ALL ABOUT THE GROTHENDIECK CONSTRUCTION

AARON MAZEL-GEE

ABSTRACT. We provide, among other things: (i) a Bousfield–Kan formula for colimits in ∞ -categories (generalizing the 1-categorical formula for a colimit as a coequalizer of maps between coproducts); (ii) ∞ -categorical generalizations of Barwick–Kan's Theorem B_n and Dwyer–Kan–Smith's Theorem C_n (regarding homotopy pullbacks in the Thomason model structure, which themselves vastly generalize Quillen's Theorem B); and (iii) an articulation of the simultaneous and interwoven functoriality of colimits (or dually, of limits) for natural transformations and for pullback along maps of diagram ∞ -categories.

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0. INTRODUCTION

0.1. **Outline.** As the title and abstract suggest, this is essentially an omnibus paper in which we collect a number of useful results in ∞ -category theory, all having to do in some way or another with the *Grothendieck construction*. This is instantiated in quasicategories by Lurie's *unstraightening construction*, although we work model-independently (see §0.2).

- In §1, we fix some notation and terminology surrounding the Grothendieck construction. We also give some examples, and we highlight some of its important features notably its *naturality* (as proved by Gepner–Haugseng–Nikolaus).
- In §2, we explore the relationship between the Grothendieck construction (and its "two-sided" generalization) and colimits in the ∞ -category of spaces. For instance, we record an ∞ -categorical version of Thomason's *homotopy colimit theorem*. Some of these results are suggestive of the (∞ , 2)-categorical functoriality of the Grothendieck construction (e.g. we say the word "modification").
- In §3, we define *lax* and *oplax* natural transformations of functors C → Cat_∞ via the Grothendieck construction. Using these, we then construct a *global colimit functor* for a cocomplete ∞-category C: this is a functor

$$\mathcal{L}ax(\mathcal{C}) \to \mathcal{C}$$

from the *lax overcategory* of \mathcal{C}

– which sends an object

 $(\mathcal{D} \xrightarrow{F} \mathcal{C}) \in \mathcal{L}ax(\mathcal{C})$

to its colimit

 $\operatorname{colim}_{\mathcal{D}}(F) \in \mathcal{C},$

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and

- which encodes the simultaneous and intervoven functoriality of colimits in C
 - * for natural transformations that is, for maps in $\operatorname{Fun}(\mathcal{D}, \mathcal{C})$ and
 - * for pullback along maps of diagram ∞ -categories that is, for maps in $(Cat_{\infty})_{/C}$.

This immediately dualizes to give an analogous *global limit functor*

 $\operatorname{op}\mathcal{L}\operatorname{ax}(\mathcal{D})^{op} \xrightarrow{\lim} \mathcal{D}$

for a complete ∞ -category \mathcal{D} (now running from the *opposite* of its *oplax* overcategory).

- In §4, we prove ∞ -categorical versions of Barwick–Kan's **Theorem B**_n and Dwyer–Kan–Smith's **Theorem C**_n, thus further extending the following sequence of increasingly general results in 1-category theory.
 - Given a functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ satisfying a certain property B, Quillen's *Theorem B* gives a simple description of the fibers



of the induced map on $(\infty$ -)groupoid completions.

- Given a functor satisfying a certain property B_n (which recovers property B when n = 1 but becomes weaker as n grows),
 - * Dwyer-Kan-Smith's *Theorem* B_n gives a description of the fibers of the induced map on groupoid completions (which recovers Quillen's description when n = 1 but in trade becomes more complicated as n grows), while
 - * their *Theorem* C_n asserts that if \mathfrak{C} satisfies a certain property C_n , then any functor $\mathfrak{D} \to \mathfrak{C}$ satisfies property B_n .
- Given a cospan $\mathcal{D} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{E}$ such that F satisfies property B_n (but without any conditions on G), Barwick-Kan's *enhanced* Theorem B_n gives a description of the pullback



of the induced cospan on groupoid completions (which likewise becomes more complicated as n grows).

- In §5, we prove a **Bousfield–Kan formula** for colimits in ∞-categories. This generalizes the 1-categorical formula for a colimit as a coequalizer of maps between coproducts. We also illustrate its application with concrete examples.
- In §A, we construct a *Thomason model structure* on the ∞-category Cat_∞ of ∞-categories.¹ Aside from its intrinsic interest, this model ∞-category (Cat_∞)_{Th} provides a convenient language for the various results which appear throughout this paper: it gives a presentation of the ∞-category S of spaces, and moreover its localization functor

$$\operatorname{Cat}_{\infty} \to \operatorname{Cat}_{\infty} \llbracket \mathbf{W}_{\mathrm{Th}}^{-1} \rrbracket \simeq S$$

¹See [MGa, $\S1$] for the definition of a model structure on an ∞ -category.

can be canonically identified with the groupoid completion functor. In particular, the subcategory $\mathbf{W}_{Th} \subset Cat_{\infty}$ of Thomason weak equivalences consists of precisely those maps which become equivalences upon groupoid completion.

This model structure is analogous to the classical Thomason model structure on the category cat of categories; however, it is (in a sense) better behaved, and moreover it completely accounts for a certain quirk that prevents the latter from being lifted directly along the nerve functor. On the other hand, this model structure bears some rather surprising features of its own.

0.2. Conventions. Though it stands alone, this paper belongs to a series on *model* ∞ -categories. These papers share many key ideas; thus, rather than have the same results appear repeatedly in multiple places, we have chosen to liberally cross-reference between them. To this end, we introduce the following "code names".

title		code
Model ∞ -categories I: some pleasant properties of the ∞ -category of simplicial spaces		S
The universality of the Rezk nerve		Ν
All about the Grothendieck construction		G
Hammocks and fractions in relative ∞ -categories		Н
$Model \infty$ -categories II: Quillen adjunctions		Q
Model ∞ -categories III: the fundamental theorem		М

Thus, for instance, to refer to [MGf, Theorem 1.9], we will simply write Theorem M.1.9. (The letters are meant to be mnemonical: they stand for "simplicial space", "nerve", "Grothendieck", "hammock", "Quillen", and "model", respectively.)

We take quasicategories as our preferred model for ∞ -categories, and in general we adhere to the notation and terminology of [Lur09] and [Lur14]. In fact, our references to these two works will be frequent enough that it will be convenient for us to adopt Lurie's convention and use the code names T and A for them, respectively.²

However, we work invariantly to the greatest possible extent: that is, we primarily work within the ∞ -category of ∞ -categories. Thus, for instance, we will omit all technical uses of the word "essential", e.g. we will use the term unique in situations where one might otherwise say "essentially unique" (i.e. parametrized by a contractible space). For a full treatment of this philosophy as well as a complete elaboration of our conventions, we refer the interested reader to §S.A. The casual reader should feel free to skip this on a first reading; on the other hand, the careful reader may find it useful to peruse that section before reading the present paper. For the reader's convenience, we also provide a complete index of the notation that is used throughout this sequence of papers in §S.B.

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1. The Grothendieck construction

In this section, we recall some basic notions involving the Grothendieck construction and the various sorts of fibrations that it involves: we discuss co/cartesian (and left/right) fibrations in §1.1, and we discuss the Grothendieck construction itself in §1.2. For background, we refer the reader to our companion paper [MGj], which contains

• model-independent definitions of co/cartesian morphisms and co/cartesian fibrations,

 $^{^{2}}$ Due to the differences in their numbering systems, no ambiguity will arise between references to [Lur14] and to §A here.

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- proofs that these model-independent definitions are suitably compatible with their quasicategorical counterparts, and
- an extended informal discussion the Grothendieck construction.

1.1. **Fibrations.** We begin by fixing the following notation (without really giving any definitions).

Notation 1.1. Let \mathcal{C} be an ∞ -category.

- We denote by *L*Fib(C) the ∞-category of *left fibrations* over C; by the dual of Corollary T.2.2.3.12 (see Remark T.2.1.4.12), this is the underlying ∞-category of the covariant model structure of Proposition T.2.1.4.7, and is well-defined by Remark T.2.1.4.11.
- We denote by coCFib(C) the ∞-category of *cocartesian fibrations* over C; by the dual of Proposition T.3.1.4.1, this is the underlying ∞-category of the cocartesian model structure which is dual to that of Proposition T.3.1.3.7 (see Remark T.3.1.3.9), and is well-defined by Proposition T.3.3.1.1.

By Theorem T.3.1.5.1, these two ∞ -categories sit in a sequence of adjunctions

$$(\operatorname{Cat}_{\infty})_{\mathcal{C}} \xleftarrow{\overset{\operatorname{L}_{\operatorname{co}\mathcal{C}}\mathfrak{F}\mathrm{ib}(\mathcal{C})}{\longleftarrow}} \operatorname{co}\mathcal{C}\mathfrak{F}\mathrm{ib}(\mathcal{C}) \xleftarrow{\overset{\operatorname{L}_{\mathcal{L}}\mathfrak{F}\mathrm{ib}(\mathcal{C})}{\longleftarrow}} \mathcal{L}\mathfrak{F}\mathrm{ib}(\mathcal{C}) \xleftarrow{\overset{\operatorname{L}_{\mathcal{L}}(\mathcal{C})}{\longleftarrow}} \mathcal{S}_{/\mathfrak{C}^{\operatorname{gpd}}}$$

in $\mathfrak{C}at_\infty.^3$

Of course, there are dual notions of *right fibrations* and of *cartesian fibrations*, defined to be those functors that respectively become left fibrations or cocartesian fibrations under the involution $(-)^{op} : \operatorname{Cat}_{\infty} \to \operatorname{Cat}_{\infty}$. These then assemble into an analogous string of adjunctions

$$(\operatorname{Cat}_{\infty})_{/\operatorname{C}} \xleftarrow{\operatorname{L}_{\operatorname{CFib}(\operatorname{C})}}_{U_{\operatorname{CFib}(\operatorname{C})}} \operatorname{CFib}(\operatorname{C}) \xleftarrow{\operatorname{L}_{\operatorname{RFib}(\operatorname{C})}}_{U_{\operatorname{RFib}(\operatorname{C})}} \operatorname{RFib}(\operatorname{C}) \xleftarrow{\operatorname{L}_{\operatorname{R}(\operatorname{C})}}_{U_{\operatorname{R}(\operatorname{C})}} \operatorname{S}_{/\operatorname{Cgpd}}$$

in Cat_{∞} . By taking opposites, it will often suffice to leave observations about these latter notions implicit.

We now assemble a number of useful observations.

Remark 1.2. The adjunctions $L_{\mathcal{LFib}(\mathcal{C})} \dashv U_{\mathcal{LFib}(\mathcal{C})}$ and $L_{\mathcal{L}(\mathcal{C})} \dashv U_{\mathcal{L}(\mathcal{C})}$ are both left localization adjunctions.⁴ However, the adjunction $L_{co\mathcal{CFib}(\mathcal{C})} \dashv U_{co\mathcal{CFib}(\mathcal{C})}$ is not: given two objects $(\mathcal{D} \xrightarrow{p} \mathcal{C}), (\mathcal{E} \xrightarrow{q} \mathcal{C}) \in co\mathcal{CFib}(\mathcal{C})$, a morphism



in $(\operatorname{Cat}_{\infty})_{/\mathbb{C}}$ must take *p*-cocartesian morphisms in \mathcal{D} to *q*-cocartesian morphisms in \mathcal{E} in order to define a morphism in $\operatorname{coCFib}(\mathbb{C})$. (However, the right adjoint $U_{\operatorname{coCFib}(\mathbb{C})}$ is nevertheless the inclusion of a (non-full) subcategory, as indicated.) In §3, we will see that its failure to be a left localization gives rise to the notion of a *lax natural transformation* between objects of Fun($\mathcal{C}, \operatorname{Cat}_{\infty}$).

Remark 1.3. Given any functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$, the resulting horizontal composite in the diagram

is a cocartesian fibration by (the dual of) Corollary T.2.4.7.12. In fact, by (the dual of) [GHN, Theorem 4.5], this is precisely the *free cocartesian fibration* on F: this construction gives the left adjoint $L_{cocfib}(c)$.

³The identification of $S_{/\text{Cgpd}}$ as the underlying ∞ -category of the corresponding model category follows from the fact that $s\text{Set}_{KQ}$ is right proper, and hence *any* weak equivalence induces a Quillen adjunction on overcategories. In particular, the overcategory of a quasicategory already has the correct underlying ∞ -category, even without replacing the quasicategory by a Kan complex.

⁴To see this, note that the first is presented by the composite of a left Bousfield localization followed by a Quillen equivalence, while the second is presented by a left Bousfield localization.

Remark 1.4. In the sequence

$$(\mathfrak{C}at_{\infty})_{/\mathfrak{C}} \xrightarrow{\mathrm{L}_{\mathrm{co}\mathfrak{C}}\mathfrak{G}\mathfrak{f}\mathrm{ib}(\mathfrak{C})} \mathrm{co}\mathfrak{C}\mathfrak{F}\mathrm{ib}(\mathfrak{C}) \xrightarrow{\mathrm{L}_{\mathcal{L}}\mathfrak{G}\mathrm{ib}(\mathfrak{C})} \mathcal{L}\mathfrak{F}\mathrm{ib}(\mathfrak{C}) \xrightarrow{\mathrm{L}_{\mathcal{L}}(\mathfrak{C})} S_{\mathfrak{C}\mathrm{gpd}}$$

of left adjoints, all of the (possibly composite) left adjoints with target $S_{/C^{\text{gpd}}}$ are given by taking an object $\mathcal{D} \to \mathcal{C}$ to the object $\mathcal{D}^{\text{gpd}} \to \mathcal{C}^{\text{gpd}}$; this follows directly from the definitions of the overlying Quillen adjunctions of Theorem T.3.1.5.1.

Remark 1.5. When in fact $\mathcal{C} \in \mathcal{S} \subset \operatorname{Cat}_{\infty}$, the adjunctions $L_{\operatorname{coCFib}(\mathcal{C})} \dashv U_{\operatorname{coCFib}(\mathcal{C})}$ and $L_{\mathcal{L}(\mathcal{C})} \dashv U_{\mathcal{L}(\mathcal{C})}$ become adjoint equivalences by Theorem T.3.1.5.1.

1.2. The Grothendieck construction. We now give the main definition of this section.

Definition 1.6. We define the *Grothendieck construction* to be either equivalence of ∞ -categories in the diagram

$$\begin{array}{c} \operatorname{Fun}({\mathfrak C},\operatorname{Cat}_{\infty}) \xrightarrow{\operatorname{Gr}} \operatorname{co}{\mathfrak C}\operatorname{Fib}({\mathfrak C}) \\ & & & & & \\ & & & & & \\ & & & & & \\ \operatorname{Fun}({\mathfrak C},{\mathfrak S}) \xrightarrow{\sim} \operatorname{L}\operatorname{Fib}({\mathfrak C}); \end{array}$$

here, the upper (resp. lower) equivalence underlies the right adjoint of the Quillen equivalence which is dual to that of Theorem T.3.2.0.1 (resp. that of Theorem T.2.2.1.2) in the special case that the functor of sSet-enriched categories is the identity. The fact that the diagram commutes follows from the construction (see Definition T.3.2.1.2). Of course, there are also Grothendieck constructions for cartesian fibrations and right fibrations; we will denote these by

When we need to distinguish between these two types of Grothendieck constructions, we will refer to the former sort as *covariant* and to the latter sort as *contravariant*.

For any $\mathcal{C} \xrightarrow{F} \mathcal{C}at_{\infty}$, when we need to refer to it we will write

$$\operatorname{Gr}(F) \xrightarrow{\operatorname{pr}_{\operatorname{Gr}(F)}} \mathcal{C}$$

for the cocartesian fibration that it classifies. Similarly, given any $\mathcal{C}^{op} \xrightarrow{G} \mathcal{C}at_{\infty}$, when we need to refer to it we will write

$$\operatorname{Gr}^{-}(G) \xrightarrow{\operatorname{pr}_{\operatorname{Gr}^{-}(G)}} \mathcal{C}$$

for the cartesian fibration that it classifies.

The following characterization of the Grothendieck construction may at first appear rather abstract. However, it gives excellent geometric intuition, as we illustrate in Example 1.8 (the first nontrivial case).

Remark 1.7. The covariant Grothendieck construction can be characterized as a *lax colimit*: by [GHN, Theorem 7.4], for any functor $\mathcal{C} \xrightarrow{F} \mathcal{C}at_{\infty}$ we have an equivalence

$$\operatorname{Gr}(F) \simeq \int^{c \in \mathcal{C}} \mathcal{C}_{c/} \times F(c)$$

of its covariant Grothendieck construction with its colimit weighted by $\mathcal{C}^{op} \xrightarrow{\mathcal{C}_{-/}} \mathcal{C}at_{\infty}$. Thus, whereas the ordinary colimit of the diagram F can be viewed as "gluing together" all of the ∞ -categories F(c) – that is, taking all of the values $F(c) \in \mathcal{C}at_{\infty}$ and, for every map $c \xrightarrow{\varphi} c'$ in \mathcal{C} adding in an *equivalence*

$$y \xrightarrow{\sim} (F(\varphi))(y)$$

for every $y \in F(c)$ –, in a lax colimit, we now only add in a noninvertible morphism

$$y \to (F(\varphi))(y)$$

corresponding to such data. Dually, the contravariant Grothendieck construction can be characterized as an *oplax colimit*: by [GHN, Corollary 7.6], for any functor $\mathcal{C}^{op} \xrightarrow{G} \mathcal{C}at_{\infty}$ we have an equivalence

$$\operatorname{Gr}^{-}(G) \simeq \int^{c^{\circ} \in \mathfrak{C}^{op}} \mathfrak{C}_{/c} \times G(c^{\circ})$$

of its contravariant Grothendieck construction with its colimit weighted by $\mathcal{C} \xrightarrow{\mathcal{C}_{/-}} \mathcal{C}at_{\infty}.^{5}$

Example 1.8. Suppose that $[1] \xrightarrow{F} Cat_{\infty}$ selects a functor $C_0 \xrightarrow{f} C_1$. Then, its covariant Grothendieck construction can be identified as

$$\operatorname{Gr}(F) \simeq \operatorname{colim} \left(\begin{array}{cc} \mathbb{C}_0 & \xrightarrow{f} & \mathbb{C}_1 \\ \\ \left| \operatorname{id}_{\mathbb{C}_0} \times \{1\} \right| & & \\ \\ \mathbb{C}_0 \times [1] & \end{array} \right),$$

a "directed mapping cylinder" for f. Dually, if $[1]^{op} \xrightarrow{G} Cat_{\infty}$ selects a functor $\mathcal{D}_0 \xleftarrow{g} \mathcal{D}_1$, then its contravariant Grothendieck construction can be identified as

$$\operatorname{Gr}^{-}(G) \simeq \operatorname{colim} \left(\begin{array}{cc} \mathcal{D}_1 & \stackrel{g}{\longrightarrow} & \mathcal{D}_0 \\ \left. \operatorname{id}_{\mathcal{D}_1} \times \left\{ 0 \right\} \right| & & \\ \mathcal{D}_1 \times [1] & & \end{array} \right),$$

a "reversed directed mapping cylinder" for g.

We now list a few more basic examples of the Grothendieck construction.

Example 1.9. The equivalence

$$\operatorname{Fun}({\mathfrak C},\operatorname{\mathfrak{Cat}}_\infty)\xrightarrow[]{\sim}{\sim}\operatorname{co}{\mathfrak{CFib}}({\mathfrak C})$$

necessarily preserves terminal objects. In the source, the terminal object is $const(pt_{Cat_{\infty}})$, while in the target, the terminal object is the identity functor $id_{\mathcal{C}}$ (which is a cocartesian fibration). Similarly, the identity functor $id_{\mathcal{C}}$ is the terminal object of $\mathcal{CFib}(\mathcal{C})$.

Example 1.10. Given any two functors $F, G \in Fun(\mathcal{C}, \mathfrak{Cat}_{\infty})$, consider the pullback diagram

in Cat_{∞} . Note first that the composite functor

$$\operatorname{Gr}(F) \times_{\mathfrak{C}} \operatorname{Gr}(G) \to \mathfrak{C}$$

is again a cocartesian fibration by the dual of (parts (2) and (3) of) Proposition T.2.4.2.3. Moreover, this is the product of the objects $\operatorname{pr}_{\operatorname{Gr}(F)}$ and $\operatorname{pr}_{\operatorname{Gr}(G)}$ in $(\operatorname{Cat}_{\infty})_{/\mathbb{C}}$, and since the inclusion $\operatorname{coCFib}(\mathbb{C}) \subset (\operatorname{Cat}_{\infty})_{/\mathbb{C}}$ is a right adjoint, this must also be their product in $\operatorname{coCFib}(\mathbb{C})$. Thus, this cocartesian fibration must be classified by the composite functor

$$\mathfrak{C} \xrightarrow{(F,G)} \mathfrak{Cat}_{\infty} \times \mathfrak{Cat}_{\infty} \xrightarrow{-\times -} \mathfrak{Cat}_{\infty},$$

i.e. the product $F \times G \in \operatorname{Fun}(\mathfrak{C}, \operatorname{Cat}_{\infty})$.

