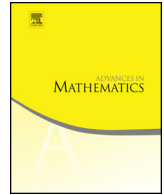Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

Goerss–Hopkins obstruction theory for ∞-categories

Aaron Mazel-Gee

ABSTRACT

Goerss–Hopkins obstruction theory is a powerful tool for constructing structured ring spectra from purely algebraic data. Using the formalism of model ∞-categories, we provide a generalization that applies in an arbitrary presentably symmetric monoidal stable ∞-category (such as that of equivariant spectra or of motivic spectra).

© 2024 Elsevier Inc. All rights are reserved, including those for text and data mining, AI training, and similar technologies.

Contents

E-mail address: aaron@etale.site.

0. Introduction

0.1. Summary

It has long been recognized that **structured ring spectra** – in particular, $\mathbb{A}_\infty$ and $\mathbb{E}_\infty$ ring spectra – are of central importance in stable homotopy theory. Indeed, such operadic structure allows for a good theory of modules, and is useful for importing such fundamental algebraic constructions as Hochschild homology and algebraic K-theory to the world of ring spectra. In a different direction, it also induces rich additional structure on the corresponding cohomology theory, namely that of power operations.

Some spectra admit enhancements to structured ring spectra for transparent reasons. For instance, the Eilenberg–Mac Lane spectrum *HR* of an associative (resp. commutative) ring R carries an $\mathbb{A}_\infty$ (resp. $\mathbb{E}_\infty$) structure essentially by construction, as the functor $H : \mathrm{Ab} \to \mathcal{S}\mathrm{p}$ is lax symmetric monoidal. And the real and complex K-theory spectra carry $\mathbb{E}_\infty$ structures as a result of the fact that the tensor product of vector bundles is associative and commutative up to natural isomorphism.

Goerss–Hopkins obstruction theory is a tool for constructing a much broader class of structured ring spectra. This has found many crucial uses in the study of structured ring spectra: its first application [24] was to show that the Morava *E*-theory spectra admit essentially unique $\mathbb{E}_\infty$ structures and to compute their automorphisms; perhaps its most spectacular application to date is the construction of the $\mathbb{E}_\infty$ ring spectrum *tmf* of topological modular forms [17]; it is a key ingredient in Galois theory for $\mathbb{E}_\infty$ ring spectra [67]; and it plays an important role in a number of other works such as [3,29,43,45,68,70,73].

It would be highly desirable to have a more general version of Goerss–Hopkins obstruction theory. In particular, this should apply in the settings of equivariant and motivic stable homotopy theory, as well as in the setting of diagrams of spectra (e.g. sheaves of spectra (e.g. over the moduli stack of elliptic curves)). The recent work [33] indicates the expected utility of an obstruction theory for logarithmic ring spectra.

The purpose of the present paper is the construction of just such a generalized obstruction theory.

Slogan 0.1. *There is a Goerss–Hopkins obstruction theory for any presentably symmetric monoidal stable ∞-category.*

We will explain Slogan 0.1 in precise detail in §1.

Remark 0.2. In forthcoming work [52], we use this obstruction theory to produce $\mathbb{E}_\infty$ structures on the *motivic* Morava *E*-theory spectra and compute their automorphisms. This is a first step towards a moduli-theoretic construction of a motivic spectrum *mmf* of **motivic modular forms**, in analogy with *tmf*.[1] As the construction of *tmf* has been highly influential in chromatic homotopy theory, so would the construction of *mmf* significantly advance the chromatic approach to motivic homotopy theory, which is a highly active area of research [1,2,10,25,32,34,35,37,39–42,44,58–60,71,72].

Remark 0.3. There has been much recent interest in "genuine" operadic structures, e.g. genuine *G*-spectra with multiplications indexed by maps of finite *G*-sets (instead of just finite sets) [6–8,13,30,31,69], as well as analogous structures in motivic homotopy theory [5]. We do not contend with such structures here, but we are optimistic that the obstruction theory we construct admits a fairly direct enhancement to one that would handle them in a formally analogous way.

0.2. Model categories and ∞-categories

Despite the demand and evident utility, Goerss–Hopkins obstruction theory has thus far resisted generalization. This is not without cause, however. Its construction is based in a carefully chosen model category of spectra – let us denote it here by $\mathtt{Sp}$ –, and rests on a plethora of delicate foundational assumptions surrounding that choice (see [23, Theorems 1.2.1 and 1.2.3]). These assumptions greatly simplify the arguments; for instance, they guarantee (among many other things) that the homotopy theory $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{Sp})$ of $\mathbb{E}_\infty$ ring spectra is presented by the model category $\mathrm{Alg}_{\mathrm{Comm}}(\mathtt{Sp})$ of strict commutative algebra objects in that model category. Thus, a direct generalization of Goerss–Hopkins obstruction theory e.g. to the motivic setting would involve obtaining a model category $\mathtt{Sp}^{\mathrm{mot}}$ of motivic spectra sharing these same point-set features and then proceeding from there.[2]

On the other hand, further reflection reveals that such a direct approach is actually less than ideal. After all, this would require a new argument for each distinct homotopy theory $\mathcal{C}$ in which one wants to obtain a version of Goerss–Hopkins obstruction theory – or at least, it would require the establishment of a suitable model category $\mathtt{C}$ presenting the homotopy theory $\mathcal{C}$. On the other hand, the obstruction theory itself is completely independent of the ambient choice of model category: it only depends on the *underlying ∞-category*. Thus, the more robust approach to obtaining a generalized Goerss–Hopkins

---

[1] The works [27,66] take a different approach, producing motivic spectra over $\mathbb{R}$ and $\mathbb{C}$ whose cohomologies coincide with that expected of *mmf* (in analogy with *tmf*). These constructions are indirect, and relatively specific to the chosen base fields; in particular, the resulting motivic spectra are not manifestly related to any theory of elliptic motivic spectra.

[2] Indeed, [36] provides such a model category of motivic spectra, but this first step towards a motivic Goerss–Hopkins obstruction theory has not been carried further.

obstruction theory is to dispense with such irrelevant point-set technicalities and work at the level of ∞-categories. This is the approach that we pursue here.[3]

0.3. Model ∞-categories

As it turns out, however, dispensing with point-set technicalities is not the same thing as dispensing with model structures. Indeed, the original construction of Goerss–Hopkins obstruction theory takes place in the **resolution model structure** on the category $s\mathrm{Sp}$ of simplicial spectra. This presents the **nonabelian derived ∞-category** of spectra, which we denote by $\mathscr{D}_{\geq 0}(\mathrm{Sp})$. Correspondingly, Goerss–Hopkins obstruction theory for a more general ∞-category $\mathcal{C}$ takes place in its nonabelian derived ∞-category $\mathscr{D}_{\geq 0}(\mathcal{C})$.

On the other hand, the arguments necessary for setting up Goerss–Hopkins obstruction theory do *not* just take place in the nonabelian derived ∞-category $\mathscr{D}_{\geq 0}(\mathrm{Sp})$. Rather, they make essential use of the resolution model structure itself, which is necessary for computing hom-spaces therein. The reason for this is that the nonabelian derived ∞-category $\mathscr{D}_{\geq 0}(\mathrm{Sp})$ enjoys a universal property *as an ∞-category*. This is one category-level removed from the hom-spaces themselves, and as a result the latter are a priori quite difficult to describe in explicit terms.[4]

Thus, in order to make computations within the nonabelian derived ∞-category $\mathscr{D}_{\geq 0}(\mathcal{C})$, we apply the theory of **model ∞-categories**, which we developed in previous work [50,51,53–56] for this purpose. Namely, in this paper we construct a resolution model structure on the ∞-*category* $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$; the fundamental theorem of model ∞-categories then implies that we can use this model structure to compute hom-spaces in the (∞-categorical) localization $\mathscr{D}_{\geq 0}(\mathcal{C}) \simeq s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ at its subcategory of weak equivalences.

In this paper, we make free use of the theory of model ∞-categories. Given a working knowledge of the classical theory of model categories, the terminology and the main theorems surrounding model ∞-categories are all as one would expect, though of course the proofs are substantially more involved; we refer the reader to [53, §0.2] for a rapid summary of the theory of model ∞-categories (or to [57, §0.2.1] for a somewhat more in-depth discussion). However, we note here that the central role of the model category $s\mathrm{Set}_{\mathrm{KQ}}$ of simplicial sets equipped with the Kan–Quillen model structure (e.g. in the definition of a simplicial model category) is played by the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$ of

---

[3] In private communication, Goerss has explained that there were two reasons that the culminating Goerss–Hopkins paper [23] was never published. Firstly, they envisioned a more comprehensive version of the obstruction theory that would apply not just to spectra but to diagrams of spectra (in particular sheaves of spectra over the moduli stack of elliptic curves, towards constructing *tmf*), but they never managed to work this out. And secondly, they came to realize that the then-nascent theory of ∞-categories would be able to elegantly handle the various technical problems with which they had wrestled.

[4] Perhaps the simplest example of this phenomenon arises in the groupoid completion of a one-object category, which corresponds to the group completion of the corresponding monoid. This groupoid is easy to characterize in terms of its universal property, but it is hopelessly difficult to describe in concrete terms: this is an intractable (in fact, computationally undecidable) task, closely related to the so-called "word problem" for generators and relations in abstract algebra.

simplicial *spaces* equipped with a likewise Kan–Quillen model structure – both of which present the ∞-category $\mathcal{S}$ of spaces.

0.4. Conventions

We take quasicategories as our preferred model for ∞-categories, and in general we adhere to the notation and terminology of [46] and [47]. In fact, our references to these two works will be frequent enough that it will be convenient for us to adopt Lurie's convention and use the "code names" T and A for them, respectively. Thus, for instance, to refer to [46, Theorem 4.1.3.1], we will simply write Theorem T.4.1.3.1.

However, we work invariantly to the greatest possible extent: that is, we primarily work *within the ∞-category of ∞-categories*. Thus, for instance, we will omit all technical uses of the word "essential", e.g. we will use the term *unique* in situations where one might otherwise say "essentially unique" (i.e. parametrized by a contractible space). For a full treatment of this philosophy as well as a complete elaboration of our conventions, we refer the interested reader to [53, §A]. The casual reader should feel free to skip this on a first reading; on the other hand, the careful reader may find it useful to peruse that section before reading the present paper. For the reader's convenience, we also provide a complete index of the notation that is used throughout this paper and the model ∞-categories papers in [53, §B].

0.5. Outline

We now provide a more detailed outline of the contents of this paper.

- In §1, we provide an informal overview of our generalized Goerss–Hopkins obstruction theory.
- In §2, we introduce resolution model structures on ∞-categories of simplicial objects, and give sufficient conditions for their existence.
- In §3, we lay out our foundations and assumptions regarding the ambient presentably symmetric monoidal stable ∞-category $\mathcal{C}$, and we construct an auxiliary resolution model structure on the ∞-category $s\mathcal{C}$.
- In §4, we add operadic structures into the story: if our main goal is to construct algebras in $\mathcal{C}$ over some operad $\mathcal{O} \in \mathrm{Op}$, we obtain a simplicial resolution $T \in s\mathrm{Op}$ of $\mathcal{O}$ and lift the above resolution model structure to one on the ∞-category $\mathrm{Alg}_T(s\mathcal{C})$ of T-algebras in $s\mathcal{C}$.
- In §5, we turn to the algebraic part of the story, introducing a certain category $\mathcal{A}$ of comodules and positing monads on the categories $\mathcal{A}$ and $s\mathcal{A}$ that respectively govern the structures present on the homologies of $\mathcal{O}$-algebras in $\mathcal{C}$ and of T-algebras in $s\mathcal{C}$.
- In §6, we study Postnikov theory, André–Quillen cohomology, and moduli spaces in the model ∞-category $s\mathcal{A}$.

- In §7, we study Postnikov theory, André–Quillen cohomology, and moduli spaces in the model ∞-category $s\mathcal{C}$.
- In §8, we prove our main theorems.

Acknowledgments

1. Informal overview

Suppose we are given a flat homotopy commutative ring spectrum

$$E \in \mathrm{CAlg}(\mathrm{ho}(\mathcal{S}\mathrm{p}))$$

satisfying *Adams's condition* (which we will describe in §1.5); we will refer to the its corresponding homology theory E_* as our "detecting" homology theory. Suppose moreover that we are given a commutative algebra

$$A \in \mathrm{CAlg}(\mathrm{Comod}_{(E_*, E_* E)})$$

in comodules. Then, Goerss–Hopkins obstruction theory provides a method for computing the **moduli space of** (E-**local**) **realizations** of A as an $\mathbb{E}_\infty$ ring spectrum – the first question being whether it is nonempty. In fact, it applies to algebras over *any* operad $\mathcal{O}$, though this changes the nature of the algebraic object in comodules that we must consider.

The purpose of the present section is to explain this story in detail, as well as the generalization from Sp to an arbitrary presentably symmetric monoidal stable ∞-category which is the purpose of this paper. We begin by explaining the obstruction theory as a black box in §§1.1–1.3, focusing for simplicity only on the $\mathbb{E}_\infty$ case. We then proceed to unpack the inner workings of the obstruction theory in §§1.4–1.9.

1.1. The moduli space of realizations

First of all, a **realization** of A is an $\mathbb{E}_\infty$ ring spectrum X for which there exists an isomorphism $E_* X \cong A$ (of algebras in comodules). These are our objects of interest. Note that we do *not* require the existence of a spectrum realizing the underlying comodule of A: that is, we start with *purely algebraic* data.

Next, an E-**equivalence** is a map $X \to Y$ of spectra that induces an isomorphism $E_* X \xrightarrow{\cong} E_* Y$ of $E_* E$-comodules (or equivalently of E_*-modules). In a universal way, we can invert the E-equivalences in the ∞-category of spectra to form the ∞-category $\mathrm{L}_E(\mathrm{Sp})$ of E-*local spectra*. The terminology stems from the fact that this localization actually participates in a reflective localization

$$\mathrm{L}_E : \mathrm{Sp} \rightleftarrows \mathrm{L}_E(\mathrm{Sp}) : \mathrm{U}_E,$$

i.e. an adjunction whose right adjoint is fully faithful; in particular, we can consider $\mathrm{L}_E(\mathrm{Sp}) \subset \mathrm{Sp}$ as a full subcategory.[5] In other words, E-local spectra are just particular sorts of spectra, but E-equivalences between them are necessarily equivalences.

Finally, the **moduli space of E-local realizations** of A is the full subgroupoid

$$\mathscr{M}_A \subset \mathrm{CAlg}(\mathrm{L}_E(\mathrm{Sp}))$$

on the E-local $\mathbb{E}_\infty$ ring spectra which are realizations of A; its morphisms are the E-equivalences (which are also equivalences) between them. As indicated above, we will generally leave the descriptor "E-local" implicit.

Remark 1.1. Of course, this necessarily only produces E-local spectra. Thus, if one is interested in obtaining an $\mathbb{E}_\infty$ ring structure on a particular spectrum $X \in \mathrm{Sp}$, one

---

[5] This is the underlying ∞-categorical content of the theory of *Bousfield localization* of spectra, as introduced in the classic paper [11].

must choose a detecting homology theory E_* for which X is E-local. On the other hand, this locality is not so hard to satisfy in practice: crucially, any E-module is necessarily E-local. Note that this is a relatively weak (and in particular, unstructured) hypothesis: we have only assumed that E is a *homotopy* commutative ring spectrum, and thus by "module" we can only possibly mean an object $X \in \mathrm{Mod}_E(\mathrm{ho}(\mathcal{S}\mathrm{p}))$.

In particular, it follows that E is E-local. This implies the nearly unbelievable conclusion that if we would like to endow a homotopy commutative ring spectrum $E \in \mathrm{CAlg}(\mathrm{ho}(\mathcal{S}\mathrm{p}))$ with an $\mathbb{E}_\infty$-structure, then E can *itself* serve as the detecting homology theory!

1.2. André–Quillen cohomology

Given our commutative algebra A in comodules, one can speak of *modules* over A (in comodules); we mention now that for any $n \geq 1$ one can define a canonical A-module $\Omega^n A$, which will play a role in our story shortly. For any A-module M and any augmented commutative algebra

$$X \in \mathrm{CAlg}(\mathrm{Comod}_{(E_*, E_*E)})_{/A}$$

in comodules, we can define the corresponding **André–Quillen cohomology groups** $H^*(X; M)$. In fact, these are given by the homotopy groups of a certain spectrum

$$\mathcal{H}(X; M) = \{\mathcal{H}^n(X; M)\}_{n \geq 0},$$

in the sense that

$$H^n(X; M) = \pi_{-n}\mathcal{H}(X; M) \cong \pi_0\mathcal{H}^n(X; M) \cong \pi_1\mathcal{H}^{n+1}(X; M) \cong \cdots$$

for any $n \geq 0$ (or really for any $n \in \mathbb{Z}$: this spectrum has vanishing positive-dimensional homotopy groups, not unlike $\underline{\mathrm{hom}}_{\mathcal{S}\mathrm{p}}(\Sigma^\infty_+ X, HM)$ for any space $X \in \mathcal{S}$ and any Eilenberg–MacLane spectrum $HM \in \mathcal{S}\mathrm{p}$). The group

$$\mathrm{Aut}(A, M)$$

of automorphisms of the pair (A, M) (whose elements are pairs of an isomorphism $\varphi : A \xrightarrow{\cong} A$ and an isomorphism $M \xrightarrow{\cong} \varphi^* M$) naturally acts on this spectrum. In particular, it acts on each constituent space $\mathcal{H}^n(X; M)$, and we write

$$\widehat{\mathcal{H}^n}(X; M) = (\mathcal{H}^n(X; M))_{\mathrm{Aut}(A, M)}$$

for the (homotopy) quotient. This action fixes the basepoint of $\mathcal{H}^n(X; M)$ (whose path component corresponds to the zero element $0 \in H^n(X; M)$), and so the inclusion of the basepoint is $\mathrm{Aut}(A, M)$-equivariant and hence determines a map

$$BAut(A, M) \to \widehat{\mathscr{H}^n}(X; M)$$

on quotients. We note for future reference that this map, whose source is connected, lands entirely in the path component selected by the composite

$$\mathrm{pt} \xrightarrow{0} \mathscr{H}^n(X; M) \to \widehat{\mathscr{H}^n}(X; M).$$

1.3. Obstructions to realization

As we will describe in more depth in §1.9, our understanding of the moduli space $\mathscr{M}_A$ actually comes from a sequence of moduli spaces $\mathscr{M}_n(A)$ of "n-stage approximations" to a realization of A. These moduli spaces are related by pullback squares

$$
\begin{array}{ccc}
\mathscr{M}_n(A) & \longrightarrow & BAut(A, \Omega^n A) \\
\downarrow & & \downarrow \\
\mathscr{M}_{n-1}(A) & \longrightarrow & \widehat{\mathscr{H}^{n+2}}(A; \Omega^n A)
\end{array}
$$

(for all $n \geq 1$), in which the left vertical map is induced by an "$(n-1)^{\mathrm{st}}$ Postnikov trunction" functor and the lower map is induced by an "n^{th} k-invariant" functor

$$\mathscr{M}_{n-1}(A) \xrightarrow{\chi_n} \mathscr{H}^{n+2}(A; \Omega^n A).$$

Moreover, under a key technical hypothesis that we will not discuss further here (Assumption 8.7 (see also Remark 8.8)), we have a canonical identification

$$\mathscr{M}_A \xrightarrow{\sim} \lim\left(\cdots \to \mathscr{M}_2(A) \to \mathscr{M}_1(A) \to \mathscr{M}_0(A)\right)$$

of our moduli space of realizations as the limit of the resulting tower (Theorem 8.9). Finally, as the base for our inductive understanding, we have an equivalence

$$\mathscr{M}_0(A) \simeq BAut(A).$$

We can now describe the sense in which we can "compute" the moduli space $\mathscr{M}_A$. Observe that the above pullback square implies that an $(n-1)$-stage X can be lifted to an n-stage if and only if the k-invariant

$$[\chi_n(X)] \in H^{n+2}(A; \Omega^n A)$$

vanishes: this is the only case in which there exists a nonempty fiber in the diagram

$$
\begin{array}{ccccccc}
& & & & & & BAut(A, \Omega^n A) \\
& & & & & & \downarrow \\
\mathrm{pt} \xrightarrow[X]{} & \mathscr{M}_{n-1}(A) & \xrightarrow[\chi_n]{} & \mathscr{H}^{n+2}(A; \Omega^n A) & \longrightarrow & \widehat{\mathscr{H}^{n+2}}(A; \Omega^n A),
\end{array}
$$

which is necessary and sufficient for there to exist a nonempty fiber in the diagram

$$
\begin{array}{c}
\mathscr{M}_n(A) \\
\downarrow \\
\mathrm{pt} \xrightarrow[X]{} \mathscr{M}_{n-1}(A).
\end{array}
$$

Remark 1.2. Of course, this is most useful in the étale case, i.e. when the relevant André–Quillen cohomology groups all vanish. Under this assumption, the entire tower collapses to an equivalence

$$
\mathscr{M}_A \xrightarrow{\sim} \mathscr{M}_0(A) \simeq B\mathrm{Aut}(A).
$$

This is visibly the case with Goerss–Hopkins's original application to the Morava E-theories. In fact, after enough algebraic manipulation, it also becomes the case in the construction of the sheaf $\mathcal{O}^{\mathrm{der}}$ of $\mathbb{E}_\infty$ ring spectra over the moduli stack of elliptic curves, whose global sections are *tmf* (but these manipulations are themselves not completely trivial) [17, §12].

In fact, this is also the case in another prominent application of Goerss–Hopkins obstruction theory as well. In his inspiring monograph [67], Rognes develops the *Galois theory* of $\mathbb{E}_\infty$ ring spectra. This may be seen as the study of *covering spaces* among affine spectral schemes, and provides a remarkably effective framework for the organization of chromatic homotopy theory from the viewpoint of spectral algebraic geometry. Just as classical Galois theory, this is governed by a Galois correspondence, i.e. a contravariant equivalence of posets. In order to prove this fundamental theorem, Rognes uses Goerss–Hopkins obstruction theory to obtain the desired intermediate Galois extension from a subgroup of the Galois group.

1.4. Nonabelian derived ∞-categories and resolution model structures

We now explain what exactly we meant by the notation "$\mathscr{D}_{\geq 0}(\mathcal{C})$" used in §0.3. In fact, this notation is slightly misleading: this construction does not depend on the ∞-category $\mathcal{C}$ alone. Rather, we must first choose a full subcategory $\mathcal{G} \subset \mathcal{C}$ which is closed under finite coproducts, which should be thought of as a subcategory of "projective generators". Given this, we define the (**nonnegatively-graded**) **nonabelian derived ∞-category** of $\mathcal{C}$ to be

$$
\mathscr{D}_{\geq 0}(\mathcal{C}) = \mathscr{D}_{\geq 0}(\mathcal{C}, \mathcal{G}) = \mathcal{P}_\Sigma(\mathcal{G}) = \mathrm{Fun}^\times(\mathcal{G}^{op}, \mathcal{S}),
$$

the ∞-category of *product-preserving presheaves of spaces* on $\mathcal{G}$, i.e. the full subcategory of $\mathrm{Fun}(\mathcal{G}^{op}, \mathcal{S})$ on those contravariant functors that take finite coproducts in $\mathcal{G}$ to finite products in $\mathcal{S}$. (We will use the various notations interchangeably, depending on our desired emphasis.)

Observe that there is a canonical functor

$$s\mathcal{C} \xrightarrow{X_\bullet \mapsto \left(Y \mapsto |\hom_{\mathcal{C}}^{\mathrm{lw}}(Y, X_\bullet)|\right)} \mathcal{P}_\Sigma(\mathcal{G})$$

from the ∞-category of simplicial objects in $\mathcal{C}$, the levelwise restricted Yoneda functor followed by geometric realization. In the case that $\mathcal{C}$ is an ordinary category and $\mathcal{G} \subset \mathcal{C}$ is a full subcategory of small projective generators, in [62, §II.4] Quillen defined a model structure on $s\mathcal{C}$ which (in hindsight) is precisely a presentation of the ∞-category $\mathcal{P}_\Sigma(\mathcal{G})$. For example, if we take $\mathcal{C} = \mathcal{S}$et to be the category of sets and $\mathcal{G} = \mathcal{F}$in to be the full subcategory of finite sets, this recovers the standard Kan–Quillen model structure $s\mathcal{S}$et$_{\mathrm{KQ}}$, which presents the ∞-category

$$\mathscr{D}_{\geq 0}(\mathcal{S}\mathrm{et}) = \mathscr{D}_{\geq 0}(\mathcal{S}\mathrm{et}, \mathcal{F}\mathrm{in}) = \mathrm{Fun}^\times(\mathcal{F}\mathrm{in}^{op}, \mathcal{S}) \simeq \mathrm{Fun}(\mathrm{pt}^{op}, \mathcal{S}) \simeq \mathcal{S}$$

of spaces as the nonabelian derived ∞-category of the category of sets. On the other hand, if $\mathcal{C}$ is an abelian category with enough projectives, then $\mathscr{D}_{\geq 0}(\mathcal{C})$ recovers the usual (nonnegatively-graded) derived ∞-category of $\mathcal{C}$, which through the Dold–Kan correspondence is equivalent to the usual definition in terms of nonnegatively-graded chain complexes in $\mathcal{C}$. In general, cofibrant replacements in these model structures may thus be thought of as **nonabelian projective resolutions**.

In fact, this same idea has been carried further in homotopy theory. In [20], Dwyer–Kan–Stover defined a **resolution model structure** on the category $s\mathcal{T}$op$_*$ of simplicial pointed topological spaces based on the set of generators

$$\{S^n \in \mathcal{T}\mathrm{op}_*\}_{n \geq 1},$$

and in [12] Bousfield generalized this to an arbitrary (pointed, right proper) model category equipped with a set of h-cogroup objects satisfying certain conditions. In both cases, the restriction to h-cogroup objects is motivated by the desire for spectral sequences converging to the "homotopy groups" (with respect to the generators and their finite coproducts) of the geometric realization of an object (in the model-categorical sense). The levelwise weak equivalences are weak equivalences in these model structures, but there are in general strictly more of the latter.

From the perspective of model ∞-categories, it is clear that these model 1-categories are fairly inefficient: it is wholly unnecessary to distinguish between objects which are levelwise weakly equivalent. On the other hand, the resolutions that these model structures afford *are* necessary – indeed, they are the entire point. Thus, one might expect to freely invert the levelwise weak equivalences while keeping track of the remaining resolution weak equivalences. To this end, we have the following theorem.

Theorem 1.3 *(2.19 and 2.23). Let $\mathcal{C}$ be a presentable ∞-category, let $\{Z_\alpha \in \mathcal{C}\}$ be a set of compact objects, and write $\mathcal{G} \subset \mathcal{C}$ for the full subcategory generated by the objects Z_α and*

*their finite coproducts. Then there exists a **resolution model structure** on the ∞-category* $s\mathcal{C}$, *denoted* $s\mathcal{C}_{\mathrm{res}}$. *This model structure is simplicial (i.e., it is compatibly enriched over* $s\mathcal{S}_{\mathrm{KQ}}$). *Moreover, it participates in a Quillen adjunction*

$$\mathrm{Fun}(\mathcal{G}, s\mathcal{S}_{\mathrm{KQ}})_{\mathrm{proj}} \rightleftarrows s\mathcal{C}_{\mathrm{res}},$$

whose derived adjunction is precisely the canonical adjunction

$$\mathcal{P}(\mathcal{G}) \rightleftarrows \mathcal{P}_{\Sigma}(\mathcal{G}).$$

Remark 1.4. The resolution model ∞-category structure of Theorem 1.3 is indeed much more efficient than its 1-categorical analogs. For example, every object in $s\mathcal{C}_{\mathrm{res}}$ is fibrant; by contrast, in the resolution model structures of Dwyer–Kan–Stover and Bousfield, the fibrant objects are precisely the Reedy fibrant objects. (This is by no means a decisive advantage, but it seems worth pointing out nonetheless.)

1.5. The detecting homology theory and resolutions

Let us fix a presentably symmetric monoidal stable ∞-category $\mathcal{C}$. This replaces a model 1-category of spectra, which in the original construction of Goerss–Hopkins obstruction theory must be assumed to satisfy a long list of technical assumptions. We assume that $\mathcal{C}$ is equipped with a full subcategory $\mathcal{G} \subset \mathcal{C}$ of generators, which we assume to be sufficiently nice (e.g. its objects must all have inverses with respect to the symmetric monoidal structure – thereafter, our assumptions will imply that its objects are compact). This generalizes the set of sphere spectra. These generators define a "homotopy groups" functor $\pi_{\circledast}$.

We now discuss our detecting homology theory, which we assume to be given by a flat homotopy commutative algebra $E \in \mathrm{CAlg}(\mathrm{ho}(\mathcal{C}))$. We can now explain the all-important **Adams's condition**. This is the requirement that E be obtainable as a filtered colimit

$$\mathrm{colim}_{\mathcal{J}} E_{\alpha} \xrightarrow{\sim} E$$

of *dualizable* objects E_{α}, such that their duals $\mathrm{D}E_{\alpha}$ have projective E-homology. This condition allows us to treat E-homology as being given by "homotopy groups with respect to these duals". More precisely, our assumptions guarantee that for any generator $S^{\beta} \in \mathcal{G}$ we have a string of isomorphisms

$$\mathrm{colim}_{\alpha \in \mathcal{J}}[\Sigma^{\beta}\mathrm{D}E_{\alpha}, X]_{\mathcal{C}} \cong \mathrm{colim}_{\alpha \in \mathcal{J}}[S^{\beta}, E_{\alpha} \otimes X]_{\mathcal{C}} \cong [S^{\beta}, \mathrm{colim}_{\alpha \in \mathcal{J}}(E_{\alpha} \otimes X)]_{\mathcal{C}}$$

$$\cong [S^{\beta}, \mathrm{colim}_{\alpha \in \mathcal{J}}(E_{\alpha}) \otimes X]_{\mathcal{C}} \cong [S^{\beta}, E \otimes X]_{\mathcal{C}} = E_{\beta}X$$

(where we suggestively write Σ^{β} for the functor $S^{\beta} \otimes -$). Therefore, if a map $X \to Y$ induces "$\mathrm{D}E_{\alpha}$-homotopy" isomorphisms

$$[\Sigma^\beta \mathrm{D}E_\alpha, X]_{\mathbb{C}} \xrightarrow{\cong} [\Sigma^\beta \mathrm{D}E_\alpha, Y]_{\mathbb{C}}$$

for all $S^\beta \in \mathcal{G}$ and all $\alpha \in \mathcal{J}$, then it induces an isomorphism on E-homology. On the other hand, the converse will not generally hold. This subtlety can be handled with a little bit of care (or with a lot of care, in the original model 1-categorical case), and we will return to it in due time.

Let us write $\mathcal{G}_E \subset \mathcal{C}$ for the full subcategory generated by the subcategory $\mathcal{G}$ and the objects $\mathrm{D}E_\alpha$ under finite coproducts and de/suspensions. Then, our resolutions will be based on the nonabelian derived ∞-category

$$\mathscr{D}_{\geq 0}(\mathcal{C}, \mathcal{G}_E).$$

However, we will need to make computations using actual simplicial resolutions (i.e. objects of $s\mathcal{C}$) instead of their images under the functor

$$s\mathcal{C} \to \mathcal{P}_\Sigma(\mathcal{G}_E) = \mathscr{D}_{\geq 0}(\mathcal{C}, \mathcal{G}_E),$$

and for this we will use the resolution model structure provided by Theorem 1.3.

As we will explain in §1.6, we will not actually be using this model ∞-category directly, but rather a generalization of it. However, even in this special case we can point out an essential feature of the story. Let us write $\tilde{\mathcal{A}}$ for the category of $(E_*, E_* E)$-comodules, and let us write $\mathcal{G}_{\tilde{\mathcal{A}}} \subset \tilde{\mathcal{A}}$ for the full subcategory on objects of the form $E_* S^\varepsilon$ for some $S^\varepsilon \in \mathcal{G}_E$; by our assumptions, these will be projective as E_*-modules. As we have assumed that $\mathcal{C}$ is presentably symmetric monoidal, it follows that the induced functor

$$E_* : \mathcal{G}_E \to \mathcal{G}_{\tilde{\mathcal{A}}}$$

preserves finite coproducts. It follows formally that the induced functor

$$\mathcal{P}_\Sigma(E_*) : \mathcal{P}_\Sigma(\mathcal{G}_E) \to \mathcal{P}_\Sigma(\mathcal{G}_{\tilde{\mathcal{A}}})$$

preserves all colimits. Ultimately, this fact will be (a shadow of) the reason that our topological obstructions can be computed purely algebraically. At the level of model ∞-categories, this can be seen as resulting from the fact that the functor

$$E_*^{\mathrm{lw}} : s\mathcal{C}_{\mathrm{res}} \to s\tilde{\mathcal{A}}_{\mathrm{res}}$$

preserves cofibrations between cofibrant objects relative to an analogous resolution model structure $s\tilde{\mathcal{A}}_{\mathrm{res}}$.

