Quasi-coherent sheaves on the Moduli Stack of Formal Groups

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For years I have been echoing my betters, especially Mike Hopkins, and telling anyone who would listen that the chromatic picture of stable homotopy theory is dictated and controlled by the geometry of the moduli stack $\mathcal{M}_{\mathbf{fg}}$ of smooth, one-dimensional formal groups. Specifically, I would say that the height filtration of $\mathcal{M}_{\mathbf{fg}}$ dictates a canonical and natural decomposition of a quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}$, and this decomposition predicts and controls the chromatic decomposition of a finite spectrum. This sounds well, and is even true, but there is no single place in the literature where I could send anyone in order for him or her to get a clear, detailed, unified, and linear rendition of this story. This document is an attempt to set that right.

Before going on to state in detail what I actually hope to accomplish here, I should quickly acknowledge that the opening sentences of this introduction and, indeed, this whole point of view is not original with me. I have already mentioned Mike Hopkins, and just about everything I'm going to say here is encapsulated in the table in section 2 of [15] and can be gleaned from the notes of various courses Mike gave at MIT. See, for example, [14]. Further back, the intellectual journey begins, for myself as a homotopy theorist, with Quillen's fundamental insight linking formal groups, complex orientable cohomology theories, and complex cobordism – the basic papers are [40] and [41]. But the theory of formal groups predates Quillen's work connecting algebraic topology and the algebraic geometry of formal groups: there was a rich literature already in place at the time he wrote his papers. Lazard did fundamental work in '50s (see [30]), and there was work of Cartier [2] on what happens when you work localized at a prime, and even a thorough treatment of the deformation theory given by Lubin and Tate [31]. In short, Quillen's work opened the door for the importation of a mature theory in geometry into homotopy theory.

It was Jack Morava, I think, who really had the vision of how this should go, but the 1970s saw a broad eruption of applications of formal groups to homotopy theory. The twin towers here are the paper of Miller, Ravenel, and Wilson [34] giving deep computations in the Adams-Novikov Spectral Sequence and Ravenel's nilpotence conjectures [42], later largely proved by Devinatz, Hopkins, and Smith in [6] and [16]. This period fundamentally changed stable

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homotopy theory. Morava himself wrote a number of papers, most notably [36] (see also Doug Ravenel's Math Review of this paper in [43]), but there are rumors of a highly-realized and lengthy manuscript on formal groups and their applications to homotopy theory. If so, it is a loss that Jack never thought this manuscript ready for prime-time viewing.[1]

Let me begin the account of what you can find here with some indication of how stacks come into the narrative. One simple observation, due originally (I think) to Neil Strickland is that stacks can calculate homology groups. Specifically, if E_* and F_* are two 2-periodic Landweber exact homology theories and if G and H are the formal groups over E_0 and F_0 respectively, then there is a 2-category pull-back square

$$\begin{tikzcd} \operatorname{Spec}(E_0F) \arrow[r] \arrow[d] & \operatorname{Spec}(F_0) \arrow[d, "H"] \\ \operatorname{Spec}(E_0) \arrow[r, "G"'] & \mathcal{M}_{\mathbf{fg}}. \end{tikzcd}$$

This can be seen in Lemma 2.11 below. I heard Mike, in his lectures at Münster, noting this fact as piquing his interest in stacks. Beyond this simple calculation, Strickland should certainly get a lot of credit for all of this: while the reference [48] never actually uses the word "stack", the point of view is clear and, in fact, much of what I say here can be found there in different – and sometimes not so different – language.

For computations, especially with the Adams-Novikov Spectral Sequence, homotopy theorists worked with the cohomology of comodules over Hopf algebroids. A succinct way to define such objects is to say that a Hopf algebroid represents an affine groupoid scheme; in particular, Quillen's theorem mentioned above amounts to the statement that the affine groupoid scheme arising from the Hopf algebroid of complex cobordism is none other than the groupoid scheme which assigns to each commutative ring A the groupoid of formal group laws and their strict isomorphisms over A. Hopf algebroids were and are a powerful computational tool – as far as I know, the calculations of [34] remain, for the combination of beauty and technical prowess, in a class with Secretariat's run at the Belmont Stakes – but an early and fundamental result was "Morava's Change of Rings Theorem", which, in summary, says that if two Hopf algebroids represent equivalent (not isomorphic) groupoid schemes, then they have isomorphic cohomology. A more subtle observation is that the change of rings results holds under weaker hypotheses: the groupoid schemes need only be equivalent "locally in the flat topology"; that is, the presheaves π_0 of components and π_1 of automorphisms induces isomorphic sheaves in the $fpqc$ topology. (See [17] and [12] for discussions of this result.) In modern language, we prove this result by combining the following three observations:

[1] My standard joke is that if you see this manuscript on eBay or somewhere, you should let me know. But, of course, it's not a joke.

- the category of comodules over a Hopf algebroid is equivalent to the category of quasi-coherent sheaves on the associated stack;
- two groupoid schemes locally equivalent in the flat topology have equivalent associated stacks; and
- equivalent stacks have equivalent categories of quasi-coherent sheaves.

Note that in the end, we have a much stronger result than simply an isomorphism of cohomology groups – we have an entire equivalence of categories.

Once we've established an equivalence between the category of comodules and the category of quasi-coherent sheaves (see Remark 3.4) we can rewrite the cohomology of comodules as coherent cohomology of quasi-coherent sheaves; for example,

$$\mathrm{Ext}^s_{MU_*MU}(\Sigma^{2t}MU_*, MU_*) \cong H^s(\mathcal{M}_{\mathbf{fg}}, \omega^{\otimes t})$$

where ω is the invertible sheaf on $\mathcal{M}_{\mathbf{fg}}$ which assigns to each flat morphism $g : \mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$ the invariant differentials ω_G of the formal group classified by G. Thus, one of our most sensitive algebraic approximations to the stable homotopy groups of spheres can be computed as the cohomology of the moduli stack $\mathcal{M}_{\mathbf{fg}}$.

There are other reasons for wanting to pass from comodules over Hopf algebroids to quasi-coherent sheaves. For example, there are naturally occuring stacks which are not canonically equivalent, even in the local sense mentioned above, to an affine groupoid scheme. The most immediate example is the moduli stack $\mathcal{U}(n)$ of formal groups of height less than or equal to some fixed integer $n \geq 0$. These stacks have affine presentations, but not canonically; the canonical presentation is a non-affine open subscheme of $\mathrm{Spec}(L)$, where L is the Lazard ring. Thus the quasi-coherent sheaves on $\mathcal{U}(n)$ are equivalent to many categories of comodules, but no particular such category is preferred (except by tradition – this is one role for the Johnson-Wilson homology theories $E(n)_*$) and the quasi-coherent sheaves themselves remain the basic object of study. The point is taken up in [21] and [37].

Here is what I hope to accomplish in these notes.

- Give a definition of formal group which evidently satisfies the effective descent condition necessary to produce a moduli stack. See Proposition 2.6. This can be done in a number of ways, but the I have chosen to use the notion of formal Lie varieties, a concept developed by Grothendeick to give a conceptual formulation of smoothness in the formal setting.
- A formal group law is equivalent to a formal group with a chosen coordinate. The scheme of all coordinates for a formal group G over a base scheme S is a torsor Coord_G over S for the group scheme Λ which assigns to each ring R the group of power series invertible under composition. Using coordinates we can identify $\mathcal{M}_{\mathbf{fg}}$ as the quotient stack of the scheme of formal groups by the algebraic group Λ. See Proposition 3.13. This makes transparent the fact that $\mathcal{M}_{\mathbf{fg}}$ is an algebraic stack (of a suitable

sort) and it makes transparent the equivalence between comodules and quasi-coherent sheaves.

- The stack $\mathcal{M}_{\mathbf{fg}}$ is not an algebraic stack in the sense of the standard literature (for example, [29]) because it does not have a presentation by a scheme locally of finite type – the Lazard ring is a polynomial ring on infinitely many generators. It is, however, pro-algebraic: it can be written as 2-category (i.e., homotopy) inverse limit of the algebraic stacks $\mathcal{M}_{\mathbf{fg}}\langle n\rangle$ of n-buds of formal groups. This result is inherent in Lazard's original work – it is the essence of the 2-cocycle lemma – but I learned it from Mike Hopkins and it has been worked out in detail by Brian Smithling [47]. An important point is that any *finitely presented* quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}$ is actually the pull-back of a quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}\langle n\rangle$ for some n. See Theorem 3.25.

- Give a coordinate-free definition of height and the height filtration. Working over $\mathbb{Z}_{(p)}$, the height filtration is a filtration by closed, reduced substacks
$$\cdots \subseteq \mathcal{M}(n) \subseteq \mathcal{M}(n-1) \subseteq \cdots \subseteq \mathcal{M}(1) \subseteq \mathcal{M}_{\mathbf{fg}}$$
so that inclusion $\mathcal{M}(n) \subseteq \mathcal{M}(n-1)$ is the effective Cartier divisor defined by a global section v_n of the invertible sheaf $\omega^{\otimes(p^n-1)}$ over $\mathcal{M}(n-1)$. This implies, among other things, that $\mathcal{M}(n) \subseteq \mathcal{M}_{\mathbf{fg}}$ is regularly embedded, a key ingredient in Landweber Exact Functor Theorem and chromatic convergence. The height filtration is essentially unique: working over $\mathbb{Z}_{(p)}$, any closed, reduced substack of $\mathcal{M}_{\mathbf{fg}}$ is either $\mathcal{M}_{\mathbf{fg}}$ itself, $\mathcal{M}(n)$ for some n, or $\mathcal{M}(\infty) = \cap\mathcal{M}(n)$. See Theorem 5.13. This is the geometric content of the Landweber's invariant prime ideal theorem. The stack $\mathcal{M}(\infty)$ is not empty as the morphism classifying the additive formal group over $\mathbb{F}_p$ factors through $\mathcal{M}(\infty)$. This point and the next can also be found in Smithling's thesis [47]. Some of this material is also in the work of Hollander [13].

- Identity $\mathcal{H}(n) = \mathcal{M}(n) - \mathcal{M}(n+1)$, the moduli stack of formal groups of exact height n, as the neutral gerbe determined by the automorphism scheme of any height n formal group Γ_n over $\mathbb{F}_p$. See Theorem 5.35. This automorphism scheme is affine and, if we choose Γ_n to be the Honda formal group of height n, well known to homotopy theorists – its ring of functions is the Morava stabilizer algebra (see [44], Chapter 6) and its group of $\mathbb{F}_{p^n}$ points is the Morava stabilizer group. This is all a restatement of Lazard's uniqueness theorem for height n formal groups in modern language; indeed, the key step in the argument is the proof, essentially due to Lazard, that given any two formal groups G_1 and G_2 over an $\mathbb{F}_p$-scheme S, then the scheme $\mathrm{Iso}_S(G_1, G_2)$ of isomorphisms from G_1 to G_2 is either empty (if they have different heights) or pro-étale and surjective over S (if they have the same height). See Theorem 5.23; we give essentially Lazard's proof, but similar results with nearly identical statements appear in [25].

- Describe the formal neighborhood $\widehat{\mathcal{H}}(n)$ of $\mathcal{H}(n)$ inside the open substack $\mathcal{U}(n)$ of $\mathcal{M}_{\mathbf{fg}}$ of formal groups of height at most n. Given a choice of Γ_n of formal group of height n over the algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$ the morphism

$$\mathbf{Def}(\bar{\mathbb{F}}_p, \Gamma_n) \longrightarrow \widehat{\mathcal{H}}(n)$$

from the Lubin-Tate deformation space to the formal neighborhood is pro-Galois with Galois group $\mathbb{G}(\bar{\mathbb{F}}_p, \Gamma_n)$ of the pair $(\bar{\mathbb{F}}_p, \Gamma_n)$. Lubin-Tate theory identifies $\mathbf{Def}(\bar{\mathbb{F}}_p, \Gamma_n)$ as the formal spectrum of a power series ring; since a power series ring can have no finite étale extensions, we may say $\mathbf{Def}(\bar{\mathbb{F}}_p, \Gamma_n)$ is the universal cover of $\widehat{\mathcal{H}}(n)$. If Γ_n is actually defined over $\mathbb{F}_p$, then $\mathbb{G}(\bar{\mathbb{F}}_p, \Gamma_n)$ is known to homotopy theorists as the big Morava stabilizer group:

$$\mathbb{G}(\bar{\mathbb{F}}_p, \Gamma_n) \cong \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \rtimes \mathrm{Aut}_{\bar{\mathbb{F}}_p}(\Gamma_n).$$

From this theory, it is possible to describe what it means to be a module on the formal neighborhood of a height n point; that is, to give a definition of the category of "Morava modules". See Remark 7.28.

- If $\mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ is a representable, separated, and flat morphism of algebraic stacks, then the induced height filtration

$$\cdots \subseteq \mathcal{N}(n) \subseteq \mathcal{N}(n-1) \subseteq \cdots \subseteq \mathcal{N}(1) \subseteq \mathcal{N}$$

with $\mathcal{N}(n) = \mathcal{M}(n) \times_{\mathcal{M}_{\mathbf{fg}}} \mathcal{N}$ automatically has that the inclusions $\mathcal{N}(n) \subseteq \mathcal{N}(n-1)$ are effective Cartier divisors. The Landweber Exact Functor Theorem (LEFT) is a partial converse to this statement. Here I wrote down a proof due to Mike Hopkins ([14]) of this fact. Other proofs abound – besides the original [28], there's one due to Haynes Miller [33], and Sharon Hollander has an argument as well [13]. The morphism from the moduli stack of elliptic curves to $\mathcal{M}_{\mathbf{fg}}$ which assigns to each elliptic curve its associated formal group is an example. It is worth emphasizing that this is a special fact about the moduli stack of formal groups – the proof uses that $\mathcal{H}(n)$ has a unique geometric point.

- Give proofs of the algebraic analogs of the topological chromatic convergence and fracture square results for spectra. Work over $\mathbb{Z}_{(p)}$ and let $i_n : \mathcal{U}(n) \to \mathcal{M}_{\mathbf{fg}}$ be the open inclusion of the moduli stack of formal groups of height at most n. If $\mathcal{F}$ is a quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}$, we can form the derived push-forward of the pull-back $R(i_n)_* i_n^* \mathcal{F}$. As n varies, these assemble into a tower of cochain complexes of quasi-coherent sheaves on $\mathcal{M}_{\mathbf{fg}}$ and there is a natural map

$$\mathcal{F} \longrightarrow \operatorname{holim} R(i_n)_* i_n^* \mathcal{F}.$$

Chromatic convergence then says that if $\mathcal{F}$ is finitely presented, this is morphism is an equivalence. The result has teeth as the $\mathcal{U}(n)$ do not exhaust $\mathcal{M}_{\mathbf{fg}}$. To examine the transitions in this tower, we note that the inclusion $\mathcal{M}(n) = \mathcal{M}_{\mathbf{fg}} - \mathcal{U}(n-1) \subseteq \mathcal{M}_{\mathbf{fg}}$ is defined by the vanishing of a sheaf of ideals $\mathcal{I}_n$ which is locally generated by regular sequence. Then for any quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}$ there is a homotopy Cartesian square (the fracture square)

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & L(\mathcal{F})^{\wedge}_{\mathcal{I}(n)} \arrow[d] \\
R(i_{n-1})_* i_{n-1}^* \mathcal{F} \arrow[r] & R(i_{n-1})_* i_{n-1}^* (L(\mathcal{F})^{\wedge}_{\mathcal{I}(n)})
\end{tikzcd}$$

where $L(\mathcal{F})^{\wedge}_{\mathcal{I}(n)}$ is the total left derived functor of the completion of $\mathcal{F}$. Both proofs use the homotopy fiber of

$$\mathcal{F} \longrightarrow R(i_n)_* i_n^* \mathcal{F}$$

which is the total local cohomology sheaf $R\Gamma_{\mathcal{M}(n)}\mathcal{F}$. This can be analyzed using the fact that $\mathcal{M}(n) \subseteq \mathcal{M}_{\mathbf{fg}}$ is a regular embedding and Greenlees-May duality [10]; the requisite arguments can be lifted nearly verbatim from [1], but see also [7] – the fracture square appears in exactly this form in this last citation. Chromatic convergence is less general – the proof I give here uses that any finitely presented sheaf can be obtained as a pull-back from the stack of n-buds $\mathcal{M}_{\mathbf{fg}}\langle m \rangle$ for some m. This allows one to show that the transition map

$$R\Gamma_{\mathcal{M}(n+1)}\mathcal{F} \to R\Gamma_{\mathcal{M}(n)}\mathcal{F}$$

between the various total local cohomology sheaves in zero in cohomology for large n.

This document begins with a compressed introduction to some of the algebraic geometry we will need. While I can bluff my way through a lot of algebraic geometry, I am not a geometer either by inclination or training. There are bound to be minor errors, but I hope there's nothing egregious. Corrections would be appreciated.

Acknowledgements: I hope I've made clear my debt to Mike Hopkins; in not, let me emphasize it again. Various people have listened to me talk on this subject in the past few years; in particular, Rick Jardine has twice offered me extended forums for this work, once at the University of Western Ontario, once at the Fields Institute. Some of my students have listened to me at length as well. Two of them – Ethan Pribble [38] and Valentina Joukhovitski [24] – have written theses on various aspects of the theory.

Contents

1 Schemes and formal schemes

This section is devoted entirely to a review of the algebraic geometry we need for the rest of the paper. It can – and perhaps should – be skipped by anyone knowledgeable in these matters.

1.1 Schemes and sheaves

We first recall some basic definitions about schemes and morphisms of schemes, then enlarge the category slightly to sheaves in the *fpqc*-topology. This is necessary as formal schemes and formal groups are not really schemes.

Fix a commutative ring R. Schemes over R can be thought of as functors from $\mathbf{Alg}_R$ to the category of sets. We briefly review this material – mostly to establish language.

The basic schemes over R are the affine schemes $\mathrm{Spec}(B)$, where B is an R-algebra. As a functor

$$\mathrm{Spec}(B) : \mathbf{Alg}_R \longrightarrow \mathbf{Sets}$$

is the representable functor determined by R; that is,

$$\mathrm{Spec}(B)(A) = \mathbf{Alg}_R(B, A).$$

If $I \subseteq B$ is an ideal we have the open subfunctor $U_I \subseteq \mathrm{Spec}(B)$ with

$$U_I(A) = \{\ f : B \to A \mid f(I)A = A\ \} \subseteq \mathrm{Spec}(B).$$

This defines the Zariski topology on $\mathrm{Spec}(B)$. The complement of U_I is defined to be the closed subfunctor $Z_I = \mathrm{Spec}(B/I)$; thus,

$$Z_I(A) = \{\ f : B \to A \mid f(I)A = 0\ \} \subseteq \mathrm{Spec}(B).$$

Note that we can guarantee that

$$U_I(A) \cup Z_I(A) = \mathrm{Spec}(B)(A)$$

only if A is a field.

If $X : \mathbf{Alg}_R \to \mathbf{Sets}$ is any functor, we define a subfunctor $U \subseteq X$ to be *open* if the subfunctor

$$U \times_X \mathrm{Spec}(B) \subseteq \mathrm{Spec}(B)$$

is open for for all morphisms of functors $\mathrm{Spec}(B) \to X$. Such morphisms are in one-to-one correspondence with $X(B)$, by the Yoneda Lemma. A collection of subfunctors $U_i \subseteq X$ is called a cover if the morphism $\sqcup U_i(\mathbb{F}) \to X(\mathbb{F})$ is onto for all *fields* $\mathbb{F}$.

As a matter of language, a functor $X : \mathbf{Alg}_R \to \mathbf{Sets}$ will be called an *R-functor*.

1.1 Definition. *An R-functor X is a scheme over R if it satisfies the following two conditions:*

1. *X is a sheaf in the Zariski topology; that is, if A is an R-algebra and $a_1, \ldots, a_n \in A$ are elements so that $a_1 + \cdots + a_n = 1$, then*

$$X(A) \longrightarrow \prod X(A[a_i^{-1}]) \rightrightarrows \prod X(A[a_i^{-1}a_j^{-1}])$$

   is an equalizer diagram; and

2. *X has an open cover by affine schemes* $\mathrm{Spec}(B)$ *where each B is an R-algebra*

A morphism $X \to Y$ of schemes over R is a natural transformation of R-functors.

An open subfunctor U of scheme X is itself a scheme; the collection of all open subfunctors defines the *Zariski topology* on X.

There is an obvious sheaf of rings $\mathcal{O}_X$ in this topology on X called the *structure sheaf* of X. If $U = \mathrm{Spec}(R) \subseteq X$ is an affine open, then $\mathcal{O}_X(U) = R$; this defintion extends to other open subsets by the sheaf condition. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$*-modules* on X is a sheaf so that

1. for all open $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module;

2. for all inclusions $V \to U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is a morphism of $\mathcal{O}_X(U)$-modules.

A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called *quasi-coherent* if, in addition, we have that

3. there is an open cover by open subschemes $U_i \subseteq X$ so that for each U_i in the cover there are sets I and J and exact sequence of sheaves

$$\mathcal{O}_{U_i}^{(J)} \longrightarrow \mathcal{O}_{U_i}^{(I)} \to \mathcal{F}|_{U_i}$$

A quasi-coherent sheaf is *locally free* if the set J can be taken to be empty. Let $\mathbf{Mod}_X$ and $\mathbf{Qmod}_X$ be the categories of $\mathcal{O}_X$-module sheaves and quasi-coherent sheaves respectively.

If $X_1 \to Y \leftarrow X_2$ is a diagram of schemes, the evident fiber product $X_1 \times_Y X_2$ of functors is again a scheme; furthermore, if $U = X_1 \to Y$ is an open subscheme, then $U \times_Y X_2 \to X_2$ is also an open subscheme. Thus, if $f : X \to Y$ is a morphism of schemes, and $\mathcal{F}$ is a sheaf in the Zariski topology on X, we get sheaf $f_*\mathcal{F}$ on Y with

$$[f_*\mathcal{F}(U) = \mathcal{F}(U \times_Y X).$$

In particular, $f_*\mathcal{O}_X$ is a sheaf of $\mathcal{O}_Y$-algebras and if $\mathcal{F}$ is an $\mathcal{O}_X$-module sheaf, $f_*\mathcal{F}$ becomes a $\mathcal{O}_Y$-module sheaf. Extra hypotheses are needed for $f_*(-)$ to send quasi-coherent sheaves to quasi-coherent sheaves. See Proposition 1.3 below.

1.2 Remark. We use this paragraph to give some standard definitions of properties of morphisms of schemes.

1.) A morphism $f : X \to Y$ of schemes is *flat* if for all open $U \subseteq Y$, $f_*\mathcal{O}_X(U)$ is a flat $\mathcal{O}_Y(U)$-algebra. The morphism f is *faithfully flat* if it is flat and surjective. Here surjective means $X(\mathbb{F}) \to Y(\mathbb{F})$ is onto for all fields.

2.) A scheme X is called *quasi-compact* if every cover by open subschemes $U_i \subseteq X$ has a finite subcover. A morphism of schemes $X \to Y$ is quasi-compact if for every quasi-compact open $V \subseteq Y$, the scheme $V \times_Y X$ is quasi-compact.

3.) A morphism $f : X \to Y$ of schemes is called *quasi-separated* if the diagonal morphism $X \to X \times_Y X$ is quasi-compact.

4.) A morphism $f : X \to Y$ of schemes is *finitely presented* if for all open $U \subseteq Y$, $f_*\mathcal{O}_X(U)$ is a finitely presented $\mathcal{O}_Y(U)$-algebra; that is, $f_*\mathcal{O}_X(U)$ is a quotient of $\mathcal{O}_Y(U)[x_1, \ldots, x_n]$ by a finitely generated ideal.

Any affine scheme $\mathrm{Spec}(R)$ is quasi-compact as the subschemes $\mathrm{Spec}(R[1/f])$ form a basis for the Zariski topology. It follows that every morphism of affine schemes is quasi-compact and quasi-separated.

The following is in [3], Proposition I.2.2.4.

1.3 Proposition. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of schemes. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module sheaf, then $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$ module sheaf.*

The functor f_* from $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules has a left adjoint, of course. If $f : X \to Y$ is a morphism of schemes and $\mathcal{F}$ is any sheaf on Y, define a sheaf $f^{-1}\mathcal{F}$ on X by

$$[f^{-1}\mathcal{F}](U) = \operatorname{colim} \mathcal{F}(V)$$

where the colimit is taken over all diagrams of the form

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ V \arrow[r] & Y \end{tikzcd}$$

with V open in Y. If $\mathcal{F}$ is a $\mathcal{O}_Y$-module sheaf, then $f^{-1}\mathcal{F}$ is a $f^{-1}\mathcal{O}_Y$-module sheaf and

$$f^*\mathcal{F} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{F}.$$

If $\mathcal{F}$ is quasi-coherent, so if $f^*\mathcal{F}$. Thus we have an adjoint pair

$$f^* : \mathbf{Mod}_Y \rightleftarrows \mathbf{Mod}_X : f_*.$$

Here and always, the left adjoint is written on top and from left to right. If f is quasi-compact and quasi-separated, this yields an adjoint pair

$$f^* : \mathbf{Qmod}_Y \rightleftarrows \mathbf{Qmod}_X : f_*.$$

1.4 Remark (Faithfully flat descent). Let $f : X \to Y$ be an morphism of schemes and let

$$X^n = X \times_Y \cdots \times_Y X$$

where the product is taken n times. If $\phi : [m] \to [n]$ is any morphism in the ordinal number category, define $\phi^* : X^n \to X^m$ by the pointwise formula

$$\phi(x_0, \ldots, x_n) = (x_{\phi(0)}, \cdots, x_{\phi(n)}).$$

In this way we obtain a simplicial R-functor $X_\bullet$ augmented to Y. This is the *coskeleton* of f.

A *descent problem* for f is a pair $(\mathcal{F}, \psi)$ where $\mathcal{F}$ is a sheaf on X and $\psi : d_1^*\mathcal{F} \to d_0^*\mathcal{F}$ is an isomorphism of sheaves on $X \times_Y X$ subject to the cocycle condition

$$d_1^*\psi = d_2^*\psi d_0^*\psi$$

over $X \times_Y X \times_Y X$. A solution to a descent problem is a sheaf $\mathcal{E}$ over Y an isomorphism $\phi_0 : f^*\mathcal{E} \to \mathcal{F}$ over X so that the following diagram commutes

$$\begin{tikzcd}
d_1^* f^*\mathcal{E} \arrow[r, "c"] \arrow[d, "d_1^*\psi_0"'] & d_0^* f^*\mathcal{E} \arrow[d, "d_0^*\psi_0"] \\
d_1^*\mathcal{F} \arrow[r, "\psi"'] & d_0^*\mathcal{F}
\end{tikzcd}$$

where c is the canonical isomorphism obtained from the equation $fd_1 = fd_0$. If $f : X \to Y$ is flat and $(\mathcal{F}, \psi)$ is a descent problem with $\mathcal{F}$ quasi-coherent, then there is at most one solution with $\mathcal{E}$ quasi-coherent. If f is faithfully flat, there is exactly one solution and we get an evident equivalence of categories. This has many refinements; for example, one could concentrate on algebra sheaves instead of module sheaves. See Proposition 1.12 below.

1.5 Definition (The *fpqc*-topology). *Let X be a scheme. Define the fpqc-topology on X as follows. The basic opens are the quasi-compact morphisms of schemes $V \to X$. A cover of $V \to X$ is a finite set of flat and quasi-compact morphisms $V_i \to V$ so that $\coprod V_i \to V$ is surjective.*

The acronym *fpqc* stands for “fidèlement plat quasi-compact”. Note that, by taking disjoint unions, we can always insist that the cover have exactly one element.

There is a structure sheaf $\mathcal{O}_X^{fqpc}$ for X is the *fpqc*-topology which assigns to each quasi-compact morphism $V \to X$ the ring

$$\mathcal{O}_X^{fpqc}(V) = H^0(V, \mathcal{O}_V). \tag{1.1}$$

The notion of an $\mathcal{O}_X$-modules extends to this topology as well. Let us write $\mathbf{Mod}_X^{fpqc}$ for the resulting category.

Let X be a quasi-compact and quasi-separated scheme. Then every open morphism $U \to X$ is flat and quasi-compact; hence, there is a restriction morphism

$$\epsilon_* : \mathbf{Mod}_X^{fpqc} \longrightarrow \mathbf{Mod}_X.$$

The functor ϵ_* has a left adjoint ϵ^*: if $\mathcal{F} \in \mathbf{Mod}_X^{Zar}$ then for each flat and quasi-compact morphism $f : U \to X$ we have a $\mathcal{O}_U$-module sheaf $f^*\mathcal{F}$ and these assemble into a presheaf. If $\mathcal{F}$ is quasi-coherent as a sheaf in the Zariski topology, this is already a sheaf, by faithfully flat descent. In the more general case, let $\epsilon^*\mathcal{F}$ be the associated sheaf.

1.6 Definition. *Let $\mathcal{F} \in \mathbf{Mod}_X^{fpqc}$ be an $\mathcal{O}_X$-module sheaf in the $fpqc$-topology. Then $\mathcal{F}$ is cartesian if for any diagram*

$$\begin{tikzcd} V \arrow[d, "f"'] \arrow[dr, "y"] & \\ U \arrow[r, "x"'] & X \end{tikzcd}$$

with all three of f, x, and y flat and quasi-compact, the natural map in $\mathbf{Mod}_V^{Zar}$

$$f^*\mathcal{F}_U \longrightarrow \mathcal{F}_V$$

is an isomorphism.

The following is clear.

1.7 Lemma. *Let $\mathcal{E} \in \mathbf{Mod}_X^{Zar}$ be a quasi-coherent sheaf on X. Then $\epsilon^*\mathcal{E} \in \mathbf{Mod}_X^{fpqc}$ is cartesian and for each flat and quasi-compact morphism $f : U \to X$, the $\mathcal{O}_U$-module sheaf $f^*\mathcal{E}$ is quasi-coherent as a sheaf in the Zariski topology.*

1.8 Proposition. *Let X be a be a quasi-compact and quasi-separated scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$ modules in the $fpqc$-topology. Then the following conditions are equivalent.*

1. *The sheaf $\mathcal{F}$ is cartesian and for each flat and quasi-compact morphism $U \to X$ the $\mathcal{O}_U$-module sheaf $\mathcal{F}_U$ is quasi-coherent in the Zariski topology.*
2. *The sheaf $\epsilon_*\mathcal{F} = \mathcal{F}_X$ is quasi-coherent and the induced morphism $\epsilon^*\epsilon_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.*
3. *There is a flat surjective quasi-compact morphism $p : Y \to X$ and sets I and J so that there is an exact sequence of sheaves in $\mathbf{Mod}_Y^{fpqc}$*

$$\mathcal{O}_Y^{(J)} \longrightarrow \mathcal{O}_Y^{(I)} \longrightarrow p^*\mathcal{F} \longrightarrow 0.$$

This has the following consequence. Let us (for the moment) write $\mathbf{Mod}_X^{cart}$ for the category of cartesian sheaves of $\mathcal{O}_X^{fpqc}$ modules. Assume X is quasi-compact and quasi-separated. Then the adjoint functors

$$\epsilon^* : \mathbf{Qmod}_X \rightleftarrows \mathbf{Mod}_X^{cart} : \epsilon_*$$

forms an equivalence of categories. As a result, we will write $\mathbf{Qmod}_X$ for either category; we also drop the notation $\mathcal{O}_X^{fpqc}$. The analogous statement for categories of modules is false; a useful example to keep in mind is the sheaf $\Omega_{X/R}$ of differentials; if $i : V \to X$ is any morphism of schemes, the map $i^*\Omega_{X/R} \to \Omega_{V/R}$ need not be an isomorphism.

1.9 Remark. Let Y be scheme and let $\mathbf{Sh}_{fpqc}(Y)$ be the category of sheaves of sets in the *fpqc*-topology on Y. If $X \to Y$ is a scheme, then X defines the representable functor which assigns to all quasi-compact morphisms $U \to Y$ the morphisms $U \to X$ of schemes over Y this defines a full and faithfully embedding of schemes over Y into $\mathbf{Sh}_{fqpc}(Y)$. A sheaf of this form is called *representable* and it is customary not to make a notational distinction between the scheme and the sheaf. Note that Y itself is the terminal object of both categories.

We finish this section with a review of an important class of morphisms.

1.10 Definition. *1.) A morphism $f : X \to Y$ of schemes is called* **affine** *if for all morphisms* $\mathrm{Spec}(B) \to Y$*, the R-functor* $\mathrm{Spec}(B) \times_Y X$ *is isomorphic to an affine scheme.*

2.) A morphism $f : X \to Y$ of schemes is a **closed embedding** *if it is affine and for all flat morphisms* $\mathrm{Spec}(B) \to Y$*, the induced morphisms of rings*

$$B = \mathcal{O}_Y(B) \longrightarrow f_*\mathcal{O}_X(B) = \mathcal{O}_X(\mathrm{Spec}(B) \times_Y X \to X)$$

is surjective.

3.) A morphism $f : X \to Y$ of schemes is **separated** *if the diagonal morphism $X \to X \times_Y X$ is a closed embedding. A scheme over a commutative R is separated if the morphism $X \to \mathrm{Spec}(R)$ is separated.*

If $f : X \to Y$ is an affine morphism of schemes, then the $\mathcal{O}_Y$ algebra sheaf $f_*\mathcal{O}_X$ is quasi-coherent. Conversely, if $\mathcal{B}$ is quasi-coherent $\mathcal{O}_Y$-algebra sheaf, define a R-functor $\mathrm{Spec}_Y(\mathcal{B})$ over Y by

$$\mathrm{Spec}_Y(\mathcal{B})(A) = \coprod_{\mathrm{Spec}(A) \to Y} \mathbf{Alg}_A(\mathcal{B}(\mathrm{Spec}(A) \to Y), A).$$

Then $q : \mathrm{Spec}_Y(\mathcal{B}) \to Y$ is an affine morphism of schemes and $q_*\mathcal{O}_{\mathrm{Spec}_Y(\mathcal{B})} \cong \mathcal{B}$. This gives an equivalence between the category of quasi-coherent $\mathcal{O}_Y$-algebras and the category of affine morphisms over Y. Restricting this equivalence gives a one-to-one correspondence between closed embeddings $X \to Y$ and ideal sheaves $\mathcal{I} \subseteq \mathcal{O}_Y$.

An analogous result with an analogous construction holds for quasi-coherent sheaves.

1.11 Proposition. *Let $f : X \to Y$ be an affine morphism of schemes. Then the push-forward functor f_* defines an equivalence of categories between quasi-coherent sheaves on X and quasi-coherent $f_*\mathcal{O}_X$-module sheaves on Y. In particular, f_* is exact.*

If $f : T \to S$ is a morphism of schemes and $X \to S$ is an affine morphism, the $f^*X = T \times_S X$ is also affine. If f is faithfully flat, we have the following result.

1.12 Proposition. *Let $f : T \to S$ be a faithfully flat morphism of schemes. The $f^*(-)$ defines and equivalence of categories from the category of schemes affine over S to the category of descent problems in schemes affine over T.*

1.2 The tangent scheme

If A is a commutative ring, let $A(\epsilon) = A[x]/(x^2)$ be the A-algebra of dual numbers. Here we have written $\epsilon = x + (x^2)$. There is an augmentation $q : A(\epsilon) \to A$ given by $\epsilon \mapsto 0$.

Let R be a commutative ring and let X be a functor from $\mathbf{Alg}_R$ to sets (an "R-functor"); for example, X could be a scheme. Define the *tangent functor* $\mathcal{T}\mathrm{an}_X \to X$ over X to be the functor

$$\mathcal{T}\mathrm{an}_X(A) = X(A(\epsilon))$$

with the projection induced by the augmentation $q : A(\epsilon) \to A$. There is a *zero section* $s : X \to \mathcal{T}\mathrm{an}_X$ induced by the unit map $A \to A(\epsilon)$. If $X \to S$ is a morphism of R-functors, then the relative tangent functor $\mathcal{T}\mathrm{an}_{X/S}$ is defined by the pull-back diagram

$$\begin{tikzcd} \mathcal{T}\mathrm{an}_{X/S} \arrow[r] \arrow[d] & \mathcal{T}\mathrm{an}_X \arrow[d] \\ S \arrow[r, "s"'] & \mathcal{T}\mathrm{an}_S \end{tikzcd}$$

If we let $A(\epsilon_1, \epsilon_2) = A[x,y]/(x^2, xy, y^2)$, then the natural A-algebra homomorphism $A(\epsilon) \to A(\epsilon_1, \epsilon_2)$ given by $\epsilon \mapsto \epsilon_1 + \epsilon_2$ defines a multiplication over X

$$\mathcal{T}\mathrm{an}_{X/S} \times_X \mathcal{T}\mathrm{an}_{X/S} \to \mathcal{T}\mathrm{an}_{X/S}$$

so that $\mathcal{T}\mathrm{an}_{X/S}$ is an abelian group R-functor over X.

If $X \to S$ is a morphism of schemes, then $\mathcal{T}\mathrm{an}_{X/S}$ is an affine scheme over X. See Proposition 1.18. We will see this once we have discussed the connection between the $\mathcal{T}\mathrm{an}_{X/S}$ and the sheaf of differentials $\Omega_{X/S}$.

Let X be an R-functor for some commutative ring R. Define the $\mathcal{O}_X$-module presheaf of differential $\Omega_{X/R}$ by the formula

$$\Omega_{X/R}(\mathrm{Spec}(B) \to X) = \Omega_{B/R}.$$

This became a quasi-coherent sheaf in the Zariski topology. If $f : X \to Y$ is a morphism of R-functors, define $\Omega_{X/Y}$ by the exact sequence of $\mathcal{O}_X$-modules (in the Zariski topology)

$$f^*\Omega_{Y/R} \to \mathcal{O}_{X/R} \to \Omega_{X/Y} \to 0.$$

Since $\Omega_{B/R} = J(B)/J(B)^2$ where $J(B)$ is the kernel of the multiplication map

$$B \otimes_R B \longrightarrow B$$

this definition can be reformulated as follows. A proof can be found in [3] §I.4.2.

1.13 Lemma. *Let $X \to S$ be a separated morphism of schemes, so that diagonal morphism : $\Delta : X \to X \times_S X$ is a closed embedding. Then there is a natural isomorphism $\Omega_{X/S}$ of quasi-coherent sheaves on X*

$$\Omega_{X/S} \cong \Delta^* \mathcal{J}/\mathcal{J}^2$$

where $\mathcal{J}$ is the module of the closed embedding Δ.

If X is not separated, we can still identify the differentials by a variation on this method: if we factor the diagonal map as a closed embedding followed by an open inclusion

$$X \xrightarrow{i} V \longrightarrow X \times_S X$$

then $\Omega_{X/S} \cong i^*\mathcal{J}/\mathcal{J}^2$ where $\mathcal{J}$ is the ideal defining j.

Needless to say, there is a closed connection between differentials and derivations.

1.14 Definition. *Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. Then we define the $\mathcal{O}_X$-algebra sheaf $\mathcal{O}_X \rtimes \mathcal{F}$ on X to be the evident square-zero extension of $\mathcal{O}_X$. Then a derivation of X with coefficients in $\mathcal{F}$ is a diagram of sheaves of commutative rings*

$$\begin{tikzcd} \mathcal{O}_X \arrow[rr, "f"] \arrow[dr, "="'] & & \mathcal{O}_X \rtimes \mathcal{F} \arrow[dl, "p_1"] \\ & \mathcal{O}_X. & \end{tikzcd}$$

If $q : X \to S$ is a scheme over S, then an S-derivation of X with coefficients in $\mathcal{F}$ is a derivation of X with coefficients in $\mathcal{F}$ so that

$$q_*f : q_*\mathcal{O}_X \longrightarrow q_*\mathcal{O}_X \rtimes q_*\mathcal{F}$$

is a morphism of $\mathcal{O}_S$-algebra sheaves. We will write $\mathrm{Der}_S(X, \mathcal{F})$ *for the set of all S-derivations of X with coefficients in $\mathcal{F}$.*

1.15 Example. Suppose $X \to S$ is a separated morphism of schemes. Then, by definition, $\Delta : X \to X \times_S X$ is a closed embedding; let $\mathcal{J}$ be the ideal of this embedding. Write $(X \times_S X)_1 \subseteq X \times_S X$ for the subscheme defined by the vanishing of $\mathcal{J}^2$. Then the splitting provided by the first projection $p_1 : X \times_S X \to X$ defines an isomorphism

$$\Delta^*\mathcal{O}_{(X \times_S X)_1} \cong \mathcal{O}_X \rtimes \Omega_{X/S}.$$

Then the second projection defines an S-derivation of X

$$f_u : \mathcal{O}_X \longrightarrow \mathcal{O}_X \rtimes \Omega_{X/S}$$

The morphisms f_u or the resulting morphism $d : \mathcal{O}_X \to \Omega_{X/S}$ is called the *universal derivation.*

The module of S-derivations is the global sections of the sheaf $\mathcal{D}er_S(X, \mathcal{F})$ which assigns to each Zariski open $U \subseteq X$ the module of derivations

$$\mathrm{Der}_S(U, \mathcal{F}|_U).$$

This is an $\mathcal{O}_X$-module sheaf, although not necessarily quasi-coherent.

1.16 Proposition. *There is a natural isomorphism of $\mathcal{O}_X$-module sheaves*

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/S},\mathcal{F})\longrightarrow \mathcal{D}er_S(X,\mathcal{F})$$

given by composing with the universal derivation.

Proof. The inverse to this this morphism is given as follows. Let $f:\mathcal{O}_X\to\mathcal{O}_X\rtimes\mathcal{F}$ be any derivation and let f_0 be the zero derivation; that is, inclusion into the first factor. Also let $p:\mathcal{O}_X\rtimes\mathcal{F}\to\mathcal{O}_X$ be the projection. Consider the lifting problem

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p"'] & (X\times_S X)_1 \arrow[d, "\subseteq"] \\
\operatorname{Spec}_X(\mathcal{O}_X\rtimes\mathcal{F}) \arrow[r, "f_0\times f"'] \arrow[ur, dashed] & X\times_S X.
\end{tikzcd}$$

Here we have written h for a morphism when we mean $\operatorname{Spec}_X(h)$. Since $\mathcal{O}_X\rtimes\mathcal{F}$ is a square-zero extension, this lifting problem has a unique solution g and that g yields a morphism

$$\Delta^*\mathcal{O}_{(X\times_S X)_1}\cong\mathcal{O}_X\rtimes\Omega_{X/S}\to\mathcal{O}_X\rtimes\mathcal{F}$$

of $\mathcal{O}_X$-algebra sheaves over $\mathcal{O}_X$ as needed. □

The following result follows immediately from the previous proposition upon taking global sections. Note that if $X=\operatorname{Spec}(B)\to\operatorname{Spec}(k)$, this amounts to the classical assertion that

$$\mathbf{Mod}_B(\Omega_{B/k},M)=\operatorname{Der}_k(B,M).$$

1.17 Corollary. *This is a natural isomorphism of modules over the global sections over X*

$$\mathbf{Mod}_X(\Omega_{X/S},\mathcal{F})\cong\operatorname{Der}_S(X,\mathcal{F})$$

given by composing with the universal derivation.

If $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules on a scheme X, we can form the symmetric algebra $\operatorname{Sym}_{\mathcal{O}_X}(\mathcal{F})$; this is a sheaf of quasi-coherent $\mathcal{O}_X$-algebras on X and we denote by

$$\mathbb{V}(\mathcal{F})\longrightarrow X$$

the resulting morphism of affine schemes. If A is an R-algebra, then

$$\mathbb{V}(\mathcal{F})(A)=\coprod_{\operatorname{Spec}(A)\to X}\mathbf{Mod}_A(\mathcal{F}(A),A).$$

The diagonal map $\mathcal{F}\to\mathcal{F}\oplus\mathcal{F}$ gives $\mathbb{V}(\mathcal{F})$ the structure of an abelian group scheme over X.

Proposition 1.16 implies the following result – in the latter proposition set $\mathcal{F}=\mathcal{O}_X$ and note that

$$\mathcal{O}_X(\epsilon)=\mathcal{O}_X\rtimes\mathcal{O}_X.$$

1.18 Proposition. *If $X \to S$ is a separated morphism of schemes, there is a natural isomorphism of abelian group schemes over X*

$$\mathbb{V}(\Omega_{X/S}) \cong \mathcal{T}\mathrm{an}_{X/S}.$$

The following standard fact is useful for calculations.

1.19 Lemma. *Let $i : X \to Y$ be a closed embedding of separated schemes over S defined by an ideal $\mathcal{I} \subseteq \mathcal{O}_Y$. Then there is an exact sequence of sheaves on X*

$$i^*\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} i^*\Omega_{Y/S} \longrightarrow \Omega_{X/S} \longrightarrow 0$$

where d is induced by the restriction of the universal derivation.

Proof. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on X. The statement of the lemma is equivalent to the exactness of the sequence

$$0 \to \mathcal{D}er_S(X, \mathcal{F}) \to \mathcal{D}er_S(Y, i_*\mathcal{F}) \to \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, i_*\mathcal{F})$$

which is easily checked. □

1.3 Formal Lie varieties

We next review the notion of a formal Lie variety, which can be interpreted as a notion of a smooth formal scheme affine over a base scheme S with a preferred section. The first concept (which appeared implicitly in Lemma 1.19) is important in its own right.

1.20 Definition. *Let $i : X \to Y$ be a closed embedding of schemes defined by an ideal $\mathcal{I} \subseteq \mathcal{O}_Y$. Then the quasi-coherent $\mathcal{O}_X$-module*

$$\omega_i \stackrel{\text{def}}{=} i^*\mathcal{I}/\mathcal{I}^2$$

is called **conormal sheaf** *or the module of the embedding i.*

Note that the canonical map $\mathcal{I}/\mathcal{I}^2 \to i_*\omega_i$ of quasi-coherent sheaves on Y is an isomorphism.

1.21 Definition. *If $i : X \to Y$ is a closed embedding of schemes defined by an ideal $\mathcal{I}$, define the nth* **infinitesimal neighborhood**

$$Y_n = \mathrm{Inf}^n_X(Y) \subseteq Y$$

of X in Y to be the closed subscheme of Y defined by the ideal $\mathcal{I}^{n+1}$.

More generally, suppose that $X \to Y$ is an injection of $fqpc$ sheaves over some base scheme S. Define $\mathrm{Inf}^n_X(Y) \subseteq Y$ to be the subsheaf with the following sections. If $U \to S$ is quasi-compact, then $[\mathrm{Inf}^n_X(Y)](U)$ is the set of all $a \in Y(U)$ which satisfy the following condition: there is an $fpqc$ cover $V \to U$ and a closed subscheme $V' \subseteq V$ defined by an ideal with vanishing $(n+1)$st power so that

$$a|_V \in X(V').$$

1.22 Remark. The proof that the notion of infinitesimal neighborhoods for sheaves generalizes that for schemes is Lemma II.1.02 of [32]. This lemma is stated for the $fppf$-topology, but uses only properties of faithfully flat maps of affine schemes, so applies equally well to the $fpqc$-topology. In the same reference, Lemma II.1.03, Messing shows that infinitesimal neighborhoods behave well with respect to base change. Specifically, if $X \subseteq Y$ is an embedding of $fpqc$-sheaves over a scheme S and $f : T \to S$ is a morphism of schemes, then

$$\mathrm{Inf}^n_{f^*X}(f^*Y) \cong f^*\mathrm{Inf}^n_X(Y). \tag{1.2}$$

If $X \to Y$ is a closed embedding of schemes, we get an ascending chain of closed subschemes

$$X = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y.$$

The conormal sheaves $X \to Y_n$ are all canonically isomorphic; hence this module depends only on Y_1. To get an invariant which depends on Y_n, filter $\mathcal{O}_Y$ by the powers of the ideal $\mathcal{I}$ to get a graded $\mathcal{O}_Y/\mathcal{I}$-algebra sheaf on Y. By Proposition 1.11 this determines a unique graded $\mathcal{O}_X$-algebra sheaf $\mathrm{gr}_*(Y)$ on X with

$$\mathrm{gr}_k(Y) = i^*(\mathcal{I}^k/\mathcal{I}^{k+1})$$

In particular, $\mathrm{gr}_1(Y) = \omega_i$. We immediately have that

$$\mathrm{gr}_k(Y_n) = \begin{cases} \mathrm{gr}_k(Y), & k \leq n; \\ 0, & k > n. \end{cases}$$

Now suppose Y is a scheme over S and $e : S \to Y$ is a closed inclusion and a section of the projection $Y \to S$. Let us define

$$\mathcal{O}_Y(e) \subseteq \mathcal{O}_Y$$

to be the ideal sheaf defining this inclusion. It can be thought of as the sheaf of functions vanishing at e. In this case the natural map of Lemma 1.19

$$d : \omega_e = e^*\mathcal{O}_Y(e)/\mathcal{O}_Y(e)^2 \longrightarrow e^*\Omega_{Y/S}$$

becomes an isomorphism; indeed, the exact sequence of the proof collapses to an isomorphism.