⁵One can also identify the ∞ -categories of sections of Grothendieck constructions with op/lax *limits* (see e.g. [GHN, Proposition 7.1]), but this is less essential for geometric intuition.

Example 1.11. In the special case that $\mathcal{C} = \operatorname{pt}_{\operatorname{Cat}_{\infty}}$, the Grothendieck construction yields an equivalence

$$\mathbb{C}\mathrm{at}_{\infty}\simeq \mathrm{Fun}(\mathrm{pt}_{\mathbb{C}\mathrm{at}_{\infty}},\mathbb{C}\mathrm{at}_{\infty})\xrightarrow{\mathrm{Gr}}\mathrm{co}\mathbb{C}\mathrm{Fib}(\mathrm{pt}_{\mathbb{C}\mathrm{at}_{\infty}}).$$

By [Toë05, Théorèm 6.3], this must be inverse to the composite

$$\operatorname{coCFib}(\operatorname{pt}_{\operatorname{Cat}_{\infty}}) \xrightarrow{\sim} (\operatorname{Cat}_{\infty})_{/\operatorname{pt}_{\operatorname{Cat}_{\infty}}} \xrightarrow{\sim} \operatorname{Cat}_{\infty}$$

of two forgetful equivalences.

Example 1.12. Let $\mathcal{C} \in \operatorname{Cat}_{\infty}$, and let $c \in \mathcal{C}$. Then, the forgetful functor $\mathcal{C}_{c/} \to \mathcal{C}$ from the undercategory is a left fibration by Corollary T.2.1.2.2; in fact, it follows from Proposition T.4.4.4.5 that we can identify this as

$$\mathcal{C}_{c/} \simeq \operatorname{Gr}\left(\mathfrak{C} \xrightarrow{\hom_{\mathfrak{C}}(c,-)} \mathfrak{S}\right).$$

Dually, we have that $(\mathcal{C}_{/c} \to \mathcal{C}) \in \mathfrak{RFib}(\mathcal{C})$, and we have the identification

$$\mathcal{C}_{/c} \simeq \operatorname{Gr}^{-} \left(\mathcal{C}^{op} \xrightarrow{\hom_{\mathfrak{C}}(-,c)} \mathcal{S} \right).$$

Remark 1.13. An important property of the Grothendieck construction is its *naturality*: it assembles to a functor

$$(\operatorname{Cat}_{\infty})^{op} \to \operatorname{Fun}([1], \operatorname{Cat}_{\infty})$$

which sends an ∞ -category \mathcal{C} to the object

$$\operatorname{Fun}(\mathfrak{C}, \operatorname{\mathfrak{Cat}}_\infty) \xrightarrow[\sim]{}_{\sim} \operatorname{co}\mathfrak{CFib}(\mathfrak{C})$$

of Fun([1], $\operatorname{Cat}_{\infty}$) by (the dual of) [GHN, Corollary A.31]. By its construction (see the proof of [GHN, Proposition A.30]), over $0 \in [1]$ this functor is given by precomposition of functors to $\operatorname{Cat}_{\infty}$, while over $1 \in [1]$ this functor is given by pullback of cocartesian fibrations. So for instance, a map $\mathfrak{C} \xrightarrow{F} \mathcal{D}$ in $\operatorname{Cat}_{\infty}$ determines a map

$$\begin{array}{c} \operatorname{Fun}(\mathcal{D},\operatorname{Cat}_{\infty}) \xrightarrow{\operatorname{Gr}} \operatorname{co}\mathcal{C}\operatorname{Fib}(\mathcal{D}) \\ & -\circ F \downarrow & \downarrow F^* \\ \operatorname{Fun}(\mathcal{C},\operatorname{Cat}_{\infty}) \xrightarrow{\sim} \operatorname{co}\mathcal{C}\operatorname{Fib}(\mathcal{C}) \end{array}$$

in $\operatorname{Fun}([1], \operatorname{Cat}_{\infty})$ (where we read this square in $\operatorname{Cat}_{\infty}$ vertically in order to consider it as a morphism between the two horizontal arrows). Of course, by duality the contravariant Grothendieck construction enjoys analogous naturality.

Since this observation will arise so frequently for us, we codify it.

Definition 1.14. We refer to the phenomenon described in Remark 1.13 as the *naturality* of the Grothendieck construction.

This has the following easy consequence.

Example 1.15. Consider the tautological factorization



By Example 1.11, we have a canonical equivalence

$$\operatorname{Gr}\left(\operatorname{pt}_{\operatorname{Cat}_{\infty}}\xrightarrow{\mathcal{D}}\operatorname{Cat}_{\infty}\right)\simeq\mathcal{D}.$$

Hence, the naturality of the Grothendieck construction implies that the pullback square



in Cat_{∞} provides a canonical equivalence

$$\operatorname{Gr}\left(\mathfrak{C}\xrightarrow{\operatorname{const}(\mathcal{D})} \mathfrak{Cat}_{\infty}\right) \simeq \mathfrak{C} \times \mathfrak{D},$$

with the cocartesian fibration down to \mathcal{C} identifying as $\mathrm{pr}_{\mathrm{Gr}(\mathrm{const}(\mathcal{D}))} \simeq \mathrm{pr}_{\mathcal{C}}$, the projection onto the first factor. (From here, we can recover Example 1.9 as the special case where $\mathcal{D} = \mathrm{pt}_{\mathrm{Cat}_{\infty}}$.) Similarly, we have that

$$\operatorname{Gr}^{-}\left(\operatorname{\mathcal{C}}^{op} \xrightarrow{\operatorname{const}(\mathcal{D})} \operatorname{\mathcal{C}at}_{\infty}\right) \simeq \operatorname{\mathcal{C}} \times \mathcal{D},$$

with $\operatorname{pr}_{\operatorname{Gr}^-(\operatorname{const}(\mathcal{D}))} \simeq \operatorname{pr}_{\mathcal{C}}$: in other words, the projection $\mathcal{C} \times \mathcal{D} \xrightarrow{\operatorname{pr}_{\mathcal{C}}} \mathcal{C}$ is simultaneously a cocartesian fibration and a cartesian fibration, in either case classified by the constant functor at the object $\mathcal{D} \in \operatorname{Cat}_{\infty}$.

Definition 1.16. Fix some $\mathcal{C} \xrightarrow{F} \mathcal{C}at_{\infty}$. By the naturality of the Grothendieck construction (and Example 1.11), for any $x \in \mathcal{C}$ there is a canonical pullback square



in $\operatorname{Cat}_{\infty}$. We refer to F(x) as the *fiber* of the cocartesian fibration over the object $x \in \mathbb{C}$, and we refer to the above commutative square as a *fiber inclusion*.⁶

Remark 1.17. A fiber inclusion will not generally be the inclusion of a full subcategory: it will not contain those morphisms covering nontrivial endomorphisms. In fact, in the extreme case that we take $\mathcal{C} = Y \in \mathcal{S} \subset Cat_{\infty}$ to be an ∞ -groupoid, the functor corepresented by an object $y \in Y$ will have

$$\operatorname{Gr}\left(Y \xrightarrow{\hom_Y(y,-)} \mathcal{S}\right) \simeq Y_{y/} \simeq \operatorname{pt}_{\mathcal{S}}$$

(recall Example 1.12); then, the fiber inclusion over the object $y \in Y$ itself will be given by the pullback square



in $S \subset Cat_{\infty}$, the "inclusion" of the based loopspace of Y at y into the terminal space.

We have the following concrete identification of the left localization which takes cocartesian fibrations to left fibrations.

Proposition 1.18. Under the equivalences given by the Grothendieck construction, the left localization

$$\operatorname{co}\mathcal{CFib}(\mathcal{C}) \xrightarrow{\operatorname{L}_{\mathcal{L}}\mathcal{F}ib(\mathcal{C})} \mathcal{L}\mathcal{F}ib(\mathcal{C})$$

corresponds to the functor $\operatorname{Fun}(\mathfrak{C}, \operatorname{Cat}_{\infty}) \to \operatorname{Fun}(\mathfrak{C}, \mathfrak{S})$ given by postcomposition with

$$\operatorname{Cat}_{\infty} \xrightarrow{(-)^{\operatorname{gpd}}} S.$$

⁶At the level of quasicategories, this can be computed by the inclusion of the fiber over a vertex corresponding to $x \in C$, which is a homotopy fiber in sSet_{Joval} by the Reedy trick (and the implications of Remark T.2.0.0.5).

Proof. This follows from the uniqueness of left adjoints and the commutativity of the diagram in Definition 1.6: the adjunction $\operatorname{Fun}(\mathcal{C}, \operatorname{Cat}_{\infty}) \rightleftharpoons \operatorname{Fun}(\mathcal{C}, \mathbb{S})$ comes from applying the functor $\operatorname{Fun}(\mathcal{C}, -) : \operatorname{Cat}_{\infty} \to \operatorname{Cat}_{\infty}$ to the adjunction $(-)^{\operatorname{gpd}} : \operatorname{Cat}_{\infty} \rightleftharpoons \mathbb{S}$.

2. The Grothendieck construction and colimits of spaces

In this section, we study the relationship between the Grothendieck construction and colimits in the ∞ -category S of spaces: in §2.1 we give some basic results, in §2.2 we extend these to the "two-sided" Grothendieck construction, and in §2.3 we collect some results which are suggestive of the $(\infty, 2)$ -categorical functoriality of the Grothendieck construction.

2.1. The Grothendieck construction and colimits. We begin with the following fundamental fact, on which all further results in this direction are based. Its 1-categorical version, Thomason's *homotopy colimit theorem*, first appeared as [Tho79, Theorem 1.2].

Proposition 2.1. For any $\mathcal{C} \xrightarrow{F} \operatorname{Cat}_{\infty}$, the Grothendieck construction computes its homotopy colimit when considered as diagram in $(\operatorname{Cat}_{\infty})_{\mathrm{Th}}$, i.e.

$$\operatorname{Gr}(F)^{\operatorname{gpd}} \simeq \operatorname{colim}\left(\mathfrak{C} \xrightarrow{F} \operatorname{Cat}_{\infty} \xrightarrow{(-)^{\operatorname{gpd}}} \mathfrak{S}\right).$$

Proof. This follows from Corollary T.3.3.4.3 and the fact that groupoid completion (being a left adjoint) commutes with colimits. \Box

Corollary 2.2. Suppose that we are given a pair of functors $F, G : \mathfrak{C} \rightrightarrows (\mathfrak{Cat}_{\infty})_{\mathrm{Th}}$ and a natural weak equivalence $F \xrightarrow{\approx} G$. Then the induced map $\mathrm{Gr}(F) \to \mathrm{Gr}(G)$ is a weak equivalence in $(\mathfrak{Cat}_{\infty})_{\mathrm{Th}}$.

Proof. Combining the equivalence

$$(-)^{\operatorname{gpd}} \circ F \xrightarrow{\sim} (-)^{\operatorname{gpd}} \circ G$$

in $\operatorname{Fun}(\mathcal{C}, S)$ with two applications of the equivalence of Proposition 2.1, we obtain a composite equivalence

$$\operatorname{Gr}(F_1)^{\operatorname{gpd}} \simeq \operatorname{colim}\left(\mathfrak{C} \xrightarrow{F_1} \operatorname{Cat}_{\infty} \xrightarrow{(-)^{\operatorname{gpd}}} \mathfrak{S}\right) \xrightarrow{\sim} \operatorname{colim}\left(\mathfrak{C} \xrightarrow{F_2} \operatorname{Cat}_{\infty} \xrightarrow{(-)^{\operatorname{gpd}}} \mathfrak{S}\right) \simeq \operatorname{Gr}(F_2)^{\operatorname{gpd}}$$

in S.

2.2. The two-sided Grothendieck construction and colimits. We will also be interested in the following variant of the Grothendieck construction and its interaction with colimits in S.

Definition 2.3. Given any (ordered) pair of functors $\mathbb{C}^{op} \xrightarrow{F} \mathbb{C}at_{\infty}$ and $\mathbb{C} \xrightarrow{G} \mathbb{C}at_{\infty}$, we define the *two-sided Grothendieck construction* of *F* and *G* to be the pullback

$$\operatorname{Gr}(F, \mathfrak{C}, G) = \lim \left(\begin{array}{c} \operatorname{Gr}^{-}(F) \\ \downarrow \\ \operatorname{Gr}(G) \longrightarrow \mathfrak{C} \end{array} \right)$$

in $\operatorname{Cat}_{\infty}$.

Proposition 2.4. Suppose that we are given

- a pair of functors $F, F' : \mathcal{C}^{op} \rightrightarrows (\operatorname{Cat}_{\infty})_{\operatorname{Th}}$ and a natural weak equivalence $F \xrightarrow{\approx} F'$, and
- a pair of functors $G, G' : \mathfrak{C} \rightrightarrows (\mathfrak{Cat}_{\infty})_{\mathrm{Th}}$ and a natural weak equivalence $G \xrightarrow{\approx} G'$. Then, the induced map

$$\operatorname{Gr}(F, \mathcal{C}, G) \to \operatorname{Gr}(F', \mathcal{C}, G')$$

is a weak equivalence in $(Cat_{\infty})_{Th}$.

Proof. The naturality of the Grothendieck construction induces an evident naturality of the two-sided Grothendieck construction; in light of this, by duality and since $\mathbf{W}_{\mathrm{Th}} \subset \operatorname{Cat}_{\infty}$ is closed under composition, it suffices to prove the claim in the special case that G = G' and that the natural weak equivalence $G \xrightarrow{\approx} G'$ is simply id_G . Then, by the naturality of the Grothendieck construction we have an equivalence

$$\operatorname{Gr}(F, \mathfrak{C}, G) = \lim \left(\begin{array}{c} \operatorname{Gr}^{-}(F) \\ \downarrow \\ \operatorname{Gr}(G) \longrightarrow \mathfrak{C} \end{array} \right) \simeq \operatorname{Gr}^{-}(F \circ \operatorname{pr}_{\operatorname{Gr}(G)}^{op})$$

in $\mathcal{CFib}(\mathrm{Gr}(G))$, and similarly we have an equivalence

$$\operatorname{Gr}(F', \mathfrak{C}, G) \simeq \operatorname{Gr}^{-}(F' \circ \operatorname{pr}^{op}_{\operatorname{Gr}(G)})$$

in $\mathcal{CFib}(\mathrm{Gr}(G))$. Moreover, by assumption, the map

$$F \circ \operatorname{pr}_{\operatorname{Gr}(G)}^{op} \to F' \circ \operatorname{pr}_{\operatorname{Gr}(G)}^{op}$$

in $\operatorname{Fun}(\operatorname{Gr}(G)^{op}, \operatorname{Cat}_{\infty})$ has that for every $y \in \operatorname{Gr}(G)^{op}$, the induced map

$$(F \circ \operatorname{pr}_{\operatorname{Gr}(G)}^{op})(y)^{\operatorname{gpd}} \to (F' \circ \operatorname{pr}_{\operatorname{Gr}(G)}^{op})(y)^{\operatorname{gpd}}$$

is an equivalence in S: in other words, the map

$$(-)^{\operatorname{gpd}} \circ F \circ \operatorname{pr}_{\operatorname{Gr}(G)}^{op} \to (-)^{\operatorname{gpd}} \circ F' \circ \operatorname{pr}_{\operatorname{Gr}(G)}^{op}$$

is an equivalence in $\operatorname{Fun}(\operatorname{Gr}(G)^{op}, \mathbb{S})$. The claim now follows from (the dual of) Corollary 2.2.

2.3. Higher-categorical functoriality of the Grothendieck construction and colimits. We now assemble a few results which are suggestive of the $(\infty, 2)$ -categorical functoriality of the Grothendieck construction. However, we do not pursue such functoriality in any systematic way.

Proposition 2.5. A diagram

$$\mathfrak{C} \underbrace{\overset{E}{\overbrace{\alpha \Downarrow}}}_{F} \mathfrak{D} \xrightarrow{G} \mathfrak{Cat}_{\infty}$$

induces a commutative triangle

$$\operatorname{colim}_{\mathbb{C}}((-)^{\operatorname{gpd}} \circ G \circ E)$$

$$\downarrow$$

$$\operatorname{colim}_{\mathbb{D}}((-)^{\operatorname{gpd}} \circ G)$$

$$\operatorname{colim}_{\mathbb{D}}((-)^{\operatorname{gpd}} \circ G)$$

 $in \ S.$

Proof. This follows by combining Proposition 2.1, Lemma 2.6, and Lemma N.1.26.

The following result, an ingredient of the proof of Proposition 2.5, uses the language of §3.1.

Lemma 2.6. A diagram

$$\mathfrak{C}\underbrace{\overset{E}{\underset{F}{\overset{\alpha\Downarrow}{\longrightarrow}}}}_{F}\mathfrak{D}\overset{G}{\overset{G}{\overset{}{\longrightarrow}}}\mathfrak{Cat}_{\infty}$$

induces a morphism



in $\mathcal{L}ax(Gr(G))$.

Proof. We begin by encoding α as a functor $[1] \times \mathcal{C} \xrightarrow{H} \mathcal{D}$. By the naturality of the Grothendieck construction, this gives rise to a diagram



in which both left vertical arrows are cocartesian fibrations (the upper by pullback, the lower the Grothendieck construction of the functor $[1] \xrightarrow{\text{const}(\mathbb{C})} \operatorname{Cat}_{\infty}$). Hence, the composite left vertical arrow is a cocartesian fibration as well, namely the one classified by the map $[1] \rightarrow \operatorname{Cat}_{\infty}$ selecting the functor

$$\operatorname{Gr}(G \circ E) \xrightarrow{\operatorname{Gr}(\operatorname{id}_G \circ \alpha)} \operatorname{Gr}(G \circ F).$$

Now, in order to extract this functor as a map *between* cocartesian fibrations (as opposed to being contained *within* a cocartesian fibration), we obtain a morphism in $\operatorname{Fun}([1], \operatorname{Cat}_{\infty})$ with this map as its target by precomposing with the map $[1] \times [1] \to [1]$ given by

$$(i,j) \mapsto \begin{cases} 0, & (i,j) \neq (1,1) \\ 1, & (i,j) = (1,1); \end{cases}$$

by adjunction, this yields a functor $[1] \to \operatorname{Fun}([1], \operatorname{Cat}_{\infty})$ which, considered as a map in $\operatorname{Fun}([1], \operatorname{Cat}_{\infty})$, has source $[1] \xrightarrow{\operatorname{const}(\operatorname{Gr}(G \circ E))} \operatorname{Cat}_{\infty}$ and has target selecting this same functor $\operatorname{Gr}(G \circ E) \xrightarrow{\operatorname{Gr}(\operatorname{id}_G \circ \alpha)} \operatorname{Gr}(G \circ F)$. From here, the equivalence $\operatorname{Fun}([1], \operatorname{Cat}_{\infty}) \xrightarrow{\operatorname{Gr}} \operatorname{co}\mathcal{CF}ib([1])$ gives rise to the diagram of Figure 1, in which for clarity we include both fiber inclusions of each of these objects of $\operatorname{co}\mathcal{CF}ib([1])$ as well as the induced maps between them. Moreover, there is a canonical natural transformation in $\operatorname{Fun}(\operatorname{Gr}(G \circ E), [1] \times \operatorname{Gr}(G \circ E))$ from the inclusion of the fiber over $0 \in [1]$ to the inclusion of the fiber over $1 \in [1]$ (selected by the identity map $\operatorname{id}_{[1] \times \operatorname{Gr}(G \circ E)})$. Taking the horizontal composite



of this natural transformation with the horizontal composite in the diagram of Figure 1 then yields the desired morphism in $\mathcal{L}ax(\operatorname{Gr}(G))$.

In manipulating colimits, we will also make use of the following notion (which is actually only a special case of a more general $(\infty, 2)$ -categorical phenomenon).

Definition 2.7. Let $\mathcal{C} \in Cat_{\infty}$, let $F, G \in Fun(\mathcal{C}, Cat_{\infty})$, and let $\alpha, \beta \in hom_{Fun(\mathcal{C}, Cat_{\infty})}(F, G)$. A *modification* from α to β is a map

$$\operatorname{const}([1]) \times F \xrightarrow{\mu} G$$



FIGURE 1. The diagram in Cat_{∞} used in the proof of Lemma 2.6.

in Fun($\mathcal{C}, \operatorname{Cat}_{\infty}$) whose restriction along $\operatorname{const}(\{0\}) \to \operatorname{const}([1])$ recovers α and whose restriction along $\operatorname{const}(\{1\}) \to \operatorname{const}([1])$ recovers β .

The following result describes the effect of the Grothendieck construction on modifications.