1.6. Operadic structures and resolutions

We use the term "operad" to refer to a (single-colored) ∞-operad; the ∞-category Op of operads is presented by the relative category $\mathrm{Op}(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})$ of operads in simplicial sets,

whose weak equivalences are determined levelwise on underlying objects (i.e. ignoring the symmetric group actions). This relative category structure enhances to a *Boardman–Vogt* model structure, which (using a generalization of Theorem 1.3) we incidentally generalize to the ∞-category $\mathrm{Op}(s\mathcal{V})$ of internal operads (for a suitable symmetric monoidal ∞-category $\mathcal{V}$) as Proposition 4.9.

Now, our obstruction theory can be used to construct (E-local) $\mathcal{O}$-algebras in $\mathcal{C}$, for any operad $\mathcal{O} \in \mathrm{Op}$. Given a choice of $\mathcal{O}$, however, we must choose a monad Φ on $\tilde{\mathcal{A}}$ which will parametrize our "algebraic structures": in other words, we must have a lift

$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) & \overset{E_*}{\dashrightarrow} & \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}}) \\
{\scriptstyle \mathrm{U}_{\mathcal{O}}} \downarrow & & \downarrow {\scriptstyle \mathrm{U}_{\Phi}} \\
\mathcal{C} & \underset{E_*}{\longrightarrow} & \tilde{\mathcal{A}}
\end{array}
$$

of our E-homology functor. For instance, in the special case where $\mathcal{O} = \mathrm{Comm} = \mathbb{E}_{\infty}$ that we described in §§1.1–1.3, we also took $\Phi = \mathrm{Comm}$. However, even in the case that we take $\mathcal{O} = \mathrm{Comm}$, it can be useful – essential, even – to have this added generality.[6]

So of course, we will not be interested in resolving objects of $\mathcal{C}$, but rather objects of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. However, it will not suffice to simply resolve them by *simplicial* objects of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$: at no point will this allow us to gain control over their levelwise E-homology (in the model category $s\tilde{\mathcal{A}}_{\mathrm{res}}$).

On the other hand, there is a special case in which we do have control over their levelwise E-homology, namely when the operad $\mathcal{O}$ is π_0-$\mathfrak{S}$-*free*: by definition, this means that for every $n \geq 0$, the symmetric group $\mathfrak{S}_n$ acts freely on the set $\pi_0(\mathcal{O}(n))$ of path components of the nth constituent space of $\mathcal{O}$. When this is the case, the "free $\mathcal{O}$-algebra" functor

$$
X \mapsto \coprod_{n \geq 0} (\mathcal{O}(n) \odot X^{\otimes n})_{\mathfrak{S}_n}
$$

simplifies dramatically. Even better, if we assume that $E_* X$ is projective – such as when $X = \mathrm{D}E_{\alpha}$ –, then the Künneth spectral sequence for the E-homology of this free $\mathcal{O}$-algebra (which is guaranteed by Adams's condition) immediately collapses!

Thus, a key insight of Goerss–Hopkins obstruction theory (over its predecessors) was, for a general operad $\mathcal{O}$, to take a simplicial resolution $T_{\bullet} \in s\mathrm{Op}$ by π_0-$\mathfrak{S}$-free operads. Amusingly, this can be achieved by choosing a cofibrant representative of $\mathcal{O}$ in the model category $\mathrm{Op}(s\mathcal{S}\mathrm{et}_{\mathrm{KQ}})_{\mathrm{BV}}$ via the embedding

---

[6] The construction of *tmf* (as the global sections of a sheaf of $\mathbb{E}_{\infty}$ ring spectra over the moduli stack of elliptic curves), which was spelled out in full detail by Behrens in [17], makes essential use of such generality. In order to construct the height-1 component of the sheaf (which is necessary in order to "interpolate" between the supersingular loci at distinct primes, and which is by far the most technical aspect of the construction), one must take the p-adic complex K-theory spectrum $KU_p^{\wedge}$ as the detecting homology theory, and one must enhance the nature of the algebraic input from a commutative algebra in comodules to what is called a θ-*algebra* (which structure is canonically present on the p-adic K-theory of an $\mathbb{E}_{\infty}$ ring spectrum).

$$\mathrm{Op}(s\mathcal{S}\mathrm{et}) \simeq s\left(\mathrm{Op}(\mathcal{S}\mathrm{et})\right) \hookrightarrow s\mathrm{Op}.$$

A simplicial operad can be made to act on simplicial objects in $\mathcal{C}$, and from here we obtain (as Theorem 4.26) a lifted resolution model structure through the adjunction

$$\mathrm{F}_T : s\mathcal{C}_{\mathrm{res}} \rightleftarrows \mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}} : \mathrm{U}_T$$

This is the model ∞-category we have been seeking. On the one hand, its objects are resolutions of $\mathcal{O}$-algebras in $\mathcal{C}$: we have a canonical lift

$$
\begin{array}{ccc}
\mathrm{Alg}_T(s\mathcal{C}) & \dashrightarrow^{|-|} & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \\
{\scriptstyle \mathrm{U}_T}\downarrow & & \downarrow{\scriptstyle \mathrm{U}_{\mathcal{O}}} \\
s\mathcal{C} & \xrightarrow{\;|-|\;} & \mathcal{C}
\end{array}
$$

of the geometric realization functor. On the other hand, we will assume enough so that there is a monad $\tilde{T}_E$ on $s\tilde{\mathcal{A}}$ admitting a lift

$$
\begin{array}{ccc}
\mathrm{Alg}_T(s\mathcal{C}) & \dashrightarrow^{E^{\mathrm{lw}}_{\divideontimes}} & \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \\
{\scriptstyle \mathrm{U}_T}\downarrow & & \downarrow{\scriptstyle \mathrm{U}_{\tilde{T}_E}} \\
s\mathcal{C} & \xrightarrow{\;E^{\mathrm{lw}}_{\divideontimes}\;} & s\tilde{\mathcal{A}}.
\end{array}
$$

Just as our unstructured functor

$$E^{\mathrm{lw}}_{\divideontimes} : s\mathcal{C}_{\mathrm{res}} \to s\tilde{\mathcal{A}}_{\mathrm{res}}$$

preserves cofibrations between cofibrant objects, so will this lifted functor $E^{\mathrm{lw}}_{\divideontimes}$ (with respect to an analogously lifted resolution model structure $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})_{\mathrm{res}}$), which crucially implies that its localization

$$E^{\mathrm{lw}}_{\divideontimes} : \mathrm{Alg}_T(s\mathcal{C})\llbracket \mathbf{W}^{-1}_{\mathrm{res}} \rrbracket \to \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})\llbracket \mathbf{W}^{-1}_{\mathrm{res}} \rrbracket$$

preserves colimits. Although there will be one more small wrinkle that must be smoothed out, this fact is very nearly the true reason that our topological obstructions can be computed purely algebraically.

1.7. $E_{\divideontimes}$-localization

Given our algebraic object $A \in \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$, we can now explain that our "n-stage approximations" to A will be objects of the ∞-category $\mathrm{Alg}_T(s\mathcal{C})\llbracket \mathbf{W}^{-1}_{\mathrm{res}} \rrbracket$, and our André–Quillen cohomology spaces will be certain mapping spaces extracted from the ∞-category $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})\llbracket \mathbf{W}^{-1}_{\mathrm{res}} \rrbracket$. However, these facts are technically true but slightly misleading.

To clarify both at once, let us recall for the sake of analogy that in the ∞-category $\mathcal{C}$, a map becoming an isomorphism under all of functors $[\Sigma^\beta \mathrm{D} E_\alpha, -]_{\mathcal{C}}$ implies that it also becomes an isomorphism under the functor E_*, but that the converse is generally false. Then, in the algebraic case, note that there exists a forgetful functor

$$\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \xrightarrow{\mathrm{U}_{\tilde{T}_E}} s\tilde{\mathcal{A}} \xrightarrow{s(\mathrm{U}_{\tilde{\mathcal{A}}})} s\mathcal{S}\mathrm{et}_*,$$

which takes the subcategory $\mathbf{W}_{\mathrm{res}} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$ into the subcategory $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}\mathrm{et}_*$, but not only this subcategory; defining

$$\mathbf{W}_{\pi_*} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$$

to be the pullback of $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}\mathrm{et}_*$, we obtain a reflective localization

$$\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \rightleftarrows \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!].$$

Similarly, in the topological case, the functor

$$E_*^{\mathrm{lw}} : \mathrm{Alg}_T(s\mathcal{C}) \to \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$$

takes the subcategory $\mathbf{W}_{\mathrm{res}} \subset \mathrm{Alg}_T(s\mathcal{C})$ into the subcategory $\mathbf{W}_{\pi_*} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$, but not only this subcategory; defining

$$\mathbf{W}_{E_*^{\mathrm{lw}}} \subset \mathrm{Alg}_T(s\mathcal{C})$$

to be the pullback of $\mathbf{W}_{\pi_*} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$, we obtain a reflective localization

$$\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \rightleftarrows \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!].$$

Now, we can clarify that the moduli spaces of n-stages for A are naturally subgroupoids

$$\mathscr{M}_n(A) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!] \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$$

of the reflective localization, while the relevant André–Quillen cohomology spaces are computed by mapping in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ to an object of the reflective subcategory $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!] \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$. Moreover, these two reflective localization functors participate as the downwards arrows in a commutative square

$$
\begin{array}{ccc}
\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] & \xrightarrow{E_*^{\mathrm{lw}}} & \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \\
\downarrow & & \downarrow \\
\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!] & \dashrightarrow{E_*^{\mathrm{lw}}} & \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!],
\end{array}
$$

in which the dotted arrow exists by the universal property of localization and preserves colimits by an easy diagram chase. This, finally, is the *true* reason that our topological obstructions can be computed purely algebraically. However, in order to explain this, we must introduce the *spiral exact sequence*.

1.8. Bigraded E-homology groups and the spiral exact sequence

Given a simplicial object $X \in s\mathcal{C}$, there are two sorts of E-homology groups that one might extract: the *classical E-homology groups*

$$\pi_n E^{\mathrm{lw}}_\beta X = \pi_n [S^\beta, E \otimes X]^{\mathrm{lw}}_{\mathcal{C}}$$

(i.e. the nth homotopy group of the simplicial abelian group $[i]^\circ \mapsto [S^\beta, E \otimes X_i]_{\mathcal{C}}$ of homotopy classes of maps) and the *natural E-homology groups*

$$E^\natural_{n,\beta} X = \pi_n \left(\hom_{\mathscr{D}_{\geq 0}(\mathcal{C}, \mathcal{G}_E)}(S^\beta, (E \otimes X)^{\mathrm{lw}}) \right)$$

(i.e. the nth homotopy group of the space of morphisms in $\mathscr{D}_{\geq 0}(\mathcal{C}, \mathcal{G}_E)$, where $(E \otimes X)^{\mathrm{lw}} \in s\mathcal{C}$ denotes the simplicial object $[i]^\circ \mapsto E \otimes X_i$ obtained by tensoring the simplicial object X levelwise with E). These serve dual purposes.

On the one hand, the classical E-homology groups assemble into the E^2 page of a spectral sequence

$$\mathrm{E}^2 = \pi_n E^{\mathrm{lw}}_\beta X \Rightarrow \mathrm{E}^\infty = E_{\beta+n}|X|,$$

where we write $S^{\beta+n} = S^\beta \otimes S^n = \Sigma^n S^\beta$. Of course, this spectral sequence allows us to obtain control over the E-homology of the geometric realization $|X|$.

On the other hand, the natural E-homology groups are by their very definition much more directly related to the ∞-category

$$\mathscr{D}_{\geq 0}(\mathcal{C}, \mathcal{G}_E) \simeq s\mathcal{C}[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!].$$

Thus, they participate in a "cells and disks" obstruction theory within this ∞-category. In order to explain this, we introduce the notation

$$D^n_{\boldsymbol{\Delta}} = \Delta^n / \Lambda^n_0 \in (s\mathcal{S}\mathrm{et}_*)_{\mathrm{KQ}}$$

and

$$S^n_{\boldsymbol{\Delta}} = \Delta^n / \partial \Delta^n \in (s\mathcal{S}\mathrm{et}_*)_{\mathrm{KQ}}.$$

There are evident cofibrations

$$S^n_{\boldsymbol{\Delta}} \rightarrowtail D^{n+1}_{\boldsymbol{\Delta}}$$

in $(s\mathcal{S}\mathrm{et}_*)_{\mathrm{KQ}}$, which present the maps

$$S^n \to D^{n+1} \simeq \mathrm{pt}$$

in $\mathcal{S}_*$. Moreover, for any $K \in s\mathcal{S}_*$ and any $X \in s\mathcal{C}$, there exists a "based tensor" object $K\overline{\odot}X \in s\mathcal{C}$, which is compatible with the canonical enrichment of $s\mathcal{C}$ over $s\mathcal{S}_*$ (where the basepoint is given by the zero morphism). Writing $S^\varepsilon \in \mathcal{G}_E$ for an arbitrary object, the fact that the model ∞-category $s\mathcal{C}_{\mathrm{res}}$ is *simplicial* implies that the "cells" given by

$$S^n_{\boldsymbol{\Delta}}\overline{\odot}\mathrm{const}(S^\varepsilon) \in s\mathcal{C}_{\mathrm{res}}$$

and the "disks" given by

$$D^n_{\boldsymbol{\Delta}}\overline{\odot}\mathrm{const}(S^\varepsilon) \in s\mathcal{C}_{\mathrm{res}}$$

together control the theory of *Postnikov towers* in $s\mathcal{C}[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$.[7]

Now, the ("*localized*") **spiral exact sequence** relates these two types of E-homology, running

$$\cdots \longrightarrow \pi_{i+1}E_\beta X \xrightarrow{\ \delta\ } E^\natural_{i-1,\beta+1}X \longrightarrow E^\natural_{i,\beta}X \longrightarrow \pi_i E_\beta X \xrightarrow{\ \delta\ } \cdots$$

$$\cdots \xrightarrow{\ \delta\ } E^\natural_{0,\beta+1}X \longrightarrow E^\natural_{1,\beta}X \longrightarrow \pi_1 E_\beta X \longrightarrow 0.$$

Note that it is two-thirds natural E-homology, and one-third classical E-homology.[8] Thus, via the spiral exact sequence, *by controlling the natural E-homology groups* (via "cells and disks") *we can also control the classical E-homology groups* (which assemble into the E^2 page of the spectral sequence).

1.9. The tower of moduli spaces

We can now explain the connection with "n-stages" for our chosen object $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$ of which we are interested in realizations. First of all, an ∞-*stage* for A is an object of $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]$ whose E^2 page is simply given by A, concentrated in the bottom row; these assemble into a moduli space

$$\mathscr{M}_\infty(A) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!].$$

We then have the following result, which cements the relationship between realizations of A and their (approximate) resolutions.

---

[7] Examining the structure maps of the simplicial sets $D^n_{\boldsymbol{\Delta}}$ and $S^n_{\boldsymbol{\Delta}}$, one sees that they may be seen as corepresenting the *nonabelian n-cycles* and *nonabelian normalized n-chains* objects of an object $X \in s\mathcal{C}$ (via a "based cotensor" bifunctor $-\overline{\pitchfork}- : s\mathcal{S}^{op}_* \times s\mathcal{C} \to \mathcal{C}$ which we will not make precise here).

[8] In fact, these long exact sequences are what organize into the exact couple defining the above spectral sequence.

Theorem 1.5 *(8.5). The geometric realization functor*

$$|-| : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_{\circledast}^{\mathrm{lw}}}^{-1}]\!] \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{L}_E(\mathcal{C}))$$

induces an equivalence

$$\mathscr{M}_\infty(A) \xrightarrow{\sim} \mathscr{M}_A.$$

We emphasize that the moduli space $\mathscr{M}_\infty(A) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_{\circledast}^{\mathrm{lw}}}^{-1}]\!]$ will *not* generally contain all of the objects whose geometric realizations are realizations of A: rather, it only contains those whose geometric realizations are realizations of A "for obvious reasons" (namely that their spectral sequences collapse immediately).

Let us now move to the bottom of the tower. A 0-stage for A is an object $X \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_{\circledast}^{\mathrm{lw}}}^{-1}]\!]$ whose natural E-homology is given by

$$E_{i,\circledast}^{\natural} X \cong \begin{cases} A, & i = 0 \\ 0, & i > 0. \end{cases}$$

As the natural E-homology groups govern cellular approximations in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_{\circledast}^{\mathrm{lw}}}^{-1}]\!]$, the following result should be plausible.

Theorem 1.6 *(8.10). The moduli space of 0-stages for A admits a canonical equivalence*

$$\mathscr{M}_0(A) \simeq B\mathrm{Aut}(A).$$

Now, if $X \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_{\circledast}^{\mathrm{lw}}}^{-1}]\!]$ is a 0-stage for A, then its natural E-homology is extremely simple. On the other hand, as dictated by the spiral exact sequence, its classical E-homology – and hence its E^2 page – is not quite correct for it to be an ∞-stage: instead, we will have

$$\pi_i E_{\circledast}^{\mathrm{lw}} X \cong \begin{cases} A, & i = 0 \\ \Omega A, & i = 2 \\ 0, & i \notin \{0, 2\}. \end{cases}$$

In fact, more generally, if X is an n-stage for A, then we will have

$$\pi_i E_{\circledast}^{\mathrm{lw}} X \cong \begin{cases} A, & i = 0 \\ \Omega^{n+1} A, & i = n+2 \\ 0, & i \notin \{0, n+2\}. \end{cases}$$

Thus, to move upwards through the tower of moduli spaces is to push the failure of X to be an ∞-stage "further and further away".[9] However, we emphasize that the above

---

[9] In fact, the spectral sequence for an n-stage will collapse after the E^{n+2} page, directly after cancelling out the entire $(n+2)^{\mathrm{nd}}$ row with the corresponding entries of the 0th row.

identification of classical E-homology does not alone imply that X is an n-stage: it must also have the correct k-invariants (or equivalently, it must also have the correct natural E-homology). Indeed, an n-stage will be in essence an object that "looks like" the Postnikov n-truncation of an ∞-stage.

We now explain why this iterative *topological* procedure is indeed governed by *algebraic* computations. (In fact, a somewhat simpler argument will also justify Theorem 1.6.) This is where we will use the cocontinuity of the functor

$$E_*^{\mathrm{lw}} : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!] \to \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$$

between presentable ∞-categories.[10]

Suppose that $X \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!]$ is an $(n-1)$-stage for A. As we have just seen, its image

$$Y = E_*^{\mathrm{lw}} X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$$

will have its homotopy concentrated in degrees 0 and $n+1$: for brevity, we simply write

$$\pi_* Y \cong A \times (\Omega^n A)[n+1].$$

We are interested in modifying X to obtain an n-stage for A: this entails simultaneously peeling off this copy of $(\Omega^n A)[n+1]$ and replacing it with a copy of $(\Omega^{n+1} A)[n+2]$, all in a way that behaves correctly with respect to the natural E-homology groups.

In order to address this task, we first examine the levelwise E-homology object $Y = E_*^{\mathrm{lw}} X$. Now, in the ∞-category $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$, homotopy groups alone do not characterize equivalence classes: just as with (based) spaces, one must also keep track of the k-invariants. In this case, since Y only has potentially nonvanishing homotopy in dimensions 0 and $(n+1)$, it participates in a uniquely determined pullback square

$$
\begin{array}{ccc}
Y & \longrightarrow & K_A \\
\downarrow & & \downarrow \\
A & \xrightarrow[\chi_n(Y)]{} & K_A(\Omega^n A, n+2)
\end{array}
$$

in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$, in which the objects on the right are **algebraic Eilenberg–Mac Lane objects** with $\pi_* K_A \cong A$ and $\pi_* K_A(\Omega^n A, n+2) \cong A \times (\Omega^n A)[n+2]$; the right vertical map between them is an isomorphism on π_0, and the map $\chi_n(Y)$ is the unique potentially nontrivial k-invariant of Y. This defines a class

$$[\chi_n(Y)] \in H^{n+2}(A; \Omega^n A)$$

in the indicated André–Quillen cohomology group, and taken over all $(n-1)$-stages $X \in \mathscr{M}_{n-1}(A)$ this defines a map

$$\mathscr{M}_{n-1}(A) \xrightarrow{\chi_n} \mathscr{H}^{n+2}(A; \Omega^n A)$$

to the indicated André–Quillen cohomology space.

Returning to topology, we now come to the crucial point: for any object $Z \in \operatorname{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})\llbracket \mathbf{W}_{\pi_*}^{-1} \rrbracket$, the composite functor

$$\operatorname{Alg}_T(s\mathcal{C})\llbracket \mathbf{W}_{E_*^{\mathrm{lw}}}^{-1} \rrbracket^{op} \xrightarrow{E_*^{\mathrm{lw}}} \operatorname{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})\llbracket \mathbf{W}_{\pi_*}^{-1} \rrbracket^{op} \xrightarrow{\hom_{\operatorname{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})\llbracket \mathbf{W}_{\pi_*}^{-1} \rrbracket}(-,Z)} \mathcal{S}$$

preserves limits (i.e. takes colimits in $\operatorname{Alg}_T(s\mathcal{C})\llbracket \mathbf{W}_{E_*^{\mathrm{lw}}}^{-1} \rrbracket$ to limits in $\mathcal{S}$) and so must be representable (by presentability). When $Z = K_A$ or $Z = K_A(\Omega^n A, n+2)$, we obtain **topological Eilenberg–Mac Lane objects**, which we respectively denote by B_A and $B_A(\Omega^n A, n+2)$.

Now, if there exists an n-stage $\widetilde{X}$ lifting X, then Postnikov theory in $\operatorname{Alg}_T(s\mathcal{C})\llbracket \mathbf{W}_{E_*^{\mathrm{lw}}}^{-1} \rrbracket$ implies that it must fit into a pullback square

$$\begin{array}{ccc}
\widetilde{X} & \longrightarrow & B_A \\
\downarrow & & \downarrow \\
X & \longrightarrow & B_A(\Omega^n A, n+2),
\end{array}$$

in which the right vertical map classifies the standard map $K_A \to K_A(\Omega^n A, n+2)$. Conversely, if we define $\widetilde{X}$ to be such a pullback, then it will be an n-stage if and only if the lower map corresponds to an equivalence

$$E_*^{\mathrm{lw}} X = Y \xrightarrow{\sim} K_A(\Omega^n A, n+2).$$

As we have just seen, the equivalence class of Y is entirely classified by a k-invariant

$$[\chi_n(Y)] \in H^{n+2}(A; \Omega^n A),$$

and it is not hard to show that such an equivalence $Y \xrightarrow{\sim} K_A(\Omega^n A, n+2)$ exists if and only this k-invariant vanishes.

All in all, an expansion of this argument can be used to prove the following.

Theorem 1.7 *([8.11]). For any $n \geq 1$, there is a natural pullback square*

$$\begin{array}{ccc}
\mathscr{M}_n(A) & \longrightarrow & B\operatorname{Aut}(A, \Omega^n A) \\
\downarrow & & \downarrow \\
\mathscr{M}_{n-1}(A) & \longrightarrow & \widehat{\mathscr{H}}^{n+2}(A; \Omega^n A).
\end{array}$$

This is the final ingredient in our generalized Goerss–Hopkins obstruction theory, which allows us to compute the purely algebraic obstructions to the inductive passage up the tower of moduli spaces

$$
\begin{array}{ccc}
\mathscr{M}_A & \xleftarrow{\ \sim\ } & \mathscr{M}_\infty(A) \\
 & & \downarrow{\scriptstyle \mathrm{lim}} \\
 & & \vdots \\
 & & \downarrow \\
 & & \mathscr{M}_n(A) \longrightarrow B\mathrm{Aut}(A, \Omega^n A) \\
 & & \downarrow \qquad\qquad\qquad \downarrow \\
 & & \mathscr{M}_{n-1}(A) \longrightarrow \widehat{\mathscr{H}}^{n+2}(A; \Omega^n A) \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
 & & B\mathrm{Aut}(A) \simeq \mathscr{M}_0(A).
\end{array}
$$

2. The resolution model structure

In this section, we lift results from [28, Chapter II] in order to provide sufficient conditions for the existence of *resolution* simplicial model ∞-category structures. We begin in §2.1 by collecting some basic observations regarding enrichment and bitensoring in the presentable setting. Using these, in §2.2 we give general criteria for lifting model ∞-category structures, for which we give various sufficient conditions in §2.3. We conclude by identifying the underlying ∞-category in §2.4.

Remark 2.1. In this section, we will be constructing certain *resolution* model structures. These are closely related to the model structures of [20] and [12]; indeed, it is straightforward (but tedious) to verify that the proof of [12, Theorem 3.3] immediately generalizes to an arbitrary right proper model ∞-category $\mathcal{M}$ equipped with a set of h-cogroup objects (in the model ∞-categorical sense). However, those model structures are in a sense more difficult: they're built by modifying $(s\mathcal{M})_{\mathrm{Reedy}}$, and in the end the fibrant objects are exactly the Reedy fibrant objects. By contrast, using model ∞-categories effectively allows us to obtain a model structure presenting the desired ∞-category by starting with a *trivial* model ∞-category (so that the Reedy model structure on simplicial objects therein will also be trivial).

2.1. Enrichments and bitensorings in the presence of presentability

We begin by providing sufficient conditions for constructing enrichments and bitensorings among presentable ∞-categories, and for lifting adjunctions between ∞-categories equipped with these to enriched adjunctions.

Proposition 2.2. *Let $\mathcal{V} \in \mathrm{Alg}(\mathrm{Pr}^L)$ be a presentably monoidal ∞-category, and let $\mathcal{D} \in \mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^L)$ be a presentable ∞-category equipped with a left action of $\mathcal{V}$. Then this action $- \odot - : \mathcal{V} \times \mathcal{D} \to \mathcal{D}$ extends to an enrichment and bitensoring of $\mathcal{D}$ over $\mathcal{V}$, encoded by a two-variable adjunction*

$$\left(\mathcal{V} \times \mathcal{D} \xrightarrow{-\odot-} \mathcal{D} \ , \ \mathcal{V}^{op} \times \mathcal{D} \xrightarrow{-\pitchfork-} \mathcal{D} \ , \mathcal{D}^{op} \times \mathcal{D} \xrightarrow{\hom_{\mathcal{D}}(-,-)} \mathcal{V} \right).$$

Proof. The fact that the action takes place in the symmetric monoidal ∞-category Pr^L guarantees that it commutes with colimits separately in each variable. From here, presentability guarantees the co/representability required by the definition of a two-variable adjunction. $\square$

Lemma 2.3. *Suppose that $\mathcal{D} \in \mathrm{Pr}^L$ and $\mathcal{I} \in \mathrm{Cat}_{\infty}$. Then the levelwise tensoring of $\mathrm{Fun}(\mathcal{I}, \mathcal{D})$ over $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$ commutes with colimits separately in each variable and extends to an action $\mathrm{Fun}(\mathcal{I}, \mathcal{D}) \in \mathrm{LMod}_{\mathrm{Fun}(\mathcal{I}, \mathcal{S})}(\mathrm{Pr}^L)$.*

Proof. The levelwise tensoring is given by the composite

$$\mathrm{Fun}(\mathcal{I}, \mathcal{S}) \times \mathrm{Fun}(\mathcal{I}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{I}, \mathcal{S} \times \mathcal{D}) \xrightarrow{\mathrm{Fun}(\mathcal{I}, -\odot-)} \mathrm{Fun}(\mathcal{I}, \mathcal{D});$$

indeed, we obtain $\mathrm{Fun}(\mathcal{I}, \mathcal{D}) \in \mathrm{LMod}_{\mathrm{Fun}(\mathcal{I}, \mathcal{S})}(\mathrm{Cat}_{\infty})$ by applying $\mathrm{Fun}(\mathcal{I}, -)$ to the data of $\mathcal{D} \in \mathrm{LMod}_{\mathcal{S}}(\mathrm{Cat}_{\infty})$. Moreover, by definition the tensoring $- \odot - : \mathcal{S} \times \mathcal{D} \to \mathcal{D}$ commutes with colimits separately in each variable; as colimits in a functor ∞-category are computed pointwise, the above composite commutes with colimits separately in each variable as well. $\square$

Corollary 2.4. *For any $\mathcal{D} \in \mathrm{Pr}^L$, the levelwise tensoring of $s\mathcal{D}$ over $s\mathcal{S}$ extends to an enrichment and bitensoring.*

Proof. By Lemma 2.3, the levelwise tensoring defines an action $s\mathcal{D} \in \mathrm{Mod}_{s\mathcal{S}}(\mathrm{Pr}^L)$, and so the claim follows from Proposition 2.2. $\square$

Observation 2.5. Given two ∞-categories $\mathcal{D}$ and $\mathcal{E}$, one can define an adjunction $\mathcal{D} \rightleftarrows \mathcal{E}$ to be a functor $A : \mathcal{D}^{op} \times \mathcal{E} \to \mathcal{S}$ satisfying certain co/representability conditions (see [53, item (25) of §A]). If for some closed monoidal ∞-category $\mathcal{V}$ these ∞-categories are equipped with lifts $\underline{\mathcal{D}}$ and $\underline{\mathcal{E}}$ to $\mathcal{V}$-enriched ∞-categories, then an enriched adjunction

$\underline{\mathcal{D}} \rightleftarrows \underline{\mathcal{E}}$ can be defined as a functor $\underline{A} : \mathcal{D}^{op} \times \mathcal{E} \to \mathcal{V}$ satisfying analogous co/representability conditions. (This recovers an ordinary adjunction between the underlying unenriched ∞-categories by postcomposition with the functor $\underline{\hom}_{\mathcal{V}}(\mathbf{1}_{\mathcal{V}}, -) : \mathcal{V} \to \mathcal{S}$.) We use this fact without further comment.

Lemma 2.6. *Let $\mathcal{V} \in \mathrm{Alg}(\mathcal{Cat}_{\infty})$ be a presentable monoidal ∞-category, suppose that two ∞-categories $\mathcal{D}$ and $\mathcal{E}$ are enriched and bitensored over $\mathcal{V}$, and suppose we are given an adjunction $F : \mathcal{D} \rightleftarrows \mathcal{E} : G$ between their underlying ∞-categories. Suppose further that we have a natural equivalence $F(- \odot_{\mathcal{D}} -) \simeq (-) \odot_{\mathcal{E}} F(-)$ in $\mathrm{Fun}(\mathcal{V} \times \mathcal{D}, \mathcal{E})$. Then the adjunction $F \dashv G$ lifts to a $\mathcal{V}$-enriched adjunction $\underline{F} : \underline{\mathcal{D}} :\rightleftarrows \underline{\mathcal{E}} : \underline{G}$, and moreover we have a natural equivalence $G(- \pitchfork_{\mathcal{E}} -) \simeq (-) \pitchfork_{\mathcal{D}} G(-)$ in $\mathrm{Fun}(\mathcal{V}^{op} \times \mathcal{E}, \mathcal{D})$.*

Proof. First of all, the final claim follows from our assumption (and the Yoneda lemma) by the string of natural equivalences

$$\hom_{\mathcal{D}}(d, G(v \pitchfork_{\mathcal{E}} e)) \simeq \hom_{\mathcal{E}}(F(d), v \pitchfork_{\mathcal{E}} e) \simeq \hom_{\mathcal{E}}(v \odot_{\mathcal{E}} F(d), e)$$
$$\simeq \hom_{\mathcal{E}}(F(v \odot_{\mathcal{D}} d), e) \simeq \hom_{\mathcal{D}}(v \odot_{\mathcal{D}} d, G(e))$$
$$\simeq \hom_{\mathcal{D}}(d, v \pitchfork_{\mathcal{D}} G(e)).$$

Now, consider the functor $\mathcal{D}^{op} \times \mathcal{E} \to \mathcal{P}(\mathcal{V})$ which takes a pair of objects $(d^{\circ}, e) \in \mathcal{D}^{op} \times \mathcal{E}$ to the presheaf taking $v^{\circ} \in \mathcal{V}^{op}$ to the space

$$\hom_{\mathcal{D}}(v \odot_{\mathcal{D}} d, Ge) \simeq \hom_{\mathcal{E}}(F(v \odot_{\mathcal{D}} d), e) \simeq \hom_{\mathcal{E}}(v \odot_{\mathcal{E}} F(d), e) \simeq \hom_{\mathcal{E}}(F(d), v \pitchfork_{\mathcal{E}} e).$$

Since $\mathcal{V}$ is presentable, this factors through the Yoneda embedding $\mathcal{V} \xrightarrow{\text{\j}_{\mathcal{V}}} \mathcal{P}(\mathcal{V})$. By construction, this defines an enriched adjunction $\underline{F} : \underline{\mathcal{D}} \rightleftarrows \underline{\mathcal{E}} : \underline{G}$ lifting the original adjunction $F \dashv G$. $\square$

Corollary 2.7. *For any $\mathcal{D} \in \mathrm{Pr}^L$ and any monad $t \in \mathrm{Alg}(\mathrm{End}(s\mathcal{D}))$, we obtain a canonical enrichment and bitensoring of $\mathrm{Alg}_t(s\mathcal{D})$ over $s\mathcal{S}$, and moreover the adjunction $\mathrm{F}_t : s\mathcal{D} \rightleftarrows \mathrm{Alg}_t(s\mathcal{D}) : \mathrm{U}_t$ is canonically enriched over $s\mathcal{S}$.*

Proof. As any object of $\mathrm{Alg}_t(s\mathcal{D})$ is a colimit of free objects, for any $K \in s\mathcal{S}$ and any $Y \in \mathrm{Alg}_t(s\mathcal{D})$ we define

$$K \odot Y = \mathrm{colim}_{(X \to \mathrm{U}_t(Y)) \in s\mathcal{N}_{/\mathrm{U}_t(Y)}} \mathrm{F}_t(K \odot X)$$

(using the action $s\mathcal{D} \in \mathrm{LMod}_{s\mathcal{S}}(\mathrm{Pr}^L)$ of Corollary 2.4). This defines a bifunctor $- \odot - : s\mathcal{S} \times \mathrm{Alg}_t(s\mathcal{D}) \to \mathrm{Alg}_t(s\mathcal{D})$ which by construction commutes with colimits separately in each variable. Thus it defines an action $\mathrm{Alg}_t(s\mathcal{D}) \in \mathrm{LMod}_{s\mathcal{S}}(\mathrm{Pr}^L)$, and so by Proposition 2.2 extends to an enrichment and bitensoring of $\mathrm{Alg}_t(s\mathcal{D})$ over $s\mathcal{S}$. Then, the enrichment of the adjunction $\mathrm{F}_t \dashv \mathrm{U}_t$ follows from Lemma 2.6. $\square$

2.2. Simplicial model structures

We now provide a lifting theorem for constructing simplicial model ∞-category structures. This requires two auxiliary pieces of terminology.