1.23 Remark. Let S be a scheme, X a sheaf in the $fpqc$-topology over S and $e : S \to X$ a section of the structure map $X \to S$. Then we can make the following definitions and constructions.

1. Let $X_n = \mathrm{Inf}^n_S(X)$. We say X is *ind-infinitesmal* if the natural map

$$\operatorname{colim} \mathrm{Inf}^n_S(X) \to X$$

is an isomorphism of sheaves.

2. Suppose each of the X_n is representable. Then ω_e can be defined as the conormal sheaf of any of the embeddings $S \to X_n$; furthermore $\omega_e \cong e^*\Omega_{X_n/S}$ for all n.

3. More generally, if each of the X_n is representable define the graded ring $\mathrm{gr}_*(X) = \lim \mathrm{gr}_*(X_n)$; then if $k \le n$

$$\mathrm{gr}_k(X) \cong \mathrm{gr}_k(X_n).$$

1.24 Definition (Formal Lie variety). *Let S be a scheme, X a sheaf in the fpqc-topology over S and $e : S \to X$ a section of the structure map $X \to S$. Then (X, e) is* **formal Lie variety** *if*

1. *X is ind-infinitesmal and $X_n = \mathrm{Inf}^n_S(X)$ is representable and affine over S for all $n \ge 0$;*

2. *the quasi-coherent sheaf ω_e is locally free of finite rank on S;*

3. *the natural map of graded rings $\mathrm{Sym}_*(\omega_e) \to \mathrm{gr}_*(X)$ is an isomorphism.*

A morphism $f : (X, e) \to (X', e')$ of formal Lie varieties is morphism of sheaves which preserves the sections.

1.25 Remark. By Remark 1.22 and Equation 1.2 formal Lie varieties behave well under base change. If (X, e) is a formal Lie variety over a base scheme S and $T \to S$ is a morphism of schemes, then $f^*X \to T$ has an induced section f^*e and

$$f^*(X, e) \overset{\mathrm{def}}{=} (f^*X, f^*e)$$

is a formal Lie variety. We will often drop the section e from the notation.

.

1.26 Remark. We show that, locally in the Zariski topology, every formal Lie variety is isomorphic to the formal spectrum of a power series ring.

1.) Let $S = \mathrm{Spec}(A)$ and let X be the formal scheme $\mathrm{Spf}(A[[x_1, \cdots, x_t]])$. Thus for an A-algebra B,

$$X(B) = \{\ (b_1, \ldots, b_t) \mid b_i \text{ is nilpotent }\} \subseteq B^n$$

Let $e : S \to X$ be the zero section, then

$$X_n = \mathrm{Spec}(A[x_1, \cdots, x_t]/(x_1, \ldots, x_t)^{n+1})$$

and ω_e is the sheaf obtained from the free A-module of rank t generated by the residue classes of $x_1, \cdots, x_t$. It follows that (X, e) is a formal Lie variety.

2.) Conversely, suppose that $S = \mathrm{Spec}(A)$ and that the global sections of ω_e on S is a free A-module with a chosen basis $x_1, \ldots, x_t$. There is an exact sequence of quasi-coherent $\mathcal{O}_{X_n}$-sheaves

$$0 \to \mathcal{O}_{X_n}(e) \to \mathcal{O}_{X_n} \to e_*\mathcal{O}_S \to 0$$

whence an exact sequences of quasi-coherent sheaves on $\mathcal{O}_S$

$$0 \to q_*\mathcal{O}_{X_n}(e) \to q_*\mathcal{O}_{X_n} \to \mathcal{O}_S \to 0.$$

Here we are writing $q : X_n \to S$ for the projection. Since $q_*\mathcal{O}_{X_n}(e) \to q_*\mathcal{O}_{X_{n-1}}(e)$ is onto for all n and $q_*\mathcal{O}_{X_1}(e) \cong \omega_e$ we may choose compatible (in n) splittings $\omega_e \to q_*\mathcal{O}_{X_n}(e)$ and get compatible maps

$$\mathrm{Sym}_S(\omega_e) \to q_*\mathcal{O}_{X_n}$$

which, by Definition 1.24.3 induce isomorphisms

$$\mathrm{Sym}_S(\omega_e)/\mathcal{J}^{n+1} \to q_*\mathcal{O}_{X_n}$$

where $\mathcal{J} = \oplus_{k>0}\mathrm{Sym}_k(\omega_e)$ is the augmentation ideal. Since the global sections of $\mathrm{Sym}_S(\omega_e)/\mathcal{J}^{n+1}$ are isomorphic to $A[x_1, \cdots, x_t]/(x_1, \cdots, x_t)^{n+1}$ we say that the choice of the basis for the global sections of ω_e and the choice of compatible splittings yield an isomorphism $X \cong \mathrm{Spf}(A[[x_1, \cdots, x_t]])$ of formal Lie varieties. This isomorphism is very non-canonical, however.

3.) Finally, for a general base scheme S, choose an open cover by affines $U_i = \mathrm{Spec}(A_i)$ so that the sections of ω_e over U_i is free. Then, after making suitable choices, we get an isomorphism

$$U_i \times_S X \cong \mathrm{Spf}(A_i[[x_1, \cdots, x_t]]).$$

1.27 Remark (The tangent scheme of a formal Lie variety). Let (X, e) be a formal Lie variety over a scheme S. Then $\mathcal{T}\mathrm{an}_{X/S}$ is not necessarily scheme, but only a sheaf in the *fpqc* topology. We'd like to give a description of $\mathcal{T}\mathrm{an}_{X/S}$ as a formal Lie variety. Define $\mathrm{Lie}_{X/S}$ as the pull-back

$$\begin{tikzcd} \mathrm{Lie}_{X/S} \arrow[r, "\varepsilon"] \arrow[d] & \mathcal{T}\mathrm{an}_{X/S} \arrow[d] \\ S \arrow[r, "e"'] & X. \end{tikzcd}$$

More generally, define $(\mathcal{T}\mathrm{an}_{X/S})_n$ by the pull-back diagram

$$\begin{tikzcd} (\mathcal{T}\mathrm{an}_{X/S})_n \arrow[r] \arrow[d] & \mathcal{T}\mathrm{an}_{X/S} \arrow[d] \\ X_n \arrow[r] & X, \end{tikzcd}$$

so that $(\mathcal{T}\mathrm{an}_{X/S})_1 = \mathrm{Lie}_{X/S}$. There are natural maps

$$\mathcal{T}\mathrm{an}_{X_n/S} \to (\mathcal{T}\mathrm{an}_{X/S})_n$$

but these are not in general isomorphisms; however, we do have that

$$\mathrm{colim}\, \mathcal{T}\mathrm{an}_{X_n/S} \to \mathrm{colim}(\mathcal{T}\mathrm{an}_{X/S})_n \to \mathcal{T}\mathrm{an}_{X/S}$$

are all isomorphisms.

To analyze the sheaves $(\mathcal{T}\mathrm{an}_{X/S})_n$ let us write $j_k : X_n \to X_{n+k}$ for the inclusion. Then Lemma 1.19 shows that for all $k > 0$, the natural map

$$j_{k+1}^*\Omega_{X_{n+k+1}/S} \to j_k^*\Omega_{X_{n+k}/S}$$

is an isomorphism of locally free sheaves on X_n. Write $(\Omega_{X/S})_n$ for this sheaf. Again Lemma 1.19 shows that there is a surjection

$$(\Omega_{X/S})_n \longrightarrow \Omega_{X_n/S}$$

but the source is a locally free sheaf and the target is not in general. For example, if $n = 1$, $(\Omega_{X/S})_1 = \omega_e$ but $\Omega_{X_1/S} = 0$. Now we check, using that $X = \operatorname{colim} X_n$ and Lemma 1.18 that there is a natural isomorphism of abelian sheaves over X_n

$$\mathbb{V}_{X_n}((\Omega_{X/S})_n) \cong (\mathcal{T}\mathrm{an}_{X/S})_n.$$

In particular

$$\mathbb{V}_S(\omega_e) \cong \mathrm{Lie}_{X/S}.$$

The natural map $\omega_e = q_*e_*\omega_e \to q_*(\Omega_{X/S})_n$ of quasi-coherent sheaves on S defines a coherent sequence of projections

$$(\mathcal{T}\mathrm{an}_{X/S})_n \longrightarrow \mathrm{Lie}_{X/S}$$

and $\varepsilon : \mathrm{Lie}_{X/S} \to (\mathcal{T}\mathrm{an}_{X/S})_n$ is a section of this projection. Local calculations, using Remark 1.26, now imply that the map $(\mathcal{T}\mathrm{an}_{X/S}, \varepsilon)$ is a formal Lie variety over $\mathrm{Lie}_{X/S}$; the scheme $(\mathcal{T}\mathrm{an}_{X/S})_n$ is the nth infinitesimal neighborhood of $\mathrm{Lie}_{X/S}$ in $\mathcal{T}\mathrm{an}_{X/S}$.

The local calculations are instructive. If $S = \mathrm{Spec}(A)$ and suppose $X = \mathrm{Spf}(A[[x_1, \ldots, x_t]]$ with $e : S \to X$ defined by the ideal $I = (x_1, \ldots, x_n)$, then

$$(\mathcal{T}\mathrm{an}_{X/S})_n \cong \mathrm{Spec}((A[[x_1, \ldots, x_]]/I^n)[dx_1, \ldots, dx_t])$$

In particular

$$\mathrm{Lie}_{X/S} \cong \mathrm{Spec}(A[dx_1, \ldots, dx_t]).$$

The projection $(\mathcal{T}\mathrm{an}_{X/S})_n \to \mathrm{Lie}_{X/S}$ is induced by the natural inclusion of A into $A[[x_1, \ldots, x_t]]/I^n$.

It is worth noting that in the case where $t = 1$,

$$\mathcal{T}\mathrm{an}_{X_n/S} = \mathrm{Spec}(A[x, dx]/(x^n, nx^{n-1}dx)).$$

1.28 Remark. Let $f : (X, e_x) \to (Y, e_Y)$ be a morphism of formal Lie varieties over a fixed base scheme S. Then f determines a sequence of morphims of schemes affine over S

$$f_n : X_n = \mathrm{Inf}_S^n(X) \to \mathrm{Inf}_S^n(Y) = Y_n$$

with the property $\mathrm{Inf}_S^n(f_k) = f_n$ when $k \geq n$. Conversely, given any such sequence of morphisms define $f : X \to Y$ by $f = \operatorname{colim} f_n$; then f is a morphism of formal Lie varieties. Thus we have a one-to-one correspondence between morphisms of Lie varieties and compatible sequences of morphisms on infinitesimal neighborhoods. This is the key to following results.

1.29 Lemma. *Let X and Y be two formal Lie varieties over a scheme S and define the presheaf $\mathrm{Iso}_S(X,Y)$ to the functor which assigns to each morphism $i : U \to S$ of schemes the set of isomorphisms $i^*X \to i^*Y$ of formal Lie varieties. Then $\mathrm{Iso}_S(X,Y)$ is a sheaf in the $fqpc$-topology.*

Proof. Suppose $f : T \to S$ is a quasi-compact and faithfully flat morphism of schemes and suppose $\phi : f^*X \to f^*Y$ is an isomorphism of formal Lie varieties so that $d_1^*\phi = d_0^*\phi$ over $T \times_S T$. Then

$$\phi_n : f^*X_n \longrightarrow f^*Y_n$$

also satisfies the sheaf condition. Thus, by faithfully flat descent for affine schemes (Proposition 1.12), there is a unique isomorphism of affine schemes $\psi_n : X_n \to Y_n$ so that $f^*\psi_n = \phi_n$. By uniqueness $\mathrm{Inf}_S^n(\phi_k) = \phi_n$. Set $\psi = \operatorname{colim} \psi_n$. Then $f^*\psi = \phi$ as needed. This argument extends to the entire site by replacing S by U and X and Y by i^*X and i^*Y respectively. □

The notion of descent problem was defined in Remark 1.4. The following result can be upgraded to an equivalence of categories, as in Proposition 1.12.

1.30 Lemma. *Let $f : T \to S$ be a faithfully flat quasi-compact morphism of schemes. Let (X, ϕ) be a descent problem in formal Lie varieties over T. Then there is a unique (up to isomorphism) solution in formal Lie varieties over S.*

Proof. This again follows from faithfully flat descent. We begin by using Proposition 1.12 to get unique (up to isomorphism) schemes Y_n affine over S and isomorphisms $\phi_n : f^*Y_n \to X_n$ solving the descent problem for X_n. Uniqueness implies that there are unique isomorphisms $S \cong Y_0 \cong S$ and $\mathrm{inf}_S^n(Y_k) \cong Y_n$. Thus $Y = \operatorname{colim} Y_n$ is the candidate for the solution to the descent problem. We must verify points (2) and (3) of Definition 1.24.

For (2) we have that $f^*\omega_{e_Y} = \omega_{e_X}$. Since a quasi-coherent sheaf $\mathcal{F}$ over Y is locally free and finitely generated if and only if $f^*\mathcal{F}$ is locally free and finitely generated. (See [9]§2.6.) For (3), the map (with $e = e_Y$)

$$\mathrm{Sym}_*(\omega_e) \to \mathrm{gr}_*(Y)$$

is an isomorphism because it becomes an isomorphism after applying $f^*(-)$. Thus point (3) is covered. □

The notion of a category fibered in groupoids is defined in Définition 2.1 of [29]. The associated notion of stack is defined in Définition 3.1 of the same reference.

Define a category $\mathcal{M}_{\mathbf{flv}}$ fibered in groupoids over schemes as follows. The objects of $\mathcal{M}_{\mathbf{flv}}$ are pairs (S, X) where S a scheme and $X \to S$ is a formal Lie variety over S. A morphism $(T, Y) \to (S, X)$ in $\mathcal{M}_{\mathbf{flv}}$ is a pair (f, ϕ) where $f : T \to S$ is a morphism of schemes and $\phi : Y \to f^*X$ is an isomorphism of formal Lie varieties over T.

1.31 Proposition. *The category $\mathcal{M}_{\mathbf{flv}}$ fibered in groupoids is a stack in the $fpqc$-topology.*

Proof. For a category fibered in groupoids to be a stack, isomorphisms must form a sheaf (Lemma 1.29) and the groupoids must satisfy effective descent (Lemma 1.30). □

2 Formal groups and coordinates

In the section, we introduce formal groups and the moduli stack $\mathcal{M}_{\mathbf{fg}}$ of formal groups – these are the basic objects of study of this monograph. Except on extremely rare occasions, "formal group" will mean a formal group $G \to S$ we mean a commutative group object in formal Lie varieties of relative dimensions 1 over S, as in Definition 2.2. Thus may think of G as affine and smooth of dimension 1 over S.

We will begin with a definition of formal group which does not depend on a theory of coordinates for formal groups; however, that theory is important, and we will spend part of the section working out the details. Specifically, we note that choices of coordinates amount to sections of scheme over S and we explore the geometry of that scheme. The main result is Theorem 2.24, which shows we are dealing with particularly simple scheme.

Part of this section also explores formal group laws, which are particulary familiar to homotopy theorists.

2.1 Formal groups

A formal group will be mean of formal Lie group of dimension 1. We first note that the category of formal Lie varieties has products. If X and Y are formal Lie varieties over a scheme S, let $X \times_S Y$ be the product be the product sheaf in the $fpqc$ topology. We have that

$$X \times_S Y = \mathrm{colim}(X_n \times_S Y_n).$$

2.1 Lemma. *The sheaf $X \times_S Y$ is a formal Lie variety and the product of X and Y in the category for formal Lie varieties.*

Proof. We leave most of this as exercise. The key observations are that

$$\mathrm{Inf}^n_S(X \times_S Y) = \cup_{p+q=n} \mathrm{Inf}^p_S(X) \times \mathrm{Inf}^q_S(Y)$$

and that

$$\omega_{(e_X, e_Y)} = \omega_{e_X} \oplus \omega_{e_Y}.$$

□

This product has a simple description Zariski locally. (Compare 1.26.) If we choose and affine open $U = \mathrm{Spec}(A) \to S$ over which the global sections of ω_{e_X} and ω_{e_Y} are free with bases $\{x_1, \cdots, x_m\}$ and $\{y_1, \cdots, y_n\}$ respectively, then there is an isomorphism of formal Lie varieties

$$(X \times_S Y)|_U \cong \mathrm{Spf}(A[[x_1, \cdots, x_m, y_1, \cdots, y_n]]).$$

2.2 Definition. *Let S be a scheme. A* **formal group** *over S is an abelian group object (G, e) in the category of formal Lie varieties over S with the property that*

$$\omega_G \stackrel{\text{def}}{=} \omega_e$$

is locally free of **rank 1**. *A homomorphism of formal groups is a morphism of group objects.*

If $f : T \to S$ is a morphism of schemes and G is a formal group over S, then $f^*G = T \times_S G$ is a formal group over T. If $i : U \to S$ is a Zariski open, we write $G|_U$ for i^*G.

2.3 Example (Formal group laws). A formal group (G, e) defines and is defined by a formal group law Zariski locally. This is an expansion of Remark 1.26.

In more detail, if A is a commutative ring, a commutative *formal group law* of dimension 1 is a power series

$$F(x_1, x_2) = x_1 +_F x_2 \in A[[x_1, x_2]]$$

so that

1. $0 +_F x = x +_F 0 = x$;
2. $x_1 +_F x_2 = x_2 +_F x_1$;
3. $(x_1 +_F x_2) +_F x_3 = x_1 +_F (x_2 +_F x_3)$.

If we think of a formal group law F as the homomorphism $F : A[[x]] \to A[[x_1, x_2]]$ sending x to $F(x_1, x_2)$, then F defines a formal group G over $S = \text{Spec}(A)$ by setting $G = \text{Spf}(A[[x]])$ with multiplication

$$G \times_S G \cong \text{Spf}(A[[x_1, x_2]]) \xrightarrow{\text{Spf}(F)} \text{Spf}(A[[x]]) = G.$$

Conversely, if G is a formal group choose a cover $U_i = \text{Spec}(A_i) \to S$ by affines so that for each i, the sections of ω_G is free of rank 1. A choice of generator x for these sections defines an isomorphism

$$G|_{U_i} \cong \text{Spf}(A_i[[x]])$$

and the multiplication on G defines a continuous morphism of power series

$$\Delta : A_i[[x]] \longrightarrow A_i[[x_1, x_2]].$$

Then

$$F_i(x_1, x_2) \stackrel{\text{def}}{=} \Delta(x)$$

is a formal group law.

2.4 Example (Homomorphisms). Homomorphisms of formal groups are determined by power series, at least Zariski locally. A **homomorphism** $\phi : F \to F'$ of formal group laws over R is a power series $\phi(x) \in xA[[x]]$ so that

$$\phi(x_1 +_F x_2) = \phi(x_1) +_{F'} f(x_2). \tag{2.1}$$

A homomorphism is an **isomorphism** if it is invertible under composition; that is, if $\phi'(0)$ is a unit in A. Any homomorphism of formal group laws induces a homomorphism of the formal groups defined by the formal group laws.

Conversely, let $\psi : G \to G'$ be a homomorphism of formal groups over S and choose a cover $U_i = \mathrm{Spec}(A_i)$ so that the global sections of both ω_G and $\omega_{G'}$ are free over A_i. Choose a generator x and y for these global sections and let F and F' be the associated formal group laws over A_i. Then we get a commutative diagram induced by ψ

$$\begin{tikzcd}
\mathrm{Spf}(A_i[[x]]) \arrow[r, "\psi"] \arrow[d, "\Delta_G"'] & \mathrm{Spf}(A_i[[y]]) \arrow[d, "\Delta'_G"'] \\
\mathrm{Spf}(A_i[[x_1, x_2]] \arrow[r, "\psi\times\psi"'] & \mathrm{Spf}(A_i[[y_1, y_2]])
\end{tikzcd}$$

If we let $\phi_i(x) = \psi^*(y) \in A_i[[x]]$, this diagram implies $\phi_i : F \to F'$ is a homomorphism of formal group laws

We now introduce the moduli stack $\mathcal{M}_{\mathbf{fg}}$ of formal groups – meaning formal Lie groups of dimension 1. This stack will be algebraic, in a sense. See Theorem 2.27 below.

2.5 Definition. *Define the* **moduli stack of formal groups** $\mathcal{M}_{\mathbf{fg}}$ *as the following category fibered in groupoids over schemes. The objects in* $\mathcal{M}_{\mathbf{fg}}$ *are pairs* (S, G) *where* S *is a scheme and* $G \to S$ *is a (commutative, 1-dimensional) formal group over* S*. A morphism* $(S, G) \to (T, H)$ *is a pair* (f, ϕ) *where* $f : S \to T$ *is a morphism of schemes and* $\phi : H \to f^*G$ *is an isomorphism of formal groups.*

Of course, we still must prove the following result.

2.6 Proposition. *The category* $\mathcal{M}_{\mathbf{fg}}$ *fibered in groupoids over schemes is a stack in the fpqc topology.*

Proof. The argument exactly as in Proposition 1.31, once we note that the proofs of Lemmas 1.29 and 1.30 immediately apply to this case. □

2.2 Formal group laws

Here we review some of the classical literature on formal group laws.

2.7 Theorem (Lazard). *1.) Let* **fgl** *denote the functor from commutative rings to sets which assigns to each ring A the set of formal group laws over A. Then* **fgl** *is affine scheme; indeed, if $L \cong \mathbb{Z}[x_1, x_2, \cdots]$ is the Lazard ring, then*

$$\mathbf{fgl} \cong \mathrm{Spec}(L).$$

2.) Let **Isofgl** *be the functor which assigns to each commutative ring A the set of isomorphisms $f : F \to F'$ of formal group laws over A. Then* **Isofgl** *is an affine scheme; indeed, if $W \cong L[a_0^{\pm 1}, a_1, a_2, \cdots]$, then*

$$\mathbf{Isofgl} \cong \mathrm{Spec}(W).$$

Put another way, the functor which assigns to any commutative ring A the groupoid of formal group laws over A and their isomorphisms is an affine groupoid scheme. In the language of stable homotopy theory, the pair (L, W) is a *Hopf algebroid*.

2.8 Remark. It is worth noting that the isomorphism $L \cong \mathbb{Z}[x_1, x_2, \ldots]$ is not canonical. The difficult part of Lazard's argument is the symmetric 2-cocycle lemma ([44] A.2.12), which we now revisit. Let

$$C_n(x, y) = \frac{1}{d}[(x + y)^n - x^n - y^n]$$

where $d = p$ if n is a power of p and $d = 1$ otherwise. This is the nth homogeneous symmetric 2-cocycle. Then Lazard proves that if $F(x_1, x_2)$ is a formal group law over a ring A, then there are elements $b_1, b_2, \ldots$ in A so that

$$F(x_1, x_2) \equiv x_1 + x_2 + b_1 C_2(x_1, x_2) + b_2 C_3(x_1, x_2) + \cdots$$

modulo $(a_1, a_2, \ldots)^2$.

The isomorphism $W = L[a_0^{\pm 1}, a_1, \ldots]$ is in better shape, as we are using the usual coordinates on power series.

We now introduce the prestack $\mathcal{M}_{\mathbf{fgl}}$ of formal group laws. It will not be a stack as it does not satisfy effective descent.

Let $\mathbf{Aff}_{\mathbb{Z}}$ be the category of affine schemes over $\mathrm{Spec}(\mathbb{Z})$. Recall from [29], Definition 3.1, that a *prestack* $\mathcal{M}$ over $\mathbf{Aff}_{\mathbb{Z}}$ is a category fibered in groupoids over $\mathbf{Aff}_{\mathbb{Z}}$ so that isomorphisms between objects form a sheaf in the *fpqc* topology.

If $F(x_1, x_2)$ is a formal group law over a ring A and $f : A \to B$ is a ring homomorphism, let $f^*F(x_1, x_2)$ be the formal group law over B obtained by pushing forward the coefficients. The resulting formal group over $\mathrm{Spec}(B)$ is the pull-back of the formal group over $\mathrm{Spec}(A)$ defined by F; hence, we will refer to f^*F as the pull-back of F along f.

2.9 Definition. *Define a category $\mathcal{M}_{\mathbf{fgl}}$ fibered in groupoids over $\mathbf{Aff}_{\mathbb{Z}}$ as follows. Then objects are the pairs $(Spec(A), F)$ where A is a commutative ring and F is a formal group law over A. A morphism $(\mathrm{Spec}(A), F) \to (\mathrm{Spec}(B), F)$ is a pair (f, ϕ) where $f : B \to A$ is a homomorphism of commutative rings and $\phi : F \to f^*F'$ is an isomorphism of formal group laws.*

2.10 Lemma. *The category $\mathcal{M}_{\mathbf{fgl}}$ fibered in groupoids over $\mathbf{Aff}_{\mathbb{Z}}$ is a prestack.*

Proof. Let $S = \operatorname{Spec}(A)$ and let F and F' be two formal group laws over S. We are asking that the functor which assigns to each morphism $\operatorname{Spec}(f) : U = \operatorname{Spec}(R) \to S$ the set of isomorphisms

$$\phi : (f^*F) \to (f^*F')$$

be a sheaf in the $fpqc$-topology. Theorem 2.7.2 gives that this functor is the the affine scheme $\operatorname{Spec}(A \otimes_L W \otimes_L A)$. $\square$

The functor which assigns to each formal group law F over a ring A the associated formal group G_F over the affine scheme $\operatorname{Spec}(A)$ defines a morphism

$$\mathcal{M}_{\mathbf{fgl}} \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

of prestacks. This is not an equivalence, but we will see that this morphism identifies $\mathcal{M}_{\mathbf{fg}}$ as the stack associated to the prestack $\mathcal{M}_{\mathbf{fgl}}$. See Theorem 2.31.

The next result, first noticed by Neil Strickland, probably was the first inkling that stacks were a good thing for stable homotopy theory.

2.11 Lemma. *Suppose F_i, $i = 1, 2$ are formal group laws over commutative rings A_i respectively. Let*

$$G_i \to S_i = \operatorname{Spec}(A_i), \qquad i = 1, 2$$

be the corresponding formal groups. Then the two category pull-back $S_1 \times_{\mathcal{M}_{\mathbf{fg}}} S_2$ is an affine scheme. Specifically, if $L \to A_i$ classifies F_i, then there is an isomorphism

$$S_1 \times_{\mathcal{M}_{\mathbf{fg}}} S_2 \cong \operatorname{Spec}(A_1 \otimes_L W \otimes_L A_2).$$

Proof. By construction we have a factoring

$$S_i \xrightarrow{F_i} \mathcal{M}_{\mathbf{fgl}} \longrightarrow \mathcal{M}_{\mathbf{fg}}.$$

of the morphism classifying G_i. By Remark 2.4, the reduction map $\mathcal{M}_{\mathbf{fgl}} \to \mathcal{M}_{\mathbf{fg}}$ is full and faithful; hence, the natural map

$$S_1 \times_{\mathcal{M}_{\mathbf{fgl}}} S_2 \to S_1 \times_{\mathcal{M}_{\mathbf{fg}}} S_2$$

is an isomorphism. If R is any commutative ring $(S_1 \times_{\mathcal{M}_{\mathbf{fgl}}} S_2)(R)$ is the trivial groupoid with object set the triples

$$(f_1, f_2, \phi : f_1^*F_1 \xrightarrow{\cong} f_2^*F_1)$$

where $f_i : A_i \to R$ are ring homomorphisms. Applying Theorem 2.7.2 now implies the result. $\square$

If G_1 and G_2 are two formal groups over a scheme S, let $\mathrm{Iso}_S(G_1, G_2)$ be the presheaf of sets which assigns to any morphims $f : U \to S$ with affine source the set of isomorphisms $f^*G_1 \to f^*G_2$. There is a pull-back diagram

$$\begin{tikzcd} \mathrm{Iso}_S(G_1,G_2) \arrow[r] \arrow[d] & S \times_{\mathcal{M}_{\mathbf{fg}}} S \arrow[d] \\ S \arrow[r, "\Delta"'] & S \times S. \end{tikzcd}$$

Proposition 2.6 we shows that $\mathrm{Iso}_S(G_1, G_2)$ is actually a sheaf. Lemma 2.11 immediately implies the following.

2.12 Lemma. *Suppose F_i, $i = 1, 2$ are formal group laws over a single commutative ring A and let*

$$G_i \to S = \mathrm{Spec}(A), \qquad i = 1, 2$$

be the corresponding formal groups. Then the sheaf $\mathrm{Iso}_S(G_1, G_2)$ is an affine scheme over S. Specifically, if $L \to A$ classifies F_i, then there is an isomorphism

$$\mathrm{Iso}_S(G_1, G_2) \cong \mathrm{Spec}(A \otimes_{A \otimes A} (A \otimes_L W \otimes_L A)).$$

2.3 Coordinates

We now begin to discuss when a formal group can arise from a formal group law. In the following definition, the base scheme need not be affine.

2.13 Definition. *Let S be a scheme and $q : G \to S$ a formal group over S with conormal sheaf ω_e. Then a* **coordinate** *for G is a global section*

$$x \in \lim H^0(S, q_*\mathcal{O}_{G_n}(e))$$

so for all affine morphisms $f : U = \mathrm{Spec}(A) \to S$, $x|_U$ generates the global sections if $(\omega_e)|_U$.

Every formal group has coordinates locally, as in Example 2.3; this definition asks for a global coordinate.

2.14 Remark. 1.) Let (G, x) be a formal group law over S with a coordinate x. Since x provides a global trivialization of the locally free sheaf ω_e, Definition 1.24.3 allows us to conclude that

$$G_n \cong \mathrm{Spec}_S(\mathcal{O}_S[x]/x^{n+1}). \tag{2.2}$$

Equivalently, we have $q_*\mathcal{O}_{G_n} = \mathcal{O}_S[x]/x^{n+1}$. In particular,

$$\lim H^0(S, q_*\mathcal{O}_{G_n}) \cong H^0(S, \mathcal{O}_S)[[x]].$$

2.) Suppose F is a formal group law over a commutative ring A and G_F is the associated formal group over $\mathrm{Spec}(A)$, as in Example 2.3. Then, as in Remark 1.26.1, G_F has a preferred coordinate x defined by the definition

$$G_F = \mathrm{Spf}(A[[x]]).$$

Then there is an equality of formal group laws

$$x_1 +_{(G_F,x)} x_2 = x_1 +_F x_2.$$

Conversely, if G is a formal group over $\mathrm{Spec}(A)$ with a coordinate x, then Equation 2.2 provides a natural isomorphism (*not* an equality) of formal groups over $\mathrm{Spec}(A)$

$$G_F \xrightarrow{\cong} G.$$

3.) If $f : (G, x) \to (H, y)$ is a homomorphism of formal groups over S with chosen coordinates, then f is defined by a morphism of $\mathcal{O}_S$-algebra sheaves

$$\phi : q_*\mathcal{O}_{H_n} \cong \mathcal{O}_S[[y]]/(y^{n+1}) \to \mathcal{O}_S[[x]]/(x^{n+1}) \cong p_*\mathcal{O}_{G_n}$$

and, thus, is defined by the power series

$$\phi(y) = f(x) \in \Gamma(\mathcal{O}_S)[[x]]$$

which is a homomorphism of formal group laws:

$$f(x_1) +_{F_H} f(x_2) = f(x_1 +_{F_G} x_2).$$

Conversely, any such power series defines a homomorphism of formal group laws.

4.) Suppose we are given a 2-commuting diagram

$$\begin{tikzcd} T \arrow[dr, "H"] \arrow[dd, "g"'] & \\ & \mathcal{M}_{\mathbf{fg}} \\ S \arrow[ur, "G"'] & \end{tikzcd}$$

and a coordinate for x for G over S. Then there is an induced coordinate for H over T. Let $\phi : H \to g^*G$ be the given isomorphism. Then the coordinate for H is the image of x under the homomorphisms

$$H^0(S, q_*\mathcal{O}_{G_n}(e)) \longrightarrow H^0(T, g^*q_*\mathcal{O}_{G_n}(e)) \xleftarrow[\cong]{\phi^*} H^0(T, q_*\mathcal{O}_{H_n}(e)).$$

The next step is to examine the geometry of the scheme of possible coordinates for G. We begin with the following result.

2.15 Lemma. *Let $q : G \to S$ be a formal group over a quasi-compact and quasi-separated scheme S. Then there is a quasi-compact and quasi-separated scheme T and a faithfully flat and quasi-compact morphism $f : T \to S$ so that f^*G has a coordinate.*

Proof. Choose a finite cover $U_i \to S$ by affine open subschemes so that the global section of $(\omega_e)|_{U_i}$ are free. Set $f : T = \sqcup U_i \to S$ to the be evident map. Then f is faifthully flat, quasi-compact and $f^*\omega_e$ is isomorphic to $\mathcal{O}_T$. Since T is a coproduct of affines, the map

$$\lim H^0(T, \mathcal{O}_{f^*G_n}(e)) \to H^0(T, f^*\omega_e)$$

is onto, and we choose as our coordinate any preimage of a generator. □

2.16 Definition. *Define a category $\mathcal{M}_{\mathbf{coord}}$ fibered in groupoids over schemes as follows. The objects of $\mathcal{M}_{\mathbf{coord}}$ are pairs $(q : G \to S, x)$ where G is a formal group over a scheme S and x is a coordinate for G. A morphism in $\mathcal{M}_{\mathbf{coord}}$*

$$(q : G \to S, x) \longrightarrow (q' : G' \to S', x')$$

is a morphism of schemes $f : S \to S'$ and an isomorphim of formal groups $\phi : G \to f^*G'$.

By forgetting the coordinate we get a projection map $\mathcal{M}_{\mathbf{coord}} \to \mathcal{M}_{\mathbf{fg}}$; if we consider this as morphism of categories fibered in groupoids over affine schemes, Remark 2.14.3, factors this projection as the composite

$$\mathcal{M}_{\mathbf{coord}} \xrightarrow{i} \mathcal{M}_{\mathbf{fgl}} \longrightarrow \mathcal{M}_{\mathbf{fg}}.$$

2.17 Proposition. *The morphism $i : \mathcal{M}_{\mathbf{coord}} \to \mathcal{M}_{\mathbf{fgl}}$ of categories fibered in groupoids is an equivalence over affine schemes; that is, for all commutative rings A, the morphism of groupoids*

$$\mathcal{M}_{\mathbf{coord}}(A) \longrightarrow \mathcal{M}_{\mathbf{fgl}}(A)$$

is an equivalence of groupoids.

Proof. This is a restatement of Remark 2.14.3 and Remark 2.14.4. □

2.18 Corollary. *The category $\mathcal{M}_{\mathbf{coord}}$ fibered in groupoids over schemes is a prestack.*

Proof. This requires only that if (G, x) and (H, Y) are two objects over a scheme S, the the isomorphisms $\mathrm{Iso}_S((G, x), (H, y))$ form a sheaf. But

$$\mathrm{Iso}_S((G, x), (H, Y)) = \mathrm{Iso}_S(G, H)$$

where $\mathrm{Iso}_S(G, H)$ is the the sheaf (by Proposition 2.6) of isomorphisms of formal groups. □

We now give extensions of Lemmas 2.12 and 2.11, in that order.

2.19 Proposition. *Let G_1 and G_2 be two formal groups over a quasi-compact and quasi-separated scheme S. Then*

$$\mathrm{Iso}_S(G_1, G_2) \to S$$

is an affine morphism of schemes.

Proof. We prove this by appealing to Lemma 2.12 and faithfully flat descent.

First, suppose G_1 and G_2 can be each given a coordinate. Then, for a fixed choice of coordinate for G_1 and G_2 and for any morphism of schemes $f : U = \mathrm{Spec}(A) \to S$, the formal groups f^*G_i over U has an induced coordinate, and Lemma 2.12 shows

$$f^*\mathrm{Iso}_S(G_1, G_2) = \mathrm{Iso}_U(f^*G_1, f^*G_2) \to U$$

is affine over U; indeed,

$$\mathrm{Iso}_U(f^*G_1, f^*G_2) \cong \mathrm{Spec}(A \otimes_{A\otimes A} (A \otimes_L W \otimes_L A)). \tag{2.3}$$

Expanding this thought, define a presheaf $\mathcal{A}(G_1, G_2)$ of $\mathcal{O}_S$ algebras by

$$\mathcal{A}(G_1, G_2)(f : U \to S) = H^0(U, \mathrm{Iso}_U(f^*G_1, f^*G_2))$$

where $f : U \to S$ runs over all flat morphisms with affine source. Then Equation 2.3 implies that $\mathcal{A}(G_1, G_2)$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. We then have

$$\mathrm{Spec}_S(\mathcal{A}(G_1, G_2)) \cong \mathrm{Iso}_S(G_1, G_2)$$

over S. If $f : T \to S$ is any morphism of schemes, then f^*G_i also can be given a coordinate, by Remark 2.14.4, and then Lemma 2.12 implies that

$$f^*\mathcal{A}(G_1, G_2) \cong \mathcal{A}(f^*G_1, f^*G_2) \tag{2.4}$$

as quasi-coherent $\mathcal{O}_T$-algebra sheaves. This is equivalent to the statement that

$$T \times_S \mathrm{Iso}_S(G_1, G_2) \cong \mathrm{Iso}_T(f^*G_1, f^*G_2). \tag{2.5}$$

For the general case, we appeal to Lemma 2.15 to choose an *fpqc*-cover $f : T \to S$ so that f^*G_i can each be given a coordinate. Then $\mathrm{Iso}_T(f^*G_1, f^*G_2) \cong \mathrm{Spec}_T(\mathcal{A}(f^*G_1, f^*G_2))$ and Equation 2.4 (or Equation 2.5) yields an isomorphism of quasi-coherent $\mathcal{O}_{T\times_S T}$-algebra sheaves

$$\phi : p_1^*\mathcal{A}(f^*G_1, f^*G_2) \longrightarrow \mathcal{A}(f^*G_1, f^*G_2).$$

We check that this isomorphism satisfies the cocycle condition and we get, by faithfully flat descent, a quasi-coherent $\mathcal{O}_S$-algebra sheaf $\mathcal{A}(G_1, G_2)$. Uniqueness of descent and Equation 2.5 imply that $\mathrm{Spec}_S(\mathcal{A}(G_1, G_2) \cong \mathrm{Iso}_S(G_1, G_2)$ over S. □

2.20 Corollary. *Let $G \to S$ and $H \to T$ be formal groups over quasi-compact and quasi-separated schemes. Then the projection morphism*

$$S \times_{\mathcal{M}_{\mathbf{fg}}} T \longrightarrow S \times T$$

is an affine morphism of schemes. In particular $S \times_{\mathcal{M}_{\mathbf{fg}}} T$ is a scheme over S and it is an affine scheme over S if T is an affine scheme.

Proof. One easily checks that there is an isomorphism

$$S \times_{\mathcal{M}_{\mathbf{fg}}} T \cong \mathrm{Iso}_{S\times T}(p_1^*G, p_2^*H).$$

Now we use Proposition 2.19. □

In the following definition we are going to have a functor F on affine schemes over a scheme S. We'll write $F|_U$ for $F(U)$ to in order to avoid too many parentheses.

2.21 Definition. *1.) Let G be a formal group over a scheme S. Define a functor* Coord(G/S) *from affine schemes over S to groupoids as follows. If $i : U \to S$ is any affine morphism, then the objects of* Coord$(G/S)|_U$ *are pairs (i^*G, x) where x is a coordinate for i^*G. The morphisms $f : (i^*G, x) \to (i^*G, y)$ of* Coord$(G/S)|_U$ *are those morphisms of formal group laws so that the underlying morphism of formal groups $f_0 = 1 : i^*G \to i^*G$ is the identity.*

2.) Let us write $\mathrm{Coord}_G \to S$ *for the functor of objects of the groupoid functor* Coord(G/S)

3.) By 2.14.4, a morphism $f : (G, x) \to (G, y)$ so that the underlying morphism of formal groups is the identity amounts to writing the coordinate y as a power series in x. We will call this a change of coordinates.

In the following result, note that we have an isomorphism, not simply an equivalence.

2.22 Lemma. *Let $G \to S$ be a formal group over a scheme S and let $S \to \mathcal{M}_{\mathbf{fg}}$ classify G. Then there is an isomorphism of groupoids over S*

$$\lambda : \mathrm{Coord}(G/S) \longrightarrow S \times_{\mathcal{M}_{\mathbf{fg}}} \mathcal{M}_{\mathbf{fgl}}.$$

Proof. First we define the morphism. Let $f : U = \mathrm{Spec}(A) \to S$ be a morphism out of an affine scheme and let $(f^*G, x) \in \mathrm{Coord}(G/S)|_U$. Define $\lambda(f^*G, x)$ to the the triple

$$(f : U \to S, F, \phi : G_F \to f^*G)$$

where F is the formal group law determined by x (Remark 2.14.2) and ϕ is the natural isomorphism from the formal group determined by F (Example 2.4) to f^*G. The inverse of λ sends (f, F, ϕ) to the pair (f^*G, x) where x is the coordinate defined by ϕ (Remark 2.14.4). □

The next result follows immediately from Proposition 2.17. Notice only have an equivalence in this case.

2.23 Corollary. *Let $G \to S$ be a formal group over a scheme S and let $S \to \mathcal{M}_{\mathbf{fg}}$ classify G. Then there is an equivalence of groupoids over S*

$$\lambda : \mathrm{Coord}(G/S) \longrightarrow S \times_{\mathcal{M}_{\mathbf{fg}}} \mathcal{M}_{\mathbf{coord}}.$$

In the following result, we will call a groupoid scheme $\mathcal{G}$ over S affine over S if both the projection maps ob$\mathcal{G} \to S$ and mor$\mathcal{G} \to S$ are affine morphisms.

2.24 Theorem. *1.) Let $G \to S$ be a formal group over a quasi-compact and quasi-separated scheme S. Then* $\mathrm{Coord}(G/S) \to S$ *is a groupoid scheme affine over S.*

2.) For all morphisms $f : T \to S$ of schemes, the groupoid $\mathrm{Coord}(G/S)|_T$ *is either empty or contractible.*

3.) The objects $\mathrm{Coord}_G \to S$ *of* $\mathrm{Coord}(G/S)$ *form an affine scheme over S.*

Proof. Lemma 2.22 and Theorem 2.7 imply together that the objects and morphisms of $\mathrm{Coord}(G/S)$ are, respectively,

$$S \times_{\mathcal{M}_{\mathbf{fg}}} \mathrm{Spec}(L)$$

and

$$S \times_{\mathcal{M}_{\mathbf{fg}}} \mathrm{Spec}(W).$$

Part (1) of the theorem follows from Corollary 2.20. Since Coord_G is the scheme of objects in $\mathrm{Coord}(G/S)$, part (3) follows from part (1).

For part (2), if f^*G has no coordinate, then $\mathrm{Coord}(G/S)|_T$ is empty. If, however, f^*G has a coordinate, then any two coordinates are connected by a unique isomorphism, by Remark 2.14.3, and the groupoid is contractible. □

The reader familiar with the language will recognize that the scheme Coord_G of objects is actually a torsor for an appropriate group scheme. See Lemma 3.11.

2.25 Remark. Since the proof of Theorem 2.24 is at the end of a logical thread which winds in way through most of this section, it might be worthwhile to consider the example where $S = \mathrm{Spec}(B)$ is affine and G is a formal group which can be given a coordinate y. Then if $f : \mathrm{Spec}(A) = U \to S$ is any morphism from an affine scheme, and $(f^*G, x) \in \mathrm{Coord}(G/S)|_U$ is any coordinate for G over U, then x can be written in terms of y; that is,

$$x = a_0 y + a_1 y^2 + a_2 y^3 + \cdots \overset{\mathrm{def}}{=} a(y)$$

where $a_i \in A$ and a_0 is invertible. From this we see that the choice of the coordinate y defines an isomorphism of schemes

$$\mathrm{Coord}_G \cong \mathrm{Spec}(B[a_0^{\pm 1}, a_1, a_2, \cdots]) \cong \mathrm{Spec}(B \otimes_L W) \cong S \times_{\mathcal{M}_{\mathbf{fg}}} \mathrm{Spec}(L).$$

An isomorphism $\phi : (f^*G, x_0) \to (f^*G, x_1)$ in $\mathrm{Coord}(G/S)|_U$ is determined by a power series

$$x_1 = \lambda_0 x_0 + \lambda_1 x_0^2 + \lambda_2 x_0^3 + \cdots = \lambda(x_0)$$

and $x_1 = \lambda(a(y))$. This shows the choice of the coordinate y defines an isomorphism of schemes from the morphisms of $\mathrm{Coord}(G/S)$ to

$$\mathrm{Spec}(B[a_0^{\pm 1}, a_1, \ldots, \lambda_0^{\pm 1}, \lambda_1, \ldots]) \cong \mathrm{Spec}(B \otimes_L W \otimes_L W).$$

2.4 $\mathcal{M}_{\mathbf{fg}}$ is an *fpqc*-algebraic stack

We begin by defining our notion of an algebraic stack.

2.26 Definition. *Let Y be a scheme and $\mathcal{M}$ any stack over Y. Then $\mathcal{M}$ is an* **algebraic stack** *in the fpqc-topology if*

1. *the diagonal morphism $\mathcal{M} \to \mathcal{M} \times_Y \mathcal{M}$ is representable, separated, and quasi-compact; and*

2. *there a scheme X and a surjective, flat, and quasi-compact morphism $X \to \mathcal{M}$. The morphism $X \to \mathcal{M}$ is called a presentation of $\mathcal{M}$.*

This is a relaxation of the usual definition of algebraic stack (as in [29], Définition 4.1) where the presentation $X \to \mathcal{M}$ is required to be smooth, so in particular flat and locally of finite type. It turns out that $\mathcal{M}_{\mathbf{fg}}$ can be approximated by such stacks, as we see in the next section.

The following result is obtained by combining Propositions 2.28 and 2.29 below.

2.27 Theorem. *The moduli stack $\mathcal{M}_{\mathbf{fg}}$ is an algebraic stack over* $\mathrm{Spec}(\mathbb{Z})$ *in the fpqc-topology. Let $\mathbf{fgl} = \mathrm{Spec}(L)$ be the affine scheme of formal group laws and let $G_F \to \mathbf{fgl}$ be the formal group arising from the universal formal group law. Then*

$$G_F : \mathbf{fgl} \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

is a presentation for $\mathcal{M}_{\mathbf{fg}}$.

Let $\mathcal{M}$ be a stack and $x_1, x_2 : S \to \mathcal{M}$ be two 1-morphisms. Then the 2-category pull-back of

$$\begin{tikzcd} & \mathcal{M} \arrow[d, "\Delta"] \\ S \arrow[r, "(x_1,x_2)"] & \mathcal{M} \times_Y \mathcal{M} \end{tikzcd}$$

is equivalent to the *fpqc* sheaf $\mathrm{Iso}_S(x_1, x_2)$ which assigns to each affine scheme $U \to S$ over S the isomorphisms $\mathrm{Iso}_U(f^*x_1, f^*x_2)$.

2.28 Proposition. *The diagonal morphism*

$$\mathcal{M}_{\mathbf{fg}} \longrightarrow \mathcal{M}_{\mathbf{fg}} \times \mathcal{M}_{\mathbf{fg}}$$

is representable, quasi-compact, and separated.