Proposition 2.8. Let $\mathcal{C} \in \operatorname{Cat}_{\infty}$, let $F, G \in \operatorname{Fun}(\mathcal{C}, \operatorname{Cat}_{\infty})$, and let $\alpha, \beta \in \operatorname{hom}_{\operatorname{Fun}(\mathcal{C}, \operatorname{Cat}_{\infty})}(F, G)$. A modification μ from α to β induces a natural transformation



in $\operatorname{Cat}_{\infty}$.

Proof. Applying the Grothendieck construction to the modification μ , we obtain a map

$$\operatorname{Gr}(\operatorname{const}([1]) \times F) \xrightarrow{\operatorname{Gr}(\mu)} \operatorname{Gr}(G)$$

in coCFib(C). But this source can be identified as

$$\operatorname{Gr}(\operatorname{const}([1]) \times F) \simeq \operatorname{Gr}(\operatorname{const}([1])) \times_{\mathfrak{C}} \operatorname{Gr}(F) \simeq ([1] \times \mathfrak{C}) \times_{\mathfrak{C}} \operatorname{Gr}(F) \simeq [1] \times \operatorname{Gr}(F),$$

where the first equivalence follows from the fact that $\operatorname{Fun}(\mathbb{C}, \operatorname{Cat}_{\infty}) \xrightarrow{\operatorname{Gr}} \operatorname{coCFib}(\mathbb{C})$ is an equivalence (so commutes with products) and the fact that the forgetful functor $\operatorname{coCFib}(\mathbb{C}) \to (\operatorname{Cat}_{\infty})_{/\mathbb{C}}$ is a right adjoint and hence commutes with products. So, this becomes a map

$$[1] \times \operatorname{Gr}(F) \xrightarrow{\operatorname{Gr}(\mu)} \operatorname{Gr}(G),$$

as desired.

3. Op/lax natural transformations and the global co/limit functor

In ordinary category theory, functors with the same source and target can be related by natural transformations between them. When the target (and possibly the source) is a 2-category, this notion can be relaxed in two different ways, yielding notions of *lax* and *oplax* natural transformations.

We will be interested in this phenomenon in the ∞ -categorical context. However, we will concern ourselves exclusively with the special case in which the source is only an ∞ -category and the target is $\operatorname{Cat}_{\infty}$, considered as an $(\infty, 2)$ -category via the closure of its symmetric monoidal structure ($\operatorname{Cat}_{\infty}, \times$). In this case, the Grothendieck construction allows us to easily and concisely define such transformations without reference to an ambient theory of $(\infty, 2)$ -categories: heuristically speaking, it "reduces category level by one".

In §3.1 we define op/lax natural transformations via the Grothendieck construction, and then in §3.2 we apply this framework to study the $((\infty, 2)$ -categorical) functoriality of colimits in an arbitrary but fixed ∞ -category.

3.1. Op/lax natural transformations. We begin with the following definition.

Definition 3.1. Suppose we are given a pair of functors $F, G : \mathbb{C} \rightrightarrows \operatorname{Cat}_{\infty}$. A *lax natural transformation* from F to G, denoted $F \rightsquigarrow G$ for short, is a map



in $(\operatorname{Cat}_{\infty})_{/\mathcal{C}}$. Meanwhile, an *oplax natural transformation* from F to G, denoted $F \stackrel{op}{\leadsto} G$ for short, is a map



in $(\operatorname{Cat}_{\infty})_{/\operatorname{C}^{op}}$.

To provide some intuition, we illustrate Definition 3.1 in the simplest nontrivial case.

Example 3.2. Recall from Example 1.8 that we can think of the equivalence

$$\operatorname{Fun}([1], \operatorname{Cat}_{\infty}) \xrightarrow{\operatorname{Gr}} \operatorname{coCFib}([1])$$

as a sort of "directed mapping cylinder" construction. Hence, if $[1] \xrightarrow{F} \operatorname{Cat}_{\infty}$ selects a functor $\mathcal{C}_0 \xrightarrow{f} \mathcal{C}_1$ and $[1] \xrightarrow{G} \operatorname{Cat}_{\infty}$ selects a functor $\mathcal{D}_0 \xrightarrow{g} \mathcal{D}_1$, then a lax natural transformation $\alpha : F \rightsquigarrow G$ is equivalent to a square

$$\begin{array}{ccc} \mathcal{C}_{0} & \stackrel{f}{\longrightarrow} & \mathcal{C}_{1} \\ \alpha_{0} & \swarrow & & \downarrow \alpha_{1} \\ \mathcal{D}_{0} & \stackrel{g}{\longrightarrow} & \mathcal{D}_{1} \end{array}$$

in Cat_{∞} , i.e. the data of

- a functor $\mathfrak{C}_0 \xrightarrow{\alpha_0} \mathfrak{D}_0$,
- a functor $\mathcal{C}_1 \xrightarrow{\alpha_1} \mathcal{D}_1$, and
- a natural transformation $g \circ \alpha_0 \to \alpha_1 \circ f$ in Fun($\mathcal{C}_0, \mathcal{D}_1$).

A similar analysis shows that an oplax natural transformation $\alpha : F \xrightarrow{op} G$ simply reverses the direction of the natural transformation (so that it runs down and to the left, instead of up and to the right).

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We will be interested in the special case of Example 3.2 in which $G = \text{const}(\mathcal{C})$ (so that $g = \text{id}_{\mathcal{C}}$) for some chosen ∞ -category \mathcal{C} . For a fixed such choice, these can be organized in the following way.

Definition 3.3. For any $\mathcal{C} \in Cat_{\infty}$, we define the *lax overcategory* of \mathcal{C} to be

$$\mathcal{L}ax(\mathfrak{C}) = \mathrm{Gr}^{-} \left((\mathfrak{C}at_{\infty})^{op} \xrightarrow{\mathrm{Fun}(-,\mathfrak{C})} \mathfrak{C}at_{\infty} \right).$$

Thus, the objects of $\mathcal{L}ax(\mathcal{C})$ are functors with target \mathcal{C} , and a morphism from $\mathcal{D} \xrightarrow{F} \mathcal{C}$ to $\mathcal{E} \xrightarrow{G} \mathcal{C}$ in $\mathcal{L}ax(\mathcal{C})$ is given by a triangle



in Cat_{∞} . We write

$$\mathcal{L}ax(\mathcal{C}) \xrightarrow{s_{\mathcal{L}ax(\mathcal{C})}} \mathcal{C}at_{\infty}$$

for the canonical projection map, and refer to it as the **source projection**; this is by definition a cartesian fibration, with fiber over $\mathcal{D} \in Cat_{\infty}$ given by $Fun(\mathcal{D}, \mathcal{C})$.

Similarly, we define the *oplax overcategory* of \mathcal{C} to be

$$\operatorname{op}\mathcal{L}\operatorname{ax}(\mathfrak{C}) = \operatorname{Gr}^{-}\left((\operatorname{Cat}_{\infty})^{op} \xrightarrow{\operatorname{Fun}((-)^{op}, \mathfrak{C}^{op})} \operatorname{Cat}_{\infty} \right).$$

Thus, the objects of $\operatorname{op}\mathcal{L}\operatorname{ax}(\mathcal{C})$ can be canonically identified (via the involution $(-)^{op} : \operatorname{Cat}_{\infty} \xrightarrow{\sim} \operatorname{Cat}_{\infty}$) with functors with target \mathcal{C} , and a morphism from (the object identified with) $\mathcal{D} \xrightarrow{F} \mathcal{C}$ to (the object identified with) $\mathcal{E} \xrightarrow{G} \mathcal{C}$ in $\operatorname{op}\mathcal{L}\operatorname{ax}(\mathcal{C})$ can be canonically identified with a triangle



in Cat_{∞} . We write

$$\operatorname{op}\mathcal{L}\operatorname{ax}(\mathcal{C}) \xrightarrow{s_{\operatorname{op}\mathcal{L}\operatorname{ax}(\mathcal{C})}} \operatorname{Cat}_{\infty}$$

for the canonical projection map, and also refer to it as the **source projection**; this is again by definition a cartesian fibration, with fiber over $\mathcal{D} \in Cat_{\infty}$ given by $Fun(\mathcal{D}^{op}, \mathcal{C}^{op}) \simeq Fun(\mathcal{D}, \mathcal{C})^{op}$.

Remark 3.4. A map $C_1 \to C_2$ induces a natural transformation $\operatorname{Fun}(-, C_1) \to \operatorname{Fun}(-, C_2)$ in $\operatorname{Fun}((\operatorname{Cat}_{\infty})^{op}, \operatorname{Cat}_{\infty})$, which in turn gives rise to a commutative triangle



in Cat_{∞} ; altogether, we obtain a functor

$$\operatorname{Cat}_{\infty} \xrightarrow{\mathcal{L}\operatorname{ax}(-)} \operatorname{CFib}(\operatorname{Cat}_{\infty}).$$

Similarly, we obtain a functor

$$\operatorname{Cat}_{\infty} \xrightarrow{\operatorname{op}\mathcal{L}\operatorname{ax}(-)} \operatorname{CFib}(\operatorname{Cat}_{\infty}).$$

Remark 3.5. Expanding out the definition, we also can write

$$op\mathcal{L}ax(\mathcal{C}) = \mathrm{Gr}^{-} \left((\mathfrak{C}at_{\infty})^{op} \xrightarrow{((-)^{op})^{op}} (\mathfrak{C}at_{\infty})^{op} \xrightarrow{\mathrm{Fun}(-,\mathfrak{C}^{op})} \mathfrak{C}at_{\infty} \right)$$

(where the first functor is obtained by applying the involution $\operatorname{Cat}_{\infty} \xrightarrow{(-)^{op}} \operatorname{Cat}_{\infty}$ to the morphism $\operatorname{Cat}_{\infty} \xrightarrow{(-)^{op}} \operatorname{Cat}_{\infty}$). By the naturality of the Grothendieck construction, it follows that we have a pullback square

$$\begin{array}{c} \operatorname{op}\mathcal{L}\operatorname{ax}(\mathbb{C}) & \xrightarrow{\sim} & \mathcal{L}\operatorname{ax}(\mathbb{C}^{op}) \\ s_{\operatorname{op}\mathcal{L}\operatorname{ax}(\mathbb{C})} & & \downarrow s_{\mathcal{L}\operatorname{ax}(\mathbb{C}^{op})} \\ & & & \downarrow s_{\mathcal{L}\operatorname{ax}(\mathbb{C}^{op})} \\ & & & & \downarrow s_{\mathcal{L}\operatorname{ax}(\mathbb{C}^{op})} \\ & & & & & \vdots \end{array}$$

in $\operatorname{Cat}_{\infty}$.

Remark 3.6. As an alternative to the construction of Definition 3.3, both $\mathcal{L}ax(\mathcal{C})$ and $op\mathcal{L}ax(\mathcal{C})$ are simultaneously encoded in the "(op)lax square" of $\operatorname{Cat}_{\infty}$ as constructed in [JFS, §5]. While this alternative construction is both clean and aesthetically pleasing, we have chosen our own exposition in pursuit of the meta-goal of this paper (namely, to connect as many different concepts as possible to the Grothendieck construction).

3.2. The global co/limit functor. Suppose we are given an arbitrary cocomplete ∞ -category C. Then, the operation of taking colimits in C should be functorial in two different senses.

• On the one hand, colimits are functorial for natural transformations. For instance, a natural transformation



induces a canonical map $\operatorname{colim}_{\mathcal{D}}(F) \to \operatorname{colim}_{\mathcal{D}}(G)$ in \mathfrak{C} : a colimiting cocone $\mathcal{D}^{\triangleright} \to \mathfrak{C}$ extending G composes to give an arbitrary cocone $\mathcal{D}^{\triangleright} \to \mathfrak{C}$ extending F, and then the desired map arises from the definition of a colimit as an initial cocone extending the given diagram. Thus, taking colimits should give rise to a functor

$$\operatorname{Fun}(\mathcal{D}, \mathfrak{C}) \xrightarrow{\operatorname{colim}} \mathfrak{C}.$$

• On the other hand, colimits are also functorial for commutative diagrams of ∞-categories over C. For instance, a commutative diagram



in $\operatorname{Cat}_{\infty}$ induces a canonical map $\operatorname{colim}_{\mathcal{D}}(F) \to \operatorname{colim}_{\mathcal{E}}(G)$ in \mathbb{C} : a colimiting cocone $\mathcal{E}^{\triangleright} \to \mathbb{C}$ extending G composes to give an arbitrary cocone $\mathcal{D}^{\triangleright} \to \mathcal{E}^{\triangleright} \to \mathbb{C}$ extending F, and then the desired map once again arises from the definition of a colimit as an initial cocone extending the given diagram. Thus, taking colimits should also give rise to a functor

$$(\operatorname{Cat}_{\infty})_{/\mathfrak{C}} \xrightarrow{\operatorname{colim}} \mathfrak{C}.$$

In fact, we have already seen a construction in Definition 3.3 which unifies these two situations: via the cartesian fibration $\mathcal{L}ax(\mathcal{C}) \xrightarrow{s_{\mathcal{L}ax}(\mathcal{C})} \mathcal{C}at_{\infty}$, an arbitrary morphism



in $\mathcal{L}ax(\mathcal{C})$ gives rise to a unique diagram



in which the inner (commutative) triangle determines a cartesian arrow and the indicated natural transformation is a fiber morphism (lying over the object $\mathcal{D} \in Cat_{\infty}$). Thus, one might expect that it is possible to unify the above two senses in which colimits are functorial by means of a single functor

 $\mathcal{L}ax(\mathcal{C}) \xrightarrow{\operatorname{colim}} \mathcal{C}.$

The purpose of this subsection is to construct precisely such a global colimit functor. In fact, we will achieve this (as Proposition 3.12) for an arbitrary ∞ -category \mathcal{C} (i.e. when \mathcal{C} is not necessarily cocomplete), although in this more general setting we will of course need to restrict to the full subcategory of $\mathcal{L}ax(\mathcal{C})$ spanned by those diagrams in \mathcal{C} which admit a colimit (see Notation 3.11).

Remark 3.7. A similar analysis suggests that when \mathcal{C} is complete, there should exist a corresponding global limit functor running

$$\operatorname{op}\mathcal{L}\operatorname{ax}(\mathfrak{C})^{op} \xrightarrow{\lim} \mathfrak{C}$$

Indeed, this can be obtained from Proposition 3.12 simply by taking opposites: if C is complete then C^{op} is cocomplete, and combining the resulting global colimit functor

$$\mathcal{L}ax(\mathcal{C}^{op}) \xrightarrow{\operatorname{colim}} \mathcal{C}^{op}$$

with the equivalence of Remark 3.5 and taking opposites yields a composite

$$\operatorname{op}\mathcal{L}\operatorname{ax}(\mathcal{C})^{op} \xrightarrow{\sim} \mathcal{L}\operatorname{ax}(\mathcal{C}^{op})^{op} \to \mathcal{C}$$

which one can easily verify is the desired global limit functor. (And of course, this equally well generalizes to the case that C only admits certain limits.) We will therefore henceforth focus our attention only on the global *colimit* functor.

The first step in constructing the global colimit functor is to make the following reidentification of the nerve of the ∞ -category Cat_{∞} of ∞ -categories.

Lemma 3.8. There is a canonical identification of $N_{\infty}(Cat_{\infty})_{\bullet} \in CSS \subset sS$ with the composite

$$\Delta^{op} \hookrightarrow (\operatorname{Cat}_{\infty})^{op} \xrightarrow{\operatorname{coCFib}(-)^{\simeq}} S$$

Proof. For any $[n] \in \Delta$, the Grothendieck construction provides an equivalence

$$\operatorname{Fun}([n], \operatorname{Cat}_{\infty}) \xrightarrow{\operatorname{Gr}} \operatorname{coCFib}([n]);$$

passing to maximal subgroupoids, we obtain an equivalence

$$N_{\infty}(\operatorname{Cat}_{\infty})_n = \operatorname{hom}_{\operatorname{Cat}_{\infty}}([n], \operatorname{Cat}_{\infty}) \simeq \operatorname{coCFib}([n])^{\simeq}.$$

The claim then follows from the naturality of the Grothendieck construction.

Next, we would like to correspondingly identify the nerve of the lax overcategory of \mathcal{C} (along with that of its source projection). For this, observe (recalling Example 3.2) that the datum of a morphism in $\mathcal{L}ax(\mathcal{C})$ is specified by the pair of

• its image $[1] \xrightarrow{H} Cat_{\infty}$ under the source projection, which by Lemma 3.8 is equivalent to specifying a point

$$\left(\operatorname{Gr}(H) \xrightarrow{\operatorname{pr}_{\operatorname{Gr}(H)}} [1]\right) \in \operatorname{coCFib}([1])^{\simeq} \in \mathcal{S},$$

along with

 a map Gr(H) → C from (the underlying ∞-category of) the "directed mapping cylinder" Gr(H) into our fixed target ∞-category C.

Moreover, a similar observation holds when we replace the object $[1] \in \Delta$ by an arbitrary object $[n] \in \Delta$. Altogether, this identifies the canonical maps

$$\mathrm{N}_{\infty}(\mathcal{L}\mathrm{ax}(\mathfrak{C}))_n \to \mathrm{N}_{\infty}(\mathfrak{C}\mathrm{at}_{\infty})_n$$

(obtained by applying the functor

$$N_{\infty}(-)_n \simeq \hom_{\operatorname{Cat}_{\infty}}([n], -) \in \operatorname{Fun}(\operatorname{Cat}_{\infty}, \mathcal{S})$$

to the source projection map $\mathcal{L}ax(\mathcal{C}) \xrightarrow{s_{\mathcal{L}ax}(\mathcal{C})} \mathcal{C}at_{\infty}$ as being the cocartesian fibration associated to a certain map

$$N_{\infty}(\operatorname{Cat}_{\infty})_n \to S.$$

This motivates our desired identification (Lemma 3.10), but in order to state it precisely we first introduce the following notation.

Notation 3.9. To ease notation, for any ∞ -category \mathcal{C} we denote by U_C any sub-composite of the composite

$$\operatorname{coCFib}(\mathcal{C})^{\simeq} \hookrightarrow \operatorname{coCFib}(\mathcal{C}) \xrightarrow{\operatorname{U_{coCFib}(\mathcal{C})}} (\operatorname{Cat}_{\infty})_{/\mathcal{C}} \to \operatorname{Cat}_{\infty}$$

of the inclusion of the maximal subgroupoid with the two evident forgetful functors. Moreover, we denote by $U_{\mathcal{C}}^{\dagger}$ any sub-composite beginning at $\operatorname{co}\mathcal{CFib}(\mathcal{C})^{\simeq}$ of the composite

$$\operatorname{coCFib}(\mathbb{C})^{\simeq} \xrightarrow{\sim} (\operatorname{coCFib}(\mathbb{C})^{\simeq})^{op} \hookrightarrow (\operatorname{coCFib}(\mathbb{C}))^{op} \xrightarrow{\operatorname{U}_{\operatorname{coCFib}}^{op}(\mathbb{C})} \left((\operatorname{Cat}_{\infty})_{/\mathbb{C}} \right)^{op} \to (\operatorname{Cat}_{\infty})^{op}$$

of the canonical equivalence followed by the opposite of the above composite. We also use this same notation when restricting to the subcategory $\mathcal{L}Fib(\mathcal{C}) \subset co\mathcal{C}Fib(\mathcal{C})$ or to its maximal subgroupoid.

Lemma 3.10. *Fix any* $C \in Cat_{\infty}$ *.*

(1) For any $n \ge 0$, there is a canonical equivalence

$$\mathcal{N}_{\infty}(\mathcal{L}ax(\mathcal{C}))_{n} \simeq \mathcal{U}_{\operatorname{coCFib}([n])^{\simeq}} \left(\operatorname{Gr} \left(\operatorname{coCFib}([n])^{\simeq} \xrightarrow{\mathcal{U}_{[n]}^{\dagger}(-)} (\operatorname{Cat}_{\infty})^{op} \xrightarrow{\operatorname{hom}_{\operatorname{Cat}_{\infty}}(-,\mathcal{C})} S \right) \right)$$

in S.

(2) The equivalences of part (1) assemble into a canonical identification of $N_{\infty}(\mathcal{L}ax(\mathbb{C}))_{\bullet} \in \mathbb{CSS} \subset sS$ with the composite

$$\Delta^{op} \hookrightarrow (\operatorname{Cat}_{\infty})^{op} \xrightarrow{\operatorname{U}_{\operatorname{coCFib}(-)}^{c} \simeq} \left(\operatorname{Gr}\left(\operatorname{coCFib}(-)^{\simeq} \xrightarrow{\operatorname{U}_{(-)}^{\dagger}(=)} (\operatorname{Cat}_{\infty})^{op} \xrightarrow{\operatorname{hom}_{\operatorname{Cat}_{\infty}}(=,\mathbb{C})} \right) \right) \\ S.$$

Proof. Part (1) follows directly from Definition 3.3, and part (2) follows from the naturality of the Grothendieck construction. \Box

Notation 3.11. We denote by $\mathcal{L}ax(\mathcal{C})^{\operatorname{colim}} \subset \mathcal{L}ax(\mathcal{C})$ the full subcategory on those functors $\mathcal{D} \xrightarrow{F} \mathcal{C}$ which admit a colimit in \mathcal{C} .