Definition 2.8. Given a set I of homotopy classes of maps in $\mathcal{C}$, the subcategory I-proj of *I-projectives* is the subcategory of maps with $\mathrm{llp}(I)$.

Definition 2.9. Let $\mathcal{V}$ be a monoidal model ∞-category, and suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{V}$-enriched model ∞-categories. Then a *$\mathcal{V}$-enriched Quillen adjunction* between $\mathcal{M}$ and $\mathcal{N}$ is a $\mathcal{V}$-enriched adjunction $\underline{F} : \underline{\mathcal{M}} \rightleftarrows \underline{\mathcal{N}} : \underline{G}$ such that the underlying adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ is a Quillen adjunction.

Theorem 2.10. *Let $\mathcal{M}$ be a bicomplete ∞-category, and let $F : s\mathcal{S} \rightleftarrows \mathcal{M} : G$ be an adjunction such that G commutes with filtered colimits. Write $\mathbf{W}^{\mathcal{M}} = G^{-1}(\mathbf{W}_{\mathrm{KQ}}^{s\mathcal{S}})$, $\mathbf{F}^{\mathcal{M}} = G^{-1}(\mathbf{F}_{\mathrm{KQ}}^{s\mathcal{S}})$, and $\mathbf{C}^{\mathcal{M}} = (\mathbf{W} \cap \mathbf{F})^{\mathcal{M}}$-proj. Suppose that the following condition holds:*

$$\left(\mathbf{C}^{\mathcal{M}} \cap (\mathbf{F}^{\mathcal{M}}\text{-proj}) \right) \subset \mathbf{W}^{\mathcal{M}}. \tag{$*$}$$

*Then $\mathcal{M}$ admits a **resolution model structure**, denoted $\mathcal{M}_{\mathrm{res}}$, with $\mathbf{W}_{\mathrm{res}}^{\mathcal{M}} = \mathbf{W}^{\mathcal{M}}$, $\mathbf{C}_{\mathrm{res}}^{\mathcal{M}} = \mathbf{C}^{\mathcal{M}}$, and $\mathbf{F}_{\mathrm{res}}^{\mathcal{M}} = \mathbf{F}^{\mathcal{M}}$, and the above adjunction becomes a Quillen adjunction $F : s\mathcal{S}_{\mathrm{KQ}} \rightleftarrows \mathcal{M}_{\mathrm{res}} : G$.*

Proof. The proof is almost identical to that of [28, Theorem II.4.1] (despite the fact that there they only work in the special case of a category of simplicial objects); the only modification which must be made is that in the proofs of [28, Lemmas II.4.2 and II.4.3] (which construct required factorizations) one must take a coproduct over *homotopy classes* of commutative squares. $\square$

Remark 2.11. In practice, there seems to more-or-less always be (at least) one thing that's difficult to check in constructing a model structure. In this case, condition $(*)$ of Theorem 2.10 effectively requires that those would-be cofibrations that moreover have the left lifting property for all would-be fibrations are also would-be weak equivalences. We will give sufficient conditions for this condition to hold in §2.3.

Remark 2.12. It follows from the proof of Theorem 2.10 that one can replace the condition $(*)$ with the following pair of conditions:

$(*')$ for every map $\Lambda_i^n \to \Delta^n$ in $J_{\mathrm{KQ}}^{s\mathcal{S}}$, the induced map $F(\Lambda_i^n) \to F(\Delta^n)$ lies in $\mathbf{W}^{\mathcal{M}} \subset \mathcal{M}$;
$(*'')$ the maps in $(\mathbf{W} \cap \mathbf{C})^{\mathcal{M}}$ are closed under coproducts, pushouts, and sequential colimits.

This is explained in [28, Remark II.4.5].

Theorem 2.13. *In the setting of Theorem 2.10, suppose that we have an action $\mathcal{M} \in$ $\mathrm{LMod}_{s\mathcal{S}}(\mathrm{Cat}_\infty)$, denoted $- \odot - : s\mathcal{S} \times \mathcal{M} \to \mathcal{M}$, such that this bifunctor commutes with colimits separately in each variable, and suppose that we have a natural equivalence $F(- \times -) \simeq (-) \odot F(-)$ in $\mathrm{Fun}(s\mathcal{S} \times s\mathcal{S}, \mathcal{M})$. Then the resolution model structure canonically enhances to a simplicial model ∞-category $\underline{\mathcal{M}}_{\mathrm{res}}$, and the Quillen adjunction canonically enhances to an $s\mathcal{S}_{\mathrm{KQ}}$-enriched Quillen adjunction $\underline{F} : \underline{s\mathcal{S}}_{\mathrm{KQ}} \rightleftarrows \underline{\mathcal{M}}_{\mathrm{res}} : \underline{G}$.*

Proof. Using Lemma 2.6, the proof is identical to that of [28, Theorem II.4.4]. □

2.3. Sufficient criteria for the satisfaction of condition $(*)$ of Theorem 2.10

We now provide various conditions guaranteeing that condition $(*)$ of Theorem 2.10 is satisfied.

The key result is the following.

Proposition 2.14. *In the setting of Theorem 2.10, suppose that there exists an endofunctor $\mathbb{R} : \mathcal{M} \to \mathcal{M}$ which factors through the subcategory $\mathcal{M}^f \subset \mathcal{M}$ of fibrant objects and which admits a map $\mathrm{id}_\mathcal{M} \to \mathbb{R}$ whose components lie in $\mathbf{W}^\mathcal{M}$. Then condition $(*)$ holds.*

Proof. The proof is identical to that of [28, Lemma II.5.1]. □

Corollary 2.15. *In the setting of Theorem 2.10, suppose that for every object $X \in \mathcal{M}$ the terminal map $X \to \mathrm{pt}_\mathcal{M}$ lies in $\mathbf{F}^\mathcal{M}$. Then condition $(*)$ holds.*

Proof. This follows from Proposition 2.14, taking $\mathbb{R} = \mathrm{id}_\mathcal{M}$ (equipped with the identity coaugmentation). □

Corollary 2.16. *Let $\mathcal{N}$ be a bicomplete ∞-category, and for any object $Z \in \mathcal{N}$ consider the adjunction*

$$- \odot \mathrm{const}(Z) : s\mathcal{S} \rightleftarrows s\mathcal{N} : \hom_\mathcal{N}^{\mathrm{lw}}(Z, -).$$

If the object $Z \in \mathcal{N}$ is small, then this adjunction satisfies condition $(*)$ of Theorem 2.10.

Proof. With the theory of the Ex^∞ functor for $s\mathcal{S}_{\mathrm{KQ}}$ of [53, §6] in hand (specifically [53, Proposition 6.22 and Remark 6.23]), this follows from Proposition 2.14 by an identical argument to that of [28, Proposition II.5.5]. □

Remark 2.17. The technique of Corollary 2.16 cannot work for a general (bicomplete) ∞-category equipped with a right adjoint functor to $s\mathcal{S}$: it must be an ∞-category of simplicial objects. In effect, this is because the endofunctor Ex is a right adjoint, but it is not an *enriched* right adjoint. Indeed, the functor $\underline{\hom}_{s\mathcal{S}}(\Delta^1, -) : s\mathcal{S} \to s\mathcal{S}$ is an example of an enriched limit and so commutes with any enriched right adjoint, but the

canonical map $\mathrm{Ex}(\underline{\mathrm{hom}}_{s\mathbb{S}}(\Delta^1, -)) \to \underline{\mathrm{hom}}_{s\mathbb{S}}(\Delta^1, \mathrm{Ex}(-))$ is not an equivalence; this can be seen by evaluating on Δ^1, since the source has three 0-simplices but the target has five.

Corollary 2.18. *Let $\mathbb{N} \in \mathrm{Pr}^L$, and let $Z \in \mathbb{N}$ be a small object. Then with the enrichment and bitensoring of $s\mathbb{N}$ over $s\mathbb{S}$ of Corollary 2.4, there exists a simplicial model structure on $s\mathbb{N}$ created by the $s\mathbb{S}$-enriched Quillen adjunction*

$$- \odot \mathrm{const}(Z) : \underline{s\mathbb{S}}_{\mathrm{KQ}} \rightleftarrows \underline{s\mathbb{N}}_{\mathrm{res}} : \mathrm{hom}_{\mathbb{N}}^{\mathrm{lw}}(Z, -).$$

Proof. By Corollary 2.16, this adjunction satisfies condition $(*)$ of Theorem 2.10 and hence creates a model structure on $s\mathbb{N}$. By Lemma 2.3, this adjunction furthermore satisfies the hypotheses of Theorem 2.13, so that $s\mathbb{N}_{\mathrm{res}}$ and the Quillen adjunction becomes compatibly $s\mathbb{S}_{\mathrm{KQ}}$-enriched. $\square$

We will also be interested in the following "many-object" version of Corollary 2.18.

Theorem 2.19. *Let $\mathbb{N} \in \mathrm{Pr}^L$, and suppose we are given a set of small objects $Z_\alpha \in \mathbb{N}$. Then with the enrichment and bitensoring of $s\mathbb{N}$ over $s\mathbb{S}$ of Corollary 2.4, there exists a simplicial model structure on $s\mathbb{N}$ created by the $s\mathbb{S}$-enriched Quillen adjunction*

$$\coprod_\alpha \mathrm{pr}_\alpha(-) \odot \mathrm{const}(Z_\alpha) : \prod_\alpha \underline{s\mathbb{S}}_{\mathrm{KQ}} \rightleftarrows \underline{s\mathbb{N}}_{\mathrm{res}} : \left(\mathrm{hom}_{\mathbb{N}}^{\mathrm{lw}}(Z_\alpha, -)\right).$$

Proof. Given the above results, the proof is essentially identical to that of [28, Proposition II.5.9]. $\square$

Remark 2.20. In Theorem 2.19, if the objects Z_α form a set of compact projective generators (in the sense of Definition T.5.5.8.23) and the ∞-category $\mathbb{N}$ has enough projectives, then weak equivalences and fibrations in $s\mathbb{N}_{\mathrm{res}}$ will be detected by *all* projective objects (see [28, Example II.5.10]).

We end this subsection with the following result, which gives a convenient class of examples for which the condition of Corollary 2.15 holds (i.e. that all objects are ("would-be") fibrant). It is an ∞-categorical analog of the classical fact that every simplicial group is in particular a Kan complex.

Lemma 2.21. *In the adjunction $\mathrm{F}_{s\mathcal{G}\mathrm{rp}(\mathbb{S})} : s\mathbb{S} \rightleftarrows s\mathcal{G}\mathrm{rp}(\mathbb{S}) : \mathrm{U}_{s\mathcal{G}\mathrm{rp}(\mathbb{S})}$, the right adjoint factors through the subcategory $s\mathbb{S}_{\mathrm{KQ}}^f \subset s\mathbb{S}$ of fibrant objects with respect to the Kan–Quillen model structure.*

Proof. Observe that the adjunction $\mathrm{F}_{\mathcal{G}\mathrm{rp}(\mathbb{S})} : \mathbb{S} \rightleftarrows \mathcal{G}\mathrm{rp}(\mathbb{S}) : \mathrm{U}_{\mathcal{G}\mathrm{rp}(\mathbb{S})}$ factors as the composite adjunction

$$\mathcal{S} \underset{U_{\mathrm{Mon}(\mathcal{S})}}{\overset{F_{\mathrm{Mon}(\mathcal{S})}}{\rightleftarrows}} \mathrm{Mon}(\mathcal{S}) \overset{(-)^{\mathrm{gp}}}{\rightleftarrows} \mathrm{Grp}(\mathcal{S}).$$

We claim that the diagram

$$
\begin{array}{ccccc}
\mathrm{Set} & \xrightarrow{\ F_{\mathrm{Mon}}\ } & \mathrm{Mon} & \xrightarrow{\ (-)^{\mathrm{gp}}\ } & \mathrm{Grp} \\
\downarrow & & & & \downarrow \\
\mathcal{S} & \xrightarrow{\ F_{\mathrm{Mon}(\mathcal{S})}\ } & \mathrm{Mon}(\mathcal{S}) & \xrightarrow{\ (-)^{\mathrm{gp}}\ } & \mathrm{Grp}(\mathcal{S})
\end{array}
$$

commutes.[11] Indeed, recall the factorization

$$
\begin{array}{ccc}
\mathrm{Mon}(\mathcal{S}) & \xrightarrow{\ (-)^{\mathrm{gp}}\ } & \mathrm{Grp}(\mathcal{S}), \\
& \searrow^{B} \quad \nearrow^{\Omega} & \\
& \mathcal{S}_* &
\end{array}
$$

and recall that the functor $\mathrm{Mon}(\mathcal{S}) \xrightarrow{B} \mathcal{S}_*$ can itself be obtained as the composite

$$\mathrm{Mon}(\mathcal{S}) \xrightarrow{\mathfrak{B}} (\mathcal{C}\mathrm{at}_\infty)_* \xrightarrow{(-)^{\mathrm{gpd}}} (\mathcal{G}\mathrm{pd}_\infty)_* \simeq \mathcal{S}_*$$

(where $\mathfrak{B}$ denotes the "categorical delooping" functor). The claim now follows from the commutativity of the diagram

$$
\begin{array}{ccccccc}
\mathrm{Set} & \xrightarrow{\ F_{\mathrm{Mon}}\ } & \mathrm{Mon} & \xrightarrow{\ \mathfrak{B}\ } & \mathrm{cat}_* & \xrightarrow{\ (-)^{\mathrm{gpd}}\ } & \mathrm{gpd}_* \\
\downarrow & & & & & & \downarrow \\
\mathcal{S} & \xrightarrow{\ F_{\mathrm{Mon}(\mathcal{S})}\ } & \mathrm{Mon}(\mathcal{S}) & \xrightarrow{\ \mathfrak{B}\ } & (\mathcal{C}\mathrm{at}_\infty)_* & \xrightarrow{\ (-)^{\mathrm{gpd}}\ } & (\mathcal{G}\mathrm{pd}_\infty)_*
\end{array}
$$

(where cat denotes the category of strict categories (i.e. Segal simplicial sets) and similarly for gpd (see [53, item (4)(c) of §A])), which itself follows from [18, 5.4].

Now, applying $\mathrm{Fun}(\mathbf{\Delta}^{op}, -)$ to the original commutative rectangle, we obtain a commutative square

$$
\begin{array}{ccc}
s\mathrm{Set} & \xrightarrow{\ F_{s\mathrm{Grp}}\ } & s\mathrm{Grp} \\
\downarrow & & \downarrow \\
s\mathcal{S} & \xrightarrow{\ F_{s\mathrm{Grp}(\mathcal{S})}\ } & s\mathrm{Grp}(\mathcal{S}).
\end{array}
$$

---

[11] If we were to add in the middle vertical inclusion $\mathrm{Mon} \hookrightarrow \mathrm{Mon}(\mathcal{S})$, the left square would commute (simply by inspection of the functor $F_{\mathrm{Mon}(\mathcal{S})}$), but the right square would not: its extreme failure to do so is encoded by [49, Theorem 1].

In particular, the image of any element $\Lambda_i^n \to \Delta^n$ of $J_{\mathrm{KQ}}^{s\mathcal{S}\mathrm{et}} = J_{\mathrm{KQ}}^{s\mathcal{S}}$ under the composite

$$s\mathcal{S}\mathrm{et} \hookrightarrow s\mathcal{S} \xrightarrow{\mathrm{F}_{s\mathcal{G}\mathrm{rp}(\mathcal{S})}} s\mathcal{G}\mathrm{rp}(\mathcal{S})$$

admits a retraction (see e.g. [28, Lemma I.3.4]). This proves the claim. $\square$

2.4. The underlying ∞-category of the resolution model structure

We conclude this section by identifying the underlying ∞-category of the resolution model structure of Theorem 2.19.

Definition 2.22. For an ∞-category $\mathcal{D}$ admitting finite coproducts, we write $\mathcal{P}_\Sigma(\mathcal{D}) = \mathrm{Fun}^\times(\mathcal{D}^{op}, \mathcal{S})$ for its **nonabelian derived ∞-category** of product-preserving presheaves (i.e. of functors taking finite coproducts in $\mathcal{D}$ to finite products in $\mathcal{S}$). We write $\mathcal{P}_\Sigma^\delta(\mathcal{D}) \subset \mathcal{P}_\Sigma(\mathcal{D})$ for its subcategory of discrete objects; thus $\mathcal{P}_\Sigma^\delta(\mathcal{D}) \simeq \mathrm{Fun}^\times(\mathcal{D}^{op}, \mathcal{S}\mathrm{et}) \simeq \mathrm{Fun}^\times(\mathrm{ho}(\mathcal{D})^{op}, \mathcal{S}\mathrm{et})$.

Theorem 2.23. *In the situation of Theorem 2.19, writing $\mathcal{G} \subset \mathcal{N}$ for the full subcategory generated by the objects Z_α under finite coproducts, we have a canonical Quillen adjunction*

$$\mathrm{Fun}(\mathcal{G}^{op}, s\mathcal{S}_{\mathrm{KQ}})_{\mathrm{proj}} \rightleftarrows s\mathcal{N}_{\mathrm{res}}$$

with the projective model structure (as in [54, §1.2]), with derived adjunction given by the canonical adjunction

$$\mathcal{P}(\mathcal{G}) \rightleftarrows \mathcal{P}_\Sigma(\mathcal{G})$$

whose right adjoint is the defining inclusion.

Proof. The projective model structure can also be seen as lifted via Theorem 2.19 from the same product of copies of the model ∞-category $s\mathcal{S}_{\mathrm{KQ}}$, which implies that this is indeed a Quillen adjunction. As the functor $|-| : s\mathcal{S} \to \mathcal{S}$ commutes with finite products, it follows that the derived right adjoint factors through the subcategory $\mathcal{P}_\Sigma(\mathcal{G}) \subset \mathcal{P}(\mathcal{G})$. Moreover, as $\mathcal{N}$ is presentable, the restricted Yoneda embedding participates in an adjunction $\mathcal{P}_\Sigma(\mathcal{G}) \rightleftarrows \mathcal{N}$, from which it follows that this derived right adjoint surjects onto $\mathcal{P}_\Sigma(\mathcal{G})$ (by taking the constant simplicial object on a given object of $\mathcal{N}$, seen as a product-preserving presheaf on $\mathcal{G}$). So, it will suffice to show that the functor $s\mathcal{N}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \to \mathcal{P}(\mathcal{G})$ is fully faithful. First of all, taking any $X \in s\mathcal{N}_{\mathrm{res}}^f$, since $s\mathcal{N}_{\mathrm{res}}$ is simplicial, for any $K \in s\mathcal{S}\mathrm{et} = s\mathcal{S}_{\mathrm{KQ}}^c$ we have that

$$\hom_{s\mathcal{N}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}(K \odot \mathrm{const}(Z_\alpha), X) \simeq \underline{\hom}_{s\mathcal{N}}(K \odot \mathrm{const}(Z_\alpha), X)$$

$$\simeq \underline{\hom}_{s\mathcal{S}}(K, \underline{\hom}_{s\mathcal{N}}(\mathrm{const}(Z_\alpha), X))$$

$$\simeq \underline{\hom}_{s\mathbb{S}}(K, \hom_{\mathcal{N}}^{\mathrm{lw}}(Z_\alpha, X))$$
$$\simeq \hom_{\mathbb{S}}(|K|, |\hom_{\mathcal{N}}^{\mathrm{lw}}(Z_\alpha, X)|)$$

(where the last equivalence uses the fact that $s\mathbb{S}_{\mathrm{KQ}}$ is a simplicial model ∞-category). The claim now follows from the fact that $I_{\mathrm{res}}^{s\mathcal{N}} = \{I_{\mathrm{KQ}}^{s\mathbb{S}} \odot \mathrm{const}(Z_\alpha)\}$ forms a set of generating cofibrations of $s\mathcal{N}_{\mathrm{res}}$, so that we can construct a cofibrant replacement of any object as a transfinite composition of pushouts of these maps. □

3. Topology

In this section, we discuss our basic topological framework, absent any operadic structure. We begin in §3.1 by discussing the ∞-category $\mathcal{C}$ in which we will be working, as well as its notion of "homotopy groups". We then discuss the resolution model structure on $s\mathcal{C}$ in §3.2, and foreshadow a further localization at its "E_*-equivalences". We discuss the spiral exact sequence (relating classical and natural homotopy groups in $s\mathcal{C}$) in §3.3, and we discuss the localized spiral exact sequence (relating classical and natural E-homology groups in $s\mathcal{C}$) in §3.4.

3.1. Foundations of topology

Assumption 3.1. We begin with a presentably symmetric monoidal stable ∞-category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1})$. By presentability, this will automatically be closed (i.e. admit an internal hom bifunctor).

Remark 3.2. When it is convenient, we will consider $\mathcal{C}$ as being enriched over the symmetric monoidal ∞-category $(\mathcal{S}_*, \wedge, S^0)$ of pointed spaces equipped with the smash product: the basepoint $0 \in \hom_{\mathcal{C}}(X, Y)$ is given by the unique "zero map" $X \to 0_{\mathcal{C}} \to Y$, and the fact that the composition maps factor through the smash products amounts to the observation that any sequence of composable maps in which at least one of the maps is a zero map composes canonically to another zero map. Moreover, $\mathcal{C}$ admits a canonical bitensoring over $\mathcal{S}_*$ which is compatible with this enrichment. (It is not hard to make these assertions precise using the formalism of [22].)

Notation 3.3. We write $\mathrm{D} = \underline{\hom}_{\mathcal{C}}(-, \mathbf{1}) : \mathcal{C}^{op} \to \mathcal{C}$ for the "linear dual" functor, and we write $\mathcal{C}^{\mathrm{inv}} \subset \mathcal{C}^{\mathrm{d}} \subset \mathcal{C}$ for the full subcategories of invertible objects and of dualizable objects.

Assumption 3.4. We assume that the unit object $\mathbf{1} \in \mathcal{C}$ is compact, i.e. that the functor $\hom_{\mathcal{C}}(\mathbf{1}, -) : \mathcal{C} \to \mathcal{S}$ commutes with filtered colimits.

Observation 3.5. It follows immediately from Assumption 3.4 that any invertible object of $\mathcal{C}$ is necessarily compact. In fact, because of the assumption that the symmetric

monoidal structure commutes with colimits separately in each variable, it follows that any dualizable object is compact as well: this is a consequence of the natural equivalence $\hom_{\mathcal{C}}(X, -) \simeq \hom_{\mathcal{C}}(\mathbf{1}, DX \otimes -)$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$.

Assumption 3.6. We assume the existence of a small subcategory $\mathcal{G} \subset \mathcal{C}$ of (strong) generators, which we generally denote by $S^\beta \in \mathcal{G}$ (with the "S" and "β" chosen to evoke the notion of a "bigraded sphere" (from motivic stable homotopy theory)); that is, we assume that the functors

$$\hom_{\mathcal{C}}(S^\beta, -) : \mathcal{C} \to \mathcal{S}$$

are jointly conservative. We moreover assume that the subcategory $\mathcal{G} \subset \mathcal{C}$

- contains the unit object $\mathbf{1} \in \mathcal{C}$,
- is closed under de/suspensions,
- consists of invertible objects, and
- is closed under the monoidal product of $\mathcal{C}$.

We write $S^{n+\beta} = \Sigma^n S^\beta$ for any $n \in \mathbb{Z}$.

Notation 3.7. We write $\mathcal{G}^\delta = \pi_0(\mathcal{G}^\simeq) \in \mathrm{Ab}\mathcal{G}\mathrm{rp}$ for the abelian group of equivalence classes of objects of $\mathcal{G}$, with addition given by the monoidal product of $\mathcal{C}$. We denote the element corresponding to $S^\beta \in \mathcal{G}$ simply by $\beta \in \mathcal{G}^\delta$.

Definition 3.8. For any $\beta \in \mathcal{G}^\delta$, we refer to the equivalence $S^\beta \otimes - : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ as the β-**fold suspension**. The ordinary notion of suspension is recovered as $(\Sigma^n \mathbf{1}) \otimes - : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$. We will henceforth refer to *any* β-fold suspension as a "suspension", and refer to this latter more restrictive notion as a ***categorical suspension***. We denote β-fold suspension by Σ^β, and categorical suspension simply by Σ^n. (Note that these conventions jibe with those of Assumption 3.6.) While through this definition the term "desuspension" technically becomes superfluous, we will nevertheless continue to employ it for aesthetic reasons.

Notation 3.9. We write $\underline{\mathcal{A}} = \mathrm{Fun}(\mathcal{G}^\delta, \mathrm{Ab})$ for the category of $\mathcal{G}^\delta$-graded abelian groups, equipped with the Day convolution monoidal structure relative to $(\mathcal{G}^\delta, +) = (\mathcal{G}^\delta, \otimes_{\mathcal{C}})$ and $(\mathrm{Ab}, \otimes_{\mathbb{Z}})$. This receives a "homotopy" functor $\pi_* : \mathcal{C} \to \underline{\mathcal{A}}$, given by $\pi_\beta X = (\pi_* X)(S^\beta) = [S^\beta, X]_{\mathcal{C}}$.[12] This functor is itself lax monoidal, and in fact descends along the monoidal functor $\mathcal{C} \to \mathrm{ho}(\mathcal{C})$ to another lax monoidal functor $\pi_* : \mathrm{ho}(\mathcal{C}) \to \underline{\mathcal{A}}$.

Remark 3.10. As a result of Assumption 3.6, to say that $\mathcal{G} \subset \mathcal{C}$ is a subcategory of strong generators is precisely to say that the functor $\pi_* : \mathcal{C} \to \underline{\mathcal{A}}$ creates the equivalences in $\mathcal{C}$.

---

[12] This is the composite of the canonical projection $\mathcal{C} \to \mathrm{ho}(\mathcal{C})$ followed by the restricted Yoneda embedding along the functor $\mathcal{G}^\delta \to \mathrm{ho}(\mathcal{C})$; note that we have a canonical equivalence $\mathcal{G}^\delta \simeq (\mathcal{G}^\delta)^{op}$ since this category has no nonidentity morphisms.

Remark 3.11. One could alternatively consider the "homotopy" functor as taking values in $\mathcal{P}^{\delta}_{\Sigma}(\mathcal{G}^{\vee}) = \mathrm{Fun}((\mathcal{G}^{\vee})^{op}, \mathcal{S}\mathrm{et})$, the category of product-preserving presheaves of sets on the closure of $\mathcal{G} \subset \mathcal{C}$ under finite coproducts (which remain coproducts in $\mathrm{ho}(\mathcal{C})$ since $\pi_0 : \mathcal{S} \to \mathcal{S}\mathrm{et}$ preserves products). This is analogous to the "Π-algebra" perspective taken by Dwyer–Kan–Stover in [21] and by Blanc–Dwyer–Goerss in [4]. However, in order to obtain a computable obstruction theory, Goerss–Hopkins take an alternative route, considering the homotopy groups of a spectrum simply as a $\mathbb{Z}$-graded abelian group (rather than as a module over the stable homotopy groups of spheres).[13]

We conclude this subsection with a few remarks concerning the choice of ambient ∞-category.

Remark 3.12. If we remove the requirement that $\mathcal{C}$ be stable, it becomes necessary to assume that the generators admit desuspensions in order for Lemma 3.45 to hold. It also becomes necessary to assume that the generators are h-cogroup objects (with respect to the wedge sum) in order to construct the relevant spectral sequence, but of course this is a strictly weaker assumption. More broadly, a great many of the arguments would become substantially more delicate.

Remark 3.13. If we only require $\mathcal{C}$ to be monoidal (instead of symmetric monoidal), then by the so-called "microcosm principle" it will only make sense to discuss associative algebras in $\mathcal{C}$, instead of commutative algebras. In the setting of ordinary spectra, associative algebras can be constructed via Hopkins–Miller obstruction theory (see [65]), which is far simpler than Goerss–Hopkins obstruction theory since it is not necessary to resolve the associative operad (see §4.1.3). On the other hand, if we set our sights lower and remove the operad from the picture entirely, we simply recover an abstract version of Blanc–Dwyer–Goerss obstruction theory (see [4]). In any case, we expect that practical examples of interest will carry symmetric monoidal structures anyways.

3.2. The resolution model structure

Notation 3.14. Let $E \in \mathrm{CAlg}(\mathrm{ho}(\mathcal{C}))$ be a homotopy commutative algebra object in $\mathcal{C}$. This induces $E_{*} = \pi_{*}E \in \mathrm{CAlg}(\underline{\mathcal{A}})$, and we write $\mathcal{A} = \mathrm{Mod}_{E_{*}}(\underline{\mathcal{A}})$ for its category of modules. Then we obtain a "homology" functor $E_{*} : \mathcal{C} \to \mathcal{A}$ by $E_{*}X = \pi_{*}(E \otimes X)$.

Definition 3.15. An E_{*}-**equivalence** in $\mathcal{C}$ is a morphism which becomes an isomorphism under the functor $E_{*} : \mathcal{C} \to \mathcal{A}$.

Notation 3.16. By definition, the E_{*}-equivalences are created by the composite $\mathcal{C} \xrightarrow{E \otimes -} \mathcal{C} \xrightarrow{\pi_{*}} \underline{\mathcal{A}}$ (as isomorphisms in $\mathcal{A}$ are created in $\underline{\mathcal{A}}$). However, Remark 3.10 implies that

---

[13] Nevertheless, product-preserving presheaves pervade this story. We will mostly suppress them, but we will need to discuss them explicitly in §5.5.

they are also created by the functor $\mathcal{C} \xrightarrow{E \otimes -} \mathcal{C}$. Our assumption that $\mathcal{C}$ is presentably symmetric monoidal immediately implies that the E_*-equivalences are strongly saturated (in the sense of Definition T.5.5.4.5), and so by Proposition T.5.5.4.15 there exists a left localization adjunction $L_{E_*} : \mathcal{C} \rightleftarrows L_E(\mathcal{C}) : U_{E_*}$.

Definition 3.17. We define the subcategory $\mathcal{A}_{\mathrm{proj}} \subset \mathcal{A}$ of ***projective*** objects just as in classical algebra.

Assumption 3.18. We assume henceforth that E satisfies ***Adams's condition***, namely that there exists a filtered diagram $E_\bullet : \mathcal{J} \to \mathcal{C}^{\mathrm{d}}_{/E} = \mathcal{C}^{\mathrm{d}} \times_{\mathcal{C}} \mathcal{C}_{/E}$ with $\mathrm{colim}(\mathcal{J} \xrightarrow{E_\bullet} \mathcal{C}) \xrightarrow{\sim} E$, such that for every $\alpha \in \mathcal{J}$,

- $E_* D E_\alpha \in \mathcal{A}_{\mathrm{proj}}$, and
- for every $M \in \mathrm{Mod}_E(\mathrm{ho}(\mathcal{C}))$, the canonical map

$$[DE_\alpha, M]_{\mathcal{C}} \to \hom_{\mathcal{A}}(E_* D E_\alpha, \pi_* M)$$

$$\left(DE_\alpha \xrightarrow{f} M \right) \mapsto \left(E_* D E_\alpha \xrightarrow{E_*(f)} E_* M = \pi_*(E \otimes M) \to \pi_* M \right)$$

  is an isomorphism.

