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Affineness and chromatic homotopy theory

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Abstract

Given an algebraic stack X, one may compare the derived category of quasi-coherent sheaves on X with the category of dg-modules over the dg-ring of functions on X. We study the analogous question in stable homotopy theory, for derived stacks that arise via realizations of diagrams of Landweber-exact homology theories. We identify a condition (quasi-affineness of the map to the moduli stack of formal groups) under which the two categories are equivalent, and study applications to topological modular forms. In particular, we provide new examples of Galois extensions of ring spectra and vanishing results for Tate spectra.

Contents

1. *Introduction*

Let X be a scheme (or, more generally, an algebraic stack). Then one has a natural abelian category $\mathrm{QCoh}^{\mathrm{ab}}(X)$ of *quasi-coherent sheaves* on X, which comes with a left exact functor

$$\Gamma\colon \mathrm{QCoh}^{\mathrm{ab}}(X) \longrightarrow \mathrm{Mod}^{\mathrm{ab}}(R),$$

into the abelian category $\mathrm{Mod}^{\mathrm{ab}}(R)$ of R-modules, where $R = \Gamma(X, \mathcal{O}_X)$ is the ring of regular functions on X. When X is affine, so that $X = \operatorname{Spec} R$, the functor Γ is an equivalence of categories (and is in particular exact). Conversely, a classical result of Serre implies that if X is a quasi-compact scheme, then the converse holds: if Γ is an equivalence, then $X \simeq \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$, so that X can be recovered as the spectrum of the ring of global sections on X, which in turn is determined by the category of $\Gamma(X, \mathcal{O}_X)$-modules.

Namely, one has the following theorem.

Theorem 1.1 (Serre). *Let X be a quasi-compact scheme. Suppose that the higher cohomologies $\{H^i(X, \mathcal{F})\}_{i \geqslant 1}$ vanish, for every quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}^{\mathrm{ab}}(X)$. Then X is affine.*

For a modern reference (and the strongest statement), we refer the reader to [**51**, Tag 01XF]. (These references can be looked up on http://stacks.math.columbia.edu/tag.) One can consider

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the equivalent question in the derived setting. Given a stack X, one has a natural *derived* category of quasi-coherent sheaves on X, which we will denote by $\mathrm{QCoh}(X)$. Rather than being an abelian category, it is a triangulated category or, better, the underlying homotopy category of a stable ∞-category in the sense of [**40**]. One has a similar (derived) global sections functor

$$\Gamma\colon \mathrm{QCoh}(X) \longrightarrow \mathrm{Mod}(R),$$

where $R = \Gamma(X, \mathcal{O}_X) = R\Gamma(X, \mathcal{O}_X)$ is now no longer a commutative ring, but itself a derived ring: it is a coconnective E_∞-ring spectrum, obtained as the homotopy limit of the discrete rings that map to X. In characteristic zero, R is a commutative, differential graded algebra such that $H^i(R) = 0$ for $i < 0$. In particular, $\mathrm{Mod}(R)$ itself is a stable ∞-category: if R is discrete, then it is the derived category of the abelian category $\mathrm{Mod}^{\mathrm{ab}}(R)$ of ordinary (that is, discrete) R-modules.

Certain phenomena work better in the derived context. For example, Γ is always 'exact' in the stable sense, which means that it respects finite homotopy limits and homotopy colimits. (Indeed, it respects arbitrary homotopy limits.) Unlike in the ordinary setting, it is possible for Γ to be an equivalence even if X is not affine, although in these cases R will usually be non-discrete.