We can now give the main result of this section, whose output we refer to as the *global colimit functor* for C.

Proposition 3.12. For any $C \in Cat_{\infty}$, there is a functor

$$\mathcal{L}ax(\mathcal{C})^{\operatorname{colim}} \xrightarrow{\operatorname{colim}} \mathcal{C}$$

which takes an object $(\mathcal{D} \xrightarrow{F} \mathcal{C}) \in \mathcal{L}ax(\mathcal{C})^{\operatorname{colim}}$ to $\operatorname{colim}_{\mathcal{D}}(F) \in \mathcal{C}$. Moreover, this functor

• restricts to the usual colimit functor on each fiber

$$\operatorname{Fun}(\mathcal{D}, \mathfrak{C})^{\operatorname{colim}} = \operatorname{Fun}(\mathcal{D}, \mathfrak{C}) \cap \mathcal{L}\operatorname{ax}(\mathfrak{C})^{\operatorname{colim}}$$

and

• takes a commutative diagram



(considered as a cartesian morphism in $Lax(\mathbb{C})^{\text{colim}}$) to the canonical induced map

$$\operatorname{colim}_{\mathcal{D}} F \to \operatorname{colim}_{\mathcal{E}} G$$

 $in \ {\mathbb C}.$

Proof. In order to prove the claim, we make the following construction. Define $\mathcal{L}ax(\mathcal{C})' \in \mathcal{C}at_{\infty}$ to be the unique ∞ -category such that $N_{\infty}(\mathcal{L}ax(\mathcal{C})')_{\bullet} \in \mathcal{CSS} \subset sS$ is given by

$$\Delta^{op} \hookrightarrow (\operatorname{Cat}_{\infty})^{op} \xrightarrow{\operatorname{U}_{\operatorname{co}\mathscr{C}\mathscr{F}\operatorname{ib}(-)}\simeq} \left(\operatorname{Gr} \left(\operatorname{co}\mathscr{C}\mathscr{F}\operatorname{ib}(-)^{\simeq} \xrightarrow{\operatorname{U}_{(-)}^{\dagger} \left(= \circ \atop (-)^{(-)} \right)} (\operatorname{Cat}_{\infty})^{op} \xrightarrow{\operatorname{hom}_{\operatorname{Cat}_{\infty}}(=, \mathfrak{C})} \mathfrak{S} \right) \right)_{!} \mathfrak{S},$$

where the subscript decorating the second functor indicates that we are restricting to the subspace corresponding to those pairs of a functor $[n] \xrightarrow{F} Cat_{\infty}$ and a map

$$\mathrm{U}_{[n]}^{\dagger}\left(\mathrm{Gr}(F)\underset{[n]}{\diamond}[n]
ight)
ightarrow \mathbb{C}$$

such that the induced composite diagram

$$\begin{array}{c} \mathbf{U}_{[n]}^{\dagger}(\mathrm{Gr}(F)) \longrightarrow \mathbb{C} \\ & \swarrow \\ \mathbf{U}_{[n]}^{\dagger} \left(\mathrm{Gr}(F) \underset{[n]}{\diamond} [n] \right) \end{array}$$

defines a left Kan extension (along the inclusion of a full subcategory). The canonical inclusions

$$\operatorname{Gr}(F) \hookrightarrow \operatorname{Gr}(F) \underset{[n]}{\diamond} [n] \hookleftarrow [n]$$

induce maps

$$N_{\infty}(\mathcal{L}ax(\mathcal{C}))_{\bullet} \leftarrow N_{\infty}(\mathcal{L}ax(\mathcal{C})')_{\bullet} \to N_{\infty}(\mathcal{C})_{\bullet} \times N_{\infty}(\mathcal{C}at_{\infty})_{\bullet}$$

in $CSS \subset sS$, where the identification of $N_{\infty}(\mathcal{L}ax(\mathcal{C}))_{\bullet}$ comes from Lemma 3.10(2) and the identification of $N_{\infty}(\mathcal{C})_{\bullet} \times N_{\infty}(Cat_{\infty})_{\bullet}$ follows from Lemma 3.8 and Example 1.15. It then follows from Proposition T.4.3.2.15 that we have an induced factorization



via an equivalence in $\operatorname{Cat}_{\infty}$. Moreover, combining this diagram with the composite $\operatorname{Lax}(\mathfrak{C})' \to \mathfrak{C} \times \operatorname{Cat}_{\infty} \to \mathfrak{C}$ with the projection map gives us a unique functor



making the diagram commute. By Proposition T.4.3.3.10, this is precisely the desired functor; moreover, it is clear from the construction that it encodes the asserted functorialities. \Box

Remark 3.13. Clearly, the global colimit functor (Proposition 3.12) is itself functorial in the following sense: if $C_1 \rightarrow C_2$ is a functor which commutes with all colimits existing in C_1 , then we obtain a commutative square



in $\operatorname{Cat}_{\infty}$.

Remark 3.14. One could also construct the global colimit functor (i.e. prove Proposition 3.12) in the following way. First of all, the Grothendieck construction of the source projection

$$\mathcal{L}ax(\mathcal{C}) \xrightarrow{s_{\mathcal{L}ax}(\mathcal{C})} \mathcal{C}at_{\infty}$$

produces a "tautological bundle" over $\mathcal{L}ax(\mathcal{C})$, a cocartesian fibration whose fiber over an object $(\mathcal{D} \xrightarrow{F} \mathcal{C}) \in \mathcal{L}ax(\mathcal{C})$ is \mathcal{D} itself. This can moreover be shown to admit a tautological map to \mathcal{C} which, restricted to such a fiber, is precisely the functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$. The global colimit functor can then be produced by appealing to Proposition T.4.2.2.7. However, this method is in fact quite a bit more involved than the route we have taken here.

4. Homotopy pullbacks in
$$(Cat_{\infty})_{Th}$$
, finality, and Theorems A, B_n , and C_n

Via the Thomason model structure on $\operatorname{Cat}_{\infty}$ of §A, we can consider ∞ -categories as "presentations of spaces"; the corresponding localization functor $\operatorname{Cat}_{\infty} \to \operatorname{Cat}_{\infty} \llbracket \mathbf{W}_{\mathrm{Th}}^{-1} \rrbracket \simeq S$ is that of groupoid completion. Being a left adjoint, this functor commutes with colimits, but in general its interplay with limits is much more complicated. In this section, we describe certain sufficient conditions under which it commutes with a given pullback.

In the 1-categorical case, there is a long history of results of this variety, going back to Quillen's celebrated [Qui73, Theorem B]. The current state of the art seems to be Barwick–Kan's pair of results [BK, Theorems B_n (5.6) and C_n (5.8)] (the former generalizing Dwyer–Kan–Smith's [DKS89, Theorem B_n (6.2)], the latter identical to their [DKS89, Theorem C_n (6.4)]), as described in §0.1.

The main goal of this section is to give ∞ -categorical generalizations of these results; these appear in §4.3. In §4.1 we work towards this goal with a pair of foundational results surrounding homotopy pullbacks in $(Cat_{\infty})_{Th}$ (whose 1-categorical analogs constitute the main input to the proof of [Qui73, Theorem B]), and in §4.2 we take a moment to briefly restate Joyal's quasicategorical analog (namely Theorem T.4.1.3.1) of Quillen's [Qui73, Theorem A] in invariant langauge (i.e. stated in Cat_{∞} instead of in $sSet_{Joyal}$).

4.1. Homotopy pullbacks in $(Cat_{\infty})_{Th}$: a first pass. In and of themselves, pullbacks among ∞ categories are relatively understandable; for instance, limits commute with the right adjoint of the composite
adjunction

$$sS \xleftarrow[U_{CSS}]{} CSS \xleftarrow[N_{\infty}^{-1}]{} Cat_{\infty}.$$

On the other hand, it is a subtle question to determine when such a pullback commutes with groupoid completion (or, working in complete Segal spaces, with geometric realization (recall Proposition N.2.4)). In this subsection, we address this question in the special case that one of the maps in the pullback is a special sort of cocartesian fibration.

We begin with the relevant definition, an analog of [Qui73, 4.4].

Definition 4.1. We say that a functor $\mathbb{C} \xrightarrow{F} \mathbb{C}at_{\infty}$ has *property* Q if it factors through $\mathbf{W}_{\mathrm{Th}} \subset \mathbb{C}at_{\infty}$.

The importance of Definition 4.1 stems from the following result, which we call *Lemma Q*, an analog of [BK, Lemma 4.5] (there called "Quillen's lemma").

Lemma 4.2. If $\mathfrak{C} \xrightarrow{F} \mathfrak{Cat}_{\infty}$ factors through $\mathbf{W}_{\mathrm{Th}} \subset \mathfrak{Cat}_{\infty}$, then for any $x \in \mathfrak{C}$, the fiber inclusion



is a homotopy pullback square in $(\operatorname{Cat}_{\infty})_{\mathrm{Th}}$, i.e. it gives rise to a pullback square



in S.

and

Proof. Let $C \in sSet_{Joyal}^{f}$ be a quasicategory presenting $C \in Cat_{\infty}$, let $D \to C$ be a JL-cocartesian fibration presenting $(Gr(F) \to C) \in coCFib(C)$, let $D' \to C$ be a JL-left fibration presenting $(L_{\mathcal{L}Fib(C)}(Gr(F)) \to C) \in \mathcal{L}Fib(C)$, and let $\mathbf{x} \in C_0$ be a vertex corresponding to $x \in C$ (see [MGj, Definition 3.2]). Let us define $\mathbf{F}, \mathbf{F}' \in sSet$ via the pullback squares



in sSet. Considered in sSet_{Joyal}, these present fiber inclusions, the first of which is

$$F(x) \longrightarrow \operatorname{Gr}(F)$$

$$\downarrow \qquad \qquad \downarrow$$

$$\{x\} \longleftrightarrow \mathcal{C}$$

and the second of which, by Proposition 1.18, we can identify as

Moreover, by Proposition T.2.1.3.1, the assertion that

$$\mathfrak{C} \xrightarrow{F} \mathfrak{Cat}_{\infty} \xrightarrow{(-)^{\mathrm{gpd}}} \mathfrak{S}$$

factors through $S^{\simeq} \subset S$ is equivalent to the assertion that the map $D' \to C$ is in fact in \mathbf{F}_{KQ} . Since $sSet_{KQ}$ is right proper (see [Hir03, Theorem 13.1.13]), it follows that the pullback square



in $sSet_{KQ}$ is also a homotopy pullback square, which implies that



is a pullback square in S. By Remark 1.4 we obtain an equivalence

$$\operatorname{Gr}(F)^{\operatorname{gpd}} \xrightarrow{\sim} (\operatorname{L}_{\mathcal{L}\operatorname{Fib}(\mathcal{C})}(\operatorname{Gr}(F)))^{\operatorname{gpd}}$$

in $S_{/Cgpd}$, which completes the proof.

We also have the following parametrized version of Lemma Q (4.2).

Corollary 4.3. If $\mathbb{C} \xrightarrow{F} \operatorname{Cat}_{\infty}$ has property Q, then for any $\mathbb{D} \xrightarrow{G} \mathbb{C}$, the resulting pullback $\mathbb{D} \times_{\mathbb{C}} \operatorname{Gr}(F)$ is a homotopy pullback in $(\operatorname{Cat}_{\infty})_{\mathrm{Th}}$.

Proof. By the naturality of the Grothendieck construction, we have a commutative square

$$\begin{array}{c} \operatorname{Gr}(F \circ G) \longrightarrow \operatorname{Gr}(F) \\ \downarrow & \downarrow \\ \mathcal{D} \xrightarrow{G} & \mathcal{C} \end{array}$$

in $\operatorname{Cat}_{\infty}$, where both of the vertical maps are cocartesian fibrations. We would like to show that this becomes a pullback square upon application of $(-)^{\operatorname{gpd}}$: $\operatorname{Cat}_{\infty} \to \mathcal{S}$. For this, note that any map $\operatorname{pt}_{\mathcal{S}} \xrightarrow{x} \mathcal{D}^{\operatorname{gpd}}$ in \mathcal{S} comes from a map $\operatorname{pt}_{\operatorname{Cat}_{\infty}} \xrightarrow{\tilde{x}} \mathcal{D}$ in $\operatorname{Cat}_{\infty}$. Then, we have the diagram



in $\operatorname{Cat}_{\infty}$, whose back face is the above commutative square and in which the upper oblique arrows are the fiber inclusions over $\tilde{x} \in \mathcal{D}$ and $G(\tilde{x}) \in \mathcal{C}$. By Lemma Q (4.2), we obtain that applying $(-)^{\operatorname{gpd}} : \operatorname{Cat}_{\infty} \to S$ to this commutative diagram yields a commutative diagram in S in which the oblique squares are pullbacks. Hence, the square



is a pullback in S, since for every point of \mathcal{D}^{gpd} the induced map on corresponding fibers is an equivalence. \Box

4.2. Finality and Theorem A. In this subsection, we briefly recall a few definitions and results from \S T.4.1, restating them in invariant language.

Definition 4.4. A functor $\mathcal{I} \xrightarrow{F} \mathcal{J}$ is called *final* if, for any functor $\mathcal{J} \xrightarrow{G} \mathfrak{C}$ such that $\operatorname{colim}_{\mathcal{J}} G$ exists, the colimit $\operatorname{colim}_{\mathcal{J}}(G \circ F)$ also exists and the natural map

$$\operatorname{colim}_{\mathcal{I}}(G \circ F) \to \operatorname{colim}_{\mathcal{I}} G$$

(in the sense of the global colimit functor (Proposition 3.12)) is an equivalence in C.

Dually, a functor $\mathfrak{I} \xrightarrow{F} \mathfrak{J}$ is called *initial* if, for any functor $\mathfrak{J} \xrightarrow{G} \mathfrak{C}$ such that $\lim_{\mathfrak{J}} G$ exists, the limit $\lim_{\mathfrak{I}} (G \circ F)$ also exists and the natural map

$$\lim_{\mathcal{A}} (G) \to \lim_{\mathcal{A}} (G \circ F)$$

(in the sense of the global limit functor (the dual of Proposition 3.12)) is an equivalence in C.

Remark 4.5. A functor $\mathcal{I} \to \mathcal{J}$ is initial if and only if its opposite $\mathcal{I}^{op} \to \mathcal{J}^{op}$ is final.

Remark 4.6. The notion of finality given in Definition 4.4 is also sometimes called "cofinality" or "right cofinality", while that initiality is also sometimes called "co-cofinality" or "left cofinality". We have chosen our terminology because it seems most natural: the simplest example of a final functor is the inclusion $\{pt_{\mathcal{J}}\} \hookrightarrow \mathcal{I}$ of a final object, while the simplest example of an initial functor is the inclusion $\{\varnothing_{\mathcal{I}}\} \hookrightarrow \mathcal{I}$ of an initial object.

Remark 4.7. By Proposition T.4.1.1.8 (see also Corollary T.4.1.1.10), the notion of finality given in Definition 4.4 is an invariant version of the quasicategorical notion of "cofinality" (given e.g. as Definition T.4.1.1.1).

Proposition 4.8. Every final functor lies in $\mathbf{W}_{Th} \subset Cat_{\infty}$.

Proof. This follows from Proposition T.4.1.1.3(3).

Proposition 4.9. If $\mathfrak{I}_1 \xrightarrow{F_1} \mathfrak{J}_1$ and $\mathfrak{I}_2 \xrightarrow{F_2} \mathfrak{J}_2$ are both final, then so is $\mathfrak{I}_1 \times \mathfrak{I}_2 \xrightarrow{F_1 \times F_2} \mathfrak{J}_1 \times \mathfrak{J}_2$.

Proof. By Corollary T.4.1.1.13, both of the functors in the composite

$$\mathbb{J}_1 \times \mathbb{J}_2 \xrightarrow{F_1 \times \mathrm{id}_{\mathbb{J}_2}} \mathbb{J}_1 \times \mathbb{J}_2 \xrightarrow{\mathrm{id}_{\mathbb{J}_1} \times F_2} \mathbb{J}_1 \times \mathbb{J}_2$$

are final, and hence by Proposition T.4.1.1.3(2) their composite $F_1 \times F_2$ is also final.⁷

The next result is the main point of this subsection, Joyal's ∞ -categorical analog of [Qui73, Theorem A]; we refer to it simply as **Theorem A**.

Theorem 4.10. A functor $\mathfrak{I} \xrightarrow{F} \mathfrak{J}$ is final iff for every object $j \in \mathfrak{J}$,

$$(\mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{i/})^{\mathrm{gpd}} \simeq \mathrm{pt}_{\mathcal{S}}.$$

Proof. This follows from Theorem T.4.1.3.1; by the Reedy trick (and the implications of Remark T.2.0.0.5 and Corollary T.2.1.2.2), the pullback given there is a homotopy pullback in $sSet_{Joyal}$.

Corollary 4.11. If $\mathcal{J} \in \operatorname{Cat}_{\infty}$ has a terminal object, then $\mathcal{J}^{\operatorname{gpd}} \simeq \operatorname{pt}_{\mathfrak{S}}$.

Proof. We apply Theorem 4.10 to deduce that the functor $\{\text{pt}_{\mathcal{J}}\} \hookrightarrow \mathcal{J}$ is final: for any $j \in \mathcal{J}$, we have that $\{\text{pt}_{\mathcal{J}}\} \times_{\mathcal{J}} \mathcal{J}_{j/} \simeq \hom_{\mathcal{J}}(j, \text{pt}_{\mathcal{J}}) \simeq \text{pt}_{\mathcal{S}}$. Hence, the claim follows from Proposition 4.8.

Remark 4.12. One could also prove Corollary 4.11 by observing that a functor $pt_{Cat_{\infty}} \to \mathcal{J}$ is a right adjoint if (and only if) it selects a terminal object, and then appealing to Corollary N.1.28.

4.3. Theorems \mathbf{B}_n and \mathbf{C}_n : a second pass at homotopy pullbacks in $(\operatorname{Cat}_{\infty})_{\mathbf{Th}}$. In this final subsection, we provide ∞ -categorical generalizations of Barwick–Kan's pair of results [BK, Theorems \mathbf{B}_n (5.6) and \mathbf{C}_n (5.8)].

Remark 4.13. The results of this subsection will be used in \S M.9. In fact, there will actually employ their dual formulations. Our choice of variances in this subsection are so that our exposition adheres as closely as possible to that of [BK, \S 5].

We begin with the following.

⁷This is of course closely related to Fubini's theorem for colimits, but to deduce it from that result would require slightly trickier manipulations depending on which of the various colimits actually exist.

Notation 4.14. For $n \ge 1$, we define $\underline{\mathbf{z}}_n \in \text{cat}$ by the pattern

$$\begin{aligned} \underline{\mathbf{z}}_1 &= (s \to t) \,, \\ \underline{\mathbf{z}}_2 &= (s \leftarrow \bullet \to t) \,, \\ \underline{\mathbf{z}}_3 &= (s \to \bullet \leftarrow \bullet \to t) \,, \\ \underline{\mathbf{z}}_4 &= (s \leftarrow \bullet \to \bullet \leftarrow \bullet \to t) \end{aligned}$$

etc. (where we have named only the leftmost and rightmost objects of these categories).

We now give the following omnibus definition.

Definition 4.15. For any $n \ge 1$ and any $\mathcal{C} \in \operatorname{Cat}_{\infty}$, we define $(\mathcal{C} \downarrow_n \mathcal{C}) = \operatorname{Fun}(\underline{\mathbf{z}}_n, \mathcal{C})$. Evaluation at the objects $s, t \in \underline{\mathbf{z}}_n$ induces maps

$$\mathfrak{C} \xleftarrow{s} (\mathfrak{C} \downarrow_n \mathfrak{C}) \xrightarrow{t} \mathfrak{C}.$$

More generally, for any functors $\mathcal{D} \xrightarrow{F} \mathcal{C}$ and $\mathcal{E} \xrightarrow{G} \mathcal{C}$ and any objects $d \in \mathcal{D}$ and $e \in \mathcal{E}$, we define new ∞ -categories and maps between them via the induced diagram

in $\operatorname{Cat}_{\infty}$ in which all squares are pullbacks. Thus, we may consider $(F(\mathcal{D}) \downarrow_n G(\mathcal{E}))$ as simultaneously generalizing all three constructions $(F(\mathcal{D}) \downarrow_n \mathbb{C})$, $(\mathbb{C} \downarrow_n G(\mathcal{E}))$ and $(\mathbb{C} \downarrow_n \mathbb{C})$, with the convention that if either F or G is simply $\operatorname{id}_{\mathbb{C}}$ then we omit it from the notation; we refer to any of these constructions (but especially to $(F(\mathcal{D}) \downarrow_n G(\mathcal{E}))$) as a **potential homotopy pullback** ∞ -category. Similarly, the construction $(F(d) \downarrow_n G(\mathcal{E}))$ generalizes the construction $(F(d) \downarrow_n \mathbb{C})$ (in the case that $G = \operatorname{id}_{\mathbb{C}}$), while the construction $(F(\mathcal{D}) \downarrow_n G(e))$ generalizes the construction $(\mathbb{C} \downarrow_n G(e))$ (in the case that $F = \operatorname{id}_{\mathbb{C}}$). Additionally, we denote all vertical maps landing at \mathcal{D} (and in particular, all vertical maps in the third column landing at \mathbb{C}) by sand refer to these as **source** maps, and we denote all horizontal maps landing at \mathcal{E} (and in particular, all horizontal maps in the third row landing at \mathbb{C}) by t and refer to these as **target** maps.