Moreover, we fix the witnessing data of such a filtered diagram $E_\bullet : \mathcal{J} \to \mathcal{C}^{\mathrm{d}}_{/E}$.[14]

Remark 3.19. The canonical map of Assumption 3.18 can be equivalently seen as the composite

$$[DE_\alpha, M]_{\mathcal{C}} \cong [E \otimes DE_\alpha, M]_{\mathrm{Mod}_E(\mathrm{ho}(\mathcal{C}))} \xrightarrow{\pi_*} \hom_{\mathcal{A}}(E_* D E_\alpha, \pi_* M).$$

Observation 3.20. For any $X \in \mathcal{C}$ and any $\beta \in \mathcal{G}^\delta$, we have the string of isomorphisms

$$\mathrm{colim}_{\alpha \in \mathcal{J}}[\Sigma^\beta DE_\alpha, X]_{\mathcal{C}} \cong \mathrm{colim}_{\alpha \in \mathcal{J}}[S^\beta, E_\alpha \otimes X]_{\mathcal{C}} \cong [S^\beta, \mathrm{colim}_{\alpha \in \mathcal{J}}(E_\alpha \otimes X)]_{\mathcal{C}}$$
$$\cong [S^\beta, \mathrm{colim}_{\alpha \in \mathcal{J}}(E_\alpha) \otimes X]_{\mathcal{C}} \cong [S^\beta, E \otimes X]_{\mathcal{C}} = E_\beta X$$

in Ab.

Notation 3.21. Strings of adjunction isomorphisms having the same flavor as that of Observation 3.20 will frequently be useful to us. Rather than spell out the isomorphisms each time, we simply refer to this line of reasoning as a ***colimit argument***.

Notation 3.22. We write $\mathcal{G}_E \subset \mathcal{C}$ for the smallest full subcategory containing $\mathcal{G}$ and $\{DE_\alpha\}_{\alpha \in \mathcal{J}}$ that is closed under de/suspension and finite coproducts. We generally write

---

[14] It will be clear from the construction that our main results do not depend on this choice.

$S^\varepsilon \in \mathcal{G}_E$ for an arbitrary object (the letter "ε" being suggestive of the letter "E"), although we continue to write $S^\beta \in \mathcal{G} \subset \mathcal{G}_E$ for an arbitrary object of $\mathcal{G}$ when considered as an object of $\mathcal{G}_E$. We write $\mathcal{G}_E^\delta = \pi_0((\mathcal{G}_E)^\simeq)$, and so (just as we write $\beta \in \mathcal{G}$) we simply write $\varepsilon \in \mathcal{G}_E^\delta$ to denote an arbitrary element.

Observation 3.23. For any $S^\varepsilon \in \mathcal{G}_E$ and any $M \in \mathrm{Mod}_E(\mathrm{ho}(\mathcal{C}))$, we have an isomorphism

$$[S^\varepsilon, M] \xrightarrow{\cong} \hom_{\mathcal{A}}(E_{\circledast} S^\varepsilon, \pi_{\circledast} M).$$

This can be seen as follows.

- For $S^\varepsilon = \mathrm{D}E_\alpha$, this follows from Assumption 3.18.
- For $S^\varepsilon = S^\beta \in \mathcal{G}$, note that $E_{\circledast} S^\beta \cong E_{\circledast} \otimes_{\mathbf{1}_{\circledast}} \pi_{\circledast} S^\beta$, and so we are interested in the composite

$$[S^\beta, M]_{\mathcal{C}} \cong [E \otimes S^\beta, M]_{\mathrm{Mod}_E(\mathrm{ho}(\mathcal{C}))}$$

$$\xrightarrow{\pi_{\circledast}} \hom_{\mathcal{A}}(E_{\circledast} S^\beta, \pi_{\circledast} M) \cong \hom_{\mathrm{Mod}_{\mathbf{1}_{\circledast}}(\underline{\mathcal{A}})}(\pi_{\circledast} S^\beta, \pi_{\circledast} M),$$

  which is an isomorphism with inverse given by evaluation at the universal element of $\pi_\beta S^\beta$.
- In general, this property is preserved both by de/suspension and by the formation of finite coproducts.

Notation 3.24. Recall that $s\mathcal{C}$ is canonically enriched and bitensored over $s\mathcal{S}$ (see Corollary 2.4); these data assemble into a two-variable adjunction, which we denote by

$$\left(s\mathcal{S} \times s\mathcal{C} \xrightarrow{-\odot-} s\mathcal{C} \ , \ s\mathcal{S}^{op} \times s\mathcal{C} \xrightarrow{-\pitchfork-} s\mathcal{C} \ , \ s\mathcal{C}^{op} \times s\mathcal{C} \xrightarrow{\hom_{\mathcal{C}}(-,-)} s\mathcal{S} \right).$$

Definition 3.25. We fix the following terminology.

(1) A morphism in $\mathrm{ho}(\mathcal{C})$ is called a $\mathcal{G}_E$-**epimorphism** if the restricted Yoneda functor $\mathrm{ho}(\mathcal{C}) \to \mathcal{P}_\Sigma^\delta(\mathcal{G}_E)$ takes it to a componentwise surjection.
(2) An object of $\mathrm{ho}(\mathcal{C})$ is called $\mathcal{G}_E$-**projective** if it has the extension property for all $\mathcal{G}_E$-epimorphisms.
(3) A morphism in $\mathrm{ho}(\mathcal{C})$ is called a $\mathcal{G}_E$-**projective cofibration** if it has the left lifting property for all $\mathcal{G}_E$-epimorphisms.

Theorem 3.26. *There is a **resolution model structure** on $s\mathcal{C}$, denoted $s\mathcal{C}_{\mathrm{res}}$, which enjoys the following properties.*

(1) *Its weak equivalences and fibrations are created by the functor*

$$s\mathcal{C} \xrightarrow{X \mapsto (S^\varepsilon \mapsto \hom_\mathcal{C}^{\mathrm{lw}}(S^\varepsilon, X))} \prod_{\mathcal{G}_E^\delta} s\mathcal{S}_{\mathrm{KQ}},$$

i.e. a morphism is a weak equivalence or a fibration iff it is carried to one in $s\mathcal{S}_{\mathrm{KQ}}$ by the functor $\hom_\mathcal{C}^{\mathrm{lw}}(S^\varepsilon, -)$ for every $\varepsilon \in \mathcal{G}_E^\delta$.

(2) *It is simplicial.*

(3) *Its cofibrations are precisely those morphisms whose relative latching maps are $\mathcal{G}_E$-projective cofibrations.*

(4) *All objects are fibrant in it.*

(5) *It is cofibrantly generated by the sets*

$$I_{\mathrm{res}}^{s\mathcal{C}} = \{I_{\mathrm{KQ}}^{s\mathcal{S}} \odot \mathrm{const}(S^\varepsilon)\}_{S^\varepsilon \in \mathcal{G}_E} = \{\partial\Delta^n \odot \mathrm{const}(S^\varepsilon) \to \Delta^n \odot \mathrm{const}(S^\varepsilon)\}_{n \geq 0, S^\varepsilon \in \mathcal{G}_E}$$

and

$$J_{\mathrm{res}}^{s\mathcal{C}} = \{J_{\mathrm{KQ}}^{s\mathcal{S}} \odot \mathrm{const}(S^\varepsilon)\}_{S^\varepsilon \in \mathcal{G}_E} = \{\Lambda_i^n \odot \mathrm{const}(S^\varepsilon) \to \Delta^n \odot \mathrm{const}(S^\varepsilon)\}_{0 \leq i \leq n \geq 1, S^\varepsilon \in \mathcal{G}_E}.$$

Proof. This follows from Theorem 2.19 and Lemma 2.21. □

Remark 3.27. It will follow from the localized spiral exact sequence of Construction 3.52 that the weak equivalences of $s\mathcal{C}_{\mathrm{res}}$ are created by the functor

$$s\mathcal{C} \xrightarrow{[=,-]_\mathcal{C}^{\mathrm{lw}}} s\mathrm{Fun}(\mathcal{G}_E, \mathrm{Ab}) \simeq \mathrm{Fun}(\mathcal{G}_E, s\mathrm{Ab}_{\mathrm{KQ}})_{\mathrm{proj}}.$$

(In fact, the fibrations are as well.)

Definition 3.28. We define the subcategory of E_*-**equivalences**, denoted $\mathbf{W}_{E_*^{\mathrm{lw}}} = \mathbf{W}_{E_*^{\mathrm{lw}}}^{s\mathcal{C}} \subset s\mathcal{C}$, to be created by pulling back the subcategory $\mathbf{W}_{\mathrm{KQ}}^{s\mathcal{A}} \subset s\mathcal{A}_{\mathrm{KQ}}$ under the functor $E_*^{\mathrm{lw}} : s\mathcal{C} \to s\mathcal{A}_{\mathrm{KQ}}$.

Notation 3.29. Rather than overburden notation, we simply write $\pi_n : s\mathrm{Ab} \to \mathrm{Ab}$ for the composite

$$s\mathrm{Ab} \xrightarrow{|-|} \mathrm{Ab}\mathcal{G}\mathrm{rp}(\mathcal{S}_*) \xrightarrow{\mathrm{Ab}\mathcal{G}\mathrm{rp}(\pi_n)} \mathrm{Ab}\mathcal{G}\mathrm{rp}(\mathcal{S}\mathrm{et}_*) = \mathrm{Ab}.$$

This can be obtained more abstractly as a "homotopy" functor from a derived ∞-category to its heart, and indeed we use this same notation π_n to denote all corresponding functors $s\mathcal{S}\mathrm{et}_* \to \mathcal{S}\mathrm{et}_*$, $s\underline{\mathcal{A}} \to \underline{\mathcal{A}}$, $s\mathcal{A} \to \mathcal{A}$, etc.

Observation 3.30. Suppose that $X \xrightarrow{\approx} Y$ is a weak equivalence in $s\mathcal{C}_{\mathrm{res}}$. By Remark 3.27, this means that for every $S^\varepsilon \in \mathcal{G}_E$ we obtain a weak equivalence $[S^\varepsilon, X]_{\mathcal{C}}^{\mathrm{lw}} \xrightarrow{\approx} [S^\varepsilon, Y]_{\mathcal{C}}^{\mathrm{lw}}$ in $s\mathrm{Ab}_{\mathrm{KQ}}$, i.e. that we obtain isomorphisms $\pi_n([S^\varepsilon, X]_{\mathcal{C}}^{\mathrm{lw}}) \xrightarrow{\cong} \pi_n([S^\varepsilon, Y]_{\mathcal{C}}^{\mathrm{lw}})$ in Ab for all $n \geq 0$. In particular, letting S^ε range over the set $\{\Sigma^\beta \mathrm{D}E_\alpha\}_{\beta \in \mathcal{G}^\delta, \alpha \in \mathcal{J}}$, by Observation 3.20 and since homotopy groups in $s\mathcal{S}et_*$ commute with filtered colimits, we obtain a weak equivalence $E_*^{\mathrm{lw}}X \xrightarrow{\approx} E_*^{\mathrm{lw}}Y$ in $s\mathcal{A}_{\mathrm{KQ}}$. In other words, we have an inclusion $\mathbf{W}_{\mathrm{res}} \subset \mathbf{W}_{E_*^{\mathrm{lw}}}$ of subcategories of $s\mathcal{C}$.

Observation 3.31. In our setting, after a colimit argument the standard filtration spectral sequence for an object $X \in s\mathcal{C}$ runs $\mathrm{E}^2 = \pi_n E_\beta^{\mathrm{lw}} X \Rightarrow E_{\beta+n}|X|$. (This agrees with the spectral sequence associated to the localized spiral exact sequence of Construction 3.52 (see [23, Lemma 3.1.5 and Remark 3.1.6]).) Thus, an E_*-equivalence in $s\mathcal{C}$ (for instance a weak equivalence in $s\mathcal{C}_{\mathrm{res}}$, by Observation 3.30) induces an isomorphism on E^2 pages of this spectral sequence. In other words, there exists a factorization

$$
\begin{array}{ccc}
s\mathcal{C} & \xrightarrow{\;|-|\;} & \mathcal{C} \xrightarrow{\;E_*\;} \mathcal{A} \\
\downarrow & & \nearrow \\
s\mathcal{C}[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!] & &
\end{array}
$$

through the localization functor.

Definition 3.32. We refer to this spectral sequence $\mathrm{E}^2 = \pi_n E_\beta^{\mathrm{lw}} X \Rightarrow \mathrm{E}^\infty = E_{\beta+n}|X|$ as the ***spiral spectral sequence***.

Remark 3.33. By Theorem 2.23, the resolution model structure presents the nonabelian derived ∞-category $\mathcal{P}_\Sigma(\mathcal{G}_E)$. Moreover, the composite $\mathcal{C} \xrightarrow{\mathrm{const}} s\mathcal{C} \to s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \simeq \mathcal{P}_\Sigma(\mathcal{G}_E)$ clearly coincides with the restricted Yoneda embedding. We will generally omit this from the notation.

3.3. The spiral exact sequence

Definition 3.34. Choose any $n \geq 0$ and any $\varepsilon \in \mathcal{G}_E^\delta$.

(1) We define the corresponding ***classical homotopy group*** functor to be the composite

$$
\pi_n \pi_\varepsilon : s\mathcal{C} \xrightarrow{\;[S^\varepsilon, -]_{\mathcal{C}}^{\mathrm{lw}}\;} s\mathrm{Ab} \xrightarrow{\;\pi_n\;} \mathrm{Ab}.
$$

(2) We define the corresponding **natural homotopy group** functor to be either equivalent composite

$$
\begin{array}{ccc}
 & s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] & \\
 \nearrow & & \searrow \mathrm{hom}_{s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}(S^{\varepsilon},-) \\
\pi_{n,\varepsilon}^{\natural} : s\mathcal{C} & & \mathcal{G}\mathrm{rp}(\mathrm{ho}(\mathcal{S}_{*})) \xrightarrow{\ \pi_n\ } \mathrm{Ab}, \\
 \searrow \underline{\mathrm{hom}}_{s\mathcal{C}}(\mathrm{const}(S^{\varepsilon}),-) & & \nearrow |-| \\
 & \mathcal{G}\mathrm{rp}(\mathrm{ho}(s\mathcal{S}_{*})) &
\end{array}
$$

where

- the commutativity of the square follows from the fact that $s\mathcal{C}_{\mathrm{res}}$
  - is simplicial,
  - has $\mathrm{const}(S^{\varepsilon}) \in s\mathcal{C}_{\mathrm{res}}^c$ cofibrant, and
  - has all objects fibrant,
  and
- the fact that the down-and-right functors land in h-group objects follows from the fact that $S^{\varepsilon} \in \mathcal{C}$ is an h-cogroup object (so that $\mathrm{const}(S^{\varepsilon}) \in s\mathcal{C}$ is as well).

Definition 3.35. Let $K \in s\mathcal{S}_{*}$, and let $X \in s\mathcal{C}$. We define the **reduced tensoring** of X over K to be the pushout

$$
\begin{array}{ccc}
\mathrm{pt}_{s\mathcal{S}} \odot X & \longrightarrow & K \odot X \\
\downarrow & & \downarrow \\
\mathrm{pt}_{s\mathcal{S}} \odot 0_{s\mathcal{C}} & \longrightarrow & K \overline{\odot} X
\end{array}
$$

in $s\mathcal{C}$. This assembles into an action $s\mathcal{S}_{*} \times s\mathcal{C} \to s\mathcal{C}$.

Notation 3.36. We write $D_{\mathbf{\Delta}}^n = \Delta^n / \Lambda_0^n \in s\mathcal{S}\mathrm{et}_{*} \subset s\mathcal{S}_{*}$ for the "reduced pointed simplicial n-disk" and $S_{\mathbf{\Delta}}^n = \Delta^n / \partial \Delta^n \in s\mathcal{S}\mathrm{et}_{*} \subset s\mathcal{S}_{*}$ for the "reduced pointed simplicial n-sphere".

Observation 3.37. The canonical composite

$$
S_{\mathbf{\Delta}}^{n-1} \to D_{\mathbf{\Delta}}^n \to S_{\mathbf{\Delta}}^n
$$

(where the first map is obtained by considering $\Delta^{n-1} \cong \Delta^{\{0,\dots,n-1\}} \subset \Delta^n$) is a cofiber sequence not just in $s\mathcal{S}\mathrm{et}_{*}$ but also in $s\mathcal{S}_{*}$.

Lemma 3.38. *For any $n \geq 0$ and any $S^{\varepsilon} \in \mathcal{G}_E$, there is a natural isomorphism*

$$
\pi_{n,\varepsilon}^{\natural}(-) \cong [S_{\mathbf{\Delta}}^n \overline{\odot} \mathrm{const}(S^{\varepsilon}), -]_{s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}
$$

in $\mathrm{Fun}(s\mathcal{C}, \mathrm{Ab})$.

Proof. In light of the facts

- that $s\mathcal{C}_{\mathrm{res}}$ is simplicial,
- that $S_{\pmb{\Delta}}^n \overline{\odot} \mathrm{const}(S^\varepsilon) \in s\mathcal{C}_{\mathrm{res}}^c$ is cofibrant, and
- that all objects of $s\mathcal{C}_{\mathrm{res}}$ are fibrant,

we have the string of natural isomorphisms

$$[S_{\pmb{\Delta}}^n \overline{\odot} \mathrm{const}(S^\varepsilon), -]_{s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}$$

$$\cong \pi_0 |\underline{\hom}_{s\mathcal{C}}(S_{\pmb{\Delta}}^n \overline{\odot} \mathrm{const}(S^\varepsilon), -)|$$

$$\cong \pi_0 \left| \lim \left(\begin{array}{c} \underline{\hom}_{s\mathcal{C}}(S_{\pmb{\Delta}}^n \odot \mathrm{const}(S^\varepsilon), -) \\ \downarrow \\ \underline{\hom}_{s\mathcal{C}}(\mathrm{pt}_{s\mathcal{S}} \odot 0_{\mathcal{C}}, -) \longrightarrow \underline{\hom}_{s\mathcal{C}}(\mathrm{pt}_{s\mathcal{S}} \odot \mathrm{const}(S^\varepsilon), -) \end{array} \right) \right|$$

$$\cong \pi_0 \left| \lim \left(\begin{array}{c} \underline{\hom}_{s\mathcal{S}}(S_{\pmb{\Delta}}^n, \underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -)) \\ \downarrow {\scriptstyle \mathrm{ev}_*} \\ \mathrm{pt}_{s\mathcal{S}} \xrightarrow{\quad \underline{0} \quad} \underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -) \end{array} \right) \right|.$$

In order to continue, we make the following observations.

- The compatibility of $s\mathcal{C}_{\mathrm{res}}$ with $s\mathcal{S}_{\mathrm{KQ}}$ implies that the vertical map in this last expression is a fibration, so that we can commute the limit with the geometric realization.
- As $\mathrm{const}(S^\varepsilon) \in s\mathcal{C}_{\mathrm{res}}^c$ is cofibrant and all objects of $s\mathcal{C}_{\mathrm{res}}$ are fibrant, then $\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -) : s\mathcal{C} \to s\mathcal{S}_{\mathrm{KQ}}^f$ takes values in fibrant objects of $s\mathcal{S}_{\mathrm{KQ}}$.
- The object $S_{\pmb{\Delta}}^n \in s\mathcal{S}_{\mathrm{KQ}}^c$ is cofibrant.

Using these and the above string of natural isomorphisms, we find that

$$[S_{\pmb{\Delta}}^n \overline{\odot} \mathrm{const}(S^\varepsilon), -]_{s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]} \cong \pi_0 \lim \left(\begin{array}{c} |\underline{\hom}_{s\mathcal{S}}(S_{\pmb{\Delta}}^n, \underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -))| \\ \downarrow {\scriptstyle |\mathrm{ev}_*|} \\ |\mathrm{pt}_{s\mathcal{S}}| \xrightarrow{\quad |\underline{0}| \quad} |\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -)| \end{array} \right)$$

$$\cong \pi_0 \lim \left(\begin{array}{c} \hom_{\mathcal{S}}(|S_{\pmb{\Delta}}^n|, |\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -)|) \\ \downarrow {\scriptstyle |\mathrm{ev}_*|} \\ \mathrm{pt}_{\mathcal{S}} \xrightarrow{\quad |\underline{0}| \quad} |\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), -)| \end{array} \right)$$

$$\cong \pi_0 \lim \left(\begin{array}{c} \hom_{\mathbb{S}}(S^n, \hom_{s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}(S^\varepsilon, -)) \\ \downarrow{\scriptstyle \mathrm{ev}_*} \\ \mathrm{pt}_{\mathbb{S}} \xrightarrow{\quad 0 \quad} \hom_{s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}(S^\varepsilon, -) \end{array} \right)$$

$$\cong \pi_0 \hom_{\mathbb{S}_*}(S^n, \hom_{s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}(S^\varepsilon, -))$$

$$\cong \pi_n \hom_{s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}(S^\varepsilon, -),$$

proving the claim.   □

Definition 3.39. Let $K \in s\mathbb{S}_*$, and let $X \in s\mathcal{C}$. We define the ***reduced cotensoring*** of K into X to be the pullback

$$\begin{array}{ccc} K \overline{\pitchfork} X & \longrightarrow & K \pitchfork X \\ \downarrow & & \downarrow \\ \mathrm{pt}_{s\mathbb{S}} \pitchfork 0_{s\mathcal{C}} & \longrightarrow & \mathrm{pt}_{s\mathbb{S}} \pitchfork X \end{array}$$

in $s\mathcal{C}$. This assembles into an action $(s\mathbb{S}_*)^{op} \times s\mathcal{C} \to s\mathcal{C}$.

Observation 3.40. The reduced co/tensoring bifunctors participate into an evident two-variable adjunction

$$\left(s\mathbb{S}_* \times s\mathcal{C} \xrightarrow{-\overline{\odot}-} s\mathcal{C} \ , \ (s\mathbb{S}_*)^{op} \times s\mathcal{C} \xrightarrow{-\overline{\pitchfork}-} s\mathcal{C} \ , \ s\mathcal{C}^{op} \times s\mathcal{C} \xrightarrow{\underline{\hom}_{\mathcal{C}}(-,-)} s\mathbb{S}_* \right),$$

obtained by recognizing that the (enriched) hom-objects of $s\mathcal{C}$ are naturally pointed since $s\mathcal{C}$ has a zero object.

Observation 3.41. If

- on the one hand we restrict the reduced tensoring bifunctor to the constant simplicial objects of $\mathcal{C}$ via the composite

$$s\mathbb{S}_* \times \mathcal{C} \xrightarrow{\mathrm{id}_{s\mathbb{S}_*} \times \mathrm{const}} s\mathbb{S}_* \times s\mathcal{C} \xrightarrow{-\overline{\odot}-} s\mathcal{C},$$

  while
- on the other hand we postcompose the reduced cotensoring bifunctor with the limit functor to obtain the composite

$$(s\mathbb{S}_*)^{op} \times s\mathcal{C} \xrightarrow{-\overline{\pitchfork}-} s\mathcal{C} \xrightarrow{(-)_0} \mathcal{C},$$

then we similarly obtain a two-variable adjunction

$$\left(s\mathcal{S}_* \times \mathcal{C} \xrightarrow{-\odot \mathrm{const}(-)} s\mathcal{C} \ , \ (s\mathcal{S}_*)^{op} \times s\mathcal{C} \xrightarrow{(-\pitchfork-)_0} \mathcal{C} \ , \ \mathcal{C}^{op} \times s\mathcal{C} \xrightarrow{\underline{\hom}_{\mathcal{C}}(-,-)} s\mathcal{S}_* \right).$$

Notation 3.42. In analogy with the "generalized matching object" bifunctor

$$\mathrm{M}_{(-)}(-) : s\mathcal{S}^{op} \times s\mathcal{C} \xrightarrow{(-\pitchfork-)_0} \mathcal{C},$$

we write

$$\overline{\mathrm{M}}_{(-)}(-) : (s\mathcal{S}_*)^{op} \times s\mathcal{C} \xrightarrow{(-\overline{\pitchfork}-)_0} \mathcal{C}$$

for the "reduced generalized matching object" bifunctor.

Definition 3.43. We define the *(nonabelian) normalized n-chains* functor to be

$$N_n : s\mathcal{C} \xrightarrow{\overline{\mathrm{M}}_{D_{\pmb{\Delta}}^n}(-)} \mathcal{C},$$

and we define the *(nonabelian) n-cycles* functor to be

$$Z_n : s\mathcal{C} \xrightarrow{\overline{\mathrm{M}}_{S_{\pmb{\Delta}}^n}(-)} \mathcal{C}.$$

Note that these would reduce to the usual notions if $\mathcal{C}$ were an abelian category.

Observation 3.44. The cofiber sequence $S_{\pmb{\Delta}}^{n-1} \to D_{\pmb{\Delta}}^n \to S_{\pmb{\Delta}}^n$ in $s\mathcal{S}_*$ of Observation 3.37 induces a fiber sequence

$$Z_n \to N_n \to Z_{n-1}$$

in $\mathrm{Fun}(s\mathcal{C}, \mathcal{C})$.

Lemma 3.45. *For any $S^\varepsilon \in \mathcal{G}_E$, there is a natural isomorphism*

$$[S^\varepsilon, N_n(-)]_{\mathcal{C}} \cong N_n[S^\varepsilon, -]_{\mathcal{C}}^{\mathrm{lw}}$$

in $\mathrm{Fun}(s\mathcal{C}, \mathrm{Ab})$.

Proof. Fix a test object $X \in s\mathcal{C}$. As by definition $N_n(X) = \overline{\mathrm{M}}_{D_{\pmb{\Delta}}^n}(X)$, we have a pullback square

$$
\begin{array}{ccc}
N_n(X) & \longrightarrow & \mathrm{M}_{D_{\pmb{\Delta}}^n}(X) \\
\downarrow & & \downarrow \\
\mathrm{M}_{\mathrm{pt}_{s\mathcal{S}}}(0_{s\mathcal{C}}) & \longrightarrow & \mathrm{M}_{\mathrm{pt}_{s\mathcal{S}}}(X)
\end{array}
$$

in $\mathcal{C}$. In light of the pushout square

$$\begin{array}{ccc} \Lambda_0^n & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow \\ \Delta^0 & \longrightarrow & D_\Delta^n \end{array}$$

both in $s\mathcal{S}et$ and in $s\mathcal{S}$, we also have a pullback square

$$\begin{array}{ccc} \mathrm{M}_{D_\Delta^n}(X) & \longrightarrow & \mathrm{M}_{\Delta^n}(X) \\ \downarrow & & \downarrow \\ \mathrm{M}_{\Delta^0}(X) & \longrightarrow & \mathrm{M}_{\Lambda_0^n}(X) \end{array}$$

in $\mathcal{C}$, which simplifies to a pullback square

$$\begin{array}{ccc} \mathrm{M}_{D_\Delta^n}(X) & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & \mathrm{M}_{\Lambda_0^n}(X) \end{array}$$

in $\mathcal{C}$. As the relevant corepresenting maps $\mathrm{pt}_{s\mathcal{S}} \to D_\Delta^n$ and $\Delta^0 \to D_\Delta^n$ in $s\mathcal{S}et \subset s\mathcal{S}$ coincide, we obtain the composite pullback square

$$\begin{array}{ccccc} N_n(X) & \longrightarrow & \mathrm{M}_{D_\Delta^n}(X) & \longrightarrow & \mathrm{M}_{\Delta^n}(X) \\ \downarrow & & & & \downarrow \\ \mathrm{M}_{\mathrm{pt}_{s\mathcal{S}}}(0_{s\mathcal{C}}) & \longrightarrow & \mathrm{M}_{\mathrm{pt}_{s\mathcal{S}}}(X) \simeq \mathrm{M}_{\Delta^0}(X) & \longrightarrow & \mathrm{M}_{\Lambda_0^n}(X) \end{array}$$

in $\mathcal{C}$, which simplifies to a pullback square

$$\begin{array}{ccc} N_n(X) & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ 0_\mathcal{C} & \longrightarrow & \mathrm{M}_{\Lambda_0^n}(X) \end{array}$$

in $\mathcal{C}$. Moreover, replacing $0 \in [n]$ with any $i \in [n]$, we obtain analogous pullback squares

$$\begin{array}{ccc} \overline{\mathrm{M}}_{(\Delta^n/\Lambda_i^n)}(X) & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ 0_\mathcal{C} & \longrightarrow & \mathrm{M}_{\Lambda_i^n}(X) \end{array}$$

in $\mathcal{C}$. From here, the (dual of the corresponding cosimplicial) double induction argument of [28, Chapter VIII, Lemma 1.8] yields the claim. $\square$

Lemma 3.46. *For any $S^\varepsilon \in \mathcal{G}_E$, there is a natural exact sequence*

$$[S^\varepsilon, N_{n+1}(-)]_\mathcal{C} \to [S^\varepsilon, Z_n(-)]_\mathcal{C} \to \pi_{n,\varepsilon}^\flat(-) \to 0$$

in $\mathrm{Fun}(s\mathcal{C}, \mathrm{Ab})$.

Proof. For any test object $X \in s\mathcal{C}$, we have

$$\pi_{n,\varepsilon}^\flat X = \pi_n \hom_{s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}(S^\varepsilon, X) \cong \pi_0 \hom_{\mathcal{S}_*}(S^n, \hom_{s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}(S^\varepsilon, X)).$$

Now, since $\mathrm{const}(S^\varepsilon) \in s\mathcal{C}_{\mathrm{res}}^c$ and $X \in s\mathcal{C}_{\mathrm{res}}^f$, we have that $\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), X) \in s\mathcal{S}_{\mathrm{KQ}}^f$ and moreover $|\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), X)| \simeq \hom_{s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}(S^\varepsilon, X)$. On the other hand, $S_\mathbf{\Delta}^n \in s\mathcal{S}_{\mathrm{KQ}}^c$. Since co/fibrancy in $(s\mathcal{S}_*)_{\mathrm{KQ}}$ is created in $s\mathcal{S}_{\mathrm{KQ}}$ and moreover $s\mathcal{S}_*[\![\mathbf{W}_{\mathrm{KQ}}^{-1}]\!] \xrightarrow{\sim} \mathcal{S}_*$ (as colimits in $\mathcal{S}_*$ are computed in $\mathcal{S}$), the fundamental theorem of model ∞-categories applied to $(s\mathcal{S}_*)_{\mathrm{KQ}}$ implies that we have a surjection

$$\hom_{s\mathcal{S}_*}(S_\mathbf{\Delta}^n, \underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), X)) \to \hom_{\mathcal{S}_*}(S^n, \hom_{s\mathcal{C}[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}(S^\varepsilon, X))$$

in $\mathcal{S}$. Applying π_0, by adjunction this yields a surjection

$$[S^\varepsilon, Z_n(X)]_\mathcal{C} \to \pi_{n,\varepsilon}^\flat X$$

in $\mathcal{S}\mathrm{et}$. As epimorphisms are Ab are created in $\mathcal{S}\mathrm{et}$, this proves exactness at $\pi_{n,\varepsilon}^\flat(-)$.

Now, suppose we are given an element of $\ker([S^\varepsilon, Z_n(X)]_\mathcal{C} \to \pi_{n,\varepsilon}^\flat X)$: this is witnessed by an extension

in $\mathcal{S}_*$. Observe that $S_\mathbf{\Delta}^n \in s\mathcal{S}_{\mathrm{KQ}}^c$, so that by [16, Corollary 7.6.13], the model ∞-category $(s\mathcal{S}_{S_\mathbf{\Delta}^n/})_{\mathrm{KQ}}$ presents the ∞-category $\mathcal{S}_{S^n/}$. Since moreover $D_\mathbf{\Delta}^{n+1} \in (s\mathcal{S}_{S_\mathbf{\Delta}^n/})_{\mathrm{KQ}}^c$ and $\underline{\hom}_{s\mathcal{C}}(\mathrm{const}(S^\varepsilon), X) \in (s\mathcal{S}_{S_\mathbf{\Delta}^n/})_{\mathrm{KQ}}^f$, the fundamental theorem of model ∞-categories applied to $(s\mathcal{S}_{S_\mathbf{\Delta}^n/})_{\mathrm{KQ}}$ implies that the above extension in $\mathcal{S}_*$ is presented by an extension

in $s\mathcal{S}_*$. This proves exactness at $[S^\varepsilon, Z_n(-)]_\mathcal{C}$. $\square$

Corollary 3.47. *There is a natural isomorphism $\pi_0 \pi_\varepsilon(-) \cong \pi_{0,\varepsilon}^\natural(-)$ in* $\mathrm{Fun}(s\mathcal{C}, \mathrm{Ab})$.