Proof. We use Proposition 2.19: for any affine scheme S and any two formal groups G_1 and G_2, the morphism

$$\mathrm{Iso}_S(G_1, G_2) \to S$$

is an affine morphism of schemes. Hence the diagonal is representable ([29], 3.13), quasi-compact, and separated ([29], 3.10). □

2.29 Proposition. *Let* $\mathbf{fgl} = \mathrm{Spec}(L)$ *be the affine scheme of formal group laws and let* $G_F \to \mathbf{fgl}$ *be the formal group arising from the universal formal group law. Then*

$$G_F : \mathbf{fgl} \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

is surjective, flat, and quasi-compact.

Proof. The morphism G_F is surjective because every formal group over a field can be given a coordinate, and hence arises from a formal group law. To check that it is quasi-compact and flat, we need to check that for all morphisms

$$G : \mathrm{Spec}(A) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

with affine, source, the resulting map

$$\mathrm{Spec}(A) \times_{\mathcal{M}_{\mathbf{fg}}} \mathbf{fgl} \to \mathrm{Spec}(A)$$

is quasi-compact and flat. It is quasi-compact because it is affine (by Proposition 2.20). To check that is flat, we choose an faithfully flat extension $A \to B$ and check that

$$\mathrm{Spec}(B) \times_{\mathcal{M}_{\mathbf{fg}}} \mathbf{fgl} \cong \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A) \times_{\mathcal{M}_{\mathbf{fg}}} \mathbf{fgl} \to \mathrm{Spec}(B)$$

is flat. Put another way, we may assume G has a coordinate and arises from a formal group law. Then, by Lemma 2.11

$$\mathrm{Spec}(A) \times_{\mathcal{M}_{\mathbf{fg}}} \mathbf{fgl} \cong \mathrm{Spec}(A \otimes_L W)$$

and $A \to A \otimes_L W \cong A[a_0^{\pm 1}, a_1, \ldots]$ is certainly faithfully flat. □

2.30 Theorem. *The* 1*-morphism of prestacks*

$$\mathcal{M}_{\mathbf{coord}} \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

identifies $\mathcal{M}_{\mathbf{fg}}$ *as the stack associated to the prestack* $\mathcal{M}_{\mathbf{coord}}$.

Proof. We begin by giving a formal description of the stack $\tilde{\mathcal{M}}_{\mathbf{coord}}$ associated to $\mathcal{M}_{\mathbf{coord}}$, then prove that there is an appropriate equivalence $\tilde{\mathcal{M}}_{\mathbf{coord}} \to \mathcal{M}_{\mathbf{fg}}$.

First, we define an equivalence class of coordinates for a formal group G over S as follows. A representative of this equivalence class will be a coordinate x for $f^*G = T \times_S G \to T$ where $f : T \to S$ is a faithfully flat and quasi-compact morphism. If x_1 and x_2 are coordinates for $T_1 \times_S G$ and $T_2 \times_S G$ respectively, then we say they are equivalent if $p_1^* x_1 = p_2^* x_2$ as coordinates for $(T_1 \times_S T_2) \times_S G$. That this is an equivalence relation follows from the fact that Coord_G is a sheaf in the *fpqc* topology.

Now define the category $\tilde{\mathcal{M}}_{\mathbf{coord}}$ fibered in groupoids over $\mathbf{Aff}_{\mathbb{Z}}$ as follows. The objects are pairs $(G \to S, [x])$ where G is a formal group over S and $[x]$ is an equivalence class of coordinates, as in the previous paragraph. A morphism $f : (G, [x]) \to (H, [y])$ is given by a morphism $f_0 : G \to H$ in $\mathcal{M}_{\mathbf{fg}}$. That

$\tilde{\mathcal{M}}_{\mathbf{coord}}$ is a stack is proved exactly as in Lemma 2.6. The projection map $\mathcal{M}_{\mathrm{Coord}} \to \mathcal{M}_{\mathbf{fg}}$ has an evident factorization

$$\mathcal{M}_{\mathbf{coord}} \longrightarrow \tilde{\mathcal{M}}_{\mathbf{coord}} \longrightarrow \mathcal{M}_{\mathbf{fg}}.$$

We will prove that the first map has the universal property necessary for the associated stack, and we will show the second map is an equivalence of stacks.

First, we must show that any factorization problem

$$\begin{tikzcd}
\mathcal{M}_{\mathbf{coord}} \arrow[r, "\lambda"] \arrow[d] & \mathcal{N} \\
\tilde{\mathcal{M}}_{\mathbf{coord}} \arrow[ur, dashed] &
\end{tikzcd}$$

has a solution $\bar{\lambda} : \tilde{\mathcal{M}}_{\mathbf{coord}} \to \mathcal{N}$ so that the triangle 2-commutes. To do this, let $(G \to S, [x])$ be an object in $\tilde{\mathcal{M}}_{\mathbf{coord}}$ and choose an $fpqc$ cover $d : T \to S$ so that d^*G has a coordinate x representing $[x]$. If we apply λ to the effective descent data

$$\phi : (d_1^* d^* G, d_1^* x) \to (d_0^* d^* G, d_0^* x)$$

we obtain a object $w \in \mathcal{N}(S)$ and a unique isomorphism

$$d^* w \cong \lambda(d^* G, x). \tag{2.6}$$

Set $\bar{\lambda}(G, [x]) = w$. In like manner, $\bar{\lambda}$ can be defined on morphisms. The unique isomorphisms of Equation 2.6 shows that the resulting diagram 2-commutes.

Second, to show that $\tilde{\mathcal{M}}_{\mathbf{coord}} \to \mathcal{M}_{\mathbf{fg}}$ is an equivalence, note that for all schemes S, the morphism of groupoids

$$\tilde{\mathcal{M}}_{\mathbf{coord}}(S) \longrightarrow \mathcal{M}_{\mathbf{fg}}(S)$$

is a fibration.That is, given any object $(H, [y])$ in $\tilde{\mathcal{M}}_{\mathbf{coord}}(S)$ and any morphims $\phi : G \to H$ in $\mathcal{M}_{\mathbf{fg}}(S)$, there is a morphism $\phi : (G : [x]) \to (H, [y])$ in $\tilde{\mathcal{M}}_{\mathbf{coord}}(S)$ whose underlying morphism is ψ. This follows from Remark 2.14.5. If $G \in \mathcal{M}_{\mathbf{fg}}(S)$ is a fixed formal group, then the fiber at G is

$$\operatorname*{colim}_{T \to S} \mathrm{Coord}(G/T)$$

where T runs over all fpqc covers of S. Combining Theorem 2.24.2 and Lemma 2.15 we see that this fiber is contractible. $\square$

The following is an immediate consequence of the previous result and Proposition 2.17.

2.31 Theorem. *The* 1*-morphism of prestacks*

$$\mathcal{M}_{\mathbf{fgl}} \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

identifies $\mathcal{M}_{\mathbf{fg}}$ *as the stack associated to the prestack* $\mathcal{M}_{\mathbf{fgl}}$.

2.5 At a prime: p-typical coordinates

When making calculations, especially with the Adams-Novikov spectral sequence, it is often very convenient to use p-typical formal group laws instead of arbitrary formal group laws. We delve a little into that theory here. A point to be made is that it is not a formal group which is p-typical, but a formal group law or, equivalently, a coordinate for a formal group.

Let A be a commutative ring over $\mathbb{Z}_{(p)}$ and let F be a formal group law over A. Then, given any integer n prime to p and a primitive nth root of unity ζ, we can form the power series

$$f_n(x) = [\frac{1}{n}]_F(x +_F \zeta x +_F \cdots +_F \zeta^{n-1}x). \tag{2.7}$$

Note that this is a power series over A.

More generally, if S is a scheme over $\mathbb{Z}_{(p)}$, G a formal group over S, and X a coordinate for G, then we have a formal group law $x_1 +_{(F,x)} x_2$ over $H^0(S, \mathcal{O}_S)$ and we can form the power series $f_n(x)$ over $H^0(S, \mathcal{O}_S)$.

2.32 Definition. *1.) A p-**typical formal group law** F over a commutative ring A is a formal group law F over A so that*

$$f_\ell(x) = 0$$

for all primes $\ell \neq p$. A homomorphism of p-typical formal group laws is simply a homomorphism of formal groups.

*2.) Let G be a formal group over a scheme S over $\mathbb{Z}_{(p)}$. Then a coordinate x for G is p-**typical** if the associated formal group law over $H^0(S, \mathcal{O}_S)$ is p-typical. A morphism $\phi : (G, x) \to (H, y)$ of formal groups with p-typical coordinates is simply a homomorphism of the underlying formal groups.*

2.33 Remark (Properties of p-typical formal group laws). Let us record some of the standard properties of p-typical coordinates.

1. Let G be a formal group over a $\mathbb{Z}_{(p)}$-algebra A with p-typical coordinate x. Then there are elements $u_i \in A$ so that

   $$[p]_G(x) = px +_G u_1x^p +_G u_2x^{p^2} +_G \cdots .$$

   The elements u_i depend on the pair (G, x), hence are not invariant under changes of coordinate. However, $f : A \to B$ is a homomorphism of $\mathbb{Z}_{(p)}$-algebras, then

   $$[p]_{f^*G}(x) = px +_G f(u_1)x^p +_G f(u_2)x^{p^2} +_G \cdots .$$

   Thus, this presentation of $[p]_G(x)$ extends to schemes: given a p-typical formal group law (G, x) over a $\mathbb{Z}_{(p)}$ scheme, there are elements $u_i \in H^0(S, \mathcal{O}_S)$ so that $[p]_G(x)$ can be written as above.

2. Let us write **pfgl** for the functor which assigns to each $\mathbb{Z}_{(p)}$ algebra A the set of p-typical formal group laws over A. Then **pfgl** is an affine scheme. Indeed, if we write $V = \mathbb{Z}_{(p)}[u_1, u_2, \ldots]$ there is a p-typical formal group law F over V so that

$$[p]_F(y) = py +_F u_1 y^p +_F u_2 y^{p^2} +_F \cdots .$$

The evident morphism of schemes $\mathrm{Spec}(V) \to \mathbf{pfgl}$ is an isomorphism.

3. Let $\phi : (G, x) \to (H, y)$ be an isomorphism of formal groups with p-typical coordinates and let $f(x) \in R[[x]]$ be the power series determined by ϕ. Then there are elements $t_i \in R$ so that

$$f^{-1}(x) = t_0 x +_G t_1 x^p +_G t_2^{p^2} +_G \cdots .$$

More its true. If x is a p-typical coordinate, then y is p-typical if and only if $f^{-1}(x)$ has this form.

As in Definitions 2.9 and 2.16 and Lemmas 2.10 and 2.18, we have prestacks of $\mathcal{M}_{\mathbf{pfgl}}$ of p-typical formal group laws and $\mathcal{M}_{\mathbf{pcoord}}$ of formal groups with p-typical coordinates. We also have the analog of Proposition 2.17:

2.34 Proposition. *The canonical morphism of prestacks*

$$\mathcal{M}_{\mathbf{pcoord}} \longrightarrow \mathcal{M}_{\mathbf{pfgl}}$$

is an equivalence.

A much deeper result is the following. If X is a sheaf over $\mathrm{Spec}(R)$ in the $fpqc$-topology and $R \to S$ is a ring homomorphism, we will write

$$X \otimes_R S \stackrel{\mathrm{def}}{=} X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S).$$

2.35 Theorem (Cartier's idempotent). *The canonical* 1*-morphism of categories fibered in groupoids over* $\mathbf{Aff}_{\mathbb{Z}_{(p)}}$

$$\mathcal{M}_{\mathbf{pfgl}} \longrightarrow \mathcal{M}_{\mathbf{fgl}} \otimes \mathbb{Z}_{(p)}$$

is an equivalence.

Proof. Let A be a commutative $\mathbb{Z}_{(p)}$-algebra. Cartier's theorem (see, for example, [44]A.2.1.18) is usually phrased as follows: Given any formal group law F over A there is a p-typical formal group law eF over A and an isomorphism $\phi_F : F \to eF$ of formal group laws; furthermore, if F is p-typical, then $eF = F$ and ϕ is the identity. This implies that if $\psi : F \to F'$ is any isomorphism of formal groups laws, then there is a unique isomorphism $e\psi$ so that the following diagram commutes:

$$\begin{tikzcd}
F \arrow[r, "\phi_F"] \arrow[d, "\psi"'] & eF \arrow[d, "e\psi"] \\
F' \arrow[r, "\phi_{F'}"'] & eF'
\end{tikzcd}$$

Rephrased, we see that we have a retraction $e : \mathcal{M}_{\mathbf{fgl}}(A) \to \mathcal{M}_{\mathbf{pfgl}}(A)$ of the inclusion of groupoids $\iota : \mathcal{M}_{\mathbf{pfgl}}(A) \to \mathcal{M}_{\mathbf{fgl}}(A)$ and a natural transformation $\phi : 1 \to \iota e$. $\square$

The following is now and immediate consequence of Theorem 2.31 and Theorem 2.35.

2.36 Corollary. *The canonical* 1*-morphism of prestacks*

$$\mathcal{M}_{\mathbf{pcoord}} \longrightarrow \mathcal{M}_{\mathbf{fg}} \otimes \mathbb{Z}_{(p)}$$

identifies $\mathcal{M}_{\mathbf{fg}} \times \mathrm{Spec}(\mathbb{Z}_{(p)})$ *as the stack associated to the prestack* $\mathcal{M}_{\mathbf{pcoord}}$.

Similarly $\mathcal{M}_{\mathbf{pcoord}} \to \mathcal{M}_{\mathbf{fg}} \otimes \mathbb{Z}_{(p)}$ identifies the target as the stack associated to the prestack source. Compare Theorem 2.30.

The following now follows from Corollary 2.36 and Remark 2.33.2 and .3.

2.37 Corollary. *Let* $V = \mathbb{Z}_{(p)}[u_1, u_2, \cdots]$ *and let* $G_F \to \mathrm{Spec}(V)$ *be the formal group formed from the univeral* p*-typical formal group law* F*. Then the map*

$$\mathrm{Spec}(V) \longrightarrow \mathcal{M}_{\mathbf{fg}} \otimes \mathbb{Z}_{(p)}$$

classifying G *is an fpqc-presentation of* $\mathcal{M}_{\mathbf{fg}} \otimes \mathbb{Z}_{(p)}$*. There is an isomorphism of affine schemes*

$$\mathrm{Spec}(V) \times_{\mathcal{M}_{\mathbf{fg}}} \mathrm{Spec}(V) \cong \mathrm{Spec}(V[t_0^{\pm 1}, t_1, t_2, \cdots]).$$

2.38 Remark (Gradings and formal group laws). There is a natural grading on the Lazard ring L and the ring $V = \mathbb{Z}_{(p)}[u_1, u_2, \cdots]$ which supports the universal p-typical formal group law. This can be useful for computations.

To get the grading, we put an action of the multiplication group $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t^{\pm 1}])$ on scheme $\mathbf{fgl} = \mathrm{Spec}(L)$ of formal group laws as follows. If

$$x +_F y = \sum c_{ij} x^i y^j$$

is a formal group law over a ring R and $\lambda \in R^\times$ is a unit in R, define a new formal group law λF over R by

$$x +_{\lambda F} y = \lambda^{-1}((\lambda x) +_F (\lambda y)).$$

This action translates into a coaction

$$\psi : L \longrightarrow \mathbb{Z}[t^{\pm 1}] \otimes L$$

and hence a grading on L: $x \in L$ is of degree n if $\psi(x) = t^n \otimes x$. Then coefficients a_{ij} of the universal formal group law have degree $i + j - 1$; since

$$x +_F y = x + y + x_1 C_2(x, y) + x_2 C_3(x, y) + \cdots$$

modulo decomposables, we have that x_i has degree i. In particular, c_{ij} is a homogeneous polynomial in x_k with $k < i + j$.

The same construction applies to p-typical formal group laws and the p-series

$$[p]_G(x) = px +_G u_1x^p +_G u_2x^{p^2} +_G \cdots$$

shows that, under the action of $\mathbb{G}_m$, u_k has degree $p^k - 1$. Since the universal p-typical formal group is defined over the ring $V = \mathbb{Z}_{(p)}[u_1, u_2, \cdots]$ we have that the coefficients c_{ij} of the universal p-typical formal group are homogeneous polynomials in the u_k where $p^k \leq i + j$.

The action of $\mathbb{G}_m$ extends to the entire groupoid scheme of formal group laws and their isomorphisms. If $\phi(x) = \sum_{i\geq 0} a_ix^i$ is an isomorphism from F to G, define

$$(\lambda\phi)(x) = \lambda^{-1}\phi(\lambda x).$$

The $\lambda\phi$ is an isomorphism from λF to λG. If ϕ is universal isomorphism over $W = L[a_0^{\pm 1}, a_1, \cdots]$, the a_i has degree i. More interesting is the case of p-typical formal group laws; if ϕ is the universal isomorphism of p-typical formal group laws over $V[t_0^{\pm 1}, t_1, t_2, \cdots]$, then

$$\phi^{-1}(x) = t_0x +_G t_1x^p +_G t_2x^{p^2} +_G \cdots$$

and we see that the degree of t_k is $p^k - 1$. Thus if a_i is the ith coefficient of this power series, we have that a_i is a homogeneous polynomial in t_k and u_k with $p^k \leq i$.

3 The moduli stack of formal groups as a homotopy orbit

3.1 Algebraic homotopy orbits

First some generalities, from [29] §§2.4.2, 3.4.1, and 4.6.1. Let Λ be an group scheme over a base scheme S. Let $X \to S$ be a right-Λ-scheme. Thus, there is an action morphism

$$X \times_S \Lambda \longrightarrow X$$

over S such that the evident diagrams commute. From this data, we construct a stack $X \times_\Lambda E\Lambda$, called the *homotopy orbits* of the action of Λ on X, as follows.

Recall that an Λ-torsor is a scheme $P \to S$ with a right action of Λ so that there is an *fpqc* cover $T \to S$ and an isomorphism of Λ-schemes over T

$$T \times_S \Lambda \cong T \times_S P.$$

If you want a choice-free way of stating this last, we remark that this is equivalent to requiring that the natural map

$$(T \times_S P) \times_T (T \times_S \Lambda) \longrightarrow (T \times_S P) \times_T (T \times_S \Lambda)$$

over $(T \times_S \Lambda)$ sending (x, g) to (xg, g) is an isomorphism.

To define $X \times_\Lambda E\Lambda$ we need to specify a category fibered in groupoids. Suppose $U \to S$ is a scheme over S. Define the objects $[X \times_\Lambda E\Lambda](U)$ to be pairs (P, α) where $P \to U$ is a $\Lambda \times_S U$-torsor and

$$\alpha : P \to U \times_S X$$

is a Λ-morphism over U. A morphism $(P, \alpha) \to (Q, \beta)$ is an equivariant isomorphism $P \to Q$ so that the evident diagram over $U \times_S X$ commutes. If $V \to U$ is a morphism of schemes over S, then the map $[X \times_\Lambda E\Lambda](U) \to [X \times_\Lambda E\Lambda](V)$ is defined by pull-back. This gives a stack (see [29], 3.4.2) ; we discuss to what extent it is an algebraic stack.

There is a natural map $X \to X \times_\Lambda E\Lambda$ defined as follows. If $f : U \to X$ is a morphism of schemes over S define $P = U \times_S \Lambda$ and let α be the evident composition over U

$$U \times_S \Lambda \xrightarrow{f \times \Lambda} U \times_S X \times_S \Lambda \longrightarrow U \times_S X$$

given pointwise by $(u, g) \mapsto (u, f(u)g)$.

Note that if $U \to X \times_\Lambda E\Lambda$ classifies $P \to U \times_S X$, then a factoring

$$\begin{tikzcd} & X \arrow[d] \\ U \arrow[ur] \arrow[r] & X \times_\Lambda E\Lambda \end{tikzcd}$$

is equivalent to a choice of section of $P \to U$ and hence an chosen equivariant isomorphism $U \times_S \Lambda \to P$ over U. The notion of an algebraic stack in the *fpqc* topology was defined in Definition 2.26.

3.1 Proposition. *Let Λ be a group scheme over S and suppose the structure morphism $\Lambda \to S$ is flat and quasi-compact. Let X be a scheme over S with a right Λ-action. Then $X \times_\Lambda E\Lambda$ is an algebraic stack in the fpqc topology and*

$$q : X \longrightarrow X \times_\Lambda E\Lambda$$

is an fpqc presesentation. There is a natural commutative diagram

$$\begin{tikzcd} X \times_S \Lambda \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "\cong"'] & X \arrow[d, "="] \\ X \times_{X \times_\Lambda E\Lambda} X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \end{tikzcd}$$

where $d_0(x, g) = x$ and $d_1(x, g) = xg$ and the vertical isomorphism sends (x, g) to the triple $(x, xg, g : xg \to x)$.

3.2 Example. There are two evident examples. First we can take $X = S$ itself with the necessarily trivial right action, and we'll write

$$B\Lambda \stackrel{\text{def}}{=} S \times_\Lambda E\Lambda.$$

This is the moduli stack fo Λ-torsors on S-schemes. The other example sets $X = \Lambda$ with the canonical right action. Let's assume Λ is an affine group scheme over S. Then the projection map

$$\Lambda \times_\Lambda E\Lambda \to S$$

is an equivalence. For if $\alpha : P \to U \times_S \Lambda$ is and morphism of Λ-torsors over U, then α becomes an isomorphism on some faithfully flat over. Since $\Lambda \to S$ is affine, α is then an isomorphism by faifthfully flat descent. It follows that the groupoid $[\Lambda \times_\Lambda E\Lambda](U)$ is contractible.

3.3 Remark. Note that the Čech cover of $X \times_\Lambda E\Lambda$ that arises from the cover $X \to X \times_\Lambda E\Lambda$ is the standard bar complex given by the action of Λ on X. Thus, $X \times_\Lambda E\Lambda$ is that analog of the geometric realization of this bar complex, whence the name homotopy orbits.

3.4 Remark. Suppose that $S = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(A)$ and $\Lambda = \mathrm{Spec}(\Gamma)$. Then the group action $X \times_S \Lambda \to X$ yields a Hopf algebroid structure on the pair $(A, A \otimes_R \Gamma)$. The previous remark implies that there is an equivalence of categories between quasi-coherent sheaves over $X \times_\Lambda E\Lambda$ and the $(A, A \otimes_k \Gamma)$-comodules. In fact, if $\mathcal{M}$ is such a sheaf, the $\mathcal{M}(X \to X \times_\Lambda E\Lambda)$ determines an A-module M and the isomorphism

$$\mathcal{M}(X \times_{X \times_\Lambda E\Lambda} X \to X \times_\Lambda E\Lambda) \cong (A \otimes_R \Lambda) \otimes_A M$$

and the diagram of Proposition 3.1 determine the comodule action map.

3.5 Remark. Let's compare this construction of $X \times_\Lambda E\Lambda$ with a construction in simplicial sets. Suppose Λ is a discrete group (in sets) and X is a discrete right Λ-set. Then the simplicial set $X \times_\Lambda E\Lambda$ is defined to be the nerve of the groupoid with object set X and morphism set $X \times G$. However, this groupoid is equivalent to the groupoid with objects $\alpha : P \to X$ where P is a free and transitive G-set; morphisms are the evident commutative triangles. This is a direct translation of the construction above. Equivalent groupoids have weakly equivalent nerves; hence, if we are only interested in homotopy type, we could define $X \times_\Lambda E\Lambda$ to be the nerve of the larger groupoid.

Next let is say some words about naturality. This is simpler if we make some assumptions on our group schemes. A group scheme Λ over S is *affine over* S if the structure map $q : \Lambda \to S$ is an affine morphism. Since affine morphisms are closed under composition and base change, the multiplication map $\Lambda \times_S \Lambda \to \Lambda$ is a morphism of schemes affine over S. Thus the quasi-coherent $\mathcal{O}_S$-algebra sheaf $q^*\mathcal{O}_\Lambda$ is a sheaf of Hopf algebras. In most of our examples, $S = \mathrm{Spec}(A)$ is itself affine; in this case, $\Lambda = \mathrm{Spec}(H)$ for some Hopf algebra H over A.

If Λ is a group scheme affine over S and $P \to S$ is a Λ-torsor, then $P \to S$ is an affine morphism by faithfully flat descent. If $\phi : \Lambda_1 \to \Lambda_2$ is a morphism of group schemes affine over S and $P \to S$ a Λ_1 torsor, let $P \times_{\Lambda_1} \Lambda_2$ be the sheaf associated to the presheaf

$$A \mapsto (P(A) \times_{S(A)} \Lambda_2(A))/\sim$$

where $\sim$ is the equivalence relation given pointwise by

$$(xb, a) \sim (x, ba)$$

with $x \in P(A)$, $a \in \Lambda_2(A)$, and $b \in \Lambda_1(A)$.

3.6 Lemma. *Let $\Lambda_1 \to \Lambda_2$ be a morphism of groups schemes affine over S and Let $P \to S$ be a Λ_1-torsor. Then $P \times_{\Lambda_1} \Lambda_2$ is actually a Λ_2-torsor over S.*

Proof. If we can choose an isomorphism $P \cong \Lambda_1$ over S, then we get an induced isomorphism $P \times_{\Lambda_1} \Lambda_2 \cong \Lambda_2$. More generally, let $f : T \to S$ be an $fpqc$-cover so that

$$T \times_S P \cong T \times_S \Lambda_1.$$

Then

$$T \times_S (P \times_{\Lambda_1} \Lambda_2) \cong (T \times_S P) \times_{T \times_S \Lambda_1} (T \times_S \Lambda_2) \cong T \times_S \Lambda_2.$$

Since Λ_2 is affine over S, $T \times_S \Lambda_2$ is affine over T and faithfully flat descent implies $P \times_{\Lambda_1} \Lambda_2$ is an affine torsor over S. $\square$

Now suppose X_1 is a right Λ_1-scheme, Λ_2 is a right Λ_2-scheme and $q : X_1 \to X_2$ is a morphism of Λ_1-schemes. Then we get a morphism of stacks

$$X_1 \times_{\Lambda_1} E\Lambda_1 \to X_2 \times_{\Lambda_2} E\Lambda_2$$

sending the pair (P, α) to the pair $(P \times_{\Lambda_1} \Lambda_2, q\alpha)$; that is, there is a commutative diagram of Λ_1-schemes

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d] & X_1 \arrow[d, "q"] \\ P \times_{\Lambda_1} \Lambda_2 \arrow[r] & X_2. \end{tikzcd}$$

Such morphisms have quite nice properties. Recall that a morphism of groupoids $f : G \to H$ is a fibration if for all $x \in H$, all $y \in G$ and all morphisms $\phi : x \to f(y)$ in H, there is a morphism $\psi : x' \to y$ in G with $f\psi = \phi$. Equivalently the morphism of nerves $BG \to BH$ is a Kan fibration of simplicial sets. We will say that a morphism of stacks $\mathcal{M} \to \mathcal{N}$ is a fibration if for all commutative rings R, the map $\mathcal{M}(R) \to \mathcal{N}(R)$ is a fibration of groupoids.[2]

A topological version of the following result can be found in Remark 3.9 below.

[2] This begs for a much more extensive and sophisticated discussion. See [23] and [12].

3.7 Proposition. *Suppose $f : \Lambda_1 \to \Lambda_2$ is a morphism group schemes affine over S, X_1 is a Λ_1-scheme, X_2 is a Λ_2-scheme, and $q : X_1 \to X_2$ is a morphism of Λ_1-schemes. Then*

$$X_1 \times_{\Lambda_1} E\Lambda_1 \longrightarrow X_2 \times_{\Lambda_2} E\Lambda_2$$

is a fibration of algebraic stacks in the $fpqc$ topology.

Proof. Suppose we are given a diagram (over a base-scheme U suppressed from the notation)

$$\begin{tikzcd} & P \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "q"] \\ Q' \arrow[r, "\phi"'] & Q \arrow[r, "\beta"'] & Y \end{tikzcd}$$

with (1) P a Λ_1-torsor and α a Λ_1-morphism; (2) Q' and Q both Λ_2-torsors, β Λ_2-map and ϕ is Λ_2-isomorphism; and (3) $P \to Q$ a morphism of Λ_1-schemes with $P \times_{\Lambda_1} \Lambda_2 \cong Q$. Then we take the pull-back

$$\begin{tikzcd} Q' \times_Q P \arrow[r, "\psi"] \arrow[d] & P \arrow[d] \\ Q' \arrow[r, "\phi"'] & Q. \end{tikzcd}$$

Then Q' is a Λ_1-torsor and ψ is a Λ_1-isomorphism. Finally, we must check that the natural map $(Q' \times_Q P) \times_{\Lambda_1} \Lambda_2 \to Q'$ is an isomorphism of Λ_2-torsors. If we can choose isomorphisms $P \cong \Lambda_1$ and $Q \cong \Lambda_2$ this is clear. The general case follows from faithfully flat descent. □

It is also relatively easy to identify fibers in this setting. We restrict ourselves to a special case.

3.8 Proposition. *Suppose $f : \Lambda_1 \to \Lambda_2$ is flat surjective morphism of group schemes affine over S with kernel K. Suppose that X_1 is a Λ_1-scheme, X_2 is a Λ_2-scheme, and $q : X_1 \to X_2$ is a morphism of Λ_1-schemes. Then there is a homotopy pull-baack diagram*

$$\begin{tikzcd} X_1 \times_K EK \arrow[r] \arrow[d] & X_1 \times_{\Lambda_1} E\Lambda_1 \arrow[d] \\ X_2 \arrow[r] & X_2 \times_{\Lambda_2} E\Lambda_2. \end{tikzcd}$$

Proof. Let $f : U \to X_2$ a morphism of schemes. Then the composition $U \to X_2 \times_{\Lambda_2} E\Lambda_2$ classifies the pair $(U \times_S \Lambda_2, \alpha)$ where α is the composition

$$U \times_S \Lambda_2 \xrightarrow{f \times \Lambda_2} U \times_S X_2 \times_S \Lambda_2 \longrightarrow U \times_S X_2.$$

The homotopy fiber at U is the groupoid with objects the commutative diagrams

$$\begin{tikzcd} P \arrow[r, "\beta"] \arrow[d, "g"] & U \times_S X_1 \arrow[d] \\ U \times_S \Lambda_2 \arrow[r, "\alpha"'] & U \times_S X_2 \end{tikzcd}$$

where (P, β) is an object in $[X_1 \times_{\Lambda_1} E\Lambda_1](U)$ and g is Λ_1 morphism so that the induced map $P \times_{\Lambda_1} \Lambda_2 \to U \times_S \Lambda_2$ is an isomorphism. Let P' be the pull-back of g at inclusions induced by the identity $U \to U \times_{\Lambda_2} \Lambda_2$. Then $P' \to U \times_S X_2$ is an equivariant morphism from a K-torsor to X_2. This defines the functor from the pull-back to $X_1 \times_K EK$.

Conversely, given a K-torsor P over U and a K-morphism $P \to X_1$ we can produce a diagram

$$\begin{tikzcd} P \times_{U \times_S K} (U \times_S \Lambda_1) \arrow[r] \arrow[d, "g"] & U \times_S X_1 \arrow[d] \\ P \times_{U \times_S K} (U \times_S \Lambda_2) \arrow[r] & U \times_S X_2. \end{tikzcd}$$

Since K is the kernel of $\Lambda_1 \to \Lambda_2$, projection gives a natural morphism of Λ_2-torsors over U

$$P \times_{U \times_S K} (U \times_S \Lambda_2) \to U \times_S \Lambda_2$$

of Λ_2 torsors over U. This defines the functor back and gives the equivalence of categories. □

3.9 Remark. In the topological setting of Remark 3.5 we gave two ways to construct $X \times_\Lambda E\Lambda$. With the smaller, and more usual construction, a morphism

$$X_1 \times_{\Lambda_1} E\Lambda_1 \longrightarrow X_2 \times_{\Lambda_2} E\Lambda_2$$

is a fibration only if $\Lambda_1 \to \Lambda_2$ is onto. However, in the larger construction using transtive and free Λ-sets, this morphism is always a fibration, by the same argument as that given for Proposition 3.7. Either model allows us to prove the analog of Proposition 3.8.

As a final generality we have:

3.10 Proposition. *Suppose $f : \Lambda_1 \to \Lambda_2$ is flat surjective morphism of group schemes affine over S and let K be the kernel. Suppose that X_1 is a Λ_1-scheme, X_2 is a Λ_2-scheme, and $q : X_1 \to X_2$ is a morphism of Λ_1-schemes. If $X_1 \to X_2$ is a K-torsor over X_2, then*

$$X_1 \times_{\Lambda_1} E\Lambda_1 \longrightarrow X_2 \times_{\Lambda_2} E\Lambda_2$$

is an equivalence of algebraic stacks.

Proof. The hypothesis of $X_1 \to X_2$ means that when the action is restricted to K, then X_1 is (after pulling back to an *fpqc*-cover of X_1) isomorphic to $X_2 \times_S K$. The result follows immediately from Propositions 3.7 and 3.8, but can also be proved directly. For if $\alpha : P \to U \times_S X_2$ is some Λ_2-equivariant morphism from a Λ_2-torsor over U, then we can form the pull back square

$$\begin{tikzcd} Q \arrow[r, "\beta"] \arrow[d] & U \times_S X_1 \arrow[d] \\ P \arrow[r, "\alpha"'] & U \times_S X_2 \end{tikzcd}$$

and $\beta : Q \to U \times_S X_1$ is a Λ_1-equivariant morphism from a Λ_1-torsor over U. This defines the necessary equivalence of categories. $\square$

3.2 Formal groups

We now specialize to the case where $S = \mathrm{Spec}(\mathbb{Z})$, $\Lambda = \mathrm{Spec}(\mathbb{Z}[a_0^{\pm 1}, a_1, \ldots])$ is the group scheme of power series invertible under composition. We set $X = \mathbf{fgl} = \mathrm{Spec}(L)$ where L is the Lazard ring. Thus for a commutative ring R

$$\Lambda(R) = xR[[x]]^{\times}$$

and $X(R) = \mathbf{fgl}(R)$ is the set of formal group laws over R. The group schemes Λ acts on $\mathbf{fgl}$ by the formula

$$(F\phi)(x_1, x_2) = \phi^{-1}(F(\phi(x_1), \phi(x_2)).$$

In Theorem 2.24 we produced, for any formal group G over an affine scheme U, an affine morphism of schemes

$$\mathrm{Coord}_G \longrightarrow S.$$

The following is essentially a combination of Lemma 2.15 and Theorem 2.24.2.

3.11 Lemma. *For a formal group G over a quasi-compact and quasi-separated scheme U, the scheme of coordinates* $\mathrm{Coord}_G \to U$ *is a Λ-torsor over U.*

Proof. The formal group G over U may not have a coordinate. However, Lemma 2.15 implies that there is an *fqpc*-cover $f : V \to U$ so that f^*G has a coordinate. Reading the proof of Lemma 2.15 we see that V can be chosen to be affine. Then

$$V \times_U \mathrm{Coord}_G = \mathrm{Coord}_{f^*G}$$

is certainly a free right Λ-scheme over V. See Remark 2.25 for explicit formulas. $\square$

The following result implies that every Λ-torsor over $\mathbf{fgl}$ arises in this way from a formal group.

3.12 Lemma. *Let S be a quasi-compact and quasi-separated scheme. Let $P \to S$ be a Λ-torsor and let $P \to S \times \mathbf{fgl}$ be a morphism Λ-schemes over S. Then there is a formal group $G \to S$ and an isomorphism $P \to \mathrm{Coord}_G$ of Λ-torsors over S. This isomorphism is stable under pull-backs in S and natural in P. Furthermore, if $P = \mathrm{Coord}_H$, then there is a natural isomorphism $G \cong H$.*

Proof. We begin with an observation. Let $f : U \to S$ be any morphism of schemes so that fiber $P(U, f)$ of $P(U) \to S(U))$ at f is a free $\Lambda(U)$-set. Then we have a commutative diagram

$$\begin{tikzcd} P(U,f) \arrow[r] \arrow[d] & \mathbf{fgl}(U) \arrow[d] \\ P(U,f)/\Lambda(U) = * \arrow[r] & \mathbf{fg}(U) \end{tikzcd}$$

and the image of the bottom map is a formal group G_f over U. Since the fiber of $\mathbf{fgl}(U) \to \mathbf{fg}(U)$ at G_f is $\mathrm{Coord}_{G_f}(U)$ we have that G_f has a coordinate and we have an isomorphism of free $\Lambda(U)$-sets

$$P(U, f) \cong \mathrm{Coord}_{G_f}(U). \tag{3.1}$$

To get a formal group over S we use descent. Choose a faithfully flat and quasi-compact map $q : T \to S$ so that fiber $P(T, q)$ is a free $\Lambda(T)$-set. This yields a formal group G_q over T as above. Next examine the commutative diagram

$$\begin{tikzcd} P(T) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P(T \times_S T) \arrow[d] \\ S(T) \arrow[r, shift left] \arrow[r, shift right] & S(T \times_S T) \end{tikzcd}$$

where the horizontal maps are given by the two projections. Since the two maps

$$\begin{tikzcd} T \times_S T \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & T \arrow[r, "q"] & S \end{tikzcd}$$

are equal the projection maps yield morphisms between fibers

$$\begin{tikzcd} P(T,q) \arrow[r, shift left, "p_1^*"] \arrow[r, shift right, "p_2^*"'] & P(T \times_S T, qp_1) \end{tikzcd}$$

and hence a unique isomorphism $p_1^* G_q \cong p_2^* G_q$. This isomorphism will satisfy the cocycle condition, using uniqueness. Now descent gives the formal group $G \to S$. Note that if $P = \mathrm{Coord}_H$, then $G_q = q^* H$; therefore, $G \cong H$.

We now define the isomorphism of torsors $P \to \mathrm{Coord}_G$ over S. Since both P and Coord_G are sheaves in the *fpqc* topology, it is sufficient to define a natural isomorphism $P(U, f) \to \mathrm{Coord}_G(U, f)$ for all $f : U \to S$ so that both $P(U, f)$

and $\mathrm{Coord}_G(U, f)$ are free $\Lambda(U)$-sets. This isomorphism is defined by Equation 3.1 using the observation that

$$\mathrm{Coord}_{f^*G}(U) = \mathrm{Coord}_G(U, f).$$

$\square$

3.13 Proposition. *This morphism*

$$\mathcal{M}_{\mathbf{fg}} \longrightarrow \mathbf{fgl} \times_\Lambda E\Lambda$$

is an equivalence of algebraic stacks.

Proof. Lemma 3.12 at once supplies the map $\mathbf{fgl} \times_\Lambda E\Lambda \to \mathcal{M}_{\mathbf{fg}}$ and the needed natural transformations from either of the two composites to the identity. $\square$

Note that Λ is an affine group scheme:

$$\Lambda = \mathrm{Spec}(\mathbb{Z}[a_0^{\pm 1}, a_1, a_2, \ldots]).$$

Let L be the Lazard ring and $W = L \otimes \mathbb{Z}[a_0^{\pm 1}, a_1, a_2, \ldots]$. Then the group action

$$\mathbf{fgl} \times \Lambda \longrightarrow \mathbf{fgl}$$

yields a Hopf algebroid (L, W). Remark 3.4 now immediately implies the following result.

3.14 Proposition. *There is an equivalence of categories between quasi-coherent sheaves on $\mathcal{M}_{\mathbf{fg}}$ and the category of (L, W)-comodules.*

3.3 Buds of formal groups

One of the difficulties with the moduli stack $\mathcal{M}_{\mathbf{fg}}$ of formal groups is that it does not have good finiteness properties. We have written $\mathcal{M}_{\mathbf{fg}}$ as $\mathbf{fgl} \times_\Lambda E\Lambda$ and neither the group Λ or the scheme $\mathbf{fgl}$ is of finite type over $\mathbb{Z}$. However, we can write $\mathcal{M}_{fg}$ as the homotopy inverse limit of stacks $\mathcal{M}_{\mathbf{fg}}\langle n\rangle$ which has an affine smooth cover of dimension n.

Let $n \geq 1$ and Λ_n be the affine group scheme over $\mathrm{Spec}(\mathbb{Z})$ which assigns to each commutative ring R, the partial power series of degree n

$$f(x) = a_0 x + a_1 x^2 + \cdots + a_{n-1} x^n \in R[[x]]/(x^{n+1})$$

with a_0 a unit. This becomes a group under composition of power series. Of course,

$$\Lambda_n = \mathrm{Spec}(\mathbb{Z}[a_0^{\pm 1}, a_1, \ldots, a_{n-1}]).$$

Similarly, let $\mathbf{fgl}_n$ be the affine scheme of *n-buds of formal group laws*

$$F(x, y) \in R[[x, y]]/(x, y)^{n+1}.$$

Thus we are requiring that $F(x,0) = x = F(0,x)$, $F(x,y) = F(y,x)$, and

$$F(x, F(y,z)) = F(F(x,y),z)$$

all modulo $(x,y)^{n+1}$. The symmetric 2-cocyle lemma [44] A.2.12 now implies that

$$\mathbf{fgl}_n = \mathrm{Spec}(\mathbb{Z}[x_1, x_2, \cdots, x_{n-1}]) \overset{\mathrm{def}}{=} \mathrm{Spec}(L_n)$$

and the modulo $(x_1, \ldots, x_n)^2$, the universal n-bud reads

$$F_u(x,y) = x + y + x_1 C_2(x,y) + \cdots x_{n-1} C_n(x,y)$$

where $C_k(x,y)$ is the kth symmetric 2-cocyle. The group Λ_n acts of $\mathbf{fgl}_n$.

3.15 Definition. *The* **moduli stack of n-buds of formal groups** *is the homotopy orbit stack*

$$\mathcal{M}_{\mathbf{fg}}\langle n\rangle = \mathbf{fgl}_n \times_{\Lambda_n} E\Lambda_n.$$

There are canonical maps

$$\mathcal{M}_{\mathbf{fg}} \longrightarrow \mathcal{M}_{\mathbf{fg}}\langle n\rangle \longrightarrow \mathcal{M}_{\mathbf{fg}}\langle n-1\rangle.$$

3.16 Example. To make your confusion specific[3], note that

$$\mathcal{M}_{\mathbf{fg}}\langle 1\rangle = B\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}) \times_{\mathbb{G}_m} E\mathbb{G}_m.$$

This is because $\Lambda_1(R) = R^\times = \mathbb{G}_m(R)$ is the group of units in R and, modulo $(x,y)^2$, the unique bud of a formal group law is $x+y$.

$$\mathcal{M}_{\mathbf{fg}}\langle 2\rangle = \mathbb{A}^1 \times_{\Lambda_2} E\Lambda_2$$

where Λ_2 acts on $\mathbb{A}^1$ by

$$(b, a_0 x + a_1 x^2) \mapsto a_0 b - 2(a_1/a_0).$$

Note that, modulo $(x,y)^3$, any bud of a l formal group law is of the form $x + y + bxy$.

The following implies that $\mathcal{M}_{\mathbf{fg}}\langle n\rangle$ is an algebraic stack in the sense of [29] Définition 4.1. See also [29], Exemple 4.6.

3.17 Proposition. *The morphism*

$$\mathrm{Spec}(L_n) \to \mathcal{M}_{\mathbf{fg}}\langle n\rangle$$

classifying the universal n-bud of a formal group law is a presentation and smooth of relative dimension n.

[3]This is quote from Steve Wilson. See [50]§?

Proof. That the morphism is a presentation follows from Proposition 3.1. To see that it is smooth of relative dimension n, we must check that for all morphisms $\mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}\langle n\rangle$ the resulting pull-back

$$\mathrm{Spec}(R) \times_{\mathcal{M}_{\mathbf{fg}}\langle n\rangle} \mathrm{Spec}(L_n) \to \mathrm{Spec}(R)$$

is smooth of relative dimension n. Since smoothness is local for the $fpqc$ topology, we may assume that $\mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}\langle n\rangle$ classifies a bud of formal group law. Then

$$\mathrm{Spec}(R) \times_{\mathcal{M}_{\mathbf{fg}}\langle n\rangle} \mathrm{Spec}(L_n) \cong \mathrm{Spec}(R[a_0^{\pm 1}, a_1, \cdots, a_{n-1}]) = \mathrm{Spec}(R) \times \Lambda_n$$

and this suffices. □

Recall that that nth symmetric 2-cocycle is

$$C_n(x,y) = \frac{1}{d_n}[(x+y)^n - x^n - y^n].$$

where

$$d_n = \begin{cases} p, & n = p^k \text{ for a prime } p; \\ \\ 1, & \text{otherwise.} \end{cases}$$

Let $\mathbb{G}_a$ be the additive group scheme and let $\mathbb{A}^1\langle n\rangle$ be the $\mathbb{G}_a$ scheme with action $\mathbb{A}^1\langle n\rangle \times \mathbb{G}_a \to \mathbb{A}^1\langle n\rangle$ given by

$$(x,a) \mapsto x - d_n a.$$

3.18 Lemma. *The morphism $\Lambda_n \to \Lambda_{n-1}$ of affine group schemes is flat and surjective with kernel $\mathbb{G}_a$. Furthermore there is an equivariant isomorphism of $\mathbb{G}_a$ schemes over $\mathbf{fgl}_{n-1}$*

$$\mathbf{fgl}_n \cong \mathbf{fgl}_{n-1} \times \mathbb{A}^1\langle n\rangle.$$

Proof. The kernel of $\Lambda_n(R) \to \Lambda_{n-1}(R)$ is all power series of the form $\phi_a(x) = x + ax^n$ modulo (x^{n+1}). Since $\phi_a(\phi_{a'}(x)) = \phi_{(a+a')}(x)$ modulo (x^{n+1}), the first statement follows. For the splitting of $\mathbf{fgl}_n$ note that if $\phi_a(x)$ is an isomorphism of buds of formal group laws $F \to F'$, then

$$\begin{aligned} F'(x,y) &= F(x,y) + a[x^n - y^n - (x+y)^n] \\ &= F(x,y) - d_n a C(x,y). \end{aligned}$$

Thus the coaction morphism on coordinate rings

$$\mathbb{Z}[x_1, \ldots, x_n] \longrightarrow \mathbb{Z}[x_1, \ldots, x_n] \otimes \mathbb{Z}[a]$$

sends x_i to x_i is $i \neq n$ and x_n to

$$x_n \otimes 1 - 1 \otimes d_n a.$$

This gives the splitting. □

3.19 Proposition. *For all $n \geq 1$ the reduction map*

$$\mathcal{M}_{\mathbf{fg}}\langle n\rangle \longrightarrow \mathcal{M}_{\mathbf{fg}}\langle n-1\rangle$$

is a fibration. If R is any commutative ring in which d_n is a unit, then

$$\mathcal{M}_{\mathbf{fg}}\langle n\rangle \otimes R \longrightarrow \mathcal{M}_{\mathbf{fg}}\langle n-1\rangle \otimes R$$

is an equivalence of algebraic stacks.

Proof. This follows immediately from Example 3.2, Propositions 3.7 and 3.10, Lemma 3.18, and the folliowing fact: if d_n is a unit in A, then $\mathbb{A}^1\langle n\rangle$ is isomorphic to $\mathbb{G}_a$ as a right $\mathbb{G}_a$-scheme. □

3.20 Theorem. *The natural map*

$$\mathcal{M}_{\mathbf{fg}} \longrightarrow \operatorname{holim} \mathcal{M}_{\mathbf{fg}}\langle n\rangle$$

is an equivalence of stacks.

Proof. We must prove that for all rings R the natural morphism of groupoids

$$\mathcal{M}_{\mathbf{fg}}(R) \longrightarrow \operatorname{holim} \mathcal{M}_{\mathbf{fg}}\langle n\rangle(R)$$

is an equivalence. By Proposition 3.19 we have that the projection map

$$\mathcal{M}_{\mathbf{fg}}\langle n\rangle(R) \longrightarrow \mathcal{M}_{\mathbf{fg}}\langle n-1\rangle(R)$$

is a fibration of groupoids for all n. Thus we need only show $\mathcal{M}_{\mathbf{fg}}(R) \cong \lim \mathcal{M}_{\mathbf{fg}}\langle n\rangle(R)$, but this is obvious. □

The next result is an incredibly complicated way to prove that every formal group over an algebra over the rationals is isomorphic to the additive formal group. It proves more, however, as it also identities the automorphisms of the additive formal group. For the proof combine Theorem 3.20 and Proposition 3.19.