Remark 4.16. To make sense of the terminology, one should think of the construction $(F(\mathcal{D}) \downarrow_n G(\mathcal{E}))$ of Definition 4.15 as a sort of "directed" analog of the standard explicit construction of a homotopy pullback of topological spaces (as the space of pairs of points in the two sources of the cospan equipped with a path between their images in the common target). The question, of course, is whether this actually computes the homotopy pullback in $(Cat_{\infty})_{Th}$ (which explains the word "potential" in the name); a sufficient condition for this to be the case is precisely the content of Theorem B_n (4.23). Continuing along these lines, one might think of $(F(\mathcal{D}) \downarrow_n \mathbb{C})$ and $(\mathbb{C} \downarrow_n G(\mathcal{E}))$ as "directed" analogs of the mapping path space construction, i.e. the standard explicit factorization of an arbitrary map of topological spaces as a weak equivalence followed by a fibration. The reader may find these analogies helpful to keep in mind while reading the rest of this subsection.

In order to simultaneously deal with the case when n is even and when n is odd, we also introduce the following.

Notation 4.17. Inspired by the half-open intervals [0, 1) and (0, 1], we will enclose an expression by [-) when we mean for it to be read only when n is even, while when we will enclose an expression by (-] when we mean for it to be read only when n is odd. So for instance, $C^{[op)}$ denotes C^{op} when n is even, and simply denotes C when n is odd.

Lemma 4.18. Let $n \ge 1$, let $\mathcal{D} \xrightarrow{F} \mathcal{C}$ and $\mathcal{E} \xrightarrow{G} \mathcal{C}$ be any functors.

(1) We have

and

$$\left((F(\mathcal{D}) \downarrow_n G(\mathcal{E})) \xrightarrow{s} \mathcal{D} \right) \in [\mathrm{co}) \mathcal{CFib}(\mathcal{D})$$

$$\left((F(\mathcal{D}) \downarrow_n G(\mathcal{E})) \xrightarrow{t} \mathcal{E} \right) \in \operatorname{coCFib}(\mathcal{E}).$$

(2) For any objects $d \in \mathcal{D}$ and $e \in \mathcal{E}$, we have

$$\left((F(\mathcal{D}) \downarrow_n G(e)) \xrightarrow{s} \mathcal{D} \right) \in [\mathrm{co}) \mathfrak{CFib}(\mathcal{D})$$

and

$$\left((F(d)\downarrow_n G(\mathcal{E})) \xrightarrow{t} \mathcal{E} \right) \in \operatorname{coCFib}(\mathcal{E}).$$

Proof. In both parts, we will only prove the second of the two claims; the first claims follow from nearly identical arguments. Moreover, since cocartesian fibrations are stable under pullback, it suffices to prove these statements in the case that $G = id_c$. For this, let us denote by $t' \in \underline{\mathbf{z}}_n$ the penultimate object (reading from left to right), and let us denote by $\underline{\mathbf{z}}'_n \subset \underline{\mathbf{z}}_n$ the full subcategory on all the objects besides $t \in \underline{\mathbf{z}}_n$, so that we can identify $\underline{\mathbf{z}}_n$ as a pushout

$$\underline{\mathbf{z}}_n \simeq \underline{\mathbf{z}}'_n \coprod_{t', \mathrm{pt}_{\mathrm{Cat}_\infty}, 0} [1]$$

in Cat_{∞} . We therefore obtain a diagram

in $\operatorname{Cat}_{\infty}$ in which all squares are pullbacks and the upper row contains as composites the two functors $(F(\mathcal{D})\downarrow_n \mathbb{C}) \xrightarrow{t} \mathbb{C}$ and $(F(d)\downarrow_n \mathbb{C}) \xrightarrow{t} \mathbb{C}$ which we would like to show are cocartesian fibrations. Both claims now follow by applying the dual of Corollary T.2.4.7.12 to the corresponding composites

$$\mathcal{D} \underset{F, \mathfrak{C}, \mathrm{ev}_s}{\times} \mathrm{Fun}(\underline{\mathbf{z}}'_n, \mathfrak{C}) \to \mathfrak{C}$$

and

$$\operatorname{pt}_{\operatorname{Cat}_{\infty}} \underset{F(d), \operatorname{C}, \operatorname{ev}_{s}}{\times} \operatorname{Fun}(\underline{\mathbf{z}}'_{n}, \operatorname{C}) \to \operatorname{C}$$

in the middle row.

Notation 4.19. We denote the functors classifying the co/cartesian fibrations of Lemma 4.18(1) by

$$\mathcal{D}^{(op]} \xrightarrow{(F(-)\downarrow_n G(\mathcal{E}))} \operatorname{Cat}_{\infty}$$

and

$$\mathcal{E} \xrightarrow{(F(\mathcal{D})\downarrow_n G(-))} \operatorname{Cat}_{\infty}$$

and we denote the functors classifying the co/cartesian fibrations of Lemma 4.18(2) by

$$\mathcal{D}^{(op]} \xrightarrow{(F(-)\downarrow_n G(e))} \operatorname{Cat}_{\infty}$$

and

$$\mathcal{E} \xrightarrow{(F(d)\downarrow_n G(-))} \operatorname{Cat}_{\sim}$$

(again omitting the functor F or G if it is just id_c). Note that this is indeed consistent with the notation given in Definition 4.15, and identifies all of the various pullbacks over $pt_{Cat_{\infty}}$ in the diagram given there as fiber inclusions. (In particular, the ∞ -category ($F(d) \downarrow_n G(e)$) includes as a fiber of two different co/cartesian fibrations.)

Remark 4.20. We may consider Lemma 4.18(2) as equipping the first two functors defined in Notation 4.19 with lifts



and

(through the evident forgetful functors). (Preservation of co/cartesian morphisms follows from Corollary T.2.4.7.12.)

We then have the following elementary but important observations.

Proposition 4.21. Let $n \ge 1$, and choose any functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ and $\mathcal{E} \xrightarrow{G} \mathcal{C}$.

(1) The unique functor $\underline{\mathbf{z}}_n \to \operatorname{pt}_{\operatorname{Cat}_{\infty}}$ induces a common section

$$(\mathcal{C}\downarrow_n \mathcal{C}) \xrightarrow[t]{q} \mathcal{C}$$

to the source and target maps.

(2) The common section of part (1) induces sections

$$(F(\mathcal{D})\downarrow_n^{\checkmark} \overset{q}{\mathbb{C}}) \xrightarrow{q} \mathcal{D}$$

and

$$(\mathcal{C}\downarrow_n G(\overset{\swarrow}{\mathcal{E})}) \xrightarrow{q} \mathcal{E}.$$

(3) The section diagrams of part (2) (and in particular, those of part (1)) define homotopy equivalences in (Cat_∞)_{Th}.

Proof. Part (1) follows from the fact that both composites $\operatorname{pt}_{\operatorname{Cat}_{\infty}} \rightrightarrows \underline{\mathbf{z}}_n \to \operatorname{pt}_{\operatorname{Cat}_{\infty}}$ are canonically equivalent to $\operatorname{id}_{\operatorname{pt}_{\operatorname{Cat}_{\infty}}}$. Then, part (2) follows from part (1) and the definitions of $(F(\mathcal{D}) \downarrow_n \mathcal{C})$ and $(\mathcal{C} \downarrow_n G(\mathcal{E}))$ as pullbacks. For part (3), observe first that either composite $\underline{\mathbf{z}}_n \to \operatorname{pt}_{\operatorname{Cat}_{\infty}} \rightrightarrows \underline{\mathbf{z}}_n$ is connected by a zigzag of natural transformations to $\operatorname{id}_{\underline{\mathbf{z}}_n}$; moreover, working with $\operatorname{pt}_{\operatorname{Cat}_{\infty}} \stackrel{s}{\to} \underline{\mathbf{z}}_n$ this zigzag can be taken in eat_{*} where we point our categories using their source objects (i.e. such that all the constituent natural transformations of the zigzag have the map id_s as their component at s), and similarly for working with $\operatorname{pt}_{\operatorname{Cat}_{\infty}} \stackrel{t}{\to} \underline{\mathbf{z}}_n$. By applying Lemma H.3.5 (where we take \mathcal{C} to be equipped with the maximal relative structure), we see that either composite $(\mathcal{C} \downarrow_n \mathcal{C}) \rightrightarrows \mathcal{C} \to (\mathcal{C} \downarrow_n \mathcal{C})$ is in turn connected to $\operatorname{id}_{(\mathcal{C}\downarrow_n\mathcal{C})}$ by a zigzag of natural transformations, such that all of the constituent natural transformations commute with the chosen projection $(\mathcal{C} \downarrow_n \mathcal{C}) \to \mathcal{C}$ (either the source or target map). Since the functor $\operatorname{Cat}_{\infty} \stackrel{-\times[1]}{\longrightarrow} \operatorname{Cat}_{\infty}$ commutes with pullbacks (being a limit), by the functoriality of pullbacks these induce zigzags of natural transformations between the composite

$$(F(\mathcal{D})\downarrow_n \mathcal{C}) \xrightarrow{s} \mathcal{D} \to (F(\mathcal{D})\downarrow_n \mathcal{C})$$

and $\mathrm{id}_{(F(\mathcal{D})\downarrow_n \mathcal{C})}$ and between the composite

$$(\mathfrak{C}\downarrow_n G(\mathfrak{E})) \xrightarrow{\iota} \mathfrak{E} \to (\mathfrak{C}\downarrow_n G(\mathfrak{E}))$$

and $\operatorname{id}_{(\mathcal{C}\downarrow_n G(\mathcal{E}))}$. Thus, the claim follows from Lemma N.1.26.

We now define the key concept of this subsection.

Definition 4.22. We say that a functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ has *property* B_n if the functor

$$\mathfrak{C} \xrightarrow{(F(\mathcal{D})\downarrow_n -)} \mathfrak{C}\mathrm{at}_{\infty}$$

has property Q.

We now give the main result of this subsection, which we refer to as **Theorem** B_n (for homotopy pullbacks (in $(Cat_{\infty})_{Th}$)).

Theorem 4.23. If the functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ has property B_n , then for any functor $\mathcal{E} \xrightarrow{G} \mathcal{C}$, the (not generally commutative) square

$$\begin{array}{c} (F(\mathcal{D}) \downarrow_n G(\mathcal{E})) \stackrel{s}{\longrightarrow} \mathcal{D} \\ \downarrow \\ \iota \\ \mathcal{E} \stackrel{f}{\longrightarrow} \mathcal{C} \end{array}$$

is a homotopy pullback square in $(Cat_{\infty})_{Th}$, i.e. it induces a (commutative) pullback square



 $in \ {\mathbb S}.$

Proof. To say that $\mathcal{D} \xrightarrow{F} \mathcal{C}$ has property B_n is to say that the functor

$$\mathfrak{C} \xrightarrow{(F(\mathfrak{D})\downarrow_n)} \mathfrak{Cat}_{\infty}$$

has property Q. By the naturality of the Grothendieck construction, the functor

$$\mathcal{E} \xrightarrow{(F(\mathcal{D})\downarrow_n G(-))} \mathfrak{Cat}_{\infty}$$

is precisely the composite

$$\mathcal{E} \xrightarrow{G} \mathcal{C} \xrightarrow{(F(\mathcal{D})\downarrow_n -)} \mathfrak{Cat}_{\infty},$$

and therefore also has property Q. Hence, by Lemma Q (4.2), for any objects $c \in \mathcal{C}$ and $e \in \mathcal{E}$ the fiber inclusions



and

$$\begin{array}{c} (F(\mathcal{D})\downarrow_n G(e)) \longrightarrow (F(\mathcal{D})\downarrow_n G(\mathcal{E})) \\ \downarrow & \qquad \qquad \downarrow^t \\ \{e\} \longrightarrow \mathcal{E} \end{array}$$

are both homotopy pullback squares in $(Cat_{\infty})_{Th}$.

Now, observe that we have a diagram



in $(Cat_{\infty})_{Th}$, in which

- the map labeled q comes from Proposition 4.21(2),
- every bounded connected region is commutative except for the one containing the symbol \approx , which is homotopy commutative (and is hence bounded by weak equivalences in $(Cat_{\infty})_{Th}$) by Proposition 4.21(3), and
- the square is by definition a pullback square in Cat_{∞} .

Our goal, then, reduces to showing that the commutative square in this diagram is also a homotopy pullback square in $(Cat_{\infty})_{Th}$. For this, it suffices to verify that in the induced commutative square

$$\begin{array}{ccc} (F(\mathcal{D})\downarrow_n G(\mathcal{E}))^{\mathrm{gpd}} & \longrightarrow (F(\mathcal{D})\downarrow_n \mathcal{C})^{\mathrm{gpd}} \\ & & \downarrow^{t^{\mathrm{gpd}}} & & \downarrow^{t^{\mathrm{gpd}}} \\ & & \mathcal{E}^{\mathrm{gpd}} & & & \mathcal{C}^{\mathrm{gpd}} \end{array}$$

in S, we obtain an equivalence on fibers over every point of \mathcal{E}^{gpd} .

Now, observe first that any point $\text{pt}_{S} \to \mathcal{E}^{\text{gpd}}$ is represented by a map $\text{pt}_{\text{Cat}_{\infty}} \xrightarrow{e} \mathcal{E}$ in $(\text{Cat}_{\infty})_{\text{Th}}$. Moreover, we have just seen that in the resulting commutative diagram

in $\operatorname{Cat}_{\infty}$, both the left square and the outer rectangle are fiber inclusions which are moreover homotopy pullback squares in $(\operatorname{Cat}_{\infty})_{\mathrm{Th}}$ (where we take $c = G(e) \in \mathcal{C}$). So we do indeed obtain an equivalence on fibers over every point of $\mathcal{E}^{\mathrm{gpd}}$ in the above commutative square in S, which completes the proof.

Remark 4.24. As [DKS89, Theorem B_n (6.2)] specializes to [Qui73, Theorem B] in the case that n = 1, our Theorem B_n (4.23) also generalizes [Bar, Theorem B for ∞ -categories (5.3)].

The following definition allows us to formulate a useful sufficient condition for a functor to have property B_n .

Definition 4.25. We say that $\mathcal{C} \in Cat_{\infty}$ has *property* C_n if every functor $pt_{Cat_{\infty}} \to \mathcal{C}$ has property B_n .

The sufficient condition is then provided by the following main supporting result of this section, which we refer to as **Theorem** C_n .

Theorem 4.26. If $\mathbb{C} \in \operatorname{Cat}_{\infty}$ has property C_n , then any functor $\mathcal{D} \xrightarrow{F} \mathbb{C}$ has property B_n .

Proof. We must show that for any map $c_1 \xrightarrow{\varphi} c_2$ in \mathcal{C} , the resulting functor

$$(F(\mathcal{D})\downarrow_n c_1) \to (F(\mathcal{D})\downarrow_n c_2)$$

is in $\mathbf{W}_{Th} \subset \operatorname{Cat}_{\infty}$. Recall from Remark 4.20 that this functor can be considered as the image of the map φ under a functor

$$\mathfrak{C} \xrightarrow{(F(\mathfrak{D})\downarrow_n -)} [\mathrm{co})\mathfrak{CFib}(\mathfrak{D}).$$

Via the Grothendieck construction

$$\operatorname{Fun}(\mathcal{D}^{(op]},\operatorname{Cat}_{\infty}) \xrightarrow{\operatorname{Gr}} [\operatorname{co})\mathfrak{CFib}(\mathcal{D})$$

this is classified by a natural transformation

$$(F(-)\downarrow_n c_1) \xrightarrow{\operatorname{Gr}^{-1}((F(\mathcal{D})\downarrow_n \varphi))} (F(-)\downarrow_n c_2)$$

in Fun $(\mathcal{D}^{(op]}, \mathfrak{Cat}_{\infty})$, and by Corollary 2.2 it suffices to show that the components of this natural transformation lie in $\mathbf{W}_{\mathrm{Th}} \subset \mathfrak{Cat}_{\infty}$. This is just the assertion that for any object $d \in \mathcal{D}$, the induced map

$$(F(d)\downarrow_n c_1) \to (F(d)\downarrow_n c_2)$$

is in $\mathbf{W}_{\mathrm{Th}} \subset \operatorname{Cat}_{\infty}$, which follows by applying the definition of property C_n to the functor $\operatorname{pt}_{\operatorname{Cat}_{\infty}} \xrightarrow{F(d)} \mathcal{C}$. \Box

5. The Bousfield-Kan colimit formula

In 1-category theory, there's an extremely useful formula which expresses an arbitrary colimit as a coequalizer of maps between coproducts (at least when the ambient category is cocomplete). Namely, if we are given $\mathcal{C}, \mathcal{D} \in \text{cat}, \mathcal{C}$ admits coproducts and coequalizers, and $\mathcal{D} \xrightarrow{F} \mathcal{C}$ is any functor, then we have an isomorphism

$$\operatorname{colim}_{\mathcal{D}} F \cong \operatorname{coeq} \left(\coprod_{(d_1 \to d_2) \in \mathcal{N}(\mathcal{D})_1} F(d_1) \rightrightarrows \coprod_{d \in \mathcal{N}(\mathcal{D})_0} F(d) \right)$$

in \mathcal{C} , where on the summand corresponding to $(d_1 \xrightarrow{\varphi} d_2) \in \mathcal{N}(\mathcal{D})_1$, one map is given by

$$F(d_1) \xrightarrow{\operatorname{id}_{F(d_1)}} F(d_1)$$

(landing on the summand corresponding to $d_1 \in \mathcal{N}(\mathcal{D})_0$) and the other map is given by

$$F(d_1) \xrightarrow{F(\varphi)} F(d_2)$$

(landing on the summand corresponding to $d_2 \in \mathcal{N}(\mathcal{D})_0$). (In fact, it follows from this formula that \mathcal{C} is cocomplete if (and only if) it admits coproducts and coequalizers.)

In this section, we generalize this colimit formula to ∞ -categories. Two things must be changed. First of all, this coequalizer will be replaced by a geometric realization.⁸ Moreover, the coproducts over the *sets* of objects and morphisms of \mathcal{D} will be replaced by the colimits over the *spaces* of objects, morphisms, pairs of two composable morphisms, etc., of the diagram ∞ -category – in other words, over the constituents of its ∞ -categorical nerve. To be more precise, the *simplicial replacement* of a diagram $\mathcal{D} \xrightarrow{F} \mathcal{C}$ in \mathcal{C} will be a simplicial object srep $(F)_{\bullet} \in s\mathcal{C}$ which in level n is the colimit of the composite

$$N_{\infty}(\mathcal{D})_n \xrightarrow{\{0\}} N_{\infty}(\mathcal{D})_0 \simeq \mathcal{D}^{\simeq} \hookrightarrow \mathcal{D} \xrightarrow{F} \mathcal{C}$$

(which of course only need exist when \mathcal{C} has a sufficient supply of colimits). Then, the geometric realization of the simplicial replacement will compute the colimit $\operatorname{colim}_{\mathcal{D}} F$ of the original diagram in \mathcal{C} .

This section is organized as follows. In §5.1, we carefully construct the simplicial replacement and prove the *Bousfield–Kan colimit formula* (Theorem 5.8). In §5.2, we provide some examples to illustrate the usage of this formula. Finally, in §5.3, we provide a functoriality result.

Remark 5.1. Of course, one can dualize this entire section to obtain analogous constructions and results concerning limits in a complete ∞ -category.

5.1. The Bousfield–Kan colimit formula. In this subsection, we construct the simplicial replacement and prove the main result of this section. We begin with the following observation.