Example 1.2. In this example, we work over the rational numbers. Let $X = B\mathbb{G}_a$ be the classifying stack of the additive group. Then $\Gamma(X, \mathcal{O}_X) = \mathbb{Q}[x_{-1}]$ is the free E_∞-algebra (over $\mathbb{Q}$) on a generator in degree -1, that is, the cochains on the circle S^1. Then it is known that taking global sections establishes an equivalence between the derived ∞-category $\mathrm{QCoh}(X)$ and the ∞-category $\mathrm{Mod}(\mathbb{Q}[x_{-1}])$ of modules (that is, module spectra) over $\mathbb{Q}[x_{-1}]$. This result can be extracted from [**37**] as follows. The stack $X = B\mathbb{G}_a$ sends a rational connective E_∞-ring R to $\Omega^\infty(\Sigma R)$. In the notation of [**37**, Section 4], one has $X = \mathrm{cSpec}\, A$ for $A = \mathbb{Q}[x_{-1}]$. Now, as in [**37**, Section 4.5] one has a t-structure on $\mathrm{Mod}(A)$ whose connective objects are those A-modules M such that $M \otimes_A \mathbb{Q}$ (for the map $A \to \mathbb{Q}$, unique up to homotopy) is connective. One also has a t-structure on $\mathrm{QCoh}(X)$. The left adjoint $\mathrm{Mod}(A) \to \mathrm{QCoh}(X)$ exhibits $\mathrm{QCoh}(X)$ as the left completion of $\mathrm{Mod}(A)$ by [**37**, Remark 4.5.6]. But we claim that $\mathrm{Mod}(A)$ is already left complete, that is, for any $M \in \mathrm{Mod}(A)$, the natural map $M \to \varprojlim_n \tau_{\leqslant n} M$ is an equivalence. In fact, if $N \in \mathrm{Mod}(A)_{\geqslant n}$, then $N \otimes_A \mathbb{Q}$ is an n-connective spectrum by definition. However, in view of the cofiber sequence of A-modules $\Omega\mathbb{Q} \to A \to \mathbb{Q}$, this implies easily that N is an $(n-1)$-connective spectrum. It follows that for any $M \in \mathrm{Mod}(A)$, the cofiber of $M \to \tau_{\leqslant n} M$ is $(n+1)$-connective as a spectrum, so that $M \to \varprojlim_n \tau_{\leqslant n} M$ is an equivalence of A-modules as desired. Thus, $\mathrm{QCoh}(X) \simeq \mathrm{Mod}(A)$. This equivalence of ∞-categories also gives us $\Gamma(X, \mathcal{O}_X) \simeq A$. Generalizations of this phenomenon have been explored in [**37**, **52**].

The purpose of this paper is to study this sort of affineness in a different setting, namely derived stacks in chromatic homotopy theory. Our motivational example is the (periodic) spectrum TMF of *topological modular forms*. It arises as the global sections of a sheaf of E_∞-ring spectra $\mathcal{O}^{\mathrm{top}}$ on the moduli stack of elliptic curves M_{ell}, constructed by Goerss, Hopkins and Miller and later by Lurie.

There are two natural ∞-categories one can associate to this construction.

(1) The ∞-category $\mathrm{QCoh}(\mathfrak{M}_{\mathrm{ell}})$ of quasi-coherent sheaves on the derived stack $\mathfrak{M}_{\mathrm{ell}} = (M_{\mathrm{ell}}, \mathcal{O}^{\mathrm{top}})$.

(2) The ∞-category $\mathrm{Mod}(\mathrm{TMF})$ of TMF-modules.

As an example of the affineness result, we prove the following theorem.

THEOREM 1.3. *The global sections functor establishes an equivalence of symmetric monoidal* ∞*-categories* $\mathrm{QCoh}(\mathfrak{M}_{\mathrm{ell}}) \simeq \mathrm{Mod}(\mathrm{TMF})$.

This theorem was originally established away from the prime 2 by the second author in [**44**], and is useful for both theoretical and computational purposes. The result was also known to Lurie in unpublished work (by a different argument). We also prove a version for the compactified moduli stack of elliptic curves $M_{\overline{\mathrm{ell}}}$, which carries a sheaf of E_∞-ring spectra $\mathcal{O}^{\mathrm{top}}$ as well, defining a derived stack $\mathfrak{M}_{\overline{\mathrm{ell}}} = (M_{\overline{\mathrm{ell}}}, \mathcal{O}^{\mathrm{top}})$. The global sections $\Gamma(\mathfrak{M}_{\overline{\mathrm{ell}}}, \mathcal{O}^{\mathrm{top}})$ are denoted by Tmf.

THEOREM 1.4. *The global sections functor establishes an equivalence of symmetric monoidal* ∞*-categories* $\mathrm{QCoh}(\mathfrak{M}_{\overline{\mathrm{ell}}}) \simeq \mathrm{Mod}(\mathrm{Tmf})$.

