IV] is very thorough. A list of properties needed for the treatment of covering homology is given in [CDD, Proposition 2.4] and since our notation agrees with the one used in op. cit., we omit further details.
Again we start with a version of the definition of the Verschiebung maps, that omits the coordinate changes:

Definition 3.6.34. Let $N \subset H \subset G$ be a sequence of subgroups and let

$$tr_N^H \colon S^W \wedge ({}^{G}\!/\!_{N})_+ \to S^W \wedge ({}^{G}\!/\!_{H})_+$$

be a model for the transfer. Define for an orthogonal G-spectrum B the *Verschiebung map* V_N^H be the natural stable map induced by the transfer on fixed points in the following way

$$\begin{tikzcd}
B^N \arrow[r, "\simeq"] \arrow[dddd, "V_N^H"'] & (\Omega^W \operatorname{sh}_W B)^N \arrow[d, "="] \\
& F_{\mathbb{S}}(S^W, \operatorname{sh}_W B)^N \arrow[d, "\cong"] \\
& F_{\mathbb{S}}(S^W \wedge {}^{G}\!/\!_{N_+}, \operatorname{sh}_W B) \arrow[d, "(tr_N^H)_*"] \\
& F_{\mathbb{S}}(S^W \wedge {}^{G}\!/\!_{H_+}, \operatorname{sh}_W B) \arrow[d, "\cong"] \\
B^H \arrow[r, "\simeq"'] & F_{\mathbb{S}}(S^W, \operatorname{sh}_W B)^H
\end{tikzcd}$$

The version including the instances of ρ is the following:

Definition 3.6.35. Let α and β be isogenies of G as above, and view $(\Lambda_G)^{L_\alpha}$ as a G-spectrum via ρ_α. Then define the *Verschiebung maps* V_β as the natural stable morphisms of G-spectra

$$V_\alpha := \rho_{\alpha\beta}^* \circ V_{L_\beta}^{L_{\alpha\beta}} \circ \phi_\beta^* \colon (\Lambda_G A)^{L_\beta} \to (\Lambda_G A)^{L_{\alpha\beta}}.$$

Again, there are various relations between the Verschiebung, Frobenius and Restriction maps, and the authors of [CDD] develop the theory nicely. Since their methods are sufficiently general to be applied to our setting, we will only give a first idea here:

Proposition 3.6.36. *Restriction and Verschiebung commute, that is for α and β in $\mathcal{I}$, the following relation is satisfied in the homotopy category $HoG\mathcal{OT}$:*

$$V_\gamma R_\alpha = R_\alpha V_\gamma.$$

Proof. The two compositions we want to compare are the outer ways around the following diagram:

$$\begin{tikzcd}
\left(A^{\wedge X}\right)^{L_{\beta\alpha}} \arrow[r] \arrow[d] & \left(\left(A^{\wedge X}\right)^{L_\alpha}\right)^{L_{\beta\alpha}/L_\alpha} \arrow[r] \arrow[d, dotted] & \left(A^{\wedge X_{L_\alpha}}\right)^{L_{\beta\alpha}/L_\alpha} \arrow[r] \arrow[d, dotted] & \left(A^{\wedge X_{L_\alpha}}\right)^{L_\beta} \arrow[d] \\
F_{\mathbb{S}}(S^W \wedge G/L_{\beta\alpha_+}, \mathrm{sh}_W\, A^{\wedge X})^G \arrow[r, dotted] \arrow[d] & {} \arrow[r, dotted] \arrow[d, dotted] & {} \arrow[r, dotted] \arrow[d, dotted] & F_{\mathbb{S}}(S^{W'} \wedge G/L_{\beta_+}, \mathrm{sh}_{W'}\, A^{\wedge X^{L_\alpha}})^G \arrow[d] \\
F_{\mathbb{S}}(S^W \wedge G/L_{\gamma\beta\alpha_+}, \mathrm{sh}_W\, A^{\wedge X})^G \arrow[r, dotted] & {} \arrow[r, dotted] & {} \arrow[r, dotted] & F_{\mathbb{S}}(S^{W'} \wedge G/L_{\gamma\beta_+}, \mathrm{sh}_{W'}\, A^{\wedge X^{L_\alpha}})^G \\
\left(A^{\wedge X}\right)^{L_{\gamma\beta\alpha}} \arrow[r] \arrow[u] & \left(\left(A^{\wedge X}\right)^{L_\alpha}\right)^{L_{\gamma\beta\alpha}/L_\alpha} \arrow[r] \arrow[u, dotted] & \left(A^{\wedge X_{L_\alpha}}\right)^{L_{\gamma\beta\alpha}/L_\alpha} \arrow[r] \arrow[u, dotted] & \left(A^{\wedge X_{L_\alpha}}\right)^{L_{\gamma\beta}} \arrow[u]
\end{tikzcd}$$

Using the isomorphism $F_{\mathbb{S}}(G/H, A)^G \cong F_{\mathbb{S}}(S^0, A)^H$ several times, we reduce to showing stable commutativity of the middle row, with the commutative center square given by :

$$\begin{tikzcd}
F_{\mathbb{S}}(S^W \wedge G/L_{\beta_+}, \mathrm{sh}_W\, F_{\mathbb{S}}(G/L_{\alpha+}, \Lambda_X A)^G)^G \arrow[r, "\gamma_*"] \arrow[d, "(tr_{L_\beta}^{L_{\gamma\beta}})^*"'] & F_{\mathbb{S}}(S^{W'} \wedge G/L_{\beta_+}, \mathrm{sh}_{W'}(\Lambda_{X_{L_\alpha}} A))^G \arrow[d, "(tr_{L_\beta}^{L_{\gamma\beta}})^*"] \\
F_{\mathbb{S}}(S^W \wedge G/L_{\gamma\beta_+}, \mathrm{sh}_W\, F_{\mathbb{S}}(G/L_{\alpha+}, \Lambda_X A)^G)^G \arrow[r, "\gamma_*"] & F_{\mathbb{S}}(S^{W'} \wedge G/L_{\gamma\beta_+}, \mathrm{sh}_{W'}(\Lambda_{X_{L_\alpha}} A))^G
\end{tikzcd}$$

The two outer squares in the middle row then both commute up to stable equivalence since the transfer is compatible with the change of groups, i.e. $\rho_\alpha^*(tr_{\beta\alpha}^{\gamma\beta\alpha})$ is a model for the transfer $tr_\beta^{\gamma\beta}$. $\square$

We close our exposition by mentioning that for the case of G a (p-adic) torus, there is a fourth kind of structure maps, the higher differentials, which are defined using a stable splitting of $S^1_+ \simeq S^0 \vee S^1$, which can again be defined in terms of the equivariant transfer above (cf. [CDD, 3]). An exhaustive list of relations between these can be found in [CDD, 3.22], but it would go too far to reformulate them here, since the indexing alone is intricate enough to require extensive study. However, we have already seen above that the setting of orthogonal spectra with the Loday functor of 3.2.1 is well equipped for the study of covering homology and the higher structure surrounding it, while having the advantage of being much more concrete than the Γ-space analogue.

Appendix A

Category Theory

A.1 Some Category Theory

A.1.1 Categories

We recall some of the basics of category theory. We assume that the reader is familiar with the notions in this chapter, but the explicit definitions allow for an easier transition to monoidal and enriched categories in the next sections. The canonical reference and source for these definitions is Chapter I of [McL], though we have allowed ourselves some reformulations for the sake of uniformity when switching to the enriched setting.

Definition A.1.1. A category $\mathcal{C}$ consists of the following data:

- A class $\mathrm{Ob}(\mathcal{C})$ of *objects of* $\mathcal{C}$.
- For every two objects A and B of $\mathcal{C}$ a class $\mathcal{C}(A, B)$ of *morphisms in* $\mathcal{C}$. An element f of this class is called *morphism from A to B*. We write $f\colon A \to B$ and say that f has source A and target B.
- For every object A of $\mathcal{C}$, a distinguished morphism $\mathrm{id}_A \in \mathcal{C}(A, A)$, called *the identity of A*.
- For every three objects A, B and C of $\mathcal{C}$ a binary operation $\mu\colon \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$, called *the composition in* $\mathcal{C}$, we often write $g \circ f$ instead of $\mu(g, f)$.

This data has to satisfy the following two conditions:

- *(identity)* For any morphism $f\colon A \to B$, the composites $\mathrm{id}_B \circ f$ and $f \circ id_A$ are equal to f.
- *(associativity)* If $f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$ and $h \in \mathcal{C}(C, D)$ then the composite $h \circ (g \circ f)$ is equal to $(h \circ g) \circ f$.

A category $\mathcal{C}$ is called *locally small* if for any two objects A and B of $\mathcal{C}$, the class $\mathcal{C}(A, B)$ is a set. It is called *small* if additionally the class $\mathrm{Ob}(\mathcal{C})$ is also a set.
Two morphisms f and g in $\mathcal{C}$ are called *composable*, if the source of g is the same object as the target of f.

Definition A.1.2. A *functor* $\mathcal{F}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ is a mapping that

- assigns to each object A of $\mathcal{C}$ an object $\mathcal{F}(A)$ of $\mathcal{D}$.
- assigns to each morphism $f\colon A \to B$ in $\mathcal{C}$ a morphism $\mathcal{F}(f)\colon \mathcal{F}(A) \to \mathcal{F}(B)$ in $\mathcal{D}$.

This assignment has to satisfy the following two conditions:

- *(identity)* For every object A in $\mathcal{C}$, $\mathcal{F}(\mathrm{id}_A)$ is equal to $\mathrm{id}_{\mathcal{F}(A)}$.
- *(composition)* For any two composable morphisms f and g in $\mathcal{C}$, $\mathcal{F}(g \circ f)$ is equal to $\mathcal{F}(g) \circ \mathcal{F}(f)$.

Example A.1.3. Note that for each category $\mathcal{C}$ there is a functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that is the identity on both objects and morphisms. Also, the composition of functors is associative, such that we get a *category* $\mathcal{C}\mathbf{at}$ *of categories*, with functors as the morphisms. $\mathcal{C}\mathbf{at}$ has products ([McL, II.3]), in particular for $\mathcal{C}$ and $\mathcal{C}'$ categories, $\mathcal{C} \times \mathcal{C}'$ is the category whose objects are pairs (C, C') with $C \in \mathcal{C}$ and $C' \in \mathcal{C}'$, and the analogous morphisms.

Definition A.1.4. A *natural transformation* η between functors $\mathcal{F}$ and $\mathcal{G}$ from $\mathcal{C}$ to $\mathcal{D}$, is a mapping that assigns to each object A of $\mathcal{C}$ a morphism $\eta_A\colon \mathcal{F}(A) \to \mathcal{G}(A)$ in $\mathcal{D}$, such that the diagram

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\eta_A"'] & \mathcal{F}(B) \arrow[d, "\eta_B"] \\ \mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(B) \end{tikzcd}$$

is commutative in $\mathcal{D}$ for all $f \in \mathcal{C}(A, B)$.
A natural transformation $\eta\colon \mathcal{F} \to \mathcal{G}$ is a *natural isomorphism* if there is a natural transformation $\eta^{-1}\colon \mathcal{G} \to \mathcal{F}$ such that both composites yield the identity functor.

Example A.1.5. Again, for any functor, there is an identity natural transformation, and composition of natural transformation is associative. Therefore the set of functors $\mathcal{C} \to \mathcal{D}$, denoted by $\mathcal{C}\mathbf{at}(\mathcal{C}, \mathcal{D})$ is itself a category with morphisms the natural transformations. A category arising in this way this is called *functor category.*

A.1.2 Monoidal Categories

Again we repeat the basic definitions as far as they will be used in the enriched setting in the next section. Again the definitions are only slight reformulations of the ones in [McL, VII], adapted to our needs.

Definition A.1.6. A *monoidal category* consists of the following data:

- An *underlying category* $\mathcal{C}$.
- A bifunctor (i.e. a functor out of the product category, see A.1.3) $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the monoidal product.
- A designated object $\mathbb{I}$ of $\mathcal{C}$, called the identity object.
- Natural isomorphisms $\lambda\colon (\mathbb{I} \otimes \mathrm{id}) \to \mathrm{id}$ and $\rho(\mathrm{id} \otimes \mathbb{I}) \to \mathrm{id}$ expressing that $\mathbb{I}$ is a left and right identity object for the monoidal product.
- A natural isomorphism $a\colon [(- \otimes -) \otimes -] \to [- \otimes (- \otimes -)]$ expressing that the monoidal product is associative.

The natural transformations have to satisfy the following two coherence conditions:

- For all objects A and B of $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}(A \otimes \mathbb{I}) \otimes B \arrow[rr, "a_{A,I,B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"'] & & A \otimes (\mathbb{I} \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda_B"] \\ & A \otimes B & \end{tikzcd}$$

- For all objects A, B, C and D of $\mathcal{C}$, the following pentagon commutes:

$$\begin{tikzcd}[column sep=small] & & (A \otimes (B \otimes C)) \otimes D \arrow[drr, "a_{A,B\otimes C,D}"] & & \\ ((A \otimes B) \otimes C) \otimes D \arrow[urr, "a_{A,B,C} \otimes \mathrm{id}_D"] \arrow[dr, "a_{A\otimes B,C,D}"'] & & & & A \otimes ((B \otimes C) \otimes D) \arrow[dl, "\mathrm{id}_A \otimes a_{B,C,D}"] \\ & (A \otimes B) \otimes (C \otimes D) \arrow[rr, "a_{A,B,C\otimes D}"'] & & A \otimes (B \otimes (C \otimes D)) & \end{tikzcd}$$

Instead of the tupel $(\mathcal{C}, \otimes, \mathbb{I}, \lambda, \rho, a)$, we often just refer to the monoidal category as $(\mathcal{C}, \otimes, \mathbb{I})$ or even just to $\mathcal{C}$, when it is clear which monoidal structure is meant.

Definition A.1.7. A *lax monoidal functor* $\mathcal{F}\colon (\mathcal{C}, \otimes, \mathbb{I}) \to (\mathcal{D}, \times, \mathbb{J})$ between monoidal categories consists of the following data:

- An underlying functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$.

- A natural transformation $\mu\colon [\mathcal{F} \times \mathcal{F}] \to \mathcal{F}(- \otimes -)$.
- A designated morphism $\iota\colon \mathbb{J} \to \mathcal{F}(\mathbb{I})$ in $\mathcal{D}$.

These have to satisfy the following coherence conditions:

- For all objects A, B and C of $\mathcal{C}$, the following diagram commutes in $\mathcal{D}$:

$$\begin{tikzcd}
(\mathcal{F}(A) \times \mathcal{F}(B)) \times \mathcal{F}(C) \arrow[r, "a_{\mathcal{D}}"] \arrow[d, "\mu_{A,B} \times \mathrm{id}"'] & \mathcal{F}(A) \times (\mathcal{F}(B) \times \mathcal{F}(C)) \arrow[d, "\mathrm{id} \times \mu_{B,C}"] \\
\mathcal{F}(A \otimes B) \times \mathcal{F}(C) \arrow[d, "\mu_{A \otimes B, C}"'] & \mathcal{F}(A) \times (\mathcal{F}(B \otimes C)) \arrow[d, "\mu_{A, B \otimes C}"] \\
\mathcal{F}((A \otimes B) \otimes C) \arrow[r, "\mathcal{F}(a_{\mathcal{C}})"'] & \mathcal{F}(A \otimes (B \otimes C))
\end{tikzcd}$$

- For every object A of $\mathcal{C}$, the following diagrams commute in $\mathcal{D}$:

$$\begin{tikzcd}
\mathcal{F}(A) \times \mathbb{J} \arrow[r, "\rho_{\mathcal{D}}"] \arrow[d, "\mathrm{id} \times \iota_{\mathcal{D}}"'] & \mathcal{F}(A) \\
\mathcal{F}(A) \times \mathcal{F}(\mathbb{I}) \arrow[r, "\mu_{A,\mathbb{I}}"'] & \mathcal{F}(A \otimes \mathbb{I}) \arrow[u, "\mathcal{F}(\rho_{\mathcal{C}})"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{J} \times \mathcal{F}(A) \arrow[r, "\lambda_{\mathcal{D}}"] \arrow[d, "\iota_{\mathcal{D}} \times \mathrm{id}"'] & \mathcal{F}(A) \\
\mathcal{F}(\mathbb{I}) \times \mathcal{F}(A) \arrow[r, "\mu_{\mathbb{I},A}"'] & \mathcal{F}(I \times A) \arrow[u, "\mathcal{F}(\lambda_{\mathcal{C}})"']
\end{tikzcd}$$

A lax monoidal functor $(\mathcal{F}, \mu, \iota)$ is *strong monoidal* if μ and ι are (natural) isomorphisms, it is *strict monoidal* if they are the identity (transformation).
Again we often only refer to $\mathcal{F}$ as the monoidal functor, suppressing μ and ι in the notation, where they are not critical to the discussion.

Definition A.1.8. A monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$ is called cartesian, if $\otimes$ is the categorical product and $\mathbb{I}$ is a terminal object.

Some monoidal categories have additional extra structure:

Definition A.1.9. A monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$ is called *closed*, if for all objects A of $\mathcal{C}$, the functor $(- \otimes A)\colon \mathcal{C} \to \mathcal{C}$ has a right adjoint ([McL, IV.1]), denoted by $\mathbf{Hom}(A, -)$.
Objects of $\mathcal{C}$ the form $\mathbf{Hom}(A, B)$ are called internal **Hom** objects, the counits of these adjunctions are usually called the evaluations $\mathbf{Hom}(A, B) \otimes A \to B$.