Remark 5.2. The structure maps of the diagrams in \mathcal{C} assembling to the simplicial replacement are not strictly compatible: we will only have a lax natural transformation $N_{\infty}(\mathcal{D})_{\bullet} \rightsquigarrow \operatorname{const}(\mathcal{D})$ in $\operatorname{Fun}(\Delta^{op}, \operatorname{Cat}_{\infty})$, i.e. a map



making the triangle commute, which by precomposition will induce the passage to the simplicial replacement. For instance, a point $\varphi \in \mathcal{N}_{\infty}(\mathcal{D})_1$ corresponds to a morphism $x \xrightarrow{\varphi} y$ in \mathcal{D} , and the above map should send this to the point $x \in \mathcal{D} \times \{[1]^\circ\}$. On the other hand, the two simplicial structure maps take the point

⁸Recall that $\Delta_{\leq 1}^{op}$ is the walking reflexive pair, so that colimits over it are precisely reflexive coequalizers. In fact, the above pair of parallel arrows in \mathcal{C} is indeed a reflexive pair: a common section is given by taking the summand F(d) over the element $d \in \mathcal{N}(\mathcal{D})_0$ to summand F(d) over the element $\mathrm{id}_d \in \mathcal{N}(\mathcal{D})_1$ via the identity map $\mathrm{id}_{F(d)}$ in \mathcal{C} .

 $\varphi \in \mathcal{N}_{\infty}(\mathcal{D})_1$ to the points $x, y \in \mathcal{N}_{\infty}(\mathcal{D})_0$, and the map $\mathcal{N}_{\infty}(\mathcal{D})_0 \to \mathcal{D} \times \{[0]^\circ\}$ is simply the inclusion of the maximal subgroupoid. So in the diagram

$$\begin{array}{c|c} \mathrm{N}_{\infty}(\mathcal{D})_{1} & \xrightarrow{\{0\}} \mathcal{D} \\ & & \delta_{0} \\ & & \downarrow \\ & & \downarrow \\ \mathrm{N}_{\infty}(\mathcal{D})_{0} & \xrightarrow{\{0\}} \mathcal{D}, \end{array}$$

going across and then down gives $\varphi \mapsto x \mapsto x$, while going down and then across gives $\varphi \mapsto y \mapsto y$. Hence, the diagram does not strictly commute. However, it will commute up to a natural transformation (running down and to the left), whose component at the point $\varphi \in N_{\infty}(\mathcal{D})_1$ is simply the map φ itself.

Remark 5.3. In light of Remark 5.2, one sees that the "decomposition of colimits" results of §T.4.2.3 do not suffice for our purposes here: they only apply to *strict* diagrams of diagram ∞ -categories lying over our diagram ∞ -category \mathcal{D} .

We now construct the lax natural transformation of Remark 5.2. In fact, we will construct it in a way which is functorial in $\mathcal{D} \in Cat_{\infty}$.⁹

Construction 5.4. We construct the map $\operatorname{Gr}(N_{\infty}(\mathcal{D})_{\bullet}) \to \mathcal{D} \times \Delta^{op}$ in $(\operatorname{Cat}_{\infty})_{/\Delta^{op}}$ as follows. By the universal property of products, it suffices to construct only the map $\operatorname{Gr}(N_{\infty}(\mathcal{D})_{\bullet}) \to \mathcal{D}$ in $\operatorname{Cat}_{\infty}$. This we construct on nerves, i.e. as a map $N_{\infty}(\operatorname{Gr}(N_{\infty}(\mathcal{D})_{\bullet}))_{\bullet} \to N_{\infty}(\mathcal{D})_{\bullet}$ in $\operatorname{CSS} \subset sS$. For this, we will unwind the definitions of these two constructions as functors of $\mathcal{D} \in \operatorname{Cat}_{\infty}$.

Of course, the target is corepresented in level n by $[n] \in \Delta \subset Cat_{\infty}$.

On the other hand, every string of composable morphisms in $\operatorname{Gr}(N_{\infty}(\mathcal{D})_{\bullet})$ is uniquely determined by its source and its image in Δ^{op} (since the functor $\operatorname{Gr}(N_{\infty}(\mathcal{D})_{\bullet}) \to \Delta^{op}$ is a left fibration and hence all maps are cocartesian), and so we obtain an equivalence

$$N_{\infty}(Gr(N_{\infty}(\mathcal{D})_{\bullet}))_n \simeq \coprod_{\alpha \in N(\mathbf{\Delta}^{op})_n} N_{\infty}(\mathcal{D})_{\alpha(0)},$$

where we consider $\alpha \in \mathcal{N}(\mathbf{\Delta}^{op})_n \cong \hom_{eat}([n], \mathbf{\Delta}^{op})$. Hence, the composite functor

$$\operatorname{Cat}_{\infty} \xrightarrow{\mathrm{N}_{\infty}(-)_{\bullet}} s\mathfrak{S} \simeq \operatorname{Fun}(\mathbf{\Delta}^{op}, \mathfrak{S}) \xrightarrow{\mathrm{Gr}} \mathcal{L}\operatorname{Fib}(\mathbf{\Delta}^{op}) \xrightarrow{\mathrm{U}_{\mathcal{L}}(\mathbf{\Delta}^{op})} (\operatorname{Cat}_{\infty})_{/\mathbf{\Delta}^{op}} \to \operatorname{Cat}_{\infty} \xrightarrow{\mathrm{N}_{\infty}(-)_{\bullet}} s\mathfrak{S},$$

considered by adjunction as a simplicial object in $\operatorname{Fun}(\operatorname{Cat}_{\infty}, S)$, is given in each level by a coproduct of corepresentable objects. In level n, it is given by

$$\coprod_{\alpha \in \mathcal{N}(\mathbf{\Delta}^{op})_n} \mathfrak{k}(\alpha(0)^\circ),$$

where we write $\mathfrak{k} = \mathfrak{k}_{(\operatorname{Cat}_{\infty})^{op}}$ for the contravariant Yoneda functor

$$(\operatorname{Cat}_{\infty})^{op} \xrightarrow{\operatorname{hom}_{\operatorname{Cat}_{\infty}}(-,=)} \operatorname{Fun}(\operatorname{Cat}_{\infty}, S)$$

for brevity. To describe its simplicial structure maps, given a map $[n] \xrightarrow{\alpha} \Delta^{op}$, for any $0 \le i \le j \le n$ let us denote the corresponding map in Δ^{op} selected by α by

$$\alpha(i) \xrightarrow{\alpha_{i,j}^{\circ}} \alpha(j)$$

(i.e. the image under α of the unique element of $\hom_{[n]}(i,j)$). Then, associated to a map $[n]^{\circ} \xrightarrow{\varphi^{\circ}} [m]^{\circ}$ in Δ^{op} , the structure map

$$\left(\coprod_{\alpha \in \mathcal{N}(\mathbf{\Delta}^{op})_n} \mathfrak{k}(\alpha(0)^\circ)\right) \to \left(\coprod_{\beta \in \mathcal{N}(\mathbf{\Delta}^{op})_m} \mathfrak{k}(\beta(0)^\circ)\right)$$

⁹This is essentially a homotopy-invariant analog of the "first vertex projection" from the opposite of the category of simplices of a quasicategory. The main difficulty lies in keeping careful track of all coherence data.

of this simplicial object in Fun($\operatorname{Cat}_{\infty}$, S) is given by taking the summand $\sharp(\alpha(0)^{\circ})$ indexed by $\alpha \in \operatorname{hom}_{\operatorname{eat}}([n], \Delta^{op})$ to the summand $\sharp(\beta(0)^{\circ})$ indexed by $\beta = \alpha \circ \varphi \in \operatorname{hom}_{\operatorname{eat}}([m], \Delta^{op})$ via the map

$$\sharp(\alpha(0)^{\circ}) \xrightarrow{\sharp(\alpha_{0,\varphi(0)}^{\circ})} \sharp(\alpha(\varphi(0))^{\circ})$$

in $\operatorname{Fun}(\operatorname{Cat}_{\infty}, S)$. Since the objects of $\operatorname{Fun}(\operatorname{Cat}_{\infty}, S)$ corepresented by gaunt categories and their finite coproducts generate a full subcategory which is just a 1-category, no higher coherence issues arise. (We will implicitly appeal to this fact for our further manipulations as well.)

We can now describe our desired map $N_{\infty}(Gr(N_{\infty}(\mathcal{D})_{\bullet}))_{\bullet} \to N_{\infty}(\mathcal{D})_{\bullet}$ in $CSS \subset sS$. In level *n*, it is obtained from the map

$$\left(\coprod_{\alpha \in \mathcal{N}(\mathbf{\Delta}^{op})_n} \mathfrak{k}(\alpha(0)^\circ)\right) \to \mathfrak{k}([n]^\circ)$$

which on the summand $\alpha \in \mathcal{N}(\mathbf{\Delta}^{op})_n$ is the map

in Fun($\operatorname{Cat}_{\infty}, S$). That is, in level n, on the summand α it is corepresented by the map $[n] \to \alpha(0)^{\circ}$ in $\Delta \subset \operatorname{Cat}_{\infty}$ given by $i \mapsto \alpha_{0,i}(0)$, i.e. the map taking $i \in [n]$ to the image in $\alpha(0)^{\circ}$ of the object $0 \in \alpha(i)^{\circ}$ under the composite

$$\alpha(0)^{\circ} \xleftarrow{\alpha_{0,1}} \cdots \xleftarrow{\alpha_{i-1,i}} \alpha(i)^{\circ}$$

in Δ .

We have now associated to the object $\mathcal{D} \in \operatorname{Cat}_{\infty}$ a map $N_{\infty}(\operatorname{Gr}(N_{\infty}(\mathcal{D})_{\bullet})) \to N_{\infty}(\mathcal{D})$ in CSS. In fact, this map clearly commutes with the induced projections to $N_{\infty}(\Delta^{op})$, and moreover, this association is clearly functorial in $\mathcal{D} \in \operatorname{Cat}_{\infty}$ since it is entirely corepresented. Hence, we obtain a functor

$$\operatorname{Cat}_{\infty} \to \operatorname{Fun}([1], \operatorname{CSS}_{/\operatorname{N}_{\infty}(\Delta^{op})})$$

which immediately (and equivalently) produces our desired functor

$$\operatorname{Cat}_{\infty} \to \operatorname{Fun}([1], (\operatorname{Cat}_{\infty})_{/\Delta^{op}})$$

which takes the object $\mathcal{D} \in \operatorname{Cat}_{\infty}$ to a commutative triangle as depicted in Remark 5.2 (considered as an object of Fun([1], $(\operatorname{Cat}_{\infty})_{/\Delta^{op}}$)).

Example 5.5. To illustrate the combinatorics of the corepresenting map in Construction 5.4, we return to the situation described in Remark 5.2. Namely, let us restrict our attention to the case that n = 1, $\alpha(0) = [1]^{\circ}$, and $\alpha(1) = [0]^{\circ}$. Then, there are two possibilities for the map $\alpha \in N(\Delta^{op})_1 = \hom_{eat}([1], \Delta^{op})$: it selects either $[1]^{\circ} \xrightarrow{\delta_0} [0]^{\circ}$ or $[1]^{\circ} \xrightarrow{\delta_1} [0]^{\circ}$. These are respectively opposite to the map $[0] \rightarrow [1]$ in Δ given by $0 \mapsto 1$ or $0 \mapsto 0$. Hence, the corresponding corepresenting map $[1] \rightarrow \alpha(0)^{\circ} = [1]$ in Δ is respectively either $id_{[1]}$ or const(0).

We now define our main object of interest, the simplicial replacement.

Definition 5.6. Suppose that \mathcal{C} is cocomplete, and let $\mathcal{D} \xrightarrow{F} \mathcal{C}$ be a functor. We construct a composite



in which the first horizontal map is given by Construction 5.4. By Proposition T.4.2.2.7, there is a unique lift in the diagram



such that the composite

$$\boldsymbol{\Delta}^{op} \to \operatorname{Gr}(\operatorname{N}_{\infty}(\mathcal{D})_{\bullet}) \underset{\boldsymbol{\Delta}^{op}}{\diamond} \boldsymbol{\Delta}^{op} \to \mathbb{C} \times \boldsymbol{\Delta}^{op}$$

takes each $[n]^{op} \in \Delta^{op}$ to a colimit of the composite

$$\mathrm{N}_{\infty}(\mathcal{D})_n \xrightarrow{\{0\}} \mathrm{N}_{\infty}(\mathcal{D})_0 \simeq \mathcal{D}^{\simeq} \hookrightarrow \mathcal{D} \xrightarrow{F} \mathcal{C}.$$

We define the object $\operatorname{srep}(F)_{\bullet} \in s\mathfrak{C}$ to be the composite

$$\Delta^{op} \to \operatorname{Gr}(\operatorname{N}_{\infty}(\mathcal{D})_{\bullet}) \underset{\Delta^{op}}{\diamond} \Delta^{op} \to \mathfrak{C} \times \Delta^{op} \to \mathfrak{C},$$

and refer to it as the *simplicial replacement* of the functor F.

Lemma 5.7. Suppose that $C_1 \xrightarrow{\chi} C_2$ is a cocontinuous functor between cocomplete ∞ -categories, and suppose that $\mathfrak{D} \xrightarrow{F} \mathfrak{C}_1$ is any functor. Then the composite

$$\Delta^{op} \xrightarrow{\operatorname{srep}(F)_{\bullet}} \mathcal{C}_1 \xrightarrow{\chi} \mathcal{C}_2$$

is canonically equivalent to the object $\operatorname{srep}(\chi \circ F)_{\bullet} \in s(\mathcal{C}_2)$.

Proof. This follows directly from Definition 5.6.

We can now give the main result of this section, the **Bousfield-Kan** colimit formula.

Theorem 5.8. Let \mathcal{C} be cocomplete, and let $\mathcal{D} \xrightarrow{F} \mathcal{C}$ be a functor. Then there is a canonical equivalence $\operatorname{colim}_{\mathcal{D}} F \simeq |\operatorname{srep}(F)_{\bullet}|$

in C.

Proof. Observe that the colimit of a \mathcal{D} -shaped diagram is functorial in cocomplete ∞ -categories under \mathcal{D} and cocontinuous functors between them. Hence, in light of Lemma 5.7, it suffices to prove the statement in the universal (i.e. initial) case, namely taking the functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ to be the canonical map from \mathcal{D} into its free cocompletion, i.e. the Yoneda embedding $\mathcal{D} \xrightarrow{\sharp} \mathcal{P}(\mathcal{D}) = \operatorname{Fun}(\mathcal{D}^{op}, \mathcal{S}).$

First of all, we claim that there is a canonical equivalence $\operatorname{colim}_{\mathcal{D}}(\mathfrak{z}) \simeq \operatorname{const}(\operatorname{pt}_{\mathfrak{S}})$ in $\mathcal{P}(\mathcal{D})$. To see this, observe that for any $d \in \mathcal{D}$, we have a string of equivalences

$$(\operatorname{colim}_{\mathcal{D}} \Bbbk)(d) \simeq \operatorname{colim}_{\mathcal{D}}(\operatorname{ev}_d \circ \Bbbk) \simeq \operatorname{colim}_{\mathcal{D}}(\operatorname{hom}_{\mathcal{D}}(d, -))$$

in S, where the first equivalence is because colimits in $\mathcal{P}(\mathcal{D})$ are computed pointwise and the second is simply because $\operatorname{ev}_d \circ \mathfrak{t} \simeq \operatorname{hom}_{\mathcal{D}}(d, -)$ in $\operatorname{Fun}(\mathcal{D}, \mathfrak{S})$. Appealing to Proposition 2.1 and the canonical equivalence

$$\operatorname{Gr}(\hom_{\mathcal{D}}(d, -)) \simeq \mathcal{D}_{d/}$$

in $\mathcal{LFib}(\mathcal{D})$ of Example 1.12, we obtain a string of equivalences

$$\operatorname{colim}_{\mathcal{D}}(\operatorname{hom}_{\mathcal{D}}(d, -)) \simeq \operatorname{Gr}(\operatorname{hom}_{\mathcal{D}}(d, -))^{\operatorname{gpd}} \simeq (\mathcal{D}_{d/})^{\operatorname{gpd}} \simeq \operatorname{pt}_{\mathfrak{S}},$$

where the last equivalence follows from the dual of Corollary 4.11. Hence, $(\operatorname{colim}_{\mathcal{D}} \Bbbk)(d) \simeq \operatorname{pt}_8$ for every $d \in \mathcal{D}$, and so it follows that the terminal map $\operatorname{colim}_{\mathcal{D}}(\mathfrak{k}) \to \operatorname{const}(\operatorname{pt}_{\mathfrak{S}})$ in $\mathcal{P}(\mathcal{D})$ is indeed an equivalence.¹⁰ On the other hand, recall that $\operatorname{srep}(\mathfrak{z})_n \in \mathfrak{P}(\mathfrak{D})$ is the colimit of the composite

$$N_{\infty}(\mathcal{D})_n \xrightarrow{\{0\}} N_{\infty}(\mathcal{D})_0 \simeq \mathcal{D}^{\simeq} \hookrightarrow \mathcal{D} \xrightarrow{*} \mathcal{P}(\mathcal{D}).$$

For any $d \in \mathcal{D}$, the composite functor

$$N_{\infty}(\mathcal{D})_n \xrightarrow{\{0\}} N_{\infty}(\mathcal{D})_0 \simeq \mathcal{D}^{\simeq} \hookrightarrow \mathcal{D} \xrightarrow{\sharp} \mathcal{P}(\mathcal{D}) \xrightarrow{\operatorname{ev}_d} S$$

classifies the left fibration which is the upper composite map

$$\{d\} \underset{\substack{\mathcal{N}_{\infty}(\mathcal{D})_{0}, \{\emptyset\}}{\overset{\times}{\underset{\qquad}{\downarrow}}} \underset{\{d\}}{\overset{\mathcal{N}_{\infty}(\mathcal{D})_{n+1}}{\overset{\mathcal{N}_{\infty}(\mathcal{D})_{n+1}}{\overset{\mathcal{N}_{0}}{\underset{\qquad}{\downarrow}}} \underset{\{0\}}{\overset{\mathcal{N}_{\infty}(\mathcal{D})_{n}}{\underset{\qquad}{\downarrow}}$$

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 $^{^{10}}$ This is also proved as Lemma T.5.3.3.2.

Again appealing to Proposition 2.1 (and the fact that colimits in $\mathcal{P}(\mathcal{D})$ are computed pointwise), it follows that

$$\operatorname{srep}(\mathfrak{k})_{n}(d) \simeq \left(\operatorname{colim}\left(\operatorname{N}_{\infty}(\mathcal{D})_{n} \xrightarrow{\{0\}} \operatorname{N}_{\infty}(\mathcal{D})_{0} \simeq \mathcal{D}^{\simeq} \hookrightarrow \mathcal{D} \xrightarrow{\mathfrak{k}} \mathcal{P}(\mathcal{D})\right)\right)(d)$$
$$\simeq \operatorname{colim}\left(\operatorname{N}_{\infty}(\mathcal{D})_{n} \xrightarrow{\{0\}} \operatorname{N}_{\infty}(\mathcal{D})_{0} \simeq \mathcal{D}^{\simeq} \hookrightarrow \mathcal{D} \xrightarrow{\mathfrak{k}} \mathcal{P}(\mathcal{D}) \xrightarrow{\operatorname{ev}_{d}} \mathcal{S}\right)$$
$$\simeq \left(\{d\} \underset{\operatorname{N}_{\infty}(\mathcal{D})_{0},\{0\}}{\times} \operatorname{N}_{\infty}(\mathcal{D})_{n+1}\right)^{\operatorname{gpd}}$$
$$\simeq \{d\} \underset{\operatorname{N}_{\infty}(\mathcal{D})_{0},\{0\}}{\times} \operatorname{N}_{\infty}(\mathcal{D})_{n+1}$$

(where the last equivalence follows from the fact that the inclusion $S \subset \operatorname{Cat}_{\infty}$ is a right adjoint and hence commutes with pullbacks (and the fact that $(-)^{\operatorname{gpd}} : \operatorname{Cat}_{\infty} \to S$ is idempotent)). Unwinding the definitions of the simplicial structure maps of $\operatorname{srep}(\mathfrak{z})_{\bullet}$, we obtain that in fact

$$\operatorname{srep}(\mathtt{k})_{\bullet}(d) \simeq \left(\{d\} \underset{\operatorname{N}_{\infty}(\mathcal{D})_{0}, \{0\}}{\times} \operatorname{N}_{\infty}(\mathcal{D})_{\bullet+1} \right) \simeq \operatorname{N}_{\infty}(\mathcal{D}_{d/})_{\bullet}.$$

Hence, it follows that

$$|\operatorname{srep}(\mathfrak{z})_{\bullet}|(d) \simeq |\operatorname{srep}(\mathfrak{z})_{\bullet}(d)| \simeq |\operatorname{N}_{\infty}(\mathcal{D}_{d/})_{\bullet}| \simeq (\mathcal{D}_{d/})^{\operatorname{gpd}} \simeq \operatorname{pt}_{\mathfrak{S}}$$

where again the last equivalence follows from the dual of Corollary 4.11, and the second-to-last equivalence follows from Proposition N.2.4. Therefore, the terminal map $|\operatorname{srep}(\mathfrak{x})_{\bullet}| \to \operatorname{const}(\operatorname{pt}_{\mathbb{S}})$ in $\mathcal{P}(\mathcal{D})$ is also an equivalence. It follows that we have a canonical equivalence

$$\operatorname{colim}_{\mathcal{D}} \mathfrak{t} \simeq \operatorname{const}(\operatorname{pt}_{\mathfrak{S}}) \simeq |\operatorname{srep}(\mathfrak{t})_{\bullet}|,$$

which proves the claim.