Proof. Fix a test object $X \in s\mathcal{C}$. Applying Lemma 3.46 in the case that $n = 0$, we obtain an isomorphism

$$\mathrm{coker}([S^\varepsilon, N_1(X)]_\mathcal{C} \to [S^\varepsilon, Z_0(X)]_\mathcal{C}) \xrightarrow{\cong} \pi_{0,\varepsilon}^\natural X$$

in Ab. Unwinding the definition of $Z_0(X)$, we see that $Z_0(X) \simeq X_0 \in \mathcal{C}$, so that

$$[S^\varepsilon, Z_0(X)]_\mathcal{C} \cong [S^\varepsilon, X_0]_\mathcal{C} = ([S^\varepsilon, X]_\mathcal{C}^{\mathrm{lw}})_0.$$

Under this identification, unwinding the definition of $N_1 X$, we see that the image of the map

$$[S^\varepsilon, N_1 X]_\mathcal{C} \to [S^\varepsilon, Z_0 X]_\mathcal{C} \cong ([S^\varepsilon, X]_\mathcal{C}^{\mathrm{lw}})_0$$

is the set of those 0-simplices in $[S^\varepsilon, X]_\mathcal{C}^{\mathrm{lw}} \in s\mathrm{Ab}$ that are the "source" of a 1-simplex with "target" the basepoint 0-simplex $0 \in ([S^\varepsilon, X]_\mathcal{C}^{\mathrm{lw}})_0 \in \mathrm{Ab}$. So we obtain an isomorphism

$$\mathrm{coker}([S^\varepsilon, N_1(X)]_\mathcal{C} \to [S^\varepsilon, Z_0(X)]_\mathcal{C}) \cong \pi_0 \pi_\varepsilon X,$$

from which the claim follows.   □

Construction 3.48. For any object $X \in s\mathcal{C}$ and any $S^\varepsilon \in \mathcal{G}_E$, by Observation 3.44 we have long exact sequences

$$\cdots \to [S^{\varepsilon+1}, Z_{n-1}(X)]_\mathcal{C} \to [S^\varepsilon, Z_n(X)]_\mathcal{C} \to [S^\varepsilon, N_n(X)]_\mathcal{C} \to [S^\varepsilon, Z_{n-1}(X)]_\mathcal{C}$$

in Ab (which actually continue indefinitely to the right as well since $\mathcal{C}$ is stable). These splice together into an exact couple

$$[S^{\varepsilon+i+1}, Z_{n-1}(X)]_\mathcal{C} \dashrightarrow^{(\varepsilon+i+1)\rightsquigarrow(\varepsilon+i)} [S^{\varepsilon+i+1}, Z_n(X)]_\mathcal{C}$$

$$[S^{\varepsilon+i+1}, N_n(X)]_\mathcal{C}.$$

Using Lemmas 3.45 and 3.46, we can identify its derived long exact sequence as

$$\cdots \longrightarrow \pi_{i+1}\pi_\varepsilon(X) \xrightarrow{\delta} \pi_{i-1,\varepsilon+1}^\natural(X) \longrightarrow \pi_{i,\varepsilon}^\natural(X) \longrightarrow \pi_i \pi_\varepsilon(X) \xrightarrow{\delta} \cdots$$

$$\cdots \xrightarrow{\delta} \pi_{0,\varepsilon+1}^\natural(X) \longrightarrow \pi_{1,\varepsilon}^\natural(X) \longrightarrow \pi_1 \pi_\varepsilon(X) \longrightarrow 0.$$

We refer to this as the ***spiral exact sequence***.

3.4. The localized spiral exact sequence

In the end, we will not be interested in the natural and classical homotopy groups, but rather in their corresponding E-homology groups.

Notation 3.49. We simply write $E : s\mathcal{C} \xrightarrow{(E\otimes-)^{\mathrm{lw}}} s\mathcal{C}$ for the "tensor levelwise with E" functor.

Definition 3.50. Choose any $n \geq 0$ and any $\beta \in \mathcal{G}^\delta$.

(1) We define the corresponding **classical E-homology group** functor to be the composite

$$\pi_n E_\beta : s\mathcal{C} \xrightarrow{E} s\mathcal{C} \xrightarrow{\pi_n \pi_\beta} \mathrm{Ab}.$$

(2) We define the corresponding **natural E-homology group** functor to be the composite

$$E_{n,\beta}^\natural : s\mathcal{C} \xrightarrow{E} s\mathcal{C} \xrightarrow{\pi_{n,\beta}^\natural} \mathrm{Ab}.$$

When considered as indexed over all $\beta \in \mathcal{G}$ simultaneously, we write these functors simply as $\pi_n E_*$ and $E_{n,*}^\natural$, respectively.

Lemma 3.51. *There is a natural isomorphism* $\pi_0 E_\beta(-) \cong E_{0,\beta}^\natural(-)$ *in* $\mathrm{Fun}(s\mathcal{C}, \mathrm{Ab})$.

Proof. This follows from Corollary 3.47 and a colimit argument. $\square$

Construction 3.52. For any $X \in s\mathcal{C}$, the spiral exact sequence for $EX \in s\mathcal{C}$ with respect to any $\beta \in \mathcal{G}^\delta$ becomes

$$\cdots \longrightarrow \pi_{i+1} E_\beta X \xrightarrow{\delta} E_{i-1,\beta+1}^\natural X \longrightarrow E_{i,\beta}^\natural X \longrightarrow \pi_i E_\beta X \xrightarrow{\delta} \cdots$$

$$\cdots \xrightarrow{\delta} E_{0,\beta+1}^\natural X \longrightarrow E_{1,\beta}^\natural X \longrightarrow \pi_1 E_\beta X \longrightarrow 0.$$

We refer to this as the **localized spiral exact sequence**.

4. Algebraic topology

In this section, we add operadic structures to the mix. We begin by discussing operads and their algebras in §4.1, and then we add a simplicial direction in §4.2.

4.1. Foundations of algebraic topology

In this subsection, we give a brief unified treatment of all of the sorts of operads and algebras thereover that we will be considering. We begin by fixing basic unenriched

notions in §4.1.1. We then describe the enriched versions in §4.1.2 (which material is undergirded by the foundational work [15]). We apply these to study simplicial resolutions of operads in §4.1.3.

4.1.1. Operads and their algebras

Definition 4.1. By **operad** we mean what might otherwise be called a "single-colored ∞-operad". These are presented by monoids for the composition product in symmetric sequences in topological spaces or in simplicial sets (via the "operadic nerve" of Definition A.2.1.1.23). We write Op for the ∞-category of operads. For any $\mathcal{O} \in \mathrm{Op}$, we write $\mathcal{O}(n) \in \mathrm{Fun}(B\mathfrak{S}_n, \mathcal{S})$ for the space of n-ary operations, equipped with its canonical action of the symmetric group $\mathfrak{S}_n$.

Notation 4.2. For any $\mathcal{O} \in \mathrm{Op}$, we write $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ for the ∞-category of $\mathcal{O}$-algebras in $\mathcal{C}$, and we write

$$\mathrm{F}_{\mathcal{O}} : \mathcal{C} \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) : \mathrm{U}_{\mathcal{O}}$$

for the corresponding free/forget monadic adjunction.

Observation 4.3. The monad corresponding to the monadic adjunction $\mathrm{F}_{\mathcal{O}} \dashv \mathrm{U}_{\mathcal{O}}$ can be computed as

$$\mathrm{U}_{\mathcal{O}}(\mathrm{F}_{\mathcal{O}}(X)) \simeq \coprod_{n \geq 0} (\mathcal{O}(n) \odot X^{\otimes n})_{\mathfrak{S}_n}$$

(where we use the diagonal action to form the quotient).

Observation 4.4. Any map $\mathcal{O} \xrightarrow{\varphi} \mathcal{O}'$ in Op determines an adjunction

$$\varphi_* : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \rightleftarrows \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) : \varphi^*$$

between ∞-categories of algebras in $\mathcal{C}$, whose right adjoint is given by restriction of structure. The assignment $\varphi \mapsto \varphi_*$ assembles into a functor

$$\mathrm{Alg}_{(-)}(\mathcal{C}) : \mathrm{Op} \to \mathrm{Pr}^L.$$

Remark 4.5. We restrict to single-colored operads for simplicity, and because most operads of interest are single-colored. However, note that if one were interested in obtaining e.g. a commutative algebra $A \in \mathrm{CAlg}(\mathcal{C})$ as well as a module $M \in \mathrm{Mod}_A(\mathcal{C})$, one might proceed in steps, first using a single-colored obstruction theory in $\mathcal{C}$ to produce A, and then using a single-colored obstruction theory in $\mathrm{Mod}_A(\mathcal{C})$ to produce M.

4.1.2. Enriched operads and their algebras

Notation 4.6. For an ∞-category $\mathcal{V}$, we write $\mathcal{V}^{\mathfrak{S}} = \mathrm{Fun}(\mathcal{S}\mathrm{et}^{\simeq}, \mathcal{V})$ for the ∞-category of symmetric sequences in $\mathcal{V}$. Given $\mathcal{O} \in \mathcal{V}^{\mathfrak{S}}$, we write $\mathcal{O}(n) = \mathcal{O}(\{1, \ldots, n\})$ for simplicity. Assuming $\mathcal{V}$ has an initial object, we consider $\mathcal{V} \subset \mathcal{V}^{\mathfrak{S}}$ via left Kan extension along $\{\mathrm{pt}_{\mathcal{S}\mathrm{et}}\} \hookrightarrow \mathcal{S}\mathrm{et}^{\simeq}$. When $\mathcal{V}$ additionally admits a symmetric monoidal structure that commutes with colimits separately in each variable (e.g. if the symmetric monoidal structure is closed), the ∞-category $\mathcal{V}^{\mathfrak{S}}$ acquires a composition product monoidal structure $(\mathcal{V}^{\mathfrak{S}}, \circ, \mathbf{1}_{\mathcal{V}})$, algebras for which are precisely ("single colored") $\mathcal{V}$-operads (a/k/a "operads internal to $\mathcal{V}$"). We denote the ∞-category of these by $\mathrm{Op}(\mathcal{V})$, and write

$$\mathrm{F}_{\mathrm{Op}(\mathcal{V})} : \mathcal{V}^{\mathfrak{S}} \rightleftarrows \mathrm{Op}(\mathcal{V}) : \mathrm{U}_{\mathrm{Op}(\mathcal{V})}$$

for the resulting monadic adjunction. For brevity, we will simply say that $\mathcal{V}$ "admits operads" in this case.

When $\mathcal{V}$ is the ∞-category $\mathcal{S}$ of spaces (equipped with the cartesian symmetric monoidal structure), we (continue to) omit it from all our notation and terminology; in particular, we (continue to) refer to the objects of Op simply as "operads". For emphasis, we may refer to objects of $\mathrm{Op}(\mathcal{V})$ for some possibly unspecified $\mathcal{V}$ as "internal operads".

Notation 4.7. Let $\mathcal{D} \in \mathrm{LMod}_{\mathcal{V}}(\mathcal{C}\mathrm{at}_{\infty})$ be an ∞-category admitting an action of $\mathcal{V}$, and assume that $\mathcal{D}$ is cocomplete and finitely complete. Then for any $\mathcal{O} \in \mathrm{Op}(\mathcal{V})$ we denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})$ the ∞-category of $\mathcal{O}$-algebras in $\mathcal{D}$. This is monadic over $\mathcal{D}$, and we write

$$\mathrm{F}_{\mathcal{O}} : \mathcal{D} \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}) : \mathrm{U}_{\mathcal{O}}$$

for the monadic adjunction.

Observation 4.8. Let $\mathcal{V}$ be an ∞-category that admits operads, and let $\mathcal{I}$ be any diagram ∞-category. Then $\mathrm{Fun}(\mathcal{I}, \mathcal{V})$ also admits operads: it inherits a componentwise symmetric monoidal structure from $\mathcal{V}$, and colimits (including the empty colimit) are computed componentwise. In fact, it is not hard to see that we have an equivalence

$$\mathrm{Op}(\mathrm{Fun}(\mathcal{I}, \mathcal{V})) \simeq \mathrm{Fun}(\mathcal{I}, \mathrm{Op}(\mathcal{V})).$$

Proposition 4.9. *Let $\mathcal{V}$ be a symmetric monoidal ∞-category that admits operads and admits finite limits, and suppose that the unit object $\mathbf{1}_{\mathcal{V}} \in \mathcal{V}$ is compact. Then there exists a **Boardman–Vogt model structure** on the ∞-category of $s\mathcal{V}$-operads, denoted $\mathrm{Op}(s\mathcal{V})_{\mathrm{BV}}$, which is simplicial and participates in a Quillen adjunction*

$$\prod_{n \geq 0} s\mathcal{S}_{\mathrm{KQ}} \rightleftarrows \mathrm{Op}(s\mathcal{V})_{\mathrm{BV}} : \left(\hom_{\mathcal{V}}^{\mathrm{lw}}(\mathbf{1}_{\mathcal{V}}, \mathrm{U}_{\mathfrak{S}_n}((-)(n))) \right)_{n \geq 0}$$

of simplicial model ∞-categories, where $\mathrm{F}_{\mathfrak{S}_n} : \mathcal{V} \rightleftarrows \mathrm{Fun}(B\mathfrak{S}_n, \mathcal{V}) : \mathrm{U}_{\mathfrak{S}_n}$ *denotes the left Kan extension adjunction for the canonical functor* $\mathrm{pt}_{\mathcal{C}\mathrm{at}_\infty} \to B\mathfrak{S}_n$. *In other words, the weak equivalences and fibrations are determined levelwise after forgetting the symmetric group actions and applying the "underlying space" functor* $\hom_{\mathcal{V}}(\mathbf{1}_{\mathcal{V}}, -) : \mathcal{V} \to \mathcal{S}$ *levelwise.*

Proof. In view of Corollary 2.15, this follows from Theorems 2.10 and 2.13 (or really Theorem 2.19). $\square$

Remark 4.10. In the end, we will only use Proposition 4.9 in situations when $\mathcal{V}$ is a 1-category. In this case, the result is ultimately more-or-less just a consequence of [62, Chapter II, §4, Theorem 4]. The name of the model structure pays homage to the foundational work [14], which introduced the study of homotopy-coherent algebraic structures. The Boardman–Vogt model structure of Proposition 4.9 is also closely related to those of [9, Theorems 3.1 and 3.2], as explained in [9, Example 3.3.1].

Observation 4.11. Let $\mathcal{V}$ and $\mathcal{V}'$ be two ∞-categories equipped with symmetric monoidal structures that commute with colimits separately in each variable. Then any lax symmetric monoidal functor $\mathcal{V} \to \mathcal{V}'$ induces a functor $\mathrm{Op}(\mathcal{V}) \to \mathrm{Op}(\mathcal{V}')$.

We single out two particular cases of interest.

- The functor $- \odot \mathbf{1} : \mathcal{S} \to \mathcal{C}$ is symmetric monoidal (with respect to $(\mathcal{S}, \times, \mathrm{pt}_{\mathcal{S}})$ and $(\mathcal{C}, \otimes, \mathbf{1})$).
- The homology functor $E_* : \mathcal{C} \to \mathcal{A}$ is lax symmetric monoidal: for any $X, Y \in \mathcal{C}$, we have a canonical map $E_* X \otimes_{E_*} E_* Y \to E_*(X \otimes Y)$ in $\mathcal{A}$, which takes the element

$$\left(S^\beta \xrightarrow{\varphi} E \otimes X \right) \otimes \left(S^{\beta'} \xrightarrow{\varphi'} E \otimes Y \right)$$

  to the element

$$\left(S^{\beta + \beta'} \simeq S^\beta \otimes S^{\beta'} \xrightarrow{\varphi \otimes \varphi'} E \otimes X \otimes E \otimes Y \simeq E^{\otimes 2} \otimes X \otimes Y \xrightarrow{\mu_E \otimes \mathrm{id}_X \otimes \mathrm{id}_Y} E \otimes X \otimes Y \right).$$

It follows that the composite functor

$$\mathcal{S} \xrightarrow{- \odot \mathbf{1}} \mathcal{C} \xrightarrow{E_*} \mathcal{A}$$

is lax symmetric monoidal, and hence induces a composite functor on internal operads, which for brevity we denote simply as

$$E_* : \mathrm{Op} = \mathrm{Op}(\mathcal{S}) \xrightarrow{\mathrm{Op}(- \odot \mathbf{1})} \mathrm{Op}(\mathcal{C}) \xrightarrow{\mathrm{Op}(E_*)} \mathrm{Op}(\mathcal{A}).$$

4.1.3. Simplicial resolutions of operads

Definition 4.12. We say that an operad $\mathcal{O} \in \mathrm{Op}$ is π_0-$\mathfrak{S}$-**free** if for each $n \geq 0$ the induced action of $\mathfrak{S}_n$ on $\pi_0(\mathcal{O}(n))$ is free.

Remark 4.13. As early in the literature as [48, Definition 1.1], the term "$\mathfrak{S}$-free" is used to describe a point-set operad (e.g. in topological spaces) whose symmetric group actions are free at the point-set level. Of course, such an operad need not present a π_0-$\mathfrak{S}$-free operad in the sense of Definition 4.12.

Lemma 4.14. *The functor* $\mathrm{F}_{\mathrm{Op}} : \mathcal{S}^{\mathfrak{S}} \to \mathrm{Op}$ *takes values in* π_0-$\mathfrak{S}$-*free operads.*

Proof. This is immediate from the explicit description of F_{Op} that follows from [64, Proposition A.0.2 and Remark A.0.1]. $\quad\square$

Notation 4.15. We simply write

$$\mathrm{Bar}(-)_\bullet : \mathrm{Op} \xrightarrow{\;\mathrm{Bar}(\mathrm{pt}_{\mathcal{S}}, \mathrm{U}_{\mathrm{Op}}\mathrm{F}_{\mathrm{Op}}, -)_\bullet\;} s\mathrm{Op}$$

for the bar construction on the monad $\mathrm{U}_{\mathrm{Op}}\mathrm{F}_{\mathrm{Op}} \in \mathrm{Alg}(\mathrm{End}(\mathcal{S}^{\mathfrak{S}}))$ with respect to the left module given by the unit $\mathrm{pt}_{\mathcal{S}} \in \mathcal{S}^{\mathfrak{S}}$ and an unspecified operad considered as a right module.

Corollary 4.16. *The functor* $\mathrm{Bar} : \mathrm{Op} \to s\mathrm{Op}$ *takes values in levelwise* π_0-$\mathfrak{S}$-*free simplicial operads, and admits a natural equivalence* $|\mathrm{Bar}(-)_\bullet| \simeq \mathrm{id}_{\mathrm{Op}}$ *in* $\mathrm{Fun}(\mathrm{Op}, \mathrm{Op})$.

Proof. This follows from Lemma 4.14. $\quad\square$

Corollary 4.17. *Given an operad* $\mathcal{O}$, *suppose that* $E_*(\mathcal{O}(n)) \in \mathcal{A}_{\mathrm{proj}}$ *for all* $n \geq 0$. *Then* $E_*^{\mathrm{lw}}\mathrm{Bar}(\mathcal{O})_\bullet \in s\mathrm{Op}(\mathcal{A}) \simeq \mathrm{Op}(s\mathcal{A})_{\mathrm{BV}}$ *is cofibrant, and the augmentation* $\mathrm{Bar}(\mathcal{O})_\bullet \to \mathrm{const}(\mathcal{O})$ *induces a weak equivalence* $E_*^{\mathrm{lw}}\mathrm{Bar}(\mathcal{O})_\bullet \xrightarrow{\approx} \mathrm{const}(E_*\mathcal{O})$ *in* $\mathrm{Op}(s\mathcal{A})_{\mathrm{BV}}$.

Proof. The asserted cofibrancy follows from the projectivity assumption, while the asserted weak equivalence follows from Corollary 4.16. $\quad\square$

Remark 4.18. While we will ultimately be interested in a simplicial operad resolving our operad of primary interest, much of the theory goes through equally well for any simplicial operad.

4.2. Simplicial algebraic topology

In this subsection, we discuss algebras over simplicial operads in more detail.

Definition 4.19. Let $T \in s\mathrm{Op}$ be a simplicial object in operads. We define the ∞-category $\mathrm{Alg}_T(s\mathcal{C})$ of **simplicial T-algebras in** $\mathcal{C}$ to be the lax limit of the composite

$$\mathbf{\Delta}^{op} \xrightarrow{T} \mathrm{Op} \xrightarrow{\mathrm{Alg}_{(-)}(\mathcal{C})} \mathrm{Pr}^L.$$

Remark 4.20. The composite

$$\mathbf{\Delta}^{op} \xrightarrow{T} \mathrm{Op} \xrightarrow{\mathrm{Alg}_{(-)}(\mathcal{C})} \mathrm{Pr}^L \xrightarrow{\mathrm{U}_{\mathrm{Pr}^L}} \mathcal{C}\mathrm{at}_\infty$$

classifies a cocartesian fibration, which is in fact a bicartesian fibration; by (the dual of) [26, Proposition 7.1] (combined with Proposition T.5.5.3.13), its ∞-category of sections is precisely $\mathrm{Alg}_T(s\mathcal{C})$. Thus, we can think of a simplicial T-algebra $X = X_\bullet \in \mathrm{Alg}_T(s\mathcal{C})$ as being specified by the following data:

- for each object $[n]^\circ \in \mathbf{\Delta}^{op}$, an object $X_n \in \mathrm{Alg}_{T_n}(\mathcal{C})$;
- for each morphism $[n]^\circ \xrightarrow{\varphi} [m]^\circ$ in $\mathbf{\Delta}^{op}$, a morphism from $X_n \in \mathrm{Alg}_{T_n}(\mathcal{C})$ to $X_m \in \mathrm{Alg}_{T_m}(\mathcal{C})$ in (the bicartesian fibration over $[1]$ corresponding to) the adjunction

$$(T_\varphi)_* : \mathrm{Alg}_{T_n}(\mathcal{C}) \rightleftarrows \mathrm{Alg}_{T_m}(\mathcal{C}) : (T_\varphi)^*$$

  arising from the induced map $T_n \xrightarrow{T_\varphi} T_m$ in Op, i.e. a point in the space

$$\hom_{\mathrm{Alg}_{T_n}(\mathcal{C})}(X_n, (T_\varphi)^* X_m) \simeq \hom_{\mathrm{Alg}_{T_m}(\mathcal{C})}((T_\varphi)_* X_n, X_m);$$

- higher coherence data for these structure maps corresponding to strings of composable morphisms in $\mathbf{\Delta}^{op}$.

Observation 4.21. Any map $T \xrightarrow{\varphi} T'$ in $s\mathrm{Op}$ determines an adjunction

$$\varphi_* : \mathrm{Alg}_T(s\mathcal{C}) \rightleftarrows \mathrm{Alg}_{T'}(s\mathcal{C}) : \varphi^*$$

between ∞-categories of simplicial algebras in $\mathcal{C}$, whose right adjoint is given by restriction of structure. In particular, taking T to be trivial yields a monadic adjunction

$$\mathrm{F}_{T'} : s\mathcal{C} \rightleftarrows \mathrm{Alg}_{T'}(s\mathcal{C}) : \mathrm{U}_{T'},$$

whose underlying monad is computed levelwise.

Observation 4.22. Let $\mathcal{O} \in \mathrm{Op}$ be an operad, and consider the corresponding constant simplicial operad $\mathrm{const}(\mathcal{O}) \in s\mathrm{Op}$. Since the resulting composite

$$\mathbf{\Delta}^{op} \xrightarrow{\mathrm{const}(\mathcal{O})} \mathrm{Op} \xrightarrow{\mathrm{Alg}_{(-)}(\mathcal{C})} \mathrm{Pr}^L$$

is constant at $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, it follows that we have a canonical equivalence

$$\mathrm{Alg}_{\mathrm{const}(\mathcal{O})}(s\mathcal{C}) \simeq s(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})).$$

Observation 4.23. For any $T \in s\mathrm{Op}$, we have a canonical composite adjunction

$$\mathrm{Alg}_T(s\mathcal{C}) \underset{(\eta_T)^*}{\overset{(\eta_T)_*}{\underset{\perp}{\rightleftarrows}}} \mathrm{Alg}_{\mathrm{const}(|T|)}(s\mathcal{C}) \simeq s(\mathrm{Alg}_{|T|}(\mathcal{C})) \underset{\mathrm{const}}{\overset{|-|}{\underset{\perp}{\rightleftarrows}}} \mathrm{Alg}_{|T|}(\mathcal{C}),$$

where

- the first adjunction follows by applying Observation 4.21 to the component $T \xrightarrow{\eta_T} \mathrm{const}(|T|)$ of the unit of the adjunction $|-| : s\mathrm{Op} \rightleftarrows \mathrm{Op} : \mathrm{const}(-)$;
- the equivalence is that of Observation 4.22; and
- the second adjunction is the colimit/constant adjunction in $\mathrm{Alg}_{|T|}(\mathcal{C})$.

Notation 4.24. For simplicity, we simply write

$$|-| : \mathrm{Alg}_T(s\mathcal{C}) \rightleftarrows \mathrm{Alg}_{|T|}(\mathcal{C}) : \mathrm{const}$$

for the composite adjunction of Observation 4.23. When convenient and unambiguous, we will omit the right adjoint from the notation.

Lemma 4.25. *The diagram*

$$
\begin{array}{ccc}
\mathrm{Alg}_T(s\mathcal{C}) & \xrightarrow{\ |-|\ } & \mathrm{Alg}_{|T|}(\mathcal{C}) \\
{\scriptstyle \mathrm{U}_T} \downarrow & & \downarrow {\scriptstyle \mathrm{U}_{|T|}} \\
s\mathcal{C} & \xrightarrow[\ |-|\ ]{} & \mathcal{C}
\end{array}
$$

commutes.

Proof. Both vertical functors are right adjoints which commute with sifted colimits. $\square$

Theorem 4.26. *There is a* **resolution model structure** *on* $\mathrm{Alg}_T(s\mathcal{C})$, *denoted* $\mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}}$; *it is obtained by lifting the resolution model structure* $s\mathcal{C}_{\mathrm{res}}$ *along the adjunction*

$$\mathrm{F}_T : s\mathcal{C} \rightleftarrows \mathrm{Alg}_T(s\mathcal{C}) : \mathrm{U}_T,$$

which therefore becomes a Quillen adjunction. It enjoys the following properties.

(1) Its weak equivalences and fibrations are created by pullback along the right adjoint U_T.

(2) It is simplicial.

(3) All objects are fibrant in it.

(4) It is cofibrantly generated by the sets

$$I_{\mathrm{res}}^{\mathrm{Alg}_T(s\mathcal{C})} = \mathrm{F}_T(I_{\mathrm{res}}^{s\mathcal{C}}) = \{\mathrm{F}_T(I_{\mathrm{KQ}}^{s\mathcal{S}} \odot \mathrm{const}(S^\varepsilon))\}_{S^\varepsilon \in \mathcal{G}_E}$$

$$= \{\mathrm{F}_T(\partial\Delta^n \odot \mathrm{const}(S^\varepsilon)) \to \mathrm{F}_T(\Delta^n \odot \mathrm{const}(S^\varepsilon))\}_{n \geq 0, S^\varepsilon \in \mathcal{G}_E}$$

and

$$J_{\mathrm{res}}^{\mathrm{Alg}_T(s\mathcal{C})} = \mathrm{F}_T(J_{\mathrm{res}}^{s\mathcal{C}}) = \{\mathrm{F}_T(J_{\mathrm{KQ}}^{s\mathcal{S}} \odot \mathrm{const}(S^\varepsilon))\}_{S^\varepsilon \in \mathcal{G}_E}$$

$$= \{\mathrm{F}_T(\Lambda_i^n \odot \mathrm{const}(S^\varepsilon)) \to \mathrm{F}_T(\Delta^n \odot \mathrm{const}(S^\varepsilon))\}_{0 \leq i \leq n \geq 1, S^\varepsilon \in \mathcal{G}_E}.$$

Proof. The model structure follows from Theorem 2.10, with condition $(*)$ being implied by Corollary 2.15. The enrichment and bitensoring over $s\mathcal{S}$ follows from Corollary 2.7, and their compatibility follows from Theorem 2.13. $\square$

5. Algebra

In this section, we discuss algebraic structures on the various homotopy and E-homology groups of various topological objects. We begin in §5.1 by discussing the relevant categories of graded abelian groups and comodules. This allows us to state some crucial compatibility assumptions between our simplicial operad and our E-homology functors in §5.2. In particular, these give us control over the E-homology localization of simplicial T-algebras, as we explain in §5.3. We conclude by discussing the module structures on the localized and unlocalized spiral exact sequences in §§5.5–5.4.

5.1. Foundations of algebra

Recall that we write $\mathcal{G}^\delta = \pi_0(\mathcal{G})$ for our chosen group of Picard elements, $\underline{\mathcal{A}} = \mathrm{Fun}(\mathcal{G}^\delta, \mathrm{Ab})$ for the category of $\mathcal{G}^\delta$-graded abelian groups, and $\mathcal{A} = \mathrm{Mod}_{E_*}(\underline{\mathcal{A}})$ for the category of E_*-modules in $\underline{\mathcal{A}}$.

Observation 5.1. Adams's condition (Assumption 3.18) implies that $E_*E \in \mathcal{A}$ is flat: this follows from the filtered colimit presentation $\mathrm{colim}_{\alpha \in \mathcal{J}}(E_*E_\alpha) \xrightarrow{\cong} E_*E$ along with the fact that E_*E_α is the (E_*-linear) dual of $E_*\mathrm{D}E_\alpha \in \mathcal{A}_{\mathrm{proj}}$.

Notation 5.2. Observe that (E_*, E_*E) is a Hopf algebroid in $\underline{\mathcal{A}}$. We write $\tilde{\mathcal{A}} = \mathrm{Comod}_{(E_*, E_*E)}$ for its category of left comodules (which in light of Observation 5.1 is abelian by [63, Theorem A1.1.3]), and we consider our homology theory as a functor $E_* : \mathcal{C} \to \tilde{\mathcal{A}}$ taking values in (E_*, E_*E)-comodules.

Remark 5.3. In general, the forgetful functor $\tilde{\mathcal{A}} \xrightarrow{\mathrm{U}_{\tilde{\mathcal{A}}}} \mathcal{A}$ does not admit a left adjoint (e.g. it does not preserve products (see [38, §1.2])).

Observation 5.4. For any $\beta \in \mathcal{G}^\delta$ we obtain an evident endofunctor $\Sigma^\beta : \tilde{\mathcal{A}} \xrightarrow{\sim} \tilde{\mathcal{A}}$. This allows us to consider $\tilde{\mathcal{A}}$ as enriched over $\mathcal{A}$, where for $M, N \in \tilde{\mathcal{A}}$ we set

$$\underline{\hom}_{\tilde{\mathcal{A}}}(M, N) = \{\hom_{\tilde{\mathcal{A}}}(\Sigma^\beta M, N)\}_{\beta \in \mathcal{G}^\delta} \in \mathcal{A}.$$

5.2. Compatibility

The resolutions of operads considered in §4.1.3 are necessary but not alone sufficient to render the obstruction theory to be tractable: we have introduced a new simplicial direction on the topology side, but we have not yet exerted any control on the simplicial direction that results on the algebra side. Indeed, this will bring our E-homology computations into the realm of homotopical algebra, with its own attendant notions of "cofibrant resolution", and we must ensure that our homology functor E_* preserves resolutions.

We introduce three increasingly general notions of compatibility; the first is merely to fix ideas, the second is auxiliary, and the last is our real goal.

Definition 5.5. We say that an operad $\mathcal{O} \in \mathrm{Op}$ is **adapted** to E if it comes with a corresponding monad $\mathcal{O}_E \in \mathrm{Alg}(\mathrm{End}(\mathcal{A}))$ admitting a lift

$$
\begin{array}{ccc}
\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) & \xdashrightarrow{\ E_* \ } & \mathrm{Alg}_{\mathcal{O}_E}(\mathcal{A}) \\
{\scriptstyle \mathrm{U}_{\mathcal{O}}}\downarrow & & \downarrow{\scriptstyle \mathrm{U}_{\mathcal{O}_E}} \\
\mathcal{C} & \xrightarrow[\ E_* \ ]{} & \mathcal{A}
\end{array}
$$

such that the following condition holds:

- for any $Z \in \mathcal{C}$ with $E_* Z \in \mathcal{A}_{\mathrm{proj}}$, the natural map $\mathrm{F}_{\mathcal{O}_E}(E_* Z) \to E_*(\mathrm{F}_{\mathcal{O}}(Z))$ is an isomorphism in $\mathrm{Alg}_{\mathcal{O}_E}(\mathcal{A})$.

Definition 5.6. We say that a simplicial operad $T \in s\mathrm{Op}$ is **adapted** to E if it comes with a corresponding monad $T_E \in \mathrm{Alg}(\mathrm{End}(s\mathcal{A}))$ admitting a lift

$$
\begin{array}{ccc}
\mathrm{Alg}_{T}(s\mathcal{C}) & \xdashrightarrow{\ E_*^{\mathrm{lw}} \ } & \mathrm{Alg}_{T_E}(s\mathcal{A}) \\
{\scriptstyle \mathrm{U}_{T}}\downarrow & & \downarrow{\scriptstyle \mathrm{U}_{T_E}} \\
s\mathcal{C} & \xrightarrow[\ E_* \ ]{} & s\mathcal{A}
\end{array}
$$

such that the following condition holds:

- for any $Z \in s\mathcal{C}$ with $E_{\ast}^{\mathrm{lw}} Z \in s\mathcal{A}_{\mathrm{KQ}}^{c}$, the natural map $\mathrm{F}_{T_E}(E_{\ast}^{\mathrm{lw}} Z) \to E_{\ast}^{\mathrm{lw}}(\mathrm{F}_T(Z))$ is an isomorphism in $\mathrm{Alg}_{T_E}(s\mathcal{A})$.

This has the following consequence.