3.21 Corollary. *The projection map*

$$\mathcal{M}_{\mathbf{fg}} \otimes \mathbb{Q} \longrightarrow \mathcal{M}_{\mathbf{fg}}\langle 1\rangle \otimes \mathbb{Q} \simeq B(\mathbb{G}_m \otimes \mathbb{Q})$$

is an equivalence.

When working at a prime p, the moduli stacks $\mathcal{M}_{\mathbf{fg}}\langle p^n\rangle \otimes \mathbb{Z}_{(p)}$ form the significant layers in the tower. These should have covers by "p-typical buds"; the next result makes that thought precise. Recall that the universal p-typical formal group law F is defined over the the ring $V \cong \mathbb{Z}_{(p)}[u_1, u_2, \cdots]$. See Corollary 2.37.

3.22 Lemma. *Let $V_n = \mathbb{Z}_{(p)}[u_1, \ldots, u_n]$ be the subring of V generated by u_k, $k \leq n$. The p^n-bud F_{p^n} of the universal p-typical formal group law F is defined over V_n and the morphism*

$$F_{p^n} : \mathrm{Spec}(V_n) \to \mathcal{M}_{\mathbf{fg}}\langle p^n \rangle \otimes \mathbb{Z}_{(p)}$$

classifying this bud is a presentation. Furthermore there is an isomorphism

$$\mathrm{Spec}(V_n) \times_{\mathcal{M}_{\mathbf{fg}}\langle n \rangle} \mathrm{Spec}(V_n) \cong \mathrm{Spec}(V_n[t_0^{\pm 1}, t_1, \cdots, t_n]).$$

Proof. We use the gradings of Remark 2.38. The n-bud of a formal group law G is given by the equation

$$G_n(x, y) = \sum_{i+j \leq n} c_{ij} x^i y^j.$$

If F is the universal p-typical formal group law, we see that F_n is defined over the subring of V generated by the u_k with $p^k \leq n$. Similarly if $\phi(x)$ is the universal isomorphism of p-typical formal group laws, then its bud

$$\phi_n(x) = \sum_{i=0}^{n-1} a_i x^{i+1}$$

is defined over the subring of $V[t_0^{\pm 1}, t_1, \cdots]$ generated by t_k and u_k with $p^k \leq n$.

To show that we have a presentation, suppose G is a p^n bud of a formal group over a field k which is a $\mathbb{Z}_{(p)}$-algebra. Since k is a field, we may assume G arises from the bud of formal group law, which we also call G. Choose any formal group law G' whose p^n-bud is G and choose an isomorphism $G' \to G''$ where G'' is p-typical. Then the p^n-bud of G'' is isomorphic to G and, by the previous paragraph, arises from a morphism $g : V_n \to k$. Thus we obtain the requisite 2-commuting diagram

$$\begin{tikzcd}
 & \mathrm{Spec}(V_n) \arrow[d, "F_{p^n}"] \\
\mathrm{Spec}(k) \arrow[ur, "\mathrm{Spec}(g)"] \arrow[r, "G"'] & \mathcal{M}_{\mathbf{fg}}\langle p^n \rangle \otimes \mathbb{Z}_{(p)}.
\end{tikzcd}$$

A similar argument computes the homotopy pull-back. $\square$

3.23 Remark. It is possible to give an intrinsic geometric definition of an n-bud of a formal group in the style of Definition 1.24 and Definition 2.2. First an *n-germ* of a formal Lie variety X over a scheme S is an affine morphism of schemes $X \to S$ with a closed section e so that

1. $X = \mathrm{Inf}^n_S(X)$;
2. the quasi-coherent sheaf ω_e is locally free of finite rank on S;

3. the natural map of graded rings $\mathrm{Sym}_*(\omega_e) \to \mathrm{gr}_*(X)$ induces an isomorphism
$$\mathrm{Sym}_*(\omega_e)/\mathcal{J}^{n+1} \to \mathrm{gr}_*(X)$$
where $\mathcal{J} = \oplus_{k>0}\mathrm{Sym}_k(\omega_e)$ is the augmentation ideal.

An n-bud of a formal groupis then an n-germ $G \to S$ so that $\omega_e = \omega_G$ is locally free of rank 1 and there is a "multiplication" map

$$\mathrm{Inf}^n_S(G \times_S G) \to G$$

over S so that the obvious diagrams commute.

3.4 Quasi-coherent sheaves over $\mathcal{M}_{\mathbf{fg}}$

Better give some theory

We now consider the categories of quasi-coherent sheaves over the stacks $\mathcal{M}_{\mathbf{fg}}$ and $\mathcal{M}_{\mathbf{fg}}\langle n\rangle$. Let m and n be integers $0 \leq n < m \leq \infty$ and let

$$q = (m,n) = q : \mathcal{M}_{\mathbf{fg}}\langle m\rangle \to \mathcal{M}_{\mathbf{fg}}\langle n\rangle$$

be the projection. To avoid cluttered notation we will write q for $q_{(m,n)}$ whenever possible. Here we are writing $\mathcal{M}_{\mathbf{fg}}\langle\infty\rangle$ for $\mathcal{M}_{\mathbf{fg}}$ itself.

Write $\mathbf{Qmod}_{\mathbf{fg}}\langle n\rangle$ for the quasi-coherent sheaves on $\mathcal{M}_{\mathbf{fg}}\langle n\rangle$. We begin by discussing the pull-back and push-forward functors

$$q^* : \mathbf{Qmod}_{\mathbf{fg}}\langle n\rangle \rightleftarrows \mathbf{Qmod}_{\mathbf{fg}}\langle m\rangle : q_*.$$

By Remark 3.4, the category of quasi-coherent sheaves on $\mathcal{M}_{\mathbf{fg}}\langle n\rangle$ is equivalent to the category of (L_n, W_n)-comodules. In fact, if $\mathcal{F}$ is a quasi-coherent sheaf, the associated comodule M is obtained by evaluating $\mathcal{F}$ at $\mathrm{Spec}(L_n) \to \mathcal{M}_{\mathbf{fg}}\langle n\rangle$, and the comodule structure is obtained by evaluating $\mathcal{F}$ on the parallel arrows

$$\mathrm{Spec}(W_n) \cong \mathrm{Spec}(L_n) \times_{\mathcal{M}_{\mathbf{fg}}\langle n\rangle} \mathrm{Spec}(L_n) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathrm{Spec}(L_n) \longrightarrow \mathcal{M}_{\mathbf{fg}}\langle n\rangle.$$

We will describe the functors q_* and q^* by giving a description on comodules.

Let $\Gamma(n,m)$ be the group scheme which assigns to each commutative ring A the invertible (under composition) power series modulo (x^{m+1})

$$x + a_n x^{n+1} + a_{n+1}x^{n+2} + \cdots + a_{n-1}x^m \quad a_i \in R.$$

Then $\Gamma(n,m) = \mathrm{Spec}(\mathbb{Z}[a_n, a_{n+1}, \ldots, a_{m-1}])$ and $\Gamma(n,m)$ is the kernel of the projection map $\Lambda_m \to \Lambda_n$.

By Proposition 3.8 there is an equivalence of algebraic stacks

$$\mathrm{Spec}(L_n) \times_{\mathcal{M}_{\mathbf{fg}}\langle m\rangle} \mathcal{M}_{\mathbf{fg}}\langle m\rangle \simeq \mathbf{fgl}_m \times_{\Gamma(n,m)} E\Gamma(n,m).$$

Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}\langle m\rangle$. Then the value of $q_*\mathcal{F}$ when evaluated at $\mathrm{Spec}(L_n) \to \mathcal{M}_{\mathbf{fg}}\langle n\rangle$ is $H^0(\mathrm{Spec}(L_n) \times_{\mathcal{M}_{\mathbf{fg}}\langle n\rangle} \mathcal{M}_{\mathbf{fg}}\langle m\rangle, \mathcal{F})$. If $M = \mathcal{F}(\mathrm{Spec}(L_m))$ is the (L_m, W_m)-comodule equivalent to $\mathcal{F}$, then these global sections are the (L_n, W_n)-comodule N defined by the equalizer diagram

$$N \longrightarrow M \rightrightarrows \mathbb{Z}[a_n, a_{n+1}, \ldots, a_{m-1}] \otimes_{L_m} M$$

where the parallel arrows are given by left inclusion and the coaction map. The assignment $M \mapsto N$ determines $q_*\mathcal{F}$.

To describe q^*, we give the left adjoint to the functor just described on comodules. If N is a (L_n, W_n)-comodule, define a (L_m, W_m) comodule $M = L_m \otimes_{L_n} N$ with coaction map

$$L_m \otimes_{L_n} N \to W_m \otimes_{L_m} \otimes L_m \otimes_{L_n} N \cong W_m \otimes_{W_n} W_n \otimes_{L_n} N$$

given by

$$\eta_R \otimes \psi : L_m \otimes_{L_n} N \longrightarrow W_m \otimes_{W_n} W_n \otimes_{L_n} N.$$

3.24 Proposition. *For all m and n, $0 \le n \le m \le \infty$, the projection morphism*

$$q : \mathcal{M}_{\mathbf{fg}}\langle m\rangle \longrightarrow \mathcal{M}_{\mathbf{fg}}\langle n\rangle$$

is faithfully flat.

Proof. The morphism q is flat if and only if the functor $\mathcal{F} \mapsto q^*\mathcal{F}$ is exact. However, since the ring homomorphism $L_n \to L_m$ is flat, the equivalent functor $N \mapsto L_m \otimes_{L_n} N$ on comodules is evidently exact. The morphism q is now faithfully flat because it is surjective. $\square$

Now consider the diagram of functors

$$\begin{tikzcd}
\mathbf{Qmod}_{\mathbf{fg}}\langle n\rangle \arrow[r, "q^*_{(n,m)}"] \arrow[dr, "q^*_n"'] & \mathbf{Qmod}_{\mathbf{fg}}\langle m\rangle \arrow[d, "q^*_m"] \\
 & \mathbf{Qmod}_{\mathbf{fg}}.
\end{tikzcd}$$

Since there is a natural isomorphism $q^*_n \cong q^*_m q^*_{(n,m)}$ we get (after identifying a category with its nerve) a functor

$$q^* : \mathrm{hocolim}\, \mathbf{Qmod}_{\mathbf{fg}}\langle n\rangle \to \mathbf{Qmod}_{\mathbf{fg}}.$$

Since the morphism $q_{(n.m)} : \mathcal{M}_{\mathbf{fg}}\langle m\rangle \to \mathcal{M}_{\mathbf{fg}}\langle n\rangle$ is faithfully flat, the morphisms

$$q^*_{(n,m)} : \mathbf{Qmod}_{\mathbf{fg}}\langle n\rangle \to \mathbf{Qmod}_{\mathbf{fg}}\langle m\rangle$$

are cofibrations; hence, we get an induced map

$$q^* : \mathrm{colim}\, \mathbf{Qmod}_{\mathbf{fg}}\langle n\rangle \to \mathbf{Qmod}_{\mathbf{fg}}.$$

For any module category $\mathbf{Qmod}_X$ let $\mathbf{Qmod}_X^{fp}$ denote the full-subcategory of *coherent* (or finitely presented) modules.

3.25 Theorem. *The induced functor*

$$q^* : \operatorname{colim} \mathbf{Qmod_{fg}}\langle n\rangle \longrightarrow \mathbf{Qmod_{fg}}$$

is faithful and induces an equivalence on the categories of coherent sheaves

$$q^* : \operatorname{colim} \mathbf{Qmod}_{\mathbf{fg}}^{fp}\langle n\rangle \longrightarrow \mathbf{Qmod}_{\mathbf{fg}}^{fp}$$

Proof. The first statement – that the functor q^* is faithful on the larger category – follows from Proposition 3.24. That result implies that each of the functors

$$q^*_{(n,m)} : \mathbf{Qmod_{fg}}\langle n\rangle \to \mathbf{Qmod_{fg}}\langle m\rangle$$

is faithful. The equivalence for coherent modules follows from the description of q^* on comodules and the facts that $L = \operatorname{colim} L_n$ and $W = \operatorname{colim} W_n$. □

4 Invariant derivations and differentials

4.1 The Lie algebra of a group scheme

We begin with a basic recapitulation of the notion of the Lie algebra of a group scheme G over a scheme S. We will assume that the structure morphism $G \to S$ is locally of finite type. The tangent scheme and the connection between the tangent scheme and differentials was discussed in §1.3.

4.1 Definition. *Let $G \to S$ be a group scheme, locally of finite type over S. Let Lie_G to be the scheme over S obtained by the pull-back diagram*

$$\begin{tikzcd}
\mathrm{Lie}_G \arrow[r] \arrow[d] & \mathcal{T}\mathrm{an}_{G/S} \arrow[d] \\
S \arrow[r, "e"'] & G.
\end{tikzcd}$$

It follows immediately from Proposition 1.18 that

$$\mathrm{Lie}_G \cong \mathbb{V}(\omega_G)$$

where ω_G is the conormal sheaf of the unit $S \to G$.

4.2 Remark. The scheme $\mathrm{Lie}_G \to S$ has a great deal of structure; we'll emphasize those points which apply most directly here.

1.) Since $\mathcal{T}\mathrm{an}_{G/S}$ is an abelian group scheme over G, Lie_G is an abelian group scheme over S. More than that, it is an $\mathbb{A}^1_S$-module; that is, there is a multiplication morphism of schemes

$$\mathbb{A}^1_S \times_S \mathrm{Lie}_G \longrightarrow \mathrm{Lie}_G$$

making Lie_G into a module over the ring scheme $\mathbb{A}^1_S$. This is a coordinate free way of saying that the abelian group $\mathrm{Lie}_G(A)$ is naturally an A-module. To get this A-module structure, let $a \in A$ and define $u_a : A(\epsilon) \to A(\epsilon)$ to be the A-algebra map determined by $u_a(\epsilon) = a\epsilon$. Then $\mathrm{Lie}_G(u_a)$ determines the multiplication by a in $\mathrm{Lie}_G(A)$.

2.) The zero section $s : G \to \mathcal{T}\mathrm{an}_{G/S}$ defines an action of G on Lie_G by conjugation; if $x \in G(R)$, this action is written

$$\mathrm{Ad}(x) : \mathrm{Lie}_G \longrightarrow \mathrm{Lie}_G.$$

The naturality of the semi-direct product construction shows that there is a natural isomorphism of group schemes over G

$$\mathcal{T}\mathrm{an}_{G/S} \cong G \rtimes_S \mathrm{Lie}_G.$$

In particular, if G is commutative we have an isomorphism

$$\mathcal{T}\mathrm{an}_{G/S} \cong G \times_S \mathrm{Lie}_G \tag{4.1}$$

which is natural with respect to homomorphisms of abelian group schemes.

3.) There is a Lie bracket

$$[\ ,\ ] : \mathrm{Lie}_G \times_S \mathrm{Lie}_G \longrightarrow \mathrm{Lie}_G.$$

Thus, Lie_G is an $\mathbb{A}^1_S$-Lie algebra. If G is commutative – as is our focus here – this bracket is zero, so we won't belabor it.

4.) The structure morphism $\mathrm{Lie}_G \to S$ is affine. Indeed, if $e : S \to G$ is the unit morphism of G, then Proposition 1.18 implies that there is an isomorphism of abelian group schemes

$$\mathrm{Lie}_G \cong \mathrm{Spec}_S(\mathrm{Sym}_{\mathcal{O}_S}(e^*\Omega_{G/S})) = \mathbb{V}(e^*\Omega_{X/S}).$$

If $G = \mathrm{Spec}(A)$ is affine over $S = \mathrm{Spec}(R)$, then A is an R-Hopf algebra and $e^*\Omega_{G/S}$ is the quasi-coherent sheaf determined by the R-module of indecomposables QA.

The natural map $d : \omega_G \to e^*\Omega_{G/S}$ of Lemma 1.19 is an isomorphism; we will also identify this sheaf as the invariant differentials for G. See §4.2 below.

4.3 Remark (Invariant derivations). In Corollary 1.17 we wrote down a natural isomorphism between the module $\mathrm{Der}_S(G, \mathcal{O}_G)$ of derivations of G over S with coefficients in $\mathcal{O}_G$ and the module of sections of $q : \mathcal{T}\mathrm{an}_{G/S} \to G$. If s' is a section of $\mathrm{Lie}_G \to S$, then we get a section

$$s = s' \times G : G = S \times_S G \longrightarrow \mathrm{Lie}_G \rtimes G \cong \mathcal{T}\mathrm{an}_{G/S}$$

of $\mathcal{T}\mathrm{an}_{G/S} \to G$ and the assignment $s' \mapsto s$ induces an isomorphism from the module of sections of Lie_G to the module of left invariant sections of $\mathcal{T}\mathrm{an}_{G/S}$. The inverse assigns to s the composition

$$S \xrightarrow{e} G \xrightarrow{s_0} \mathrm{Lie}_G(S).$$

There is a sheaf version of this which defines an isomorphism from the local sections of $\mathrm{Lie}_G \to G$ to an appropriate sheaf of invariant derivations in $\mathcal{D}er_S(G, \mathcal{O}_G)$.

Now let $G \to S$ be a formal group over S; we define Lie_G exactly as above:

$$\mathrm{Lie}_G = e^*\mathcal{T}\mathrm{an}_{G/S} \to S.$$

Let $\varepsilon : \mathrm{Lie}_G \to \mathcal{T}\mathrm{an}_{G/S}$ be the induced map. In Remark 1.27 we showed that $(\mathcal{T}\mathrm{an}_{G/S}, \varepsilon)$ is a formal Lie variety over Lie_G and that there is a natural isomorphism of abelian group schemes

$$\mathbb{V}(\omega_G) \cong \mathrm{Lie}_G$$

over S. Exactly as in Equation 4.1 we have an isomorphism (now as *fpqc* sheaves)

$$\mathcal{T}\mathrm{an}_{G/S} \cong G \times_S \mathrm{Lie}_G$$

over S.

4.4 Remark. Let $f : G \to H$ be homomorphism of smooth, commutative formal groups over S. In the presence of coordinates, it is possible to give a concrete formula for computing $\mathrm{Lie}(f)$ and $\mathcal{T}\mathrm{an}(f)$.

First suppose that we choose can choose a coordinate y for G. Then y determines an isomorphism

$$\lambda_y : \mathbb{G}_a \longrightarrow \mathrm{Lie}_G$$

from the additive group over S to Lie_G sending $a \in \mathbb{G}_a(R)$ to $\epsilon a \in \mathrm{Lie}_G$.

Next suppose that we also choose a coordinate x for H. Then the image of y under f is a power series $f(x)$ and we get a commutative diagram

$$\begin{tikzcd}
G \times_S \mathbb{G}_a \arrow[r] \arrow[d, "G\times\lambda_y"'] & H \times_S \mathbb{G}_a \arrow[d, "H\times\lambda_x"] \\
\mathcal{T}\mathrm{an}_{G/S} \arrow[r, "\mathcal{T}\mathrm{an}(f)"'] & \mathcal{T}\mathrm{an}_{H/S}
\end{tikzcd}$$

where the top morphism is given pointwise by

$$(a, b) \mapsto (f(a), bf'(a)).$$

Restricting to the Lie schemes, we get a commutative diagram of schemes over S

$$\begin{tikzcd}
\mathbb{G}_a \arrow[r, "f'(0)"] \arrow[d, "\lambda_y"'] & \mathbb{G}_a \arrow[d, "\lambda_x"] \\
\mathrm{Lie}_G \arrow[r, "\mathrm{Lie}(f)"'] & \mathrm{Lie}_{H/S}.
\end{tikzcd}$$

Note that we have also effectively proved the following result.

4.5 Proposition. *Let G be a smooth one-dimensional, commutative formal groups over S. Then Lie_G is a naturally a $\mathbb{G}_a$-torsor in the fpqc topology.*

Proof. The scheme $\mathrm{Lie}_G \to S$ is a $\mathbb{G}_a$-scheme because it is an $\mathbb{A}^1_S$-module. If we choose an fpqc cover $f : T \to S$ so that f^*G can be given a coordinate, then we have just shown, in Remark 4.4, that a choice of coordinate defines an isomorphism

$$f^*\mathbb{G}_a \longrightarrow \mathrm{Lie}_T(f^*G) \cong f^*\mathrm{Lie}_G.$$

∎

4.2 Invariant differentials

Let $q : G \to S$ be a group scheme over S with identity $e : S \to G$. Let us assume that G is flat – and hence faithfully flat – and quasi-compact over S. Then we have a diagram

$$\begin{tikzcd} G \times_S G \arrow[r, shift left, "p_1"] \arrow[r, shift right, "m"'] \arrow[d, "f"'] & G \arrow[d, "="] \\ G \times_S G \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & G \end{tikzcd}$$

where f is an isomorphism give pointwise by $f(x, y) = (x, xy)$ and m is the multiplication map. From this we conclude that we have a modified version of descent for $q : G \to S$: the category of quasi-coherent sheaves on S is equivalent to the category of quasi-coherent sheaves $\mathcal{F}$ on G equipped with an isomorphism

$$(p_1)^*\mathcal{F} \to m^*\mathcal{F}$$

satisfying a suitable cocycle condition we leave the reader to formulate.

To apply this, we note that we have diagram

$$\begin{tikzcd} G \times_S G \arrow[r, shift left, "p_1"] \arrow[r, shift right, "m"'] \arrow[d, "p_2"'] & G \arrow[d, "q"] \\ G \arrow[r, "q"'] & S \end{tikzcd}$$

and both the squares are Cartesian. This supplies an isomorphism

$$(p_1)^*\Omega_{G/S} \cong \Omega_{G\times_S G/G} \cong m^*\Omega_{G/S}$$

which satisfies the necessary cocycle condition. The resulting quasi-coherent sheaf ω_G on S is the *sheaf of invariant differentials* on G. Since ω_G is already the name we've given to the conormal sheaf of the unit $e : S \to G$ we have to justify this notion. So for the next sentence, let's write ω_G for the invariant differentials and $e^*\Omega_{G/S}$ for the conormal sheaf. Then, by construction, we have that

$$q^*\omega_G \cong \Omega_{G/S} \tag{4.2}$$

from which it follows that

$$e^*\Omega_{G/S} = e^*q^*\omega_G \cong \omega_G. \tag{4.3}$$

Thus, from now on, we make no distinction between the two.

4.6 Example. This definition is less arcane that it seems. Unwinding the proof of faithfully flat descent, we see that there is an equalizer diagram of sheaves of S

$$\omega_G \longrightarrow q_*\Omega_{G/S} \overset{dp_1}{\underset{dm}{\rightrightarrows}} q_*\Omega_{(G\times_S G)/G}$$

where I have written q for the canonical projections to S. To be even more concrete, suppose $S = \mathrm{Spec}(R)$ and G is affine over R; that is, $G = \mathrm{Spec}(A)$ for some Hopf algebra A over R. Then ω_G is determined by the R-module ω_A defined by the equalizer diagram

$$\omega_A \longrightarrow \Omega_{A/R} \overset{di_1}{\underset{d\Delta}{\rightrightarrows}} \Omega_{(A\otimes_R A)/A}.$$

For example, if $G = \mathbb{G}_m$, then $A = R[x^{\pm 1}]$ with $\Delta(x) = x \otimes x$ and we calculate that ω_A is the free R-module on dx/x.

4.7 Remark. As $\mathrm{Lie}_G \cong \mathbb{V}(\omega_G)$, the sheaf *dual* to ω_G is the quasi-coherent sheaf which assigns to each Zariski open $U \subseteq S$ the sections of $\mathrm{Lie}_G|_U \to U$. In particular, the global sections of this dual sheaf are exactly invariant derivations of G. If we need a name for this sheaf we will call it $\mathrm{lie}_{G/S}$.

These notions extend to formal groups, with a little care. In this case we don't have a sheaf $\Omega_{G/S}$ defined – although we could produce it if need be. However, in Remark 1.27, we did define sheaves $(\Omega_{G/S})_n$ over G_n and we *define*

$$q^*\Omega_{G/S} \overset{\text{def}}{=} \lim q_*(\Omega_{G/S})_n \cong \lim q^*\Omega_{G_n/S}$$

over S, where $q : G_n \to S$ is any of the projections. Similarly

$$q_*\Omega_{G\times G/G} = \lim q_*\Omega_{G_n\times G_n/G_n}.$$

The following allows us to call ω_G the sheaf of invariant differentials for G.

4.8 Proposition. *Let $G \to S$ be a formal group over S. Then there is an equalizer diagram of sheaves on S*

$$\omega_G \longrightarrow q_*\Omega_{G/S} \overset{dp_1}{\underset{dm}{\rightrightarrows}} q_*\Omega_{(G\hat{\times}_S G)/G}$$

4.9 Example. Suppose that $S = \mathrm{Spec}(A)$ is affine and that G can be given a coordinate x. Then ω_G is determined by its S-module of global sections over S and we we have an equalizer diagram of A-modules

$$H^0(S, \omega_G) \longrightarrow A[[x]]dx \overset{di_1}{\underset{d\Delta}{\rightrightarrows}} A[[x, y]]dx.$$

Let's write $F(x,y) = \Delta(x)$ for the resulting formal group law and $F_x(x,y)$ for the partial derivative of that power series with respect to x. Then an invariant differential $f(x)dx$ must satisfy the equality

$$f(x)dx = f(F(x,y))F_x(x,y)dx.$$

Setting $x = 0$ and then setting $y = x$ we get that

$$f(x) = \frac{f(0)}{F_x(0,x)}.$$

Since $F_x(0,0) = 1$, we conclude that ω_G is the quasi-coherent sheaf on $\mathrm{Spec}(A)$ determined by the free A-module of rank 1 with generator

$$\eta = \frac{dx}{F_x(0,x)}.$$

4.10 Example. Calculating with Lie_G and ω_G is standard, at least locally. Compare Remark 4.4. Suppose $S = \mathrm{Spec}(A)$ and $f : G \to H$ is a homorphism of formal groups over S. By passing to a faithfhully flat extension, we may as well assume that G and H can be given coordinates x and y respectively; then f is determined by a power series $f(x) \in A[[x]]$ and the induced morphism

$$df : \omega_H \longrightarrow \omega_G$$

is multiplication by $f'(0)$.

4.3 Invariant differentials in characteristic p

As a warm-up for the next section, we will isolate some of the extra phenomena that occurs when we are working over a base scheme S which is itself a scheme over $\mathrm{Spec}(\mathbb{F}_p)$. In this case there is a Frobenius morphism $f : S \to S$. Indeed, if R is an $\mathbb{F}_p$ algebra, the Frobenius $x \mapsto x^p$ defines a natural morphism $f_R : R \to R$ of $\mathbb{F}_p$ algebras and

$$f_S(R) = S(f_R) : S(R) \longrightarrow S(R),$$

If $X \to S$ is any scheme over S, we define $X^{(p)}$ to be the pull-back

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r, "f"'] & S \end{tikzcd}$$

and the relative Frobenius $F : X \to X^{(p)}$ to the unique morphism of schemes over S so that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "F"'] \arrow[rr, bend left, "f"] \arrow[dr] & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ & S \arrow[r, "f"'] & S. \end{tikzcd}$$

The following is an exercise in definitions and the universal properties of pull-backs.

4.11 Lemma. *Let $X \to S$ be a scheme over a scheme S over $\mathrm{Spec}(\mathbb{F}_p)$.*

1.) There is a natural isomorphism $\mathcal{T}\mathrm{an}^{(p)}_{X/S} \cong \mathcal{T}\mathrm{an}_{X^{(p)}/S}$.

2.) If $G \to S$ is a group scheme over S, then there is a natural isomorphism $\mathrm{Lie}^{(p)}_G \cong \mathrm{Lie}_{G^{(p)}/S}$.

3.) The relative Frobenius $F : X \to X^{(p)}$ induces the the zero homomorphism $\mathcal{T}\mathrm{an}_{X/S}(F) : \mathcal{T}\mathrm{an}_{X/S} \to \mathcal{T}\mathrm{an}^{(p)}_{X/S}$; that is, $\mathcal{T}\mathrm{an}_{X/S}(F)$ can be factored

$$\mathcal{T}\mathrm{an}_{X/S} \longrightarrow X \xrightarrow{F} X^{(p)} \xrightarrow{s} \mathcal{T}\mathrm{an}^{(p)}_{X/S}$$

where s is the zero section.

4.) If $G \to S$ is a group scheme, the relative Frobenius $F : G \to G^{(p)}$ induces the the zero homomorphism $\mathrm{Lie}(F) : \mathrm{Lie}_G \to \mathrm{Lie}_{G^{(p)}/S}$; that is, $\mathrm{Lie}(F)$ can be factored

$$\mathrm{Lie}_G \longrightarrow S \xrightarrow{s} \mathrm{Lie}_{G^{(p)}/S}.$$

Proof. The first of two of these statements are an exercise in definitions and the universal properties of pull-backs. The second two follow from the fact that if R is an $\mathbb{F}_p$-algebra, the $f_{R(\epsilon)}(-) = (-)^p : R(\epsilon) \to R(\epsilon)$ factors

$$R(\epsilon) \xrightarrow{\epsilon=0} R \xrightarrow{f_R} R \longrightarrow R(\epsilon).$$

$\square$

While the morphism $\mathrm{Lie}(F)$ induced by the relative Frobenius $F : G \to G^{(p)}$ is the zero map, the relative Frobenius

$$F : \mathrm{Lie}_G \to \mathrm{Lie}^{(p)}_G \cong \mathrm{Lie}_{G^{(p)}/S}$$

is not. This is the map on affine schemes over S

$$\mathbb{V}(\omega_G) \longrightarrow \mathbb{V}(\omega_G)^{(p)} \cong \mathbb{V}(\omega_{G^{(p)}})$$

induced by the Frobenius morphism on algebra sheaves

$$\mathrm{Sym}_S(\omega_{G^{(p)}}) \to \mathrm{Sym}_S(\omega_G).$$

By restricting to the sub-$\mathcal{O}_S$-module $\omega_{G^{(p)}}$ of $\mathrm{Sym}_S(\omega_{G^{(p)}})$ we get the map needed for the following result. $\mathrm{Sym}_p(-)$ is the pth symmetric power functor.

4.12 Lemma. *Let G be an algebraic group over S and S a scheme over $\mathbb{F}_p$. Then the pth power map induces a natural homomorphism of quasi-coherent sheaves over S*

$$\omega_{G^{(p)}} \to \mathrm{Sym}_p(\omega_G)$$

which, if G is smooth of dimension 1, becomes an isomorphism

$$\omega_{G^{(p)}} \cong \mathrm{Sym}_p(\omega_G) \cong \omega_G^{\otimes p}.$$

Proof. The last statement follows because ω_G is locally free of rank 1. $\square$

The exact same argument now proves:

4.13 Lemma. *Let G be a formal group over S and S a scheme over $\mathbb{F}_p$. Then the pth power map induces a natural homomorphism of quasi-coherent sheaves over S*

$$\omega_{G^{(p)}} \to \mathrm{Sym}_p(\omega_G)$$

yieds an is0morphism

$$\omega_{G^{(p)}} \cong \mathrm{Sym}_p(\omega_G) \cong \omega_G^{\otimes p}.$$

5 The height filtration

The theory of formal groups in characteristic zero is quite simple: in Corollary 3.21 we saw that over $\mathbb{Q}$, we are reduced to studying the additive formal group law and its automorphisms. In characteristic $p > 0$ (and hence over the integers) the story is quite different. Here formal groups are segregated by height and it is the height filtration which is at the heart of the geometry of $\mathcal{M}_{\mathbf{fg}}$. The point of this section is to spell this out in detail.

5.1 Height and the elements v_n

We are going to study formal groups G over schemes S which are themselves scheme over $\mathrm{Spec}(\mathbb{F}_p)$. In Lemma 4.11 we introduced and discussed the relative Frobenius F and its effect on tangent and Lie schemes. The following is a standard lemma for formal groups.

5.1 Lemma. *Let $f : G \to H$ be a homomorphism of smooth one-dimensional, commutative formal groups over S which is a scheme over* $\mathrm{Spec}(\mathbb{F}_p)$*. If*

$$0 = \mathrm{Lie}(f) : \mathrm{Lie}_G \to \mathrm{Lie}_H.$$

then there is a unique morphism $g : G^{(p)} \to H$ so that

$$f = gF : G \longrightarrow H.$$

Proof. It follows immediately from the natural decomposition $\mathcal{T}\mathrm{an}_{G/S} \cong G \times \mathrm{Lie}_G$ that the induced map

$$\mathcal{T}\mathrm{an}(f) : \mathcal{T}\mathrm{an}_{G/S} \to \mathcal{T}\mathrm{an}_{H/S}$$

is the zero homomorphism as well. Because of the uniqueness of g it is sufficient to prove the result locally, so choose an fqpc-cover $q : T \to S$ so that q^*G and q^*H can each be given a coordinate. As in Remark 4.4, we write f as a power series $f(x)$ and because $\mathcal{T}\mathrm{an}(f) = 0$ we conclude that $f'(x) = 0$. Because we are working over $\mathbb{F}_p$, we may write $f(x) = g(x^p)$ for some unique $g(x)$ and we let g define the needed homomorphism $g^* : G^{(p)} \to H$. $\square$

Let G be a formal group over S, with S a scheme over $\mathrm{Spec}(\mathbb{F}_p)$. Since G is commutative the pth power map

$$[p] : G \longrightarrow G$$

is a homomorphism of formal groups over S. If G can be given a coordinate, then Remark 4.4 implies that $\mathrm{Lie}([p]) = 0$. More generally, we choose an fpqc cover $f : T \to S$ so that f^*G has a coordinate. Then, since f is faithfully flat and $f^*\mathrm{Lie}([p]) = 0$, we have $\mathrm{Lie}([p]) = 0$. Therefore, Lemma 5.1 implies there is a unique homomorphism $V : G^{(p)} \to G$ so that we have a factoring

$$\begin{tikzcd} G \arrow[r, "F"] \arrow[dr, "{[p]}"'] & G^{(p)} \arrow[d, "V"] \\ & G. \end{tikzcd}$$

The induced morphism $\mathrm{Lie}(V) : \mathrm{Lie}_G^{(p)} \to \mathrm{Lie}_G$ may itself be zero; if so, we obtain a factoring

$$\begin{tikzcd} G^{(p)} \arrow[r, "F^{(p)}"] \arrow[dr, "V"'] & G^{(p^2)} \arrow[d, "V_2"] \\ & G. \end{tikzcd}$$

We may continue if $\mathrm{Lie}(V_2) = 0$.

5.2 Definition (The height of a formal group). *Let G be a formal group over a scheme S which is itself a scheme over* $\mathrm{Spec}(\mathbb{F}_p)$. *Define G to have* **height** *at least n if there is a factoring*

$$\begin{tikzcd} G \arrow[r, "F"] \arrow[drrrr, "{[p]}"'] & G^{(p)} \arrow[r, "F^{(p)}"] & G^{(p^2)} \arrow[r, "F^{(p^2)}"] & \cdots \arrow[r, "F^{(p^{n-1})}"] & G^{(p^n)} \arrow[d, "V_n"] \\ & & & & G. \end{tikzcd}$$

Define G to have height exactly n if $\mathrm{Lie}(V_n) \neq 0$.

Note that a formal group may not have finite height; for example, if $\hat{\mathbb{G}}_a$ is the formal additive group, then $0 = [p] : \hat{\mathbb{G}}_a \to \hat{\mathbb{G}}_a$ so it must have infinite height. It follows from Lazard's uniqueness theorem (Corollary 5.24) that every infinite height formal group is locally isomorphic to the additive group.

5.3 Proposition. *Let G be a formal group over a scheme S which is itself a scheme over* $\mathrm{Spec}(\mathbb{F}_p)$. *Suppose that G has height at least n. Then there is a global section*

$$v_n(G) \in H^0(S, \omega_G^{\otimes(p^n-1)})$$

so that G has height at least $n+1$ if and only if $v_n(G) = 0$. The element $v_n(G)$ is natural; that is, if $H \to T$ is another formal group, $f : T \to S$ is a morphism of schemes and $\phi : H \to f^*G$ is an isomorphism of formal groups, then

$$f^* v_n(G) = v_n(H).$$

Proof. Since G is of height at least n, we have the morphism $V_n : G^{(p^n)} \to G$ and G has height at least $n+1$ if and only if $\mathrm{Lie}(V_n) = 0$. This will happen if and only if the induced map

$$dV_n : \omega_G \longrightarrow \omega_{G^{(p^n)}} \cong \omega_G^{\otimes p^n}.$$

is zero. The last isomorphism uses Lemma 4.13. Since ω_G is an invertible sheaf, dV_n corresponds to a unique morphism

$$v_n(G) : \mathcal{O}_S \longrightarrow \omega_G^{\otimes(p^n-1)}.$$

This defines the global section. The naturality statement follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{Lie}_{H^{(p^n)}} \arrow[r, "V_n"] \arrow[d, "\mathrm{Lie}(\phi^{(p^n)})"'] & \mathrm{Lie}_H \arrow[d, "\mathrm{Lie}(\phi)"] \\
f^*\mathrm{Lie}_{G^{(p^n)}} \arrow[r, "V_n"] & f^*\mathrm{Lie}_G
\end{tikzcd}$$

□

5.4 Remark. The global section v_n can be computed locally as follows. Let $S = \mathrm{Spec}(R)$ be affine and suppose $G \to S$ can be given a coordinate x. Then if G is of height at least n, the power series expansion of $[p] : G \to G$ gives the p-series:

$$[p](x) = a_n x^{p^n} + a_{2n} x^{2p^n} + \cdots$$

If $\eta(G, x) = dx/F_x(0, x)$ is the invariant differential associated to this coordinate, then

$$v_n(G) = a_n \eta(G, x)^{\otimes p^n - 1} \in \omega_G^{\otimes(p^n-1)}.$$

In particular, $v_n(G) = 0$ if and only if $a_n = 0$.

We wish to define a descending chain of closed substacks

$$\cdots \subseteq \mathcal{M}(3) \subseteq \mathcal{M}(2) \subseteq \mathcal{M}(1) \subseteq \mathcal{M}_{\mathbf{fg}}$$

with $\mathcal{M}(n)$ the moduli stack of formal groups of height greater than or equal to n. Of course, $\mathcal{M}(n)$ will be defined by the vanishing of $p, v_1, \ldots, v_{n-1}$, but it's worth dwelling on the definition so that the behavior of $\mathcal{M}(n)$ under base change is transparent.

Let X be an $fqpc$-algebraic stack over a base scheme S. Recall that an *effective Cartier divisor* $D \subseteq X$ is closed subscheme so that the ideal sheaf $\mathcal{I}(D) \subseteq \mathcal{O}_X$ defining D is locally free of rank 1.[4] If we tensor the exact sequence

$$0 \to \mathcal{I}(D) \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_D \to 0$$

of sheaves on $\mathcal{X}$ with the dual sheaf $\mathcal{I}(D)^{-1}$, then we get an exact seqeunce

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} \mathcal{I}(D)^{-1} \longrightarrow \mathcal{O}_D \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{I}(D)^{-1} \longrightarrow 0$$

with s a section of $\mathcal{I}(D)^{-1}$. Conversely, given an invertible sheaf $\mathcal{L}$, a section s of $\mathcal{L}$, and an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{X}} \xrightarrow{s} \mathcal{L} \longrightarrow \mathcal{L}/\mathcal{O}_X \longrightarrow 0$$

then the substack of zeros of s is an effective Cartier divisor with ideal sheaf defined by the image of the injection

$$s : \mathcal{L}^{-1} \longrightarrow \mathcal{O}_{\mathcal{X}}.$$

This establishes a one-to-one correspondence between effective Cartier divisors and isomorphism classes of pairs $(\mathcal{L}, s)$ as above. We will say that the divisor is defined by the pair $(\mathcal{L}, s)$. For example

$$\mathcal{M}(1) \subseteq \mathcal{M}_{\mathbf{fg}}$$

is the effective Cartier divisor defined by $(\mathcal{O}_{\mathbf{fg}}, p)$. Suppose $\mathcal{M}(n)$ has been defined and classifies formal groups of height at least n.

5.5 Definition. *1.) Define the closed substack $\mathcal{M}(n+1) \subseteq \mathcal{M}(n)$ to be the effective Cartier divisor defined by the pair (ω^{p^n-1}, v_n).*

2.) Let $\mathcal{H}(n) = \mathcal{M}(n) - \mathcal{M}(n+1)$ be the open complement of $\mathcal{M}(n+1)$ in $\mathcal{M}(n)$. Then $\mathcal{H}(n)$ classifies formal groups of **exact height** *n or simply of height n.*

Then Proposition 5.3 implies that $\mathcal{M}(n+1)$ classifies formal goups of height at least $n+1$. The inclusion $\mathcal{M}(n) \subseteq \mathcal{M}_{\mathbf{fg}}$ is closed; let $\mathcal{I}_n \subseteq \mathcal{O}_{\mathbf{fg}}$ be the ideal sheaf defining this inclusion. Thus we have an ascending sequence of ideal sheaves

$$0 \subseteq \mathcal{I}_1 = (p) \subseteq \mathcal{I}_2 \subseteq \cdots \mathcal{O}_{\mathbf{fg}}$$

and an isomorphism

$$v_n(G) : \omega^{-(p^n-1)} \longrightarrow \mathcal{I}_{n+1}/\mathcal{I}_n$$

on $\mathcal{M}(n)$.

[4]Some authors (*cf* [26], Chapter 1) require also that D be flat over S. This implies that if $f : T \to S$ is a morphism of schemes, then $T \times_S D$ is an effective Cartier divisor for $T \times_S X$. But is also means that if X is a scheme with $\mathcal{O}_X$ torsion free then the closed subscheme obtained from setting $p = 0$ is not an effective Cartier divisor for X over $\mathbb{Z}$.

5.6 Remark. A formal group $G \to S$ has exact height n if the global section $v_n(G) \in H^0(S, \omega_G^{p^n-1})$ is invertible in the sense that

$$v_n : \omega_G^{-(p^n-1)} \longrightarrow \mathcal{O}_S$$

is an isomorphism. This makes sense even if $n = 0$, where would have p invertible in $H^0(S, \mathcal{O}_S)$. This *defines* the notion of a formal group of height 0.

5.7 Remark. We can follow up Remark 5.4 with a local description of $\mathcal{I}_n$ and the process of defining $\mathcal{M}(n)$. If $G \to \mathrm{Spec}(R)$ can be given a coordinate x and G has height a least n, then we can write

$$v_n(G) = u_n \eta(G,x)^{\otimes p^n-1}.$$

The choice of generator $\eta(G,x)^{\otimes p^n-1}$ for $\omega_G^{p^n-1}$ defines an isomorphism $R \cong \omega_G^{p^n-1}$ and the section $v_n : R \to \omega_G^{p^n-1}$ becomes isomorphic to multiplication by u_n. Thus

$$\mathcal{I}_{n+1}(G)/\mathcal{I}_n(G) = (u_n)$$

is the principal ideal generated by u_n. Note that u_n is not an isomorphism invariant, but the ideal is.

It is tempting to write, for a general formal group G with a coordinate, that there is an isomorphism

$$\mathcal{I}_n(G) = (p, u_1, \ldots, u_{n-1}).$$

In general, u_{n-1} is only well-defined modulo $\mathcal{I}_{n-1}(G)$, so we must be careful with this notation. It is possible to choose a sequence $p, u_1, \ldots, u_{n-1}$ defining the ideal $\mathcal{I}_n(R)$, but the choices make the sequence unpleasant. In the presence of a p-typical coordinate, the situation improves. See the next remark.

5.8 Remark. The form v_n is defined globally only when $p = v_1 = \cdots v_{n-1} = 0$. But if G is a formal group with a coordinate over a $\mathbb{Z}_{(p)}$ algebra R, then Cartier's theorem gives a p-typical coordinate x for G. Let F be the resulting formal group law for G. Then we can write the p-series

$$[p](x) = px +_F u_1 x^p +_F u_2 x^{p^2} +_F \cdots$$

Remark 5.4 implies that if $p = u_1 = \cdots u_{n-1} = 0$, then

$$v_n(H) = u_n \eta(G,x)^{\otimes p^{n-1}}$$

and we really can write $\mathcal{I}_n(G) = (p, u_1, \ldots, u_{n-1})$.

Note that $v_n(G) = 0$ if and only if $u_n = 0$. Since the morphism

$$\mathrm{Spec}(\mathbb{Z}_{(p)}[u_1, u_2, \ldots]) \longrightarrow \mathcal{M}_{\mathbf{fg}} \otimes \mathbb{Z}_{(p)}$$

classifying the universal p-typical formal group is an $fpqc$-cover, this remark implies the following result.

5.9 Proposition. *For all primes p and all $n \geq 1$, there is an fpqc-cover*

$$X(n) \overset{\text{def}}{=} \operatorname{Spec}(\mathbb{F}_{(p)}[u_n, u_{n+1}, \ldots]) \to \mathcal{M}(n).$$

Furthemore,

$$p_1^*(p, u_1, \ldots, u_{n-1}) = p_2^*(p, u_1, \ldots, u_{n-1}) \subseteq \mathcal{O}_{X(n)\times_{\mathcal{M}(n)}X(n)}$$

and

$$X(n) \times_{\mathcal{M}(n)} X(n) = \operatorname{Spec}(\mathbb{F}_p[u_n, u_{n+1}, \cdots][t_0^{\pm 1}, t_1, t_2, \cdots]).$$

If $q : Y \to X$ is a representable and flat morphism of algebraic stacks and $D \subseteq X$ is an effective Cartier divisor defined by $(\mathcal{L}, s)$, then

$$f^*D \overset{\text{def}}{=} D \times_X Y \subseteq Y$$

is an effective Cartier divisor defined by $(q^*\mathcal{L}, q^*s)$. To see this, note that because f is flat, we have an exact sequence

$$0 \to f^*\mathcal{I}(D) \longrightarrow f^*\mathcal{O}_X \longrightarrow f^*\mathcal{O}_D \to 0$$

which is isomorphic to

$$0 \to f^*\mathcal{I}(D) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{f^*D} \to 0.$$

Thus $f^*\mathcal{I}(D) \cong \mathcal{I}(f^*D)$. From this we can immediately conclude the following.

5.10 Proposition. *Let $q : \mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ be a representable and flat morphism of stacks and define*

$$\mathcal{N}(n) = \mathcal{M}(n) \times_{\mathcal{M}_{\mathbf{fg}}} \mathcal{N}.$$

Then

$$\cdots \subseteq \mathcal{N}(2) \subseteq \mathcal{N}(1) \subseteq \mathcal{N}$$

is a descending chain of closed substacks so that

$$\mathcal{N}(n+1) \subseteq \mathcal{N}(n)$$

is the effective Cartier divisor defined by (ω^{p^n-1}, v_n).

This implies that for all n the section v_n defines an injection

$$v_n : \mathcal{O}_{\mathcal{N}} \longrightarrow \omega^{p^n-1}.$$

If $\mathcal{N} = \operatorname{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$ classifies a formal group for which we can choose a coordinate, this implies that each of the ideals $\mathcal{I}_n(R)$ is generated by a regular sequence. The Landweber Exact Functor Theorem 6.18 is a partial converse to this result.

In these examples, the closed embedding $\mathcal{N}(n) \subseteq \mathcal{N}$ is a regular embedding; that is, the ideal sheaf defining the embedding is locally generated by a regular sequence. But we're actually given more data. We crystalize this in the following definition which generalizes the notion of a regular scale given in [33].

5.2 Geometric points and reduced substacks of $\mathcal{M}_{\mathbf{fg}}$

Recall that we are working at a prime p, so that $\mathcal{M}_{\mathbf{fg}} = \mathcal{M}_{\mathbf{fg}} \otimes \mathbb{Z}_{(p)}$. We will show that $\mathcal{M}_{\mathbf{fg}}$ has exactly one geometric point for each height n, $0 \leq n \leq \infty$ and use this to show that the substacks $\mathcal{M}(n) \subseteq \mathcal{M}_{\mathbf{fg}}$ give a complete list of the reduced substacks of $\mathcal{M}_{\mathbf{fg}}$.

We begin with the following definition.