Lemma A.1.10. *If* $(\mathcal{C}, \otimes, \mathbb{I})$ *is closed monoidal, then there is a natural isomorphism:*

$$\mathbf{Hom}(A \otimes B, C) \cong \mathbf{Hom}(A, \mathbf{Hom}(B, C)).$$

Proof. Let Z be any object of $\mathcal{C}$, then we have the following chain of natural isomorphisms:

$$\begin{aligned} \mathcal{C}(Z, \mathbf{Hom}(A \otimes B, C)) &\cong \mathcal{C}(Z \otimes (A \otimes B), C) \\ &\cong \mathcal{C}((Z \otimes A) \otimes B, C) \\ \mathcal{C}(Z, \mathbf{Hom}(A, \mathbf{Hom}(B, C))) &\cong \mathcal{C}(Z \otimes A, \mathbf{Hom}(B, C)) \end{aligned}$$

Hence the Yoneda lemma ([McL, III.2.1]) gives the desired result. □

Construction A.1.11. Note that for any locally small monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$, we get a lax monoidal functor $\mathcal{C}(\mathbb{I}, -)\colon \mathcal{C} \to \mathcal{S}\mathrm{et}$, where the category of sets has the cartesian monoidal structure. The unit morphism ι sends the terminal one-point set to the identity morphism of $\mathbb{I}$, whereas the natural transformation

$$\mu\colon \mathcal{C}(\mathbb{I}, A) \times \mathcal{C}(\mathbb{I}, B) \xrightarrow{\otimes} \mathcal{C}(\mathbb{I} \otimes \mathbb{I}, A \otimes B) \cong \mathcal{C}(\mathbb{I}, A \otimes B),$$

uses the isomorphism $\lambda_{\mathbb{I}}\colon \mathbb{I} \otimes \mathbb{I} \to \mathbb{I}$. This functor assigns to objects of $\mathcal{C}$ their *underlying sets*.
If $\mathcal{C}$ is additionally closed and A and B are objects of $\mathcal{C}$, then the adjunction

$$(- \otimes A)\colon \mathcal{C} \rightleftarrows \mathcal{C}\colon \mathbf{Hom}(A, -)$$

gives the following natural isomorphism:

$$\mathcal{C}(\mathbb{I} \otimes A, B) \cong \mathcal{C}(\mathbb{I}, \mathbf{Hom}(A, B))$$

Since $\mathbb{I} \otimes A$ is isomorphic to A via λ_A, this implies that

$$\mathcal{C}(A, B) \cong \mathcal{C}(\mathbb{I}, \mathbf{Hom}(A, B)),$$

i.e. the underlying set of the internal **Hom** object $\mathbf{Hom}(A, B)$ is indeed naturally isomorphic to the morphism set $\mathcal{C}(A, B)$.
Considerations in this spirit lead to the study of enriched categories. We will discuss these further in Section A.2.

For any monoidal category, there are categories of *monoids* and (*left* or *right*) *modules* over such. Definitions can for example be found in [McL, VII.3,4], and will be omitted here.

A.1.3 Symmetric Monoidal Categories

We repeat more of the definitions from [McL, XI], adapted to our notation.

Definition A.1.12. A *symmetric monoidal category* is a monoidal category $(\mathcal{C}, \otimes, \mathbb{I}, \lambda, \rho, a)$, together with a natural isomorphism τ

$$\tau : \otimes \to \otimes \circ \mathrm{twist},$$

where twist is the bifunctor that permutes the two inputs.
This data has to satisfy additional coherence conditions:

- The composition of τ with itself is the identity, i.e.

  $$\tau_{B,A} \circ \tau_{A,B} = \mathrm{id}_{A \otimes B},$$

  for all objects A and B of $\mathcal{C}$.
- Compatibility with the unit, i.e.

  $$\rho = \lambda \circ \tau_{-,\mathbb{I}}.$$

- For all objects A, B, and C of $\mathcal{C}$, the following hexagon commutes:

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "a"] \arrow[d, "\tau \otimes \mathrm{id}_C"'] & A \otimes (B \otimes C) \arrow[r, "\tau"] & (B \otimes C) \otimes A \arrow[d, "a"] \\
(B \otimes A) \otimes C \arrow[r, "a"'] & B \otimes (A \otimes C) \arrow[r, "\mathrm{id}_B \otimes \tau"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

Example A.1.13. All cartesian or cocartesian monoidal categories are symmetric, making use of the universal properties of (co-) products and the projections to respectively inclusions of coproduct factors.

Since the following definition is widely used, but is only implicit in [McL], we give a few more details:

Definition A.1.14. A *commutative monoid* in a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{I}, \tau)$ consists of the following data:

- An object M of $\mathcal{C}$.
- A morphism $\eta : \mathbb{I} \to M$ in $\mathcal{C}$, called the unit of M.
- A morphism $\mu : M \otimes M \to M$ in $\mathcal{C}$, called the multiplication of M.

Such that the following diagrams are commutative:

- *unit:*

$$\begin{tikzcd} \mathbb{I} \otimes M \arrow[r, "\eta\otimes \mathrm{id}_M"] \arrow[dr, "\lambda"'] & M \otimes M \arrow[d, "\mu"] & M \otimes \mathbb{I} \arrow[l, "\mathrm{id}_M \otimes \eta"'] \arrow[dl, "\rho"] \\ & M & \end{tikzcd}$$

- *associativity:*

$$\begin{tikzcd} (M \otimes M) \otimes M \arrow[r, "a"] \arrow[d, "\mu\otimes \mathrm{id}_M"'] & M \otimes (M \otimes M) \arrow[r, "\mathrm{id}_M \otimes \mu"] & M \otimes M \arrow[d, "\mu"] \\ M \otimes M \arrow[rr, "\mu"'] & & M \end{tikzcd}$$

- *commutativity:*

$$\begin{tikzcd} M \otimes M \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\ & M & \end{tikzcd}$$

Again we can define categories of commutative monoids and modules over such as in the non-symmetric case. We work extensively with these in the case of $\mathcal{C}$ being the category $\mathcal{S}p^{\mathcal{O}}$ of orthogonal spectra. The following lemmas are well known, but it seems hard to find explicit references:

Lemma A.1.15. *Let M be a commutative monoid, then the categories of left M-modules and right M-modules are isomorphic.*

Proof. For V a right M module with action map $\nu : V \otimes M \to V$, define a left module structure on V in the following way:

$$\begin{tikzcd} M \otimes V \arrow[r, "\tau"] \arrow[dr, "\nu'"'] & V \otimes M \arrow[d, "\nu"] \\ & V \end{tikzcd}$$

There are coherence diagrams to be checked:

$$\begin{tikzcd} & V \otimes \mathbb{I} \arrow[r, "\eta"] \arrow[dd] & V \otimes M \arrow[dd] \arrow[dr] & \\ V \arrow[ur, "\rho^{-1}"] \arrow[dr, "\lambda^{-1}"'] & & & V \\ & \mathbb{I} \otimes V \arrow[r, "\eta"'] & M \otimes V \arrow[ur] & \end{tikzcd}$$

The left triangle commutes by the unit axiom for the symmetric monoidal structure, the middle square because the twist isomorphism is natural. Hence the lower composite is the identity since the upper one was. For associativity of the new multiplication map, check commutativity of the outermost two ways around the following diagram. We omit the categorical associativity isomorphisms, hence brackets, from the notation.

$$
\begin{tikzcd}
 & & & M_1 \otimes V \arrow[dr] & \\
 & M_1 \otimes V \otimes M_2 \arrow[urr, "\nu"] \arrow[rr] & & V \otimes M_2 \otimes M_1 \arrow[r, "\nu"] \arrow[dd, "\mu"] & V \otimes M_1 \arrow[dd, "\nu"] \\
M_1 \otimes M_2 \otimes V \arrow[ur] \arrow[rr] \arrow[d, "\mu"] & & V \otimes M_1 \otimes M_2 \arrow[ur] \arrow[dr] & & \\
M \otimes V \arrow[rrr] & & & V \otimes M \arrow[r, "\nu"'] & V
\end{tikzcd}
$$

Here the center triangle commutes because M was commutative, and the lower right square does so because V was a right M-module. The center parallelogram is an instance of the hexagon coherence. The other subdiagrams both commute because of the naturality of the twist isomorphism.
An analogous argument gives a functor from left to right M-modules, and they are obviously inverses to each other. $\square$

Lemma A.1.16. *Let M be a commutative monoid in the closed symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{I}, \tau)$. Assume that $\mathcal{C}$ has equalizers and coequalizers. Then the category of (right) M-modules inherits a closed symmetric monoidal structure.*

Proof. For M-modules V and W define the monoidal product $V \otimes_M W$ of as the coequalizer

$$V \otimes M \otimes W \rightrightarrows V \otimes W \to V \otimes_M W, \tag{A.1.17}$$

where one of the arrows uses the action map on V, and the other the action on W precomposed with the twist $V \otimes \tau_{M,W}$. The internal **Hom** object $\mathbf{Hom}_M(V, W)$ is the equalizer

$$\mathbf{Hom}_M(V, W) \to \mathbf{Hom}(V, W) \rightrightarrows \mathbf{Hom}(V \otimes M, W),$$

where one of the arrows is induced by the action map of V, and the other one is induced by the adjoint of the action map of W, using the isomorphism $\mathbf{Hom}(V \otimes M, W) \cong \mathbf{Hom}(V, \mathbf{Hom}(M, W))$ (cf. A.1.10). Checking coherence diagrams is then done using the universal properties of (co-)equalizers as well as the corresponding diagrams in $\mathcal{C}$ together with the same isomorphism A.1.10. $\square$

Lemma A.1.18. *Let* $(\mathcal{C}, \otimes, \mathbb{I})$ *be a symmetric monoidal category. Then* $\otimes$ *is the coproduct in the category of commutative monoids in* $\mathcal{C}$.

Proof. The monoidal product of two commutative monoids M and N is again a commutative monoid using the unit map

$$\eta : \mathbb{I} \xrightarrow{\lambda^{-1}} \mathbb{I} \otimes \mathbb{I} \xrightarrow{\eta_M \otimes \eta_N} M \otimes N,$$

and the multiplication

$$\mu : M \otimes N \otimes M \otimes N \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} M \otimes M \otimes N \otimes N \xrightarrow{\mu_M \otimes \mu_N} M \otimes N.$$

Then given maps of commutative monoids $M \to C$ and $N \to C$, we get a map

$$M \otimes N \to C \otimes C \xrightarrow{\mu_C} C.$$

On the other hand given a map of commutative monoids $M \otimes N \to C$, precomposition with the units of N and M, respectively, yields maps $M \to C$ and $N \to C$. These two constructions are obviously inverse to each other, hence $M \otimes N$ satisfies the universal property of the coproduct ([McL, III.3]). □

We will discuss monoids, (commutative) algebras and modules over such in various (symmetric) monoidal categories $(\mathcal{C}, \otimes, \mathbb{I})$. Often we use that the forgetful functors to $\mathcal{C}$ have left adjoints, and hence we recall how these adjoints are formed in general:

Lemma A.1.19. *Let* R *be a monoid in* $(\mathcal{C}, \otimes, \mathbb{I})$,

- *the functor* $- \otimes R$ *is left adjoint to the forgetful functor from* $\mathcal{C}$ *to right* R*-modules.*
- *the functor* $R \otimes -$ *is left adjoint to the forgetful functor from* $\mathcal{C}$ *to left* R*-modules.*

In both cases the action of R *simply uses the multiplication* R. *If* R *is commutative, and the category of* R*-modules has coproducts, then*

- *the functor* $\mathbb{A} := \coprod_{i \in \mathbb{N}} (-)^{\otimes_R i}$ *is left adjoint to the forgetful functor from* R*-algebras to* R*-modules.*

If the category of R *modules is cocomplete, then*

- *the functor* $\mathbb{E} := \coprod_{i \in \mathbb{N}} [(-)^{\otimes_R i}]_{\Sigma_i}$ *is left adjoint to the forgetful functor from commutative* R*-algebras to* R*-modules.*

In both cases multiplication is by simply concatenating coproduct factors and the unit map is the inclusion of R as the factor indexed by 0*.*

Here $(-)^{\otimes_R i}$ denotes the i-fold tensor power over R. Then $[-]_{\Sigma_i}$ is taking the orbits of the action of Σ_i that permutes tensor factors, i.e. the action induces a functor from Σ_i viewed as a one-object category (cf. A.2.16) and $[-]_{\Sigma_i}$ denotes its colimit. We could of course have given each of these functor in terms of the monads that the unit of the adjunction induces on $\mathcal{C}$.

A.2 Enriched Category Theory

A.2.1 Enriched Categories

Let $(\mathcal{V}, \otimes, \mathbb{I})$ be a monoidal category.

Definition A.2.1. A *category* $\mathcal{C}$ *enriched over* $\mathcal{V}$, is a $\mathcal{V}$-category in the sense of [K, 1.2]. This amounts to the following structure:

- A class $\mathrm{Ob}(\mathcal{C})$ of *objects of* $\mathcal{C}$.
- For every two objects A and B of $\mathcal{C}$ an object $\mathcal{C}(A, B)$ of $\mathcal{V}$ called the Hom-object of A and B.
- For every object A of $\mathcal{C}$, a distinguished morphism $\mathrm{id}_A \colon \mathbb{I} \to \mathcal{C}(A, A)$ in $\mathcal{V}$, called *the identity of* A.
- For every three objects A, B and C of $\mathcal{C}$ a morphism $\gamma \colon \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \to \mathcal{C}(A, C)$, called *the composition in* $\mathcal{C}$.

This data has to satisfy the following two conditions:

- For all objects A,B,C and D of $\mathcal{C}$, the following diagram commutes in $\mathcal{V}$:

$$\begin{tikzcd}
(\mathcal{C}(C,D) \otimes \mathcal{C}(B,C)) \otimes \mathcal{C}(A,B) \arrow[rr, "a"] \arrow[d, "\gamma\otimes\mathrm{id}"] & & \mathcal{C}(C,D) \otimes (\mathcal{C}(B,C) \otimes \mathcal{C}(A,B)) \arrow[d, "\mathrm{id}\otimes\gamma"'] \\
\mathcal{C}(B,D) \otimes \mathcal{C}(A,B) \arrow[dr, "\gamma"'] & & \mathcal{C}(C,D) \otimes \mathcal{C}(A,C) \arrow[dl, "\gamma"] \\
& \mathcal{C}(A,D) &
\end{tikzcd}$$

- For all objects A and B in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
\mathbb{I} \otimes \mathcal{C}(A,B) \arrow[d, "\mathrm{id}_B \otimes \mathrm{id}"'] \arrow[dr, "\lambda"] & & \mathcal{C}(A,B) \otimes \mathbb{I} \arrow[dl, "\rho"'] \arrow[d, "\mathrm{id} \otimes \mathrm{id}_A"] \\
\mathcal{C}(B,B) \otimes \mathcal{C}(A,B) \arrow[r, "\gamma"'] & \mathcal{C}(A,B) & \mathcal{C}(A,B) \otimes \mathcal{C}(A,A) \arrow[l, "\gamma"]
\end{tikzcd}$$

Example A.2.2. Note that the usual notion of a category generalizes to this context, if we restrict ourselves to locally small categories. One checks easily that those are the same as categories enriched over the cartesian monoidal category $\mathcal{S}$et of sets.

Example A.2.3. For any $(\mathcal{V}, \mathbb{I})$, we can consider the *trivial $\mathcal{V}$-category* $\star$ with one object C, and the morphism object $\star(C, C) = \mathbb{I}$.

Definition A.2.4. A *functor $\mathcal{F}$ enriched over* $\mathcal{V}$ between two categories $\mathcal{D}$ and $\mathcal{C}$ enriched over $\mathcal{V}$ consists of the following data:

- A mapping $\mathcal{F}\colon \mathrm{Ob}(\mathcal{D}) \to \mathrm{Ob}(\mathcal{C})$.
- For all objects A and B of $\mathcal{D}$, a morphism $\mathcal{F}_{A,B}\colon \mathcal{D}(A, B) \to \mathcal{C}(\mathcal{F}(A), \mathcal{F}(B))$ in $\mathcal{V}$.

These have to satisfy the following coherence conditions:

- *(identity)* For all objects A of $\mathcal{D}$ the following diagram commutes in $\mathcal{V}$:

$$\begin{tikzcd}
 & \mathcal{D}(A,A) \arrow[d, "\mathcal{F}_{A,A}"] \\
\mathbb{I} \arrow[ur, "\mathrm{id}_A"] \arrow[r, "\mathrm{id}_{\mathcal{F}(A)}"'] & \mathcal{C}(\mathcal{F}(A), \mathcal{F}(A))
\end{tikzcd}$$

- *(composition)* For all objects A, B and C of $\mathcal{D}$ the following diagram commutes in $\mathcal{V}$:

$$\begin{tikzcd}
\mathcal{D}(B,C) \otimes \mathcal{D}(A,B) \arrow[r, "\gamma"] \arrow[d, "\mathcal{F} \otimes \mathcal{F}"'] & \mathcal{D}(A,C) \arrow[d, "\mathcal{F}"] \\
\mathcal{C}(\mathcal{F}(B), \mathcal{F}(C)) \otimes \mathcal{C}(\mathcal{F}(A), \mathcal{F}(B)) \arrow[r, "\gamma"'] & \mathcal{C}(\mathcal{F}(A), \mathcal{F}(C))
\end{tikzcd}$$

Definition A.2.5. For two functors $\mathcal{F}, \mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ enriched over $\mathcal{V}$, an *enriched natural transformation* $\alpha\colon \mathcal{F} \to \mathcal{G}$ consists of morphisms $\mathbb{I} \to \mathcal{C}(\mathcal{F}(A), \mathcal{G}(A))$ in $\mathcal{V}$ for all objects A of $\mathcal{D}$, such that the following coherence diagrams commute in $\mathcal{V}$ for all objects A and B of $\mathcal{D}$:

$$\begin{tikzcd}
 & \mathbb{I}\otimes\mathcal{D}(A,B) \arrow[r, "\alpha_B\otimes\mathcal{F}"] & \mathcal{C}(\mathcal{F}(B),\mathcal{G}(B))\otimes\mathcal{C}(\mathcal{F}(A),\mathcal{F}(B)) \arrow[dr, "\gamma"] & \\
\mathcal{D}(A,B) \arrow[ur, "\lambda^{-1}"] \arrow[dr, "\rho^{-1}"'] & & & \mathcal{C}(\mathcal{F}(A),\mathcal{G}(B)) \\
 & \mathcal{D}(A,B)\otimes\mathbb{I} \arrow[r, "\mathcal{G}\otimes\alpha_A"'] & \mathcal{C}(\mathcal{G}(A),\mathcal{G}(B))\otimes\mathcal{C}(\mathcal{F}(A),\mathcal{G}(A)) \arrow[ur, "\gamma"'] &
\end{tikzcd} \tag{A.2.6}$$

Remark A.2.7. This definition gives the class $[\mathcal{D},\mathcal{C}]_0(\mathcal{F},\mathcal{G})$ of functors enriched over $\mathcal{V}$ the structure of a category (one checks that there is an identity transformation and that composition of natural transformations is associative). This makes the category $\mathcal{V}$-$\mathcal{C}\mathbf{at}$ of $\mathcal{V}$-enriched categories and $\mathcal{V}$-enriched functors into a 2-category, i.e. into a category enriched over $\mathcal{C}\mathbf{at}$.

Remark A.2.8. Under more assumptions, one can also define a $\mathcal{V}$-enriched functor category $[\mathcal{D},\mathcal{C}]$. Let $\mathcal{V}$ be closed and complete and $\mathcal{D}$ be equivalent to a small category. Then for two enriched functors $\mathcal{F}$ and $\mathcal{G}$, the following end exists and forms the morphism $\mathcal{V}$-space $[\mathcal{D},\mathcal{C}](\mathcal{F},\mathcal{G})$:

$$\int_{d\in\mathcal{D}} \mathcal{C}(\mathcal{F}(d),\mathcal{G}(d)) \to \prod_{d\in\mathcal{D}} \mathcal{C}(\mathcal{F}(d),\mathcal{G}(d)) \rightrightarrows \prod_{d,d'\in\mathcal{D}} \mathbf{Hom}(\mathcal{D}(d,d'),\mathcal{C}(\mathcal{F}(d),\mathcal{G}(d'))).$$

As indicated it can be expressed as the equalizer along two maps adjoint to the two ways around diagram (A.2.6) above. Composition and identities are then inherited from $\mathcal{C}$ (cf. [K, 2.1]).