Lemma 5.7. *If $T \in s\mathrm{Op}$ is adapted to E, then any cofibration between cofibrant objects in $\mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}}$ is a retract of a map $X \xrightarrow{\varphi} Y$ such that the underlying map of degeneracy diagrams of $E_{\ast}^{\mathrm{lw}}(\varphi)$ is isomorphic to one of the form $E_{\ast}^{\mathrm{lw}}(X) \to E_{\ast}^{\mathrm{lw}}(X) \coprod T_E(M)$, where M is s-free on an object of $\mathcal{A}_{\mathrm{proj}}$.*

Proof. The proof is identical to that of [23, Lemma 1.4.15], except that we use the description of the generating (resp. acyclic) cofibrations in $\mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}}$ of Theorem 4.26 rather than any facts about the positive model structure on the category of symmetric spectra in topological spaces. $\square$

Definition 5.8. Suppose that the simplicial operad $T \in s\mathrm{Op}$ is adapted to E. We then say that T is **homotopically adapted** to E if there exists a monad $\tilde{T}_E \in \mathrm{Alg}(\mathrm{End}(s\tilde{\mathcal{A}}))$ which lifts the monad $T_E \in \mathrm{Alg}(\mathrm{End}(s\mathcal{A}))$ (i.e. they're intertwined by $s(\mathrm{U}_{\tilde{\mathcal{A}}})$) and which admits a lift

$$
\begin{array}{ccc}
\mathrm{Alg}_T(s\mathcal{C}) & \overset{E_{\ast}^{\mathrm{lw}}}{\dashrightarrow} & \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \\
{\scriptstyle \mathrm{U}_T} \downarrow & & \downarrow {\scriptstyle \mathrm{U}_{\tilde{T}_E}} \\
s\mathcal{C} & \underset{E_{\ast}^{\mathrm{lw}}}{\longrightarrow} & s\tilde{\mathcal{A}}
\end{array}
$$

such that the following conditions hold:

- the adjunction $\mathrm{F}_{T_E} : s\mathcal{A} \rightleftarrows \mathrm{Alg}_{T_E}(s\mathcal{A}) : \mathrm{U}_{T_E}$ creates a simplicial model structure on $\mathrm{Alg}_{T_E}(s\mathcal{A})$; and
- there exists a simplicial model structure on $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$ such that the forgetful functor $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \to \mathrm{Alg}_{T_E}(s\mathcal{A})$ creates weak equivalences and preserves fibrations.

Building on Lemma 5.7, this has the following key consequence.

Lemma 5.9. *If $T \in s\mathrm{Op}$ is homotopically adapted to E, then the induced functor $E_{\ast}^{\mathrm{lw}} : \mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}} \to \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})_{\pi_{\ast}}$ preserves both weak equivalences as well as cofibrations between cofibrant objects.*

Proof. Analogously to [23, Corollary 1.4.18], this follows from Lemma 5.7 given the fact that cofibrations are retracts of (relatively) free maps. $\square$

This result, in turn, has the following ∞-categorical significance.

Corollary 5.10. *If $T \in s\mathrm{Op}$ is homotopically adapted to E, then the functor E_*^{lw} :* $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \to \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ *preserves colimits.*

Proof. This follows by combining Lemma 5.9 with the theory of homotopy colimits in model ∞-categories of [54, §1.2]; more specifically, the model ∞-categories $\mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}}$ and $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})_{\pi_*}$ are both cofibrantly generated and hence admit projective model structures, and the functor of model ∞-categories preserves projective cofibrancy by Lemma 5.9. □

Remark 5.11. Given two ∞-categories that admit finite coproducts and a functor between them that preserves these, applying the functor $\mathcal{P}_\Sigma$ automatically gives a cocontinuous functor: up to further left localizations (which commute with colimits), this is precisely the situation that Corollary 5.10 addresses. However, it is only through Theorem 2.23 that we can identify it as such.

Assumption 5.12. We henceforth assume that T is homotopically adapted to E, and fix the corresponding monad $\tilde{T}_E \in \mathrm{Alg}(\mathrm{End}(s\tilde{\mathcal{A}}))$.

Example 5.13. For any $\mathcal{O} \in \mathrm{Op}$, we can take T to be a cofibrant object of $\mathrm{Op}(s\mathcal{S}\mathrm{et})_{\mathrm{BV}}$ which presents it: each $T(n)$ will have a free $\mathfrak{S}_n$-action (as a simplicial set), and we can take $\tilde{T}_E$ to be the monad corresponding to the operad $E_*T \in \mathrm{Op}(s\tilde{\mathcal{A}})$.

5.3. Localization of simplicial T-algebras

Now that we have stipulated some compatibility between our simplicial operad T and the E-homology functor, we return to our study of simplicial T-algebras. Recall that Theorem 4.26 gives a resolution model structure $\mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}}$, which allows us to compute hom-spaces in the localization $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$. However, most natively, our moduli spaces of interest will not be subgroupoids of this localization, but rather of a further localization. We indicate the latter in Notation 5.14, and then we establish Proposition 5.15 allowing us to use the model ∞-category $\mathrm{Alg}_T(s\mathcal{C})_{\mathrm{res}}$ to make computations therein.

Notation 5.14. Extending Definition 3.28, we write $\mathbf{W}_{E_*^{\mathrm{lw}}} = \mathbf{W}_{E_*^{\mathrm{lw}}}^{\mathrm{Alg}_T(s\mathcal{C})} \subset \mathrm{Alg}_T(s\mathcal{C})$ for the preimage of $\mathbf{W}_{E_*^{\mathrm{lw}}}^{s\mathcal{C}} \subset s\mathcal{C}$ under the forgetful functor $\mathrm{U}_T : \mathrm{Alg}_T(s\mathcal{C}) \to s\mathcal{C}$. Since $\mathbf{W}_{\mathrm{res}}^{s\mathcal{C}} \subset \mathbf{W}_{E_*^{\mathrm{lw}}}^{s\mathcal{C}}$ by Observation 3.30, then also $\mathbf{W}_{\mathrm{res}}^{\mathrm{Alg}_T(s\mathcal{C})} \subset \mathbf{W}_{E_*^{\mathrm{lw}}}^{\mathrm{Alg}_T(s\mathcal{C})}$.

Proposition 5.15. *The canonical functor $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \to \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!]$ is the left adjoint in a left localization adjunction*

$$\mathrm{L}_{E_*^{\mathrm{lw}}} : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \rightleftarrows \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!] : \mathrm{U}_{E_*^{\mathrm{lw}}}.$$

Proof. It follows from Theorem 2.23 (and the monadic derived adjunction underlying the monadic Quillen adjunction $F_T \dashv U_T$) that $\text{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\text{res}}^{-1}]\!]$ is presentable. Hence, we can apply the recognition result Proposition T.5.5.4.15: it suffices to show that the image in $\text{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\text{res}}^{-1}]\!]$ of $\mathbf{W}_{E_*^{\text{lw}}} \subset \text{Alg}_T(s\mathcal{C})$ is strongly saturated and of small generation (se Definition T.5.5.4.5 and Remark T.5.5.4.7). For strong saturation, conditions (1) and (2) follow from Lemma 5.9 (similarly to Corollary 5.10 – again, note that the model ∞-category $\text{Alg}_T(s\mathcal{C})_{\text{res}}$ is cofibrantly generated and hence admits projective model structures), and condition (3) follows from the fact that $\mathbf{W}_{E_*^{\text{lw}}}$ is ultimately pulled back from a subcategory $\mathbf{W}_{\text{KQ}} \subset s\mathcal{A}$ which has the two-out-of-three property. Given the presentability of $\text{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\text{res}}^{-1}]\!]$, small generation follows the same cardinality argument as that of [23, Theorem 1.5.1] giving the existence of a small set of generating acyclic cofibrations. $\square$

Remark 5.16. In [23, Theorem 1.5.1], Goerss–Hopkins establish a "semi-model structure" that presents the localization $\text{Alg}_T(s\mathcal{S}\text{p})[\![\mathbf{W}_{E_*^{\text{lw}}}^{-1}]\!]$. (Its failure to be a model structure in general is explained in [23, Remark 1.5.9].) Of course, since we are working at the level of underlying ∞-categories, as a result of Proposition 5.15 we have no need for such a (semi-)model structure. The existence of the fully faithful right adjoint $U_{E_*^{\text{lw}}}$ should not be surprising: the semi-model structure of [23, Theorem 1.5.1] is constructed as a left Bousfield localization of the model structure of [23, Theorem 1.4.9].

Remark 5.17. Whereas we have identified $s\mathcal{C}[\![\mathbf{W}_{\text{res}}^{-1}]\!]$ as a nonabelian derived ∞-category, it appears that $s\mathcal{C}[\![\mathbf{W}_{E_*^{\text{lw}}}^{-1}]\!]$ does not generally take this form. It will become clear over the course of the construction that we really do need to be working in a nonabelian derived ∞-category.

5.4. The module structure on the localized spiral exact sequence

In this subsection, we indicate certain additional algebraic structures present on the classical and natural E-homology of simplicial T-algebras. We can summarize the key statements as follows (using notation terminology that will be introduced shortly). First of all, given a simplicial $\tilde{T}_E$-algebra in $\tilde{\mathcal{A}}$, its π_0 is naturally a Φ-algebra, and for $n > 0$ its π_n are naturally modules thereover. Hence, given a simplicial T-algebra in $\mathcal{C}$, its E_0 is naturally a Φ-algebra, and for $n > 0$ its $\pi_n E_*^{\text{lw}}$ and $E_{n,*}^{\natural}$ are naturally modules thereover.

Definition 5.18. An ***augmentation*** of the monad $\tilde{T}_E \in \text{Alg}(\text{End}(s\tilde{\mathcal{A}}))$ is the data of a monad $\Phi \in \text{Alg}(\text{End}(\tilde{\mathcal{A}}))$ and a natural isomorphism making the diagram

$$
\begin{array}{ccc}
s\tilde{\mathcal{A}} & \xrightarrow{\tilde{T}_E} & s\tilde{\mathcal{A}} \\
{\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_0} \\
\tilde{\mathcal{A}} & \xrightarrow{\Phi} & \tilde{\mathcal{A}}
\end{array}
$$

commute, satisfying the diagrammatic coherence conditions of [23, Definition 2.5.7]. We write this as $\tilde{T}_E \downarrow \Phi$, though note that this does not depict a morphism in any category.

Assumption 5.19. We henceforth assume the existence of an augmentation $\tilde{T}_E \downarrow \Phi$.

In order to describe the key consequence of Assumption 5.19, we must introduce some terminology.

Definition 5.20. For any $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$, we define the category of *A-modules* (*relative to* Φ) as the category $\mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}}) = \mathrm{Ab}(\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{/A})$ of abelian group objects in its overcategory. To align our notation with standard intuition, we write

$$\tilde{\mathcal{A}} \xleftarrow{\quad \mathrm{U}_A \quad} \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}}) \xrightarrow{\quad -\ltimes A \quad} \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$$

$$\ker^{\tilde{\mathcal{A}}}(\varphi) \longleftarrow\!\!\mid (B \xrightarrow{\varphi} A) \longmapsto B$$

for the two forgetful functors.

Lemma 5.21 *([23, Propositions 2.5.9 and 2.5.10]). There exists a canonical lift*

$$
\begin{array}{c}
& & & \mathrm{Alg}_\Phi(\tilde{\mathcal{A}}) \\
& \overset{\pi_0}{\nearrow} & & \big\downarrow {\scriptstyle \mathrm{U}_\Phi} \\
\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \xrightarrow[\mathrm{U}_{\tilde{T}_E}]{} s\tilde{\mathcal{A}} \xrightarrow[\pi_0]{} \tilde{\mathcal{A}},
\end{array}
$$

and this lift is the left adjoint in an adjunction

$$\pi_0 : \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \rightleftarrows \mathrm{Alg}_\Phi(\tilde{\mathcal{A}}) : \mathrm{const}.$$

Moreover, for any $X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$ and any $n \geq 1$, the object $\pi_n X \in \tilde{\mathcal{A}}$ admits a canonical lift through the functor

$$\mathrm{Mod}_{\pi_0 X}^\Phi(\tilde{\mathcal{A}}) \xrightarrow{\mathrm{U}_{\pi_0 X}} \tilde{\mathcal{A}}. \quad \square$$

Corollary 5.22. *There exists a canonical lift*

$$
\begin{array}{c}
& & & & \mathrm{Alg}_\Phi(\tilde{\mathcal{A}}) \\
& \overset{\pi_0 E_*^{\mathrm{lw}}}{\nearrow} & & & \big\downarrow {\scriptstyle \mathrm{U}_\Phi} \\
\mathrm{Alg}_T(s\mathcal{C}) \xrightarrow[E_*^{\mathrm{lw}}]{} \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \xrightarrow[\mathrm{U}_{\tilde{T}_E}]{} s\tilde{\mathcal{A}} \xrightarrow[\pi_0]{} \tilde{\mathcal{A}}.
\end{array}
$$

Moreover, for any $X \in \mathrm{Alg}_T(s\mathcal{C})$ and any $n \geq 1$, the object $\pi_n E_*^{\mathrm{lw}} X \in \tilde{\mathcal{A}}$ admits a canonical lift through the functor

$$\mathrm{Mod}^{\Phi}_{\pi_0 E^{\mathrm{lw}}_* X}(\tilde{\mathcal{A}}) \xrightarrow{\mathrm{U}_{\pi_0 E^{\mathrm{lw}}_* X}} \tilde{\mathcal{A}}. \quad \square$$

We record a useful fact about the adjunction of Lemma 5.21.

Lemma 5.23. *The adjunction of Lemma 5.21 lifts to a Quillen adjunction*

$$\pi_0 : \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})_{\pi_*} \rightleftarrows \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})_{\mathrm{triv}} : \mathrm{const},$$

whose derived adjunction is a left localization adjunction.

Proof. To see that this is a Quillen adjunction, we observe that the left adjoint

- trivially preserves cofibrations, and
- preserves acyclic cofibrations by definition of the subcategory $\mathbf{W}_{\pi_*} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})$.

Then, to see that the derived adjunction is a left localization adjunction, we check that its counit is a componentwise equivalence. Since every object of $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})_{\mathrm{triv}}$ is fibrant, the composite

$$\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}}) \xrightarrow{\mathrm{const}} \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}}) \to \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}^{-1}_{\pi_*}]\!]$$

computes the derived right adjoint $\mathbb{R}\mathrm{const}$. Now, let

$$\varnothing_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})} \rightarrowtail \mathbb{Q}\mathrm{const}(A) \xrightarrow{\approx} \mathrm{const}(A)$$

be a cofibrant replacement in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})_{\pi_*}$. Then by definition the induced map

$$\pi_0(\mathbb{Q}\mathrm{const}(A)) \to \pi_0(\mathrm{const}(A)) \cong A$$

is an isomorphism in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$. So the counit is indeed an equivalence. $\quad \square$

Notation 5.24. As both functors in the Quillen adjunction of Lemma 5.23 preserve all weak equivalences, we will simply write

$$\pi_0 : \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}^{-1}_{\pi_*}]\!] \rightleftarrows \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}}) : \mathrm{const}$$

for its derived adjunction (as opposed to $\mathbb{L}\pi_0 \dashv \mathbb{R}\mathrm{const}$). Moreover, we will often leave implicit both the right Quillen functor as well as its derived right adjoint.

We have just seen that classical homology groups admit certain algebraic structure. In fact, natural homology groups do too.

Lemma 5.25 *([23, Examples 3.1.14 and 3.1.17]). There exists a canonical lift*

$$
\begin{array}{ccc}
 & & \mathrm{Alg}_\Phi(\tilde{\mathcal{A}}) \\
 & \nearrow^{\!\!E_{0,*}^\natural} & \big\downarrow{\scriptstyle \mathrm{U}_\mathcal{A} \circ \mathrm{U}_{\tilde{\mathcal{A}}} \circ \mathrm{U}_\Phi} \\
\mathrm{Alg}_T(s\mathcal{C}) & \xrightarrow[E_{0,*}^\natural]{} & \underline{\mathcal{A}}.
\end{array}
$$

Moreover, for any $X \in \mathrm{Alg}_T(s\mathcal{C})$ and any $n \geq 1$, the object $E_{n,*}^\natural X \in \underline{\mathcal{A}}$ admits a canonical lift through the functor

$$
\mathrm{Mod}_{E_{0,*}^\natural X}^\Phi(\tilde{\mathcal{A}}) \xrightarrow{\mathrm{U}_\mathcal{A} \circ \mathrm{U}_{\tilde{\mathcal{A}}} \circ \mathrm{U}_{E_{0,*}^\natural X}} \underline{\mathcal{A}}. \quad \square
$$

Moreover, these algebraic structures are compatible in the following way.

Lemma 5.26 *([23, Corollary 3.1.18]). The isomorphism $\pi_0 E_*^{\mathrm{lw}}(-) \cong E_{0,*}^\natural(-)$ in* $\mathrm{Fun}(\mathrm{Alg}_T(s\mathcal{C}), \underline{\mathcal{A}})$ *of Lemma 3.51 is compatible with the lifts to* $\mathrm{Fun}(\mathrm{Alg}_T(s\mathcal{C}), \mathrm{Alg}_\Phi(\tilde{\mathcal{A}}))$ *of Corollary 5.22 and Lemma 5.25.* $\quad\square$

Notation 5.27. In order to keep our notation unbiased, we may simply write

$$
E_0 = \pi_0 E_*^{\mathrm{lw}} \cong E_{0,*}^\natural \in \mathrm{Fun}(\mathrm{Alg}_T(s\mathcal{C}), \mathrm{Alg}_\Phi(\tilde{\mathcal{A}}))
$$

(where the isomorphism is that of Lemma 5.26).

Lemma 5.28 *([23, Example 3.1.13]). For any $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$ and any $n \geq 1$, the endofunctor $\Omega^n : \tilde{\mathcal{A}} \xrightarrow{\sim} \tilde{\mathcal{A}}$ lifts to an endofunctor $\Omega^n : \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}}) \xrightarrow{\sim} \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$.* $\quad\square$

Remark 5.29. In fact, if we define $\Sigma_+^\beta S^\varepsilon = (\mathbf{1} \oplus S^\beta) \otimes S^\varepsilon$, then the construction of [23, Example 3.1.13] generalizes to define lifted endofunctors $\Omega^\beta : \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}}) \xrightarrow{\sim} \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$ for any $\beta \in \mathcal{G}^\delta$.

We can now give the module structure on the localized spiral exact sequence.

Proposition 5.30 *([23, Corollary 3.1.18]). For any $X \in \mathrm{Alg}_T(s\mathcal{C})$, assembling the localized spiral exact sequence in* Ab *over all $\beta \in \mathcal{G}^\delta$, we obtain an exact sequence*

$$
\cdots \longrightarrow \pi_{i+1} E_* X \xrightarrow{\delta} \Omega(E_{i-1,*}^\natural X) \longrightarrow E_{i,*}^\natural X \longrightarrow \pi_i E_* X \xrightarrow{\delta} \cdots
$$

$$
\cdots \xrightarrow{\delta} \Omega(E_{0,*}^\natural X) \longrightarrow E_{1,*}^\natural X \longrightarrow \pi_1 E_* X \longrightarrow 0
$$

in $\mathrm{Mod}_{E_0 X}^\Phi(\tilde{\mathcal{A}})$. $\quad\square$

5.5. The module structure on the spiral exact sequence

We will make certain computations *before* appealing to a colimit argument, and for these we will need to obtain analogous structure on the unlocalized spiral exact sequence. In fact, this is an input to the module structure on the localized spiral exact sequence (via a colimit argument, as always), but the algebraic objects at play are slightly less familiar so we have reversed their order here. However, the story is nearly identical to that of §5.4, and so we only highlight the key points.

Notation 5.31. We write $T(\mathcal{G}_E) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ for the full subcategory spanned by the image of the composite

$$\mathcal{G}_E \hookrightarrow \mathcal{C} \xrightarrow{\mathrm{const}} s\mathcal{C} \xrightarrow{\mathrm{F}_T} \mathrm{Alg}_T(s\mathcal{C}) \to \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!].$$

Observation 5.32. The functor $\mathcal{G}_E \xrightarrow{\mathrm{F}_T} T(\mathcal{G}_E)$ preserves coproducts, and so induces a forgetful functor $\mathcal{P}_\Sigma^\delta(T(\mathcal{G}_E)) \xrightarrow{\mathrm{U}_{T(\mathcal{G}_E)}} \mathcal{P}_\Sigma^\delta(\mathcal{G}_E)$.

Definition 5.33. For any $A \in \mathcal{P}_\Sigma^\delta(T(\mathcal{G}_E))$, we define the category of *A-modules* (*relative to $T(\mathcal{G}_E)$*) as the category $\mathrm{Mod}_A^{T(\mathcal{G}_E)}(\mathcal{P}_\Sigma^\delta(\mathcal{G}_E)) = \mathrm{Ab}(\mathcal{P}_\Sigma^\delta(T(\mathcal{G}_E))_{/A})$ of abelian group objects in its overcategory. This admits two forgetful functors, which we denote by

$$\mathcal{P}_\Sigma^\delta(\mathcal{G}_E) \xleftarrow{\quad \mathrm{U}_A \quad} \mathrm{Mod}_A^{T(\mathcal{G}_E)}(\mathcal{P}_\Sigma^\delta(\mathcal{G}_E)) \xrightarrow{\quad -\ltimes A \quad} \mathcal{P}_\Sigma^\delta(T(\mathcal{G}_E)).$$

$$\ker^{\mathcal{P}_\Sigma^\delta(\mathcal{G}_E)}(\varphi) \longmapsfrom (B \xrightarrow{\varphi} A) \longmapsto B$$

The following example will be of use later.

Notation 5.34. Let $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$. Then we obtain an object $\natural^E(A) \in \mathcal{P}_\Sigma^\delta(T(\mathcal{G}_E))$ by declaring that

$$\natural^E(A)(\mathrm{F}_T(S^\varepsilon)) = \hom_{\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})}(\pi_0 E_*^{\mathrm{lw}} \mathrm{F}_T(S^\varepsilon), A).$$

Similarly, if $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$, we obtain an object $\natural^E(M) \in \mathrm{Mod}_{\natural^E(A)}^{T(\mathcal{G}_E)}(\mathcal{P}_\Sigma^\delta(\mathcal{G}_E))$ by declaring that

$$\natural^E(M)(S^\varepsilon) = \hom_{\tilde{\mathcal{A}}}(\pi_0 E_*^{\mathrm{lw}} S^\varepsilon, M);$$

technically, the *A*-action arises through Definitions 5.20 and 5.33 (in terms of abelian objects in overcategories), but morally it just comes from postcomposition.

Notation 5.35. In order to keep our notation unbiased, we may simply write

$$\pi_0 = \pi_0 \pi_*^{\mathrm{lw}} \cong \pi_{0,*}^\natural \in \mathrm{Fun}(\mathrm{Alg}_T(s\mathcal{C}), \mathrm{Fun}(\mathcal{G}_E^\delta, \mathrm{Ab}))$$

(where the isomorphism is that of Corollary 3.47).

Observation 5.36. As the functor $\mathrm{Alg}_T(s\mathcal{C}) \to \mathrm{ho}(\mathrm{Alg}_T(s\mathcal{C}))$ preserves finite coproducts, by adjunction there exists a canonical lift

$$
\begin{array}{ccc}
& & \mathcal{P}_\Sigma^\delta(T(\mathcal{G}_E)) \\
& \nearrow^{\pi_0} & \downarrow^{\mathrm{U}_{T(\mathcal{G}_E)}} \\
\mathrm{Alg}_T(s\mathcal{C}) \xrightarrow{\;\mathrm{U}_T\;} s\mathcal{C} \xrightarrow{\;\pi_0\;} & & \mathcal{P}_\Sigma^\delta(\mathcal{G}_E).
\end{array}
$$

Proposition 5.37 *([23, Theorem 3.1.15]). For any $X \in \mathrm{Alg}_T(s\mathcal{C})$, assembling the spiral exact sequence in* Ab *over all* $\varepsilon \in \mathcal{G}_E^\delta$, *we obtain an exact sequence*

$$
\cdots \longrightarrow \pi_{i+1}\pi_{*}X \xrightarrow{\;\delta\;} \Omega(\pi^\natural_{i-1,*}X) \longrightarrow \pi^\natural_{i,*}X \longrightarrow \pi_i\pi_{*}X \xrightarrow{\;\delta\;} \cdots
$$

$$
\cdots \xrightarrow{\;\delta\;} \Omega(\pi^\natural_{0,*}X) \longrightarrow \pi^\natural_{1,*}X \longrightarrow \pi_1\pi_{*}X \longrightarrow 0
$$

in $\mathrm{Mod}^{T(\mathcal{G}_E)}_{\pi_0 X}(\mathcal{P}_\Sigma^\delta(\mathcal{G}_E))$. $\square$

6. Homotopical algebra

In this section, we discuss the inductive construction of simplicial $\tilde{T}_E$-algebras in $\tilde{\mathcal{A}}$, foreshadowing an approximately parallel topological discussion in §7. We begin with their Postnikov theory in §6.1. This leads in to our discussion in §6.2 of a classification of extensions in terms of André–Quillen cohomology, which we observe to be represented by certain Eilenberg–Mac Lane objects. A functorial upgrade of this classification leads to a study of moduli spaces of simplicial $\tilde{T}_E$-algebras in §6.3, regarding which we also establish a number of results that will be used in §8.

6.1. Postnikov towers in algebra

Definition 6.1. For any $n \geq 0$, an object $X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ is called *n-truncated* if $\pi_{>n}X = 0$. Such objects form a full subcategory $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]^{\leq n} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$, and as n varies these subcategories are evidently nested as

$$
\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!] \hookleftarrow \cdots \hookleftarrow \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]^{\leq 1} \hookleftarrow \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]^{\leq 0}.
$$

By presentability, these inclusions admit left adjoints, and we denote the corresponding left localization adjunctions by

$$
P_n^{\mathrm{alg}} : \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!] \rightleftarrows \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]^{\leq n} : \mathrm{U}_n^{\mathrm{alg}}.
$$

We therefore obtain a tower of functors

$$\text{id}_{\text{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]} \to \cdots \to P_1^{\text{alg}} \to P_0^{\text{alg}}.$$

We refer to its value on an object of $\text{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ as its **Postnikov tower**. We write

$$\text{id}_{\text{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]} \xrightarrow{\tau_n^{\text{alg}}} P_n^{\text{alg}}$$

for the natural transformation (or for its composite with U_m^{alg} for any $m \geq 0$), which we refer to as the n-**truncation map**.

6.2. Cohomology

Our obstructions will take place in (André–Quillen) cohomology groups in $\text{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. We will only need to consider them with respect to a base object lying in $\text{Alg}_\Phi(\tilde{\mathcal{A}})$, so we restrict to this special case.

We begin by defining the representing objects for cohomology.

Definition 6.2. Let $A \in \text{Alg}_\Phi(\tilde{\mathcal{A}})$, let $M \in \text{Mod}_A^\Phi(\tilde{\mathcal{A}})$, and let $n \geq 1$.

(1) We say that an object $X \in \text{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ is of **type** K_A if there exists an equivalence $X \simeq A$, i.e. if
  - there exists an isomorphism $\pi_0 X \cong A$ in $\text{Alg}_\Phi(\tilde{\mathcal{A}})$, and
  - $\pi_i X = 0$ for $i > 0$.
(2) We say that an object $Y \in \text{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ is of **type** $K_A(M, n)$ if
  - there exists an isomorphism $\pi_0 Y \cong A$ in $\text{Alg}_\Phi(\tilde{\mathcal{A}})$,
  - there exists an isomorphism $\pi_n Y \cong M$ via the resulting equivalence of categories $\text{Mod}_{\pi_0 Y}^\Phi(\tilde{\mathcal{A}}) \simeq \text{Mod}_A^\Phi(\tilde{\mathcal{A}})$, and
  - $\pi_i Y = 0$ for $i \notin \{0, n\}$.
(3) We say that a morphism $X \to Y$ in $\text{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ is of **type** $\vec{K}_A(M, n)$ if
  - X is of type K_A,
  - Y is of type $K_A(M, n)$, and
  - the map $\pi_0 X \to \pi_0 Y$ is an isomorphism in $\text{Alg}_\Phi(\tilde{\mathcal{A}})$.
(4) We say that an object is of **type** $K_A(M, 0)$ in $\text{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ if it is of type $K_{M \ltimes A}$, and we say that a morphism in $\text{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ is of **type** $\vec{K}_A(M, 0)$ if it admits an equivalence to the map $\text{const}(A \to M \ltimes A)$.

We refer to objects of type K_A and $K_A(M, n)$ collectively as **algebraic Eilenberg–Mac Lane objects**, and to morphisms of type $\vec{K}_A(M, n)$ collectively as **algebraic Eilenberg–Mac Lane morphisms**. We will see that these all exist and are unique in Propositions 6.25 and 6.26; justified by this, we may simply write K_A or $K_A(M, n)$ for convenience when referring to an algebraic Eilenberg–Mac Lane object of the indicated type.

Observation 6.3. Suppose that $X \to Y$ is morphism in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ of type $\vec{K}_A(M, n)$ for some $n \geq 1$. Then $P_0^{\mathrm{alg}}(Y)$ is of type K_A, and the composite

$$X \to Y \xrightarrow{\tau_0^{\mathrm{alg}}} P_0^{\mathrm{alg}}(Y)$$

with the canonical 0-truncation map is an equivalence. Fixing an equivalence $X \simeq A$ then allows us to consider

$$K_A(M, n) \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{A/\!/A}.$$

Of course, such consideration is immediate for $n = 0$.

Observation 6.4. For any $n \geq 0$, taking the pullback of a map of type $\vec{K}_A(M, n+1)$ with itself yields a fiber square

$$\begin{array}{ccc} K_A(M, n) & \xrightarrow{\tau_0^{\mathrm{alg}}} & A \\ {\scriptstyle \tau_0^{\mathrm{alg}}} \downarrow & & \downarrow \\ A & \longrightarrow & K_A(M, n+1) \end{array}$$

in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. Hence, the objects

$$\left\{ K_A(M, n) \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{A/\!/A} \right\}_{n \geq 0}$$

assemble into an Ω-spectrum object

$$K_A M \in \mathrm{Stab}\left(\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{A/\!/A} \right).$$

Definition 6.5. Let $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$, let $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$, and let $n \geq 0$. Suppose that $k \to A = \mathrm{const}(A)$ is a morphism in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$, and use this to consider $K_A(M, n) \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/\!/A}$. Then, choose any object $X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/\!/A}$.

(1) We define the $n^{\mathbf{th}}$ *(André–Quillen) cohomology group* of X with coefficients in M to be the abelian group

$$H_{\tilde{T}_E}^n(X/k; M) = [X, K_A(M, n)]_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/\!/A}} \in \mathrm{Ab}.$$

(2) We define the $n^{\mathbf{th}}$ *(André–Quillen) cohomology space* of X with coefficients in M to be the based space

$$\mathscr{H}_{\tilde{T}_E}^n(X/k; M) = \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/\!/A}}(X, K_A(M, n)) \in \mathcal{S}_*.$$

Thus, we have that

$$H_{\tilde{T}_E}^n(X/k; M) = \pi_0(\mathscr{H}_{\tilde{T}_E}^n(X/k; M)),$$

and moreover it follows from Observation 6.4 that

$$H_{\tilde{T}_E}^{n-i}(X/k; M) = \pi_i(\mathscr{H}_{\tilde{T}_E}^n(X/k; M))$$

for $0 \leq i \leq n$. (In particular, cohomology groups are indeed abelian groups, and cohomology spaces are infinite loopspaces.)

Observation 6.6. In the setting of Definition 6.5, there is an evident pullback square

$$
\begin{array}{ccc}
\hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{X//A}}(A, K_A(M, n)) & \longrightarrow & \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k//A}}(A, K_A(M, n)) \\
\downarrow & & \downarrow \\
\{X \to A \to K_A(M, n)\} & \longrightarrow & \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k//A}}(X, K_A(M, n))
\end{array}
$$

in $\mathcal{S}_*$, which is by definition a pullback square

$$
\begin{array}{ccc}
\mathscr{H}_{\tilde{T}_E}^n(A/X; M) & \longrightarrow & \mathscr{H}_{\tilde{T}_E}^n(A/k; M) \\
\downarrow & & \downarrow \\
\{0\} & \longrightarrow & \mathscr{H}_{\tilde{T}_E}^n(X/k; M).
\end{array}
$$

This gives rise to a long exact sequence

$$0 \longrightarrow H_{\tilde{T}_E}^0(A/X; M) \longrightarrow H_{\tilde{T}_E}^0(A/k; M) \longrightarrow H_{\tilde{T}_E}^0(X/k; M) \xrightarrow{\delta} \cdots$$

$$\cdots \xrightarrow{\delta} H_{\tilde{T}_E}^n(A/X; M) \longrightarrow H_{\tilde{T}_E}^n(A/k; M) \longrightarrow H_{\tilde{T}_E}^n(X/k; M) \xrightarrow{\delta} H_{\tilde{T}_E}^{n+1}(A/X; M) \longrightarrow \cdots$$

in Ab; exactness at $H_{\tilde{T}_E}^0(A/X; M)$ follows from the fact that the space

$$\hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k//A}}(X, K_A(M, 0)) \simeq \hom_{\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{\pi_0 k//A}}(\pi_0 X, M \ltimes A)$$

is discrete (and so in particular has vanishing π_1). We refer to this as the **transitivity sequence**.

Remark 6.7. When $M \in \tilde{\mathcal{A}}$ is an *extended* comodule, these cohomology computations reduce to analogous ones in $\mathrm{Alg}_{T_E}(s\mathcal{A})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ (see [23, Proposition 2.4.7]).