5.11 Definition. *Let $\mathcal{M}$ be an algebaic stack.*

1.) A **geometric** *point ξ of X is an equivalence class of the morphisms $x : \mathrm{Spec}(\mathbb{F}) \to \mathcal{M}$ where $\mathbb{F}$ is a field. Two such morphisms $(x', \mathbb{F}')$ and $(x'', \mathbb{F}'')$ are equivalent if $\mathbb{F}'$ and $\mathbb{F}''$ have a common extension $\mathbb{F}$ and the evident diagram*

$$\begin{tikzcd}
\mathrm{Spec}(\mathbb{F}) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbb{F}'') \arrow[d, "x''"] \\
\mathrm{Spec}(\mathbb{F}') \arrow[r, "x'"'] & \mathcal{M}
\end{tikzcd}$$

2-*commutes.*

2.) The set of geometric points $|X|$ has a topology with open sets $|\mathcal{U}|$ where $\mathcal{U} \subseteq \mathcal{M}$ is an open substack. When we write $|X|$ we will mean this set with this topology.

The following result implies that the topology of $|\mathcal{M}_{\mathbf{fg}}|$ is quite simple.

5.12 Proposition. *Let $\mathcal{U} \subseteq \mathcal{M}_{\mathbf{fg}}$ be an open substack and suppose that $\mathcal{U}$ has a geometric point of height n. Then it has a geometric point of height k for all $k \leq n$.*

Proof. Let $G : \mathrm{Spec}(k) \to U$ represent the geometric point of height n and $\mathrm{Spec}(L) \to \mathcal{M}_{\mathbf{fg}}$ be the cover by the Lazard ring. Then we have a 2-commutative diagram

$$\begin{tikzcd}
 & \mathcal{U} \times_{\mathcal{M}_{\mathbf{fg}}} \mathrm{Spec}(L) \arrow[r, "j"] \arrow[d, "q"] & \mathrm{Spec}(L) \arrow[d] \\
\mathrm{Spec}(k) \arrow[ur, "F"] \arrow[r, "G"'] & \mathcal{U} \arrow[r] & \mathcal{M}_{\mathbf{fg}}
\end{tikzcd}$$

obtained by choosing a coordinate for G. The morphism j is open and the morphism q is flat, as it is the pull-back of a flat map. Choose an affine open $\mathrm{Spec}(R) \subseteq \mathcal{U} \times_{\mathcal{M}_{\mathbf{fg}}} \mathrm{Spec}(L)$ so that the morphism F factors through $\mathrm{Spec}(R)$. Let G_0 be the resulting formal group over R.

By localizing R if necessary, we may assume that $R \to k$ is onto. Choose an element $w \in R$ which reduces to $v_n(G) \in k$. Since G has height n, $v_n(G) \neq 0$; thus, w is not nilpotent. By forming $R[w^{-1}]$ if necessary, we may assume that w is a unit. From this we conclude that $\mathcal{I}_{n+1}(G_0) = R$. Since $\mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$ is flat, Proposition 5.10 implies that the ideals $\mathcal{I}_k(G_0)$, $k \leq n+1$, is generated by a regular sequence. (Note that G_0 has a canonical coordinate by construction.)

Let $k \leq n$ and and let $q_k : R \to R_k \overset{\text{def}}{=} R/\mathcal{I}_k(G_0)$. We conclude immediately that $v_k(q_k^* G_0)$ is not nilpotent in R_k. Choose a prime ideal $\mathfrak{p}$ in R_k so that $v_k(q_k^* G_0) \neq 0$ in $R/\mathfrak{p}$ and let K be the field of fractions of of $R/\mathfrak{p}$. Then

$$\mathrm{Spec}(K) \to \mathrm{Spec}(R) \to \mathcal{U}$$

represents a geometric point of height k. □

The importance of the closed substacks $\mathcal{M}(n)$ is underlined by the following result.

5.13 Theorem. *For all n, $1 \leq n \leq \infty$ the algebraic stack $\mathcal{M}(n)$ is reduced. Furthermore if $\mathcal{N} \subseteq \mathcal{M}_{\mathbf{fg}}$ is any closed, reduced substack, then either $\mathcal{N} = \mathcal{M}_{\mathbf{fg}}$ or there is an n so that*

$$\mathcal{M}(n) = \mathcal{N}.$$

Before proving this result, we need to recall what it means for an algebraic stack to be reduced and how to produce the reduced substack of a stack, assuming it exists.

Fix an algebraic stack $\mathcal{M}$. We define a diagram $\mathcal{C}$ of closed substacks of $\mathcal{M}$ as follows:

1. An object of $\mathcal{C}$ is a closed substack $\mathcal{N} \subseteq \mathcal{M}$ so that the induced inclusion on geometric points $|\mathcal{N}| \to |\mathcal{M}|$ is an isomorphism;
2. A morphism $\mathcal{N}_1 \subseteq \mathcal{N}_2$ is an inclusion of closed substacks.

This diagram $\mathcal{C}$ of closed substacks is filtered; furthermore it determines and is determined by a filtered (or cofiltered) diagram $\{\mathcal{I}_{\mathcal{N}}\}$ of quasi-coherent ideals in $\mathcal{O}_{\mathcal{M}}$. Define

$$\mathcal{I}_{\mathrm{red}} = \mathrm{colim}_{\mathcal{C}^{\mathrm{op}}} \mathcal{I}_{\mathcal{N}}.$$

The colimit is taken pointwise and, since tensor products commute the colimits, $\mathcal{I}_{\mathrm{red}}$ is a quasi-coherent ideal. Let

$$\mathcal{M}_{\mathrm{red}} \subseteq \mathcal{M}$$

be the resulting closed substack. Note that $\mathcal{M}_{\mathrm{red}}$ is the initial closed substack $\mathcal{N} \subseteq \mathcal{M}$ so that $|\mathcal{N}| = |\mathcal{M}|$. We say that $\mathcal{M}$ is reduced if $\mathcal{M}_{\mathrm{red}} = \mathcal{M}$ or, equivalently, if $\mathcal{I}_{\mathrm{red}} = 0$.

The sheaf $\mathcal{I}_{\mathrm{red}}$ should be closely related to the ideal of nilpotents in $\mathcal{O}_{\mathcal{M}}$. Some care is required here, however. If we define $\mathcal{N}il_{\mathcal{M}}(U) = \mathcal{N}il_U$ for any $fpqc$-morphism $U \to \mathcal{M}$, the resulting ideal sheaf may not be cartesian in the $fqpc$-topology; thus it is not evidently quasi-coherent. (If $R \to S$ is a faithfully flat morphism of rings, then it is not necessarily true that $\mathcal{N}il_S = S \otimes_R \mathcal{N}il_R$.) However it is a sheaf in more restrictive topologies, such as the "smooth-étale" topology central to [29].

5.14 Definition. *Let $\mathcal{M}$ be an algebraic stack in $fpqc$-topology and suppose that $X \to \mathcal{M}$ is an $fpqc$-presentation so that*

$$p_1^*\mathcal{N}il_X \cong \mathcal{N}il_{X\times_{\mathcal{M}}X} \cong p_2^*\mathcal{N}il_X$$

as ideal sheaves in $\mathcal{O}_{X\times_{\mathcal{M}}X}$. Then descent theory yields a quasi-coherent ideal sheaf $\mathcal{N}il_{\mathcal{M}} \subseteq \mathcal{O}_{\mathcal{M}}$. This is the **sheaf of nilpotents** *for $\mathcal{M}$.*

5.15 Remark. 1.) It is not immediately clear that $\mathcal{N}il_{\mathcal{M}}$ does not depend on the choice of cover $X \to \mathcal{M}$; however, this will follow from Proposition 5.17 to follow.

2.) If $\mathcal{M}$ has a *smooth* cover, then $\mathcal{N}il_{\mathcal{M}}$, when restricted to the smooth-étale topology, agrees with the sheaf $\mathcal{N}il_{\mathcal{M}}$ as defined in [29].

3.) In many of our standard examples, $\mathcal{N}il_X = 0 = \mathcal{N}il_{X\times_S X}$. In particular, this applies to $\mathcal{M}_{\mathbf{fg}}$ and $\mathcal{M}(n)$, by Proposition 5.9.

We need the following preliminary result before preceding.

5.16 Lemma. *Let $\mathcal{M}$ be an algebraic stack and $\mathcal{N} \subseteq \mathcal{M}$ a closed substack. Let $X \to \mathcal{M}$ be an $fqpc$-cover. Then the natural map*

$$|X \times_{\mathcal{M}} \mathcal{N}| \longrightarrow |X| \times_{|\mathcal{M}|} |\mathcal{N}|$$

is an isomorphism.

Proof. This morphism is onto for a general pull-back; that is, we don't need $\mathcal{N} \to \mathcal{M}$ to be a closed inclusion. To see that is one-to-one, note that $X \times_{\mathcal{M}} \mathcal{N}$ is equivalent to closed subscheme $Y \subseteq X$ and that, hence, the composite

$$|Y| = |X \times_{\mathcal{M}} \mathcal{N}| \longrightarrow |X| \times_{|\mathcal{M}|} |\mathcal{N}| \to |X|$$

is an injection. $\square$

5.17 Proposition. *Suppose that $\mathcal{M}$ is an algebraic stack in the $fpqc$ topology and there is an $fpqc$-presentation $X \to \mathcal{M}$ so that $\mathcal{N}il_{\mathcal{M}}$ is defined. Then*

$$\mathcal{N}il_{\mathcal{M}} = \mathcal{I}_{\mathrm{red}}.$$

Proof. Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be the closed substack defined by $\mathcal{N}il_X$. Then $X_{\mathrm{red}} \to \mathcal{M}_0$ is a cover. Since $|X_{\mathrm{red}}| = |X|$ and $|X_{\mathrm{red}}| \to |\mathcal{M}_0|$ is surjective, we can conclude that $|\mathcal{M}_0| = |\mathcal{M}|$. This shows that $\mathcal{M}_{\mathrm{red}} \subseteq \mathcal{M}_0$ or, equivalently, that $\mathcal{I}_{\mathrm{red}} \subseteq \mathcal{N}il_{\mathcal{M}}$.

For the other inclusion, let $\mathcal{N} \subseteq \mathcal{M}$ be a closed inclusion defined by an ideal $\mathcal{J}$ and suppose $|\mathcal{N}| = |\mathcal{M}|$ and let $Y = \mathcal{N} \times_{\mathcal{M}} X \to \mathcal{N}$ be the resulting cover. Then Y is the closed subscheme of X defined by $\mathcal{J}|_X$ and the natural map

$$|Y| \longrightarrow |X| \times_{|\mathcal{M}|} |\mathcal{N}|$$

is an isomorphism, by Lemma 5.16. Thus, $|Y| = |X|$, which implies that $X_{\mathrm{red}} \subseteq Y$, or $\mathcal{N}il_X \subseteq \mathcal{J}|_X$. Since $\mathcal{N}il_X = (\mathcal{N}il_{\mathcal{M}})|_X$ and X is a cover $\mathcal{M}$, this implies that $\mathcal{N}il_{\mathcal{M}} \subseteq \mathcal{J}$. In particular, $\mathcal{N}il_{\mathcal{M}} \subseteq \mathcal{I}_{\mathrm{red}}$. $\square$

5.18 Corollary. *Suppose that $\mathcal{M}$ is an algebraic stack in the fpqc topology and there is an fpqc-presentation $X \to \mathcal{M}$ so that X and $X \times_{\mathcal{M}} X$ are reduced. Then $\mathcal{M}$ is reduced.*

We next begin an investigation of the closed substacks of $\mathcal{M}_{\mathbf{fg}}$. Recall that $\mathcal{M}(1) = \mathcal{M}_{\mathbf{fg}} \otimes \mathbb{F}_p$.

5.19 Proposition. *Let $\mathcal{N} \subseteq \mathcal{M}_{\mathbf{fg},p}$ be a closed substack. If $\mathcal{N}$ has a geometric point of height n, then*

$$\mathcal{M}(n) \subseteq \mathcal{N}.$$

Proof. We begin with the following observation: suppose that $\mathcal{N}_1$ and $\mathcal{N}_2$ are closed substacks of an algebraic stack, that $\mathcal{N}_1$ is reduced in the strong sense of Proposition 5.17, and that $|\mathcal{N}_1| \subseteq |\mathcal{N}_2|$. Then $\mathcal{N}_1 \subseteq \mathcal{N}_2$. For if we let $X \to \mathcal{M}$ be a cover and $Y_i \subseteq X$, $i = 1, 2$ the resulting closed subscheme which covers of $\mathcal{N}_i$, then Y_1 is reduced and Lemma 5.16 implies that $|Y_1| \subseteq |Y_2|$. Then $Y_1 \subseteq Y_2$ and arguing as the end of the proof of Proposition 5.17, we have $\mathcal{N}_1 \subseteq \mathcal{N}_2$.

To prove the result, then, we need only show that there is an n so that $|\mathcal{M}(n)| \subseteq |\mathcal{N}|$. Thus we must prove that if $\mathcal{N} \subseteq \mathcal{M}_{\mathbf{fg}}$ is closed and contains a geometric point of height n, then it contains a geometric point of height k for all $k \geq n$. This can be rephrased in terms of the complementary open $\mathcal{U} = \mathcal{M}_{\mathbf{fg}} - \mathcal{N}$ as follows: if $\mathcal{U}$ does not have a geometric point of height n, it does not have a geometric point of height k, $k \geq n$. But, by rephrasing this as a positive statement, this becomes exactly Proposition 5.12. □

5.20 Proof of the Theorem 5.13. Suppose $\mathcal{N} \subseteq \mathcal{M}_{\mathbf{fg}}$ is closed and reduced. If $\mathcal{N} \neq \mathcal{M}_{\mathbf{fg}}$, then we have $\mathcal{N} \subseteq \mathcal{M}_{\mathbf{fg},p} = \mathcal{M}(1)$. Indeed, if $\mathcal{U} = \mathcal{M}_{\mathbf{fg}} - \mathcal{N}$ is not empty, then it must contain a geometric point of height 0, by Proposition 5.12. Let n be the smallest integer $1 \leq n \leq \infty$ so that $\mathcal{N}$ has a geometric point of height n. Then Proposition 5.19 implies that $\mathcal{M}(n) \subseteq \mathcal{N}$. Furthermore $|cM(n)| = |\mathcal{N}|$. The argument in the first paragraph of Proposition 5.19 shows that $\mathcal{M}(n) = \mathcal{N}$.

5.3 Isomorphisms and layers

In this section we discuss the difference between the closed substacks $\mathcal{M}(n)$ and $\mathcal{M}(n+1)$; that is, we discuss the geometry of

$$\mathcal{H}(n) \overset{\text{def}}{=} \mathcal{M}(n) - \mathcal{M}(n+1).$$

and the geometry of

$$\mathcal{M}(\infty) \overset{\text{def}}{=} \cap_n \mathcal{M}(n).$$

In both case we will find that we have stacks of the form $B\Lambda$ where Λ is the group of automorphisms of some height n formal group law. The group Λ is not an algebraic group as it is not finite; however, it will pro-étale in an appropriate sense. See Theorem 5.23.

Here is a preliminary result.

5.21 Lemma. *The inclusion*

$$f_n : \mathcal{H}(n) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

is an affine morphism of algebraic stacks.

Proof. Suppose $\mathrm{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$ is classifies a formal group G with a chosen coordinate x. Then the 2-category pull-back $\mathrm{Spec}(R) \times_{\mathcal{M}_{\mathbf{fg}}} \mathcal{H}(n)$ is the groupoid scheme which assigns to each commutative ring S the triples (f, Γ, ϕ) where $f : R \to S$ is a morphism of commutative rings, Γ is formal group of exact height n over S and $\phi : \Gamma \to f^*G$ is an isomorphism of formal groups. An isomorphism of triples $(f, \Gamma, \phi) \to (f, \Gamma', \phi')$ is an isomorphism of formal groups $\psi : \Gamma \to \Gamma'$ so that $\phi'\psi = \phi$. Given such a triple, (f, Γ, ϕ), the existence of ϕ forces f to factor as a composition

$$R \xrightarrow{q} R/\mathcal{I}_n(G))[u_n^{-1}] \xrightarrow{g} S$$

where $[p]_G(x) = u_n x^{p^n} + \cdots$ modulo $\mathcal{I}_n(G)$. We now check that the morphism of groupoid schemes

$$\mathrm{Spec}(R/\mathcal{I}_n(G))[u_n^{-1}]) \to \mathrm{Spec}(R) \times_{\mathcal{M}_{\mathbf{fg}}} \mathcal{H}(n)$$

sending g to $(gq, (gq)^*G, 1)$ is an equivalence. For more general G, we use faithfully flat descent to describe the pullback as an affine scheme. □

5.22 Remark. From this result and Proposition 5.9 we have that there is an *fpqc*-cover

$$Y(n) \overset{\mathrm{def}}{=} \mathrm{Spec}(\mathbb{F}_p[u_n^{\pm 1}, u_{n+1}, u_{n+2}, \ldots]) \to \mathcal{H}(n).$$

and that

$$Y(n) \times_{\mathcal{H}(n)} Y(n) \cong \mathrm{Spec}(\mathbb{F}_p[u_n^{\pm 1}, u_{n+1}, u_{n+2}, \ldots][t_0^{\pm 1}, t_1, t_2, \ldots]).$$

Now let S be a scheme and let G_1 and G_2 be two formal groups over S. Define the scheme of isomorphisms from G_1 to G_2 by the 2-category pull-back

$$\begin{tikzcd} \mathrm{Iso}_S(G_1, G_2) \arrow[r] \arrow[d] & \mathcal{M}_{\mathbf{fg}} \arrow[d, "\Delta"] \\ S \arrow[r, "G_1 \times G_2"'] & \mathcal{M}_{\mathbf{fg}} \times \mathcal{M}_{\mathbf{fg}}. \end{tikzcd}$$

Thus if $f : T \to S$ is a morphism of schemes, then a T-point of $\mathrm{Iso}_S(G_1, G_2)$ is an isomorphism $\phi : f^*G_1 \to f^*G_2$ of formal groups over T. By Proposition 2.19, $\mathrm{Iso}_S(G_1, G_2)$ is affine over S.

If G_3 is another formal group over S, then there is a composition

$$\mathrm{Iso}_S(G_2, G_3) \times_S \mathrm{Iso}_S(G_1, G_2) \longrightarrow \mathrm{Iso}_S(G_1, G_3).$$

In particular, $\mathrm{Aut}_S(G_1) = \mathrm{Iso}_S(G_1, G_1)$ acts on the right on $\mathrm{Iso}_S(G_1, G_2)$.

Because isomorphisms are locally given by power series, it is fairly clear that $\mathrm{Iso}_S(G_1, G_2) \to S$ does not have good finiteness properties. To get well-behaved approximations we assume S is a scheme over $\mathbb{Z}_{(p)}$ and define

$$\begin{tikzcd}
\mathrm{Iso}_S(G_1, G_2)_k \arrow[r] \arrow[d] & \mathcal{M}_{\mathbf{fg}}\langle p^k \rangle \arrow[d, "\Delta"] \\
S \arrow[r, "G_1 \times G_2"'] & \mathcal{M}_{\mathbf{fg}}\langle p^k \rangle \times \mathcal{M}_{\mathbf{fg}}\langle p^k \rangle.
\end{tikzcd}$$

Thus, for $f : T \to S$, a T-point of $\mathrm{Iso}_S(G_1, G_2)_k$ is an isomorphism of their p^k-buds $\phi : (G_1)_{p^k} \to (G_2)_{p^k}$.

Let $\mathrm{Iso}_S(G_1, G_2)_\infty = \mathrm{Iso}_S(G_1, G_2)$ and let $\mathrm{Iso}_S(G_1, G_2)_0 = S$; then there is a tower under $\mathrm{Iso}_S(G_1, G_2)$ and over S with transition morphisms

$$\mathrm{Iso}_S(G_1, G_2)_k \longrightarrow \mathrm{Iso}_S(G_1, G_2)_{k-1}.$$

Pointwise, these maps are fibrations, so we have that

$$\mathrm{Iso}_S(G_1, G_2) \to \operatorname{holim} \mathrm{Iso}_S(G_1, G_2)_k$$

is an equivalence.

The following is a refined version of Lazard's uniqueness theorem. See Corollary 5.24 below.

5.23 Theorem. *Let S be a scheme over $\mathbb{F}_p$ and let G_1 and G_2 be two formal groups of strict height n, $1 \leq n < \infty$ over S. Then*

$$\mathrm{Iso}_S(G_1, G_2)_1 \longrightarrow S$$

is surjective and étale of degree $p^n - 1$. For all $k > 1$, the morphism

$$\mathrm{Iso}_S(G_1, G_2)_k \longrightarrow \mathrm{Iso}_S(G_1, G_2)_{k-1}$$

is surjective and étale of degree p^n. Finally, the morphism

$$\mathrm{Iso}_S(G_1, G_2) \longrightarrow S$$

is surjective and pro-étale.

The proof is below in 5.27.

5.24 Corollary (Lazard). *Let $\mathbb{F}$ be a field of characteristic p and G_1 and G_2 two formal groups of strict height n. Then there is a separable extension $f : \mathbb{F} \to L$ so that f^*G_1 and f^*G_2 are isomorphic. In particular, if $\mathbb{F}$ is separably closed, then G_1 and G_2 are isomorphic.*

Proof. If the height $n < \infty$, this follows from the surjectivity statement of Theorem 5.23. If $n = \infty$, then the p-series of G_i must be zero; hence, a choice of p-typical coordinate for G_i defines an isomorphism from G_i to the additive formal group. $\square$

5.25 Remark. If G_1 and G_2 are two formal groups over a scheme S classified by maps $G_i : S \to \mathcal{M}_{\mathbf{fg}}$, we have a pull-back diagram

$$\begin{tikzcd} \mathrm{Iso}_S(G_1, G_2) \arrow[r] \arrow[d] & S \times_{\mathcal{M}_{\mathbf{fg}}} S \arrow[d] \\ S \arrow[r, "\Delta"'] & S \times S. \end{tikzcd}$$

If $S = \mathrm{Spec}(A)$ is affine and each of the formal groups G_i can be given a coordinate, then this writes (by Lemma 2.12) $\mathrm{Iso}_S(G_1, G_2)$ as the spectrum of the ring

$$A \otimes_{(A \otimes A)} A \otimes_L W \otimes_S A.$$

Thus if $x \in A$ we have (using the standard notation for Hopf algebroids)

$$x = \eta_R(x)$$

in this commutative ring. This makes it very unusual that

$$\mathrm{Iso}_S(G_1, G_2) \longrightarrow S$$

is flat, let alone étale. Thus the Theorem 5.23 is something of a surprise.

5.26 Example. We can be very concrete about the scheme $\mathrm{Aut}_{\mathbb{F}}(\Gamma_n)$ where Γ_n is of strict height n over over a field $\mathbb{F}$. The formal group Γ_n can be given a coordinate and we can dispaly $\mathrm{Aut}_{\mathbb{F}}(\Gamma_n)$ as

$$\mathrm{Aut}_{\mathbb{F}}(\Gamma_n) = \mathrm{Spec}(\mathbb{F} \otimes_L W \otimes_L \mathbb{F})$$

where $L \to \mathbb{F}$ classifies Γ_n with a choice of coordinate. For example, if $1 \leq n < \infty$ and if Γ_n is the Honda formal group over $\mathbb{F}_p$ with coordinate so that $[p](x) = x^{p^n}$, then we have an isomorphism of Hopf algebras

$$\mathbb{F}_p \otimes_L W \otimes_L \mathbb{F}_p = \mathbb{F}_p[a_0^{\pm 1}, a_1, a_2, \cdots]/(a_i^{p^n} - a_i). \tag{5.1}$$

This is the Hopf algebra analyzed by Ravenel in Chapter 6 of [44]. The automorphisms of the the p^k buds are displayed as

$$\mathrm{Aut}_{\mathbb{F}}(\Gamma_n)_{p^k} = \mathrm{Spec}(F\mathbb{F}_p[a_0^{\pm 1}, a_1, \cdots, a_k]/(a_i^{p^n} - a_i).$$

In the infinite height case, the failure to be étale can be easily seen: if we take $\Gamma_\infty = \hat{\mathbb{G}}_a$ with its standard coordinate, then

$$\mathbb{F}_p \otimes_L W \otimes_L \mathbb{F}_p = \mathbb{F}_p[a_0^{\pm 1}, a_1, a_2, \cdots]. \tag{5.2}$$

This is closely related to the mod p dual Steenrod algebra.

5.27 Proof of the Theorem 5.23. This argument is a rephrasing of an argument I learned from Neil Strickland [48]. But see also [25]. The properties listed – étale, degree, and surjectivity – are all local in the $fpqc$-topology on S; thus we may assume that $S = \mathrm{Spec}(A)$ is affine and that G_1 and G_2 can be given a simultaneous coordinate x. Furthermore, since all of these conditions remain invariant under isomorphisms of the formal groups involved, we may assume that G_1 and G_2 are p-typical. This implies that we may write the p-series of of the formal groups

$$[p]_{G_1}(x) = u_n x^{p^n} +_{G_1} u_{n+1} x^{p^{n+1}} + \cdots$$
$$[p]_{G_2}(x) = u'_n x^{p^n} +_{G_2} u'_{n+1} x^{p^{n+1}} + \cdots$$

and, hence, that the Verschiebungs may be written

$$V_{G_1}(x) = u_n x +_{G_1} u_{n+1} x^p + \cdots$$
$$V_{G_2}(x) = u'_n x +_{G_2} u'_{n+1} x^p + \cdots .$$

Because the formal groups have strict height n, u_n and u'_n are units.

First assume $k = 0$. Then an isomorphism $\phi : (G_1)_p \to (G_2)_p$ of p-typical formal group buds can be written $\phi(x) = b_0 x$ modulo (x^p). Since $V_{G_2}(\phi^{(p^n)}(x)) = \phi(V_{G_1}(x))$ we have $u'_n b_0^{p^n} x = u_n b_0 x$. Since b_0, u_n, and u'_n are all units we get an equation

$$b_0^{p^n-1} - v = 0 \tag{5.3}$$

where $v = u_n/u'_n$ is a unit. Thus,

$$\mathrm{Iso}_S(G_1, G_2)_1 = \mathrm{Spec}(A[b_0]/(b_0^{p^n-1} - v)).$$

This is étale of degree $p^n - 1$ over $\mathrm{Spec}(A)$ since b_0 is a unit in $A[b_0]/(b_0^{p^n-1} - v)$. Surjectivity follows from the fact that $A \to A[b_0]/(b_0^{p^n-1} - v)$ is faithfully flat.

Now assume $k > 0$ and keep the notation above. We make the inductive assumption that $\mathrm{Iso}_S(G_1, G_2)_{k-1} = \mathrm{Spec}(A_{k-1})$ for some A-algebra A_{k-1}. Suppose we have an isomorphism

$$\phi_0(x) : (G_1)_{p^{k-1}} \to (G_2)_{p^{k-1}}$$

of p^{k-1}-buds over some A-algebra R. We want to lift this to an isomorphism

$$\phi : (G_1)_{p^k} \longrightarrow (G_2)_{p^k}$$

so that $\phi \equiv \phi_0$ as isomorphisms of $(G_1)_{p^{k-1}}$ to $(G_2)_{p^{k-1}}$. We may write $\phi(x) = \phi_0(x) +_{G_1} b_k x^{p^k}$. Then again we must have

$$V_{G_2}(\phi^{(p^n)}(x)) = \phi(V_{G_1}(x))$$

and, equating the coefficients of x^{p^k} we get an equation

$$b_k^{p^n} - v b_k + w = 0 \tag{5.4}$$

where $v = u_n^{p^k}/u'_n$ is a unit. Thus,

$$\mathrm{Iso}_S(G_1, G_2)_k = \mathrm{Spec}(A_{k-1}[b_k]/(b_k^{p^n} - vb_k + w)).$$

This is faithfully flat, étale and of degree p^n over $\mathrm{Spec}(A_{k-1})$. $\square$

The projection morphism $\mathrm{Iso}_S(G_1, G_2)_k \to S$ has a right action by the étale group scheme $\mathrm{Aut}_S(G_1)_k \to S$, $1 \leq k \leq \infty$. The action induces a diagram of schemes over $\mathrm{Iso}_S(G_1, G_2)$

$$\begin{tikzcd}
\mathrm{Iso}_S(G_1, G_2)_k \times_S \mathrm{Aut}_S(G_1)_k \arrow[r] \arrow[d, "p_1"'] & \mathrm{Iso}_S(G_1, G_2)_k \times_S \mathrm{Iso}_S(G_1, G_2)_k \arrow[d, "p_1"] \\
\mathrm{Iso}_S(G_1, G_2)_k \arrow[r, "="'] & \mathrm{Iso}_S(G_1, G_2)_k
\end{tikzcd}$$

where the top map is given pointwise by

$$(\phi, \psi) \mapsto (\phi, \phi\psi).$$

This map is evidently an isomorphism; hence we have proven the following result.

5.28 Proposition. *The morphism* $\mathrm{Iso}_S(G_1, G_2)_k \to S$ *is an* $\mathrm{Aut}_S(G_1)_k$*-torsor.*

We can specialize this result even further, but first some notation and definitions.

5.29 Remark. If X is a finite set, define $X_{\mathbb{Z}}$ to be the scheme $\mathrm{Spec}(\mathrm{map}(X, \mathbb{Z}))$. Then for any category Y fibered in groupoids over affine schemes we get a new functor $Y \times X_{\mathbb{Z}}$. If $Y = \mathrm{Spec}(R)$, then

$$Y \times X_{\mathbb{Z}} = \mathrm{Spec}(\mathrm{map}(X, R)) \stackrel{\mathrm{def}}{=} X_R.$$

If G is a finite group, the $\mathbb{G}_{\mathbb{Z}}$ is a finite group scheme over $\mathbb{Z}$ and the action of G on itself extends to a right action on $Y \times G_{\mathbb{Z}}$.

If $X = \lim X_k$ is a profinite set, define

$$X_Z = \lim (X_k)_{\mathbb{Z}} = \mathrm{Spec}(\mathrm{colim}\, \mathrm{map}(X_k, \mathbb{Z})).$$

If $G = \lim G_k$ is a profinite group, then $\mathbb{G}_{\mathbb{Z}}$ is a profinite group scheme over $\mathbb{Z}$.

The notation $G_{\mathbb{Z}}$ is cumbersome; we will drop it if G is evidently a profinite group.

Now suppose $X \to S$ is a finite and étale morphism of schemes; let $\mathrm{Aut}_S(X)$ denote the automorphisms of X over S. This is finite group. Then X is *Galois* over S if the natural map

$$\mathrm{Aut}_S(X) \times_S X \longrightarrow X \times_S X$$

given pointwise by $(\phi, x) \mapsto (x, \phi(x))$ is an isomorphism. If $X = \lim X_k \to S$ where $\{X_k\}$ is a tower of finite and étale maps over S, then X is *pro-Galois* if there there is a coherent set of morphisms $\mathrm{Aut}_S(X) \to \mathrm{Aut}_S(X_k)$ so that

$$\mathrm{Aut}_S(X) = \lim \mathrm{Aut}_S(X_k)$$

and each of the morphism $X_k \to S$ is Galois.

Suppose that Γ is a formal group of height n over a separably closed field $\mathbb{F}$ and let $\mathbb{G}_k(\Gamma)$ be the $\mathbb{F}$-points of $\mathrm{Aut}_{\mathbb{F}}(\Gamma)_k$. If $k < \infty$, then $\mathbb{G}_k(\Gamma)$ has order $p^n k - 1$. The equations 5.3 and 5.4 imply that the natural map

$$\mathbb{G}_k(\Gamma)_{\mathbb{F}} \longrightarrow \mathrm{Aut}_{\mathbb{F}}(\Gamma), \quad k < \infty$$

is an isomorphism. Furthermore,

$$\mathbb{G}(\Gamma) \overset{\mathrm{def}}{=} \mathbb{G}_\infty(\Gamma) \cong \lim \mathbb{G}_k(\Gamma).$$

This displays $\mathbb{G}(\Gamma)$ as a profinite group. Note that the equations 5.3 and 5.4 also imply that

$$\mathbb{G}_1(\Gamma) \cong \mathbb{F}_{p^n}^\times \quad \text{and} \quad \mathbb{G}_k(\Gamma)/\mathbb{G}_{k-1}(\Gamma) \cong \mathbb{F}_{p^n}.$$

5.30 Theorem. *Let S be a scheme over a separably closed field $\mathbb{F}$ and let G_1 and G_2 be two formal groups of strict height n, $1 \leq n < \infty$ over S. Suppose that G_1 obtained by base change from a formal group Γ of height n over $\bar{\mathbb{F}}_p$. Then for all $k < \infty$, the morphism*

$$\mathrm{Iso}_S(G_1, G_2)_k \longrightarrow S$$

is Galois with Galois group $\mathbb{G}_k(\Gamma)$. Finally, the morphism

$$\mathrm{Iso}_S(G_1, G_2) \longrightarrow S$$

is pro-Galois with Galois group $\mathbb{G}(\Gamma)$.

Proof. Let $f : T \to S$ be any morphism of schemes. Then

$$T \times_S \mathrm{Iso}(G_1, G_2)_k \cong \mathrm{Iso}_T(f^*G_1, f^*G_2)_k.$$

In particular

$$\mathrm{Aut}_S(G_1)_k \cong S \times_{\mathrm{Spec}(\mathbb{F})} \mathrm{Aut}_{\mathbb{F}}(\Gamma)_k \cong \mathbb{G}_k(\Gamma)_S$$

and the result now follows from Proposition 5.28. □

The étale extensions we produced in the proof of Theorem 5.23 were of a very particular type. See Equations 5.3 and 5.4. This can be rephrased Proposition 5.32 below, which can be proved by examining the proof just given. Here, however, we give a more conceptual proof, based on the following observation.

If R is an $\mathbb{F}_p$-algebra, let us write $f_R : R \to R$ for the Frobenius homomorphism sending x to x^p. Then $\mathcal{M}$ is any stack over $\mathrm{Spec}(\mathbb{F}_p)$, we get a Frobenius homomorphism

$$f_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M}$$

of stacks over $\mathrm{Spec}(\mathbb{F}_p)$ which, upon evaluating at an $\mathbb{F}_p$ algebra R is given by

$$f_{\mathcal{M}} = \mathcal{M}(f_R) : \mathcal{M}(R) \longrightarrow \mathcal{M}(R).$$

For example if $\mathcal{M}(1) = \mathcal{M}_{\mathbf{fg}} \otimes \mathrm{Spec}(\mathbb{F}_p)$ is the moduli stack of formal group over schemes over $\mathbb{F}_p$ then

$$f_{\mathcal{M}}(G \to S) = G^{(p)} \to S.$$

5.31 Remark (The Frobenius trick). Let $\mathcal{N}(n) = \mathcal{M}(n)[v_n^{\pm 1}]$ be the moduli stack of formal groups of exact height n, with $1 \leq n < \infty$. For all formal groups G of exact height n the *natural* factoring of the morphism $[p] : G \to G$ in Definition 5.2 yields a natural isomorphism

$$V_n = V_n^G : G^{(p^n)} \longrightarrow G.$$

Thus, if $f_{\mathcal{N}} : \mathcal{N}(n) \longrightarrow \mathcal{N}(n)$ is the Frobenius – which, as we have just seen, assigns to each $G \to S$ the formal group $G^{(p)} \to S$ – we get a natural transformation

$$V_n : f_{\mathcal{N}}^n \longrightarrow 1$$

from $f_{\mathcal{N}}^n$ to the identity of $\mathcal{N}(n)$.

5.32 Proposition. *Let $0 \leq k_1 \leq k_2 \leq \infty$. Then the relative Frobenius*

$$\begin{tikzcd} \mathrm{Iso}_S(G_1, G_2)_{k_2} \arrow[rr, "F"] \arrow[dr] & & \mathrm{Iso}_S(G_1, G_2)_{k_2}^{(p)} \arrow[dl] \\ & \mathrm{Iso}_S(G_1, G_2)_{k_1} & \end{tikzcd}$$

is an isomorphism.

Proof. We will first do the absolute case of $\mathrm{Iso}_S(G_1, G_2) \to S$ – that is, $k_1 = 0$ and $k_2 = \infty$ – and indicate at the end of the argument what changes are needed in general.

The scheme $\mathrm{Iso}_S(G_1, G_2)^{(p)}$ over S assigns to each morphism $f : T \to S$ of schemes the set of isomorphisms

$$\psi : f^* G_1^{(p)} \longrightarrow f^* G_2^{(p)}.$$

The relative Frobenius $F : \mathrm{Iso}_S(G_1, G_2) \to \mathrm{Iso}_S(G_1, G_2)^{(p)}$ over S sends a T-point $\phi : f^* G_1 \to f^* G_2$ to the T-point $\phi^{(p)} : f^* G_1^{(p)} \to G_2^{(p)}$. It is this we must show is an isomorphism. However, if we are given ψ, may produce ϕ using the following commutative diagram of isomorphisms

$$\begin{tikzcd}
f^*G_1^{(p^n)} \arrow[r, "\psi^{(p^{n-1})}"] \arrow[d, "V_{G_1}"'] & f^*G_2^{(p^n)} \arrow[d, "V_{G_2}"] \\
f^*G_1 \arrow[r, "\phi"'] & f^*G_2.
\end{tikzcd}$$

As $V_{G^{(p)}} = (V_G)^{(p)}$ we may conclude that $\phi^{(p)} = \psi$ from the diagram

$$\begin{tikzcd}
f^*G_1^{(p)} \arrow[d, "\phi^{(p)}"'] & f^*G_1^{(p^{n+1})} \arrow[l, "V_{G_1}^{(p)}"'] \arrow[r, "V_{G_1}^{(p)}"] \arrow[d, "\psi^{(p^n)}"] & f^*G_1^{(p)} \arrow[d, "\psi"] \\
f^*G_2^{(p)} & f^*G_2^{(p^{n+1})} \arrow[l, "V_{G_2}^{(p)}"] \arrow[r, "V_{G_2}^{(p)}"'] & f^*G_2^{(p)}.
\end{tikzcd}$$

The same proof works in the relative case, but the notation is more complicated. The scheme

$$\mathrm{Iso}_S(G_1, G_2)_{k_2}^{(p)} \longrightarrow \mathrm{Iso}_S(G_1, G_2)_{k_1}$$

consists of pairs (ψ_1, ψ_2) where ψ_1 and ψ_2 are isomorphisms on buds

$$\psi_2 : f^*(G_1)_{p^{k_2}}^{(p)} \longrightarrow f^*(G_2)_{p^{k_2}}^{(p)}$$

and

$$\psi_1 : f^*(G_1)_{p^{k_1}} \longrightarrow f^*(G_2)_{p^{k_1}}.$$

so that

$$\psi_2 \equiv \psi_1^{(p)} : f^*(G_1)_{p^{k_1}}^{(p)} \longrightarrow f^*(G_2)_{p^{k_1}}^{(p)}.$$

The relative Frobenius sends

$$\phi : f^*(G_1)_{p^{k_2}} \longrightarrow f^*(G_2)_{p^{k_2}}$$

to the pair $(\bar{\phi}, \phi^{(p)})$ where $\bar{\phi}$ is the reduction of ϕ. The argument given in the absolute case now adapts to show this is an isomorphism. □

5.33 Proposition. *Let $\mathcal{N}(n) = \mathcal{H}(n)$ if $1 \leq n < \infty$ and let $\mathcal{N}(\infty) = \mathcal{M}(\infty)$. Let*

$$g : \mathrm{Spec}(A) \to \mathcal{N}(n), \qquad 1 \leq n \leq \infty$$

be any morphism. Then g is an fqpc-cover. In particular, g is faithfully flat.

Proof. If $g_i : \mathrm{Spec}(A_i) \to \mathcal{M}(n)[v_n^{\pm 1}]$, then we have a 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r, "f_2"] \arrow[d, "f_1"'] & \mathrm{Spec}(A_2) \arrow[d, "g_2"] \\ \mathrm{Spec}(A_1) \arrow[r, "g_1"'] & \mathcal{M}(n)[v_n^{\pm 1}] \end{tikzcd}$$

where both f_1 and f_2 are faithfully flat: we take $A = A_1 \otimes_{\mathbb{F}_p} A_2$ in Theorem 5.24.

Now take g_1 to be faithfully flat and g_2 to be arbitrary. Then $g_1 f_1$ is faithfully flat and, since f_1 is faithfully flat, g_2 must be faithfully flat as well. Since

$$\mathrm{Spec}(A_1) \otimes_{\mathcal{M}_{\mathbf{fg}}} \mathrm{Spec}(A_2) \to \mathrm{Spec}(A_1)$$

is affine; it follows by descent that $\mathrm{Spec}(A_2) \to \mathcal{M}(n)[v_n^{\pm 1}]$ is affine as well. In particular, it is quasi-compact. □

The following is now immediate. Of course, we will not go wild in our choice of field $\mathbb{F}$, almost always taking an algebraic extension of $\mathbb{F}_p$.

5.34 Corollary. *Let $\mathcal{N}(n) = \mathcal{H}(n)$ if $1 \leq n < \infty$ and let $\mathcal{N}(\infty) = \mathcal{M}(\infty)$. Let $\mathbb{F}$ be a field of characteristic p and $G \to \mathrm{Spec}(\mathbb{F})$ any formal group of height n, $1 \leq n \leq \infty$. Then the classifying map for G*

$$\mathrm{Spec}(\mathbb{F}) \longrightarrow \mathcal{N}(n)$$

is a cover in the fpqc-topology. In particular, $\mathcal{H}(n)$ and $\mathcal{M}(\infty)$ each has a single geometric point.

Now fix a formal group Γ_n over $\mathbb{F}_p$ of height n; for example, the Honda formal group. If $n = \infty$ we may as well fix $\Gamma_n = \hat{\mathbb{G}}_a$, the completion of the additive group. Define $\mathrm{Aut}(\Gamma_n)$ to be the group scheme which assigns to each $\mathbb{F}_p$-algebra $i : \mathbb{F}_p \to A$ the automorphisms of the formal group $i^*\Gamma_n$ over A. See Example 5.26 for a concrete discussion.

5.35 Theorem. *Let $\mathcal{H}(n) = \mathcal{M}(n) - \mathcal{M}(n+1)$ be the open substack of $\mathcal{M}(n)$ complementary to $\mathcal{M}(n+1)$. Then $\mathcal{H}(n)$ has a single geometric point represented by any formal group Γ_n of height n over $\mathbb{F}_p$. Furthermore, the map*

$$\mathcal{H}(n) \longrightarrow B\,\mathrm{Aut}(\Gamma_n)$$

sending a formal group G of height n over an $\mathbb{F}_p$-algebra A to the $\mathrm{Aut}(\Gamma_n)$*-torsor* $\mathrm{Iso}(\Gamma_n, G)$ *is an equivalence of stacks.*

5.36 Theorem. *Let $\mathcal{M}(\infty) = \cap \mathcal{M}(n)$. Then $\mathcal{M}(\infty)$ has a single geometric point represented by the formal additive group $\hat{\mathbb{G}}_a$ over $\mathbb{F}_p$. Furthermore, the map*

$$\mathcal{M}(\infty) \longrightarrow B\,\mathrm{Aut}(\hat{\mathbb{G}}_a)$$

sending a formal group G of infinite height over an $\mathbb{F}_p$-algebra A to the $\mathrm{Aut}(\Gamma_n)$*-torsor* $\mathrm{Iso}(\hat{\mathbb{G}}_a, G)$ *is an equivalence of stacks.*

The apparent choice of the formal group Γ_n makes this result a bit puzzling. This can be rectified by coming to terms with the notion of a gerbe. Here we appeal to [29] §3.15ff.

5.37 Definition. *1.) Let S be a scheme and $X \to S$ a scheme over S. Then a* **gerbe** *over X is a stack $q : \mathcal{G} \to X$ over X with the properties that*

i.) for all affine $U \to S$ and all pairs of morphisms $x_1, x_2 : U \to \mathcal{G}$ so that $qx_1 = qx_2 : U \to X$, there is an faithfully flat covering $f : V \to U$ by an affine so that there is an isomorphism $f^*x_1 \cong f^*x_2$;

ii.) for all affine $U \to S$ and all $f : U \to X$ over S, there is an faithfully flat covering $f : V \to U$ by an affine so that there is a morphism $x : V \to \mathcal{G}$ with $qx = fx$.

2.) A gerbe $q : \mathcal{G} \to X$ is **neutral** *if there is a section $s : X \to \mathcal{G}$ of q.*

The following is exactly Lemma 3.21 of [29]

5.38 Lemma. *Suppose $q : \mathcal{G} \to X$ is a neutral gerbe over X. Then a section s of q determines an equivalence of stacks over X*

$$\mathcal{G} \simeq B\,\mathrm{Aut}_{\mathcal{G}}(s/X)$$

where $\mathrm{Aut}(s/X)$ *is the group scheme which assigns to each $f : U \to X$ the group* $\mathrm{Aut}_{\mathcal{G}}(f^*x \to U)$. *This equivalence sends $g \in \mathcal{G}(U)$ to the torsor* $\mathrm{Iso}(sq(g), g)$.

5.39 Proof of Theorems 5.35 and 5.36. Let $\mathcal{N}(n) = \mathcal{H}(n)$ if $1 \leq n < \infty$ and let $\mathcal{N}(\infty) = \mathcal{M}(\infty)$. We claim that $\mathcal{N}(n) \to \mathrm{Spec}(\mathbb{F}_p)$ is a neutral gerbe. Then the result follows from Lemma 5.38. The two conditions to be gerbe are easily satisfied in this case: (1) any two height n formal groups over an $\mathbb{F}_p$-algebra A become isomorphic after a faithfully flat extension and (2) every $\mathbb{F}_p$ algebra A has a height n formal group over it. Finally the choice of Γ_n shows that we have a neutral gerbe.

5.40 Remark. The *Morava stabilizer group* $\mathbb{S}_n$ is defined to be the $\bar{\mathbb{F}}_p$ points of the algebraic group $\mathrm{Aut}(\Gamma_n)$; that is, if : $i : \mathbb{F}_p \to \bar{\mathbb{F}}_p$ is the inclusion, then $\mathbb{S}_n$ is the automorphisms over $\bar{\mathbb{F}}_p$ of the formal group $i^*\Gamma_n$. By Theorem 5.24, this is independent of the choice of Γ_n. The *big Morava stabilizer group* $\mathbb{G}_n$ is the group of 2-commuting diagrams

$$\begin{tikzcd}
\mathrm{Spec}(\bar{\mathbb{F}}_p) \arrow[rr] \arrow[dr, "i^*\Gamma_n"'] & & \mathrm{Spec}(\bar{\mathbb{F}}_p) \arrow[dl, "i^*\Gamma_n"] \\
& \mathcal{M}_{\mathbf{fg}}. &
\end{tikzcd}$$

There is a semi-direct product decomposition

$$\mathbb{G}_n \cong \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \rtimes \mathbb{S}_n.$$

6 Localizing sheaves at a height n point

In this section we define and discuss the sheaves $\mathcal{F}[v_n^{-1}]$ when $\mathcal{F}$ is an $\mathcal{I}_n$-torsion quasi-coherent sheaf on the moduli stack $\mathcal{M}_{\mathbf{fg}}$ of formal groups. This is largely groundwork for the results on chromatic convergence to be proved later, but we do revisit the Landweber Exact Functor Theorem here, using a proof due to Mike Hopkins [14]. We begin with a discussion of the derived tensor product and derived completions, which – by results of Hovey [18] – have a particularly nice form for the stacks under consideration here.

6.1 Derived tensor products and derived completion

We will want to control the derived tensor product of two quasi-coherent sheaves on an algebraic stack $\mathcal{M}$. While this works particularly well if $\mathcal{M}$ is quasi-compact and separated, for the stacks encountered in homotopy theory we can do even better: by using results of Mark Hovey, it is possible to give completely functorial construction using resolutions by locally free sheaves. This is because we will be able to assume that the category $\mathbf{Qmod}_{\mathcal{M}}$ of quasi-coherent sheaves is generated by the finitely generated locally free sheaves on $\mathcal{M}$. There is no reason to expect this assumption to hold in great generality, of course, but it holds when $\mathcal{M}$ is one the stacks that arises in the chromatic picture. We will discuss this below in Proposition 6.6.

The following a restatement of some of the results of [18] §2, especially Theorem 2.13 of that paper. Indeed, that result is a statement about the cofibrant objects in a model category structure on chain complexes of quasi-coherent sheaves. Weak equivalences can be deduced from point (2) of the statement.