Construction A.2.9. Note that if we have a distinguished (lax) monoidal functor $(\mathbb{M},\mu,\iota)\colon(\mathcal{V},\otimes,\mathbb{I})\to(\mathcal{W},\times,\mathbb{J})$, any category $\mathcal{C}$ enriched over $\mathcal{V}$ gives a category enriched over $\mathcal{W}$, by just applying $\mathbb{M}$ to all the Hom objects. The identity morphisms are defined as the composites

$$\mathrm{id}'_A\colon \mathbb{J}\xrightarrow{\iota}\mathbb{M}[\mathbb{I}]\xrightarrow{\mathbb{M}[\mathrm{id}_A]}\mathbb{M}[\mathcal{C}(A,A)].$$

The composition is given by

$$\gamma'\colon\ \mathbb{M}[\mathcal{C}(B,C)]\times\mathbb{M}[\mathcal{C}(A,B)]\xrightarrow{\mu}\mathbb{M}[\mathcal{C}(B,C)\otimes\mathcal{C}(A,B)]\xrightarrow{\mathbb{M}[\gamma]}\mathbb{M}[\mathcal{C}(A,C)].$$

One checks that the coherence diagrams still commute.
Also, in the same way $\mathcal{V}$-enriched functors give $\mathcal{W}$-enriched functors and $\mathcal{V}$-enriched natural transformations give $\mathcal{W}$-enriched natural transformations via the monoidal

functor $\mathbb{M}$. (One checks that $\mathbb{M}(\mathcal{F})$ still takes identities to identities and respects composition, and that the appropriate diagram for $\mathbb{M}\,\alpha$ still commutes using the monoidal structure maps of $\mathbb{M}$).

Remark A.2.10. In the spirit of the above Remark A.2.7, one can check that $\mathbb{M}$ induces a $\mathcal{C}\mathbf{at}$-enriched, or 2-functor $\mathbb{M}\colon \mathcal{V}\text{-}\mathcal{C}\mathbf{at} \to \mathcal{W}\text{-}\mathcal{C}\mathbf{at}$, i.e. that $\mathbb{M}$ takes the identity $\mathcal{V}$-enriched natural transformations to the identity $\mathcal{W}$-enriched natural transformations, and that it respects composition of enriched natural transformations.

Example A.2.11. In this way, if $\mathcal{V}$ is a locally small monoidal category, every category enriched over $\mathcal{V}$ has a canonical underlying "normal" category, i.e. one enriched over $\mathcal{S}$et, with the same objects. The morphism sets are obtained by using the monoidal functor $\mathcal{V}(\mathbb{I}, -)$ from A.1.11 in the way described above.

Remark A.2.12. For $\mathcal{D}$ and $\mathcal{C}$ categories enriched over $\mathcal{V}$ as in A.2.8, the underlying underlying $\mathcal{S}$et-category of the functor $\mathcal{V}$-category $[\mathcal{D}, \mathcal{C}]$ is $[\mathcal{D}, \mathcal{C}]_0$, if the former exists.

Example A.2.13. Let $(\mathcal{V}_0, \otimes, \mathbb{I})$ be a closed monoidal category. Then there is a $\mathcal{V}_0$-category $\mathcal{V}$ that restricts to $\mathcal{V}_0$ along the monoidal functor $\mathcal{V}(\mathbb{I}, -)$.
Define $\mathcal{V}$ as having the same objects as $\mathcal{V}_0$ and for morphism objects set $\mathcal{V}(A, B) = \mathbf{Hom}(A, B)$. Then composition is adjoint to iterated evaluation, and the axioms for an enriched category trivially hold. When discussing a specific category $\mathcal{V}_0$, we will often identify $\mathcal{V}$ and $\mathcal{V}_0$ and therefore say that $\mathcal{V}$ is enriched over itself, but there are also important cases where we explicitly keep the notation separate (e.g. A.2.19).

Remark A.2.14. When viewing $\mathcal{V}$ as enriched over itself in this sense, Lemma A.1.10 can be reformulated to state that the adjunctions between $- \otimes A$ and $\mathbf{Hom}(A, -)$ are actually enriched, i.e. imply natural isomorphisms even on morphism objects.

Example A.2.15. Let $\mathcal{V} = \mathcal{T}op$ the cartesian monoidal category of topological spaces. Then a category $\mathcal{C}$ enriched over $\mathcal{T}op$ is a usual category, with a choice of topology on each morphism set, such that the composition law gives continuous (!) maps. More important for us is the closed monoidal variation $\mathcal{U}$, containing only the *compactly generated weak Hausdorff spaces.*
For another example let $\mathcal{V}$ be the category $\mathcal{T}$ *of based compactly generated weak Hausdorff spaces*, i.e. objects of $\mathcal{U}$ with a distinguished basepoint. We will usually drop the extra adjectives and just call these *spaces.*
Since $\mathcal{T}$ has products and coproducts, it is monoidal in several ways: with the cartesian product $\times$ and unit a one point space $\{*\}$, or, more importantly for us, with respect to the smash product $\wedge$ and unit S^0, the 0-sphere. The latter choice makes $\mathcal{T}$ closed monoidal, and we will denote the internal $\mathbf{Hom}$ spaces merely as $\mathcal{T}(-, -)$ in agreement with A.2.13. The identity functor $(\mathcal{T}, \wedge, S^0) \to (\mathcal{T}, \times, \{*\})$ is

lax monoidal, just as the functor $\mathcal{T} \to \mathcal{U}$ that forgets the basepoints. The monoidal structure maps are given by the projections $X \times Y \to X \wedge Y$ and the inclusion of $\{*\}$ as the non-basepoint of S^0. These functors give us a canonical way to view a category enriched over $(\mathcal{T}, \wedge, S^0)$ as one enriched over $(\mathcal{T}, \times, \{*\})$, or $\mathcal{U}$. The forgetful functor from $\mathcal{U}$ to $\mathcal{S}$et preserves products and is therefore strict monoidal, indeed it is isomorphic to the functor described in A.1.11. Hence a category enriched over either monoidal structure on $\mathcal{T}$ (or $\mathcal{U}$) is a category. In the other directions, including sets as discrete topological spaces and adding disjoint basepoints to spaces in $\mathcal{U}$ give left adjoints to the forgetful functors and are also (strong) monoidal. Hence together with A.2.13 we can view $\mathcal{U}$ and $\mathcal{T}$ as enriched over either themselves or each other. Generally, categories enriched over any of the above are called *topological categories*. Enriched functors between both $\mathcal{T}op$- $\mathcal{U}$- and $\mathcal{T}$-categories are usually called the continuous functors.

Definition A.2.16. For G a group, there is a *category* $\mathcal{G}$ *associated to* G. It consists of one object $\star$, and the morphism set $\mathcal{G}(\star, \star)$ is given as the group G. The neutral element of the group is the identity morphism and the group multiplication gives composition of morphisms. Often we use the group G and its associated category synonymously.

If G is a topological group, its associated category is canonically a topological category. If G is in $\mathcal{U}$, its associated category is canonically enriched over $\mathcal{U}$, and adding a disjoint basepoint as above, enriched over $\mathcal{T}$. The latter viewpoint is often more useful than considering a specific element of G as the basepoint of the morphism space.

Definition A.2.17. We denote the category of functors $G \to \mathcal{S}$et and natural transformations between them by $G\mathcal{S}$et instead of $[G, \mathcal{S}\mathrm{et}]_0$, its objects are called G-sets. Note that a G-set is the same as a set with a (left) action of G, and a morphism of G-sets is a G-equivariant map.

Just like $\mathcal{S}$et, the category $G\mathcal{S}$et is a cartesian monoidal category with respect to the usual cartesian product of sets, which is given the diagonal G-action. The unit object is the trivial G-set consisting of only one point. Note that there are **two** obvious monoidal functors $G\mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$. One is the forgetful functor, which is obviously product preserving, but this is *not* the functor described in A.1.11. In fact, $G\mathcal{S}\mathrm{et}(\star, X)$ assigns to a G-set X its set of G-fixed points X^G, and this gives the second monoidal functor. We distinguish this in language by saying X *is* a set, but *has* X^G as its underlying set (of G-fixed points).

Definition A.2.18. Let G be a group, a *G-category* is a category enriched over $G\mathcal{S}$et. We call the elements of the morphism G-sets *morphisms*, whereas the elements of the underlying G-fixed point sets are called *G-maps*. As above, every

G-category is also a category, and has an underlying G-fixed category.
A *G-functor* $\mathcal{F}\colon \mathcal{D} \to \mathcal{C}$ between G-categories is an enriched functor of enriched categories, i.e. the induced maps on morphism G-sets

$$\mathcal{F}\colon \mathcal{D}(X,Y) \to \mathcal{C}(\mathcal{F}X, \mathcal{F}Y)$$

have to be G-equivariant.
Two types of natural transformations are important for us: A *natural G-transformation* $\alpha\colon \mathcal{F} \to \mathcal{G}$ between two G-functors, is an enriched natural transformation of enriched functors, i.e. it consists of a G-map $\alpha_X \in \mathcal{C}(\mathcal{F}X, \mathcal{G}X)$ for every object X of $\mathcal{D}$ such that the diagrams

$$\begin{tikzcd} \mathcal{F}X \arrow[r, "\mathcal{F}f"] \arrow[d, "\alpha_X"'] & \mathcal{F}Y \arrow[d, "\alpha_Y"] \\ \mathcal{G}X \arrow[r, "\mathcal{G}f"'] & \mathcal{G}Y \end{tikzcd},$$

commute in $\mathcal{C}$ for all $f \in \mathcal{D}(X,Y)$. These are the morphisms in the functor category $[\mathcal{D},\mathcal{C}]_0$.
On the other hand, there are the natural tranformations, given as collections of maps $\alpha_X \in \mathcal{C}(\mathcal{F}X, \mathcal{G}X)$. On the set of these transformation G again acts by conjugation. Then as indicated in A.2.12, the G-natural transformations are exactly the G-fixed natural transformations, so that the functor G-category $[\mathcal{D},\mathcal{C}]$ has the functor category $[\mathcal{D},\mathcal{C}]_0$ as its underlying (G-fixed) category.

The following combination of the above definitions will be important in our studies of equivariant orthogonal spectra. Let G be a (compactly generated weak Hausdorff) topological group, respectively the associated one object $\mathcal{T}$-category with morphism space G_+

Definition A.2.19. The category of G-spaces $G\mathcal{T}$, consists of functors $G \to \mathcal{T}$ and natural transformations between them. In particular, objects of $G\mathcal{T}$ are spaces with a (left) action of G and morphisms are G-equivariant continuous maps.
Giving smash products the diagonal G-action, $G\mathcal{T}$ inherits a closed symmetric monoidal structure from $\mathcal{T}$. Again this allows us to view $G\mathcal{T}$ as enriched over itself, and we shall use the notation $\mathcal{T}_G$ for the ensuing enriched category (A.2.13), as well as $\mathcal{T}_G(-,-)$ for the internal **Hom**-functor of $G\mathcal{T}$. Then $\mathcal{T}_G$ has G-spaces as objects, and morphisms are (not necessarily G-equivariant) continuous maps.

Definition A.2.20. A category $\mathcal{C}_G$ is called a *topological G-category* if it is enriched over $G\mathcal{T}$. Such a $\mathcal{C}_G$ has a *G-fixed category* $G\mathcal{C}$ that is obtained by applying the fixed point functor to the morphism G-spaces.
The appropriate functors enriched over $G\mathcal{T}$ are called *continuous G-functors*. The

appropriate enriched natural transformations are called *continuous natural G-transformations.* ([MM, p. 27] calls these natural G-maps between functors.) We will often drop the extra adjective "continuous" in the future.

Remark A.2.21. Note that the fixed point functor $(-)^G \colon G\mathscr{T} \to \mathcal{T}$ has a left adjoint giving a space the trivial G-action. As $(-)^G$, this preserves (smash-) products and is therefore strict monoidal.

As above monoidal functors starting in $G\mathscr{T}$ allow us to transport enrichments as in A.2.9. Transportation along functors in the commutative diagram

$$\begin{tikzcd}
{} & \mathcal{T} \arrow[r] & \mathcal{U} \arrow[r] & \mathcal{S}\mathrm{et} \\
{} \arrow[u, "\text{forget.}"] & G\mathscr{T} \arrow[r] \arrow[u] \arrow[d] & G\mathcal{U} \arrow[r] \arrow[u] \arrow[d] & G\mathcal{S}\mathrm{et} \arrow[u] \arrow[d] \\
{} \arrow[from=u, "(-)^G"'] & \mathcal{T} \arrow[r] & \mathcal{U} \arrow[r] & \mathcal{S}\mathrm{et},
\end{tikzcd} \tag{A.2.22}$$

as well as their left adjoints, and even variations only using subgroups of G (C.1)appears at various points when doing equivariant homotopy theory. Usually this is omitted in notation, as the sheer amount of viewpoints does not lend itself well to readable notation. However, at least the author has fought with misunderstandings coming from such omission, so throughout this thesis we try not to hide the enrichments completely.

Example A.2.23. As it is defined, the $G\mathscr{T}$-category $\mathscr{T}_G$ has the underlying G-fixed $\mathcal{T}$-category $G\mathscr{T}$, which is closed symmetric monoidal. Also, $\mathscr{T}_G$ is closed symmetric monoidal itself, when viewing it as a mere category using the upper way through diagram A.2.22, using the same smash product and internal hom functor as in $\mathcal{T}$. One choice of internal **Hom**-functor for $\mathscr{T}_G$ is $\mathscr{T}_G(-,-)$, and we agree to use this choice.

A.2.2 Tensors and Cotensors

Detailed treatment of the concepts of *(indexed) limits and colimits* in $\mathscr{V}$-enriched categories can be found in Chapter 3 of [K]. We will not give explicit definitions, since we will mainly be concerned with the special case of tensors and cotensors:

Definition A.2.24. Let $\mathscr{C}$ be enriched over the closed symmetric monoidal category $\mathscr{V}$. Let V be an object of $\mathscr{V}$ and A an object of $\mathscr{C}$. Then their *tensor product* $V \otimes A$ is an object of C, such that for objects B in $\mathscr{C}$, there is a $\mathscr{V}$-natural isomorphism:

$$\mathscr{C}(V \otimes A, B) \cong \mathbf{Hom}(V, \mathscr{C}(A, B)),$$

where **Hom** denotes the internal **Hom**-object in $\mathcal{V}$.
Their *cotensor product* $V \pitchfork A$ is an object of $\mathcal{C}$, such that again for objects B in $\mathcal{C}$, there is a $\mathcal{V}$-natural isomorphism:

$$\mathcal{C}(B, V \pitchfork A) \cong \mathbf{Hom}(V, \mathcal{C}(B, A)).$$

If all such (co-)tensor products exist we call $\mathcal{C}$ *(co-)tensored.* If we consider $\mathcal{C}$ as enriched over different monoidal categories, we clarify the one used for (co-)tensors by saying it is *(co-)tensored over* $\mathcal{V}$.

Remark A.2.25. Note that for $\mathcal{V} = \mathcal{S}\mathrm{et}$, being tensored and cotensored over $\mathcal{S}\mathrm{et}$ is equivalent to having all small copowers $\coprod\limits_X A$. Dually, being cotensored over $\mathcal{S}\mathrm{et}$ is equivalent to having all small powers $\prod\limits_X A$.

Example A.2.26. Considering the closed symmetric monoidal category $\mathcal{V}$ as enriched over itself (cf. Example A.2.13), it is both tensored and cotensored over itself, by the defining adjunction of the internal **Hom**-space A.1.9.

Example A.2.27. As mentioned in Remark A.2.23, the category $\mathfrak{T}_G$ is enriched over $G\mathfrak{T}$, but also over itself, i.e. $\mathbf{Hom}(X, Y) = \mathfrak{T}_G(X, Y)$. This immediately implies that $\mathfrak{T}_G$ is both tensored and cotensored over both itself and $G\mathfrak{T}$, where both are displayed by the same natural isomorphisms, considered either in $G\mathfrak{T}$ or $\mathfrak{T}_G$:

$$\mathfrak{T}_G(D, \mathfrak{T}_G(A, B)) \cong \mathfrak{T}_G(D \wedge A, B) \cong \mathfrak{T}_G(A, \mathfrak{T}_G(D, B)).$$

Since $\mathfrak{T}_G$ has $G\mathfrak{T}$ as its underlying G-fixed category, this implies natural isomorphisms in $\mathcal{T}$:

$$G\mathfrak{T}(S, \mathfrak{T}_G(A, B)) \cong G\mathfrak{T}(S \wedge A, B) \cong G\mathfrak{T}(A, \mathfrak{T}_G(S, B)).$$

For S any object of $\mathcal{T}$, i.e. with trivial G-action, this reduces to:

$$\mathcal{T}(S, G\mathfrak{T}(A, B)) \cong G\mathfrak{T}(S \wedge A, B) \cong G\mathfrak{T}(A, \mathfrak{T}_G(S, B)),$$

which shows that $G\mathfrak{T}$ is tensored and cotensored over $\mathcal{T}$.