The main purpose of this paper is to prove these theorems in a more general context, as a consequence of nilpotence technology.

Given any noetherian and separated Deligne–Mumford stack X with a flat morphism $X \to M_{FG}$ to the moduli stack M_{FG} of formal groups, one can construct a presheaf of even periodic Landweber-exact homology theories on X. Sometimes, it can be lifted to E_∞-rings to produce a derived stack $\mathfrak{X}$, as in the case $X = M_{\mathrm{ell}}$. If it can be lifted, then one can ask the same question as above: is the ∞-category $\mathrm{QCoh}(\mathfrak{X})$ of quasi-coherent sheaves on $\mathfrak{X}$ equivalent to the ∞-category of modules over $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$? This is certainly true when X is affine. We show that the same conclusion holds in the following setting.

MAIN THEOREM. *If* $X \to M_{FG}$ *is quasi-affine, then the global sections functor establishes an equivalence of symmetric monoidal* ∞*-categories* $\mathrm{QCoh}(\mathfrak{X}) \simeq \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}))$.

Recall here that a map $X \to M_{FG}$ is *quasi-affine* if for every map $\mathrm{Spec}\, A \to M_{FG}$ the pullback $\mathrm{Spec}\, A \times_{M_{FG}} X$ is quasi-affine, that is, a quasi-compact open subscheme of an affine scheme. The result is proved via a consequence of derived Morita (Schwede–Shipley) theory together with a version of the Hopkins–Ravenel smash product theorem. The latter states that the localization functor L_n commutes with homotopy colimits. Likewise, a crucial part of our main theorem is that the global sections functor commutes with homotopy colimits. This turns out to be true even if $X \to M_{FG}$ is not quasi-affine, but only *tame*, that is, the order of every automorphism of a point of X not detected by the formal group is invertible on X.

We apply our main theorem to the study of Galois extensions of E_∞-rings (in the sense of Rognes [**47**]) and to vanishing results about Tate spectra. As an example, we consider the moduli stack of elliptic curves with $\Gamma(n)$-level structure $M_{\mathrm{ell}}(n)$ and its compactified version $M_{\overline{\mathrm{ell}}}(n)$. These classify (generalized) elliptic curves with a chosen isomorphism between the n-torsion points and $(\mathbb{Z}/n\mathbb{Z})^2$. The action of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ on $(\mathbb{Z}/n\mathbb{Z})^2$ defines $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-actions on $M_{\mathrm{ell}}(n)$ and $M_{\overline{\mathrm{ell}}}(n)$. Both of these stacks carry sheaves of E_∞-ring spectra $\mathcal{O}^{\mathrm{top}}$, whose global sections are denoted by $\mathrm{TMF}(n)$ and $\mathrm{Tmf}(n)$, respectively. The latter was recently defined by work of Goerss–Hopkins and Hill–Lawson [**23**].

We can prove the following two theorems.

THEOREM 1.5. *For every* n, *the* E_∞*-ring spectrum* $\mathrm{TMF}(n)$ *is a faithful* $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$*-Galois extension of* $\mathrm{TMF}[1/n]$.

THEOREM 1.6. *For every* n, *the norm map* $\mathrm{Tmf}(n)_{hGL_2(\mathbb{Z}/n\mathbb{Z})} \to \mathrm{Tmf}(n)^{hGL_2(\mathbb{Z}/n\mathbb{Z})}$ *is an equivalence. Equivalently, the Tate spectrum* $\mathrm{Tmf}(n)_{tGL_2(\mathbb{Z}/n\mathbb{Z})}$ *vanishes.*

Note that the vanishing of Tate spectra is automatic for faithful Galois extensions, but $\mathrm{Tmf}[1/n] \to \mathrm{Tmf}(n)$ is not a Galois extension. Note furthermore that the second theorem was proved by Stojanoska in [**50**] in the case $n = 2$ in her investigation of the Anderson self-duality of Tmf. We hope that our results about the vanishing of Tate spectra will have future applications to duality.