6.3. Moduli spaces in algebra

We will be interested in various moduli spaces of algebraic objects: ultimately, our obstruction theory will be based on homotopy groups in the ∞-category $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$.

In order to be able to effectively control these homotopy groups, we need to make the following assumption.

Assumption 6.8. We assume that $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ has **Blakers–Massey excision**: for any pushout square

$$
\begin{array}{ccc}
X & \xrightarrow{\psi} & Z \\
{\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\rho} \\
Y & \longrightarrow & W
\end{array}
$$

such that $\pi_{<m}(\mathrm{fib}(\varphi)) = \pi_{<n}(\mathrm{fib}(\psi)) = 0$, the map $\pi_k(\mathrm{fib}(\varphi)) \to \pi_k(\mathrm{fib}(\rho))$ is an isomorphism for $k < m + n$ and is surjective for $k = m + n$.

Corollary 6.9 *([23, Corollary 2.3.15]). Suppose that*

$$
\begin{array}{ccc}
X & \xrightarrow{\psi} & Z \\
{\scriptstyle\varphi}\downarrow & & \downarrow \\
Y & \longrightarrow & W
\end{array}
$$

is a pushout square in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ such that $\pi_{<m}(\mathrm{fib}(\varphi)) = \pi_{<n}(\mathrm{fib}(\psi)) = 0$. Then there is an induced partial long exact sequence

$$
\pi_{m+n}(Y) \oplus \pi_{m+n}(Z) \longrightarrow \pi_{m+n}(W) \xrightarrow{\delta} \pi_{m+n-1}(X) \longrightarrow \cdots
$$

$$
\cdots \xrightarrow{\delta} \pi_0(X) \longrightarrow \pi_0(Y) \oplus \pi_0(Z) \longrightarrow \pi_0(W) \longrightarrow 0
$$

*in $\tilde{\mathcal{A}}$, which we refer to as the **Blakers–Massey long exact sequence**.*

Remark 6.10. Assumption 6.8 holds in examples of interest, e.g. when $\tilde{T}_E$ is the monad corresponding to an operad $E_{\circledast}(T) \in \mathrm{Op}(s\tilde{\mathcal{A}})$ for any $T \in \mathrm{Op}(s\mathrm{Set})$ (see [23, Theorem 2.3.13 and Remark 2.3.14]).

Our moduli spaces will be related by the following natural construction.

Construction 6.11. Let $X \xrightarrow{\varphi} Y$ be a map in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$, and write

$$
\mathrm{p}_0^{\mathrm{alg}}(\varphi) = Y \coprod_X P_0^{\mathrm{alg}} X = \mathrm{colim}\left(
\begin{array}{ccc}
X & \xrightarrow{\tau_0^{\mathrm{alg}}} & P_0^{\mathrm{alg}}(X) \\
{\scriptstyle\varphi}\downarrow & & \\
Y & &
\end{array}
\right)
$$

for the indicated pushout. For any $n \geq 0$ we obtain a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\tau_0^{\mathrm{alg}}} & P_0^{\mathrm{alg}}(X) \\
\varphi \downarrow & & \downarrow \quad \searrow{\delta_n(\varphi)} \\
Y & \xrightarrow[\tau_{n+1}^{\mathrm{alg}}]{} \mathrm{p}_0^{\mathrm{alg}}(\varphi) & \xrightarrow{} P_{n+1}^{\mathrm{alg}}(\mathrm{p}_0^{\mathrm{alg}}(\varphi))
\end{array}
$$

in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$, and we refer to the map $\delta_n(\varphi)$ as the $n^{\mathbf{th}}$ **difference construction** on the map φ. This defines an augmented endofunctor on $\mathrm{Fun}([1], \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!])$. We will generally only apply this in the case that $n \geq 1$, and in the case that $\pi_{<n}(\varphi)$ is an isomorphism.

Lemma 6.12. *Suppose that the map* $X \xrightarrow{\varphi} Y$ *in* $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ *is an isomorphism on* $\pi_{<n}$ *for some* $n \geq 1$. *Write* $A = \pi_0 X \cong \pi_0 Y \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$ *and* $M = \pi_n \mathrm{fib}(\varphi) \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$. *Then, the map*

$$
P_0^{\mathrm{alg}}(X) \xrightarrow{\delta_n(\varphi)} P_{n+1}^{\mathrm{alg}}(\mathrm{p}_0^{\mathrm{alg}}(\varphi))
$$

is of type $\vec{K}_A(M, n+1)$.

Proof. This follows from Assumption 6.8. $\square$

Corollary 6.13 *([23, Proposition 2.5.13]).* *Let* $X \xrightarrow{\varphi} Y$ *be a map in* $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. *Suppose that* $\pi_* \mathrm{fib}(\varphi)$ *is concentrated in degree* n. *The square*

$$
\begin{array}{ccc}
X & \xrightarrow{\tau_0^{\mathrm{alg}}} & P_0^{\mathrm{alg}}(X) \\
\varphi \downarrow & & \downarrow{\delta_n(\varphi)} \\
Y & \xrightarrow{} & P_{n+1}^{\mathrm{alg}}(\mathrm{p}_0^{\mathrm{alg}}(\varphi))
\end{array}
$$

is a pullback in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. $\square$

Observation 6.14. In the setting of Corollary 6.13, if additionally X (and hence Y) is n-truncated, then we can identify the map $X \to Y$ as $\tau_{\leq n}^{\mathrm{alg}} X \to \tau_{\leq(n-1)}^{\mathrm{alg}} X$, and from here Lemma 6.12 allows us to identify the pullback square of Corollary 6.13 as

$$
\begin{array}{ccc}
P_n^{\mathrm{alg}} X & \xrightarrow{} & K_A \\
\tau_{n-1}^{\mathrm{alg}} \downarrow & & \downarrow \\
P_{n-1}^{\mathrm{alg}} X & \xrightarrow{} & K_A(M, n+1)
\end{array}
$$

(in which the right vertical map is of type $\vec{K}_A(M, n+1)$). This is a functorial construction of k-invariants in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$.

Notation 6.15. We fix an object $k \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. We will generally work in its undercategory $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$; in particular, we will generally have fixed a map $k \to A = \mathrm{const}(A)$. Everything will take place in this undercategory, so that e.g. a morphism in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$ of type $\vec{K}_A(M, n)$ will be understood to mean a commutative triangle

$$
\begin{array}{ccc}
 & k & \\
 \swarrow & & \searrow \\
 K_A & \longrightarrow & K_A(M, n)
\end{array}
$$

in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ in which the left vertical arrow identifies with the fixed map.

Notation 6.16. Suppose that $Y \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$ is $(n-1)$-truncated for some $n \geq 1$, write $A = \pi_0 Y \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$, and suppose $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$. We write

$$
\mathscr{M}_k(Y \oplus (M, n)) \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}
$$

for the moduli space of those objects X such that

- X is n-truncated,
- there exists an equivalence $P_{n-1}^{\mathrm{alg}} X \xrightarrow{\sim} Y$, and
- there exists an isomorphism $\pi_n X \cong M$ via the resulting equivalence $\mathrm{Mod}_{\pi_0 X}^\Phi(\tilde{\mathcal{A}}) \simeq \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$.

Notation 6.17. In our moduli spaces, we will use the symbol $\looparrowright$ to denote a restriction to morphisms which are isomorphisms on homotopy groups in those dimensions for which *both* the source and the target have nonvanishing homotopy. So for instance, we would write $\mathscr{M}(K_A \looparrowright K_A(M, n))$ for the moduli space of morphisms of type $\vec{K}_A(M, n)$.

We now arrive at our first equivalence of moduli spaces, which expresses "extensions of Y by M in dimension n" (as in Notation 6.16) in terms of the nth difference construction.

Proposition 6.18 ([23, Theorem 2.5.16]). *Suppose that* $Y \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$ *is* $(n-1)$-*truncated for some* $n \geq 1$, *write* $A = \pi_0 Y \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$, *and suppose* $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$. *Then the functor*

$$
X \mapsto \left(P_{n-1}^{\mathrm{alg}}(X) \to P_{n+1}^{\mathrm{alg}}(\mathrm{p}_0^{\mathrm{alg}}((\tau_{n-1}^{\mathrm{alg}})X)) \xleftarrow{\delta_n((\tau_{n-1}^{\mathrm{alg}})X)} P_0^{\mathrm{alg}}(X) \right)
$$

determines an equivalence

$$\mathscr{M}_k(Y \oplus (M,n)) \xrightarrow{\sim} \mathscr{M}_k(Y \hookrightarrow K_A(M, n+1) \hookleftarrow K_A)$$

in $\mathcal{S}$.

Proof. An inverse is provided by the pullback functor. $\quad\square$

Notation 6.19. For any $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$, we write

$$\mathscr{M}_{A/k} \subset \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$$

for the moduli space of objects of type $K_{A/k}$. For any $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$ and any $n \geq 1$, we write

$$\mathscr{M}_{A/k}(M,n) \subset \mathrm{Fun}([1], \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/})$$

for the moduli space of morphisms of type $\vec{K}_{A/k}(M,n)$.

Notation 6.20. It will be of auxiliary use to write

$$\mathscr{M}_{A/k}(M,0)$$

for the moduli space of pairs of an object $X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ and an abelian (∞-)group object $Y \in \mathrm{Ab}(\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{/X})$ in its overcategory which are in the image of (A, M) under the derived right adjoint

$$\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{/A} \xrightarrow{\mathrm{const}} \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{/A}$$

of the Quillen adjunction of Lemma 5.23.

Our next equivalence of moduli spaces shows that $\mathscr{M}_{A/k}(M,n)$ is in fact independent of n; we will give a more explicit identification in Proposition 6.26.

Proposition 6.21. *Let* $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$, *let* $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$, *and let* $n \geq 0$. *Then the functor*

$$(X \to Y) \mapsto \left(X \to \lim^{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}} \left(\begin{array}{cc} & X \\ & \downarrow \\ X \longrightarrow & Y \end{array} \right) \right)$$

defines an equivalence

$$\mathscr{M}_{A/k}(M, n+1) \xrightarrow{\sim} \mathscr{M}_{A/k}(M, n)$$

in $\mathcal{S}$.

Proof. For $n \geq 1$, an inverse is provided by the functor

$$(Z \to W) \mapsto \delta_n(W \to P_0^{\mathrm{alg}}(W)).$$

For $n = 0$, an inverse is provided by the functor taking the pair

$$\left(W \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!] \ , \ Z \in \mathrm{Ab}(\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{/W}) \right),$$

say with structure map $Z \xrightarrow{\varphi} W$, to the map

$$K_{\pi_0 W} \to K_{\pi_0 W}(\ker(\pi_0(\varphi)), 1)$$

(which is evidently of type $\vec{K}_A(M, 1)$). $\square$

We now identify homotopy classes of maps into algebraic Eilenberg–Mac Lane objects in terms of cohomology groups.

Proposition 6.22 *([23, Lemma 2.5.18]).* *Let* $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$, *let* $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$, *let* $X \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$, *and let* $n \geq 0$. *Then there exists a natural isomorphism*

$$[X, K_A(M, n)]_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}} \cong \coprod_{\mathrm{hom}_{\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}}(\pi_0 X, A)} H_{\tilde{T}_E}^n(X/k; M)$$

in Ab *(where the implicit structure map* $X \to A = \mathrm{const}(A)$ *in* $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})_{k/}$ *necessary for defining the cohomology of* X *varies over the indexing set).* $\square$

Notation 6.23. Given an ∞-category $\mathcal{D}$ and objects $d_1, d_2 \in \mathcal{D}$, we write $\mathrm{hom}_{\mathcal{D}}^{\simeq}(d_1, d_2) \subset \mathrm{hom}_{\mathcal{D}}(d_1, d_2)$ for the subspace of equivalences. For any other sort of decoration denoting a certain property of a morphism, we use corresponding exponent notation to denote the subspace of the hom-space corresponding to morphisms having this property.

Notation 6.24. For any $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$, we write $\mathrm{Aut}_k(A) = \mathrm{Aut}_{\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}}(A)$. Moreover, for any $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$, we write $\mathrm{Aut}_k(A, M)$ for the group of pairs

$$\left(\varphi \in \mathrm{Aut}_k(A) \ , \ \psi \in \mathrm{hom}_{\mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})}^{\simeq}(M, \varphi^*(M)) \right).$$

Proposition 6.25 *([23, Proposition 2.5.19(1)]).* *For any* $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$, *we have an equivalence* $\mathscr{M}_{A/k} \simeq B\mathrm{Aut}_k(A)$ *in* $\mathcal{S}$.

Proof. This is the assertion that the canonical map

$$\operatorname{Aut}_{\operatorname{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}}(A) \to \operatorname{Aut}_{\operatorname{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}}(\operatorname{const}(A))$$

induced by the functor

$$\operatorname{Alg}_\Phi(\tilde{\mathcal{A}}) \xrightarrow{\operatorname{const}} \operatorname{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$$

is an equivalence, which follows from Lemma 5.23 since it implies that this functor is a full inclusion. $\square$

We now give a straightforward description of the moduli spaces $\mathscr{M}_{A/k}(M, n)$.

Proposition 6.26 ([23, Proposition 2.5.19(2)]). *Suppose that $A \in \operatorname{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$ and that $M \in \operatorname{Mod}_A^\Phi(\tilde{\mathcal{A}})$. Then for any $n \geq 0$ we have an equivalence $\mathscr{M}_{A/k}(M, n) \simeq B\operatorname{Aut}_k(A, M)$.*

Proof. This follows from combining Proposition 6.21 with the essentially definitional equivalence $\mathscr{M}_{A/k}(M, 0) \simeq B\operatorname{Aut}_k(A, M)$. $\square$

Notation 6.27. Given an object $X \in \operatorname{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$, we write

$$\mathscr{M}_k(X) \subset \operatorname{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$$

for the full subgroupoid generated by it.

The following fiber sequence in particular expresses the moduli spaces appearing in Proposition 6.18 as extensions by cohomology spaces.

Lemma 6.28 ([23, Proposition 2.5.22]). *For any $X \in \operatorname{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$, there exists a canonical pullback square*

$$
\begin{array}{ccc}
\coprod_{\operatorname{hom}^{\simeq}_{\operatorname{Alg}_\Phi(\tilde{\mathcal{A}})}(\pi_0 X, A)} \mathscr{H}^n_{\tilde{T}_E}(X/k; M) & \longrightarrow & \mathscr{M}_k(X \looparrowright K_A(M, n) \looparrowleft A) \\
\downarrow & & \downarrow \\
\operatorname{pt}_\mathcal{S} & \xrightarrow{(X, \operatorname{id}_{(A,M)})} & \mathscr{M}_k(X) \times B\operatorname{Aut}_k(A, M)
\end{array}
$$

in $\mathcal{S}$.

Proof. This is immediate from the definitions. $\square$

Notation 6.29. We write

$$\widehat{\mathscr{H}}^n_{\tilde{T}_E}(A/k; M) = \left(\mathscr{H}^n_{\tilde{T}_E}(A/k; M) \right)_{\mathrm{Aut}_k(A,M)} \in \mathcal{S}_*$$

for the based space of coinvariants of the canonical action of $\mathrm{Aut}_k(A, M)$ on $\mathscr{H}^n_{\tilde{T}_E}(A/k; M) \in \mathcal{S}_*$.

Corollary 6.30. *There exists a canonical pullback square*

$$
\begin{array}{ccc}
\mathscr{H}^n_{\tilde{T}_E}(A/k; M) & \longrightarrow & \mathscr{M}_k(A \hookleftarrow K_A(M, n) \hookrightarrow A) \\
\downarrow & & \downarrow \\
\mathrm{pt}_{\mathcal{S}} & \longrightarrow & B\mathrm{Aut}_k(A, M)
\end{array}
$$

in $\mathcal{S}$, whose induced action of $\mathrm{Aut}_k(A, M)$ on $\mathscr{H}^n_{\tilde{T}_E}(A/k; M)$ is the natural one, and which induces an equivalence

$$\mathscr{M}_k(A \hookleftarrow K_A(M, n) \hookrightarrow A) \simeq \widehat{\mathscr{H}}^n_{\tilde{T}_E}(A/k; M)$$

in $\mathcal{S}$.

Proof. First of all, applying Lemma 6.28 in the case that $X = A$ yields a pullback square

$$
\begin{array}{ccc}
\coprod_{\hom^{\simeq}_{\mathrm{Alg}_\Phi(\tilde{A})}(A,A)} \mathscr{H}^n_{\tilde{T}_E}(A/k; M) & \longrightarrow & \mathscr{M}_k(A \hookleftarrow K_A(M, n) \hookrightarrow A) \\
\downarrow & & \downarrow \\
\mathrm{pt}_{\mathcal{S}} & \xrightarrow[(X,\mathrm{id}_{(A,M)})]{} & \mathscr{M}_k(A) \times B\mathrm{Aut}_k(A, M)
\end{array}
$$

in $\mathcal{S}$. By Proposition 6.25, we have $\mathscr{M}_k(A) \simeq B\mathrm{Aut}_k(A) = \mathrm{Aut}_{\mathrm{Alg}_\Phi(\tilde{A})_{k/}}(A)$, and the action on the fibers is clearly the canonical one and is hence free on its path components. Thus, pulling back along the map

$$B\mathrm{Aut}_k(A, M) \simeq \{A\} \times B\mathrm{Aut}_k(A, M) \to \mathscr{M}_k(A) \times B\mathrm{Aut}_k(A, M)$$

yields a pullback square

$$
\begin{array}{ccc}
\mathscr{H}^n_{\tilde{T}_E}(A/k; M) & \longrightarrow & \mathscr{M}_k(A \hookleftarrow K_A(M, n) \hookrightarrow A) \\
\downarrow & & \downarrow \\
\mathrm{pt}_{\mathcal{S}} & \xrightarrow[\mathrm{id}_{(A,M)}]{} & B\mathrm{Aut}_k(A, M)
\end{array}
$$

in $\mathcal{S}$. The claim now follows readily from [50, Proposition 2.1]. $\quad\square$

7. Homotopical topology

In this section, we discuss the inductive construction of simplicial T-algebras in $\mathcal{C}$, in approximate parallel with our algebraic discussion in §6. We begin with their Postnikov theory in §7.1. In §7.2, we establish the existence of Eilenberg–Mac Lane objects that represent the André–Quillen cohomology of their levelwise E-homology. This leads to a preliminary discussion of moduli spaces of simplicial T-algebras in §7.3, although we defer a deeper analysis thereof to §8.

7.1. Postnikov towers in topology

We now study the homotopy theory of the ∞-category $\mathrm{Alg}_T(s\mathcal{C})$ of simplicial T-algebras; we will mostly work in its localization $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$, but we will ultimately be interested in deducing results about its further localization $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!]$ via Proposition 5.15.

Definition 7.1. For any $n \geq 0$, an object $X \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ is called n-**truncated** if $\pi_{>n,\varepsilon}^\natural X = 0$ for all $\varepsilon \in \mathcal{G}_E^\delta$. Such objects form a full subcategory $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]^{\leq n} \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$, and as n varies these subcategories are evidently nested as

$$\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \hookleftarrow \cdots \hookleftarrow \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]^{\leq 1} \hookleftarrow \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]^{\leq 0}.$$

By presentability, these inclusions admit left adjoints, and we denote the corresponding left localization adjunctions by

$$P_n^{\mathrm{top}} : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \rightleftarrows \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]^{\leq n} : \mathrm{U}_n^{\mathrm{top}}.$$

We therefore obtain a tower of functors

$$\mathrm{id}_{\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]} \to \cdots \to P_1^{\mathrm{top}} \to P_0^{\mathrm{top}}.$$

We refer to its value on an object of $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ as its **Postnikov tower**. We write

$$\mathrm{id}_{\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]} \xrightarrow{\tau_n^{\mathrm{top}}} P_n^{\mathrm{top}}$$

for the natural transformation (or its for composite with $\mathrm{U}_m^{\mathrm{top}}$ for any $m \geq 0$), which we refer to as the n-**truncation map**.

Observation 7.2. By a colimit argument, if $X \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ is n-truncated then $E_{>n,*}^\natural X = 0$ as well.

7.2. Topological Eilenberg–Mac Lane objects

We now define certain objects of $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ which will represent the various functors "apply E_*^{lw}, then take cohomology".

Definition 7.3. Let $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$, let $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$, and let $n \geq 1$.

(1) We say that an object $X \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ is of **type B_A** if there exists a universal map $E_*^{\mathrm{lw}} X \to K_A$ inducing natural equivalences

$$\hom_{\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}(Z, X) \xrightarrow{\sim} \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]}(E_*^{\mathrm{lw}} Z, K_A)$$

for all $Z \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$.

(2) We say that an object $Y \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ is of **type $B_A(M, n)$** if there exists a universal map $E_*^{\mathrm{lw}} Y \to K_A(M, n)$ inducing natural equivalences

$$\hom_{\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}(Z, Y) \xrightarrow{\sim} \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]}(E_*^{\mathrm{lw}} Z, K_A(M, n))$$

for all $Z \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$.

(3) We say that a map $X \to Y$ in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ is of **type $\vec{B}_A(M, n)$** if X is of type B_A, Y is of type $B_A(M, n)$ and the map $\pi_0 E_*^{\mathrm{lw}} X \to \pi_0 E_*^{\mathrm{lw}} Y$ is an isomorphism in $\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$.

We refer to objects of type B_A and $B_A(M, n)$ collectively as **topological Eilenberg–Mac Lane objects**, and to morphisms of type $\vec{B}_A(M, n)$ collectively as **topological Eilenberg–Mac Lane morphisms**.

Lemma 7.4. *For any $A \in \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$, and $M \in \mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$, and any $n \geq 1$, there exist objects of type B_A and $B_A(M, n)$, and there exist morphisms of type $\vec{B}_A(M, n)$.*

Proof. By the presentability of $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ (which follows from Theorem 2.23 and the derived monadic adjunction underlying the monadic Quillen adjunction $\mathrm{F}_T \dashv \mathrm{U}_T$), this follows from Corollary 5.10. $\square$

Notation 7.5. Justified by Lemma 7.4, we may simply write B_A or $B_A(M, n)$ for convenience when referring to a topological Eilenberg–Mac Lane object of the indicated type.

Observation 7.6. If $X \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ is an object of type B_A, it follows immediately that $\pi_{0,*}^\natural X \cong \downarrow^{\pi_0 E_*^{\mathrm{lw}}}(A)$ in $\mathcal{P}_\Sigma^\delta(T(\mathcal{G}_E))$ and that $\pi_{>0,*}^\natural X = 0$. By the spiral exact sequence, it follows that

$$\pi_i \pi_{\underline{*}} X \cong \begin{cases} \natural^E(A), & i = 0 \\ \natural^E(\Omega A), & i = 2 \\ 0, & i \notin \{0, 2\}. \end{cases}$$

For convenience, we simply write $\pi_* \pi_{\underline{*}} X \cong \natural^E(A) \times \natural^E(\Omega A)[2]$.

Now, suppose that $X \to Y$ is a map of type $\vec{B}_A(M, n)$. It follows that $\pi_{0,\underline{*}}^{\natural} Y \cong \natural^E(A)$ in $\mathcal{P}_\Sigma^\delta(T(\mathcal{G}_E))$ and that for $i \geq 1$,

$$\pi_{i,\underline{*}}^{\natural} Y \cong \begin{cases} \natural^E(M), & i = n \\ 0, & i \neq n \end{cases}$$

in $\mathrm{Mod}_A^\Phi(\tilde{\mathcal{A}})$, and furthermore that the composite $X \to Y \to P_0^{\mathrm{top}}(Y)$ is an equivalence. Combining this with the spiral exact sequence yields that $\pi_* \pi_{\underline{*}} Y \cong \pi_* \pi_{\underline{*}} X \times \natural^E(M)[n] \times \natural^E(\Omega M)[n+2]$.

7.3. Moduli spaces in topology

We begin by mimicking Construction 6.11.

Construction 7.7. Let $X \xrightarrow{\varphi} Y$ be a map in $\mathrm{Alg}_T(s\mathcal{C})\llbracket \mathbf{W}_{\mathrm{res}}^{-1} \rrbracket$, and write

$$\mathrm{p}_0^{\mathrm{top}}(\varphi) = Y \coprod_X P_0^{\mathrm{top}} X = \mathrm{colim} \left(\begin{array}{ccc} X & \xrightarrow{\tau_0^{\mathrm{top}}} & P_0^{\mathrm{top}}(X) \\ \varphi \downarrow & & \\ Y & & \end{array} \right)$$

for the indicated pushout. For any $n \geq 0$ we obtain a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\tau_0^{\mathrm{top}}} & P_0^{\mathrm{top}}(X) \\ \varphi \downarrow & & \downarrow \quad \searrow^{\delta_n(\varphi)} \\ Y & \longrightarrow & \mathrm{p}_0^{\mathrm{top}}(\varphi) \xrightarrow{\tau_{n+1}^{\mathrm{top}}} P_{n+1}^{\mathrm{top}}(\mathrm{p}_0^{\mathrm{top}}(\varphi)) \end{array}$$

in $\mathrm{Alg}_T(s\mathcal{C})\llbracket \mathbf{W}_{\mathrm{res}}^{-1} \rrbracket$, and we refer to the map $\delta_n(\varphi)$ as the $n^{\mathbf{th}}$ **difference construction** on the map φ. This defines an augmented endofunctor on $\mathrm{Fun}([1], \mathrm{Alg}_T(s\mathcal{C})\llbracket \mathbf{W}_{\mathrm{res}}^{-1} \rrbracket)$. We will generally only apply this in the case that $n \geq 1$, and in the case that $\pi_{<n,\underline{*}}^{\natural}(\varphi)$ is an isomorphism.

We now employ our assumption that T is homotopically adapted to E, which provides a fundamental link between our computations in homotopical topology and homotopical algebra.

Proposition 7.8. *Let $X \xrightarrow{\varphi} Y$ be a map in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$, let $n \geq 1$, and suppose that $E_{<n,*}^{\natural}(\varphi)$ is an isomorphism and that $E_{n,*}^{\natural}(\varphi)$ is surjective. Write $A = E_{0,*}^{\natural} X \cong E_{0,*}^{\natural} Y$ in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$, and write $M = \mathrm{fib}(E_{n,*}^{\natural}(\varphi)) \in \tilde{\mathcal{A}}$.*

(1) We can canonically consider $M \in \mathrm{Mod}_A^{\Phi}(\tilde{\mathcal{A}})$.

(2) The map $\delta_n(\varphi)$ becomes equivalent to a morphism of type $\vec{B}_A(M, n)$ under the localization functor $\mathrm{L}_{E_*^{\mathrm{lw}}} : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \to \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!]$.

(3) If $\pi_{i,*}^{\natural}(\mathrm{fib}(\varphi)) = 0$ for $i \neq n+1$, then the square

$$
\begin{array}{ccc}
X & \xrightarrow{\ \tau_0^{\mathrm{top}}\ } & P_0^{\mathrm{top}}(X) \\
{\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\delta_n(\varphi)} \\
Y & \longrightarrow & P_{n+1}^{\mathrm{top}}(\mathrm{p}_0^{\mathrm{top}}(\varphi))
\end{array}
$$

becomes a pullback under the localization functor $\mathrm{L}_{E_*^{\mathrm{lw}}} : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \to \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!]$.

Proof. It follows from Corollary 5.10 that the functor

$$
\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \xrightarrow{\ E_*^{\mathrm{lw}}\ } \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]
$$

preserves pushouts. Thus, the square

$$
\begin{array}{ccc}
E_*^{\mathrm{lw}} X & \xrightarrow{\ E_*^{\mathrm{lw}}(\tau_0^{\mathrm{top}})\ } & E_*^{\mathrm{lw}}(P_0^{\mathrm{top}}(X)) \\
{\scriptstyle E_*^{\mathrm{lw}}(\varphi)}\downarrow & & \downarrow \\
E_*^{\mathrm{lw}} Y & \longrightarrow & E_*^{\mathrm{lw}}(\mathrm{p}_0^{\mathrm{top}}(\varphi))
\end{array}
$$

is a pushout in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. From here, the proof is essentially identical to that of [23, Proposition 3.2.9]. $\square$

In order to work in a relative setting, we fix the following.

Notation 7.9. We assume we are given an object $Y \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ equipped with an isomorphism $E_*^{\mathrm{lw}} Y \cong k$ in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$ for some chosen object $k \in \mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$ (specialized via the derived right adjoint $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}}) \xrightarrow{\mathrm{const}} \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ from our previous assumption from Notation 6.15 that $k \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$). A map $k \to A$ in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$ gives rise to a composite

$$
E_*^{\mathrm{lw}}\mathrm{const}(Y) \xrightarrow{\sim} k \to A
$$

in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$, via which for any choice of topological Eilenberg–Mac Lane object B_A we obtain a canonical map $\mathrm{const}(Y) \to B_A$. We will simply write $Y = \mathrm{const}(Y) \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$, and we will work in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y//B_A}$.

Observation 7.10. Fix any morphism $B_A \to B_A(M, n)$ in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ of type $\vec{B}_A(M, n)$. From Observation 7.6 and Notation 7.9, we obtain a sequence of composable morphisms

$$Y \to B_A \to B_A(M, n) \to B_A$$

(in which the composite of all but the first map is an equivalence). For any $X \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y//B_A}$ and as soon as $n \geq 2$, we immediately obtain equivalences

$$\hom_{\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y/}}(X, B_A) \xrightarrow{\sim} \hom_{\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}}(\pi_0 E_*^{\mathrm{lw}} X, A)$$

and

$$\hom_{\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y//B_A}}(X, B_A(M, n)) \xrightarrow{\sim} \mathscr{H}_{\tilde{T}_E}^n(E_*^{\mathrm{lw}}(X)/k; M)$$

in $\mathcal{S}_*$.

Notation 7.11. We write $\mathscr{M}_Y(A) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y/}$ for the moduli space of objects $Y \to X$ such that X is of type B_A and moreover the map $E_0^{\mathrm{lw}} Y \to E_0^{\mathrm{lw}} X$ is equivalent to the map $k \to A$ in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. Moreover, we write $\mathscr{M}_{A/Y}(M, n) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y/}$ for the moduli space of morphisms $Z \to W$ of type $\vec{B}_A(M, n)$ under Y such that $(Y \to Z) \in \mathscr{M}_Y(A)$.

Proposition 7.12. *The functor*

$$X \mapsto P_0^{\mathrm{alg}} E_*^{\mathrm{lw}}(X)$$

defines an equivalence

$$\mathscr{M}_Y(A) \xrightarrow{\sim} \mathscr{M}_k(A),$$

and the functor

$$\varphi \mapsto \delta_{n-1}(E_*(\varphi))$$

defines an equivalence

$$\mathscr{M}_{A/Y}(M, n) \xrightarrow{\sim} \mathscr{M}_{A/k}(M, n) \simeq B\mathrm{Aut}_k(A, M).$$

Proof. These assertions both follow immediately from the functors that topological Eilenberg–Mac Lane objects are defined to represent, just as in the proof of [23, Proposition 3.2.17]. $\square$

8. Decomposition of moduli spaces

In this section we prove our main theorems. We introduce the various objects of interest in §8.1, and proceed to study the relationships between their moduli spaces in §8.2.

8.1. Realizations and n-stages

We finally come to our main theorems: these provide an inductive procedure for understanding our moduli space of ultimate interest, which we begin by introducing.

Definition 8.1. With respect to

- our fixed base object $Y \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$,
- our chosen morphism $k \to A$ in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$, and
- our chosen isomorphism $E_{\circledast} Y \cong k$ in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})$,

we define a ***realization*** to be an object $(Y \xrightarrow{\varphi} X) \in \mathrm{Alg}_{\mathcal{O}}(\mathrm{L}_E(\mathcal{C}))_{Y/}$ such that there exists an isomorphism $E_{\circledast} X \cong A$ in $\mathrm{Alg}_{\Phi}(\tilde{\mathcal{A}})_{k/}$. We write

$$\mathscr{M}_{A/Y} \subset \mathrm{Alg}_{\mathcal{O}}(\mathrm{L}_E(\mathcal{C}))_{Y/}$$

for the moduli space of realizations (and $E_{\circledast}$-equivalences between them).

Before diving in, we provide a bit of big-picture intuition.

Remark 8.2. Given a simplicial T-algebra Z, a good way to control $E_{\circledast}|Z|$ is to control its spiral spectral sequence. More to the point, the easiest way to ensure that $|Z|$ be a realization is to demand that $\mathrm{E}^2 = \pi_* E_{\circledast}^{\mathrm{lw}} Z \cong \pi_0 E_{\circledast}^{\mathrm{lw}} Z \cong A$, so that the spectral sequence collapses immediately.

However, it is not so straightforward to obtain such an object or understand its automorphisms: the E^2 page consists of *classical E*-homology groups, but it is the *natural E*-homology groups that are more closely connected to the actual homotopy theory of the ∞-category $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$.

Luckily, however, we have a tool that relates these two types of *E*-homology groups: the localized spiral exact sequence. As it is one-third classical and two-thirds natural, it allows us to exert control over the classical *E*-homology groups by manipulating the natural *E*-homology groups.