The following construction is important for the compatibility of an enrichment and the model structures on the involved categories, and also appears prominently in a lot of our constructions of cellular filtrations:

Definition A.2.28. Let $(\mathcal{V}, \wedge, \mathbb{I})$ be a closed symmetric monoidal category. Let $\mathcal{C}$ be enriched and tensored over $\mathcal{V}$, and have pushouts. For $i : A \to B$ a morphism in

$\mathcal{V}$, $j : X \to Y$ a morphism in the underlying category $\mathcal{C}_0$ of $\mathcal{C}$. Define the *pushout product* $i\square j$ to be the dotted map from the pushout in the diagram:

$$\begin{tikzcd}
A\otimes X \arrow[r, "\mathrm{id}\otimes j"] \arrow[d, "i\otimes\mathrm{id}"'] & A\otimes Y \arrow[d] \arrow[ddr, bend left, "i\otimes\mathrm{id}"] & \\
B\otimes X \arrow[r] \arrow[drr, bend right, "\mathrm{id}\otimes j"'] \arrow[ur, phantom, "\ulcorner", very near end] & P \arrow[dr, dotted, "i\square j"] & \\
& & B\otimes Y
\end{tikzcd}$$

The dual construction is the following:

Definition A.2.29. Let $(\mathcal{V}, \wedge, \mathbb{I})$ be a closed symmetric monoidal category. Let $\mathcal{C}$ be enriched and cotensored over $\mathcal{V}$, and have pushouts. For $i : A \to B$ a morphism in $\mathcal{V}$, $p : E \to F$ a morphism in the underlying category $\mathcal{C}_0$ of $\mathcal{C}$. Define the map $\pitchfork_\square(i,p)$ to be the dotted map to the pullback in the diagram:

$$\begin{tikzcd}
B\pitchfork X \arrow[drr, bend left, "i^*"] \arrow[dr, dotted, "\pitchfork_\square(i,p)"] \arrow[ddr, bend right, "p_*"'] & & \\
& Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A\pitchfork X \arrow[d, "p_*"] \\
& B\pitchfork Y \arrow[r, "i^*"'] & A\pitchfork Y
\end{tikzcd}$$

This again has an analogue living in the category $\mathcal{V}$:

Definition A.2.30. Let $(\mathcal{V}, \wedge, \mathbb{I})$ be a closed symmetric monoidal category having pullbacks. Let $\mathcal{C}$ be enriched over $\mathcal{V}$. For $j : X \to Y$ and $p : E \to F$ be morphisms in the underlying category $\mathcal{C}_0$ of $\mathcal{C}$. Define the map $\mathcal{C}(j^*, p_*)$ to be the dotted map to the pullback in the diagram in $\mathcal{V}$:

$$\begin{tikzcd}
\mathcal{C}(Y,E) \arrow[drr, bend left, "j^*"] \arrow[dr, dotted, "\mathcal{C}(j^*,p_*)"] \arrow[ddr, bend right, "p_*"'] & & \\
& R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(X,E) \arrow[d, "p_*"] \\
& \mathcal{C}(Y,F) \arrow[r, "j^*"'] & \mathcal{C}(X,F)
\end{tikzcd}$$

This construction can be used to characterize lifting properties in the enriched setting:

Lemma A.2.31. *Assume that $\mathcal{V}$ is locally small, then in the situation of Definition A.2.30, the pair (j, p) has the lifting property in $\mathcal{C}_0$, if and only if the map of sets $\mathcal{V}(\mathbb{I}, \mathcal{C}(j^*, p_*)))$ is surjective.*

Proof. Recall that morphisms $X \to Y$ in $\mathcal{C}_0$ correspond to elements of $\mathcal{V}(\mathbb{I}, \mathcal{C}(X, Y))$ from A.1.11. Then the universal property of the pullback gives that elements of $\mathcal{V}(\mathbb{I}, R)$ correspond exactly to commutative diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & E \arrow[d, "p"] \\ B \arrow[r] & F \end{tikzcd}$$

in $\mathcal{C}_0$. Then $\mathcal{V}(\mathbb{I}, \mathcal{C}(j^*, p_*)))$ sends maps $f : B \to E$ in $\mathcal{C}_0$ to the diagram with $f \circ i$ as the top and $p \circ f$ as the bottom horizontal arrow, so that surjectivity indeed corresponds exactly to the existence of the lift. $\square$

Given that all of the three above constructions are defined, there is the following crucial relation between them:

Lemma A.2.32. *Let $(\mathcal{V}, \wedge, \mathbb{I})$ be closed symmetric monoidal and have small limits. Let $\mathcal{C}$ be enriched, tensored and cotensored over $\mathcal{V}$ and have pullbacks and pushouts. Let $i : A \to B$ a morphism in $\mathcal{V}$ and $j : X \to Y$ and $p : E \to F$ morphisms in the underlying category $\mathcal{C}_0$ of $\mathcal{C}$. Then the following maps in $\mathcal{V}$ are naturally isomorphic:*

$$\mathcal{C}((i\square j)^*, p_*) \cong \mathcal{V}(i^*, \mathcal{C}(j^*, p_*)_*) \cong \mathcal{C}(j^*, \pitchfork_\square (i, p))$$

Proof. Note that for the middle map we considered $\mathcal{V}$ as enriched over itself as in A.2.13. By careful use of the universal properties of pushouts and pullbacks as well as the defining adjunctions for tensors and cotensors A.2.24, one observes that all three maps are naturally isomorphic to the map from $\mathcal{V}(B, \mathcal{C}(Y, E))$ to the limit of

$$\begin{tikzcd} \mathcal{V}(A, \mathcal{C}(Y, E)) \arrow[d] \arrow[dr] & \mathcal{V}(B, \mathcal{C}(Y, F)) \arrow[dl] \arrow[dr] & \mathcal{V}(B, \mathcal{C}(X, E)) \arrow[dl] \arrow[d] \\ \mathcal{V}(A, \mathcal{C}(Y, E)) \arrow[dr] & \mathcal{V}(A, \mathcal{C}(X, E)) \arrow[d] & \mathcal{V}(B, \mathcal{C}(X, F)) \arrow[dl] \\ & \mathcal{V}(A, \mathcal{C}(X, F)) & \end{tikzcd}$$

$\square$

These two lemmas allow us to characterize lifting properties in $\mathcal{C}_0$ in terms of those in $\mathcal{V}$, which is of course of particular interest when $\mathcal{C}_0$ and $\mathcal{V}$ are model categories (cf. B.1.10).

A.2.3 Kan Extensions

The discussion about enriched Kan extensions in [K, 4] is, due to its generality rather technical. As in the case of enriched (co-) limits, extra care has to be taken in several places. Since we do not need the full generality, we state a slightly simpler definition and list only the explicit properties we make use of, without going into much detail. We concentrate on the case of left Kan extensions, since the dual notion will not appear outside of pure existence statements.
Let $\mathcal{V}$ be closed symmetric monoidal and consider the solid arrow diagram of $\mathcal{V}$-categories and $\mathcal{V}$-functors:

$$\begin{tikzcd} & \mathcal{C} \arrow[dd, dotted, "\mathrm{Lan}_K G"] \\ \mathcal{A} \arrow[ur, "K"] \arrow[dr, "G"'] & \\ & \mathcal{B}, \end{tikzcd}$$

where $\mathcal{A}$ is equivalent to a small $\mathcal{V}$-category and $\mathcal{B}$ is cotensored over $\mathcal{V}$.

Definition A.2.33. In the above situation, a *left Kan extension* $\mathrm{Lan}_K G$ *of* G *along* K is a $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{B}$, together with a $\mathcal{V}$-natural isomorphism

$$[\mathcal{C}, \mathcal{B}](\mathrm{Lan}_K G, S) \cong [\mathcal{A}, \mathcal{B}](G, S \circ K).$$

The image of the identity transformation for $S = \mathrm{Lan}_K G$ is a $\mathcal{V}$-natural transformation $\phi : G \to \mathrm{Lan}_K G \circ K$ and is called the *unit* of $\mathrm{Lan}_K G$.

It is important to note, that in a situation where $\mathcal{B}$ is not cotensored, this definition is not adequate, in the sense that it does not describe the left Kan extension in the sense of Kelly, but rather a weaker notion. For counterexamples see the discussion after [K, 4.43].The following proposition will give us the existence of left Kan extensions in all the cases that we will consider:

Proposition A.2.34. *[K, 4.33]* $\mathcal{B}$ *admits all left Kan extensions* $\mathrm{Lan}_K G$*, where* $K : \mathcal{A} \to \mathcal{C}$ *and* $G : \mathcal{A} \to \mathcal{B}$*, where* $\mathcal{A}$ *is equivalent to a small* $\mathcal{V}$*-category, if and only if it is enriched cocomplete.*

To check the required cocompleteness, we will generally be able to use the following characterization, which is a combination of several statements in [K]:

Theorem A.2.35. *Let* $\mathcal{B}$ *be enriched over* $\mathcal{V}$*:*

(i) $\mathcal{B}$ *is cocomplete in the enriched sense, if and only if is tensored and admits all small conical (enriched) colimits.*

(ii) *$\mathcal{B}$ is complete in the enriched sense, if and only if is cotensored and admits all small conical (enriched) limits.*

(iii) *Assuming $\mathcal{B}$ is cotensored, it admits all small conical colimits, if and only if its underlying ordinary category $\mathcal{B}_0$ is cocomplete.*

(iv) *Assuming $\mathcal{B}$ is tensored, it admits all small conical limits, if and only if its underlying ordinary category $\mathcal{B}_0$ is complete.*

In particular for tensored and cotensored $\mathcal{B}$, the conical (co-)limits are the ones created in $\mathcal{B}_0$.

Proof. The precise references in [K] are: Theorem 3.73 for (ii), dualize for (i). The discussion between 3.53 and 3.54 for conical (co-)-limits in $\mathcal{B}$ or $\mathcal{B}_0$, and the discussion between 3.33 and 3.34 for the connection to classical (co-)completeness, □

Since it is not always the enriched functor category from A.2.8 that is of interest for us, we would also like a characterization of the left Kan extension in terms of the underlying category of enriched functors and enriched transformations. Luckily our assumption that $\mathcal{B}$ is cotensored allows us to use the following universal property from [K, 4.43] and the discussion that follows it:

Theorem A.2.36. *In the situation of Definition A.2.33, i.e. if $\mathcal{B}$ is cotensored, a $\mathcal{V}$-functor L is a left Kan extension of G along K, if and only if there is a natural bijection of sets*

$$[\mathcal{C},\mathcal{B}]_0(\mathrm{Lan}_K G, S) \cong [\mathcal{A},\mathcal{B}]_0(G, S\circ K).$$

In particular a $\mathcal{V}$-functor L equipped with a $\mathcal{V}$-natural transformation $\phi : G \to L\circ K$ is a left Kan extension of G along K, if and only if any $\mathcal{V}$-natural transformation $\alpha : G \to L\circ K$ factors uniquely as $\alpha = \beta\circ\phi$.

Hence in the case of $\mathcal{B}$ tensored, cotensored and cocomplete, the two characterizations together with the existence result A.2.34, allow us to state the following:

Proposition A.2.37. *In the situation of Definition A.2.33, precomposition with K defines a $\mathcal{V}$-functor $K^*\colon [\mathcal{C},\mathcal{B}] \to [\mathcal{A},\mathcal{B}]$. The left Kan extension provides a left adjoint, both in the enriched sense, and on underlying ordinary categories.*

Finally, the following property helps to compute the Kan extensions in a lot of interesting special cases:

Proposition A.2.38. *[K, 4.23] In the situation of Proposition A.2.37, the $\mathcal{V}$-functor K is fully faithful if and only if the unit $\mathrm{id}_{[\mathcal{A},\mathcal{B}]} \to K^*\mathrm{Lan}_K-$ of the adjunction is a natural isomorphism.*

Appendix B

Model Categories

We assume that the reader is familiar with the basic theory of model categories, an introductory account can for example be found in [DS]. A more exhaustive source is [H] or [Hir].

B.1 Recollections

Almost all of the model structures appearing in this thesis are *cofibrantly generated*, we recall the definition and state the main theorem we use to recognize such model structures from [H, 2.1.3]:

Definition B.1.1. Let $\mathcal{C}$ be a model category. It is called *cofibrantly generated* if there are sets I and J of maps, such that:

(i) The domains of the maps of I are small with respect to $I-$cell,

(ii) The domains of the maps of J are small with respect to $J-$cell,

(iii) The class of fibrations is $J-$inj,

(iv) The class of acyclic fibrations is $I-$inj.

Theorem B.1.2 (Recognition Theorem [H, 2.1.19])**.** *Suppose $\mathcal{C}$ is a category with all small colimits and limits. Suppose W is a subcategory of $\mathcal{C}$ and I and J are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating acyclic cofibrations, and W as the subcategory of weak equivalences if and only if the following conditions are satisfied:*

(i) The subcategory W has the two out of three property and is closed under retracts.

*(ii) The domains of I are small relative to $I-$*cell.

*(iii) The domains of J are small relative to $J-$*cell*.*

(iv) $J-\text{cell} \subset W \cap I-\text{cof}$.

(v) $I-\text{inj} \subset W \cap J-\text{inj}$.

(vi) Either $W \cap I-\text{cof} \subset J-\text{cof}$ *or* $W \cap J-\text{inj} \subset I-\text{inj}$.

The following lemmas are applicable in any model category. These are well known and often used without further mention in the literature, but since they lie at the heart of the homotopy theory we need, we recall the exact statements:

Lemma B.1.3 (Generalized Whitehead Theorem (cf. [DS, 4.24]))**.** *Let $\mathcal{C}$ be a model category. Suppose that $f : X \to Y$ is a map in $\mathcal{C}$ with X and Y both fibrant and cofibrant objects. Then f is a weak equivalence if and only if it is a homotopy equivalence.*

Recall the following definition from [GJ, II 8.5]):

Definition B.1.4. A *category of cofibrant objects* is a category $\mathcal{D}$ with all finite coproducts, with two classes of maps, called weak equivalences and cofibrations, such that the following axioms are satisfied:

(i) The weak equivalences satisfy the 2 out of 3 property.

(ii) The composite of two cofibrations is a cofibration. Any isomorphism is a cofibration.

(iii) Pushouts along cofibrations exist. Cobase changes of cofibrations (that are weak equivalences) are cofibrations (and weak equivalences).

(iv) All maps from the initial object are cofibrations.

(v) Any object X has a cylinder object $\mathrm{Cyl}(X)$, i.e. a factorization of the fold map $\nabla : X \coprod X \to X$ as

$$X \coprod X \xrightarrow{i} \mathrm{Cyl}(X) \xrightarrow{\sigma} X,$$

with i a cofibration and σ a weak equivalence.

In any model category, the cofibrant objects form a *category of cofibrant objects* in the sense of [GJ, II.8]. This in particular lets us apply the following two important lemmas:

Lemma B.1.5 (Generalized Cobase Change Lemma (cf. [GJ, II.8.5])). *Let $\mathcal{C}$ be a category of cofibrant objects. Suppose*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ Y \arrow[r, "g"'] & P \end{tikzcd} \tag{B.1.6}$$

is a pushout diagram in $\mathcal{C}$, such that i is a cofibration and f is a weak equivalence. Then g is also a weak equivalence.

Lemma B.1.7 (Generalized Cube Lemma (cf. [GJ, II.8.8])). *Let $\mathcal{C}$ be a category of cofibrant objects. Suppose given a commutative cube*

$$\begin{tikzcd} & A_0 \arrow[rr] \arrow[dl, "i_0"'] \arrow[dd, "f_A"] & & X_0 \arrow[dl] \arrow[dd, "f_X"] \\ Y_0 \arrow[rr, crossing over] \arrow[dd, "f_Y"'] & & P_0 & \\ & A_1 \arrow[rr] \arrow[dl, "i_1"'] & & X_1 \arrow[dl] \\ Y_1 \arrow[rr] & & P_1 \arrow[from=uu, crossing over, "f_P"'] & \end{tikzcd} \tag{B.1.8}$$

in $\mathcal{C}$. Suppose further that the top and bottom faces are pushouts, that i_0 and i_1 are cofibrations and that the vertical maps f_A, f_X and f_Y are weak equivalences. Then the induced map of pushouts f_P is also a weak equivalence.

Definition B.1.9. Inspired by these, we say that *the generalized (cube) cobase change lemma holds for a class of objects, a class of cofibrations and a class of weak equivalences*, if the analogous statements hold, without explicitly referring to a category of cofibrant objects.

Note that not all examples of categories of cofibrant objects come from model structures, in particular we will want to apply Lemma B.1.7 to cases where the cofibrations and weak equivalences come from different model structures on the same category in B.1.21. In the case of topological model categories (cf. B.1.10), May and Sigurdsson propose a more general treatment in [MS, 5.4], using so called well-grounded categories of weak equivalences. There the statement of Lemma B.1.7 appears as one of the axioms of such categories. We will try to avoid to drag our treatment into this generality though, since setting up the whole machinery of [MS] would require significantly more effort and offer only slightly more insight. We will however handpick some of the statements in our Subsection B.1.1 on topological model categories.

Definition B.1.10. Let $(\mathcal{V}, \otimes, \mathbb{I})$ be a locally small closed symmetric monoidal category, that is also a model category. Let $\mathcal{C}$ be a category enriched, tensored and cotensored over $\mathcal{V}$. Further let the underlying category $\mathcal{C}_0$ of $\mathcal{C}$ (cf. A.2.11) have a model structure. Then this model structure on $\mathcal{C}$ is called *enriched over* $\mathcal{V}$, if the following two axioms hold:

(i) *Pushout product axiom* Let i be a cofibration in $\mathcal{C}_0$ and let j be a cofibration in $\mathcal{V}$. Then the map $i \square j$ in $\mathcal{C}_0$ is also a cofibration. If in addition either one of i or j is acyclic, so is $i \square j$.

(ii) *Unit axiom* Let $q : \mathbb{I}^c \xrightarrow{\sim} \mathbb{I}$ be a cofibrant replacement of the unit object of $\mathcal{V}$. Then for every cofibrant object A in $\mathcal{C}$, the morphism

$$q \otimes \mathrm{id} : \mathbb{I}^c \otimes A \to \mathbb{I} \otimes A \cong A$$

is a weak equivalence.

If $\mathcal{C}$ is equal to $\mathcal{V}$, i.e. we consider $\mathcal{V}$ as enriched over itself (cf. A.2.13), a model structure satisfying the above axioms is called *monoidal*.

Note that the unit axiom is redundant, if the unit object of $\mathcal{V}$ is itself cofibrant, since it is then implied by the pushout product axiom. For monoidal model categories, there is an additional important axiom:

Definition B.1.11. A monoidal model category $(\mathcal{C}, \otimes)$ satisfies the *monoid axiom*, if every map in

$$(\{\text{acyclic cofibrations}\} \otimes \mathcal{C})-\text{cell}$$

is a weak equivalence.

The pushout product axiom has several adjoint formulations:

Lemma B.1.12. *In the situation of Definition B.1.10, the pushout product axiom is equivalent to both of the following formulations:*

- *Let p be a fibration in $\mathcal{C}_0$ and let j be a cofibration in $\mathcal{V}$, then $\pitchfork_\square (j, p)$ is a fibration in $\mathcal{C}_0$, which is acyclic if either of p or j was.*
- *Let p be a fibration in $\mathcal{C}_0$ and let i be a cofibration in $\mathcal{C}_0$, then $\mathcal{C}(i^*, p_*)$ is a fibration in $\mathcal{D}$, which is acyclic if either of p or i was.*

Proof. This is immediate from lemmas A.2.31 and A.2.32. $\square$

Example B.1.13. Taking $\mathcal{V}$ to be the categories of simplicial sets, spaces, symmetric spectra or G-spaces, yields, under the choice of the usual model structures, the well known notions of simplicial, topological, spectral and G-topological model categories.

In particular the example of topological and G-topological model categories will be very important for us. We discuss some of their distinct features in the following subsection.