Remark 5.9. Our Bousfield–Kan colimit formula (Theorem 5.8) is certainly inspired by the classical Bousfield– Kan formula (see the original source [BK72, Chapter XII, §5], or e.g. [Hir03, §18.1] for a more modern treatment), but it is actually rather different: the latter computes a homotopy colimit in a simplicial model category. Moreover, even in the case that both \mathcal{C} and \mathcal{D} are only 1-categories, it does not generally agree with the formula for a colimit as a coequalizer of coproducts: this additionally requires that \mathcal{D} be gaunt. (Note that that formula is actually evil: it is not invariant under replacing \mathcal{D} by an equivalent category, referring as it does to its actual sets $N(\mathcal{D})_0$ and $N(\mathcal{D})_1$ of objects and of morphisms.) On the other hand, when \mathcal{D} is a (1- or ∞ -)groupoid, then our simplicial replacement (Definition 5.6) is trivial: in that case we have a canonical equivalence $N_{\infty}(\mathcal{D})_{\bullet} \simeq \operatorname{const}(\mathcal{D})$ in $s\mathcal{C}$, whence it follows that the simplicial replacement $\operatorname{srep}(F)_{\bullet} \in s\mathcal{C}$ of a diagram $\mathcal{D} \xrightarrow{F} \mathcal{C}$ is already constant at the object $\operatorname{colim}_{\mathcal{D}} F \in \mathcal{C}$.

5.2. Examples of the Bousfield–Kan colimit formula. The Bousfield–Kan colimit formula (Theorem 5.8) is most interesting (and novel) when the diagram ∞ -category \mathcal{D} is not merely a 1-category. For instance, applying it to a pushout diagram and canceling out the redundancies in the resulting geometric realization yields nothing but the original pushout diagram. We therefore give two inherently ∞ -categorical examples to illustrate its application.

Example 5.10. Choose any space $Y \in S$, and let us take \mathcal{D} to be its "categorical suspension": this is an ∞ -category with two objects d_1 and d_2 , which is determined by the prescriptions that

- $\hom_{\mathcal{D}}(d_1, d_1) \simeq \hom_{\mathcal{D}}(d_2, d_2) \simeq \operatorname{pt}_{\mathcal{S}},$
- $\hom_{\mathcal{D}}(d_2, d_1) \simeq \varnothing_{\mathcal{S}}$, and
- $\hom_{\mathcal{D}}(d_1, d_2) \simeq Y.$

Then, a functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ selects the data of

- a pair of objects $c_1, c_2 \in \mathcal{C}$, and
- a map $Y \to \hom_{\mathcal{S}}(c_1, c_2)$ in S.

Canceling out redundancies (and assuming C is cocomplete), the Bousfield–Kan colimit formula (Theorem 5.8) then gives equivalences

$$\operatorname{colim}_{\mathcal{D}}(F) \simeq |\operatorname{srep}(F)_{\bullet}| \simeq \operatorname{colim} \left(\begin{array}{c} c_1 \odot Y \longrightarrow c_1 \\ \downarrow \\ c_2 \end{array} \right)$$

in \mathcal{C} , where in the pushout

- the horizontal map is given by $c_1 \odot (Y \to \text{pt}_8)$, and
- the vertical map is the adjunct of the chosen map $Y \to \hom_{\mathcal{C}}(c_1, c_2)$.

Example 5.11. Choose any space $Y \in S$, and let us take $\mathcal{D} = Y^{\triangleleft} \in Cat_{\infty}$ to be the left cone on it (considered as an ∞ -groupoid). Assume for simplicity that \mathcal{C} is bicomplete. Then, a functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$ is determined by

- its restriction $Y \xrightarrow{F'} \mathcal{C}$, and
- a map $c \to \lim_{Y} (F')$ from some object $c \in \mathcal{C}$ (selected by the cone point) to the limit of this restriction.¹¹

Canceling out redundancies, the Bousfield-Kan colimit formula (Theorem 5.8) then gives equivalences

$$\operatorname{colim}_{\mathcal{D}}(F) \simeq |\operatorname{srep}(F)_{\bullet}| \simeq \operatorname{colim} \left(\begin{array}{c} c \odot Y \longrightarrow c \\ \downarrow \\ \operatorname{colim}_{Y}(F') \end{array} \right)$$

in \mathcal{C} , where in the pushout

- the horizontal map is given by $c \odot (Y \to \text{pt}_8)$, and
- the vertical map is induced by the natural transformation $\operatorname{const}_Y(c) \to F'$ in $\operatorname{Fun}(Y, \mathbb{C})$ which is adjunct to the chosen map $c \to \lim_Y (F')$.

On an object $c' \in \mathcal{C}$, this pushout corepresents the data of

- a morphism $\operatorname{colim}_Y(F') \to c'$ in \mathcal{C} (or equivalently, a morphism $F' \to \operatorname{const}_Y(c')$ in $\operatorname{Fun}(Y, \mathcal{C})$, along with
- a trivialization



of the adjunct to the induced composite

$$c \odot Y \to \operatorname{colim}_Y(F') \to c'.$$

5.3. Functoriality of the Bousfield–Kan colimit formula. Of course, it is perfectly reasonable to expect that the Bousfield–Kan colimit formula enjoys good functoriality properties, along the lines of those explored in §3.2. However, rather than pursue a full treatment, in this subsection we exhibit only the mere shadow of such functoriality that we will actually need.

We begin by identifying the simplicial replacement as a left Kan extension.

¹¹If $Y \xrightarrow{F'} \mathbb{C}$ is constant, then $\lim_{Y} (F')$ reduces to a cotensor. However, there are interesting examples where this is not the case: for instance, if Y is a 1-type, this limit computes the *fixed points* of the corresponding group action. Of course, a dual observation applies to $\operatorname{colim}_Y(F')$, which appears here as well. (Another interesting example of a colimit over an ∞ -groupoid is the *Thom spectrum* construction.)

Lemma 5.12. Suppose that \mathcal{C} is cocomplete, let $\mathcal{D} \xrightarrow{F} \mathcal{C}$ be any diagram. Then, in the commutative diagram



containing the simplicial replacement $\Delta^{op} \xrightarrow{\operatorname{srep}(F)_{\bullet}} \mathbb{C}$ as a composite, the vertical functor is a full inclusion and the dotted arrow is a left Kan extension along it.

Proof. To see that the vertical functor is a full inclusion, we simply unwind the definition of its target to obtain

$$\begin{split} \operatorname{Gr}(\operatorname{N}_{\infty}(\mathcal{D})_{\bullet}) & \stackrel{\diamond}{\overset{\diamond}{\operatorname{\Delta}^{op}}} \operatorname{\Delta}^{op} \\ &= \operatorname{Gr}(\operatorname{N}_{\infty}(\mathcal{D})_{\bullet}) \prod_{\left(\operatorname{Gr}(\operatorname{N}_{\infty}(\mathcal{D})_{\bullet}) \underset{\operatorname{\Delta}^{op}}{\times} \operatorname{\Delta}^{op} \times \{0\}\right)} \left(\operatorname{Gr}(\operatorname{N}_{\infty}(\mathcal{D})_{\bullet}) \underset{\operatorname{\Delta}^{op}}{\times} \operatorname{\Delta}^{op} \times [1]\right) \prod_{\left(\operatorname{Gr}(\operatorname{N}_{\infty}(\mathcal{D})_{\bullet}) \times \{1\}\right)} \left(\operatorname{Gr}(\operatorname{N}_{\infty}(\mathcal{D})_{\bullet}) \underset{\operatorname{\Delta}^{op}}{\times} \operatorname{\Delta}^{op} \times \{1\}\right) \\ &\simeq \left(\operatorname{Gr}(\operatorname{N}_{\infty}(\mathcal{D})_{\bullet}) \times [1]\right) \prod_{\left(\operatorname{Gr}(\operatorname{N}_{\infty}(\mathcal{D})_{\bullet}) \times \{1\}\right)} \operatorname{\Delta}^{op}. \end{split}$$

That is, this target is precisely the cocartesian fibration over [1] classified by the functor $[1] \to \operatorname{Cat}_{\infty}$ which selects the projection $\operatorname{Gr}(N_{\infty}(\mathcal{D})_{\bullet}) \to \Delta^{op}$, and our vertical functor is the fiber inclusion over the object $0 \in [1]$; in particular, it is full (as the object $0 \in [1]$ admits no nontrivial automorphisms).

Now, to check that the dotted arrow is a left Kan extension along this full inclusion, by Remark T.4.3.2.3 it suffices to show that for any object

$$[n]^{\circ} \in \mathbf{\Delta}^{op} \subset \left(\operatorname{Gr}(\mathcal{N}_{\infty}(\mathcal{D})_{\bullet}) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op} \right),$$

the corresponding fiber inclusion $N_{\infty}(\mathcal{D})_n \hookrightarrow Gr(N_{\infty}(\mathcal{D})_{\bullet})$ induces a functor

$$N_{\infty}(\mathcal{D})_{n} \to \left(\operatorname{Gr}(N_{\infty}(\mathcal{D})_{\bullet}) \underset{\left(\operatorname{Gr}(N_{\infty}(\mathcal{D})_{\bullet})_{\Delta^{\diamond p}}^{\diamond} \Delta^{\circ p}\right)}{\times} \left(\operatorname{Gr}(N_{\infty}(\mathcal{D})_{\bullet})_{\Delta^{\diamond p}}^{\diamond} \Delta^{\circ p} \right)_{/[n]^{\circ}} \right)$$

which is final. This is straightforwardly verified using Theorem A (4.10): all of the comma ∞ -categories whose groupoid completions must be shown to be contractible are easily seen to possess initial objects, and hence the equivalent condition follows from the opposite of Corollary 4.11.

We can now describe our desired shadow of functoriality.

Proposition 5.13. Let C be cocomplete, and suppose that



is a commutative diagram in Cat_{∞} .

(1) There is a canonical induced map

$$\operatorname{srep}(F)_{\bullet} \to \operatorname{srep}(G)_{\bullet}$$

in sC, which is functorial in the variable C for cocontinuous functors between cocomplete ∞ -categories.

(2) We have a commutative square



in C, in which

- the upper map is the induced map on colimits of the global colimit functor (Proposition 3.12),
- the lower map is the geometric realization of the canonical map of part (1), and
- the vertical equivalences are those of Theorem 5.8.

Proof. For part (1), by Lemma 5.12 and the functoriality provided by Construction 5.4, we have a commutative diagram



in which the vertical arrows are full inclusions and the dotted arrows are left Kan extensions therealong. As the composite

$$\left(\operatorname{Gr}(\operatorname{N}_{\infty}(\mathcal{D})_{\bullet}) \underset{\boldsymbol{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op}\right) \to \left(\operatorname{Gr}(\operatorname{N}_{\infty}(\mathcal{E})_{\bullet}) \underset{\boldsymbol{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op}\right) \to \mathfrak{C}$$

also extends the map $Gr(N_{\infty}(\mathcal{D})_{\bullet}) \to \mathcal{C}$ along the given full inclusion, it therefore admits a canonical natural transformation from the dotted map

$$\operatorname{Gr}(\operatorname{N}_{\infty}(\mathcal{D})_{\bullet}) \underset{\Delta^{op}}{\diamond} \Delta^{op} \to \mathfrak{C}.$$

Restricting to the full subcategory

$$\mathbf{\Delta}^{op} \subset \left(\mathrm{Gr}(\mathrm{N}_{\infty}(\mathcal{D})_{\bullet}) \underset{\mathbf{\Delta}^{op}}{\diamond} \mathbf{\Delta}^{op} \right),$$

we obtain a natural transformation

$$\Delta^{op} \underset{\operatorname{srep}(G)_{\bullet}}{\overset{\operatorname{srep}(F)_{\bullet}}{\overset{\operatorname{rep}(F)_{\bullet}}{\overset{\operatorname{rep}(G)_{\bullet}}{\overset{\operatorname{srep}(F)_{\bullet}}{\overset{srep}(F)_{\bullet}}{\overset{srep}}}{\overset{srep}(F)_{\bullet}}{\overset{srep}(F)_{\bullet}}{\overset{srep}}}{\overset{srep}(F)_{\bullet}}{\overset{srep}(F)_{\bullet}}{\overset{srep}(F)_{\bullet}}{\overset{srep}}}}{\overset{srep}(F)_{\bullet}}{\overset{srep}}}{\overset{srep}(F)_{\bullet}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}(F)_{\bullet}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}(F)_{\bullet}}{\overset{srep}}}}{\overset{srep}(F)_{\bullet}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}(F)_{\bullet}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}}}{\overset{srep}}}{\overset{$$

which is precisely our desired map in $s\mathcal{C}$. Moreover, the asserted functoriality follows easily from this argument (recall Remark 3.13).

For part (2), note that there is a unique cocontinuous functor $\mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{E})$ making the diagram

$$\begin{array}{c} \mathcal{D} & \longrightarrow \mathcal{E} \\ \mathbb{D} & & \downarrow \mathbb{P}(\mathcal{D}) & \longrightarrow \mathcal{P}(\mathcal{E}) \end{array}$$

commute.¹² From here, by Lemma 5.7 and the functoriality asserted in part (1), it suffices to verify the claim in the case that the functor $\mathcal{E} \xrightarrow{G} \mathcal{C}$ is the Yoneda embedding $\mathcal{E} \xrightarrow{\pm \varepsilon} \mathcal{P}(\mathcal{E})$, so that the functor $\mathcal{D} \xrightarrow{F} \mathcal{C}$

¹²In fact, by Lemma T.5.1.5.5(1), this functor must be precisely the left Kan extension $(\sharp_{\mathcal{D}})_!(\sharp_{\mathcal{E}} \circ H)$.

is the composite $\mathcal{D} \xrightarrow{\mathfrak{k}_{\mathcal{E}} \circ H} \mathcal{P}(\mathcal{E})$. Hence, it remains to show that we have a commutative square

$$\begin{array}{cc} \operatorname{colim}_{\mathcal{D}}(\mathfrak{z}_{\mathcal{E}}\circ H) \longrightarrow \operatorname{colim}_{\mathcal{E}}\mathfrak{z}_{\mathcal{E}} \\ \swarrow & & \downarrow^{\mathfrak{z}} \\ |\operatorname{srep}(\mathfrak{z}_{\mathcal{E}}\circ H)_{\bullet}| \longrightarrow |\operatorname{srep}(\mathfrak{z}_{\mathcal{E}})_{\bullet}| \end{array}$$

in $\mathcal{P}(\mathcal{E})$ satisfying the described criteria. But as we have seen in the proof of Theorem 5.8, if we consider these two vertical maps as objects of Fun([1], $\mathcal{P}(\mathcal{E})$), then the one on the right determines a terminal object. As the datum of this commutative square is equivalent to that of the corresponding morphism in Fun([1], $\mathcal{P}(\mathcal{E})$) (reading the square from left to right), it follows that in fact the square must commute in a canonical and unique way.

Appendix A. The Thomason model structure on the ∞ -category of ∞ -categories

In this appendix, we equip the ∞ -category $\operatorname{Cat}_{\infty}$ of ∞ -categories with a *Thomason model structure* analogous to the classical Thomason model structure on cat (though see Remark A.6) and observe some of its basic features. (We refer the reader to §S.1 for the definition of a model structure on an ∞ -category, and to §S.4 for the definition of the Kan–Quillen model structure on the ∞ -category sS of simplicial spaces.) This model structure provides a convenient language for a number of the results in the main body of the paper.

We begin by constructing it.

Theorem A.1. The Kan-Quillen model structure on sS lifts along the composite adjunction

$$sS \xrightarrow[U_{ess}]{} CSS \xrightarrow[N_{\infty}^{-1}]{} Cat_{\infty}.$$

Moreover, this Quillen adjunction is a Quillen equivalence.

Proof. For notational convenience, we prove the statement for the adjunction $L_{CSS} \dashv U_{CSS}$ (which is of course equivalent).

For the first claim, we verify the hypotheses of the lifting theorem for cofibrantly generated model ∞ -categories (S.3.12) in turn.

First of all, for any $Y \in sS$ and $Z \in CSS$ we have that

$$\hom_{\mathfrak{CSS}}(\mathcal{L}_{\mathfrak{CSS}}(Y), Z) \simeq \hom_{s\mathfrak{S}}(Y, \mathcal{U}_{\mathfrak{CSS}}(Z)),$$

and so the fact that the sets $L_{CSS}(I_{KQ})$ and $L_{CSS}(J_{KQ})$ of homotopy classes of maps in CSS permit the small object argument follows from Lemma S.5.2.

Next, we show that the right adjoint $sS \xleftarrow{Uess} CSS$ takes relative $L_{CSS}(J_{KQ})$ -cell complexes into W_{KQ} . For this, let us begin by supposing that

$$\begin{array}{c} \mathcal{L}_{\mathcal{CSS}}(\Lambda_i^n) \longrightarrow Y \\ & \downarrow \\ \mathcal{L}_{\mathcal{CSS}}(\Delta^n) \longrightarrow Z \end{array}$$

is a pushout square in CSS. Since $sS \xrightarrow{\text{Less}} CSS$ commutes with pushouts (being a left adjoint), we have that

$$Z \simeq \operatorname{colim}^{\mathfrak{CSS}} \left(\begin{array}{c} \mathcal{L}_{\mathfrak{CSS}}(\Lambda_i^n) \longrightarrow \mathcal{L}_{\mathfrak{CSS}}(\mathcal{U}_{\mathfrak{CSS}}(Y)) \\ \downarrow \\ \mathcal{L}_{\mathfrak{CSS}}(\Delta^n) \end{array} \right) \simeq \mathcal{L}_{\mathfrak{CSS}} \left(\begin{array}{c} \Lambda_i^n \longrightarrow \mathcal{U}_{\mathfrak{CSS}}(Y) \\ \downarrow \\ \Delta^n \end{array} \right) \right).$$

Using Proposition N.2.4 and the fact that colimits commute with colimits, we then obtain the string of equivalences

$$\begin{aligned} |\mathbf{U}_{\mathrm{CSS}}(Z)| &\simeq \left| \mathbf{U}_{\mathrm{CSS}} \left(\mathbf{L}_{\mathrm{CSS}} \left(\operatorname{colim}^{sS} \left(\begin{array}{c} \Lambda_i^n \longrightarrow \mathbf{U}_{\mathrm{CSS}}(Y) \\ \downarrow \\ \Delta^n \end{array} \right) \right) \right) \right) \right| \\ &\simeq \left| \operatorname{colim}^{sS} \left(\begin{array}{c} \Lambda_i^n \longrightarrow \mathbf{U}_{\mathrm{CSS}}(Y) \\ \downarrow \\ \Delta^n \end{array} \right) \right| \\ &\simeq \operatorname{colim}^{s} \left(\begin{array}{c} |\Lambda_i^n| \longrightarrow |\mathbf{U}_{\mathrm{CSS}}(Y)| \\ \downarrow \\ |\Delta^n| \end{array} \right) \\ &\simeq \operatorname{colim}^{s} \left(\begin{array}{c} |\Lambda_i^n| \longrightarrow |\mathbf{U}_{\mathrm{CSS}}(Y)| \\ \downarrow \\ |\Delta^n| \end{array} \right) \\ &\simeq |\mathbf{U}_{\mathrm{CSS}}(Y)|. \end{aligned} \end{aligned}$$

Hence, the map

$$U_{CSS}(Y) \to U_{CSS}(Z)$$

lies in $\mathbf{W}_{\mathrm{KQ}} \subset s\mathcal{S}$. Now, to prove the claim for more general transfinite compositions, it then suffices to observe that the composite

$$\mathbb{CSS} \xrightarrow{\mathrm{U}_{\mathbb{CSS}}} sS \xrightarrow{|-|} S$$

is a left adjoint and hence commutes with colimits. Thus, U_{CSS} does indeed take relative $L_{CSS}(J_{KQ})$ -cell complexes into W_{KQ} .

So, the Kan–Quillen model structure sS_{KQ} does indeed lift along the adjunction as claimed, and it remains to check that the resulting Quillen adjunction is a Quillen equivalence. For this, suppose we are given *any* objects $Y \in sS$ and $Z \in CSS$. Then, a map

$$Y \to U_{CSS}(Z)$$

in sS corresponds via the adjunction to the map

$$L_{CSS}(Y) \to L_{CSS}(U_{CSS}(Z)) \simeq Z$$

in CSS. Since the lifting theorem (S.3.12) produces a model structure for which the right adjoint creates the weak equivalences, the claim follows from Proposition N.2.4. \Box

Definition A.2. We refer to the model structure on Cat_{∞} defined by Theorem A.1 as the **Thomason** model structure, denoted $(Cat_{\infty})_{Th}$.