6.1 Proposition. *Let $\mathcal{M}$ be an algebraic stack so that the finitely generated locally free sheaves generate the category of $\mathbf{Qmod}_{\mathcal{M}}$ of quasi-coherent sheaves on $\mathcal{M}$. Then for any chain complex of quasi-coherent sheaves $\mathcal{F}$ on $\mathcal{M}$ there is a natural quasi-isomorphism of chain complexes*

$$\mathcal{P}_\bullet^{\mathcal{M}}(\mathcal{F}) = \mathcal{P}_\bullet \to \mathcal{F}_\bullet$$

with the properties that

1. *for all n, the sheaf $\mathcal{P}_n$ is a coproduct of finitely generated locally free sheaves;*

2. *for all finitely generated locally free sheaves $\mathcal{V}$ on $\mathcal{M}$ the morphism of function complexes*

$$\mathrm{Hom}(\mathcal{V}, \mathcal{P}_\bullet) \longrightarrow \mathrm{Hom}(\mathcal{V}, \mathcal{F}_\bullet)$$

   is a quasi-isomorphim.

Let $\mathcal{F}$ be a sheaf on $\mathcal{M}$. In much of the sequel we will write $\mathcal{F}(R)$ for $\mathcal{F}(\mathrm{Spec}(R) \to \mathcal{M})$. We note that the tensor product of quasi-coherent sheaves behaves particularly well.

6.2 Lemma. *Let $\mathcal{E}$ and $\mathcal{F}$ be two quasi-coherent sheaves in the fpqc-topology on an algebraic stack $\mathcal{M}$. Then presheaf $\mathcal{E} \otimes \mathcal{F} = \mathcal{E} \otimes_{\mathcal{O}} \mathcal{F}$ which assigns to each flat and quasi -compact morphism* $\mathrm{Spec}(R) \to \mathcal{M}$ *the tensor product*

$$\mathcal{E}(R) \otimes_R \mathcal{F}(R)$$

is a quasi-coherent sheaf.

Proof. To see that we actually have a sheaf, let $R \to S$ be faithfully flat extension. Then

$$\mathcal{E}(R) \otimes_R \mathcal{F}(R) \longrightarrow \mathcal{E}(S) \otimes_S \mathcal{F}(S) \rightrightarrows \mathcal{E}(S \otimes_R S) \otimes_{(S \otimes_R S)} \mathcal{F}(S \otimes_R S)$$

is, up to isomorphism,

$$\mathcal{E}(R) \otimes_R \mathcal{F}(R) \longrightarrow S \otimes_R (\mathcal{E}(R) \otimes_R \mathcal{F}(R)) \rightrightarrows S \otimes_R S \otimes_R (\mathcal{E}(R) \otimes_R \mathcal{F}(R)).$$

If we apply $S \otimes_R (-)$ to the later sequence it becomes exact, as it has a retraction. Since $R \to S$ is faithfully flat, it was exact to begin with. This proves we have a sheaf; it is quasi-coherent because (as we have already noted)

$$S \otimes_R \mathcal{E}(R) \otimes_R \mathcal{F}(R) \xrightarrow{\cong} \mathcal{E}(S) \otimes_S \mathcal{F}(S).$$

$\square$

6.3 Definition. *Suppose $\mathcal{M}$ is an algebraic stack in the fqpc-topology so that the finitely generated locally free sheaves generated* $\mathbf{Qmod}_{\mathcal{M}}$*. Let $\mathcal{E}$ and $\mathcal{F}$ be two quasi-coherent sheaves on $\mathcal{M}$. Define their* **derived tensor product** $\mathcal{E} \otimes^L \mathcal{F} = \mathcal{E} \otimes^L_{\mathcal{O}_{\mathcal{M}}} \mathcal{F}$ *to be the chain complex of quasi-coherent sheaves (for the fpqc-topology) with values at* $\mathrm{Spec}(R) \to \mathcal{M}$ *given by*

$$\mathcal{E}(R) \otimes_R \mathcal{P}_\bullet(R)$$

where $\mathcal{P}_\bullet \to \mathcal{F}$ is the natural resolution of Proposition 6.1.

Many of the usual properties of tensor product apply to this construction. For example, if

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

is a short exact sequence of quasi-coherent sheaves, then we get a distinquished triangle in the derived category of quasi-coherent sheaves

$$\mathcal{E}_1 \otimes^L \mathcal{F} \to \mathcal{E}_2 \otimes^L_{\mathcal{O}_{\mathcal{M}}} \mathcal{F} \to \mathcal{E}_3 \otimes^L_{\mathcal{O}_{\mathcal{M}}} \mathcal{F} \to (\mathcal{E}_1 \otimes^L_{\mathcal{O}_{\mathcal{M}}} \mathcal{F})[-1].$$

This definition and the distinguished triangle generalize to the case when $\mathcal{E}$ and $\mathcal{F}$ are bounded below complex chain complexes of quasi-coherent sheaves.

Closely related to the derived tensor product is the derived completion.

6.4 Definition. *Let $\mathcal{Z} \subseteq \mathcal{M}$ be a closed substack defined by a quasi-coherent ideal sheaf $\mathcal{I}$. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{M}$. Then the* **derived completion** *of $\mathcal{F}$ at $\mathcal{Z}$ by*

$$L(\mathcal{F})^{\wedge}_{\mathcal{I}} = L(\mathcal{F})^{\wedge}_{\mathcal{Z}} = \mathrm{holim}(\mathcal{O}/\mathcal{I}^n \otimes^L \mathcal{F}).$$

Thus, if $\mathrm{Spec}(R) \to \mathcal{M}$ is faithfully flat and quasi-compact, we can set

$$L(\mathcal{F})^{\wedge}_{\mathcal{Z}}(R) = \lim P_{\bullet}(R)/\mathcal{I}^n(R)\mathcal{P}_{\bullet}(R).$$

This is an $\mathcal{O}$-module sheaf, but not necessarily quasi-coherent, as inverse limit and tensor product need not commute. If $j_n : \mathcal{Z}^{(n)} \subseteq \mathcal{M}$ are the infinitesimal neighborhoods of $\mathcal{Z}$ defined by the powers of $\mathcal{I}$, then

$$L(\mathcal{F})^{\wedge}_{\mathcal{Z}} = \mathrm{holim}(j_n)_*(Lj_n)^*\mathcal{F}$$

where $(Lj_n)^*$ is the total left derived functor of j_n^*.

We now turn to the question of when the hypotheses of Proposition 6.1 apply. There is a classical and useful notion from stable homotopy theory which guarantees that the finitely generated locally free sheaves generate the category of quasi-coherent sheaves.

6.5 Definition. *1.) A Hopf algebroid (A, Λ) is an* **Adams Hopf algebroid** *if the left unit $A \to \Lambda$ is flat and the (A, Λ)-comodule Λ can be written as a filtered colimit of comodules Λ_i each of which is a finitely generated projective A-module.*

2.) An algebraic stack $\mathcal{M}$ will be an **Adams stack** *if there is an fpqc-presentation $\mathrm{Spec}(A) \to \mathcal{M}$ so that*

$$\mathrm{Spec}(A) \times_{\mathcal{M}} \mathrm{Spec}(A) \cong \mathrm{Spec}(\Lambda)$$

is itself affine and the resulting Hopf algebroid (A, Λ) is an Adams Hopf algebroid.

This definition has a curious and unfortunate feature. We would like to assert that if $\mathcal{M}$ has one *fpqc*-presentation $\mathrm{Spec}(A) \to \mathcal{M}$ so that (A, Λ) is an Adams Hopf algebroid then any other *fpqc* presentation would have the same property. But this is not known. See [18], Question 1.4.2.[5] However, we do have the following rephrasing of Proposition 1.4.4 of [18].

6.6 Proposition. *Let $\mathcal{M}$ be an Adams stack. The the finitely generated locally free sheaves generate the category* $\mathbf{Qmod}_{\mathcal{M}}$ *of quasi-coherent sheaves on $\mathcal{M}$.*

We now make good on our claim that most of the stacks in this monograph are of this kind.

[5] This problem could be avoided by working with resolutions by appropriately flat modules; these exist over any quasi-compact and separated stack. See [1] §1. I have chosen to use the Adams condition only because it fits better with the culture of homotopy theory.

6.7 Proposition. *For all n, $0 \leq n \leq \infty$, the moduli stack $\mathcal{M}_{\mathbf{fg}}\langle n\rangle$ of n-buds of formal groups is an Adams stack.*

Proof. We show that the evident presentation

$$\mathrm{Spec}(L_n) \longrightarrow \mathcal{M}_{\mathbf{fg}}\langle n\rangle$$

has the desired property. Recall from Proposition 3.1 that

$$\mathrm{Spec}(L_n) \times_{\mathcal{M}_{\mathbf{fg}}\langle n\rangle} \mathrm{Spec}(L_n) \cong \mathrm{Spec}(W_n) \cong \mathbf{fgl}_n \times \Lambda_n.$$

While we've not emphasized the fact, the algebraic groups Λ_n and the schemes $\mathbf{fgl}_n$ have an evident action of the multiplicative group $\mathbb{G}_m$ – and, hence, a grading on the rings of functions. Indeed, if $a \in R^\times = \mathbb{G}_m(R)$ and $f(x) = a_0x + \cdots a_{n-1}x^n \in \Lambda_n(R)$, then we define $(af)(x)$ by

$$(af)(x) = a^{-1}f(ax).$$

Similarly if $x +_F y \in \mathbf{fgl}_n(R)$, then we get a new element of $\mathbf{fgl}_n(R)$ via the formula

$$x +_{aF} y = a^{-1}(ax +_F ay).$$

Writing $\Lambda_n = \mathrm{Spec}(\mathbb{Z}[a_0^{\pm 1}, \ldots, a_{n-1}])$ and $L_n = \mathrm{Spec}(\mathbb{Z}[x_1, \cdots, x_{n-1}]$ we immediately have that the degree of a_i is i and the degree of x_i is i.

The diagonal on W_n is induced from the diagonal of $\mathbb{Z}[a_0^{\pm 1}, \ldots, a_{n-1}]$ which is determined by the composition of power series. Let

$$A_{n,i} \subseteq \mathbb{Z}[a_1, \ldots, a_{n-1}]$$

be the elements of degree less than or equal to i, let

$$B_{n,i} = \bigoplus_{-i \leq s \leq i} A_{n,i}a_0^s \subseteq \mathbb{Z}[a_0^{\pm 1}, \ldots, a_{n-1}]$$

and let

$$W_{n,i} = L_n \otimes B_{n,i}.$$

Then $W_{n,i}$ is a finitely generated free L_n module, a sub-comodule of W_n, and $\mathrm{colim}_i\, W_{n,i} = W_n$. □

6.8 Proposition. *The following stacks are Adams stacks.*

1. *$\mathcal{M}(n)$, the closed substack of $\mathcal{M}_{\mathbf{fg}} \otimes \mathbb{Z}_{(p)}$ of formal groups of height at least n;*

2. *$\mathcal{H}(n) = \mathcal{M}(n)[v^{-1}]$, the open substack of $\mathcal{M}(n)$ of formal groups of exactly height n;*

3. *$\mathcal{U}(n)$, the open substack of $\mathcal{M}_{\mathbf{fg}} \otimes \mathbb{Z}_{(p)}$ of formal groups of height at most n.*

Proof. Because we have base-changed over $\mathbb{Z}_{(p)}$, we can choose the morphism

$$\operatorname{Spec}(\mathbb{Z}_{(p)}[u_1, u_2, \cdots]) \to \mathcal{M}_{\mathbf{fg}} \otimes \mathbb{Z}_{(p)}$$

representing the universal p-typical formal group as the presentation. Then we have presentations

$$\operatorname{Spec}(\mathbb{F}_p[u_n, u_{n-1}, \cdots]) \to \mathcal{M}(n)$$

and

$$\operatorname{Spec}(\mathbb{F}_p[u_n^{\pm 1}, u_{n-1}, \cdots]) \to \mathcal{H}(n)$$

and

$$\operatorname{Spec}(\mathbb{Z}_{(p)}[u_1, \ldots, u_{n-1}, u_n^{\pm 1}]) \to \mathcal{U}(n).$$

Then we appeal to Theorem 1.4.9 and Proposition 1.4.11. of [18]. $\square$

6.2 Torsion modules and inverting v_n

In the next section on Landweber exactness, and later when we discuss chromatic convergence, we are are going to need some technical lemmas about inverting v_n for $\mathcal{I}_n$-torsion sheaves on $\mathcal{M}_{\mathbf{fg}}$. We begin with some definitions so that we can work in some generality with algebraic stacks $\mathcal{N}$ flat over $\mathcal{M}_{\mathbf{fg}}$. The following definition generalizes the definition of regular scale given in [33].

6.9 Definition. *Let $\mathcal{N}$ be an algebraic stack and*

$$0 = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \cdots \subseteq \mathcal{O}_{\mathcal{N}}$$

be an ascending sequence of ideal sheaves. Then the sequence $\{\mathcal{J}_n\}$ forms a **regular scale** *for $\mathcal{N}$ if for all n, the ideal sheaf $\mathcal{J}_{n+1}/\mathcal{J}_n$ is locally free of rank 1 as a $\mathcal{O}/\mathcal{J}_n$ module. A regular scale is a* **finite** *if $\mathcal{J}_n = \mathcal{O}$ for some n.*

6.10 Remark. Given a regular scale on $\mathcal{N}$, let $\mathcal{N}(n)$ denote that closed substack defined by $\mathcal{J}_n$. Then $\mathcal{N}(n) \subseteq \mathcal{N}(n-1)$ is an effective Cartier divisor for $\mathcal{N}(n-1)$. An embedding $\mathcal{Z} \subseteq \mathcal{N}$ of a closed substack is called *regular* if the ideal defining the embedding is locally generated by a regular sequence. Thus a regular scale produces regular emeddings $\mathcal{N}(n) \subseteq \mathcal{N}$, but it is does more: it specifies the terms in the regular sequence modulo the lower terms.

6.11 Example. Fix a prime p, let $\mathcal{M} = \mathcal{M}_{\mathbf{fg}}$ and let

$$0 \subseteq \mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{O}_{\mathbf{fg}}$$

be the ascending chain of ideals giving the closed substacks $\mathcal{M}(n) \subseteq \mathcal{M}_{\mathbf{fg}}$ classifying formal groups of height greater than or equal to n. This, of course, is the basic example of a regular scale. This scale is not finite; however, if we let $i_n : \mathcal{U}(n) \to \mathcal{M}_{\mathbf{fg}}$ be the open substack classifying formal groups of height less than or equal to n, then

$$i_n^*\mathcal{I}_0 \subseteq i_n^*\mathcal{I}_1 \subseteq i_n^*\mathcal{I}_2 \subseteq \cdots \subseteq i_n^*\mathcal{O}_{\mathbf{fg}} = \mathcal{O}_{\mathcal{U}(n)}$$

is a finite regular scale as $i_n^*\mathcal{I}_n = i_n^*\mathcal{I}_k = \mathcal{O}_{\mathcal{U}(n)}$ for $k \geq n$.

This example can be generalized to stacks $\mathcal{N}$ representable and flat over $\mathcal{M}_{\mathbf{fg}}$. See Proposition 5.10.

We now come to torsion modules and inverting v_n. Let $\mathcal{N}$ be an algebraic stack and let $\{\mathcal{J}_n\}$ be a scale for $\mathcal{N}$. Let $j_n : \mathcal{N}(n) \subseteq \mathcal{N}$ be the closed inclusion defined by $\mathcal{J}_n$ and let $i_{n-1} : \mathcal{V}(n-1) \to \mathcal{M}_{\mathbf{fg}}$ be the open complement. (The numerology is chosen to agree with case of $\mathcal{I}_n \subseteq \mathcal{O}_{\mathbf{fg}}$.) Let's write $\mathcal{O}$ for $\mathcal{O}_{\mathcal{N}}$.

6.12 Definition. *An $\mathcal{O}$-module sheaf $\mathcal{F}$ is* **supported on** $\mathcal{N}(n)$ *if $i_{n-1}^*\mathcal{F} = 0$. We also say that $\mathcal{F}$ is $\mathcal{J}_n$-***torsion** *if for any flat and quasi-compact morphism* $\operatorname{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$*, the R-module $\mathcal{F}(R)$ is $\mathcal{I}_n(R)$-torsion.*

In Definition 6.12 we do not assume that $\mathcal{F}$ is quasi-coherent; however, the next result shows that the two notions defined there are equivalent for quasi-coherent sheaves.

6.13 Lemma. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}$-module sheaf. Then $\mathcal{F}$ is supported on $\mathcal{N}(n)$ if and only if $\mathcal{F}$ is $\mathcal{J}_n$-torsion.*

Proof. This is a consequence of the fact that $\mathcal{J}_n$ defines a regular embedding. For each flat and quasi-compact morphism $\operatorname{Spec}(R) \to \mathcal{M}_{\mathbf{fg}}$, choose – by passing to a faithfully flat extension if necessary – generators $(p, u_1, \ldots, u_{n-1})$ of $\mathcal{J}_n(R)$.

First suppose $\mathcal{F}$ is supported on $\mathcal{N}(n)$. Then there are commutative diagrams

$$\begin{tikzcd} \operatorname{Spec}(R[u_i^{-1}]) \arrow[r, "\subseteq"] \arrow[d] & \operatorname{Spec}(R) \arrow[d] \\ V(n-1) \arrow[r, "\subseteq"] & \mathcal{N}. \end{tikzcd}$$

Thus $R[u_i^{-1}] \otimes_R \mathcal{F}(R) \cong \mathcal{F}(R[u_i^{-1}]) = 0$, and we may conclude that $\mathcal{F}(R)$ is $\mathcal{J}_n(R)$-torsion.

Conversely, suppose $\mathcal{F}$ is $\mathcal{J}_n$-torsion and $\operatorname{Spec}(R) \to V(n-1)$ is any flat and quasi-compact morphism. Then $\mathcal{J}_n(R) = R$, so $\mathcal{J}_n(R)^k = R$ for all $k > 0$. If $x \in \mathcal{F}(R)$, then $Rx = \mathcal{J}_n(R)^k x = 0$ for some k, whence $x = 0$. Thus $i_{n-1}^*\mathcal{F} = 0$. $\square$

6.14 Lemma. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{J}_n$-torsion sheaf. Then the evaluation defines a natural isomorphism*

$$\operatorname{colim} \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathcal{J}_n^k, \mathcal{F}) \xrightarrow{\cong} \mathcal{F}.$$

If $f_k : \mathcal{N}(n)_k \subseteq \mathcal{N}$ is the inclusion of the kth infinitesimal neighborhood of $\mathcal{N}(n)$ defined by the vanishing of $\mathcal{I}_n^k$, then there is a quasi-coherent sheaf $\mathcal{F}_k$ on $\mathcal{N}(n)_k$ and a natural isomorphism

$$(f_k)_*\mathcal{F}_k \cong \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}/\mathcal{I}_n^k, \mathcal{F}).$$

Proof. The first statement can be check locally, and there it follows from the fact that $\mathcal{J}_n$ if finitely generated. For the second statement, we use the fact that any closed inclusion is affine (see 1.10). From this it follows that $(f_k)_*$ induces an equivalence between the categories of quasi-coherent $\mathcal{O}/\mathcal{J}_n^k$-modules on $\mathcal{N}$ and the category of quasi-coherent modules on $\mathcal{N}(n)_k$. See Proposition 1.11. $\square$

Suppose $\mathcal{F}$ is a quasi-coherent $\mathcal{J}_n$-torsion sheaf. The Lemma 6.13 implies that $i_{n-1}^*\mathcal{F} = 0$. We next consider $i_n^*\mathcal{F}$ or, more exactly, the resulting push-forward $(i_n)_*i_n^*\mathcal{F}$, which is a sheaf on $\mathcal{N}$. The next result shows that the natural map

$$(i_n)_*i_n^*\mathcal{F} \to R(i_n)_*i_n^*\mathcal{F}$$

is an equivalence and gives a local description of $(i_n)_*i_n^*\mathcal{F}$.

6.15 Proposition. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{J}_n$-torsion sheaf on $\mathcal{N}$. Let* $\mathrm{Spec}(R) \to \mathcal{N}$ *be any flat and quasi-compact morphism so that $\mathcal{J}_n(R)/\mathcal{J}_{n-1}(R)$ is free of rank one over $R/\mathcal{J}_{n-1}(R)$. Then we have an isomorphism*

$$[(i_n)_*i_n^*\mathcal{F}](R) \cong \mathcal{F}[u_n^{-1}]$$

where $u_n \in \mathcal{J}_n(R)$ is any element so that $u_n + \mathcal{J}_{n-1}(R)$ generates the R-module $\mathcal{J}_n(R)/\mathcal{J}_{n-1}(R)$. Furthermore,

$$R^s(i_n)_*i_n^*\mathcal{F} = 0, \;\; s > 0.$$

Proof. By Lemma 6.14 and a colimit argument, we may assume that $\mathcal{F} = f_*\mathcal{E}$ for some quasi-coherent sheaf $\mathcal{E}$ on the kth infinitesimal neighborhood $f : \mathcal{N}(n)_k \to \mathcal{N}$ of $\mathcal{N}(n)$.

Consider the pull-back square

$$\begin{tikzcd}
\mathcal{N}(n)_k \times_{\mathcal{N}} \mathcal{V}(n) \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{V}(n) \arrow[d, "i_n"] \\
\mathcal{N}(n)_k \arrow[r, "f"'] & \mathcal{N}.
\end{tikzcd}$$

(In the case where $\mathcal{N} = \mathcal{M}_{\mathbf{fg}}$ and $\mathcal{J}_n = \mathcal{I}_n$, we have that $\mathcal{N}(n)_k \times_{\mathcal{N}} \mathcal{V}(n)$ is the kth infinitesimal neighborhood of $\mathcal{H}(n)$.) Then $i_n^*(f_k)_*\mathcal{E} = (p_2)_*p_1^*\mathcal{E}$; thus, we may conclude that we have an equivalence in the derived category

$$R(i_n)_*i_n^*f_*\mathcal{E} \simeq f_*R(p_1)_*p_1^*\mathcal{E}. \tag{6.1}$$

The open inclusion $\mathcal{N}(n)_k \times_{\mathcal{N}} \mathcal{V}(n) \subseteq \mathcal{N}(n)_k$ is the complement of the closed inclusion $\mathcal{N}(n+1) \subseteq \mathcal{N}(n)_k$. Locally, this closed inclusion is defined by the vanishing of u_n. We see that this implies

$$R(p_1)_*p_1^*\mathcal{E} \simeq (p_1)_*p_1^*\mathcal{E} \cong \mathcal{E}[u_n^{-1}]. \tag{6.2}$$

The result now follows because f_* is exact. $\square$

6.16 Remark. Now let $f : \mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ be a representable and flat morphism of algebraic stacks and let $\{\mathcal{J}_n\} = \{f^*\mathcal{I}_n\}$ be the resulting scale. See Proposition 5.10. Regard v_n as a global section of ω^{p^n-1} considered as a sheaf over $\mathcal{N}(n)$. Suppose $\mathcal{F}$ is actually an $\mathcal{O}/\mathcal{J}_n$-module sheaf; that is, suppose $\mathcal{F} = (j_n)_*\mathcal{E}$

for some quasi-coherent sheaf $\mathcal{E}$ on $\mathcal{N}(n)$. Then we can form the colimit sheaf $\mathcal{F}[v_n^{-1}]$ of the sequence

$$\mathcal{F} \xrightarrow{v_n} \mathcal{F} \otimes \omega^{p^n-1} \xrightarrow{v_n} \mathcal{F} \otimes \omega^{2(p^n-1)} \xrightarrow{v_n} \cdots .$$

We claim that $\mathcal{F}[v_n^{-1}] \cong (i_n)_* i_n^* \mathcal{F}$.

By Equations 6.1 and 6.2, and because $(j_n)_*$ is exact, it is sufficient to show

$$\mathcal{E}[v_n^{-1}] \cong (p_1)_* p_1^* \mathcal{E}.$$

Since multiplication by v_n is invertible for sheaves on

$$\mathcal{N}(n) \times_{\mathcal{N}} V(n) \cong \mathcal{H}(n) \times_{\mathcal{M}_{\mathbf{fg}}} \mathcal{N},$$

the natural map $\mathcal{E} \to (p_1)_* p_1^* \mathcal{E}$ factors as a map $\mathcal{E}[v_n^{-1}] \to (p_1)_* p_1^* \mathcal{E}$. To show this is an isomorphism we need only work locally. Let

$$G : \operatorname{Spec}(R) \longrightarrow \mathcal{M}(n)$$

be a flat and quasi-compact morphism classifying a formal group G. Taking a faithfully flat extension if needed, we may choose an invariant derivation $u \in \omega_G^{-1}$ for G generating the free R-module ω_G^{-1}; then the element

$$u_n \overset{\text{def}}{=} u^{(p^n-1)} v_n(G) \in R = \omega_G^0$$

generates $\mathcal{J}_n(R) = \mathcal{J}_n(R)/\mathcal{J}_{n-1}(R)$. Then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}(G) \arrow[r, "v_n"] \arrow[d, "="'] & \mathcal{E}(G) \otimes \omega^{p^n-1} \arrow[r, "v_n"] \arrow[d, "u^{p^n-1}"] & \mathcal{E}(G) \otimes \omega^{2(p^n-1)} \arrow[r, "v_n"] \arrow[d, "u^{2(p^n-1)}"] & \cdots \\
\mathcal{E}(G) \arrow[r, "u_n"] & \mathcal{E}(G) \arrow[r, "u_n"] & \mathcal{E}(G) \arrow[r, "u_n"] & \cdots .
\end{tikzcd}$$

Since the vertical maps are isomorphisms, the claim follows from Proposition 6.15.

This observation can be easily be generalized to the case where $\mathcal{F}$ is an $\mathcal{O}/\mathcal{J}_n^k$-module sheaf for any $k \geq 1$ because a power of v_n is a global section of the appropriate power of ω.

Because of the previous remark, we feel moved to make the following definition.

6.17 Definition. *Let $f : \mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ be a representable and flat morphism of algebraic stacks and let $\{\mathcal{J}_n\} = \{f^*\mathcal{I}_n\}$ be the resulting scale for $\mathcal{N}$. If $\mathcal{F}$ be a quasi-coherent $\mathcal{J}_n$-torsion sheaf on $\mathcal{N}$ define*

$$\mathcal{F}[v_n^{-1}] = (i_n)_* i_n^* \mathcal{F}.$$

6.3 LEFT: A condition for flatness

Let $f : \mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ be a representable morphism of algebraic stacks. We would like to give a concrete and easily checked condition on this morphism to guarantee that it be flat. This condition is a partial converse to Proposition 5.10 and a version of the Landweber Exact Functor Theorem (LEFT). This theorem has a variety of avatars; the one we give here is due to Hopkins and Miller. See [14] and [33]. The original source is [28].

In this section we will work over $\mathcal{M}_{\mathbf{fg}}$ over $\mathrm{Spec}(\mathbb{Z})$, rather than at a given prime.

Now let $f : \mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ be a representable, quasi-compact, and quasi-separated morphism of stacks. (The hypotheses on the morphism guarantee that if $\mathcal{F}$ is a quasi-coherent sheaf on $\mathcal{N}$, then $f_*\mathcal{F}$ is a quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}$.) Let $\mathcal{J}_n \subseteq \mathcal{O}_{\mathcal{N}}$ be the kernel of the morphism

$$\mathcal{O}_{\mathcal{N}} = f^*\mathcal{O}_{\mathbf{fg}} \to f^*(\mathcal{O}_{\mathbf{fg}}/\mathcal{I}_n).$$

Thus $\mathcal{J}_n$ defines the closed inclusion

$$j_n : \mathcal{N}(n) = \mathcal{M}(n) \times_{\mathcal{M}_{\mathbf{fg}}} \mathcal{N} \xrightarrow{\subseteq} \mathcal{N}.$$

Thus there is a surjection $f^*\mathcal{I}_n \to \mathcal{J}_n$ which becomes an isomorphism if f is flat. Also note that

$$(j_n)_*\mathcal{O}_{\mathcal{N}(n)} = \mathcal{O}_{\mathcal{N}}/\mathcal{J}_n = \mathcal{O}_{\mathcal{N}}/f^*\mathcal{I}_n.$$

From this we can conclude that the form v_n defines a surjection

$$v_n : \mathcal{O}_{\mathcal{N}}/\mathcal{J}_{n-1} \to \mathcal{J}_n/\mathcal{J}_{n-1} \otimes \omega^{p^n-1}.$$

This includes the case $n = 0$; we set $v_0 = p$. The basic criterion of flatness is the following. Note that if $\mathcal{N}$ is a stack over $\mathbb{Z}_{(\ell)}$ for some prime ℓ, then the hypotheses automatically true for all prime $p \neq \ell$. This remark will have a variant for all our other versions of Landweber exactness below.

6.18 Theorem (Landweber Exactness I). *Let $f : \mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ be a representable, quasi-compact, and quasi-separated morphism of stacks. Suppose that for all primes p,*

1. *$v_n : \mathcal{O}_{\mathcal{N}}/\mathcal{J}_{n-1} \to \mathcal{J}_n/\mathcal{J}_{n-1} \otimes \omega^{p^n-1}$ is an isomorphism, and*
2. *$\mathcal{J}_n = \mathcal{O}_{\mathcal{N}}$ for large n.*

Then f is flat. Conversely, if for all primes p, $\mathcal{J}_n = \mathcal{O}_{\mathcal{N}}$ for some n, then f is flat only if condition (1) holds.

6.19 Remark. The hypotheses of Theorem 6.18 imply, in particular, that the ideals $\mathcal{J}_n$ form a finite regular scale for $\mathcal{N}$; in particular, in the descending chain of closed substacks

$$\cdots \subseteq \mathcal{N}(n) \subseteq \mathcal{N}(n-1) \subseteq \cdots \subseteq \mathcal{N}(1) \subseteq \mathcal{N}$$

each of the inclusions is that of an effective Cartier divisor and that there is an n so that $\mathcal{N}(k)$ is empty for $k > n$. Furthermore, an inductive argument shows that the natural surjections $f^*\mathcal{I}_n \to \mathcal{J}_n$ are, in fact, isomorphisms. Indeed, if $f^*\mathcal{I}_{n-1} \cong \mathcal{J}_{n-1}$, then we obtain a diagram

$$\begin{tikzcd} & f^*\mathcal{I}_n/f^*\mathcal{I}_{n-1} \arrow[dd] \\ \mathcal{O}_{\mathcal{N}}/f^*\mathcal{I}_{n-1} \arrow[ur, two heads, "v_n"] \arrow[dr, "\cong", "v_n"'] & \\ & \mathcal{J}_n/f^*\mathcal{I}_{n-1} \end{tikzcd}$$

and we can conclude $f^*\mathcal{I}_n/f^*\mathcal{I}_{n-1} \to \mathcal{J}_n/\mathcal{J}_{n-1}$ is an isomorphism.

By specializing to the affine case and using Remark 5.7, we obtain a more classical version of Landweber exactness.

6.20 Corollary. *Let $g : \mathrm{Spec}(A) \to \mathcal{M}_{\mathbf{fg}}$ classify a formal group G with a coordinate x. For all primes p, let $p, u_1, u_2, \ldots$ be elements of A so that the p-series can be written*

$$[p](x) = u_k x^{p^k} + \cdots$$

modulo $(p, u_1, \ldots, u_{k-1})$. Suppose the elements $p, u_1, \ldots$ form a regular sequence and suppose there is some n so that

$$(p, u_1, \ldots, u_{n-1}) = A$$

Then g is flat.

In Landweber's original paper [28] the hypothesis that $\mathcal{I}_n(G) = A$ for some n was not required.

6.21 Remark. We can reformulate the hypotheses of Theorem 6.18 as conditions on the quasi-coherent algebra sheaf $f_*\mathcal{O}_{\mathcal{N}}$ on $\mathcal{M}_{\mathbf{fg}}$. As a matter of notation, let's write

$$\mathcal{F}/\mathcal{I}_n \stackrel{\mathrm{def}}{=} (j_n)_* j_n^* \mathcal{F}$$

for any $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}$. We will say that the regular scale $\{\mathcal{I}_n\}$ acts **regularly and finitely** on $\mathcal{F}$ if for all n

$$v_n : \mathcal{F}/\mathcal{I}_n \longrightarrow \mathcal{F}/\mathcal{I}_n \otimes \omega^{p^n-1}$$

is injective and $\mathcal{F}/\mathcal{I}_n = 0$ for large n. Because have a pull-back square for all n

$$\begin{tikzcd} \mathcal{M}(n) \times_{\mathcal{M}_{\mathbf{fg}}} \mathcal{N} \arrow[r] \arrow[d] & \mathcal{N} \arrow[d, "f"] \\ \mathcal{M}(n) \arrow[r, "j_n"'] & \mathcal{M}_{\mathbf{fg}} \end{tikzcd}$$

we have that $f_*(\mathcal{O}_\mathcal{N}/\mathcal{J}_n) = (f_*\mathcal{O}_\mathcal{N})/\mathcal{I}_n$.

Suppose the hypotheses of Theorem 6.18 hold. Then

$$v_n : \mathcal{O}_\mathcal{N}/\mathcal{J}_{n-1} \to \mathcal{O}_n/\mathcal{J}_{n-1} \otimes \omega^{p^n-1}$$

is injective and $\mathcal{O}_\mathcal{N}/\mathcal{J}_n = 0$ for large n. Since f_* is left exact, we have that $\{\mathcal{I}_n\}$ acts regularly and finitely on $f_*\mathcal{O}_\mathcal{N}$.

Conversely, suppose $\{\mathcal{I}_n\}$ acts regularly and finitely on $f_*\mathcal{O}_\mathcal{N}$. We will see below in Theorem 6.25 that this implies that f is a flat morphism, which in turn implies that $f^*\mathcal{I}_n = \mathcal{J}_n$ and, hence, that

$$f^*(\mathcal{I}_n/\mathcal{I}_{n-1}) = \mathcal{J}_n/\mathcal{J}_{n-1}.$$

This, in its turn, implies that the hypotheses of Theorem 6.18 hold. Thus, Theorem 6.18 is equivalent to the following result.

6.22 Theorem (Landweber Exactness II). *Let $f : \mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ be a representable, quasi-compact, and quasi-separated morphism of stacks. Suppose that for all primes, the set of ideals $\{\mathcal{I}_n\}$ acts regularly and finitely on $f_*\mathcal{O}_\mathcal{N}$. Then f is flat.*

Conversely, if for all primes p, $f_*\mathcal{O}_\mathcal{N}/\mathcal{I}_n = 0$ for some n, then f is flat only if the set of ideals $\{\mathcal{I}_n\}$ acts regularly and finitely on $f_*\mathcal{O}_\mathcal{N}$.

This, in turn, is a corollary Proposition 5.10 and the following result.

6.23 Theorem (Landweber Exactness III). *Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}$. Suppose that for all primes p, the set of ideals $\{\mathcal{I}_n\}$ acts regularly and finitely on $\mathcal{F}$. Then $\mathcal{F}$ is flat as an $\mathcal{O}_{\mathbf{fg}}$ module; that is,*

$$\mathrm{Tor}_s^{\mathcal{O}}(\mathcal{F}, \mathcal{E}) = 0, \quad s > 0.$$

Conversely, if for all primes p, $\mathcal{F}/\mathcal{I}_n\mathcal{F} = 0$ for some n, then $\mathcal{F}$ is flat only if the set of ideals $\{\mathcal{I}_n\}$ acts regularly and finitely on $\mathcal{F}$.

Theorem 6.23 was proved by Mike Hopkins in [14]; the proofs here are the same.

Let $j_n : \mathcal{M}(n) \to \mathcal{M}_{\mathbf{fg}}$ be the inclusion. The first result is this. The argument requires careful organization of exact sequences.

6.24 Proposition. *Suppose that for each prime p, the scale $\{\mathcal{I}_n\}$ acts regularly and finitely on $\mathcal{F}$ and that for each n,*

$$\mathrm{Tor}_s^{\mathcal{O}}((\mathcal{F}/\mathcal{I}_n)[v_n^{-1}], -) = 0, \quad s > n.$$

Then $\mathcal{F}$ is a flat $\mathcal{O}_{\mathbf{fg}}$-module sheaf.

Proof. By hypothesis, we have that for all large k,

$$\mathrm{Tor}_s^{\mathcal{O}}(\mathcal{F}/\mathcal{I}_k, -) = 0.$$

This begins a downward induction, where the induction hypothesis is that

$$\mathrm{Tor}_s^{\mathcal{O}}((\mathcal{F}/\mathcal{I}_{n+1}),-) = 0, \qquad s > n+1.$$

The final result is the case $n = -1$.

We make the argument for the induction step, using the following fact: if $\mathcal{L}$ is any locally free sheaf, then

$$\mathrm{Tor}_s^{\mathcal{O}}(\mathcal{E} \otimes \mathcal{L}, \mathcal{E}') \cong \mathrm{Tor}_s^{\mathcal{O}}(\mathcal{E}, \mathcal{E}') \otimes \mathcal{L}$$

for any quasi-coherent sheaves $\mathcal{E}$ and $\mathcal{E}'$.

Since the scale $\{\mathcal{I}_n\}$ acts regularly, we have an exact sequence

$$0 \to \mathcal{F}/\mathcal{I}_n \otimes \omega^{-(p^n-1)} \xrightarrow{v_n} \mathcal{F}/\mathcal{I}_n \to \mathcal{F}/\mathcal{I}_{n+1} \to 0.$$

The induction hypothesis implies that for any quasi-coherent sheaf $\mathcal{E}$

$$v_n : \mathrm{Tor}_s^{\mathcal{O}}(\mathcal{F}/\mathcal{I}_n, \mathcal{E}) \longrightarrow \mathrm{Tor}_s^{\mathcal{O}}(\mathcal{F}/\mathcal{I}_n, \mathcal{E}) \otimes \omega^{p^n-1}$$

is an injection for $s > n$.

Now recall that in Remark 6.16 we showed that $\mathcal{F}/\mathcal{I}_n[v_n^{-1}]$ can be written as the colimit of the sequence

$$\mathcal{F}/\mathcal{I}_n \xrightarrow{v_n} \mathcal{F}/\mathcal{I}_n \otimes \omega^{p^n-1} \xrightarrow{v_n} \mathcal{F}/\mathcal{I}_n \otimes \omega^{2(p^n-1)} \xrightarrow{v_n} \cdots.$$

Thus, we have for $s > n$,

$$0 = \mathrm{Tor}_s(\mathcal{F}/\mathcal{I}_n[v_n^{-1}], \mathcal{E}) \cong \operatorname{colim} \mathrm{Tor}_s(\mathcal{F}/\mathcal{I}_n, \mathcal{E}) \otimes \omega^{t(p^n-1)}.$$

Since each of the maps in the sequence is an injection, the induction step follows. $\square$

Now we must check the hypothesis of Proposition 6.24 in order to prove Theorem 6.23. Recall from Definition 6.17, that if $\mathcal{E}$ is any $\mathcal{I}_n$-torsion sheaf, then

$$\mathcal{E}[v_n^{-1}] = (i_n)_* i_n^* \mathcal{E}$$

where $i_n : \mathcal{U}(n) \to \mathcal{M}_{\mathbf{fg}}$ is the inclusion. In the case where $\mathcal{E} = \mathcal{F}/\mathcal{I}_n$, we have that $\mathcal{E}$ is the push-forward of the sheaf $j_n^* \mathcal{F}$ on $\mathcal{M}(n)$. Since there is a pull-back diagram

$$\begin{tikzcd}
\mathcal{H}(n) \arrow[r, "g_n"] \arrow[d, "k_n"'] & \mathcal{M}(n) \arrow[d, "j_n"] \\
\mathcal{U}(n) \arrow[r, "i_n"'] & \mathcal{M}_{\mathbf{fg}}
\end{tikzcd}$$

we have

$$i_n^* \mathcal{F}/\mathcal{I}_n = (k_n)_* g_n^* j_n^* \mathcal{F}.$$

Write $f_n : \mathcal{H}(n) \to \mathcal{M}_{\mathbf{fg}}$ for the inclusion Thus we can conclude that

$$\mathcal{F}/\mathcal{I}_n[v_n^{-1}] = (f_n)_*(f_n)^* \mathcal{F}.$$

The next result then verifies the hypothesis of Proposition 6.24.

6.25 Proposition. *Let $\mathcal{F}$ be any quasi-coherent sheaf on $\mathcal{H}(n)$ and let $\mathcal{E}$ be any quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}$. Then*

$$Tor_s^{\mathcal{O}}((f_n)_*\mathcal{F}, \mathcal{E}) = 0, \qquad s > n.$$

Proof. Recall from Lemma 5.21, that the inclusion $f_n : \mathcal{H}(n) \to \mathcal{M}_{\mathbf{fg}}$ is affine. This implies that the category of quasi-coherent sheaves on $\mathcal{H}(n)$ is equivalent, via the the push-forward $(f_n)_*$, to the category of quasi-coherent $(f_n)_*\mathcal{O}_{\mathcal{H}(n)}$ modules on $\mathcal{M}_{\mathbf{fg}}$. (Compare Proposition 1.11.) In particular, $(f_n)_*$ is exact on quasi-coherent sheaves. Also, for all quasi-coherent sheaves $\mathcal{E}$ on $\mathcal{M}_{\mathbf{fg}}$, there is a natural isomorphism

$$(f_n)_* f_n^* \mathcal{E} \cong (f_n)_*\mathcal{O}_{\mathcal{H}(n)} \otimes_{\mathcal{O}_{\mathbf{fg}}} \mathcal{E}.$$

It follows that there is a natural isomorphism

$$(f_n)_*(\mathcal{F}) \otimes \mathcal{E} \cong (f_n)_*(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{H}(n)}} f_n^*\mathcal{E}) \tag{6.3}$$

which becomes an equivalence of derived sheaves

$$(f_n)_*(\mathcal{F}) \otimes^L \mathcal{E} \cong (f_n)_*(\mathcal{F} \otimes^L_{\mathcal{O}_{\mathcal{H}(n)}} L(f_n^*)\mathcal{E}).$$

By Theorem 5.35 we have that the morphism $\Gamma_n : \mathrm{Spec}(\mathbb{F}_p) \to \mathcal{H}(n)$ classifying any height n formal group over $\mathbb{F}_p$ is an $fqpc$-cover; hence, the category of quasi-coherent sheaves on $\mathcal{H}(n)$ is equivalent to the category of $(\mathbb{F}_p, \mathcal{O}_{\mathrm{Aut}(\Gamma_n)})$ comodules. Here we have written $\mathrm{Spec}(\mathcal{O}_{\mathrm{Aut}(\Gamma_n)}) = \mathrm{Spec}(\mathbb{F}_p) \times_{\mathcal{H}(n)} \mathrm{Spec}(\mathbb{F}_p)$. From this we have that the functor $\mathcal{F} \otimes_{\mathcal{H}(n)} (-)$ is exact, since the corresponding functor on comodules is simply

$$\mathcal{F}(\Gamma_n : \mathrm{Spec}(\mathbb{F}_p) \to \mathcal{H}(n)) \otimes_{\mathbb{F}_p} (-).$$

Thus we need only show that

$$H_s L(f_n^*)\mathcal{E} = 0$$

for $s > n$. Since $(f_n)_*$ is exact, we need only check that $H_s(f_n)_* L(f_n^*)\mathcal{E} = 0$ for $s > n$. We need only check this equation locally, thus we may evaluate at any morphism

$$G : \mathrm{Spec}(R) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

classifying a formal group with a coordinate. Applying the formula of Equation 6.3 we see that locally these homology sheaves are given by

$$\mathrm{Tor}_s^R(u_n^{-1}R/\mathcal{I}_n(G), \mathcal{E}(G)).$$

The result now follows from the fact that $\mathcal{I}_n(G)$ is locally generated by a regular sequence of length n. □

6.26 Remark. Almost all of the argument for Theorem 6.23 uses only that we have a sequence of regularly embedded closed substacks $\{\mathcal{M}(n)\}$ of $\mathcal{M}_{\mathbf{fg}}$. However, in the proof Proposition 6.25 we used Theorem 5.35 which, in turn, ultimately depends on Lazard's proof of the result that, over a separably closed field of characteristic p, all formal groups of height n are isomorphic. Thus, it does not appear to me that the Landweber exact functor theorem is a generality – it seems quite specific to formal groups.

7 The formal neighborhood of a height n point

In this section we make the following slogan precise: the formal neighborhood in $\mathcal{M}_{\mathbf{fg}}$ of a height n formal group Γ over a perfect field of characteristic p is the Lubin-Tate space of the deformations of Γ. The exact statement is given in Theorem 7.23 below.

Let $\mathcal{U}_{\mathbf{fg}}(n) = \mathcal{M}_{\mathbf{fg}} - \mathcal{M}(n+1)$ be the open substack of $\mathcal{M}_{\mathbf{fg}}$ classifying formal groups of height less than or equal to n. Then

$$\mathcal{H}(n) = \mathcal{U}_{\mathbf{fg}}(n) - \mathcal{U}_{\mathbf{fg}}(n-1)$$

is a closed substack of $\mathcal{U}_{\mathbf{fg}}(n)$ defined by the vanishing of the ideal $\mathcal{I}_n$. Recall the $\mathcal{H}(n)$ has a single geometric point, but that this point has plenty of automorphisms. See Theorem 5.35. We wish to write down a description of the formal neighborhood $\widehat{\mathcal{H}}(n)$ of $\mathcal{H}(n) \subseteq \mathcal{U}_{\mathbf{fg}}(n)$. The answer is intimately connected with the Lubin-Tate space of deformations of a height n formal group.

By definition, $\widehat{\mathcal{H}}(n)$ is the category fibered in groupoids over $\mathbf{Aff}_{\mathbb{Z}_{(p)}}$ which assigns to each $\mathbb{Z}_{(p)}$-algebra B the groupoid with objects the formal groups G over B so that

1. $\mathcal{I}_n(G) \subseteq B$ is nilpotent; and
2. $\mathcal{I}_{n+1}(G) = B$.

Thus, if $q : B \to B/\mathcal{I}_n(G)$ is the quotient map, the formal group q^*G has strict height n in the sense that $v_i(G) = 0$ for $i < n$ and $v_n(G)$ is invertible. A great many examples of such formal groups can be obtained as deformations of a height n formal group; thus, we now discuss deformations and Lubin-Tate space.

7.1 Deformations of height n formal groups over a field

Fix a formal group Γ of height n over a perfect field $\mathbb{F}$ of characteristic p. (In practice, $\mathbb{F}$ will be an algebraic extension of a the prime field $\mathbb{F}_p$). Recall that an Artin local ring is a Noetherian commutative ring with a unique nilpotent maximal ideal. Let $\mathbf{Art}_{\mathbb{F}}$ denote the category of Artin local rings $(A, \mathfrak{m})$ so that we can choose an isomorphism $A/\mathfrak{m} \cong \mathbb{F}$ from the residue field of A to $\mathbb{F}$. The isomorphism is not part of the data. Morphisms in $\mathbf{Art}_{\mathbb{F}}$ are ring homomorphisms which induce an isomorphism on residue fields.

7.1 Definition. *A* **deformation** *of the pair* $(\mathbb{F}, \Gamma)$ *to an object* A *of* $\mathbf{Art}_{\mathbb{F}}$ *is a Cartesian square*

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & G \arrow[d] \\ \mathrm{Spec}(\mathbb{F}) \arrow[r, "f"] & \mathrm{Spec}(A) \end{tikzcd}$$

where G is a formal group over A and f induces an isomorphism $\mathrm{Spec}(\mathbb{F}) \cong \mathrm{Spec}(A/\mathfrak{m})$.

Deformations become a category $\mathbf{Def}(\mathbb{F},\Gamma)$ *fibered in groupoids over* $\mathbf{Art}_{\mathbb{F}}$ *by setting a morphism to be a commutative cube*

$$\begin{tikzcd}
& \Gamma \arrow[rr] \arrow[dd] & & G' \arrow[dd] \\
\Gamma \arrow[ur, "="] \arrow[rr] \arrow[dd] & & G \arrow[ur] \arrow[dd] & \\
& \mathrm{Spec}(\mathbb{F}) \arrow[rr] & & \mathrm{Spec}(A') \\
\mathrm{Spec}(\mathbb{F}) \arrow[ur, "="] \arrow[rr] & & \mathrm{Spec}(A) \arrow[ur] &
\end{tikzcd}$$

where the right face is also a Cartesian.