B.1.1 Topological Model Categories

In this subsection we have to discuss two different categories of topological spaces. We distinguish between the category $\mathcal{U}$ of compactly generated weak Hausdorff spaces, and the category $\mathcal{T}$ of such spaces with a distinguished basepoint. Alternatively one can think of $\mathcal{T}$ as the under-category $* \to \mathcal{U}$ for $*$ any one-point object in $\mathcal{U}$.
Let I denote the unit interval in $\mathcal{U}$, as usual it comes equipped with the two inclusions of the endpoints. For any category $\mathcal{C}$ enriched and tensored over $\mathcal{U}$, we can then form *homotopies in* $\mathcal{C}_0$ in terms of the tensor with I:

$$\begin{tikzcd}
\{0\} \otimes X \arrow[r, "\cong"] \arrow[d] & X \arrow[dr, "h_0"] & \\
I \otimes X \arrow[rr, "h"] & & Y \\
\{1\} \otimes X \arrow[r, "\cong"'] \arrow[u] & X \arrow[ur, "h_1"'] &
\end{tikzcd}$$

Analogously for $\mathcal{C}$ enriched over $\mathcal{T}$, we can add a disjoint basepoints and use the tensor with I_+ to define *(based) homotopies*.
There are two classical model structures on $\mathcal{U}$ that are important for us, the Strøm- or h-model structure and the Quillen- or q-model structure. Especially the cofibrations of the former have very favorable properties, the defining one being the homotopy extension property:

Definition B.1.14. Let $\mathcal{C}$ be enriched and tensored over $\mathcal{U}$. A map $i : A \to X$ in $\mathcal{C}_0$ is a *free h-cofibration* if it satisfies the *free homotopy extension property*. That is, for every map $f : X \to Y$ and homotopy $h : I \otimes A \to Y$ such that $h_0 = f \circ i$, there is a homotopy $H : I \otimes X \to Y$ such that $H_0 = f$ and $H \circ (i \otimes \mathrm{id}) = h$.

The universal test case for this property is the mapping cylinder $Y = Mi = X \cup_i (I \otimes A)$, with the obvious f and h. The exact statement is the following lemma.

Lemma B.1.15. *Let $\mathcal{C}$ be enriched and tensored over $\mathcal{U}$ and have pushouts. A map $i : A \to X$ in $\mathcal{C}_0$ is a free h-cofibration if and only if the canonical map $Mi \to I \times X$ has a retraction.*

Proof. The HEP gives the dotted arrow in the following diagram, which is a retraction by the universal property of Mi:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d] & X \arrow[d] \arrow[ddr, bend left] & \\
I \otimes A \arrow[r, "I\otimes i"] \arrow[drr, bend right] & I \otimes X \arrow[dr, dotted] & \\
 & & Mi
\end{tikzcd}$$

On the other hand, if the retraction is given, the universal property of the pushout Mi gives the desired extensions. □

Remark B.1.16. This implies a variety of closure properties of the class of free h-cofibrations. In particular any functor that preserves pushouts and the tensor with the interval also preserves h-cofibrations, since any functor preserves retractions.

Theorem B.1.17. *[Str, Theorem 3], cf. [MS, 4.4.4] The homotopy equivalences, Hurevicz fibrations and free h-cofibrations give a proper model structure on $\mathcal{U}$.*

Note that Strøm originally works in the category of all topological spaces, but the intermediate objects for the factorizations he constructs are in $\mathcal{U}$ if source and target were. Properness is not mentioned in the original article, but is implied by the fact that all objects are fibrant and cofibrant.

Definition B.1.18. Let f be a map in $\mathcal{U}$. Then f is a *weak equivalence* if it induces isomorphisms on all homotopy groups. Call f a *q-cofibration* if it has the left lifting property with respect to all Serre fibrations that are weak equivalences.

Remark B.1.19. Recall that every Hurevicz fibration is a Serre fibration and every homotopy equivalence is a weak equivalence. Hence in particular any q-cofibration is a free h-cofibration.

Theorem B.1.20. *[Q, II.3.1], cf. [H, 2.4.25] The weak equivalences, Serre fibrations and q-cofibrations give a proper model structure on $\mathcal{U}$.*

Again note that Quillen also works with general topological spaces, the transition to $\mathcal{U}$ is well documented in [H, 2.4]. Properness is proved using that every object is fibrant as well as the following lemma:

Lemma B.1.21. *The category $\mathcal{U}$ is a category of cofibrant objects (B.1.4) with respect to the h-cofibrations and the weak equivalences. In particular the generalized cobase change (B.1.5) and cube lemma (B.1.7) hold for these choices.*

Moving to the context of based spaces, we can for example follow the discussion after Remark 1.1.7 of [H] to transport both model structures from $\mathcal{U}$ to $\mathcal{T}$. This proves satisfactory in case of the Quillen model structure:

Theorem B.1.22. *The category $\mathcal{T}$ is a proper model category using those based maps that are q-cofibrations, Serre fibrations respectively weak equivalences in $\mathcal{U}$, i.e. when forgetting the basepoints. Similarly the underlying free h-cofibrations, Hurevicz fibrations and (free) homotopy equivalences give a proper model structure on $\mathcal{T}$.*

Remark B.1.23. We will often make use of the the fact that the Quillen model structures on $\mathcal{U}$ and $\mathcal{T}$ are cofibrantly generated. Generating sets of cofibrations and acyclic cofibrations are given in the pointed case by:

$$I := \{i : S^{n-1}_+ \to D^n_+, n \geq 0\} \text{ and}$$

$$J := \{i_0 : D^n_+ \to (D^n \times [0,1])_+, n \geq 0.\}$$

Remark B.1.24. Note that not all spaces in $\mathcal{T}$ are cofibrant with respect to the second model structure in the above theorem. In particular the theorem only implies pointed analogues to the versions of the generalized cube and cobase change lemmas from B.1.21 above for so called well based spaces:

Definition B.1.25. An object X of $\mathcal{T}$ is called *well based* or *well pointed* if the inclusion of the basepoint is a free h-cofibration.

We need a stronger version of the cube lemma when we work in the $\mathcal{T}$-enriched setting:

Definition B.1.26. Let $\mathcal{C}$ be enriched and tensored over $\mathcal{T}$. A map $i : A \to X$ in $\mathcal{C}_0$ is a *based h-cofibration* if it satisfies the *based homotopy extension property*. That is, for every map $f : X \to Y$ and based homotopy $h : I_+ \wedge A \to Y$ such that $h_0 = f \circ i$, there is a based homotopy $H : I_+ \wedge X \to Y$ such that $H_0 = f$ and $H \circ (\mathrm{id} \wedge i) = h$. In cases where no confusion is possible, we will usually omit the adjective based.

Remark B.1.27. Again there is a recognition lemma analogous to B.1.15 in terms of a reduced mapping cylinder, implying a similar closure property as in Lemma B.1.15. Also note that all (free or based) h-cofibrations are closed inclusions (cf. [M, § 6, Ex 1,], [MMSS, 5.2 ff.]).

The following proposition is a combination of Proposition 9 in [Str] and the proposition on page 44 of [M], both are proved by explicitly constructing the required homotopies, respectively retractions.

Proposition B.1.28. *Let $f : X \to Y$ be a map between well based spaces in $\mathcal{T}$. Then f is a based homotopy equivalence if and only if it is a free homotopy equivalence and it is a based h-cofibration if and only if it is a free h-cofibration.*

Note that being a weak equivalence in $\mathcal{T}$ and $\mathcal{U}$ is always equivalent, so we have the following corollary:

Corollary B.1.29. *If all involved spaces are well based the generalized cube lemma and the generalized cobase change lemma hold for based h-cofibrations and homotopy equivalences. Also, they hold for h-cofibrations and weak equivalences if all the spaces A_i and Y_i in the diagrams B.1.6 and B.1.8 are well based.*

Finally we record the following property from [MMSS, 6.8(v)]:

Lemma B.1.30. *Transfinite composition of h-cofibrations that are weak equivalences are weak equivalences.*

The following condition on sets of maps in a topological category has proven very helpful in several contexts. We use the formulation from [MMSS, 5.3], and hence use $\mathcal{T}$ for the enrichment. Let $\mathcal{A}$ and $\mathcal{C}$ be categories enriched over $\mathcal{T}$ that are (enriched) bicomplete and in particular tensored and cotensored. Let $\mathcal{A}$ be equipped with a continuous and faithful forgetful functor $\mathcal{A} \to \mathcal{C}$.

Condition B.1.31. Cofibration Hypothesis Let I be a set of maps in $\mathcal{A}$. We say that I satisfies the *cofibration hypothesis* if it satisfies the following two conditions.

(i) Let $i : A \to B$ be a coproduct of maps in I. Then any cobase change of i in $\mathcal{A}$ is an underlying h-cofibration, i.e. an h-cofibration in $\mathcal{C}$ after use of the forgetful functor.

(ii) Viewed as an object of $\mathcal{C}$, the colimit of a sequence of maps in $\mathcal{A}$ that are underlying h-cofibrations is their colimit as a sequence of maps in $\mathcal{C}$.

Remark B.1.32. In particular I-cell complexes in $\mathcal{A}$ are underlying sequential colimits along h-cofibrations in $\mathcal{C}$.

The smallness conditions in the definition of a cofibrantly generated model category are as lax as possible. In many of the topological examples, we can actually be more strict, in order to get around having to deal with transfinite inductions as much as possible. A convenient condition is the following, again taken from [MMSS, 5.6, ff.], with $\mathcal{A}$ and $\mathcal{C}$ as above:

Definition B.1.33. An object X of $\mathcal{A}$ is *compact* if

$$\mathcal{A}(X, \operatorname{colim} Y_n) \cong \operatorname{colim} \mathcal{A}(X, Y_n),$$

whenever $Y_n \to Y_{n+1}$ is a sequence of maps in $\mathcal{A}$ that are h-cofibrations in $\mathcal{C}$.

Definition B.1.34. Let $\mathcal{A}$ be a model category. Then $\mathcal{A}$ is *compactly generated*, if it is cofibrantly generated with generating sets of (acyclic) cofibrations I and J, such that the domains of all maps in I or J are compact, and I and J both satisfy the cofibration hypothesis B.1.31.

B.1.2 Simplicial Objects in Topological Categories

In this section, we recall some basic simplicial techniques. A convenient reference for a lot of the following discussion is [GJ, VII.3], but we need some rather specific technical lemmas which to the author's knowledge have not been formulated similarly before. We start by reminding the reader of the basic definitions:

Definition B.1.35. The *simplicial category* $\mathbf{\Delta}$ has the finite ordinal numbers as objects and order preserving maps as morphisms between them.

To be more specific, we will denote objects of $\mathbf{\Delta}$ by $\mathbf{n}$, i.e.

$$\mathbf{n} := \{0 < 1 < \ldots < n\}.$$

Recall the generating morphisms s_i and d_i in $\mathbf{\Delta}$ and the relations between them from [GJ, I.1.2].

Definition B.1.36. Let $\mathcal{C}$ be a category. The *category $s\mathcal{C}$ of simplicial objects in* $\mathcal{C}$ is the functor category $[\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}]$.

Let from now on $\mathcal{C}$ be enriched and tensored over the category of simplicial sets.

Definition B.1.37. The *geometric realization* $|X|_{\mathcal{C}}$ of a simplicial object $X \in s\mathcal{C}$ is the coend

$$|X|_{\mathcal{C}} := \int^{\mathbf{k} \in \Delta^{\mathrm{op}}} X_k \otimes \Delta^k,$$

where Δ^k is the simplicial n-simplex given by $\Delta^k_n = \mathbf{\Delta}(\mathbf{n}, \mathbf{k})$. With the obvious extension on morphisms, this defines a functor $| \cdot |_{\mathcal{C}} \colon s\mathcal{C} \to \mathcal{C}$.

We will often drop the subscript from $| \cdot |_{\mathcal{C}}$ when the category is clear. Note that any functor $\mathcal{C} \to \mathcal{C}'$ that preserves colimits and tensors preserves the geometric realization.

Definition B.1.38. If $\mathcal{C}$ is also cotensored over simplicial sets, the geometric realization has a right adjoint given by the functor that assigns to an object Y of $\mathcal{C}$ the simplicial object $\mathbf{\Delta} \pitchfork Y$ which is given in level $\mathbf{k}$ by

$$(\mathbf{\Delta} \pitchfork Y)_k := \Delta^k \pitchfork Y.$$

Remark B.1.39. The most important special case for our applications will be when the category $\mathcal{C}$ is actually enriched and tensored over $\mathcal{T}$. In this case, we can first transport the enrichment to $\mathcal{U}$ along the forgetful functor and then to simplicial sets via the singular set functor as in A.2.9 since both of these are (lax) monoidal. Then the defining adjunctions immediately give an isomorphism

$$X_k \otimes_{s\mathcal{S}\mathrm{et}} \Delta^k \cong X_k \otimes_{\mathcal{U}} |\Delta^k| \cong X_k \otimes_{\mathcal{T}} |\Delta^k|_+,$$

where the $|\Delta^k|$ denotes the topological k-simplex (with a disjoint basepoint on the right).
Classical realization of simplicial sets is then a special case of the above by viewing sets as discrete objects of $\mathcal{T}$.

We want to filter the geometric realization, in analogy to the classical case where the geometric realization admits a filtration via its structure as a CW-complex. For this purpose consider for a natural number n the full subcategory $\mathbf{\Delta}_n$ of $\mathbf{\Delta}$ consisting of all objects $\mathbf{k}$ with $k \leq n$.

Definition B.1.40. For $X \in s\mathcal{C}$ a simplicial object, define the *n-skeleton* $\mathrm{sk}_n |X|_{\mathcal{C}}$ as the coend

$$\mathrm{sk}_n |X|_{\mathcal{C}} := \int^{\mathbf{k} \in \Delta_n^{\mathrm{op}}} X_k \otimes \Delta^k.$$

Again, we will often simplify notation and just write $\mathrm{sk}\, X$ instead of $\mathrm{sk}\, |X|_{\mathcal{C}}$ when the context does not allow confusion.

Lemma B.1.41. *The inclusions of categories $\mathbf{\Delta}_n \to \mathbf{\Delta}_{n+1} \to \ldots \to \mathbf{\Delta}$ induce morphisms of coends and we get*

$$\operatorname*{colim}_n \mathrm{sk}_n X \cong |X|.$$

Proof. We can rewrite the defining coends via coequalizer diagrams:

$$\coprod_{\alpha: \mathbf{k} \to \mathbf{l} \in \Delta^{\mathrm{op}}} X_k \otimes \Delta^l \rightrightarrows \coprod_{k \in \mathbf{\Delta}} X_k \otimes \Delta^k \longrightarrow |X|_{\mathcal{C}},$$

and analogously for $\mathrm{sk}_n |X|_{\mathcal{C}}$. In the resulting diagram of coequalizer sequences, one easily constructs a pair of inverse morphisms between the two different colimits. $\square$

We define an analogue to the degenerate simplices, or rather the latching spaces in the classical setting:

Definition B.1.42. Let $X \in s\mathcal{C}$ be a simplicial object. The *latching object* L_nX comes together with a distinct map $L_nX \to X_n$ and is defined inductively as follows: Let L_0X be the initial object of $\mathcal{C}$. Assuming L_nX and $L_nX \to X_n$ already defined, let $L_{n+1}X$ be an $(n+1)$-fold pushout of $L_nX \to X_n$, i.e. the colimit of the following solid arrow diagram:

$$\begin{tikzcd}
L_nX \arrow[rrr] \arrow[drr] \arrow[dddr] & & & X_n \arrow[ddd, dotted, "s_0"] \\
& \cdots & X_n \arrow[ddr, dotted, "s_1"] & \\
& & \cdots & \\
& X_n \arrow[rr, dotted, "s_n"] & & X_{n+1}
\end{tikzcd} \tag{B.1.43}$$

The map $L_{n+1}X \to X_{n+1}$ is then induced by the pointed instances of the simplicial degeneracy maps $s_i : X_n \to X_{n+1}$.

The importance of the latching objects lies in the following proposition:

Proposition B.1.44. *[GJ, VII.3.8] Let $X \in s\mathcal{C}$ be a simplicial object. Then for all $n \geq 0$ there is a pushout diagram in $\mathcal{C}$:*

$$\begin{tikzcd}
X_n \otimes \partial\Delta^n \cup_{L_nX \otimes \partial\Delta^n} L_nX \otimes \Delta^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \operatorname{sk}_{n-1} X \arrow[d] \\
X_n \otimes \Delta^n \arrow[r] & \operatorname{sk}_n X,
\end{tikzcd}$$

where the left vertical map is the pushout product of $L_nX \to X_n$ with the inclusion of the boundary $\partial\Delta^n \to \Delta^n$.

Definition B.1.45. Fix a class of morphisms called C-cofibrations in $\mathcal{C}$. We call a simplicial object $X \in s\mathcal{C}$ C-proper, if all the maps $L_nX \to X_n$ are C-cofibrations.

We finally turn to the case of $\mathcal{C}$ being a topological model category, i.e. a model category enriched over $\mathcal{U}$ in the sense of B.1.10.

Proposition B.1.46. *Let $\mathcal{C}$ be enriched and tensored over $\mathcal{U}$. Let the C-cofibrations be a class of maps that is closed under cobase change and satisfies the pushout product axiom with respect to the Quillen model structure on $\mathcal{U}$ (e.g. if $\mathcal{C}$ is a model category* *enriched over $\mathcal{U}$ in the sense of B.1.10). Then for any C-proper simplicial object X in $s\mathcal{C}$, the skeleton filtration of $|X|_{\mathcal{C}}$ consists of C-cofibrations.*

$$\begin{tikzcd} X_n \otimes |\partial\Delta^n| \cup_{L_nX\otimes|\partial\Delta^n|} L_nX \otimes |\Delta^n| \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{sk}_{n-1} X \arrow[d, "C-\mathrm{cof}"] \\ X_n \otimes |\Delta^n| \arrow[r] & \mathrm{sk}_n X, \end{tikzcd}$$

The next proposition concerns interactions of simplicial objects with weak equivalences. Assume that $\mathcal{C}$ is enriched and tensored over $\mathcal{U}$, with chosen classes of cofibrations and weak equivalences, such that the cofibrant objects form a category of cofibrant objects in the sense of B.1.4. Assume that the cofibrations and weak equivalences are compatible with the enrichment in the sense that the pushout product axiom B.1.10(i) is satisfied.

Proposition B.1.47. *Let $\mathcal{C}$ as above. Let X and Y in $s\mathcal{C}$ be proper simplicial objects such that X_0 and Y_0 are cofibrant. If $f : X \to Y$ is a morphism of simplicial objects that is a weak equivalence in each simplicial degree, then the induced map of realizations*

$$|f|_{\mathcal{C}} : |X|_{\mathcal{C}} \to |Y|_{\mathcal{C}}$$

is a weak equivalence.