In Section 2, we will discuss various background material. This includes the relationship between formal groups and even periodic ring spectra and, furthermore, derived stacks and coarse moduli spaces. The knowledgeable reader may just want to pick up our definitions of an even periodic refinement (Definition 2.5), of a derived stack (Definition 2.6) and of a tame morphism (Definition 2.28). In Section 3, we discuss first an abstract characterization of derived stacks for which the global sections functor is an equivalence (via the Schwede–Shipley theorem). Then we show certain descent and ascent properties of this class of derived stacks. In Section 4, we specialize these abstract theorems to chromatic homotopy theory and obtain our main theorem. Section 5 contains our abstract theorems about Galois extensions and the vanishing of Tate spectra, which are then applied to examples in Sections 6 and 7. Theorem 7.12 discusses the behavior of ∞-category-valued sheaves with respect to a finite open cover of a topological space (as used in Section 3).

Throughout this paper, we use the language of quasi-categories (that is, ∞-categories) of [**28**, **33**], and the theory of structured ring spectra as developed originally in [**17**], and formulated in ∞-categorical terms in [**40**]. We will let $\mathcal{S}$ denote the ∞-category of spaces, Sp the ∞-category of spectra and we will write $\otimes$ for the smash product in the latter.

2. *Derived stacks*

We will take a naive approach to derived stacks in this paper, and avoid the most general theory. In this section, we summarize what we need, and briefly review the role of formal groups. Furthermore, we will review the theory of coarse moduli spaces and the Zariski topology for algebraic stacks.

2.1. *Even periodic ring spectra and formal groups*

Recall first the following definition.

Definition 2.1. Let M_{FG} be the *moduli stack of formal groups*: that is, it is the (infinite-dimensional) stack assigning to a commutative ring R the groupoid of one-dimensional, commutative formal groups over R and isomorphisms between them.

Define $MUP = \bigvee_{k\in\mathbb{Z}} \Sigma^{2k} MU$ to be a periodic version of complex bordism MU. A theorem of Quillen (see, for example, [**2**]) shows that $MU_* = MUP_0$ is isomorphic to the Lazard ring L, which carries the universal formal group law. Even more is true: the simplicial scheme $\operatorname{Spec}\pi_0(MUP^{\otimes \bullet+1})$ is isomorphic to the simplicial scheme $(\operatorname{Spec} L)^{\times_{M_{FG}} \bullet+1}$. In particular, if we use the notation $\operatorname{Spec} W = \operatorname{Spec} L \times_{M_{FG}} \operatorname{Spec} L$, it is true that the Hopf algebroids $(MUP_0, MUP_0 MUP)$ and (L, W) are isomorphic.

Construction 2.2. Given a spectrum X, both $MUP_0(X)$ and $MUP_1(X)$ are comodules over the Hopf algebroid (MUP_0, MUP_0MUP). Via the equivalence between (MUP_0, MUP_0MUP)-comodules and quasi-coherent sheaves on M_{FG}, this defined a $\mathbb{Z}/2$-graded sheaf $\mathcal{F}_*(X)$. This sheaf is characterized by the property that the evaluation of $\mathcal{F}_0(X)$ on $\operatorname{Spec} L$ agrees with $MUP_0(X)$ as comodules over $(MUP_0, MUP_0MUP) \cong (L, W)$, and evaluation of $\mathcal{F}_1(X)$ on $\operatorname{Spec} L$ agrees with $MUP_1(X)$.

The $\mathbb{Z}/2$-graded sheaf $\mathcal{F}_*(X)$ can often be described explicitly. For example, the sphere S^{-2} is associated to a line bundle $\omega \in \mathrm{Pic}(M_{FG})$ which assigns to every formal group the dual of its *Lie algebra*. Moreover, the theorem above by Quillen implies that $\mathcal{F}_0(MUP) = (\phi_L)_*\mathcal{O}_{\mathrm{Spec}\, L}$ and $\mathcal{F}_0(MUP \otimes MUP) = (\phi_W)_*\mathcal{O}_{\mathrm{Spec}\, W}$, where $\phi_L \colon \mathrm{Spec}\, L \to M_{FG}$ and $\phi_L \colon \mathrm{Spec}\, W \to M_{FG}$ are the obvious maps. This point of view has been very fruitful in describing large-scale features of stable homotopy theory via the special geometry of M_{FG}.