7.2 Remark. We can rephrase this as follows. A deformation of Γ to A is a triple (G,i,ϕ) where G is a formal group over A, $i : \mathrm{Spec}(\mathbb{F}) \to \mathrm{Spec}(A/\mathfrak{m})$ is an isomorphism and $\phi : \Gamma \to i^*G_0$ is an isomorphism of formal groups over $\mathbb{F}$. Here and always we write G_0 for the *special fiber* of G; that is, the induced formal group over $A/\mathfrak{m}$. There is an isomorphism of deformations $\psi : (G,i,\phi) \to (G,i',\phi')$ to A if $i = i'$ and $\psi : G \to G'$ is an isomorphism of formal groups so that

$$\begin{tikzcd}
& i^*G_0 \arrow[dd, "i^*\psi_0"] \\
\Gamma \arrow[ur, "\phi"] \arrow[dr, "\phi'"'] & \\
& i^*G_0'
\end{tikzcd}$$

commutes.

In either formulation of a deformation, we note that if G is a deformation of Γ to $(A,\mathfrak{m})$, then $\mathcal{I}_n(G) \subseteq \mathfrak{m}$ and $\mathcal{I}_{n+1}(G) = A$.

7.3 Remark. Let $R = (R,\mathfrak{m}_R)$ be a complete local ring so that $R/\mathfrak{m}_R \cong \mathbb{F}$. We write $\mathrm{Spf}(R)$ equally for the resulting formal scheme and for the the category fibered in groupoids over $\mathbf{Art}_{\mathbb{F}}$ that assigns to each object $(A,\mathfrak{m})$ of $\mathbf{Art}_{\mathbb{F}}$ the discrete groupoid of all ring homomorphisms which induce an isomorphism $R/\mathfrak{m}_R \cong A/\mathfrak{m}$; so, in particular, $f(\mathfrak{m}_R) \subseteq \mathfrak{m}$. This is an abuse of notation, but a mild one, and should cause no confusion. Indeed, the formal scheme $\mathrm{Spf}(R)$ is the left Kan extension of the functor $\mathrm{Spf}(R)$ on $\mathbf{Art}_{\mathbb{F}}$ along the inclusion of $\mathbf{Art}_{\mathbb{F}}$ into all rings.

7.4 Theorem (Lubin-Tate). *The category fibered in groupoids* $\mathbf{Def}(\mathbb{F},\Gamma)$ *is discrete and pro-representable; that is, there is a complete local ring* $R(\mathbb{F},\Gamma)$ *and a deformation*

$$\begin{tikzcd}
\Gamma \arrow[r] \arrow[d] & H \arrow[d] \\
\mathrm{Spec}(\mathbb{F}) \arrow[r] & \mathrm{Spf}(R(\mathbb{F},\Gamma))
\end{tikzcd}$$

of Γ *to* $R(\mathbb{F},\Gamma)$ *so that the induced morphism*

$$\mathrm{Spf}(R(\mathbb{F},\Gamma)) \longrightarrow \mathbf{Def}(\mathbb{F},\Gamma)$$

is an equivalence of categories fibered in groupoids over $\mathbf{Art}_{\mathbb{F}}$.

7.5 Remark. The induced morphism $\mathrm{Spf}(R(\mathbb{F},\Gamma)) \longrightarrow \mathbf{Def}(\mathbb{F},\Gamma)$ is not completely trivial to define. Given a homomorphism $R(\mathbb{F},\Gamma) \to A$ to an Artin local ring which induces an isomorphism of residue fields, we are asserting there is a unique way to complete the back square of the following diagram so that it commutes.

$$\begin{tikzcd}
& \Gamma \arrow[rr] \arrow[dl, "="'] \arrow[dd] & & f^*H \arrow[dl] \arrow[dd] \\
\Gamma \arrow[rr, crossing over] \arrow[dd] & & H & \\
& \mathrm{Spec}(\mathbb{F}) \arrow[rr] \arrow[dl, "="'] & & \mathrm{Spec}(A) \arrow[dl] \\
\mathrm{Spec}(\mathbb{F}) \arrow[rr] & & \mathrm{Spf}(R(\mathbb{F},\Gamma)) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

Thus, given $f : \mathrm{Spec}(A) \to \mathrm{Spf}(R(\mathbb{F},\Gamma))$, the universal deformation (H, j, ϕ_u) gets sent to

$$f^*(H, j, \phi_u) \overset{\text{def}}{=} (f^*H, f_0^{-1}j, \phi_u)$$

where $f_0 : \mathrm{Spec}(A/\mathfrak{m}_A) \to \mathrm{Spf}(R(\mathbb{F},\Gamma)/\mathfrak{m})$ is the induced isomorphism and we have written ϕ_u for both the universal isomorphism

$$\phi_u : \Gamma \longrightarrow j^*H_0$$

and the induced isomorphism

$$\phi_u : \Gamma \longrightarrow (f_0^{-1}j)^*(f^*H)_0 \cong j^*H_0.$$

In this language, the theorem of Lubin and Tate reads as follows: given a deformation (G, i, ϕ) of Γ to $A \in \mathbf{Art}_{\mathbb{F}}$, there is a homomorphism $f : R(\mathbb{F},\Gamma) \to A$ inducing an isomorphism on residue fields and a unique isomorphism of deformations

$$\psi : (G, i, \phi) \longrightarrow f^*(H, j, \phi_u).$$

The main lemma of Lubin and Tate is to calculate the deformations of Γ to the ring of dual numbers $\mathbb{F}[\epsilon]$, where $\epsilon^2 = 0$. Indeed, they show for that ring there is a non-canonical isomorphism

$$\pi_0 \mathbf{Def}(\mathbb{F},\Gamma)_{\mathbb{F}[\epsilon]} \cong (\mathbb{F}\epsilon)^{n-1}$$

where $(-)^k$ means the kth Cartesian power and

$$\pi_1(\mathbf{Def}(\mathbb{F},\Gamma), G)_{\mathbb{F}[\epsilon]} = \{1\}$$

for any deformation G. The general theory of deformations (see [46], Proposition 3.12) then shows that there is a (non-canonical) isomorphism

$$\pi_0\mathbf{Def}(\mathbb{F},\Gamma) \cong \mathfrak{m}^{n-1} \tag{7.1}$$

where we are write $\mathfrak{m}$ for the functor which assigns to an Artin local ring A its maximal ideal $\mathfrak{m}_A$. For any deformation G,

$$\pi_1(\mathbf{Def}(\mathbb{F},\Gamma), G) = \{1\}. \tag{7.2}$$

It immediately follows that the ring $R(\mathbb{F},\Gamma)$ is a power series ring. More explicitly, since $R(\mathbb{F},\Gamma)$ is local, Noetherian, and p-complete, the universal deformation H can be given a p-typical coordinate x for which the p-series of H becomes

$$[p]_H(x) = px +_H u_1x^p +_H \cdots +_H u_{n-1}x^{p^{n-1}} +_H u_nx^{p^n} +_H \cdots .$$

Then there is an isomorphism

$$R(\mathbb{F},\Gamma) \cong W(\mathbb{F})[[u_1,\ldots,u_{n-1}]] \tag{7.3}$$

where $W(\mathbb{F})$ is the Witt vectors of $\mathbb{F}$ and the maximal ideal $\mathfrak{m} = \mathcal{I}_n(H) = (p, u_1, \ldots, u_{n-1})$. Note that u_n is a unit. This isomorphism is non-canonical as it depends on a choice of p-typical coordinate.

Equation 7.2 can be deduced from the following result.

7.6 Lemma. *Let $(A,\mathfrak{m})$ be an Artin local ring with $A/\mathfrak{m}$ of characteristic p. Let G_1 and G_2 be two formal groups over B so that $(G_1)_0$ and $(G_2)_0$ are of height $n < \infty$. Then the affine morphism*

$$\mathrm{Iso}_B(G_1,G_2) \longrightarrow \mathrm{Spec}(A)$$

is unramified. In particular, if we are given a choice of isomorphism

$$\phi : (G_1)_0 \longrightarrow (G_2)_0$$

over $(A/\mathfrak{m})$, then there is at most one isomorphism $\psi : G_1 \to G_2$ over A so that $(\psi)_0 = \phi$.

Proof. By Theorem 5.23 the morphism

$$\mathrm{Iso}_{A/\mathfrak{m}}((G_1)_0,(G_2)_0) \longrightarrow \mathrm{Spec}(A/\mathfrak{m})$$

is pro-étale; that is, flat and unramified. Since $\mathrm{Spec}(A/\mathfrak{m}) \to \mathrm{Spec}(A)$ is the unique point of $\mathrm{Spec}(A)$, the main statement follows. (See Proposition I.3.2 of [35].) The statement about the unique lifting follows from one of the characterizations of unramified: there is at most one way to complete the diagram

$$\begin{tikzcd}
\mathrm{Spec}(A/\mathfrak{m}) \arrow[r, "\phi"] \arrow[d] & \mathrm{Iso}_B(G_1,G_2) \arrow[d] \\
\mathrm{Spec}(A) \arrow[r, "="'] \arrow[ur, dashed] & \mathrm{Spec}(A)
\end{tikzcd}$$

so that both triangles commute. ∎

7.7 Remark. We can give an alternate description of deformations in terms of formal group laws. Fix a coordinate x for Γ and let F_Γ be the resulting formal group law over $\mathbb{F}$. Define $\mathbf{Def}_*(\mathbb{F},\Gamma)$ to be the groupoid-valued functor on $\mathbf{Art}_\mathbb{F}$ which assigns to each Artin local ring $(A,\mathfrak{m})$ of $\mathbf{Art}_\mathbb{F}$ the groupoid with objects all pairs (i,F) where $i : A/\mathfrak{m} \to \mathbb{F}$ is an isomorphism and F is a formal group law over A so that

$$i^*F_0(x,y) = F_\Gamma(x,y) \in \mathbb{F}[[x,y]].$$

Here we've written $F_0(x,y)$ for the reduction of F to $A/\mathfrak{m}$. There is a morphism $\psi : (i,F) \to (i',F')$ if $i = i'$ and $\psi : F \to F'$ is an isomorphism of formal group laws so that

$$i^*\psi_0(x) = x \in \mathbb{F}[[x]].$$

The set $\pi_0\mathbf{Def}_*(\mathbb{F},\Gamma)$ is the the set of $\star$-isomorphism classes of deformations of the formal group law F_Γ.

There is a natural transformation of groupoid functors

$$\mathbf{Def}_*(\mathbb{F},\Gamma)\longrightarrow\mathbf{Def}(\mathbb{F},\Gamma).$$

This is a equivalence. It is obviously full and faithful, so we need only show that every object in the target is isomorphic to some object from the source.

To see this, we'll use the notation of Remark 7.2. Let $(A,\mathfrak{m})$ be an Artin local ring over $\mathbb{F}$ and G a deformation of Γ to A. Since $\mathbb{F}$ is a field, G_0 can be given a coordinate; since A is local Noetherian, G can be given a coordinate which reduces to a chosen coordinate for G_0. The isomorphism $\phi : \Gamma \to G_0$ determines an isomorphism of formal group laws

$$\phi : F_{G_0}(x,y)\longrightarrow(i^*)^{-1}F_\Gamma(x,y).$$

Lift the power series $\phi(x)$ to a power series $\psi(x) \in A[[x]]$ so that $\psi_0(x) = \phi(x)$ and define a formal group law $F(x,y)$ over A by requiring that

$$\psi(x) : F_G(x,y)\longrightarrow F(x,y)$$

is an isomorphism. Then $F(x,y)$ is the required formal group law.

7.8 Remark. Following the example of Remark 7.3 we extend the notion of deformations to all commutative rings by left Kan extensions along the forgetful functor from $\mathbf{Art}_\mathbb{F}$ to rings. In more detail, let B be a commutative ring. Define the category $\mathbf{Art}_\mathbb{F}/B$ to have objects the morphisms

$$A\longrightarrow B$$

of commutative rings where $(A,\mathfrak{m})$ is an Artin local ring in $\mathbf{Art}_\mathbb{F}$. Morphisms in $\mathbf{Art}/B$ are commutative triangles. Since the tensor product $A\otimes_\mathbb{Z} A'$ of Artin local rings is an Artin local ring, the category $\mathbf{Art}_\mathbb{F}/B$ is filtered and has a cofinal subcategory consisting of those morphisms which are injections.

Define the groupoid $\mathbf{Def}(\mathbb{F},\Gamma)_B$ of deformations of Γ over B to be the colimit

$$\mathbf{Def}(\mathbb{F},\Gamma)_B = \mathrm{colim}_{\mathbf{Art}_\mathbb{F}/B}\,\mathbf{Def}(\mathbb{F},\Gamma)_A.$$

Thus a generalized deformation of Γ to B is a deformation of Γ to an Artin local subring $A \subseteq B$. This is probably easiest to understand using formal group laws.

Fix a coordinate of Γ and let $\mathbf{Def}_*(\mathbb{F}, \Gamma)$ be the groupoid defined in Remark 7.7. Then by that remark, there is an equivalence

$$\mathrm{colim}_{\mathbf{Art}_{\mathbb{F}}/B}\, \mathbf{Def}_*(\mathbb{F}, \Gamma)_A \to \mathbf{Def}(\mathbb{F}, \Gamma)_B$$

and the elements of the source are easily described. The objects are equivalence classes of pairs (F, i) where F is a formal group law

$$F(x, y) = \sum a_{ij} x^i y^j \in B[[x, y]]$$

so that the coefficients a_{ij} lie in an Artin local subring $A \subseteq B$ and so that the pair $(F|_A, i) \in \mathbf{Def}_*(\mathbb{F}, \Gamma)_A$. Isomorphisms in $\mathbf{Def}(\mathbb{F}, \Gamma)_B$ must similarly lie over Artlin local subrings.

The following result extends and follows immediately from Remark 7.3 and Theorem 7.4.

7.9 Theorem. *The natural isomorphism of functors on commutative rings*

$$\mathrm{Spf}(R(\mathbb{F}, \Gamma)) \longrightarrow \pi_0 \mathbf{Def}(\mathbb{F}, \Gamma)$$

is an isomorphism and for all deformation G of Γ over B

$$\pi_1(\mathbf{Def}(\mathbb{F}, \Gamma)_B, G) = \{\ 1\ \}.$$

Now let $\mathcal{H}(n) \subseteq \mathcal{U}_{\mathbf{fg}}(n)$ be the closed substack of formal groups of exact height n and let $\widehat{\mathcal{H}}(n)$ denote its formal neighborhood. There is a 1-morphism of groupoid schemes

$$\mathbf{Def}(\mathbb{F}, \Gamma) \longrightarrow \widehat{\mathcal{H}}(n)$$

which sends a deformation $(G/B, \phi)$ to the formal group G.

Given a formal group G over B so that $\mathcal{I}_n(G)$ is nilpotent and $\mathcal{I}_{n+1}(G) = B$, it is not necessarily true that G arises from a deformation; that would amount to a choice of 2-commuting diagram

$$\begin{tikzcd} & \mathbf{Def}(\mathbb{F}, \Gamma) \arrow[d] \\ \mathrm{Spec}(B) \arrow[ur] \arrow[r, "G"'] & \widehat{\mathcal{H}}(n). \end{tikzcd}$$

Nonetheless, we have the following two results. Recall from Corollary 5.34 that if Γ is a height n formal group over a field $\mathbb{F}$, then the induced map $\mathrm{Spec}(\mathbb{F}) \to \mathcal{H}(n)$ is a presentation. Such a formal group also defines a trivial deformation; that is, Γ is itself a deformation of Γ to $\mathbb{F}$.

7.10 Proposition. *Let Γ be a height n formal group law over an algebraic extension of $\mathbb{F}_p$. Then the 2-commuting square*

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{F}) \arrow[r] \arrow[d] & \mathbf{Def}(\mathbb{F},\Gamma) \arrow[d] \\ \mathcal{H}(n) \arrow[r] & \widehat{\mathcal{H}}(n) \end{tikzcd}$$

is 2-category pull-back square.

Proof. Write P for the 2-category pull-back. By Theorem 7.9, an object in P over a ring B is a triple (G, f, ϕ) where G is formal group of exact height n over B, $f : R(\mathbb{F},\Gamma) \to B$ is a ring homomorphism so that $B \cdot f(\mathfrak{m})$ is nilpotent, and $\phi : G \to f^*H$ is an isomorphism of formal groups. Here H is the universal deformation as in Theorem 7.4 and $\mathfrak{m} \subseteq R(\mathbb{F},\Gamma)$ is the maximal ideal. A morphism $\psi : (G, f, \phi) \to (G', f, \phi')$ is an isomorphism $\psi : G \to G'$ so that $\phi'\psi = \phi$. In particular, such a triple has no non-identity automorphisms and P is discrete.

Given a triple (G, f, ϕ), we have that

$$0 = \mathcal{I}_n(G) = \mathcal{I}_n(f^*H) = B \cdot f(\mathfrak{m});$$

hence, the morphism $f : R(\mathbb{F},\Gamma) \to B$ factors through $\mathbb{F}$. Furthermore ϕ itself defines an isomorphism

$$(G, f, \phi) \to (f^*H, f, 1).$$

It follows that there is an equivalence $\mathrm{Spec}(\mathbb{F}) \to P$ sending $g : \mathbb{F} \to B$ to the triple $(f^*H, f, 1)$ with f the composite $R(\mathbb{F},\Gamma) \to \mathbb{F} \to B$. □

7.11 Proposition. *The morphism $q : \mathbf{Def}(\mathbb{F},\Gamma) \to \widehat{\mathcal{H}}(n)$ is representable, flat, and surjective.*

Proof. We first show it is representable; in fact, we will show that given a diagram

$$\mathrm{Spec}(B) \xrightarrow{G} \widehat{\mathcal{H}}(n) \longleftarrow \mathrm{Spf}(R(\mathbb{F},\Gamma))$$

then

$$p_1 : \mathrm{Spec}(B) \times_{\widehat{\mathcal{H}}(n)} \mathrm{Spf}(R(\mathbb{F},\Gamma)) \to \mathrm{Spec}(B)$$

is a formal affine scheme over B. By a descent argument, we may assume that G has a coordinate. Then, arguing as in Lemma 2.11, we see that the pull-back is equivalent to

$$\mathrm{Spf}(B \otimes_L W \otimes_L R(\mathbb{F},\Gamma)),$$

the formal neighborhood of $B \otimes_L W \otimes_L R(\mathbb{F},\Gamma)$ at the ideal $\mathcal{I}_n(p_1^*G) = \mathcal{I}_n(p_2^*H)$ where p_i, $i = 1, 2$ are the projections onto the two factors.

To see that q is flat, we apply Theorem 6.18 (the Landweber Exact Functor Theorem) and the descripition of $R(\mathbb{F},\Gamma)$ given in Equation 7.3.

For surjectivity, note that if k is any field and $g : \mathrm{Spec}(k) \to \widehat{\mathcal{H}}(n)$ classifies a formal group G, then $\mathcal{I}_n(G) = 0$ and G is a height n; that is, g factors through $\mathcal{H}(n)$. The result now follows from the first part and the fact that $\mathrm{Spec}(\mathbb{F}) \to \mathcal{H}(n)$ is surjective – see Corollary 5.34. □

7.2 The action of the automorphism group

Let $\mathbb{G}(\mathbb{F},\Gamma) = \mathrm{Aut}(\mathbb{F},\Gamma)$, the automorphism of the pair $(\mathbb{F},\Gamma)$. An element of $\mathbb{G}(\mathbb{F},\Gamma)$ is a pull-back diagram

$$\begin{tikzcd} \Gamma \arrow[r, "g"] \arrow[d] & \Gamma \arrow[d] \\ \mathrm{Spec}(\mathbb{F}) \arrow[r, "\sigma"'] & \mathrm{Spec}(\mathbb{F}). \end{tikzcd}$$

where σ is induced by a field automorphism. For historical and topological reasons we call this the *Morava stabilizer group* of the pair $(\mathbb{F},\Gamma)$. We may write such a diagram as a pair (σ,λ) where $\lambda : \Gamma \to \sigma^*\Gamma$ is the isomorphism induced by g. If $\mathbb{F}$ is an algebraic extension of $\mathbb{F}_p$ and Γ is defined over $\mathbb{F}_p$, this yields an isomorphism

$$\mathbb{G}(\mathbb{F},\Gamma) \cong \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p) \ltimes \mathrm{Aut}(\Gamma)$$

where $\mathrm{Aut}(\Gamma)$ is the group of automorphisms of Γ defined over $\mathbb{F}$.

The group $\mathbb{G}(\mathbb{F},\Gamma)$ acts on the groupoid functor $\mathbf{Def}(\mathbb{F},\Gamma)$ on the right by sending a diagram

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & G \arrow[d] \\ \mathrm{Spec}(\mathbb{F}) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

to the outer square of the diagram

$$\begin{tikzcd} \Gamma \arrow[r, "g"] \arrow[d] & \Gamma \arrow[r] \arrow[d] & G \arrow[d] \\ \mathrm{Spec}(\mathbb{F}) \arrow[r, "\sigma"'] & \mathrm{Spec}(\mathbb{F}) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

This action commutes with the isomorphisms in $\mathbf{Def}(\mathbb{F},\Gamma)$; thus we obtain a right action on $\pi_0\mathbf{Def}(\mathbb{F},\Gamma)$ and hence a left action on $R(\mathbb{F},\Gamma)$.

7.12 Remark. If, following Remark 7.2, we think of a deformation of Γ to A as a triple, (G,i,ϕ) and an element of $\mathbb{G}(\mathbb{F},\Gamma)$ as a pair (σ,λ) as above, then the action of (σ,λ) on (G,i,ϕ) yields the triple

$$(G, i\sigma, (\sigma^*\phi)\lambda).$$

Because $\pi_0\mathbf{Def}(\mathbb{F},\Gamma)$ is a set of equivalence classes, we must take a little care in interpreting this action on $R(\mathbb{F},\Gamma)$. See, for example, [5].

The following is the key lemma about this action.

7.13 Lemma. *Let A be a Artin local ring with residue field isomorphic to $\mathbb{F}$ and let (G, i, ϕ) and (G', i', ϕ') be two deformations of $(\mathbb{F}, \Gamma)$. Suppose there is an isomorphism of formal groups $\psi : G \to G'$ over A. Then there is a unique pair $(\sigma, \lambda) \in \mathrm{Aut}(\mathbb{F}, \Gamma)$ so that ψ induces an isomorphism*

$$\psi : (G, i\sigma, \sigma^*(\phi)\lambda) \longrightarrow (G', i', \phi').$$

Proof. This is simply the assertion that there is a unique way to fill in left face of the following diagram so that it commutes

$$\begin{tikzcd}
& \Gamma \arrow[rr] \arrow[dl, "g"'] \arrow[dd] & & G_0 \arrow[dl, "\psi_0"'] \arrow[dd] \\
\Gamma \arrow[rr] \arrow[dd] & & G_0' \arrow[dd] & \\
& \mathrm{Spec}(\mathbb{F}) \arrow[rr, "i"] \arrow[dl, "\sigma"'] & & \mathrm{Spec}(A/\mathfrak{m}) \arrow[dl] \\
\mathrm{Spec}(\mathbb{F}) \arrow[rr, "i'"'] & & \mathrm{Spec}(A/\mathfrak{m}) &
\end{tikzcd}$$

Alternatively, writing down the equations provides both the pair (σ, λ) and its uniqueness. Indeed, we need an equality of isomorphisms from $\mathrm{Spec}(\mathbb{F})$ to $\mathrm{Spec}(A/\mathfrak{m})$

$$i\sigma = i'$$

and a commutative diagram of isomorphisms of formal groups

$$\begin{tikzcd}
& i^*G_0 \arrow[dd, "(i\sigma)^*\psi_0"] \\
\Gamma \arrow[ur, "\sigma^*(\phi)\lambda"] \arrow[dr, "\phi'"'] & \\
& i^*G_0'.
\end{tikzcd}$$

□

The action of the Morava stabilizer group is actually continuous in a sense we now make precise. Assume now that $\mathbb{F}$ is an algebraic extension of $\mathbb{F}_p$. First we notice that the extended automorphism group $\mathbb{G}(\mathbb{F}, \Gamma)$ is profinite. Define normal subgroups $\mathbb{G}_k(\mathbb{F}, \Gamma)$ of $\mathbb{G}(\mathbb{F}, \Gamma)$ as follows. The group $\mathbb{G}(\mathbb{F}, \Gamma)$ is the set of Cartesian squares

$$\begin{tikzcd}
\Gamma \arrow[r, "\lambda"] \arrow[d] & \Gamma \arrow[d] \\
\mathrm{Spec}(\mathbb{F}) \arrow[r, "\sigma"'] & \mathrm{Spec}(\mathbb{F})
\end{tikzcd}$$

under composition. The subgroup $\mathbb{G}_i(\mathbb{F}, \Gamma)$ is the set of those squares so that σ is the identity and λ induces the identity on the p^k-bud of the formal group Γ. Then $\mathbb{G}_0(\mathbb{F}, \Gamma) = \mathrm{Aut}_{\mathbb{F}}(\Gamma)$,

$$\mathbb{G}(\mathbb{F}, \Gamma)/\mathbb{G}_0(\mathbb{F}, \Gamma) = \mathrm{Gal}(\mathbb{F}, \mathbb{F}_p)$$

and

$$\mathbb{G}(\mathbb{F},\Gamma) \cong \lim \mathbb{G}_k(\mathbb{F},\Gamma).$$

If $\mathbb{F} \to \mathbb{F}'$ is an extension of subfields of $\bar{\mathbb{F}}_p$, then we get an injection of groups $\mathbb{G}_0(\mathbb{F},\Gamma) \to \mathbb{G}_0(\mathbb{F},\Gamma)$ which preserves the subgroups above. Thus the following result displays $\mathbb{G}(\mathbb{F},\Gamma)$ as a profinite group. We remark that the equations developed in the proof make it possible to actually calculate $\mathbb{G}_k(\mathbb{F},\Gamma)/\mathbb{G}_{k+1}(\mathbb{F},\Gamma)$ for all $\mathbb{F} \subseteq \bar{\mathbb{F}}_p$.

7.14 Lemma. *There is an isomorphism*

$$\mathbb{G}_0(\bar{\mathbb{F}}_p,\Gamma)/\mathbb{G}_1(\bar{\mathbb{F}}_p,\Gamma) \cong \mathbb{F}_{p^n}^{\times}$$

and for $k > 0$ a non-canonical isomorphism

$$\mathbb{G}_k(\bar{\mathbb{F}}_p,\Gamma)/\mathbb{G}_{k+1}(\bar{\mathbb{F}}_p,\Gamma) \cong \mathbb{F}_{p^n}$$

Proof. Choose a p-typical coordinate x for Γ and write the Verschiebung $V : \Gamma^{(p^n)} \to \Gamma$ out in this coordinate as

$$V(x) = u_n x +_\Gamma u_{n+1} x^p +_\Gamma \cdots .$$

(Compare the argument of 5.27.) Note that u_n is a unit. If $\lambda \in \mathbb{G}_k(\mathbb{F},\Gamma)$, then we have, for $k = 0$,

$$\lambda(x) = ax \mod x^2$$

where a is a unit and for $k > 0$

$$\lambda(x) = x + ax^{p^k} \mod x^{p^k+1}.$$

By examining the equality $\lambda(V(x)) = V(\lambda^{(p^n)}(x))$ we see that in either case we an equation

$$u_n a^{p^n} = u_n^{p^k} a.$$

The result follows; indeed, when $k = 0$ we simply have the equation $a^{p^n-1} = 1$ and if $k > 0$ we are looking for the roots of

$$u_n a^{p^n} - u_n^{p^k} a = 0$$

under addition. If we choose a non-zero root r then the assignment $b \mapsto rb$ for $b \in \mathbb{F}_{p^n}$ defines the necessary isomorphism. □

7.15 Remark (The continuity of the action). Define $\mathbf{Def}_k(\mathbb{F},\Gamma)$ to be the groupoid of triples (G,i,ϕ) where G is a formal group over an Artin local ring A, $i : \mathrm{Spec}(\mathbb{F}) \to \mathrm{Spec}(A/\mathfrak{m})$ is an isomorphism and

$$\phi : \Gamma_{p^k} \longrightarrow i^*(G_0)_{p^k}$$

is an isomorphism of p^k-buds. The morphisms in $\mathbf{Def}_k(\mathbb{F},\Gamma)$ are isomorphisms $\psi : G \to G'$ which induce the appropriate commutative triangle over $\mathbb{F}$. (Note that $\mathbf{Def}_f(\mathbb{F},\mathbb{G})$ is *not* the deformation of the buds, as these isomorphisms are defined over the whole group.) Since every isomorphism of buds over a Noetherian local ring can be lifted to an isomorphism of the formal groups, we have that the evident map

$$\mathbf{Def}(\mathbb{F},\Gamma)\longrightarrow\mathbf{Def}_k(\mathbb{F},\Gamma)$$

is surjective on objects and $\mathbb{G}(\mathbb{F},\Gamma)$-equivariant; furthermore, the induced map

$$\mathbf{Def}(\mathbb{F},\Gamma)\longrightarrow\lim\mathbf{Def}_k(\mathbb{F},\Gamma) \tag{7.4}$$

is an isomorphism. The action of $\mathbb{G}(\mathbb{F},\Gamma)$ on $\mathbf{Def}_k(\mathbb{F},\Gamma)$ factors through the quotient group $\mathbb{G}(\mathbb{F},\Gamma)/\mathbb{G}_k(\mathbb{F},\Gamma)$. If we give $\mathbf{Def}_k(\mathbb{F},\Gamma)$ the discrete topology and $\mathbf{Def}(\mathbb{F},\Gamma)$ the topology defined by the natural isomorphism 7.4, then the action of $\mathbb{G}(\mathbb{F},\Gamma)$ on $\mathbf{Def}(\mathbb{F},\Gamma)$ is continuous.

7.16 Lemma. *There is a natural isomorphism of functors on Artin local rings*

$$[\pi_0\mathbf{Def}(\mathbb{F},\Gamma)]/\mathbb{G}_k(\mathbb{F},\Gamma)\xrightarrow{\cong}\pi_0\mathbf{Def}_k(\mathbb{F},\Gamma)$$

and for all (G,i,ϕ) *in* $\mathbf{Def}_k(\mathbb{F},\Gamma)_A$,

$$\pi_1(\mathbf{Def}_k(\mathbb{F},\Gamma)_A,G)=\mathbb{G}_k(\mathbb{F},\Gamma).$$

Proof. This is a direct consequence of Lemma 7.13. The natural transformation

$$\pi_0\mathbf{Def}(\mathbb{F},\Gamma)/\mathbb{G}_k(\mathbb{F},\Gamma)\longrightarrow\pi_0\mathbf{Def}_k(\mathbb{F},\Gamma)$$

is onto. If (G,i,ϕ) and (G',i,ϕ') are two deformations and

$$\psi:(G,i,\phi)\longrightarrow(G',i,\phi')$$

is an isomorphism in the $\mathbf{Def}_k(\mathbb{F},\Gamma)$ there is a *unique* automorphism λ of Γ over $\mathbb{F}$ so that

$$\psi:(G,i,\phi\lambda)\longrightarrow(G',i,\phi')$$

is an isomorphism in $\mathbf{Def}(\mathbb{F},\Gamma)$. Note that λ is necessarily in $\mathbb{G}_k(\mathbb{F},\Gamma)$.

Almost the same proof gives the statement about π_1. Indeed, if $(G,i,\bar{\phi})$ is any lift of (G,i,ϕ) to $\mathbf{Def}(\mathbb{F},\Gamma)$, then an automorphism $\psi:(G,i,\phi)\to(G,i,\phi)$ determines a unique element in $\mathbb{G}_k(\mathbb{F},\Gamma)$ so that ψ induces an isomorphism

$$\psi:(G,i,\bar{\phi}\lambda)\longrightarrow(G,i,\bar{\phi}).$$

The assignment $\psi\mapsto\lambda$ induces the requisite isomorphism. □

The functor $\mathbf{Def}_k(\mathbb{F},\Gamma)$ from Artin rings to groupoids can be extended to all commutative rings using a left Kan extension as in Remark 7.8. Since we are taking a filtered colimit, the natural transformation $\mathbf{Def}(\mathbb{F},\Gamma)\to\mathbf{Def}_k(\mathbb{F},\Gamma)$ remains onto for all commutative rings. We get a natural sequence of maps

$$\pi_0\mathbf{Def}(\mathbb{F},\Gamma)_B\longrightarrow\lim\pi_0\mathbf{Def}(\mathbb{F},\Gamma)_B/\mathbb{G}_k(\mathbb{F},\Gamma)\xrightarrow{\cong}\lim\mathbf{Def}_k(\mathbb{F},\Gamma)_B.$$

The first map, which is an isomorphism for Artin rings, is not immediately an isomorphism in this generality because colimits don't commute with limits in general; however it is continuous and, as a result, an injection.

7.3 Deformations are the universal cover

If X is a finite set, let's also write $X_{\mathbb{Z}}$ for the scheme $\mathrm{Spec}(\mathrm{map}(X,\mathbb{Z}))$. Then for any category Y fibered in groupoids over affine schemes we get a new functor $Y \times X_{\mathbb{Z}}$. If $Y = \mathrm{Spec}(R)$, then

$$Y \times X_{\mathbb{Z}} = \mathrm{Spec}(\mathrm{map}(X,R)) \overset{\mathrm{def}}{=} X_R.$$

If G is a finite group, then $G_{\mathbb{Z}}$ is a finite group scheme over $\mathbb{Z}$ and the right action of G on itself extends to a right action on $Y \times G_{\mathbb{Z}}$.

If $X = \lim X_k$ is a profinite set, define

$$X_Z = \lim (X_i)_{\mathbb{Z}} = \mathrm{Spec}(\mathrm{colim}\,\mathrm{map}(X_i,\mathbb{Z})).$$

If $G = \lim G_k$ is a profinite group, then $G_{\mathbb{Z}}$ is a profinite group scheme over $\mathbb{Z}$. A right action of $G_{\mathbb{Z}}$ on a category Y fibered in groupoids over affine schemes will be continuous if $Y = \lim Y_k$, where $G_{\mathbb{Z}}$ acts on Y_k through $(G/G_k)_{\mathbb{Z}}$.

The notation $G_{\mathbb{Z}}$ is cumbersome; we will drop it if G is evidently a profinite group.

7.17 Theorem. *The natural transformations of groupoids over Artin local rings*

$$\mathbf{Def}(\mathbb{F},\Gamma) \times \mathbb{G}(A,\mathbb{F}) \longrightarrow \mathbf{Def}(\mathbb{F},\Gamma) \times_{\widehat{\mathcal{H}}(n)} \mathbf{Def}(\mathbb{F},\Gamma)$$

given by

$$((G,i,\phi),(\sigma,\lambda)) \mapsto ((G,i,\phi),(G,i\sigma,\phi\lambda), 1 : G \to G)$$

is an equivalence.

Proof. A typical element in the pull-back is a triple

$$((G,i,\phi),(G',i',\phi'),\psi : G \to G') \tag{7.5}$$

where the first two terms are deformations and ψ is any isomorphism of formal groups. A morphism in the pull-back

$$\begin{aligned}(\gamma,\gamma') : ((G_1,i,\phi),(G_1',i',\phi'),\psi_1 : G_1 \to G_1') \to \\ ((G_2,j,\phi),(G_2',i',\phi'),\psi_2 : G_2 \to G_2')\end{aligned}$$

are isomorphisms γ and γ' of deformations so that $\psi_2\gamma = \gamma'\psi_1$. Now we apply Lemma 7.13. Given the typical element, as in 7.5, we get a unique pair (σ,λ) in $\mathbb{G}(\mathbb{F},\Gamma)$ so that

$$(1,\psi) : ((G,i,\phi),(G,i\sigma,\phi\lambda),1_G) \to ((G,i,\phi),(G',i',\phi'),\psi) \tag{7.6}$$

is an isomorphism in the pull-back. The assignment

$$((G,i,\phi),(G',i',\phi'),\psi) \mapsto ((G,i,\phi),(G,i\sigma,\phi\lambda),1_G)$$

becomes a natural transformation of groupoids sending a morphism (γ,γ') to (γ,γ). Then 7.6 displays the necessary contraction. $\square$

7.18 Definition. *Let $q : Y \to X$ be a morphism of categories fibered in groupoids over some base category. The group $\mathrm{Aut}_Y(X)$ of automorphisms of Y over X consists of equivalence classes pairs (f, ψ) where $f : Y \to Y$ is a 1-morphism of groupoids and $\psi : q \to qf$ is a 2-morphism. Two such pairs (f, ψ) and (f', ψ') are equivalent if there is a 2-morphism $\phi : f \to f'$ so that $\psi'\phi = \psi$. The composition law reads*

$$(g, \psi)(f, \phi) = (gf, (f^*\psi)\phi).$$

There is a homomorphism

$$\mathbb{G}(\mathbb{F}, \Gamma) \longrightarrow \mathrm{Aut}_{\widehat{\mathcal{H}}(n)}(\mathbf{Def}(\mathbb{F}, \Gamma))$$

sending (σ, λ) to the pair $(f_{(\sigma,\lambda)}, 1)$ where $f_{(\sigma,\lambda)}$ is the transformation

$$(G, i, \phi) \mapsto (G, i\sigma, \phi\lambda).$$

7.19 Theorem. *This homomorphism*

$$\mathbb{G}(\mathbb{F}, \Gamma) \longrightarrow \mathrm{Aut}_{\widehat{\mathcal{H}}(n)}(\mathbf{Def}(\mathbb{F}, \Gamma)).$$

is an isomorphism and induces isomorphisms

$$\mathbb{G}(\mathbb{F}, \Gamma)/\mathbb{G}_{k+1}(\mathbb{F}, \Gamma) \longrightarrow \mathrm{Aut}_{\widehat{\mathcal{H}}(n)}(\mathbf{Def}_k(\mathbb{F}, \Gamma)).$$

Proof. That the map is an injection is clear from the definitions. We now prove that it's surjective. Let (f, ψ) be an element of the automorphisms of $\mathbf{Def}(\mathbb{F}, \Gamma)$ over $\widehat{\mathcal{H}}(n)$. Let's write

$$f(G, i, \phi) = (G_f, i_f, \phi_f).$$

Then ψ gives isomorphism of formal groups $\psi_G : G \to G_f$. By Lemma 7.13 there is a unique pair $(\sigma, \lambda) \in \mathbb{G}(\mathbb{F}, \Gamma)$ so that

$$\psi_G : (G, i\sigma, \phi\lambda) \to (G_f, i_f, \phi_f)$$

is an isomorphism of deformations. The uniqueness of (σ, λ) and this equation give us the needed 2-morphism

$$\phi : f_{(\sigma,\lambda)} \longrightarrow f$$

□

We wish to show that the isomorphism of Theorem 7.19 is appropriately continuous.

7.20 Lemma. *There is a surjective homomorphism of groups*

$$q_k : \mathrm{Aut}_{\widehat{\mathcal{H}}(n)}(\mathbf{Def}(\mathbb{F}, \Gamma)) \to \mathrm{Aut}_{\widehat{\mathcal{H}}(n)}(\mathbf{Def}_k(\mathbb{F}, \Gamma))$$

which induces a commutative diagram of groups

$$\begin{array}{ccc} \mathbb{G}(\mathbb{F},\Gamma) & \longrightarrow & \mathrm{Aut}_{\widehat{\mathcal{H}}(n)}(\mathbf{Def}(\mathbb{F},\Gamma)) \\ \downarrow & & \downarrow{\scriptstyle q_k} \\ \mathbb{G}(\mathbb{F},\Gamma)/\mathbb{G}_{k+1}(\mathbb{F},\Gamma) & \longrightarrow & \mathrm{Aut}_{\widehat{\mathcal{H}}(n)}(\mathbf{Def}_k(\mathbb{F},\Gamma)). \end{array}$$

Proof. We use Theorem 7.19. Any automorphism of $\mathbf{Def}(\mathbb{F},\Gamma)$ of the form $f_{(\sigma,\lambda)}$ immediately induces an automorphism of $\mathbf{Def}_k(\mathbb{F},\Gamma)$. This defines a morphism

$$\mathbb{G}(\mathbb{F},\Gamma)\longrightarrow \mathrm{Aut}_{\widehat{\mathcal{H}}(n)}(\mathbf{Def}_k(\mathbb{F},\Gamma))$$

which factors through $\mathbb{G}(\mathbb{F},\Gamma)/\mathbb{G}_{k+1}(\mathbb{F},\Gamma)$. It remains only to show that it's onto. To show this, we use a variant of the argument in the proof of Theorem 7.19. If (f,ψ) is an automorphism of $\mathbf{Def}_k(\mathbb{F},\Gamma)$ over $\widehat{\mathcal{H}}(n)$, we again write

$$f(G,i,\phi) = (G_f, i_f, \phi_f).$$

choose isomorphisms $\bar{\phi}$ and $\bar{\phi}_f$ lifting ϕ and ϕ_f respectively. Then Lemma 7.13 supplies an element $(\sigma,\lambda)\in\mathbb{G}(\mathbb{F},\Gamma)$ so that

$$\psi_G : (G, i\sigma, \bar{\phi}\lambda) \to (G_f, i_f, \bar{\phi}_f).$$

The class of (σ,λ) in $\mathbb{G}(\mathbb{F},\Gamma)/\mathbb{G}_{k+1}(\mathbb{F},\Gamma)$ is independent of the choice and ψ supplies the needed 2-morphism to show surjectivity. □

The groupoid $\mathbf{Def}_0(\mathbb{F},\Gamma)$ has a simple description. Indeed, $\mathbf{Def}_0(\mathbb{F},\Gamma)$ assigns to each Artin local ring A the pairs (G,i) where $i:\mathbb{F}\to\mathbb{A}/\mathfrak{m}$ is an isomorphism. Since $\mathbb{F}$ is perfect, the universal property of Witt vectors ([4] §III.3) implies there is a unique homomorphism of rings $W(\mathbb{F})\to A$ which reduces to $\mathbb{F}$ modulo maximal ideals. Thus, we conclude that $\mathbf{Def}_0(\mathbb{F},\Gamma)$ is the functor from groupoids to which assigns to each Artin local $W(\mathbb{F})$-algebra A so that

$$W(\mathbb{F})/(p)\longrightarrow A/\mathfrak{m}$$

is an isomorphism the groupoid of formal groups G over A so that $\mathcal{I}_n(G)\subseteq\mathfrak{m}$ and $\mathcal{I}_{n+1}(G)=A$. Thus we have proved:

7.21 Lemma. *There is a natural isomorphism of categories fibered in groupoids over* $\mathbf{Art}_{\mathbb{F}}$

$$\mathbf{Def}_0(\mathbb{F},\Gamma)\xrightarrow{\cong} W(\mathbb{F})\otimes_{\mathbb{Z}_p}\widehat{\mathcal{H}}(n).$$

7.22 Definition. *A representable morphism $q:X\to Y$ of sheaves of groupoids in the fpqc-topology is* **Galois** *if q is representable, faithfully flat, and if the natural map*

$$X\times\mathrm{Aut}_Y(X)\longrightarrow X\times_Y X$$

is an equivalence of groupoids over X.

The main result of the section is now as follows.

7.23 Theorem. *Let $\mathbb{F} = \bar{\mathbb{F}}_p$ be the algebraic closure of the prime field and let Γ be any height n-formal group over $\mathbb{F}_p$. Then*

$$q : \mathbf{Def}(\mathbb{F}, \Gamma) \longrightarrow \widehat{\mathcal{H}}(n)$$

is Galois with Galois-group

$$\mathbb{G}(\mathbb{F}, \Gamma) = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p) \ltimes \mathrm{Aut}_{\bar{\mathbb{F}}_p}(\Gamma).$$

The discrete groupoid $\mathbf{Def}(\mathbb{F}, \Gamma) \simeq \mathrm{Spf}(R(\mathbb{F}, \Gamma))$ itself has no non-trivial étale covers, so the morphism q is the universal cover.

Proof. To get that q is Galois, combine Proposition 7.11, Theorem 7.17, Theorem 7.19, Lemma 7.20, and Lemma 7.21. That $R(\mathbb{F}, \Gamma)$ has no non-trivial étale extensions follows from the fact that this ring is complete, local, and has an algebraically closed residue field. $\square$

7.24 Remark. All of these results can be rewritten in terms of the Lubin-Tate ring $R(\mathbb{F}, \Gamma)$ of Theorem 7.4 if we wish. For example, we can define a homomorphism

$$\mathbb{G}(\mathbb{F}, \Gamma) \longrightarrow \mathrm{Aut}_{\widehat{\mathcal{H}}(n)}(\mathrm{Spf}(R(\mathbb{F}, \Gamma))$$

as follows. We refer to Remark 7.5. Let (H, j, ϕ_u) be the universal deformation over $\mathrm{Spf}(R(\mathbb{F}, \Gamma))$ and let (σ, λ) be in $\mathbb{G}(\mathbb{F}, \Gamma)$. Then we get a new deformation $(H, j\sigma, \phi_u \lambda)$ over $\mathrm{Spf}(R(\mathbb{F}, \Gamma)$, classified by a map

$$f = f_{(\sigma,\lambda)} : \mathrm{Spf}(R(\mathbb{F}, \Gamma)) \to \mathrm{Spf}(R(\mathbb{F}, \Gamma)).$$

Thus there is a unique isomorphism of deformations

$$\psi = \psi_{(\sigma,\lambda)} : (H, j, \phi_u) \to f^*(H, j, \phi_u).$$

The pair $(f_{(\sigma,\lambda)}, \psi_{(\sigma,\lambda)})$ now produces the 2-commuting diagram

$$\begin{tikzcd} \mathrm{Spf}(R(\mathbb{F}, \Gamma)) \arrow[rr, "f_{(\sigma,\lambda)}"] \arrow[dr] & & \mathrm{Spf}(R(\mathbb{F}, \Gamma)) \arrow[dl] \\ & \widehat{\mathcal{H}}(n). & \end{tikzcd}$$

and the assignment

$$(\sigma, \lambda) \longmapsto (f_{(\sigma,\lambda)}, \psi_{(\sigma,\lambda)})$$

defines a group homomorphism

$$\mathbb{G}(\mathbb{F}, \Gamma) \longrightarrow \mathrm{Aut}_{\widehat{\mathcal{H}}(n)}(\mathrm{Spf}(R(\mathbb{F}, \Gamma))).$$

Theorem 7.19 then becomes the following result.

7.25 Proposition. *This homomorphism*

$$\mathbb{G}(\mathbb{F},\Gamma)\longrightarrow \mathrm{Aut}_{\widehat{\mathcal{H}}(n)}(\mathrm{Spf}(R(\mathbb{F},\Gamma))).$$

is an isomorphism.

Theorem 7.23 then reads as follows:

7.26 Theorem. *Let Γ be a formal group of height n over $\mathbb{F}_p$ and let*

$$q : \mathrm{Spf}(R(\bar{\mathbb{F}}_p,\Gamma))\longrightarrow\widehat{\mathcal{H}}(n)$$

classify a universal deformation of Γ regarded as a formal group over $\bar{\mathbb{F}}_p$. Then q is the universal cover of the formal neighborhood $\widehat{\mathcal{H}}(n)$ of Γ; specifically, q is pro-étale and Galois with Galois group the big Morava stabilizer group

$$\mathbb{G}(\bar{\mathbb{F}}_p,\Gamma)) \cong \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \rtimes \mathrm{Aut}_{\bar{\mathbb{F}}_p}(\Gamma).$$

7.27 Remark. Theorem 7.17 is the restatement of the well-known calculation of the homology cooperations in Lubin-Tate theory. There is a 2-periodic homology theory $E(\bar{\mathbb{F}}_p,\Gamma)$ with $E(\bar{\mathbb{F}}_p,\Gamma)_0 = R(\bar{\mathbb{F}}_p,\Gamma)$ and whose associated formal group is a choice of universal deformation of Γ. The $E(\bar{\mathbb{F}}_p,\Gamma)$ is Landweber exact and

$$\begin{aligned} E(\bar{\mathbb{F}}_p,\Gamma)_0E(\bar{\mathbb{F}}_p,\Gamma) &\overset{\mathrm{def}}{=} \pi_0 L_{K(n)}(E(\bar{\mathbb{F}}_p,\Gamma)\wedge E(\bar{\mathbb{F}}_p,\Gamma)) \\ &\cong \mathrm{map}(\mathbb{G}(\bar{\mathbb{F}}_p,\Gamma), R(\bar{\mathbb{F}}_p,\Gamma)) \end{aligned}$$

where $\mathrm{map}(-,-)$ is the set of continuous maps. Proofs of this statement can be found in [49] and [19]; indeed, the argument here is very similar to Hovey's.