Proof. We begin with showing that all the X_n, Y_n and L_nX and L_nY are cofibrant. $L_1X = X_0$ is cofibrant by hypothesis, so assume inductively that $L_{n-1}X$ is cofibrant. Since X is proper, the solid arrow part of diagram B.1.43 consists only of cofibrations, hence in particular X_n is cofibrant. Since $L_{n+1}X$ is an iterated pushout of X_n along cofibrations it is cofibrant itself. We continue by induction on the skeleton filtration of B.1.46 to show that the maps $sk_nX \to \mathrm{sk}_n Y$ are weak equivalences. Note that the tensor with a cofibrant space preserves weak equivalences between cofibrant objects by [GJ, II.8.4]. Hence by the generalized cube lemma we only need to show that the maps

$$X_n \otimes |\partial\Delta^n| \cup_{L_nX\otimes|\partial\Delta^n|} L_nX \otimes |\Delta^n| \longrightarrow Y_n \otimes |\partial\Delta^n| \cup_{L_nY\otimes|\partial\Delta^n|} L_nY \otimes |\Delta^n|$$

are weak equivalences between cofibrant objects. Again using the generalized cube lemma on the defining diagram for the pushout product of $L_nX \to X$ and $\partial\Delta^n \to \Delta^n$, this reduces to showing that $L_nX \to L_nY$ is a weak equivalence. As above this is proven inductively, by comparing the diagrams B.1.43 for X and Y and applying the generalized cube lemma to each of the iterated pushouts. □

Remark B.1.48. A very obvious example for categories $\mathcal{C}$ which satisfy the requirements of the above proposition is given by a model category enriched over $\mathcal{U}$ in the sense of B.1.10. However, we will in particular want to apply the proposition to (levelwise) h-cofibrations and π_*-isomorphisms of orthogonal spectra, so the more general formulation is necessary.

It can be hard to verify the properness of a simplicial object. Sometimes the following is easier to check:

Definition B.1.49. Fix a class of morphisms called C-cofibrations in $\mathcal{C}$. We call a simplicial object $X \in s\mathcal{C}$ *C-good*, if for all n all the degeneracy maps $s_i^* \colon X_n \to X_{n+1}$ are C-cofibrations.

In particular in $\mathcal{T}$ and $\mathcal{U}$, there is Lillig's Union Theorem [Li], which implies the following helpful statement:

Lemma B.1.50. *For simplicial objects in the categories $\mathcal{T}$ or $\mathcal{U}$, h-proper and h-good are equivalent notions. Since colimits and tensors are computed levelwise, the same is true for levelwise h-cofibrations of (equivariant) orthogonal spectra.*

The following is an important example of a simplicial object, and in particular comes up in the proofs of the convenience property for the $\mathbb{S}$-model structures:

Definition B.1.51 (cf. [EKMM, IV.7.2])**.** Let $(\mathcal{C}, \wedge, \mathbb{I})$ be symmetric monoidal, let (R, ϕ, η) be a monoid in $\mathcal{C}$, (M, μ) a right R-module, (N, ν) a left R-module. Define *the bar construction* $\mathbf{B}_*(M, R, N) \in s\mathcal{C}$ by setting

$$\mathbf{B}_p(M, R, N) = M \wedge R^{\wedge p} \wedge N,$$

where $R^{\wedge 0} = \mathbb{I}$. The face and degeneracy operators on $\mathbf{B}_p(M, R, N)$ are

$$d_i = \begin{cases} \mu \wedge \mathrm{id}_R^{\wedge(p-1)} \wedge \mathrm{id}_N & \text{if i = 0} \\ \mathrm{id}_m \wedge \mathrm{id}_R^{\wedge(i-1)} \wedge \phi \wedge \mathrm{id}_R^{\wedge(p-i-1)} \wedge \mathrm{id}_N & \text{if 0 < i < p} \\ \mathrm{id}_M \wedge \mathrm{id}_R^{\wedge(p-1)} \wedge \nu & \text{if i = p} \end{cases}$$

and $s_i = \mathrm{id}_m \wedge \mathrm{id}_R^{\wedge i} \wedge \eta \wedge \mathrm{id}_R^{p-i} \wedge \mathrm{id}_N$ if $0 \leq i \leq p$.
Note that if M was an (R', R)-bimodule, then $\mathbf{B}_*(M, R, N)$ is a simplicial R'-module. In the case that $\mathcal{C}$ is enriched and tensored over simplicial sets, so that geometric realization makes sense, we will usually denote the realization by

$$\mathbf{B}(M, R, N) := |\mathbf{B}_*(M, R, N)|.$$

B.2 Assembling Model Structures

Given a model structure on a category $\mathcal{C}$, one often wants to give corresponding structures to categories of functors $\mathcal{D} \to \mathcal{C}$ for some diagram category $\mathcal{D}$. Theorems on the possibility and methods to do this are well studied in many cases, examples can be found in [Hir, 14.2.1] for cases of cofibrantly generated structures on $\mathcal{C}$, in [H, Chapter 5] for the case of $\mathcal{D}$ a Reedy category. More recently Angeltveit has studied the Reedy approach in an enriched setting ([A]). The result of this section is more in the direction of the former, in particular as a special case we will get an enriched version of Hirschhorns Theorem [Hir, 11.6.1]. However, the significant difference in our approach is, that we lift not just a single model structure on the target category, but rather assemble a new model structure from several given ones. Hirschhorns method uses the evaluation functors that any diagram category is equipped with; we give a short recollection: Let $\mathcal{D}$ be small. Consider the trivial category $\star$ with one object $*$, and only one (identity) morphism. For each object d of $\mathcal{D}$ it embeds into $\mathcal{D}$ sending $*$ to d. Then the evaluation functor ev_d assigns to a functor $X\colon \mathcal{D} \to \mathcal{C}$ the precomposition with the inclusion of $\star$:

$$\begin{tikzcd} \star \arrow[d, "\mathrm{inc}_d"'] \arrow[dr, dotted, "\mathrm{ev}_d X"] & \\ \mathcal{D} \arrow[r, "X"'] & \mathcal{C} \end{tikzcd}$$

Left Kan extension provides left adjoints to all these evaluation functors, denoted by $\mathcal{F}_d(-)$, i.e. we have adjoint pairs:

$$\mathcal{F}_d : \mathcal{C} \cong [\star, \mathcal{C}] \leftrightarrows [\mathcal{D}, \mathcal{C}] : \mathrm{ev}_d.$$

Then given a cofibrantly generated model structure on $\mathcal{C}$, with generating sets of (acyclic) cofibrations I and J, we can form the sets

$$\mathcal{F}I := \bigcup_{d \in \mathcal{D}} \mathcal{F}_d I$$

and $\mathcal{F}J$ analogous.

Theorem B.2.1. *[Hir, 11.6.1] Let $\mathcal{D}$ be a small category, and let $\mathcal{C}$ be a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. Then the category $[\mathcal{D}, \mathcal{C}] = [\mathcal{D}, \mathcal{C}]_0$ of $\mathcal{D}$-diagrams in $\mathcal{C}$ is a cofibrantly generated model category in which a map $f : X \to Y$ is*

- *a weak equivalence if* $\mathrm{ev}_d(f) : X_d \to Y_d$ *is a weak equivalence in* $\mathcal{C}$ *for every object* $d \in \mathcal{D}$*,*
- *a fibration if* $\mathrm{ev}_d(f) : X_d \to Y_d$ *is a fibration in* $\mathcal{C}$ *for every object* $d \in \mathcal{C}$*, and*
- *an (acyclic) cofibration if it is a retract of a transfinite composition of cobase changes of maps in* $\mathcal{F}I$ *(*$\mathcal{F}J$*).*

Let us now move to an enriched setting. Let $(\mathcal{V}, \wedge, \mathbb{I})$ be a locally small complete closed symmetric monoidal category, and let $\mathcal{C}$ and $\mathcal{D}$ be enriched over $\mathcal{V}$, such that $\mathcal{D}$ is equivalent to a small category, hence the enriched functor category $[\mathcal{D}, \mathcal{C}]$ exists (A.2.8). Consider this time $\star$ as the trivial $\mathcal{V}$-category, i.e. as the $\mathcal{D}$-category with one object $*$ such that the morphism object $\star(*, *)$ is initial in $\mathcal{D}$. Then analogous to the discussion above, the inclusion of $\star$ at any object of $\mathcal{D}$ yields evaluation functors by precomposition. Under favorable conditions on $\mathcal{C}$, these have left adjoints which we again denote by $\mathcal{F}_d$ (e.g. if $\mathcal{C}$ is tensored, cotensored and cocomplete, cf. A.2.33). However this time, we want to consider an intermediate functor category: Given an object $d \in \mathcal{D}$, denote by $\mathcal{E}_d$ the full subcategory containing only that object. Then the inclusion of $\star$ at $d \in \mathcal{D}$ factors in the following way

$$\begin{tikzcd} \star \arrow[d, "\mathrm{inc}"'] \arrow[ddr, dotted, "\mathrm{ev}_d X"] & \\ \mathcal{E}_d \arrow[d, "\mathrm{inc}_d"'] \arrow[dr, "\mathrm{ev}'_d X"'] & \\ \mathcal{D} \arrow[r, "X"'] & \mathcal{C}, \end{tikzcd} \tag{B.2.2}$$

and hence we have a factorization of evaluation functors

$$\begin{tikzcd} \mathcal{C} & {[\star, \mathcal{C}]} \arrow[l, "\cong"'] & {[\mathcal{E}_d, \mathcal{C}]} \arrow[l] & {[\mathcal{D}, \mathcal{C}]} \arrow[l, "\mathrm{ev}'_d"'] \arrow[ll, bend right, "\mathrm{ev}_d"'] \end{tikzcd} . \tag{B.2.3}$$

Each of the functors in this factorization has an (enriched) left adjoint if and only if the appropriate left Kan extensions exist (A.2.34), and in that case we denote them in the following way:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\cong"] & {[\star, \mathcal{C}]} \arrow[r, "\mathcal{E}_d \otimes -"] \arrow[rr, bend left, "\mathcal{F}_d"] & {[\mathcal{E}_d, \mathcal{C}]} \arrow[r, "\mathcal{G}_d"] & {[\mathcal{D}, \mathcal{C}]} \end{tikzcd} . \tag{B.2.4}$$

We call objects of the form $\mathcal{G}_d X$ *semi-free*, in analogy to the term *free* for objects $\mathcal{F}_d Y$. Note that the notation $\mathcal{E}_d \otimes -$ is not accidental, as it is in fact given by the categorical tensor with the endomorphism $\mathcal{D}$-object of d if it exists.

Remark B.2.5. We will from now on assume that $\mathcal{D}$ is small itself, for the general case chose a small equivalent subcategory $\hat{\mathcal{D}}$, and only use objects $d \in \hat{\mathcal{D}}$ in the constructions throughout.

Assume that for each $d \in \mathcal{D}$, there is a cofibrantly generated model structure $\mathcal{M}_d$ on the underlying ordinary category $[\mathcal{E}_d, \mathcal{C}]_0$ of $[\mathcal{E}_d, \mathcal{C}]$, with generating (acyclic) cofibrations I_d and J_d, and classes of weak equivalences $\mathcal{W}_d$, respectively. Assume further that each $[\mathcal{E}_d, \mathcal{C}]$ is tensored and cotensored over $\mathcal{V}$, such that the semi-free functors $\mathcal{G}_d$ all exist. Define the sets of maps $\mathcal{G}I$ and $\mathcal{G}J$ in $[\mathcal{D}, \mathcal{C}]_0$ as

$$\mathcal{G}I := \bigcup_{d \in \mathcal{D}} \mathcal{G}_d I_d \qquad\qquad \mathcal{G}J := \bigcup_{d \in \mathcal{D}} \mathcal{G}_d J_d. \tag{B.2.6}$$

Define the class $\mathcal{W}$ of maps in $[\mathcal{D}, \mathcal{C}]_0$ as

$$\mathcal{W} := \{f \in [\mathcal{D}, \mathcal{C}]_0, \text{ s.t. } \mathrm{ev}'_d(f) \in \mathcal{W}_d \; \forall \, d \in \mathcal{D}\}. \tag{B.2.7}$$

Then the assembling theorem is the following

Theorem B.2.8. *Let $\mathcal{V}$ be a complete closed symmetric monoidal category and let $\mathcal{C}$ and $\mathcal{D}$ be enriched over $\mathcal{V}$ such that $\mathcal{D}$ is equivalent to a small subcategory. Assume that each of the functor categories $[\mathcal{E}_d, \mathcal{C}]$ is tensored and cotensored over $\mathcal{V}$ and that we have a family of cofibrantly generated model structures $\{\mathcal{M}_d\}$ as above.*
Assume that the domains of the maps in $\mathcal{G}I$ are small relative to $\mathcal{G}I$-cell, the domains of the maps in $\mathcal{G}J$ are small with respect to $\mathcal{G}J$-cell and that $\mathcal{G}J$-cell $\subset \mathcal{W}$.
Then the underlying category $[\mathcal{D}, \mathcal{C}]_0$ of $[\mathcal{D}, \mathcal{C}]$ is a cofibrantly generated model category where a map $f : X \to Y$ is a fibration, if and only if each $\mathrm{ev}'_d f$ is a fibration in the model structure $\mathcal{M}_d$ on $[\mathcal{E}_d, \mathcal{C}]_0$, and a weak equivalence if and only if it is in $\mathcal{W}$. The generating cofibrations are given by $\mathcal{G}I$ and the generating acyclic cofibrations are given by $\mathcal{G}J$.

Proof. We check the conditions from the recognition theorem B.1.2. First of all, enriched limits and colimits in $[\mathcal{D}, \mathcal{C}]$ are calculated pointwise by [K, 3.3], i.e. the (co-)limit of a diagram exists if and only if it does so after evaluating to the $[\mathcal{E}_d, \mathcal{C}]$ or equivalently to $[\star, \mathcal{C}]$. Since all the $[\mathcal{E}_d, \mathcal{C}]$ had model structures, they were in particular bicomplete. As they were also tensored and cotensored, they were enriched bicomplete hence so is $[\mathcal{D}, \mathcal{C}]$. The class $\mathcal{W}$ is a subcategory satisfying the 2 out of 3 axiom since it is defined by a levelwise property. By assumption, $\mathcal{G}J$-cell is in $\mathcal{W}$, and since as a left adjoint $\mathcal{G}_d$ preserves retracts and cell complexes, $\mathcal{G}_d J_d$-cell $\subset \mathcal{G}_d I_d$-cof, hence $\mathcal{G}J$-cell $\subset \mathcal{G}I$-cof. Since $\mathcal{G}_d$ is left adjoint to ev'_d, a map has the right lifting property with respect to $\mathcal{G}I$ if and only if for each d its evaluation is an acyclic fibration, in particular if and only if it is in $\mathcal{W}$ and has the lifting property with respect to $\mathcal{G}J$. $\square$

Remark B.2.9. Similarly to the argument for the bicompleteness of $[\mathcal{D}, \mathcal{C}]$, [K, 3.3] implies that the assumption, that each of the $[\mathcal{E}_d, \mathcal{C}]$ is tensored and cotensored, is immediately satisfied if $\mathcal{C}$ was so itself.

Proposition B.2.10. *In the situation of Theorem B.2.8 assume that* $[\mathcal{D}, \mathcal{C}]$ *is itself tensored and cotensored over* $\mathcal{V}$. *If each of the model structures* $\mathcal{M}_d$ *satisfies the pushout product axiom (B.1.10(i)), then so does the assembled model structure on* $[\mathcal{D}, \mathcal{C}]_0$.

Proof. By the adjoint formulations in B.1.12, it suffices to check the pushout product axiom for i a generating cofibration. But $\mathcal{G}_d$ commutes with tensors and pushouts, hence $j \square \mathcal{G}_d i \cong \mathcal{G}_d(j \square i)$. Since $\mathcal{G}_d$ also preserves cell complexes and retracts, that is indeed a cofibration. The case of i or j being acyclic is exactly the same. $\square$

Remark B.2.11. Hence if we can guarantee the analogous proposition for the Unit axiom, a family $\{M_d\}$ of enriched model assembles puzzles together to an enriched model structure on $\mathcal{C}^{\mathcal{D}}$. In particular if the unit object of $\mathcal{V}$ is cofibrant this is trivial. A common other way to ensure this is demanding some sort of *cofibration hypothesis*, cf. B.1.31 and a sufficiently general version of the cube lemma B.1.7.

Depending on the setting, the condition $\mathcal{G}J$-cell $\subset \mathcal{W}$ in can be hard to verify. A way around this is using Shipley and Schwede's lifting lemma [SS, 2.3] instead of the recognition Theorem B.1.2. However, for that result to be applicable in our case, we require another layer of constructions:
In the situation of Theorem B.2.8, consider the subcategogry $\mathcal{E}_{\mathcal{D}}$ of $\mathcal{D}$, consisting of all objects but only the endomorphisms. More precisely, define $\mathcal{E}_{\mathcal{D}}(d, d) := \mathcal{D}(d, d)$ but let $\mathcal{E}(d, e)$ be initial in $\mathcal{D}$ for $d \neq e$. Then the inclusions B.2.2 factor through $\mathcal{E}_{\mathcal{D}}$ and hence we get further factorizations of the evaluation functors from B.2.3

$$\begin{tikzcd} \mathcal{C} & {[\star, \mathcal{C}]} \arrow[l, "\cong"'] & {[\mathcal{E}_d, \mathcal{C}]} \arrow[l] & {[\mathcal{E}_{\mathcal{D}}, \mathcal{C}]} \arrow[l] & {[\mathcal{D}, \mathcal{C}]} \arrow[l, "\mathrm{ev}''"'] \arrow[lll, bend right, "\mathrm{ev}_d"'] \arrow[ll, bend left, "\mathrm{ev}'_d"] \end{tikzcd} .$$

The corresponding diagram of left adjoints B.2.4

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\cong"] & {[\star, \mathcal{C}]} \arrow[r, "\mathcal{E}_d \otimes -"] \arrow[rrr, bend left, "\mathcal{F}_d"] & {[\mathcal{E}_d, \mathcal{C}]} \arrow[r, "\mathcal{G}_d^{\mathcal{E}}"] \arrow[rr, bend right, "\mathcal{G}_d"'] & {[\mathcal{E}_{\mathcal{D}}, \mathcal{C}]} \arrow[r] & {[\mathcal{D}, \mathcal{C}]} \end{tikzcd} .$$

The induced functor pair $[\mathcal{D}, \mathcal{C}]_0 \leftrightarrows [\mathcal{E}_{\mathcal{D}}, \mathcal{C}]_0$ induces a monad T on $[\mathcal{E}_{\mathcal{D}}, \mathcal{C}]_0$ (cf. [McL, IV.1]) and we claim that the associated category of T-algebras is isomorphic to $[\mathcal{D}, \mathcal{C}]_0$. To prove this we check the prerequisites of Beck's Theorem in its weak form from [B, Theorem 1] (cf. [McL, Ex. VI.7. 1-3]). Indeed, since $[\mathcal{D}, \mathcal{C}]$ is enriched cocomplete with colimits calculated pointwise by [K, 3.3], $[\mathcal{D}, \mathcal{C}]_0$ has all coequalizers and they are preserved under evaluation to $[\mathcal{E}_{\mathcal{D}}, \mathcal{C}]_0$. Furthermore the evaluation *reflects* isomorphisms, since a $\mathcal{V}$-natural transformation $\{\alpha_d\}_{d\in\mathcal{D}}$ is an isomorphism if and only if each α_d is.
Further note that $[\mathcal{E}_{\mathcal{D}}, \mathcal{C}]_0 \cong \prod_{d\in\mathcal{D}}[\mathcal{E}_d, \mathcal{C}]_0$ since the $\mathcal{V}$-naturality condition A.2.6 is void when $\mathcal{E}_{\mathcal{D}}(d, e)$ is initial. Hence given the family $\{\mathcal{M}_d\}_{d\in\mathcal{D}}$ we get the product model structure on $[\mathcal{E}_{\mathcal{D}}, \mathcal{C}]_0$:

Proposition B.2.12. *In the situation of Theorem B.2.8, there is a cofibrantly generated model structure on $[\mathcal{E}_{\mathcal{D}}, \mathcal{C}]_0$, where a map is a fibration, cofibration or weak equivalence if and only if it is one in $\mathcal{M}_d$, for all $d \in \mathcal{D}$. The generating sets of cofibrations and acyclic cofibrations are given by the sets $\mathcal{G}^{\mathcal{E}}I$ and $\mathcal{G}^{\mathcal{E}}J$, respectively, which are defined analogous to B.2.6.*

Definition B.2.13. In the situation of theorem B.2.8, an object P of $[\mathcal{D}, \mathcal{C}]$ is a *path object* of an object X of $[\mathcal{D}, \mathcal{C}]$ if there is a factorization of the diagonal map

$$\begin{tikzcd} X \arrow[r, "w"] \arrow[rr, bend left, "\Delta"] & P \arrow[r, "p"] & X \coprod X, \end{tikzcd}$$

with $w \in \mathcal{W}$ and p a pointwise fibration, i.e. a fibration in $\mathcal{M}_d$ after evaluating to $[\mathcal{E}_d, \mathcal{C}]_0$ for all $d \in \mathcal{D}$.