An example of this connection is the partial correspondence between certain ring spectra and formal groups: one can associate a formal group to certain ring spectra, and in some cases one can recover the value of the associated cohomology theory in terms of the $\mathbb{Z}/2$-graded quasi-coherent sheaf $\mathcal{F}_*(X)$ on M_{FG}. In this way, complex bordism allows one to manufacture a great deal of new ring spectra.

Definition 2.3 ([**3**]). A homotopy commutative ring spectrum E is said to be *even periodic* if $\pi_i E = 0$ for i odd and if $\pi_2 E$ is an invertible module over $\pi_0 E$, with inverse $\pi_{-2}E$, such that

$$\pi_{2k}E \simeq (\pi_2 E)^{\otimes k}, \quad k \in \mathbb{Z},$$

under multiplication. This is slightly weaker than the definition in [**3**], which requires $\pi_2 E$ to be the trivial invertible module, that is, to contain a unit. We will refer to such ring spectra as *strongly even periodic*.

Given an even periodic ring spectrum E, the Atiyah–Hirzebruch spectral sequence for the E-cohomology of any *even* space X (that is, with integral homology free and concentrated in even dimensions) degenerates. For example, if E is strongly even periodic, then there is an isomorphism of rings $E^0(\mathbb{CP}^\infty) = \pi_0 E[[x]]$, where the generator $x \in \widetilde{E}^0(\mathbb{CP}^\infty)$ is non-canonical (and called a *complex orientation* of E). The multiplication on $\mathbb{CP}^\infty$ is dual to a *comultiplication* in $E^0(\mathbb{CP}^\infty)$, which gives $E^0(\mathbb{CP}^\infty)$ the structure of a (continuous) commutative, cocommutative Hopf algebra. Equivalently, the formal scheme $\mathrm{Spf}\, E^0(\mathbb{CP}^\infty)$ is canonically a *formal group* over $\pi_0 E$. This persists for a general even periodic ring spectrum, although the formal group need only admit a coordinate Zariski locally on $\pi_0 E$, and we get a map $\mathrm{Spec}\, \pi_0 E \to M_{FG}$. This is one direction of the correspondence between ring spectra and formal groups.

In some cases, one can reconstruct the cohomology theory (and even the ring spectrum) from the formal group. For example, the *Landweber exact functor theorem* [**30**] gives a concrete and often easily checked criterion for a map $\phi \colon \mathrm{Spec}\, R \to M_{FG}$ to be flat. Given such a flat map, and a spectrum X, one can pullback the $\mathbb{Z}/2$-graded quasi-coherent sheaf $\mathcal{F}_*(X)$ to $\mathrm{Spec}\, R$ to define an invariant of X, which is in fact an even periodic homology theory E. More precisely, we define

$$E_{2k}(X) := \Gamma(\mathrm{Spec}\, R, \phi^*(\mathcal{F}_0(X) \otimes \omega^{\otimes k})) = (\mathcal{F}_0(X) \otimes \omega^{\otimes k})(\mathrm{Spec}\, R),$$
$$E_{2k+1}(X) := \Gamma(\mathrm{Spec}\, R, \phi^*(\mathcal{F}_1(X) \otimes \omega^{\otimes k})) = (\mathcal{F}_1(X) \otimes \omega^{\otimes k})(\mathrm{Spec}\, R).$$

Given a flat morphism $\phi \colon \mathrm{Spec}\, R \to M_{FG}$, the formal group of the Landweber-exact even periodic cohomology theory that one obtains is precisely classified by the map ϕ. Conversely, given an even periodic ring spectrum E, one obtains a map $\phi \colon \mathrm{Spec}\, \pi_0 E \to M_{FG}$ classifying the formal group $\mathrm{Spf}\, E^0(\mathbb{CP}^\infty)$; if this map is flat, then E is the Landweber-exact theory obtained from ϕ. An important example of such a theory is given by complex K-theory KU, as was first shown (without using Landweber's theorem) by Conner and Floyd.

The following proposition is well known.