Thus, our method will be to attempt to interpolate one stage at a time from

- objects which are easy to understand (read: have controlled natural E-homology) but do not have the correct E^2 pages (read: have the wrong classical E-homology), towards
- objects which are somewhat more difficult to understand (read: have more complicated natural E-homology) but have E^2 pages which are closer and closer to collapsing at A (read: their classical E-homology is equivalent to A itself (concentrated in degree 0) in an increasingly large range).

Of course, such interpolations will not always be possible, but in the course of our attempt we will discover the precise cohomological obstructions to their possibility.

We now define certain objects which, via geometric realization, provide approximations to realizations.

Definition 8.3. For $0 \leq n \leq \infty$, we say that an object $Z \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!]_{Y/}$ is an n-**stage** if the following conditions hold:

(1) there exists an isomorphism $\pi_0 E_*^{\mathrm{lw}} Z \cong A$ in $\mathrm{Alg}_\Phi(\tilde{\mathcal{A}})_{k/}$;
(2) $\pi_{>n,*}^\natural Z = 0$; and
(3) $\pi_i E_*^{\mathrm{lw}} Z = 0$ for $1 \leq i \leq n+1$.

We write

$$\mathscr{M}_n(A/Y) \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!]_{Y/}$$

for the moduli space of n-stages (and E_*-equivalences between them).

Observation 8.4. Suppose that $Z \in \mathscr{M}_n(A/Y)$. By condition (3), the tail end of the localized spiral exact sequence degenerates into a sequence of isomorphisms. By induction, this implies that $E_{i,*}^\natural Z \cong \Omega^i A$ for all $i \leq n$: the base case of $i = 0$ follows from condition (1) and Lemma 5.26. Then, after a colimit argument, condition (2) implies that we have an isomorphism $\pi_{n+2} E_*^{\mathrm{lw}} Z \xrightarrow{\cong} \Omega(E_{n,*}^\natural Z)$ and that $\pi_{>n+2} E_*^{\mathrm{lw}} Z = 0$. The table of Fig. 1 summarizes these computations. Moreover, the same argument shows that if $n = \infty$ then $E_{i,*}^\natural Z \cong \Omega^i A$ for all $i \geq 0$ and that $\pi_* E_*^{\mathrm{lw}} Z \cong \pi_0 E_*^{\mathrm{lw}} Z \cong A$.

We now provide the connection between realizations and n-stages.

Theorem 8.5. *Geometric realization induces an equivalence*

$$\mathscr{M}_\infty(A/Y) \xrightarrow{\sim} \mathscr{M}_{A/Y}.$$

i	0	1	2	$\cdots$	$n-1$	n	$n+1$	$n+2$	$n+3$	$\cdots$
$\pi_i E^{\mathrm{lw}}_* Z$	A	0	0	$\cdots$	0	0	0	$\Omega^{n+1}A$	0	$\cdots$
$E^\flat_{i,*} Z$	A	ΩA	$\Omega^2 A$	$\cdots$	$\Omega^{n-1}A$	$\Omega^n A$	0	0	0	$\cdots$

Fig. 1. The classical and natural E-homology groups of an n-stage $Z \in \mathscr{M}_n(A/Y)$.

Proof. The adjunction $|-| : \mathrm{Alg}_T(s\mathcal{C}) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) :$ const evidently descends (or perhaps rather restricts) to an adjunction $|-| : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!] \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathrm{L}_E(\mathcal{C})) :$ const by the universal property of localization. In turn, the spiral spectral sequence implies that (after taking undercategories of Y) this latter adjunction restricts to give the desired equivalence. $\quad\square$

Remark 8.6. Note that commutative square

$$
\begin{array}{ccc}
\mathscr{M}_\infty(A/Y) & \longrightarrow & \mathscr{M}_{A/Y} \\
\downarrow & & \downarrow \\
\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]_{Y/} & \xrightarrow{\;|-|\;} & \mathrm{Alg}_{\mathcal{O}}(\mathrm{L}_E(\mathcal{C}))_{Y/}
\end{array}
$$

is *not* generally a pullback. Rather, as alluded to in Remark 8.2, an ∞-stage is exactly an object whose spiral spectral sequence has $\mathrm{E}^2 = \pi_* E^{\mathrm{lw}}_* X \cong \pi_0 E^{\mathrm{lw}}_* \cong A$, so that in particular it collapses immediately.

Assumption 8.7. We assume that Postnikov towers in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]$ converge.

Remark 8.8. Assumption 8.7 is a key technical assumption, whose necessity was first observed by Pstragowski–VanKoughnett [61, Remark 6.11]; they also proved that it holds e.g. in the case that E is Morava E-theory [61, Theorem 7.4]. Note that we are applying Postnikov theory in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!]$ to study objects in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]$, and it is not a priori clear that the functor $\mathrm{L}_{E^{\mathrm{lw}}_*} : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{\mathrm{res}}]\!] \to \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]$ preserves n-truncatedness.

Theorem 8.9. *For any $0 \leq n \leq m \leq \infty$, the n-truncation functor*

$$
\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!] \xrightarrow{P^{\mathrm{top}}_n} \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}^{-1}_{E^{\mathrm{lw}}_*}]\!]
$$

induces a map

$$
\mathscr{M}_m(A/Y) \to \mathscr{M}_n(A/Y),
$$

and these assemble to give an equivalence

$$\mathscr{M}_\infty(A/Y) \xrightarrow{\sim} \lim\left(\cdots \xrightarrow{P_2^{\mathrm{top}}} \mathscr{M}_2(A/Y) \xrightarrow{P_1^{\mathrm{top}}} \mathscr{M}_1(A/Y) \xrightarrow{P_0^{\mathrm{top}}} \mathscr{M}_0(A/Y) \right).$$

Proof. First of all, it is immediate from the localized spiral exact sequence that the n-truncation of an m-stage is an n-stage. From here, by Assumption 8.7, the asserted equivalence follows from an (∞-categorical but otherwise) identical argument to that of [19, 4.6]. $\square$

Theorem 8.10. *The functor*

$$\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!] \xrightarrow{\pi_0 E_*^{\mathrm{lw}}} \mathrm{Alg}_\Phi(\tilde{\mathcal{A}})$$

induces an equivalence

$$\mathscr{M}_0(A/Y) \xrightarrow{\sim} \mathscr{M}_{A/k}.$$

Proof. Inspection of the definitions reveals an equivalence $\mathscr{M}_0(A/Y) \simeq \mathscr{M}_Y(A)$ with the moduli space of objects under Y of type B_A, and from here the claim follows from Proposition 7.12. $\square$

8.2. Climbing the tower

We now come to the essential result, which explains how to move up the tower of moduli spaces.

Theorem 8.11. *For any $n \geq 1$, there is a natural pullback square*

$$\begin{array}{ccc} \mathscr{M}_n(A/Y) & \longrightarrow & B\mathrm{Aut}_k(A, \Omega^n A) \\ {\scriptstyle P_{n-1}^{\mathrm{top}}}\downarrow & & \downarrow \\ \mathscr{M}_{n-1}(A/Y) & \longrightarrow & \widehat{\mathscr{H}}_{\tilde{T}_E}^{n+2}(A/k; \Omega^n A) \end{array}$$

in $\mathcal{S}$.

In order to prove this, we will first develop an understanding of the object-by-object passage between $(n-1)$-stages and n-stages, and then we will analyze how this behaves in families.

Observation 8.12. Directly from the definitions, topological Eilenberg–Mac Lane objects are local with respect to the left localization adjunction $\mathrm{L}_{E_*^{\mathrm{lw}}} : \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!] \rightleftarrows \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!] : \mathrm{U}_{E_*^{\mathrm{lw}}}$. We use this fact without further comment.

Notation 8.13. Since we are working in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!]$ (as opposed to $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$), we henceforth simply omit the localization functor $\mathrm{L}_{E_*^{\mathrm{lw}}}$ from the notation.

Observation 8.14. Suppose first that $Z \in \mathscr{M}_n(A/Y)$. Then $P_{n-1}^{\mathrm{top}}(Z) \in \mathscr{M}_{n-1}(A/Y)$ by Theorem 8.9, and moreover Proposition 7.8(3) implies that we have a pullback square

$$
\begin{array}{ccc}
Z & \longrightarrow & B_A \\
{\scriptstyle \tau_{n-1}^{\mathrm{top}}} \downarrow & & \downarrow \\
P_{n-1}^{\mathrm{top}}(Z) & \longrightarrow & B_A(\Omega^n A, n+1)
\end{array}
$$

in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_{\circledast}^{\mathrm{lw}}}^{-1}]\!]$.

Let us attempt to reverse this process. Suppose that $W \in \mathscr{M}_{n-1}(A/Y)$, and suppose that we form a pullback

$$
\begin{array}{ccc}
\widetilde{W} & \longrightarrow & B_A \\
\downarrow & & \downarrow \\
W & \underset{\varphi}{\longrightarrow} & B_A(\Omega^n A, n+1)
\end{array}
$$

in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_{\circledast}^{\mathrm{lw}}}^{-1}]\!]$. Then, $\widetilde{W} \in \mathscr{M}_n(A/Y)$ if and only if the induced composite

$$
E_{\circledast}^{\mathrm{lw}} W \xrightarrow{E_{\circledast}(\varphi)} E_{\circledast}^{\mathrm{lw}}(B_A(\Omega^n A, n+1)) \to K_A(\Omega^n A, n+1)
$$

with the universal map is an equivalence in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$: this follows from the long exact sequence in classical E-homology induced by a pullback square.

Observation 8.15. We can interpret the conclusion of Observation 8.14 as follows. By Observation 8.4, the object $E_{\circledast}^{\mathrm{lw}} W \in \mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ has homotopy concentrated in degrees 0 and $n+1$ and moreover $P_n^{\mathrm{alg}}(E_{\circledast}^{\mathrm{lw}} W) \simeq A$. By Proposition 6.18, this object therefore corresponds to a unique pullback square

$$
\begin{array}{ccc}
E_{\circledast}^{\mathrm{lw}} W & \longrightarrow & K_A \\
\downarrow & & \downarrow \\
A & \underset{\chi}{\longrightarrow} & K_A(\Omega^n A, n+2)
\end{array}
$$

in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$.

Recall from Observation 6.4 that we have a pullback square

$$
\begin{array}{ccc}
K_A(\Omega^n A, n+1) & \longrightarrow & K_A \\
\downarrow & & \downarrow \\
K_A & \longrightarrow & K_A(\Omega^n A, n+2)
\end{array}
$$

in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$. Now, we claim that there exists an equivalence $E_*^{\mathrm{lw}}W \xrightarrow{\sim} K_A(\Omega^n A, n+1)$ in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]$ if and only if χ represents the zero element $0 \in H_{\tilde{T}_E}^{n+2}(A/k; \Omega^n A)$.

- Indeed, if $[\chi] = 0$, then the existence of an equivalence is manifest.
- Conversely, if such an equivalence exists, then by Proposition 6.18 there exists an equivalence between these two pullback squares, implying that $[\chi] = 0$.

Thus, the obstructions to a given $(n-1)$-stage lifting to an n-stage are given by elements of $H_{\tilde{T}_E}^{n+2}(A/k; \Omega^n A)$. In particular, if this group vanishes then *every* $(n-1)$-stage lifts to an n-stage.

We now provide the key piece of input to the proof of Theorem 8.11: in effect, we work with $\mathscr{M}_{n-1}(A/Y)$ one path component at a time.

Notation 8.16. For any $Z \in \mathscr{M}_{n-1}(A/Y)$, we write $\mathscr{M}_{n/Z}(A/Y) \subset \mathscr{M}_n(A/Y)$ for the subspace of those n-stages $W \in \mathscr{M}_n(A/Y)$ such that there exists an equivalence $P_{n-1}^{\mathrm{top}}(W) \simeq Z$ in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_*^{\mathrm{lw}}}^{-1}]\!]_{Y/}$.

Observation 8.17. Note that the space $\mathscr{M}_{n/Z}(A/Y)$ may well be empty; indeed, by Observation 8.15 it will be empty if and only if $\mathscr{M}_k(E_*^{\mathrm{lw}}Z \hookrightarrow K_A(\Omega^n A, n+1))$ is empty.

Notation 8.18. For any $Z \in \mathscr{M}_{n-1}(A/Y)$, we write $Z \xrightarrow{\approx} B_A(\Omega^n A, n)$ for a morphism in $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]_{Y/}$ which classifies an equivalence $E_*^{\mathrm{lw}}Z \xrightarrow{\sim} K_A(\Omega^n A, n)$ in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}_{\pi_*}^{-1}]\!]_{k/}$.

Lemma 8.19. *Suppose that* $Z \in \mathscr{M}_{n-1}(A/Y)$ *for some* $n \geq 1$. *Then there is a natural pullback square*

$$
\begin{array}{ccc}
\mathscr{M}_{n/Z}(A/Y) & \longrightarrow & \mathscr{M}_k(E_*^{\mathrm{lw}}Z \hookrightarrow K_A(\Omega^n A, n+1) \hookleftarrow K_A) \\
{\scriptstyle P_{n-1}^{\mathrm{top}}}\Big\downarrow & & \Big\downarrow \\
\mathscr{M}_Y(Z) & \xrightarrow[\quad E_*^{\mathrm{lw}} \quad]{} & \mathscr{M}_k(E_*^{\mathrm{lw}}Z)
\end{array}
$$

in $\mathcal{S}$.

Proof. The difference construction provides a map $\mathscr{M}_{n/Z}(A/Y) \to \mathscr{M}_Y(Z \xrightarrow{\approx} B_A(\Omega^n A, n+1) \hookleftarrow B_A)$, which is an equivalence by Observation 8.14. Thus we obtain a commutative diagram

$$\mathscr{M}_{n/Z}(A/Y) \xrightarrow{\sim} \mathscr{M}_Y(Z \xrightarrow{\approx} B_A(\Omega^n A, n+1) \hookleftarrow B_A) \longrightarrow \mathscr{M}_k(E_\ast^{\mathrm{lw}} Z \hookrightarrow K_A(\Omega^n A, n+1) \hookleftarrow K_A)$$

$$\downarrow{\scriptstyle P_{n-1}^{\mathrm{top}}} \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\mathscr{M}_Y(Z) \xrightarrow[\sim]{} \mathscr{M}_Y(Z) \xrightarrow[E_\ast^{\mathrm{lw}}]{} \mathscr{M}_k(E_\ast^{\mathrm{lw}} Z)$$

in $\mathcal{S}$, in which

- the right square is obtained by applying $E_\ast^{\mathrm{lw}}$ and using the universal characterization of topological Eilenberg–Mac Lane objects,
- the left square is tautologically a pullback, and
- our goal is to show that the outer rectangle is a pullback;

thus, it suffices to show that the right square is a pullback.

In the right square, both downwards maps are obtained by forgetting certain data: a morphism of type $\vec{B}_A(\Omega^n A, n+1)$ on the left, and a morphism of type $\vec{K}_A(\Omega^n A, n+1)$ on the right. Thus, it is convenient to use the equivalence $\mathscr{M}_{A/Y}(\Omega^n A, n+1) \xrightarrow{\sim} \mathscr{M}_{A/k}(\Omega^n A, n+1)$ of Proposition 7.12 (between the moduli spaces of such Eilenberg–Mac Lane morphisms) to obtain a larger commutative square

$$\mathscr{M}_Y(Z \xrightarrow{\approx} B_A(\Omega^n A, n+1) \hookleftarrow B_A) \longrightarrow \mathscr{M}_k(E_\ast^{\mathrm{lw}} Z \hookrightarrow K_A(\Omega^n A, n+1) \hookleftarrow K_A)$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\mathscr{M}_Y(Z) \times \mathscr{M}_{A/Y}(\Omega^n A, n+1) \longrightarrow \mathscr{M}_k(E_\ast^{\mathrm{lw}} Z) \times \mathscr{M}_{A/k}(\Omega^n A, n+1)$$

which it then suffices to show is a pullback.

Now, observe that both spaces on the bottom row are connected (by definition and by Propositions 6.26 and 7.12). So for any basepoint of $\mathscr{M}_Y(Z) \times \mathscr{M}_{A/Y}(\Omega^n A, n+1)$, it suffices to check that the induced map on fibers is an equivalence. Unwinding the definitions, we see that this is the map

$$\hom_{\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_\ast^{\mathrm{lw}}}^{-1}]\!]}^{\approx}(Z, B_A(\Omega^n A, n+1)) \to \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{A})[\![\mathbf{W}_{\pi_\ast}^{-1}]\!]}^{\sim}(E_\ast^{\mathrm{lw}} Z, K_A(\Omega^n A, n+1)).$$

As $\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{E_\ast^{\mathrm{lw}}}^{-1}]\!] \subset \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$ is a full subcategory, we see that this is by definition an equivalence of subspaces of the equivalence

$$\hom_{\mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]}(Z, B_A(\Omega^n A, n+1)) \xrightarrow{\sim} \hom_{\mathrm{Alg}_{\tilde{T}_E}(s\tilde{A})[\![\mathbf{W}_{\pi_\ast}^{-1}]\!]}(E_\ast^{\mathrm{lw}} Z, K_A(\Omega^n A, n+1))$$

characterizing the object $B_A(\Omega^n A, n+1) \in \mathrm{Alg}_T(s\mathcal{C})[\![\mathbf{W}_{\mathrm{res}}^{-1}]\!]$. $\quad\square$

We can now prove our main decomposition theorem.

Proof of Theorem 8.11. We begin with the commutative square

$$
\begin{array}{ccc}
\mathscr{M}_k(K_A(\Omega^n A, n+1) \hookleftarrow K_A) & \xrightarrow{\ \sim\ } & \mathscr{M}_k(K_A(\Omega^n, n+2) \hookleftarrow K_A) \\
\downarrow & & \downarrow \\
\mathscr{M}_k(K_A \oplus (\Omega^n A, n+1)) & \xrightarrow{\ \sim\ } & \mathscr{M}_k(K_A \hookrightarrow K_A(\Omega^n A, n+2) \hookleftarrow K_A)
\end{array}
$$

in $\mathcal{S}$, in which

- the upper horizontal map is (the inverse of) the equivalence of Proposition 6.21,
- the left vertical map is forgetful,
- the right vertical map repeats the given morphism,
- the lower horizontal map is the equivalence of Proposition 6.18.

This is tautologically a pullback square.

Now, suppose that $Z \in \mathscr{M}_{n-1}(A/Y)$. We claim that there exists a pullback square

$$
\begin{array}{ccc}
\mathscr{M}_{n/Z}(A/Y) & \longrightarrow & \mathscr{M}_k(K_A(\Omega^n A, n+2) \hookleftarrow K_A) \\
\downarrow & & \downarrow \\
\mathscr{M}_Y(Z) & \longrightarrow & \mathscr{M}_k(K_A \hookrightarrow K_A(\Omega^n A, n+2) \hookleftarrow K_A)
\end{array}
$$

in $\mathcal{S}$. To see this, we separate the argument into two cases, depending on whether or not there exists an equivalence $E^{\mathrm{lw}}_{\circledast} Z \xrightarrow{\sim} K_A(\Omega^n A, n+1)$ in $\mathrm{Alg}_{\tilde{T}_E}(s\tilde{\mathcal{A}})[\![\mathbf{W}^{-1}_{\pi_*}]\!]$.

- Suppose that no such equivalence exists. Then $\mathscr{M}_{n/Z}(A/Y)$ is empty by Observation 8.17. In this case, the subspace $\mathscr{M}_k(E^{\mathrm{lw}}_{\circledast} Z) \subset \mathscr{M}_k(K_A \oplus (\Omega^n A, n+1))$ is not in the image of the left vertical map of our original tautological pullback square. These facts imply that the above square is indeed (equally tautologically) a pullback.
- Suppose that such an equivalence exists. In this case, we obtain an evident forgetful equivalence

$$
\mathscr{M}_k(E^{\mathrm{lw}}_{\circledast} Z \hookrightarrow K_A(\Omega^n A, n+1) \hookleftarrow K_A) \xrightarrow{\sim} \mathscr{M}_k(K_A(\Omega^n A, n+1) \hookleftarrow K_A)
$$

in $\mathcal{S}$, which reduces the pullback square of Lemma 8.19 to a pullback square

$$
\begin{array}{ccc}
\mathscr{M}_{n/Z}(A/Y) & \longrightarrow & \mathscr{M}_k(K_A(\Omega^n A, n+1) \hookleftarrow K_A) \\
\downarrow & & \downarrow \\
\mathscr{M}_Y(Z) & \longrightarrow & \mathscr{M}_k(K_A(\Omega^n A, n+1)).
\end{array}
$$

The right vertical arrow of this pullback square includes as a subobject of the left vertical arrow of our original tautological pullback square, yielding the claim.

Now, assembling this pullback square over all $Z \in \mathscr{M}_{n-1}(A/Y)$, we obtain a pullback square

$$
\begin{array}{ccc}
\mathscr{M}_n(A/Y) & \longrightarrow & \mathscr{M}_k(K_A(\Omega^n, n+2) \looparrowleft K_A) \\
\downarrow & & \downarrow \\
\mathscr{M}_{n-1}(A/Y) & \longrightarrow & \mathscr{M}_k(K_A \looparrowright K_A(\Omega^n A, n+2) \looparrowleft K_A).
\end{array}
$$

From here, the equivalence

$$
\mathscr{M}_k(K_A(\Omega^n A, n+2) \looparrowleft K_A) = \mathscr{M}_{A/k}(\Omega^n A, n+2) \simeq B\mathrm{Aut}_k(A, \Omega^n A)
$$

of Proposition 6.26 and the equivalence

$$
\mathscr{M}_k(K_A \looparrowright K_A(\Omega^n A, n+2) \looparrowleft K_A) \simeq \widehat{\mathscr{H}}_{\tilde{T}_E}^{n+2}(A/k; \Omega^n A)
$$

of Corollary 6.30 allow us to rewrite this as a pullback square

$$
\begin{array}{ccc}
\mathscr{M}_n(A/Y) & \longrightarrow & B\mathrm{Aut}_k(A, \Omega^n A) \\
\downarrow & & \downarrow \\
\mathscr{M}_{n-1}(A/Y) & \longrightarrow & \widehat{H}_{\tilde{T}_E}^{n+2}(A/k; \Omega^n A),
\end{array}
$$

which completes the proof. $\square$

References

[1] M. Andrews, New families in the homotopy of the motivic sphere spectrum, available at http://math.mit.edu/~mjandr/Self_Map.pdf.
[2] P. Balmer, Spectra, spectra, spectra—tensor triangular spectra versus Zariski spectra of endomorphism rings, Algebraic Geom. Topol. 10 (3) (2010) 1521–1563.
[3] H.O. Bayindir, Topological equivalences of E-infinity differential graded algebras, available at arXiv:1706.08554.
[4] D. Blanc, W.G. Dwyer, P.G. Goerss, The realization space of a Π-algebra: a moduli problem in algebraic topology, Topology 43 (4) (2004) 857–892.
[5] T. Bachmann, M. Hoyois, Norms in motivic homotopy theory, available at arXiv:1711.03061.
[6] A.J. Blumberg, M.A. Hill, g-symmetric monoidal categories of modules over equivariant commutative ring spectra, available at arXiv:1511.07363.
[7] A.J. Blumberg, M.A. Hill, Incomplete Tambara functors, available at arXiv:1603.03292.
[8] A.J. Blumberg, M.A. Hill, Operadic multiplications in equivariant spectra, norms, and transfers, Adv. Math. 285 (2015) 658–708, MR 3406512.
[9] C. Berger, I. Moerdijk, Axiomatic homotopy theory for operads, Comment. Math. Helv. 78 (4) (2003) 805–831.
[10] S. Borghesi, Algebraic Morava K-theories, Invent. Math. 151 (2) (2003) 381–413.

[11] A.K. Bousfield, The localization of spectra with respect to homology, Topology 18 (4) (1979) 257–281.
[12] A.K. Bousfield, Cosimplicial resolutions and homotopy spectral sequences in model categories, Geom. Topol. 7 (2003) 1001–1053 (electronic).
[13] P. Bonventre, L.A. Pereira, Genuine equivariant operads, available at arXiv:1707.07226.
[14] J.M. Boardman, R.M. Vogt, Homotopy Invariant Algebraic Structures on Topological Spaces, Lecture Notes in Mathematics, vol. 347, Springer-Verlag, Berlin-New York, 1973.
[15] H. Chu, R. Haugseng, Enriched ∞-operads, available at arXiv:1707.08049.
[16] D.-C. Cisinski, Higher Categories and Homotopical Algebra, Cambridge Studies in Advanced Mathematics, vol. 180, Cambridge University Press, Cambridge, 2019.
[17] C. Douglas, J. Francis, A.G. Henriques, M.A. Hill (Eds.), Topological Modular Forms, Mathematical Surveys and Monographs, vol. 201, American Mathematical Society, Providence, RI, 2014, with contributions by Vigleik Angeltveit, Tilman Bauer, Mark Behrens, Christopher L. Douglas, André G. Henriques, Michael A. Hill, Henning Hohnhold, Michael J. Hopkins, Jacob Lurie, Mark Mahowald, Carl Mautner, Haynes R. Miller, and Corbett Redden.
[18] W.G. Dwyer, D.M. Kan, Simplicial localizations of categories, J. Pure Appl. Algebra 17 (3) (1980) 267–284.
[19] W.G. Dwyer, D.M. Kan, A classification theorem for diagrams of simplicial sets, Topology 23 (2) (1984) 139–155.
[20] W.G. Dwyer, D.M. Kan, C.R. Stover, An E^2 model category structure for pointed simplicial spaces, J. Pure Appl. Algebra 90 (2) (1993) 137–152.
[21] W.G. Dwyer, D.M. Kan, C.R. Stover, The bigraded homotopy groups $\pi_{i,j}X$ of a pointed simplicial space X, J. Pure Appl. Algebra 103 (2) (1995) 167–188.
[22] D. Gepner, R. Haugseng, Enriched ∞-categories via non-symmetric ∞-operads, available at arXiv:1312.3178.
[23] P.G. Goerss, M.J. Hopkins, Moduli problems for structured ring spectra, available at www.math.northwestern.edu/~pgoerss.
[24] P.G. Goerss, M.J. Hopkins, Moduli spaces of commutative ring spectra, in: Structured Ring Spectra, in: London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, 2004, pp. 151–200.
[25] B. Gheorghe, Exotic motivic periodicities, available at arXiv:1709.00915.
[26] D. Gepner, R. Haugseng, T. Nikolaus, Lax colimits and free fibrations in ∞-categories, available at arXiv:1501.02161.
[27] B. Gheorghe, D.C. Isaksen, A. Krause, N. Ricka, $\mathbb{C}$-motivic modular forms, available at arXiv:1810.11050.
[28] P.G. Goerss, J.F. Jardine, Simplicial Homotopy Theory, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 2009, Reprint of the 1999 edition, MR1711612.
[29] D. Gepner, T. Lawson, Brauer groups and Galois cohomology of commutative ring spectra, available at arXiv:1607.01118.
[30] J.J. Gutiérrez, D. White, Encoding equivariant commutativity via operads, Algebraic Geom. Topol. 18 (5) (2018) 2919–2962.
[31] M.A. Hill, M.J. Hopkins, Equivariant symmetric monoidal structures, available at arXiv:1610.03114.
[32] P. Hu, I. Kriz, Some remarks on real and algebraic cobordism, K-Theory 22 (4) (2001) 335–366.
[33] M. Hill, T. Lawson, Topological modular forms with level structure, Invent. Math. 203 (2) (2016) 359–416.
[34] J. Heller, K. Ormsby, Primes and fields in stable motivic homotopy theory, available at arXiv:1608.02876.
[35] J. Hornbostel, Chromatic motivic homotopy theory, available at https://faculty.math.illinois.edu/K-theory/0642/Motchrom2.pdf.
[36] J. Hornbostel, Preorientations of the derived motivic multiplicative group, Algebraic Geom. Topol. 13 (5) (2013) 2667–2712.
[37] J. Hornbostel, Some comments on motivic nilpotence, Trans. Am. Math. Soc. 370 (4) (2018) 3001–3015.
[38] M. Hovey, Homotopy theory of comodules over a Hopf algebroid, in: Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-Theory, in: Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 261–304.
[39] M. Hoyois, From algebraic cobordism to motivic cohomology, J. Reine Angew. Math. 702 (2015) 173–226.
[40] D.C. Isaksen, The homotopy of $\mathbb{C}$-motivic modular forms, available at arXiv:1811.07937.

[41] D.C. Isaksen, The cohomology of motivic $A(2)$, Homol. Homotopy Appl. 11 (2) (2009) 251–274.
[42] R. Joachimi, Thick ideals in equivariant and motivic stable homotopy categories, available at arXiv: 1503.08456.
[43] T. Lawson, Calculating obstruction groups for E_∞ ring spectra, available at arXiv:1709.09629.
[44] M. Levine, F. Morel, Algebraic Cobordism, Springer Monographs in Mathematics, Springer, Berlin, 2007.
[45] T. Lawson, N. Naumann, Commutativity conditions for truncated Brown-Peterson spectra of height 2, available at arXiv:1101.3897.
[46] J. Lurie, Higher Topos Theory, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009, also available at http://math.harvard.edu/~lurie and at arXiv:math/0608040, v4.
[47] J. Lurie, Higher algebra, available at http://www.math.harvard.edu/~lurie, version dated September 14, 2014.
[48] J.P. May, The Geometry of Iterated Loop Spaces, Lectures Notes in Mathematics, vol. 271, Springer-Verlag, Berlin-New York, 1972.
[49] D. McDuff, On the classifying spaces of discrete monoids, Topology 18 (4) (1979) 313–320.
[50] A. Mazel-Gee, All about the Grothendieck construction, available at arXiv:1510.03525.
[51] A. Mazel-Gee, Hammocks and fractions in relative ∞-categories, available at arXiv:1510.03961.
[52] A. Mazel-Gee, $\mathbb{E}_\infty$ automorphisms of motivic Morava E-theories, in press.
[53] A. Mazel-Gee, Model ∞-categories I: some pleasant properties of the ∞-category of simplicial spaces, available at arXiv:1412.8411.
[54] A. Mazel-Gee, Model ∞-categories II: Quillen adjunctions, available at arXiv:1510.04392.
[55] A. Mazel-Gee, Model ∞-categories III: the fundamental theorem, available at arXiv:1510.04777.
[56] A. Mazel-Gee, The universality of the Rezk nerve, available at arXiv:1510.03150.
[57] A. Mazel-Gee, Goerss-Hopkins obstruction theory via model infinity-categories, ProQuest LLC, Ann Arbor, MI, 2016, Thesis (Ph.D.)–University of California, Berkeley, available at etale.site/writing/ GHOsT-thesis.pdf.
[58] N. Naumann, M. Spitzweck, P.A. Østvær, Chern classes, K-theory and Landweber exactness over nonregular base schemes, in: Motives and Algebraic Cycles, in: Fields Inst. Commun., vol. 56, Amer. Math. Soc., Providence, RI, 2009, pp. 307–317.
[59] N. Naumann, M. Spitzweck, P.A. Østvær, Motivic Landweber exactness, Doc. Math. 14 (2009) 551–593.
[60] I. Panin, K. Pimenov, O. Röndigs, A universality theorem for Voevodsky's algebraic cobordism spectrum, Homol. Homotopy Appl. 10 (2) (2008) 211–226.
[61] P. Pstragowski, P. VanKoughnett, Abstract Goerss–Hopkins theory, available at arXiv:1904.08881.
[62] D.G. Quillen, Homotopical Algebra, Lecture Notes in Mathematics, vol. 43, Springer-Verlag, Berlin-New York, 1967.
[63] D.C. Ravenel, Complex Cobordism and Stable Homotopy Groups of Spheres, Pure and Applied Mathematics, vol. 121, Academic Press, Inc., Orlando, FL, 1986.
[64] C.W. Rezk, Spaces of algebra structures and cohomology of operads, ProQuest LLC, Thesis (Ph.D.), Massachusetts Institute of Technology, Ann Arbor, MI, 1996.
[65] C. Rezk, Notes on the Hopkins-Miller theorem, in: Homotopy Theory via Algebraic Geometry and Group Representations, Evanston, IL, 1997, in: Contemp. Math., vol. 220, Amer. Math. Soc., Providence, RI, 1998, pp. 313–366.
[66] N. Ricka, Motivic modular forms from equivariant stable homotopy theory, available at arXiv: 1704.04547.
[67] J. Rognes, Galois Extensions of Structured Ring Spectra. Stably Dualizable Groups, Mem. Amer. Math. Soc., vol. 192, 2008, no. 898, viii+137.
[68] F. Roth, Galois and Hopf–Galois theory for associative S-algebras, PhD thesis.
[69] J. Rubin, On the realization problem for N_∞ operads, available at arXiv:1705.03585.
[70] M. Szymik, Crystals and derived local moduli for ordinary K3 surfaces, Adv. Math. 228 (1) (2011) 1–21.
[71] G. Vezzosi, Brown-Peterson spectra in stable $\mathbb{A}^1$-homotopy theory, Rend. Semin. Mat. Univ. Padova 106 (2001) 47–64.
[72] V. Voevodsky, $\mathbf{A}^1$-homotopy theory, in: Proceedings of the International Congress of Mathematicians, vol. I, Berlin, 1998, 1998, pp. 579–604.
[73] C. Westerland, A higher chromatic analogue of the image of J, Geom. Topol. 21 (2) (2017) 1033–1093.