Remark A.3. Proposition N.2.4 implies that the subcategory $\mathbf{W}_{\mathrm{Th}} \subset \operatorname{Cat}_{\infty}$ is created by the groupoid completion functor $(-)^{\mathrm{gpd}} : \operatorname{Cat}_{\infty} \to S$. As this functor is a left localization, it therefore induces an equivalence $\operatorname{Cat}_{\infty}[\![\mathbf{W}_{\mathrm{Th}}^{-1}]\!] \xrightarrow{\sim} S$. Moreover, it is not hard to see directly that the adjoint functors $N_{\infty}^{-1} \circ L_{\mathrm{CSS}} \dashv U_{\mathrm{CSS}} \circ N_{\infty}$ induce inverse equivalences $s S[\![\mathbf{W}_{\mathrm{KQ}}^{-1}]\!] \simeq \operatorname{Cat}_{\infty}[\![\mathbf{W}_{\mathrm{Th}}^{-1}]\!]$.

Remark A.4. Let us explore what it means for an object $\mathcal{C} \in \operatorname{Cat}_{\infty}$ to be fibrant in the Thomason model structure. By definition, this means that $N_{\infty}(\mathcal{C}) \in \operatorname{CSS} \subset sS$ has the extension property for the set $J_{\mathrm{KQ}} = \{\Lambda_i^n \to \Delta^n\}_{0 \le i \le n \ge 1}$ of horn inclusions in $s\operatorname{Set} \subset sS$.

In fact, the Segal condition on a simplicial space implies that it admits *unique* fillers for the set $\{\Lambda_i^n \to \Delta^n\}_{0 \le i \le n \ge 2}$ of *inner* horn inclusions. To see this, observe that the inclusion

$$\left(\Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \cdots \coprod_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}\right) \to \Lambda^n_i$$

of subobjects of $\Delta^n \in s$ Set can be constructed as a (finite) composition of pushouts of inner horn inclusions $\Lambda_j^k \to \Delta^k$ for k < n, and the Segal condition is precisely the assertion that a given simplicial space admits unique extensions for the inclusion of the above source into Δ^n (the "nth spine inclusion"). From here, the assertion follows by induction.

On the other hand, it is already clear that if a (complete) Segal space has the extension property for the outer horn inclusion $\Lambda_0^2 \to \Delta^2$, then it must be the nerve of an ∞ -category whose morphisms are all invertible. Hence, the fibrant objects in the model structure on CSS of Theorem A.1 are precisely the *constant* complete Segal spaces (i.e. those that are constant as simplicial spaces). It follows that we can identify the subcategory of fibrant objects in the Thomason model structure as $(\operatorname{Cat}_{\infty})_{\mathrm{Th}}^f = S \subset \operatorname{Cat}_{\infty}$, the subcategory of ∞ -groupoids (i.e. spaces).

In fact, we have the following strengthening of Remark A.4.^{13,14}

Proposition A.5. The Thomason model structure on Cat_{∞} is a left localization model structure (in the sense of Example S.2.12) with respect to the left localization adjunction

$$\mathfrak{C}at_{\infty} \xrightarrow[]{(-)^{\mathrm{gpd}}}_{U_{\mathfrak{S}}} \mathfrak{S}.$$

Proof. Observe that a model structure on an ∞ -category is clearly determined by its subcategories of weak equivalences and of cofibrations (just as a model structure on a 1-category). So, it only remains to check that all maps in $(\operatorname{Cat}_{\infty})_{\mathrm{Th}}$ are cofibrations. For this, it suffices to present any map in CSS as the image under $sS \xrightarrow{\mathrm{Less}} CSS$ of a cofibration in sS_{KQ} . Note that the inclusion $sSet \subset sS$ induces an inclusion $\mathbf{C}_{\mathrm{KQ}}^{sSet} \subset \mathbf{C}_{\mathrm{KQ}}^{sS}$; in fact, we will present any map in CSS as the image of a cofibration in $sSet_{\mathrm{KQ}}$.

To accomplish this, we begin by observing that the composite functor

$$sSet \hookrightarrow sS \xrightarrow{L_{CSS}} CSS$$

consists of a right adjoint followed by a left adjoint. In fact, these are both presented by Quillen functors: the first functor is presented by the right Quillen functor in the evident Quillen adjunction

 $\pi_0^{\mathrm{lw}}: s(s\mathfrak{Set}_{\mathrm{KQ}})_{\mathrm{Reedy}} \rightleftarrows s\mathfrak{Set}_{\mathrm{triv}}: \mathrm{const}^{\mathrm{lw}}$

(where in the left adjoint we slightly abuse notation by using the symbol π_0 to refer to the composite sSet $\rightarrow s$ Set $[[\mathbf{W}_{KQ}^{-1}]] \simeq S \xrightarrow{\pi_0}$ Set), while the second functor is presented by the left Quillen functor in the left Bousfield localization

 $\mathrm{id}_{ss\mathrm{Set}} : s(s\mathrm{Set}_{\mathrm{KQ}})_{\mathrm{Reedy}} \rightleftharpoons ss\mathrm{Set}_{\mathrm{Rezk}} : \mathrm{id}_{ss\mathrm{Set}}$

(see (the proof of) [Rez01, Theorem 7.2]). Moreover, all objects of $sSet_{triv}$ are fibrant while all objects of $s(sSet_{KQ})_{Reedy}$ are cofibrant, so this composite of Quillen functors does not need to be corrected at either stage in order to compute the value of the corresponding adjoint functor of ∞ -categories. On the other hand, this composite functor

$$sSet_{triv} \xrightarrow{const^{iw}} s(sSet_{KQ})_{Reedy} \xrightarrow{id_{ssSet}} ssSet_{Rez}$$

of model categories is precisely the functor underlying the left Quillen equivalence const^{lw} : $sSet_{\text{Joyal}} \rightarrow ssSet_{\text{Rezk}}$ of [JT07, Theorem 4.11]. It therefore follows that the above composite functor of ∞ -categories induces an equivalence

$$sSet[\![\mathbf{W}_{Joyal}^{-1}]\!] \xrightarrow{\sim} CSS$$

from the localization of sSet at the subcategory $\mathbf{W}_{\text{Joyal}} \subset sSet$ to the ∞ -category of complete Segal spaces.¹⁵ We can now easily achieve our goal. Since $\mathbf{C}_{\text{KQ}}^{sSet} = \mathbf{C}_{\text{Joyal}}^{sSet}$, it follows that we can simply choose a cofibration in $sSet_{\text{Joyal}}$ presenting our given map in $CSS \simeq Cat_{\infty}$: considering this map of simplicial sets as a map of discrete simplicial spaces, its image under the functor $sS \xrightarrow{\text{Less}} CSS$ is precisely the chosen map.

¹³Proposition A.5 immediately implies the conclusion of Remark A.4 (that $(\operatorname{Cat}_{\infty})_{Th}^{f} = S \subset \operatorname{Cat}_{\infty}$), but we will want to build on the explicit geometric arguments given there in Remark A.8.

 $^{^{14}}$ Proposition A.5 does *not* follow from the discussion of Example S.2.12. For instance, this left localization is certainly not left exact; the entire point of §4 is to obtain sufficient conditions under which it commutes with pullbacks. See also Remark A.9.

¹⁵Of course, we already knew that sSet $[\![\mathbf{W}_{Joyal}^{-1}]\!]$ was equivalent to CSS (since it is equivalent to Cat $_{\infty}$); the new information here is that this particular composite functor (of ∞ -categories) *also* induces this equivalence.

Remark A.6. We observe that the Thomason model structure on $\operatorname{Cat}_{\infty}$ does *not* extend the original Thomason model structure on cat: the model category cat_{Th} is not a model subcategory of the model ∞ -category $(\operatorname{Cat}_{\infty})_{Th}$ (in the sense of Definition S.4.11, and ignoring the fact that cat is not, strictly speaking, a subcategory of $\operatorname{Cat}_{\infty}$ at all). However, the weak equivalences remain unchanged: the subcategory $\mathbf{W}_{Th}^{\text{eat}} \subset \operatorname{cat}$ is pulled back from the subcategory $\mathbf{W}_{Th}^{\text{Cat}_{\infty}} \subset \operatorname{Cat}_{\infty}$ along the composite functor $\operatorname{cat} \to \operatorname{Cat} \hookrightarrow \operatorname{Cat}_{\infty}$. To illustrate this, we recall the history of this classical model category.

To begin, we recall the main point: categories can individually be considered as "presentations of spaces" (via simplicial sets) via the nerve functor N : eat $\rightarrow s$ Set. Thus, it is natural to wonder whether this can be extended to a global presentation of the homotopy category of spaces (i.e. ho(\mathcal{S}) $\simeq s$ Set[$\mathbf{W}_{\mathrm{KQ}}^{-1}$]) as some localization of the category cat of categories. As a first step in this direction, it was proved in [III72, 3.3.1] (but attributed there to Quillen) that the nerve functor eat $\xrightarrow{\mathrm{N}} s$ Set does indeed induce an equivalence eat[$\mathbf{W}_{\mathrm{Th}}^{-1}$] $\xrightarrow{\sim} s$ Set[$\mathbf{W}_{\mathrm{KQ}}^{-1}$] on (1-categorical) localizations. In other words, the relative category (eat, \mathbf{W}_{Th}) \in \mathfrak{s} eleat has as its (1-categorical) localization the homotopy category ho(\mathcal{S}) of spaces, as desired.

On the other hand, relative categories are not so easy to work with, and so one might then further wonder whether this relative category structure can be promoted to a model category structure. Now, the most obvious way that one might hope to obtain this would be to simply lift the classical Kan–Quillen model structure (as in Theorem S.3.12, but of course really just using [Hir03, Theorem 11.3.2]) along the adjunction $L_{eat} : sSet \rightleftharpoons cat : N$. However, it is easy to see that such a Quillen adjunction could not possibly be a Quillen equivalence: the fibrant objects of cat would be precisely the subcategory $spd \subset$ cat of groupoids, but these (or rather their nerves) do not model all objects of $sSet[\mathbf{W}_{KQ}^{-1}]$, and so the derived right adjoint could not possibly be surjective.

But in [Tho80], Thomason showed that if we instead lift the Kan–Quillen model structure along the composite right adjoint

$$sSet_{KQ} \xleftarrow{Ex} sSet \xleftarrow{Ex} sSet \xleftarrow{N} cat_{M}$$

then the resulting Quillen adjunction is a Quillen equivalence.^{16,17} This defines what is now called the *Thomason model structure* on cat. From this description, it is clear that the model structure $(Cat_{\infty})_{Th}$ does not extend the model structure eat_{Th} . For instance, it follows from Proposition A.5 that all objects of $(Cat_{\infty})_{Th}$ are cofibrant, whereas [Tho80, Proposition 5.7] asserts that all cofibrant objects of eat_{Th} are in fact posets. Moreover, that same result also implies that their notions of fibrancy disagree: it follows from it that any bifibrant object of eat_{Th} is a poset, whereas according to Remark A.4 the fibrant objects of $(Cat_{\infty})_{Th}$ are precisely the ∞ -groupoids.¹⁸

Remark A.7. One way to interpret Remark A.6 is to say that the model ∞ -category $(\operatorname{Cat}_{\infty})_{\mathrm{Th}}$ entirely accounts for the quirky definition of the model category $\operatorname{cat}_{\mathrm{Th}}$: in this sense, the only "obstruction" to obtaining a model structure on cat presenting S by lifting directly along the nerve functor is the lack of would-be fibrant objects in cat. While not every space is presented by a groupoid, certainly every space is presented by an ∞ -groupoid (!), and so this obstruction vanishes when we pass from cat to $\operatorname{Cat}_{\infty}$.

Remark A.8. In contrast with Remark A.4, it is not so straightforward to characterize which maps in $(\operatorname{Cat}_{\infty})_{\mathrm{Th}}$ are fibrations.¹⁹ By definition, this would be a functor $\mathbb{C} \xrightarrow{F} \mathcal{D}$ in $\operatorname{Cat}_{\infty}$ such that the corresponding map

$$N_{\infty}(\mathcal{C}) \xrightarrow{N_{\infty}(F)} N_{\infty}(\mathcal{D})$$

on nerves in $CSS \subset sS$ has $rlp(I_{KQ})$. On the one hand, arguments similar to those of Remark A.4 imply that these maps likewise admit *unique* lifts for the *inner* horn inclusions. On the other hand, the condition

¹⁶Heuristically, one might say that the Ex functor "makes more simplicial sets fibrant". Indeed, recall that it comes equipped with a natural transformation $id_{sSet} \rightarrow Ex$, and the resulting transfinite composition defines a fibrant replacement functor on $sSet_{KQ}$ (see [GJ99, Chapter III, §4]).

¹⁷The Ex functor is not only a right Quillen equivalence from $sSet_{KQ}$ to itself, but it is also a relative functor – indeed, it defines a weak equivalence in $\pi eleat_{BK}$. Thus, even though here we are (crucially!) not only applying it to fibrant objects of $sSet_{KQ}$, in the end this composite still presents an equivalent map in Cat_{∞} (namely $S \xrightarrow{id_S} S$).

¹⁸As posets are gaunt, the composites eat $\rightarrow Cat_{\infty} \xrightarrow{N_{\infty}} sS$ and eat $\xrightarrow{N} sSet \hookrightarrow sS$ are equivalent on such objects.

¹⁹Nor should this necessarily be expected to be straightforward: by the lifting axiom $M_{\infty}4$ for model ∞ -categories and Proposition A.5, we have $\mathbf{F}_{Th} = rlp((\mathbf{W} \cap \mathbf{C})_{Th}) = rlp(\mathbf{W}_{Th})$.

that this map have the right lifting property against the outer horn inclusions seems to be a good deal more subtle.

- At n = 1, the requirement that our map have the right lifting property against the outer horn inclusion $\Lambda_0^1 \to \Delta^1$ (resp. the outer horn inclusion $\Lambda_1^1 \to \Delta^1$) is equivalent to the condition that for all objects $c \in \mathbb{C}$, the functor $\mathbb{C}_{/c} \to \mathcal{D}_{/F(c)}$ (resp. the functor $\mathbb{C}_{/c} \to \mathcal{D}_{/F(c)}$) is surjective.
- At n = 2, even just the requirement that our map have the right lifting property against the outer horn inclusion $\Lambda_0^2 \to \Delta^2$ already implies that the equivalences in \mathbb{C} are created by the functor $\mathbb{C} \xrightarrow{F} \mathcal{D}$. However, this does not appear to be a sufficient condition. Of course, one can at least rephrase the condition as follows: for our map to have the right lifting property against $\Lambda_0^2 \to \Delta^2$ (resp. $\Lambda_2^2 \to \Delta^2$), it must be the case that, given any two maps φ and ψ in \mathbb{C} whose sources (resp. targets) have been identified, then any factorization of one of the maps $F(\varphi)$ or $F(\psi)$ in \mathcal{D} through the other must already exist in \mathbb{C} .
- Requiring that our map have the right lifting property against the higher outer horn inclusions appears to demand similar but even more exotic properties of our original functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$.

Remark A.9. Combining Proposition A.5 with the discussion of Example S.2.12, we obtain that $(\mathbf{W} \cap \mathbf{F})_{\mathrm{Th}} = (\operatorname{Cat}_{\infty})^{\simeq} \subset \operatorname{Cat}_{\infty}$; in other words, any fibration in $(\operatorname{Cat}_{\infty})_{\mathrm{Th}}$ which induces an equivalence on groupoid completions must in fact itself be an equivalence. In light of the discussion of Remark A.8 regarding the complexities of the subcategory $\mathbf{F}_{\mathrm{Th}} \subset \operatorname{Cat}_{\infty}$, this appears to be a rather nontrivial fact.²⁰

Remark A.10. Both Thomason model structures $(\operatorname{Cat}_{\infty})_{Th}$ and cat_{Th} are rather quirky in their own ways. On the other hand, recalling Corollary 4.3, it appears that there should exist the structure of an " ∞ -category of fibrant objects" structure on $\operatorname{Cat}_{\infty}$ (or a "category of fibrant objects" structure on eat), in which the co/cartesian fibrations $\mathcal{D} \to \mathcal{C}$ classified by functors $\mathcal{C} \to \operatorname{Cat}_{\infty}$ that have property Q are among the fibrations.²¹ In some vague sense, this would appear to be a more "true" articulation of the role of $\operatorname{Cat}_{\infty}$ (or of eat) as a presentation of S than either of the corresponding Thomason model structures.

References

- [Bar] C. Barwick, On the Q construction for exact ∞ -categories, available at arXiv:1301.4725, v2.
- [BK] C. Barwick and D. M. Kan, From partial model categories to ∞-categories, available at http://math.mit.edu/~clarkbar (and assembled from arXiv:1101.4879, v1 and arXiv:1102.2512, v2).
- [BK72] A. K. Bousfield and D. M. Kan, Homotopy limits, completions and localizations, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin-New York, 1972.
- [Bro71] Kenneth Stephen Brown, Abstract homotopy theory and generalized sheaf cohomology, ProQuest LLC, Ann Arbor, MI, 1971, Thesis (Ph.D.)-Massachusetts Institute of Technology.
- [DKS89] W. G. Dwyer, D. M. Kan, and J. H. Smith, Homotopy commutative diagrams and their realizations, J. Pure Appl. Algebra 57 (1989), no. 1, 5–24.
- [GHN] David Gepner, Rune Haugseng, and Thomas Nikolaus, Lax colimits and free fibrations in ∞ -categories, available at arXiv:1501.02161, v2.
- [GJ99] Paul G. Goerss and John F. Jardine, Simplicial homotopy theory, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999.
- [Hir03] Philip S. Hirschhorn, Model categories and their localizations, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.
- [III72] Luc Illusie, Complexe cotangent et déformations. II, Lecture Notes in Mathematics, Vol. 283, Springer-Verlag, Berlin-New York, 1972.
- [JFS] Theo Johnson-Freyd and Claudia Scheumabuer, (Op)lax natural transformations, relative field theories, and the "even higher" Morita category of E_d-algebras, available at arXiv:1502.06526, v2.
- [JT07] André Joyal and Myles Tierney, Quasi-categories vs Segal spaces, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 277–326.
- [Lur09] Jacob Lurie, Higher topos theory, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009, also available at http://math.harvard.edu/~lurie and at arXiv:math/0608040, v4.
- [Lur14] _____, Higher algebra, available at http://www.math.harvard.edu/~lurie, version dated September 14, 2014.

²⁰It is not so hard to see directly that if $\mathcal{D} \in \mathcal{S} \subset \operatorname{Cat}_{\infty}$, then a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is in \mathbf{F}_{Th} if and only if we also have $\mathcal{C} \in \mathcal{S} \subset \operatorname{Cat}_{\infty}$. (Thus, $\mathcal{S}_{\mathrm{triv}} \subset (\operatorname{Cat}_{\infty})_{\mathrm{Th}}$ is a model subcategory.) From here, it is clear at least that $((\mathbf{W} \cap \mathbf{F})_{\mathrm{Th}} \cap \mathcal{S}) = \mathcal{S}^{\simeq} \subset \operatorname{Cat}_{\infty}$.

²¹With the model ∞ -category (Cat_{∞})_{Th} in hand, this should be obtainable from arguments along the lines of those contained in [Bro71].

- $[MGa] A aron Mazel-Gee, Model <math>\infty$ -categories I: some pleasant properties of the ∞ -category of simplicial spaces, available at arXiv:1412.8411, v2.
- [MGb] _____, The universality of the Rezk nerve, available at arXiv:1510.03150, v1.
- [MGd] _____, Hammocks and fractions in relative ∞ -categories, to appear.
- [MGe] _____, Model ∞ -categories II: Quillen adjunctions, to appear.
- [MGf] _____, Model ∞ -categories III: the fundamental theorem, to appear.
- [MGj] _____, A user's guide to co/cartesian fibrations, available at arXiv:1510.02402, v1.
- [Qui73] Daniel Quillen, Higher algebraic K-theory. I, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.
- [Rez01] Charles Rezk, A model for the homotopy theory of homotopy theory, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007 (electronic).
- [Tho79] R. W. Thomason, Homotopy colimits in the category of small categories, Math. Proc. Cambridge Philos. Soc. 85 (1979), no. 1, 91–109.
- [Tho80] _____, Cat as a closed model category, Cahiers Topologie Géom. Différentielle 21 (1980), no. 3, 305–324.
- [Toë05] Bertrand Toën, Vers une axiomatisation de la théorie des catégories supérieures, K-Theory 34 (2005), no. 3, 233–263.