Then hypothesis (2) of [SS, 2.3] allows the following variation of Theorem B.2.8

Theorem B.2.14. *The assembling Theorem B.2.8 still holds if we replace the assumption $\mathcal{G}J$-cell $\subset \mathcal{W}$, with the following:*
In each of the model structures $\mathcal{M}_d$, every object is fibrant and every object of $X \in [\mathcal{D}, \mathcal{C}]$ has a path object.

Remark B.2.15. We should note, that the category $\mathcal{E}$ as defined above is not a complete analogue of the category $\mathcal{I}$ from 1.2.16. Indeed, $\mathcal{I}$ contains isomorphisms between different objects, if they are of the same dimension. In both places, we could have made due with the respective variation. Since the definition of $\mathcal{I}$ seems very natural, while an analogue in the general setting would complicate our approach here, we hope the reader can forgive this. Of course, restricting to skeleta of $\mathcal{D}$, i.e. looking at I instead of $\mathcal{I}$, this distinction vanishes.

As promised we study an enriched version of Theorem B.2.1:

Proposition B.2.16. *Theorem B.2.1 holds in the case of categories enriched over $\mathcal{V}$, if we additionally assume that the domains of the maps in $\mathcal{F}I$ are small relative to $\mathcal{F}I$-cell, the domains of the maps in $\mathcal{F}J$ are small with respect to $\mathcal{F}J$-cell and that $\mathcal{F}J$-cell consists of maps that are level weak equivalences.*

The proof works entirely analogous to the one of B.2.8. Note that we can reformulate the extra assumptions slightly in the following way:

Lemma B.2.17. *If tensoring with morphism objects of $\mathcal{D}$ preserves cofibrations and acyclic cofibrations, the extra assumptions in B.2.16 are satisfied. In particular this is true if there is a model structure on $\mathcal{V}$, such that all the morphism objects of $\mathcal{D}$ are cofibrant, and the model structure on $\mathcal{C}_0$ satisfies the pushout product axiom.*

Proof. This is immediate once one checks that for objects d and e in $\mathcal{D}$, the composition $\mathrm{ev}_e \circ \mathcal{F}_d$ is isomorphic to tensoring with $\mathcal{D}(d, e)$. Since colimits in $[\mathcal{D}, \mathcal{C}]$ are calculated pointwise ([K, 3.3]), maps in $\mathcal{F}J$ cell are levelwise retracts of J-cell complexes, and the same for I. Then all three extra assumptions follow immediately from the axioms of a cofibrantly generated model category. □

Corollary B.2.18. *Since in every model structure cofibrations and weak equivalences are preserved under coproducts, in the case $\mathcal{V} = \mathcal{S}et$ Theorem B.2.16 reduces to B.2.1.*

B.2.1 Special Case: Orthogonal Spectra

The case of (equivariant) orthogonal spectra will be of particular interest for us, and we state consequences and simplifications of the general version of Theorem B.2.8 that become available to us once we can impose a cofibration hypothesis in the sense of B.1.31. In particular, this takes care of all smallness concerns by the following lemma.

Lemma B.2.19. *Any orthogonal spectrum is small with respect to levelwise inclusions, in particular with respect to all h-cofibrations or levelwise h-cofibrations.*

Proof. This must be well known, probably even in a far more general case of functor categories, but an adequate reference escapes the author. We work with with $\mathcal{S}p^{\mathcal{O}}$ for simplicity, but smallness is preserved under equivalences of categories so we do not lose generality. Let X be an orthogonal spectrum, then for each n, there exists κ_n such that both X_n and all $\mathcal{O}(n, m) \wedge X$ are κ_n-small with respect to inclusions for some cardinal κ_n by [H, 2.4.1]. Let κ be a cardinal with $\kappa > \kappa_n$ for $n \geq 0$ and $\kappa > \aleph_0$. We claim that X is κ-small. So let

$$Y^0 \to Y^1 \to \ldots \to Y^\beta \to \ldots$$

be a λ-sequence of levelwise inclusions for some κ-filtered ordinal λ. In particular λ is κ_n-filtered for $n \geq 0$. We want to show that the map

$$\operatorname*{colim}_{\beta<\lambda} \mathcal{S}p^{\mathcal{O}}(X, Y^\beta) \to \mathcal{S}p^{\mathcal{O}}(X, \operatorname*{colim}_{\beta<\lambda} Y^\beta)$$

is an isomorphism. For injectivity consider two elements on the left represented by $f : X \to Y^\alpha$ and $g : X \to Y^\beta$ that are mapped to the same morphism $h : X \to Y$ on the right side, where Y denotes the colimit. That implies that for each n, f_n and g_n induce the same map $X_n \to Y_n$, which implies that there is a $\gamma_n < \kappa_n$ such that the composites $X_n \xrightarrow{f} Y_n^\alpha \to Y_n^{\gamma_n}$ and $X_n \xrightarrow{g} Y_n^\beta \to Y_n^{\gamma_n}$ are equal. Hence for $\gamma := \sup \gamma_n < \lambda$, f and g already induced the same element $X \to Y^\gamma$ as desired. For surjectivity let $f : X \to Y$ be a map to the colimit. As before we get an ordinal γ, such that all the maps $f_n : X_n \to Y_n$ and $f_m \circ \sigma, \sigma \circ f_n : \mathcal{O}(n, m) \wedge X_n \to Y_m$ factor through Y^γ. Then for each pair (n, m), there is a cardinal $\gamma < \delta_{n,m} < \lambda$ such that $\sigma \circ f_n$ and $f_m \circ \sigma$ are factor through the same map $f_{n,m} : \mathcal{O}(n, m) \wedge X_n \to Y_m^{\delta_{n,m}}$ by the argument for injectivity discussed before. Hence there is a factorization of maps of spectra $X \to Y^\delta \to Y$ for $\delta := \sup \delta_{n,m} < \lambda$ as desired. □

Remark B.2.20. Hence any set I of maps in $\mathcal{S}p^{\mathcal{O}}$ that consists of cofibrations, in particular has the property that domains of maps in I are small with respect to I-cell. The same holds for $\mathcal{A}$ a category with a faithful forgetful functor to $\mathcal{S}p^{\mathcal{O}}$ and I a set of maps in $\mathcal{A}$ satisfying the cofibration hypothesis in the sense of B.1.31.

Now let us state the version of Theorem B.2.8 that we are going to use in 1.3.5. Let $\{\mathcal{M}_n\}_{n\geq 0}$ be a family of cofibrantly generated model structures on $\mathbf{O}_n$-spaces, each satisfying the cofibration hypothesis. Let I_n, J_n and $\mathcal{W}_n$ denote the respective generating cofibrations, acyclic cofibrations and classes of weak equivalences. Define $\mathcal{G}I$, $\mathcal{G}J$ and $\mathcal{W}$ as above.

Corollary B.2.21. *If $\mathcal{G}GJ \subset \mathcal{W}$, then there is a cofibrantly generated model structure on $\mathcal{S}p^{\mathcal{O}}$ with generating (acyclic) cofibrations GI (GJ) and $\mathcal{W}$ as its class of weak equivalences.*

Proof. Using Theorem B.2.8, the only thing left to check is the smallness conditions on $\mathcal{G}I$ and $\mathcal{G}J$. These are satisfied as in Remark B.2.20, since all maps in $\mathcal{G}I$ and $\mathcal{G}J$ are h-cofibrations by B.1.16. □

Appendix C

Some Equivariant Homotopy Theory

In this section we will recall some of the results from (non stable) equivariant homotopy theory needed throughout the thesis. We begin with a recollection on model structures on G-spaces and will continue with some consequences of the results of Illman from [Ill83]. We work in the pointed setting (cf. B.1.1) as this is the more important case for us, but all results could be stated in $\mathcal{U}$ as well.

C.1 G-Spaces

Let, as always, $\mathcal{T}$ denote the category of (based, compactly generated, weak Hausdorff) spaces (cf. A.2.15). Let G be an arbitrary topological group in $\mathcal{T}$, and let $G\mathfrak{T}$ be the category of spaces with a (continuous) action of G (cf. A.2.19). Recall that limits and colimits in $G\mathfrak{T}$ can be formed in $\mathcal{T}$ and then given the induced G-action.

Definition C.1.1. Define a continuous functor $\mathcal{T} \to G\mathfrak{T}$ by equipping any space with the trivial G-action. This functor has both a left and a right adjoint, which are of particular importance for us: The left adjoint

$$(-)_G : G\mathfrak{T} \to \mathcal{T},$$

assigning to a G-space its *orbit space*, and the right adjoint

$$(-)^G : G\mathfrak{T} \to \mathcal{T},$$

assigning the subspace of G-fixed points.

The fact that they are adjoints, implies in particular that $(-)_G$ preserves colimits and $(-)^G$ preserves limits, but even more is true (cf. [MM, III.1.6]):

Lemma C.1.2. *The functor $(-)^G$ preserves coproducts, pushouts of diagrams one leg of which is a closed inclusion, and colimits along sequences of closed inclusions. For X and Y in $G\mathfrak{T}$, we have $(X \wedge Y)^G = X^G \wedge Y^G$.*

For subgroups H of G, we can use the forgetful functors induced by the inclusion of one point categories $i : H \to G$ to define fixed point functors

$$G\mathfrak{T} \xrightarrow{i^*} H\mathfrak{T} \xrightarrow{(-)^H} \mathcal{T},$$

and analogous for orbit spaces. It is often convenient to factor these functors in a different way, so that not all of the group action is forgotten: Let N be a normal subgroup of G, then for a G-space X, the quotient group $J := {}^{G}\!/_{N}$ acts on the N-fixed points X^N, and we can redefine the functor $(-)^N : G\mathfrak{T} \to J\mathfrak{T}$. The slight double use of notation is remedied by the fact that the following diagram of functors then commutes:

$$\begin{tikzcd} G\mathfrak{T} \arrow[r, "i_1^*"] \arrow[d, "(-)^N"'] & N\mathfrak{T} \arrow[d, "(-)^N"] \\ J\mathfrak{T} \arrow[r, "i_2^*"] & \mathcal{T}, \end{tikzcd}$$

where $i_1 \colon N \to G$ and $i_2 \colon \{e\} \to J$ are the inclusions.
Similarly we can consider the N-orbit functors $G\mathfrak{T} \to J\mathfrak{T}$.

Adding to the established properties of the fixed point functors, the following technicality proves to be helpful in several places:

Lemma C.1.3. *The fixed point functors $(-)^H$ preserve h-cofibrations.*

Proof. Use the characterization of h-cofibrations via the mapping cylinder (cf. B.1.15), i.e. for $i : A \to X$ an h-cofibration of G-spaces the dotted arrow in the following diagram is a retraction:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & X \arrow[d] \arrow[ddr, bend left] & \\ A \wedge I_+ \arrow[r, "i \wedge I_+"] \arrow[drr, bend right] & X \wedge I_+ \arrow[dr, dotted] & \\ & & Mi \end{tikzcd}$$

Since $(-)^H$ preserves smashing with the G-trivial interval as well as the mapping cylinder by C.1.2, the induced diagram on H-fixed points gives the proof. The same arguments also apply to the case where H is a normal subgroup and we consider $(-)^H$ as a functor to ${}^{G}\!/_{H}\mathfrak{T}$. $\square$

Remark C.1.4. The forgetful functor $i^*\colon G\mathfrak{T} \to H\mathfrak{T}$ we used above is an important example of a change of groups fuctor. As in several places before, their left adjoints given by left Kan extension are just as crucial for our work.

Definition C.1.5. For an H-space $Y \in H\mathfrak{T}$ the smash product $G_+ \wedge Y$ has an action of $G \times H$, with G acting from the left on G_+ and H acting diagonally, from the right on G_+ and from the left on Y. Define the *induced G-space* as

$$G_+ \wedge_H Y,$$

i.e. the orbit G-space of $G_+ \wedge Y$ with respect to the H-action.

Note that the inducing up functor is generally not symmetric monoidal. However, there is an important compatibility property with the smash product of spaces:

Lemma C.1.6. *[Kr, 3.8.1] Let $X \in H\mathfrak{T}$, $Y \in G\mathfrak{T}$. Considering as always smash products with the diagonal actions, there is a natural G-equivariant homeomorphism*

$$(G_+ \wedge_H X) \wedge Y \cong G_+ \wedge_H (i^* Y \wedge X).$$

Proof. Define the map via $([g,x],y) \mapsto [g,(x,g^{-1}y)]$. One easily checks that this assignment is well-defined, equivariant and a homeomorphism. □

C.1.1 Model Structures

There are several well known model structures on $G\mathfrak{T}$ that we are going to make use of. The two most important ones are the genuine and the naive structure:

Definition C.1.7. A map $f : X \to Y$ in $G\mathfrak{T}$ is called a *naive fibration (or weak equivalence)*, if it is a fibration (or weak equivalence) in the usual model structure for $\mathcal{T}$. It is called a *genuine fibration (weak equivalence)*, if for all closed subgroups $H \subset G$ the restriction $f^H : X^H \to Y^H$ to H-fixed points is a fibration (weak equivalence).

The following theorem is then a combination of the statements in [MM, III.1.8] and [P, Theorem 6.3]:

Theorem C.1.8. *The category $G\mathfrak{T}$, is a model category with respect to both the naive and the genuine fibrations and weak equivalences. The cofibrations are given in both cases by the left lifting property with respect to those maps that are both fibrations and weak equivalences.*
Furthermore both of these model structures are compactly generated with the generating (acyclic) cofibrations given by the sets of h-cofibrations

$$I'_G := \{i : (S^{n-1} \times G)_+ \to (D^n \times G)_+, n \geq 0\} \textit{ and}$$

$$J'_G := \{i_0 : (D^n \times G)_+ \to (D^n \times [0,1] \times G)_+, n \geq 0\}$$

for the naive model structure (spheres and disks with the trivial G-action) respectively

$$I_G := \{i : (S^{n-1} \times {}^{G}\!/\!_{H})_+ \to (D^n \times {}^{G}\!/\!_{H})_+, n \geq 0, H \subset G\} \text{ and}$$

$$J_G := \{i_0 : (D^n \times {}^{G}\!/\!_{H})_+ \to (D^n \times [0,1] \times {}^{G}\!/\!_{H})_+, n \geq 0, H \subset G\}$$

for the genuine model structure. Here $H \subset G$ indicates that H should vary over all closed subgroups of G. Both of these model structures satisfy the cofibration hypothesis B.1.31.

Remark C.1.9. Both of these model structures are monoidal (cf. C.2.7). We will use the term *G-topological* for model structures enriched over the genuine structure in the sense of B.1.10

Since these form the basis for the mixed model structure of Theorem 1.3.2, we need to study some more of the technical properties.

Proposition C.1.10. *(i) The naive model structure satisfies the monoid axiom with respect to the smash product.*

(ii) Let j be a naive acyclic cofibration, and i an h-cofibration, then $i\Box j$ is a weak equivalence.

(iii) The functor $(-)_G \colon G\mathcal{T} \to \mathcal{T}$, sending a G space to its orbit space, is a left Quillen functor, considered both from the naive and the genuine structure.

(iv) Genuinely cofibrant G-spaces with a free action, are already naively cofibrant.

Proof. We check that the maps in $J'_G \wedge G\mathcal{T}$ are weak equivalences. By the associativity of the smash product, this follows from the fact that the Quillen model structure on $\mathcal{T}$ satisfies the monoid axiom [H04, 1.5, 1.7]. Since cell complexes built from maps that are h-cofibrations and weak equivalences are again weak equivalences, (i) holds. Then (i) implies (ii), and (iii) follows from the fact that the adjoint is obiously a right Quillen functor. Use lemma C.1.2 to prove (iv): Since taking fixed points commutes with the I_G-cell complex construction, and since a space X has free (based) G-action if and only if its fixed points with respect to all subgroups of G are trivial, there appear only free G-cells in the cell structure of X. □

Families

Definition C.1.11. Let G be a group. A family $\mathscr{F}$ of subgroups of G is called *closed*, if it is closed under taking subgroups and conjugates. It is called *open* if it is the complement of a closed family.

Note that closed and open families sometimes appear with the reversed meaning, depending on which part of the literature one looks at. Other conventions include calling our closed families merely families, and then referring to our open families as cofamilies.

The following model structures are mentioned already in [MM, IV.6.5].

Definition C.1.12. Given a family $\mathscr{F}$ of subgroups of G, a map $f : X \to Y$ in $G\mathfrak{T}$ is called a *$\mathscr{F}$-fibration (-equivalence)*, if the restriction to H-fixed points is a fibration (weak equivalence) for all subgroups $H \in \mathscr{F}$.

Theorem C.1.13. *Let $\mathscr{F}$ be a closed family of subgroups of G. The category $G\mathfrak{T}$ is a compactly generated, proper, monoidal model category with the fibrations and weak equivalences given by the $\mathscr{F}$-fibrations and $\mathscr{F}$-equivalences. The generating (acyclic) cofibrations are given by restricting the sets I_G and J_G, using only the orbit types ${}^{G}\!/_{H}$ with $H \in \mathscr{F}$.*

The proof of the Theorem proceeds analogous to that of [MM, III.1.8], restricting to the diagram category of orbits to those coming from subgroups in $\mathscr{F}$. The pushout product axiom and the condition of being G-topological, is an immediate consequence of Illman's triangulation theorem, and will be discussed in more detail in the next section.