7.28 Remark. Theorem 7.26 allows us to interpret quasi-coherent sheaves on $\widehat{\mathcal{H}}(n)$ as quasi-coherent sheaves on $\mathrm{Spf}(R(\bar{\mathbb{F}}_p,\Gamma))$ with a suitable $\mathbb{G}(\mathbb{F},\Gamma)$ action. Let's spell this out in more detail.

Let $\mathfrak{m} = \mathcal{I}_n(H) \subseteq R(\bar{\mathbb{F}}_p,\Gamma)$, where H is any choice of the universal deformation. Recall that a a quasi-coherent sheaf on $\mathrm{Spf}(R(\bar{\mathbb{F}}_p,\Gamma))$ is determined by a tower

$$\cdots \to M_k \to M_{k-1} \to \cdots \to M_1$$

where M_k is an $R(\bar{\mathbb{F}}_p,\Gamma)/\mathfrak{m}^k$-module, $M_k \to M_{k-1}$ is a $R(\bar{\mathbb{F}}_p,\Gamma)/\mathfrak{m}^k$-module homomorphism and

$$R(\bar{\mathbb{F}}_p,\Gamma)/\mathfrak{m}^{k-1} \otimes_{R(\bar{\mathbb{F}}_p,\Gamma)/\mathfrak{m}^k} M_k\longrightarrow M_{k-1}$$

is an isomorphism. Under appropriate finiteness conditions, this tower is determined by its inverse limit $\lim M_k$ regarded as a continuous $R(\bar{\mathbb{F}}_p,\Gamma)$-module.

A quasi-coherent sheaf on

$$\mathrm{Spf}(R(\bar{\mathbb{F}}_p,\Gamma)) \times_{\widehat{\mathcal{H}}(n)} \mathrm{Spf}(R(\bar{\mathbb{F}}_p,\Gamma)) \cong \mathrm{Spf}(\mathrm{map}(\mathbb{G}(\bar{\mathbb{F}}_p,\Gamma), R(\bar{\mathbb{F}}_p,\Gamma)))$$

has a similar description as modules over the tower

$$\{\mathrm{map}(\mathbb{G}(\bar{\mathbb{F}}_p,\Gamma), R(\bar{\mathbb{F}}_p,\Gamma)/\mathfrak{m}^n)\}.$$

A *Morava module* is a tower of $R(\bar{\mathbb{F}}_p, \Gamma)$-modules

$$\cdots \to M_k \to M_{k-1} \to \cdots \to M_1$$

so that

1. M_k is an $R(\bar{\mathbb{F}}_p, \Gamma)/\mathfrak{m}^k$-module and the induced map

$$R(\bar{\mathbb{F}}_p, \Gamma)/\mathfrak{m}^{k-1} \otimes_{R(\bar{\mathbb{F}}_p,\Gamma)/\mathfrak{m}^k} M_k \longrightarrow M_{k-1}$$

   is an isomorphism;

2. M_k has a continuous $\mathbb{G}(\bar{\mathbb{F}}_p, \Gamma)$-action, where M_k has the discrete topology;

3. the action of $\mathbb{G}(\bar{\mathbb{F}}_p, \Gamma)$ is twisted over $R(\bar{\mathbb{F}}_p, \Gamma)$ in the sense that if $a \in R(\bar{\mathbb{F}}_p, \Gamma)$, $x \in M_k$, and $g \in \mathbb{G}(\bar{\mathbb{F}}_p, \Gamma)$, then

$$g(ax) = g(a)g(x).$$

Now Theorem 7.26 implies there in equivalence of categories between quasi-coherent sheaves on $\widehat{\mathcal{H}}(n)$ and Morava modules.

8 Completion and chromatic convergence

In this section we give the recipe for recovering a coherent sheaf on $\mathcal{M}_{\mathbf{fg}}$ (over $\mathbb{Z}_{(p)}$) from its restrictions to each of the open substacks of formal groups of height less than or equal to n. This has two steps: passing from one height to the next via a fracture square (Theorem 8.18) and then taking a derived inverse limit (Theorem 8.22). The latter theorem has particular teeth as the union of the open substacks of finite height is not all of $\mathcal{M}_{\mathbf{fg}}$.

The reader of the homotopy theory literature will see that, in the end, our arguments, are not so different from the Hopkins-Ravenel Chromatic Convergence of [45].

8.1 Local cohomology and scales

We begin by recalling some notation from Definition 6.9 and Proposition 5.10. Let $f : \mathcal{N} \to \mathcal{M}_{\mathbf{fg}}$ be a representable, separated, and flat morphism of algebraic stacks. We will confuse the ideal sheaves $\mathcal{I}_n$ defining the height filtration with the pull-backs $f^*\mathcal{I}_n$, which induce the height filtration on $\mathcal{N}$. Thus, we let

$$0 = \mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{O}_{\mathcal{N}}$$

denote the resulting scale on $\mathcal{N}$. Let $\mathcal{N}(n) = \mathcal{M}(n) \times_{\mathcal{M}_{\mathbf{fg}}} \mathcal{N} \subseteq \mathcal{N}$ be the closed substack defined by $\mathcal{I}_n$ and let $\mathcal{V}(n-1)$ be the open complement. We will write $i_n : \mathcal{V}(n) \to \mathcal{N}$ and $j_n : \mathcal{N}(n) \to \mathcal{N}$ for the inclusions. Finally, let's write $\mathcal{O}$ for $\mathcal{O}_{\mathcal{N}}$.

If $\mathcal{F}$ is a quasi-coherent $\mathcal{I}_n$-torsion sheaf, we defined (in 6.17)

$$\mathcal{F}[v_n^{-1}] = (i_n)_* i_n^* \mathcal{F}.$$

The notation was justified in Remark 6.16. A local description of this sheaf was given in Proposition 6.15.

We wish to recursively define quasi-coherent sheaves $\mathcal{O}/\mathcal{I}_n^\infty$ on $\mathcal{N}$ by setting $\mathcal{O}/\mathcal{I}_0^\infty = \mathcal{O}$ and then defining $\mathcal{O}/\mathcal{I}_{n+1}^\infty$ by the short exact sequence

$$0 \to \mathcal{O}/\mathcal{I}_n^\infty \to \mathcal{O}/\mathcal{I}_n^\infty[v_n^{-1}] \to \mathcal{O}/\mathcal{I}_{n+1}^\infty \to 0. \tag{8.1}$$

In order to do this, we must prove the following lemma. In the process, give local descriptions on these sheaves. See Equations 8.2 and 8.3.

8.1 Lemma. *For all $n \geq 0$, the sheaf $\mathcal{O}/\mathcal{I}_n^\infty$ is an $\mathcal{I}_n$-torsion sheaf and the unit of the adjunction*

$$\mathcal{O}/\mathcal{I}_n^\infty \to (i_n)_* i_n^* \mathcal{O}/\mathcal{I}_n^\infty = \mathcal{O}/\mathcal{I}_n^\infty[v_n^{-1}]$$

is injective.

Proof. Both statements are local, so can be proved by evaluating on an affine morphism $\mathrm{Spec}(R) \to \mathcal{M}$ which is flat and quasi-compact. By taking a faithfully flat extension if necessary, we may assume that are elements $u_n \in R$ so that

$$u_n + \mathcal{I}_n(R) \in \mathcal{O}/\mathcal{I}_n(R) \cong R/(u_0, \cdots, u_{n-1})$$

is a generator of $\mathcal{I}_n(R)/\mathcal{I}_{n-1}(R)$. Since we have a scale, multiplication by u_n on $R/(u_0, \cdots, u_{n-1})$ is injective. Define R-modules $R/(u_0^\infty, \cdots, u_{n-1}^\infty)$ inductively by beginning with R and by insisting there be a short exact sequence

$$0 \to R/(u_0^\infty, \cdots, u_{n-1}^\infty) \to R/(u_0^\infty, \cdots, u_{n-1}^\infty)[u_n^{-1}] \to R/(u_0^\infty, \cdots, u_n^\infty) \to 0.$$

Then inductively we have, using Proposition 6.15

$$\mathcal{O}/\mathcal{I}_n^\infty(R) = R/(u_0^\infty, \cdots, u_{n-1}^\infty) \tag{8.2}$$

and

$$(i_n)_* i_n^* \mathcal{O}/\mathcal{I}_n^\infty(R) = R/(u_0^\infty, \cdots, u_{n-1}^\infty)[u_n^{-1}]. \tag{8.3}$$

The result now follows. □

We note that Proposition 6.15 also implies:

8.2 Lemma. *For all $n > 0$ and all $s > 0$*

$$R^s(i_n)_* i_n^* \mathcal{O}/\mathcal{I}_n^\infty = 0.$$

8.3 Remark (Triangles and fiber sequences). In the rest of the section, we are going to use a shift functor on (co-)chain complexes of sheaves determined by the following equation. If C is a cochain complex and n is an integer, then

$$H^s C[n] = H^{s+n} C.$$

If C is a chain complex, then we regard it as a cochain complex by the equation $H^s C = H_{-s} C$; thus, $H_s C[n] = H_{s-n} C$. A distinguished triangle of cochain complexes

$$A \to B \to C \to A[1]$$

induces a long exact sequence in cohomology

$$\cdots \to H^s A \to H^s B \to H^s C \to H^s A[1] = H^{s+1} A \to \cdots$$

To shorten notation we may revert to the homotopy theory conventions and say that $A \to B \to C$ is a fiber sequence in cochain complexes.

If M is a sheaf, we may regard it as a cochain complex in degree zero; hence

$$H^s M[-n] = \begin{cases} M, & s = n; \\ \\ 0, & s \neq n. \end{cases}$$

We now introduce local cohomology, which will be an important tool for the rest of this section.

8.4 Definition. *Let $Z \subseteq \mathcal{N}$ be any closed substack with open complement $i : U \to \mathcal{N}$. If $\mathcal{F}$ is a quasi-coherent sheaf on $\mathcal{N}$, define the derived* **local cohomology sheaf** *of $\mathcal{F}$ by the the distinguished triangle*

$$R\Gamma_Z(\mathcal{N}, \mathcal{F}) \to \mathcal{F} \to Ri_* i^* \mathcal{F} \to R\Gamma_Z(\mathcal{N}, \mathcal{F})[1]. \tag{8.4}$$

Put another way, $R\Gamma_Z(\mathcal{N}, \mathcal{F})$ is the homotopy fiber of the map $\mathcal{F} \to Ri_* i^* \mathcal{F}$.

If $\mathcal{N}$ is understood, we may write $R\Gamma_Z \mathcal{F}$ for $R\Gamma_Z(\mathcal{N}, \mathcal{F})$; if Z is defined by an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}$, we may write $R\Gamma_{\mathcal{I}} \mathcal{F}$ for $R\Gamma_Z(\mathcal{N}, \mathcal{F})$.

The *local cohomology* of $\mathcal{F}$ at Z is then the graded cohomology sheaf

$$H^*_Z(\mathcal{N}, \mathcal{F}) \overset{\text{def}}{=} H^* R\Gamma_Z(\mathcal{N}, \mathcal{F}).$$

If $V \to \mathcal{N}$ is an open morphism in our topology, then

$$\Gamma_Z(\mathcal{N}, \mathcal{F})(V) = H^0_Z(\mathcal{N}, \mathcal{F})(V)$$

is the set of sections $s \in \mathcal{F}(V)$ which vanish when restricted to $\mathcal{F}(U \times_{\mathcal{N}} V)$. If $\mathcal{I}$ is locally generated by a regular sequence, then we can give the following local description of $\Gamma_Z(\mathcal{N}, \mathcal{F})$. Let $\mathrm{Spec}(R) \to \mathcal{N}$ be any morphism so that $\mathcal{I}(R)$ is generated by a regular sequence $u_0, \ldots, u_{n-1}$. Then there is an exact sequence

$$\Gamma_Z(\mathcal{N}, \mathcal{F})(R) \to \mathcal{F}(R) \to \prod_i \mathcal{F}(R)[u_i^{-1}]. \tag{8.5}$$

This has the following consequence. See Corollary 3.2.4 of [1] for a generalization.

8.5 Lemma. *Suppose that the ideal $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{N}} = \mathcal{O}$ defining the closed substack $Z \subseteq \mathcal{N}$ is locally generated by a regular sequence. Then for any quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{N}$ there is a natural equivalence*

$$\operatorname{colim}_k R\operatorname{hom}(\mathcal{O}/\mathcal{I}^k, \mathcal{F}) \xrightarrow{\simeq} R\Gamma_Z(\mathcal{N}, \mathcal{F}).$$

Proof. Before taking derived functors, we note that there is certainly a natural map

$$\operatorname{colim}_k \operatorname{Hom}(\mathcal{O}/\mathcal{I}^k, \mathcal{F}) \longrightarrow \Gamma_Z(\mathcal{N}, \mathcal{F})$$

given by evaluating at the unit. We first prove that this is an isomorphism; for this it is sufficient to work locally. Let $\operatorname{Spec}(R) \to \mathcal{M}$ where $\mathcal{I}(R)$ is generated by the regular sequence $u_0, \cdots, u_{k-1}$. Then the exact sequence of 8.5 implies that $x \in \mathcal{F}(R)$ is in $\Gamma_Z(\mathcal{M}, \mathcal{F})(R)$ if and only if for all i there is a t_i so that $u_i^{t_i} x = 0$. This yields the desired (underived) isomorphism. Since colimit is exact on filtered diagrams, the derived version follows. □

We now set $Z = \mathcal{N}(n+1)$, defined by $\mathcal{I}_{n+1}$, so that $\mathcal{V}(n) = \mathcal{N} - \mathcal{N}(n+1)$ and there is a distinguished triangle

$$R\Gamma_{\mathcal{N}(n+1)}\mathcal{F} \to \mathcal{F} \to R(i_n)_* i_n^* \mathcal{F} \to R\Gamma_{\mathcal{N}(n+1)}\mathcal{F}[1].$$

If $\mathcal{F}$ is a quasi-coherent $\mathcal{I}_n$-torsion sheaf, then Proposition 6.15 applies and $(i_n)_* i_n^* \mathcal{F} = \mathcal{F}[v_n^{-1}]$.

The exact sequence defining $\mathcal{O}/\mathcal{I}_n^\infty$ and Lemmas 8.1 and 8.2 imply following result.

8.6 Lemma. *For all $n \geq 1$ there is an isomorphism in the derived category*

$$R\Gamma_{\mathcal{N}(n)}(\mathcal{N}, \mathcal{O}/\mathcal{I}_{n-1}^\infty) \cong \mathcal{O}/\mathcal{I}_n^\infty[-1].$$

We also have the following key calculation.

8.7 Proposition. *For all $n \geq 1$ there is an equivalence in the derived category of quasi-coherent sheaves*

$$R\Gamma_{\mathcal{N}(n)}(\mathcal{N}, \mathcal{O}) \simeq \mathcal{O}/\mathcal{I}_n^\infty[-n].$$

That is,

$$H^s_{\mathcal{N}(n)}(\mathcal{N}, \mathcal{O}) \cong \begin{cases} 0, & s \neq n; \\ \mathcal{O}/\mathcal{I}_n^\infty, & s = n. \end{cases}$$

Proof. We proceed by induction to show that

$$R\Gamma_{\mathcal{N}(n)}(\mathcal{N}, \mathcal{O}/\mathcal{I}_{n-k}^\infty) \simeq \mathcal{O}/\mathcal{I}_n^\infty[-k].$$

Lemma 8.6 is the case $k = 1$. To get the inductive case, we have an exact sequence

$$0 \to \mathcal{O}/\mathcal{I}_{n-(k+1)}^\infty \to (i_{n-k})_* i_{n-k}^* \mathcal{O}/\mathcal{I}_{n-(k+1)}^\infty \to \mathcal{O}/\mathcal{I}_{n-k}^\infty \to 0.$$

Hence we need to show that

$$R\Gamma_{\mathcal{N}(n)}(i_{n-k})_*i_{n-k}^*\mathcal{O}/\mathcal{I}_{n-(k+1)}^\infty = 0,$$

or equivalently that

$$(i_{n-k})_*i_{n-k}^*\mathcal{O}/\mathcal{I}_{n-(k+1)}^\infty \to R(i_{n-1})_*i_{n-1}^*(i_{n-k})_*i_{n-k}^*\mathcal{O}/\mathcal{I}_{n-(k+1)}^\infty$$

is an equivalence. Consider the sequence of inclusions

$$\begin{tikzcd} \mathcal{V}(n-k) \arrow[r, "f"] \arrow[rr, bend right, "i_{n-k}"'] & \mathcal{V}(n-1) \arrow[r, "i_{n-1}"] & \mathcal{N} \end{tikzcd}$$

We easily check that $i_{n-1}^*(i_{n-k})_* = f_*$; since i_{n-1}^* is exact we have an equivalence

$$R(i_{n-k})_*i_{n-k}^*\mathcal{O}/\mathcal{I}_{n-(k+1)}^\infty \to R(i_{n-1})_*i_{n-1}^*(i_{n-k})_*i_{n-k}^*\mathcal{O}/\mathcal{I}_{n-(k+1)}^\infty$$

The result now follows from Lemma 8.2. □

8.8 Theorem. *Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{N}$. Then there are natural equivalences in the derived category*

$$R\Gamma_{\mathcal{N}(n)}(\mathcal{N},\mathcal{F}) \simeq \mathcal{O}/\mathcal{I}_n^\infty[-n] \otimes_{\mathcal{O}}^L \mathcal{F}$$

Proof. This follows immediately from Lemma 8.5 and Proposition 8.7; indeed, since $\mathcal{O}/\mathcal{I}_n^k$ is locally finitely presented

$$\begin{aligned} R\Gamma_{\mathcal{N}(n)}(\mathcal{N},\mathcal{F}) &\simeq \operatorname{colim} R\hom(\mathcal{O}/\mathcal{I}_n^k,\mathcal{F}) \\ &\simeq \operatorname{colim} R\hom(\mathcal{O}/\mathcal{I}_n^k,\mathcal{O}) \otimes^L \mathcal{F} \\ &\simeq R\Gamma_{\mathcal{N}(n)}(\mathcal{N},\mathcal{O}) \otimes^L \mathcal{F}. \end{aligned}$$

□

Another consequence of Lemma 8.5 and Proposition 8.7 is the following result.

8.9 Proposition. *There is an equivalence*

$$\operatorname{colim}_k R\hom(\mathcal{O}/\mathcal{I}_n^k,\mathcal{O}) \xrightarrow{\cong} \mathcal{O}/\mathcal{I}_n^\infty[-n].$$

We also will be interested in what happens if we vary n. Consider the sequence of inclusions

$$\begin{tikzcd} \mathcal{V}(n-1) \arrow[r, "f"] \arrow[rr, bend right, "i_{n-1}"'] & \mathcal{V}(n) \arrow[r, "i_n"] & \mathcal{N} \end{tikzcd}$$

Recall that $\mathcal{V}(n)$ is the complement of $\mathcal{N}(n+1)$. In the case where $\mathcal{N} = \mathcal{M}_{\mathbf{fg}}$, $\mathcal{N}(n) = \mathcal{M}(n)$ classifies formal groups of height at least n, $\mathcal{V}(n) = \mathcal{U}(n)$ classifies formal groups of height at most n and $\mathcal{H}(n) = \mathcal{M}(n) \cap \mathcal{U}(n)$ classifies formal groups of exact height n.

8.10 Lemma. *For all quasi-coherent $\mathcal{F}$ on $\mathcal{N}$, there are fiber sequences of cochain complexes of quasi-coherent sheaves*

$$R(i_n)_* i_n^* R\Gamma_{\mathcal{N}(n)}\mathcal{F} \to R(i_n)_*(i_n)^*\mathcal{F} \to R(i_{n-1})_*(i_{n-1})^*\mathcal{F}$$

and

$$R\Gamma_{\mathcal{N}(n+1)}\mathcal{F} \to R\Gamma_{\mathcal{N}(n)}\mathcal{F} \to R(i_n)_* i_n^* R\Gamma_{\mathcal{N}(n)}\mathcal{F}$$

Proof. The fiber sequence which defines local cohomology (see Definition 8.4) yields that these sequences are equivalent; so, we prove the first.

For any quasi-coherent sheaf on $\mathcal{N}$, we have a fiber sequence

$$R\Gamma_{\mathcal{H}(n)}(\mathcal{V}(n), i_n^*\mathcal{F}) \to i_n^*\mathcal{F} \to Rf_* i_{n-1}^*\mathcal{F}. \tag{8.6}$$

Here we have taken the liberty of writing $\mathcal{H}(n)$ for $\mathcal{V}(n) \cap \mathcal{N}(n)$ and we have used $f^* i_n^* \cong i_{n-1}^*$.

Next note that the adjoint to the equivalence $R(i_{n-1})_* \cong R(i_n)_* Rf_*$ yields a commutative diagram

$$\begin{tikzcd}
i_n^* R\Gamma_{\mathcal{N}(n)}(\mathcal{N}, \mathcal{F}) \arrow[r] \arrow[d] & i_n^*\mathcal{F} \arrow[r] \arrow[d, "="] & i_n^* R(i_{n-1})_* i_{n-1}^*\mathcal{F} \arrow[d] \\
R\Gamma_{H(n)}(\mathcal{V}(n), i_n^*\mathcal{F}) \arrow[r] & i_n^*\mathcal{F} \arrow[r] & Rf_* i_{n-1}^*\mathcal{F}.
\end{tikzcd}$$

Finally, for all quasi-coherent sheaves $\mathcal{E}$ on $\mathcal{V}(n-1)$, the natural map

$$i_n^* R(i_{n-1}^*)\mathcal{E} \longrightarrow Rf_*\mathcal{E}$$

is an equivalence; indeed, we easily check that $i_n^*(i_{n-1})_*\mathcal{E} \to f_*\mathcal{E}$ is an isomorphism and then we use that i_n^* is exact. From this we conclude that

$$R\Gamma_{H(n)}(\mathcal{V}(n), i_n^*\mathcal{F}) \longrightarrow i_n^* R\Gamma_{\mathcal{N}(n)}(\mathcal{N}, \mathcal{F})$$

is an equivalence. We feed this into Equation 8.6 and apply $R(i_n)_*$ to get the result. $\square$

Applying the second of the fiber sequences of Lemme 8.10 to $\mathcal{F} = \mathcal{O}$ itself and using Theorem 8.8, we get the fiber sequence

$$\mathcal{O}/\mathcal{I}_{n+1}^\infty[-n-1] \to \mathcal{O}/\mathcal{I}_n^\infty[-n] \to \mathcal{O}/\mathcal{I}_n^\infty[v^{-1}][-n] \tag{8.7}$$

which is the evident shift of the defining sequence 8.1. From this we obtain the following result.

8.11 Lemma. *There is a natural commutative diagram*

$$\begin{tikzcd}
R\Gamma_{\mathcal{N}(n+1)}(\mathcal{N}, \mathcal{F}) \arrow[r] \arrow[d, "\simeq"'] & R\Gamma_{\mathcal{N}(n)}(\mathcal{N}, \mathcal{F}) \arrow[d, "\simeq"] \\
\mathcal{O}/\mathcal{I}_{n+1}^\infty[-n-1] \otimes_{\mathcal{O}}^L \mathcal{F} \arrow[r] & \mathcal{O}/\mathcal{I}_n^\infty[-n] \otimes_{\mathcal{O}}^L \mathcal{F}
\end{tikzcd}$$

where the bottom morphism is the boundary morphism induced by the short exact sequence

$$0 \to \mathcal{O}/\mathcal{I}_n^\infty \to \mathcal{O}/\mathcal{I}_n^\infty[v_n^{-1}] \to \mathcal{O}/\mathcal{I}_{n+1}^\infty \to 0.$$

8.2 Greenlees-May duality

There is a remarkable duality between local cohomology and completion first noticed by Greenlees and May [10] and globalized in [1]. Similar results appear in [3], which also has the general version of the fracture square we will write down below in Theorem 8.17. The techniques of [1] apply directly to the case of a qausi-compact and separated stack $\mathcal{N}$ and the closed substacks $\mathcal{N}(n) \subseteq \mathcal{N}$ arising from a scale. The main result we'll use is the following. Derived completion was defined in Definition 6.4.

8.12 Proposition. *For all quasi-coherent sheaves $\mathcal{F}$ on $\mathcal{M}$ there is a natural equivalence*

$$L(\mathcal{F})^{\wedge}_{\mathcal{N}(n)} \simeq R\,\mathrm{hom}(\mathcal{O}/\mathcal{I}_n^{\infty}[-n], \mathcal{F}).$$

This result is actually equivalent to an apparently stronger result – *Greenlees-May duality*:

8.13 Theorem. *Let $\mathcal{E}$ and $\mathcal{F}$ be two chain complexes of quasi-coherent sheaves on $\mathcal{M}$. Then there is a natural equivalence*

$$R\,\mathrm{hom}(R\Gamma_{\mathcal{N}(n)}\mathcal{E}, \mathcal{F}) \simeq R\,\mathrm{hom}(\mathcal{E}, L(\mathcal{F})^{\wedge}_{\mathcal{N}(n)}).$$

Certainly Theorem 8.13 implies Proposition 8.12 by setting $\mathcal{E} = \mathcal{O}_{\mathcal{M}}$ and applying Proposition 8.7. Conversely, Theorem 8.8 gives a natural isomorphism

$$\begin{aligned} R\,\mathrm{hom}(R\Gamma_{\mathcal{N}(n)}\mathcal{E}, \mathcal{F}) &\cong R\,\mathrm{hom}(\mathcal{O}/\mathcal{I}_n^{\infty}[-n] \otimes^L \mathcal{E}, \mathcal{F}) \\ &\cong R\,\mathrm{hom}(\mathcal{E}, R\,\mathrm{hom}(\mathcal{O}/\mathcal{I}_n^{\infty}[-n], \mathcal{F})). \end{aligned}$$

Hence Theorem 8.13 follows from Proposition 8.12.

The argument to prove Proposition 8.12 goes exactly as in [1]; hence we will content ourselves with giving an outline.

Lemma 8.5 allows us to define a natural map

$$\Phi : L(\mathcal{F})^{\wedge}_{\mathcal{N}(n)} \longrightarrow R\,\mathrm{hom}(R\Gamma_{\mathcal{N}(n)}(\mathcal{N}, \mathcal{O}), \mathcal{F})$$

as follows. First note that for $\mathcal{O}$-module sheaves $\mathcal{E}$ and $\mathcal{F}$, there is a natural map

$$\mathcal{E} \otimes \mathcal{F} \longrightarrow \mathrm{Hom}(\mathrm{Hom}(\mathcal{E}, \mathcal{O}), \mathcal{F}) \tag{8.8}$$

given pointwise by sending $x \otimes y$ to the homomorphism $\phi_{x\otimes y}$ with

$$\phi_{x\otimes y}(f) = f(x)y.$$

The morphism of Equation 8.8 can be derived to an morphism

$$\mathcal{E} \otimes^L \mathcal{F} \longrightarrow R\mathrm{Hom}(R\mathrm{Hom}(\mathcal{E}, \mathcal{O}), \mathcal{F}).$$

Now Φ is defined as the composition

$$\begin{aligned} L(\mathcal{F})^{\wedge}_{\mathcal{N}(n)} = \mathrm{holim}(\mathcal{F} \otimes^L \mathcal{O}/\mathcal{I}_n^k) &\to \mathrm{holim}\, R\,\mathrm{hom}(R\,\mathrm{hom}(\mathcal{O}/\mathcal{I}_n^k, \mathcal{O}), \mathcal{F}) \\ &\cong R\,\mathrm{hom}(\mathrm{colim}\, R\,\mathrm{hom}(\mathcal{O}/\mathcal{I}_n^k, \mathcal{O}), \mathcal{F}) \\ &\cong R\,\mathrm{hom}(R\Gamma_{\mathcal{N}(n)}(\mathcal{N}, \mathcal{O}), \mathcal{F}). \end{aligned}$$

Proposition 8.12 now can be restated as

8.14 Proposition. *For all quasi-coherent sheaves $\mathcal{F}$, the natural map*

$$\Phi : L(\mathcal{F})^{\wedge}_{\mathcal{N}(n)} \longrightarrow R\hom(R\Gamma_{\mathcal{N}(n)}(\mathcal{N}, \mathcal{O}), \mathcal{F})$$

is an equivalence.

The first observation is that the question is local; that is, it is sufficient to show that there Φ is an equivalence when evaluated at any flat and quasi-compact morphism $\mathrm{Spec}(R) \longrightarrow \mathcal{M}$ for which $\mathcal{I}_n(R)$ is generated by a regular sequence. This follows readily from the definition of completion (6.4) and the remarks immediately afterwards. If we write $I = \mathcal{I}_n(R)$ and $M = \mathcal{F}(R)$, then we are asking that the map

$$\Phi_V : L(M)^{\wedge}_{I} \longrightarrow R\hom(R\Gamma_I(R), M)$$

be an equivalence. This is exactly what Greenlees and May prove. There is a finiteness condition in the argument which is worth emphasizing: for all i. the R-module $R/(u_0^i, \cdots, u_{n-1}^i)$ has a finite resolution by finitely generated free R-modules. The usual such resolution is the Koszul complex, which we now review.

Let R be a commutative ring and let $u \in R$. Define $K(u)$ to the chain complex

$$R \xrightarrow{u} R$$

concentrated in degrees 0 and 1. If $\mathbf{u} = (u_0, \ldots, u_{n-1})$ is an ordered n-tuple of elements in R, define

$$K(\mathbf{u}) = K(u_0) \otimes \cdots \otimes K(u_{n-1}).$$

Note that if $\mathbf{u}$ is a regular sequence in R and I is the ideal generated by $u_0, \ldots, u_{n-1}$, then $K(\mathbf{u})$ is the Kozsul resolution of R/I and

$$H_s(K(\mathbf{u}) \otimes M) \cong \mathrm{Tor}_s^R(R/I, M).$$

Now fix the n-tuple $\mathbf{u}$ and define $\mathbf{u}^i = (u_0^i, \ldots, u_{n-1}^i)$. The commutative squares

$$\begin{tikzcd} R \arrow[r, "u_j^i"] \arrow[d, "u_j"'] & R \arrow[d, "="] \\ R \arrow[r, "u_j^{i-1}"] & R \end{tikzcd}$$

combine to give morphisms $f_i : K(\mathbf{u}^i) \to K(\mathbf{u}^{i-1})$. Thus if the element of $\mathbf{u}$ form a regular sequence,[6] then a simple bicomplex arguments shows that for any R-module M there is an homology isomorphism

$$L(M)^{\wedge}_{I} \simeq \mathrm{holim}_j(K(\mathbf{u}^j) \otimes M).$$

[6]Or, more generally, if the elements of $\mathbf{u}$ are *pro-regular* as in [10].

This equivalence is natural in M, although it doesn't look very natural in R or I.

The dual complex

$$K^*(\mathbf{u}) \stackrel{\text{def}}{=} \operatorname{Hom}_R(K(\mathbf{u}), R)$$

is a chain complex concentrated in degrees s, $-n \le s \le 0$. Note that if the u_i form a regular sequence

(8.9)

$$H_s K^*(\mathbf{u}) = \operatorname{Ext}_R^{-s}(R/(u_0, \ldots, u_{n-1}), R) \cong \begin{cases} R/(u_0, \ldots, u_{n-1}), & s = -n; \\ 0, & s \neq -n. \end{cases}$$

The dual of the maps f_i give maps $f_i^* : K^*(\mathbf{u}^{i-1}) \to K(\mathbf{u}^i)$. Define

$$K^*(\mathbf{u}^\infty) = \operatorname{colim} K^*(\mathbf{u}^i).$$

We have have natural homology equivalences, assuming the elements in $\mathbf{u}$ form a regular sequence:

$$\begin{aligned} K^*(\mathbf{u}^\infty) \otimes M &\simeq \operatorname{colim} K^*(\mathbf{u}^i) \otimes M \\ &\simeq \operatorname{colim} \operatorname{Hom}_R(K(\mathbf{u}^i), M) \\ &\simeq \operatorname{colim} R \operatorname{hom}_R(R/(x_0^i, \ldots, x_{n-1}^i), M) \\ &\simeq R\Gamma_I(M). \end{aligned}$$

More is true, because $K(\mathbf{u}^i)$ is finitely generated as a chain complex of R-modules we have that the map of Equation 8.8

$$K(\mathbf{u}^i) \otimes M \longrightarrow \operatorname{Hom}(K^*(\mathbf{u}^i), M)$$

is an isomorphism, natural in M. Then the local version of Greenlees-May duality follows:

$$\begin{aligned} L(M)_I^\wedge &\simeq \operatorname{holim}(K(\mathbf{u}^j) \otimes R) \\ &\simeq \operatorname{holim} \operatorname{Hom}(K^*(\mathbf{u}^i), M) \\ &\simeq R \operatorname{hom}(\operatorname{colim} K^*(\mathbf{u}^i), M) \\ &\simeq R \operatorname{hom}(R\Gamma_I(R), M). \end{aligned}$$

It is an exercise is bicomplexes to show that this is, up to natural homology equivalence, the map Φ of Proposition 8.14.

8.15 Remark. The isomorphism

$$H_{-s}(K^*(\mathbf{u}^\infty) \otimes M) \cong H_I^s(R, M)$$

developed above extends the exact sequence of Equation 8.5. Indeed, $K^*(\mathbf{u}^\infty)$ is exactly the chain complex

$$M \to \prod_i M[u_i^{-1}] \to \prod_{i_1 < i_2} M[u_{i_1}^{-1} u_{i_2}^{-1}] \to \cdots \to M[u_0^{-1} \cdots u_{k-1}^{-1}] \to 0 \ \cdots .$$

8.3 Algebraic chromatic convergence

We now supply the two results we promised: a fracture square for reconstructing quasi-coherent sheaves for the completions and a decomposition of a coherent sheaf as a homotopy inverse limit.

We begin with a preliminary calculation. Compare Corollary 5.1.1 of [1].

8.16 Theorem. *Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{N}$. Then the natural map*

$$R\Gamma_{\mathcal{N}(n)}\mathcal{F} \longrightarrow R\Gamma_{\mathcal{N}(n)}L(\mathcal{F})^{\wedge}_{\mathcal{N}(n)}$$

is an equivalence.

Proof. The question is local (again) and, therefore, reduces to the following assertion. Let $\mathbf{u} = (u_0, \ldots, u_{n-1})$ be a regular sequence in R, let I be the ideal generated by this regular sequence, and let P be a projective R-module. Then

$$K^*(\mathbf{u}^\infty) \otimes P \longrightarrow K^*(\mathbf{u}^\infty) \otimes (P)^{\wedge}_I$$

is an equivalence. Indeed, if we apply homology to the map

$$K^*(\mathbf{u}^i) \otimes P \longrightarrow K^*(\mathbf{u}^i) \otimes (P)^{\wedge}_I$$

then, by Equation 8.9 we obtain the maps

$$\mathrm{Ext}^s_R(R/(u_0^i, \ldots, u_{n-1}^i), P) \longrightarrow \mathrm{Ext}^s_R(R/(u_0^i, \ldots, u_{n-1}^i), (P)^{\wedge}_I).$$

Both source and target are zero if $s \neq n$ and if $s = n$ we have the map

$$P/(u_0^i, \ldots, u_{n-1}^i) \to (P)^{\wedge}_I/(u_0^i, \ldots, u_{n-1}^i)$$

which is an isomorphism. □

This result has the following fracture square as a consequence. Recall that the open inclusion $i_{n-1} : \mathcal{V}(n-1) \to \mathcal{N}$ is complementary to the closed inclusion $j_n : \mathcal{N}(n) \to \mathcal{N}$.

8.17 Theorem (The fracture squares). *Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{N}$. Then there is a homotopy cartesian square in the derived category*

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & L(\mathcal{F})^{\wedge}_{\mathcal{N}(n)} \arrow[d] \\
R(i_{n-1})_* i_{n-1}^* \mathcal{F} \arrow[r] & R(i_{n-1})_* i_{n-1}^* L(\mathcal{F})^{\wedge}_{\mathcal{N}(n)}.
\end{tikzcd}$$

Proof. The induced morphisms on fibers of the vertical maps is exactly

$$R\Gamma_{\mathcal{N}(n)}\mathcal{F} \longrightarrow R\Gamma_{\mathcal{N}(n)}L(\mathcal{F})^{\wedge}_{\mathcal{N}(n)}$$

which is an equivalence by Proposition 8.16. □

8.18 Remark. An important special case is worth isolating. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{N}$ and consider the sequence of inclusions

(8.10)
$$
\begin{tikzcd}
\mathcal{V}(n-1) \arrow[r, "f"] \arrow[rr, bend right, "i_{n-1}"'] & \mathcal{V}(n) \arrow[r, "i_n"] & \mathcal{N}
\end{tikzcd}
$$

Applying Theorem 8.17 to a complex of sheaves of the form $R(i_n)_* i_n^* \mathcal{F}$ where $\mathcal{F}$ is quasi-coherent on $\mathcal{N}$, we get a homotopy cartesian square

$$
\begin{tikzcd}
R(i_n)_* i_n^* \mathcal{F} \arrow[r] \arrow[d] & L(R(i_n)_* i_n^* \mathcal{F})^{\wedge}_{\mathcal{N}(n)} \arrow[d] \\
R(i_{n-1})_* i_{n-1}^* \mathcal{F} \arrow[r] & R(i_{n-1}) * i_{n-1}^* L(R(i_n)_* i_n^* \mathcal{F})^{\wedge}_{\mathcal{N}(n)}.
\end{tikzcd}
$$

The right hand vertical column of this diagram seems excessively complicated, but expected to those familiar with the results if [20] §7.3. The topological analog of these calculations supplies a fracture square of spectra

$$
\begin{tikzcd}
L_n X \arrow[r] \arrow[d] & L_{K(n)} X \arrow[d] \\
L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X.
\end{tikzcd}
$$

The connection to completion is somewhat less than straightforward and given by the equations

$$L_{K(n)} X \simeq L_{K(n)} L_n X = \operatorname{holim} S/I \wedge L_n X$$

where $\{S/I\}$ is a suitable family of type n complexes.

Despite the unwieldy nature of the diagram of 8.10, the induced map on the homotopy fibers of the vertical map actually simplies somewhat, as the following result shows. Compare Proposition 8.16.

8.19 Proposition. *For all quasi-coherent sheaves $\mathcal{F}$ on $\mathcal{N}$, the natural map*

$$R\Gamma_{\mathcal{N}(n)} \mathcal{F} \to R\Gamma_{\mathcal{N}(n)} R(i_n)_* i_n^* \mathcal{F}$$

is an equivalence.

Proof. This follows from the fact that

$$R(i_{n-1})_* i_{n-1}^* \mathcal{F} \to R(i_{n-1})_* i_{n-1}^* R(i_n)_* i_n^* \mathcal{F}$$

is an equivalence, which in turn follows from the fact that

$$i_{n-1}^* (i_n)_* = f^*$$

and the fact that i_{n-1}^* is exact. $\square$

Now let's specialize to the case where $\mathcal{N} = \mathcal{M}_{\mathbf{fg}}$ itself and $i_n : \mathcal{U}(n) \to \mathcal{M}_{\mathbf{fg}}$ be the inclusion of the open moduli substack of formal groups of height at most n. Then we will show that if $\mathcal{F}$ is a *coherent* sheaf on $\mathcal{M}_{\mathbf{fg}}$, then the natural map

$$\mathcal{F} \to \operatorname{holim} R(i_n)_* i_n^* \mathcal{F}$$

is an isomorphism in the derived category of quasi-coherent sheaves. This is an algebraic analog of chromatic convergence. The statement has teeth, because the open substacks $\mathcal{U}(n)$ do not exhaust $\mathcal{M}_{\mathbf{fg}}$; indeed, the morphism

$$\mathbb{G}_a : \operatorname{Spec}(\mathbb{F}_p) \longrightarrow \mathcal{M}_{\mathbf{fg}}$$

classifying the additive formal group (which has infinite height) does not factor through $\mathcal{U}(n)$ for any n.

The proof is below in Theorem 8.22. The observation that drives the argument in this: recall from Theorem 3.25 that if $\mathcal{F}$ is a coherent sheaf on $\mathcal{M}_{\mathbf{fg}}$, then there is an integer r and a coherent sheaf $\mathcal{F}_0$ on the moduli stack of buds $\mathcal{M}_{\mathbf{fg}}\langle p^r \rangle$ so that $\mathcal{F} \cong q^* \mathcal{F}_0$. Thus we begin with the next computation.

8.20 Theorem. *Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}\langle p^r \rangle$. Then for all $n > r$ and all s, the map on local cohomology groups*

$$H^s_{\mathcal{M}(n+1)}(\mathcal{M}_{\mathbf{fg}}, q^*\mathcal{F}) \to H^s_{\mathcal{M}(n)}(\mathcal{M}_{\mathbf{fg}}, q^*\mathcal{F})$$

is zero.

Proof. We apply Lemma 8.11 and show that the induced map

$$\mathcal{O}/\mathcal{I}_{n+1}^{\infty} \otimes^L_{\mathcal{O}} \mathcal{F} \to \mathcal{O}/\mathcal{I}_n^{\infty}[1] \otimes^L_{\mathcal{O}} \mathcal{F}$$

is zero in homology. It is sufficient to prove this after evaluation at any affine presentation $f : X \to \mathcal{M}_{\mathbf{fg}}$. Let

$$X = \operatorname{Spec}(\mathbb{Z}_{(p)}[u_1, u_2, \ldots]) \overset{\text{def}}{=} \operatorname{Spec}(V)$$

and let f classify the formal group obtained from the universal p-typical formal group law. Similarly, let

$$X_r = \operatorname{Spec}(\mathbb{Z}_{(p)}[u_1, u_2, \ldots, u_r]) \overset{\text{def}}{=} \operatorname{Spec}(V_r) \to \mathcal{M}_{\mathbf{fg}}\langle p^r \rangle$$

classify the resulting bud. This, too, is a presentation, by Lemma 3.22. Let $M = \mathcal{F}(X_r \to \mathcal{M}_{\mathbf{fg}}\langle p^r \rangle)$. Then

$$V \otimes_{V_r} M \cong (q^*\mathcal{F})(X \to \mathcal{M}_{\mathbf{fg}})$$

and we are trying to calculate

$$\operatorname{Tor}^V_s(V/(p^\infty, \ldots, u_n^\infty), V \otimes_{V_r} M) \to \operatorname{Tor}^V_{s-1}(V/(p^\infty, \ldots, u_{n-1}^\infty), V \otimes_{V_r} M).$$

Since V is a free V_r-module, to see this homomorphism is zero it is sufficient to note that

$$V/(p^\infty, \ldots, u_{n-1}^\infty) \to V/(p^\infty, \ldots, u_{n-1}^\infty)[u_n^{-1}]$$

is split injective as a V_r-module as long as $n > r$. □

8.21 Corollary. *Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}\langle p^r\rangle$. Then*

$$H_s(\mathcal{O}/\mathcal{I}_n^\infty \otimes^L q^*\mathcal{F}) = 0$$

for $s > r$.

Proof. Again one can work locally, using the presentations of the previous proof. We prove the result by induction on n. If $n \leq r$ the chain complex $K^*(\mathbf{u}^\infty)$ of V-modules supplies a resolution of length n of $V/(p^\infty, \ldots, u_{n-1}^\infty)$ by flat V-modules; therefore,

$$H_s(\mathcal{O}/\mathcal{I}_n^\infty \otimes^L q^*\mathcal{F}) = 0, \qquad s > n.$$

So we may assume $n > r$. Then the previous result and the induction hypothesis imply that

$$H_s(R(i_n)_* i_n^* \mathcal{O}/\mathcal{I}_{n-1}^\infty \otimes^L q^*\mathcal{F}) \cong H_s(\mathcal{O}/\mathcal{I}_n^\infty \otimes^L q^*\mathcal{F})$$

for $s > r$. Evaluated at $\mathrm{Spec}(V) \to \mathcal{M}_{\mathbf{fg}}$ this is an isomorphism

$$\mathrm{Tor}_s^V(V/(p^\infty, \ldots, u_{n-1}^\infty)[u_n^{-1}], V \otimes_{V_r} M) \cong \mathrm{Tor}_s^V(V/(p^\infty, \ldots, u_n^\infty), V \otimes_{V_r} M).$$

Since $n > r$, we have

$$\begin{aligned} \mathrm{Tor}_s^V(V/(p^\infty, \ldots, &u_{n-1}^\infty)[u_n^{-1}], V \otimes_{V_r} M) \\ &\cong \mathrm{Tor}_s^V(V/(p^\infty, \ldots, u_{n-1}^\infty), V \otimes_{V_r} M)[u_n^{-1}]. \end{aligned}$$

Since $s > r$, the latter group is zero by the induction hypothesis. □

8.22 Theorem (Chromatic Convergence). *Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{M}_{\mathbf{fg}}$. Then the natural map*

$$\mathcal{F} \longrightarrow \operatorname{holim} R(i_n)_* i_n^* \mathcal{F}$$

is a quasi-isomorphism.

Proof. There are distinquished triangles

$$R\Gamma_{\mathcal{M}(n)}\mathcal{F} \to \mathcal{F} \to R(i_n)_* i_n^* \mathcal{F} \to R\Gamma_{\mathcal{M}(n)}\mathcal{F}[1];$$

therefore, it is sufficient to show that

$$\operatorname{holim} R\Gamma_{\mathcal{M}(n)}\mathcal{F} \simeq 0.$$

But this follows from Theorem 8.20. □

8.23 Remark. The reader will have noticed that chromatic convergence holds in slightly greater generality: if $\mathcal{F}_0$ is any quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}\langle p^r\rangle$ for some $r < \infty$, then the natural map

$$q^*\mathcal{F}_0 \longrightarrow \operatorname{holim} R(i_n)_* i_n^* q^* \mathcal{F}_0$$

is a quasi-isomorphism.

8.24 Theorem. *Let $\mathcal{F}_0$ be a quasi-coherent sheaf on $\mathcal{M}_{\mathbf{fg}}\langle p^r\rangle$ and let $\mathcal{F} = q^*\mathcal{F}_0$ be the pull-back to $\mathcal{M}_{\mathbf{fg}}$. Then the natural map*

$$H^s(\mathcal{M}_{\mathbf{fg}}, \mathcal{F}) \longrightarrow H^s(\mathcal{U}(n), i_n^*\mathcal{F})$$

is an isomorphism for $s < n - r$ and injective for $s = n - r - 1$.

Proof. The failure of this map to be an isomorphism is measured by the long exact sequence

$$\cdots \to H^s(\mathcal{M}_{\mathbf{fg}}, R\Gamma_{\mathcal{M}(n+1)}\mathcal{F}) \to H^s(\mathcal{M}_{\mathbf{fg}}, \mathcal{F}) \\ \to H^s(\mathcal{U}(n), i_n^*\mathcal{F}) \to H^{s+1}(\mathcal{M}_{\mathbf{fg}}, R\Gamma_{\mathcal{M}(n+1)}\mathcal{F}) \to \cdots$$

where $H^*(\mathcal{M}_{\mathbf{fg}}, R\Gamma_{\mathcal{M}(n+1)}\mathcal{F})$ is the hyper-cohomology of the derived local cohomology sheaf $R\Gamma_{\mathcal{M}(n+1)}\mathcal{F}$. This can be computed via the spectral sequence

$$H^p(\mathcal{M}_{\mathbf{fg}}, H^q R\Gamma_{\mathcal{M}(n+1)}\mathcal{F}) \Longrightarrow H^{p+q}(\mathcal{M}_{\mathbf{fg}}, R\Gamma_{\mathcal{M}(n+1)}\mathcal{F}).$$

The isomorphism of Theorem 8.8

$$H^q R\Gamma_{\mathcal{M}(n+1)}\mathcal{F} \cong H_{n+1-q}(\mathcal{O}/\mathcal{L}_{n+1}^{\infty} \otimes^L \mathcal{F})$$

and Corollary 8.21 now give the result. $\square$

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