Remark C.1.14. The naive and genuine model structures of C.1.8, are special cases of this, with the naive structure coming from the family only containing the trivial subgroup, and the genuine structure coming from the family of all closed subgroups of G.

Remark C.1.15. Since the smash product with the interval I_+ provides cylinder objects for all these model structures, the generalized Whitehead theorem B.1.3 yields honest G-homotopy equivalences from $\mathscr{F}$-equivalences between $\mathscr{F}$-cofibrant G-spaces for all closed families $\mathscr{F}$. This result has classically been called the $\mathscr{F}$-Whitehead theorem.

As mentioned before, it is often very convenient to have homotopical properties of h-cofibrations at hand when working with topological model categories. The following lemma sums up what we will use in the G-equivariant context:

Lemma C.1.16. *Let $\mathscr{F}$ be a family of subgroups of G.*

(i) sequential colimits of $\mathscr{F}$-equivalences along h-cofibrations are $\mathscr{F}$-equivalences

(ii) pushouts of $\mathscr{F}$-equivalences along h-cofibrations between well based G-spaces are $\mathscr{F}$-equivalences

(iii) the cube lemma holds for $\mathscr{F}$-equivalences and h-cofibrations between well-based G-spaces

(iv) the cube lemma holds for G-homotopy equivalences and h-cofibrations.

Proof. All points follow directly from the analogous statement about weak equivalences in $\mathcal{T}$, and the fact that all the fixed point functors preserve h-cofibrations, as well as sequential colimits and pushouts along h-cofibrations. □

For $i\colon H \subset G$ an inclusion of a closed subgroup, recall the restriction and induction functors from above (C.1.5).

Lemma C.1.17. *Let $\mathscr{F}_H$ be a closed family of subgroups of H and $\mathscr{F}_G$ a closed family of subgroups of G. If $\mathscr{F}_H \subset \mathscr{F}_G$, then the restriction and induction functor form a Quillen pair*

$$G_+\wedge_H(-)\colon G\mathfrak{T} \leftrightarrows H\mathcal{T} \ : i^*.$$

Proof. The right adjoint obviously preserves fibrations and acyclic fibrations. □

Definition C.1.18. Let $\mathscr{F}$ be a closed family of subgroups of G. A universal $\mathscr{F}$-space is a $\mathscr{F}$-cofibrant G-space $E\mathscr{F}$, such that the H-fixed points $E\mathscr{F}^H$ are contractible for $H \in \mathscr{F}$ and empty for $H \notin \mathscr{F}$.

A discussion of existence results for universal $\mathscr{F}$-spaces can be found in the survey article [Lü]. We will therefore not discuss existence but prove the following statement which goes into our construction of the mixed model structures: 2.3.1

Lemma C.1.19. *Let $\mathscr{F}$ be a closed family of subgroups of G. The projection p : $E\mathscr{F} \to \star$ induces an $\mathscr{F}$-equivalence*

$$\pi_H\colon G_+\wedge_H E\mathscr{F}_+ \to {}^{G}\!/\!_{H_+},$$

for all H closed subgroups of G.

Proof. By C.2.5, we can filter G_+ as a naive H-complex. The proof proceeds by induction along this filtration: assume $X_n\wedge_H E\mathscr{F}_+ \to {}^{X_n}\!/\!_H$ is a $\mathscr{F}$-equivalence and X_{n+1} is constructed from X_n by attaching a free H-cell. Then the attaching diagram for the cell yields the commutative cube:

$$\begin{tikzcd}
& S^{k-1}_+\wedge H_+\wedge_H E\mathscr{F}_+ \arrow[rr] \arrow[dl, "i\wedge E\mathscr{F}_+"'] \arrow[dd] & & X_n\wedge_H E\mathscr{F}_+ \arrow[dl] \arrow[dd] \\
D^k_+\wedge H_+\wedge_H E\mathscr{F}_+ \arrow[rr] \arrow[dd] & & X_{n+1}\wedge_H E\mathscr{F}_+ \arrow[dd] & \\
& S^{k-1}_+ \arrow[rr] \arrow[dl, "i"'] & & {}^{X_n}\!/\!_H \arrow[dl] \\
D^k_+ \arrow[rr] & & {}^{X_{n+1}}\!/\!_H &
\end{tikzcd}$$

Since smashing with CW-complexes and preserves $\mathscr{F}$-equivalences and since i and $i \wedge E\mathscr{F}_+$ are h-cofibrations, the generalized cube lemma from C.1.16(iii) gives the induction step. $\square$

C.2 Illman's Triangulation Theorems

In several places we will need to check cofibrancy with respect to the genuine model structure. By the general theory for model categories it will usually suffice to understand the class of I_G-cell complexes in $G\mathfrak{T}$. From here on for the rest of the section, we restrict to the case of G a compact Lie group, where the reference of choice for such questions is [Ill83], in particular Theorems 5.5, 6.1 and 7.1. For convenience, we will recall the statements and the relevant definition from [Ill83] before we give some important corollaries.

Definition C.2.1. Let X be a G-space. For $[x] \in X_G$, define the *G-isotropy type* of $[x]$ as the conjugacy class of the stabilizer subgroup Stab_x of G.
Since the stabilizer subgroups of elements in the same orbit are all conjugate, this indeed only depends on the element $[x] \in X_G$.

Then one of the main results of [Ill83] is the following theorem:

Theorem C.2.2 ([Ill83, 5.5], paraphrased)**.** *Let X be a G-space such that there exists a G-simplicial complex K and a triangulation $t : K \to X_G$ of the orbit space X_G, such that the G-isotropy type is constant on open simplices, i.e. for each open simplex $\mathring{s}$ of K the G-isotropy type is constant on $t(\mathring{s}) \subset X_G$. Then X admits an equivariant triangulation.*

We will not give the explicit definition of an equivariant triangulation, since we will only use the above theorem with the following results in mind:

Proposition C.2.3 ([Ill83, 6.1], paraphrased)**.** *An equivariant triangulation of a G-space X gives X the structure of a G-equivariant CW-complex, in particular X is an I_G-cell complex.*

Theorem C.2.4 ([Ill83, 7.1])**.** *Let M be a smooth G-manifold with or without boundary. Then there exists an equivariant triangulation of M.*

Proof. Note that Illman's proof, uses the above Theorem C.2.2, after establishing that M_G admits a triangulation with constant isotropy type along open simplices. $\square$

As mentioned above, we will make frequent use of these theorems throughout the thesis. The following corollaries are of particular importance:

Corollary C.2.5. *Let G be a compact Lie group and H a closed subgroup. Then a genuinely cofibrant X in $G\mathfrak{T}$ is also genuinely cofibrant in $H\mathfrak{T}$.*

Proof. We restrict to the case where X is an I_G-cell complex. By induction on the cell structure, and since smashing with a space preserves colimits, it suffices to show that for any closed subgroup $K \subset G$ the orbit space ${}^{G}\!/\!_{K}$ is an I_H-cell complex. Indeed, we will show that $\left({}^{G}\!/\!_{K}\right)_H$ admits a triangulation with the H-isotropy type constant on open simplices. Consider the compact Lie group $H \times K^{\mathrm{op}}$ which acts on G via $(h,k).g := hgk^{-1}$. This makes G into an $H \times K^{\mathrm{op}}$-manifold, hence the orbit space $\left({}^{G}\!/\!_{K}\right)_H$ is triangulable where the $H \times K^{\mathrm{op}}$-isotropy type is constant along open simplices. We claim that this triangulation is the desired one: Let e be an element of the orbit space and $[g] \in {}^{G}\!/\!_{K}$ one of its preimages. The stabilizer subgroup of H for $[g]$ is the given by all $h \in H$ such that there exists some $k \in K$ with $hg = gk$ or equivalently $hgk^{-1} = g$. But this is exactly the projection to H of the stabilizer subgroup of g in $H \times K^{\mathrm{op}}$. Hence the H isotropy type of e depends only on the $H \times K^{\mathrm{op}}$-isotropy type of e and is therefore constant along open simplices. □

Corollary C.2.6. *Let G be a compact Lie group and H and K closed subgroups of G. Then the product ${}^{G}\!/\!_{H} \times {}^{G}\!/\!_{K}$ is again an I_G-cell complex, and the only orbit types that appear are ${}^{G}\!/\!_{L}$, with L subconjugate to both H and K.*

Proof. Note that ${}^{G}\!/\!_{H} \times {}^{G}\!/\!_{K}$ is isomorphic to ${}^{G \times G}\!/\!_{H \times K}$ and embed G into $G \times G$ as the diagonal (closed) subgroup. Then Corollary C.2.5 gives the result. For the statement about orbit types, we check what kind of stabilizer subgroups can appear in the product ${}^{G}\!/\!_{H} \times {}^{G}\!/\!_{K}$. In fact if L is the stabilizer of $[g_1], [g_2]$, then we have that $Lg_1 \subset g_1 H$ or equivalently that L is subconjugate to H. The analogous argument for K finishes the proof. □

A first consequence of these is another lemma on properties of the model structures for $G\mathfrak{T}$ from C.1.13

Lemma C.2.7. *Let $\mathscr{F}$ and $\mathscr{F}'$ be closed families of subgroups of G.*

(i) The $\mathscr{F}$-model structure on $G\mathfrak{T}$ is G-topological.

(ii) The $\mathscr{F}$-model structure on $G\mathfrak{T}$ is monoidal with respect to the smash product.

(iii) The smash product of an $\mathscr{F}$-cofibrant G-space with an $\mathscr{F}'$-cofibrant G-space is $\mathscr{F} \cap \mathscr{F}'$-cofibrant.

Proof. All three points are proved in the same way. We check the generating cofibrations. So let

$$i : (S^{n-1} \times {}^{G}\!/\!_{H})_+ \to (D^n \times {}^{G}\!/\!_{H})_+$$

and

$$j : (S^{m-1} \times {}^{G}\!/\!_{H'})_+ \to (D^m \times {}^{G}\!/\!_{H'})_+,$$

with $H \in \mathscr{F}$ and $H' \in \mathscr{F}'$. Then their pushout product is G-homeomorphic to

$$(S^{n+m-1} \times {}^{G}\!/\!_{H} \times {}^{G}\!/\!_{H'})_+ \to (D^{n+m} \times {}^{G}\!/\!_{H} \times {}^{G}\!/\!_{H'})_+,$$

which is a $\mathscr{F} \cap \mathscr{F}'$-cofibration by Corollary C.2.6. If either of i or j is a generating acyclic cofibration the proof is similar with the appropriate spheres replaced by discs. $\square$

Bibliography

[A] V. Angeltveit, 'Enriched Reedy Categories', *Proc. Amer. Math. Soc.* ***136*** (2008), 2323/2332.

[BM] M. Basterra, M.A. Mandell, 'Homology of E_n ring spectra and iterated THH', (2010), `arXiv:1007.5315v2[math.AT]` .

[B] J.M. Beck, 'Triples, Algebras and Cohomology', *Reprints in Theory and Applications of Categories* ***2*** (2003), 1-59.

[Bö] M. Bökstedt, 'Topological Hochschild homology', *Preprint, Bielefeld* (1986).

[BHM] M. Bökstedt, W.C. Hsiang, I. Madsen, 'The cyclotomic trace and algebraic K-theory of spaces', *Invent. Math.* ***111****, Issue 3,* (1993), 465-539.

[BF] A.K. Bousfield, E.M. Friedlander, 'Homotopy theory of Γ-spaces, spectra, and bisimplicial sets', *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill. 1977) II,* Lecture Notes in Mathematics 658 (eds. M.G. Barratt, M.E. Mahowald, Springer, Berlin, 1978), 80-130.

[BCD] M. Brun et al., 'Covering Homology', *Adv. Math.* ***225*** (2010), 3166-3213.

[BMMS] R.R. Bruner, J.P. May, J.E. McClure, M. Steinberger, *H_∞ ring spectra and their applications,* Lecture Notes in Mathematics 1176 (Springer, 1986).

[CDD] G. Carlsson et al., 'Higher topological cyclic homology and the Segal conjecture for tori', *Adv. Math.* ***226*** (2011), 1823-1874.

[C] M. Cole, 'Mixing model structures', *Topology Appl.* ***153****, Issue 7,* (2006), 1016-1032.

[D] B. Day, 'On closed categories of functors', *Reports of the Midwest Category Seminar IV,* Lecture Notes in Mathematics 137 (ed. S. MacLane, Springer, Berlin, 1970), 1-38.

[DGM] B.I. Dundas, T.G. Goodwillie, R. McCarthy, 'The local structure of algebraic K-theory', *(Preprint Version)* (2010), `http://www.uib.no/People/nmabd/` .

[DS] W.G. Dwyer, J. Spalinski, 'Homotopy theories and model categories', *Handbook of algebraic topology* (ed. I.M. James, North-Holland, Amsterdam, 1995), 73-126.

[EKMM] A. D. Elmendorf, I. Kriz, M. A. Mandell, J.P. May, *Rings, modules and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47 (American Mathematical Society, Providence, RI, 1997).

[GJ] P. Goerss, J.F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, 174. Birkhäuser Verlag, Basel, 1999.

[GM] J.P.C. Greenlees, J.P. May, 'Generalized Tate cohomology', *Mem. Amer. Math. Soc.,* ***113(543)*** (1995), viii+178.

[HM] L. Hesselholt, I. Madsen, 'Witt vectors of non-commutative rings and topological cyclic homology', *Acta Math.,* ***178(1)*** (1997), 109-141.

[HRR] M. A. Hill, M. J. Hopkins, D. C. Ravenel 'On the non-existence of elements of Kervaire invariant one', (2010), `arXiv:0908.3724v2` .

[Hir] P.S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99 (American Mathematical Society, Providence, RI, 2003).

[HH] G. Hochschild, 'On the cohomology groups of an associative algebra', *Annals of Math.,* ***46*** (1945), 58-67.

[H] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63 (American Mathematical Society, Providence, RI, 1999).

[H04] M. Hovey, 'Monoidal Model Categories', (1998), `http://math.wesleyan.edu/~mhovey/papers/mon-mod.pdf` .

[HSS] M. Hovey, B. Shipley, J. Smith, 'Symmetric Spectra', *J. Amer. Math. Soc.* ***13*** (2000), 149-208.

[Ill] S. Illman, 'Smooth equivariant triangulations of G-manifolds for G a finite group', *Math. Ann.* ***233*** (1978),199-220.

[Ill83] S. Illman, 'The equivariant triangulation theorem for actions of compact Lie groups', *Math. Ann.* ***262*** (1983), 487-501.

[K] G. M. Kelly, 'Basic Concepts of Enriched Category Theory', *Cambridge University Press, London Math. Soc. Lecture Notes Series* **64** (1982), 97-140.

[KP] D.S. Kahn, S.B. Priddy 'Applications of the transfer to stable homotopy theory', *Bull. Amer. Math. Soc.* ***78*** (1972), 981-987.

[KP78] D.S. Kahn, S.B. Priddy 'The transfer and stable homotopy theory', *Math. Proc. Camb. Phil. Soc.* ***83*** (1978), 103-110.

[Kr] T. A. Kro, 'Involutions on $S[\Omega M]$', (2008), `arXiv:math/0510221v1[math.AT]` .

[LMS] L.G. Lewis, Jr., J.P. May, M. Steinberger, *Equivariant stable homotopy theory,* Lecture Notes in Mathematics 1213 (Springer, 1986).

[Li] J. Lillig, 'A union theorem for cofibrations', *Arch. Math.* ***24*** (1973), 410-415.

[L] J.-L. Loday, *Cyclic Homology*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 301, (Springer, Berlin 1998), Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.

[L89] J.-L. Loday, 'Opérations sur l'homologie cyclique des algèbres commutatives', *Invent. Math.* ***96*** (1989), 205-230.

[Lü] W. Lück, 'Survey on classifying spaces for families of subgroups', *Infinite groups: geometric, combinatorial and dynamical aspects,* Progr. Math. 248, Birkhäuser, Basel (2005), 269-322.

[McL] S. MacLane, *Categories for the working Mathematician,*
(second ed., Springer, Berlin 1997).

[Mad] I. Madsen, 'Algebraic K-theory and traces', in *Current developments in mathematics, 1995 (Cambridge, MA),* Internat. Press, Cambridge, MA (1994), 191-321.

[MM] M.A. Mandell, J.P. May, 'Equivariant orthogonal spectra and S-modules', *Mem. Amer. Math. Soc.* ***159*** (2002).

[MMSS] M.A. Mandell, J.P. May, S. Schwede, B. Shipley, 'Model categories of diagram spectra', *Proc. London Math. Soc* ***83*** (2001), 441-512.

[M] J. P. May, *A concise course in algebraic topology,* University of Chicago Press (1999).

[MS] J. P. May, J. Sigurdsson , *'Parametrized Homotopy Theory'* , American Mathematical Society, Providence, RI (2006).

[MSV] J. McClure, R. Schwänzl, R. Vogt, 'THH(R) $\cong R \otimes S^1$ for E_∞-ring spectra', *J. Pure Appl. Algebra* ***121*** (1997), 137-159.

[P] R. J. Piacenza, 'Homotopy theory of diagrams and CW-complexes over a category', *Canad. J. Math.* ***43*** (1991), no. 4, 814-824.

[Q] D. G. Quillen, *Homotopical algebra,* Lecture Notes in Mathematics 43 (Springer, 1967).

[Schl] C. Schlichtkrull, 'Higher topological Hochschild homology of Thom spectra', *J. Topol.* ***4*** no.1 (2011), 161-189.

[SS] S. Schwede, B. Shipley, 'Algebras and modules in monoidal model categories', *Proc. London Math. Soc* ***80*** (2000), 491-511.

[SS03] S. Schwede, B. Shipley, 'Equivalences of monoidal model categories', *Algebraic & Geometric Topology* ***3*** (2003), 287-334.

[S] S. Schwede, 'Untitled book project about symmetric spectra', *(Preprint Version)* (2007),
`http://www.math.uni-bonn.de/people/schwede` .

[S11] S. Schwede, 'Lecture notes on equivariant stable homotopy theory', *(unpubl.)* (2011),
`http://www.math.uni-bonn.de/people/schwede` .

[Sh] B. Shipley, 'Symmetric spectra and topological Hochschild homology', *K-theory* ***19*** (2000), 155-183.

[Sh04] B. Shipley, 'A convenient model category for commutative ring spectra', in *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory* AMS Bookstore (2004), 473-492.

[St] N.P. Strickland, 'The category of CGWH spaces', *(unpubl.)* (2009),
`http://neil-strickland.staff.ac.uk/courses/homotopy/` .

[Str] A. Strøm, 'The homotopy category is a homotopy category', *Arch. Math.* ***23*** (1972), 435-441.

[tD] T. tomDieck, *Transformation Groups*, de Gruyter Studies in Mathematics, 8 (Walter de Gruyter & Co., Berlin 1987).