Proposition 2.4. *Given two flat morphisms* $\phi_R \colon \mathrm{Spec}\, R \to M_{FG}$ *and* $\phi_{R'} \colon \mathrm{Spec}\, R' \to M_{FG}$, *we denote the corresponding Landweber exact spectra by* E_R *and* $E_{R'}$. *With this* *notation, we have an isomorphism*

$$\pi_{2k}(E_R \otimes E_{R'}) \cong \omega^{\otimes k}(\operatorname{Spec} R \times_{M_{FG}} \operatorname{Spec} R').$$

Proof. We first investigate the situation $\phi_{R'} = \phi_L$ so that $E_{R'} = MUP$. By definition,

$$MUP_0(E_R) = (E_R)_0(MUP) = (\mathcal{F}_0(MUP))(\operatorname{Spec} R).$$

The latter agrees with

$$((\phi_L)_*\mathcal{O}_{\operatorname{Spec} L})(\operatorname{Spec} R) = \mathcal{O}_{M_{FG}}(\operatorname{Spec} L \times_{M_{FG}} \operatorname{Spec} R) = ((\phi_R)_*\mathcal{O}_{\operatorname{Spec} R})(\operatorname{Spec} L).$$

Similarly, $(MUP \otimes MUP)_0(E_R) = ((\phi_R)_*\mathcal{O}_{\operatorname{Spec} R})(\operatorname{Spec} W)$, which implies $\mathcal{F}(E_R) = (\phi_R)_* \mathcal{O}_{\operatorname{Spec} R}$.

In the general case, we get now:

$$\begin{aligned}\pi_{2k}(E_R \otimes E_{R'}) &= (E_{R'})_{2k}(E_R)\\ &= ((\phi_R)_*\mathcal{O}_{\operatorname{Spec} R} \otimes \omega^{\otimes k})(\operatorname{Spec} R')\\ &= ((\phi_R)_*(\phi_R)^*\omega^{\otimes k})(\operatorname{Spec} R')\\ &= \omega^{\otimes k}(\operatorname{Spec} R \times_{M_{FG}} \operatorname{Spec} R').\end{aligned}$$

□

2.2. *Even periodic enhancements and derived stacks*

The upshot of the discussion of the previous section is that there is a *presheaf of even periodic homology theories* on the affine flat site of M_{FG}. Equivalently, for every commutative ring R with a formal group over R classified by a flat map $\operatorname{Spec} R \to M_{FG}$, one obtains an even periodic homology theory E with E_0 given by R, and one obtains morphisms between homology theories from morphisms of formal groups. One can show that one actually gets a homotopy commutative ring spectrum from each such Landweber-exact homology theory, and that each map of formal groups gives a map of ring spectra, such that all functoriality holds up to homotopy, albeit not coherent homotopy (see [**27**, Theorem 2.8] or [**35**, Lecture 18]).

For the purposes of homotopy theory, a diagram such as above, which takes values in the *homotopy category* of spectra, is insufficient to make many natural constructions, such as homotopy limits and colimits. For example, there is no way to extend the above construction to a non-affine scheme (or stack) flat over M_{FG}. Given a (discrete) group acting on a formal group, that does not produce a strict group action on the associated spectrum. Moreover, the ring spectra one obtains do not have the structure needed to perform algebraic constructions with them: for example, one cannot generally obtain a good theory of modules (for example, a triangulated category or stable ∞-category) over an unstructured ring spectrum.

However, in certain restricted cases, it is possible to realize diagrams of homology theories much more rigidly. A survey of this problem, including a general result of Lurie, is in [**19**].

Let X be a Deligne–Mumford stack together with a flat map

$$X \longrightarrow M_{FG},$$

so that, as above, one obtains a presheaf of multiplicative homology theories on the affine flat site of X. Here a map $X \to M_{FG}$, not necessarily representable, is *flat* if for every étale covering $\operatorname{Spec} R \to X$, the composite $\operatorname{Spec} R \to X \to M_{FG}$ is flat in the sense that for every map $\operatorname{Spec} A \to M_{FG}$, the pullback $\operatorname{Spec} A \times_{M_{FG}} \operatorname{Spec} R \to \operatorname{Spec} A$ (which is a map of schemes) is flat. Let $\operatorname{Aff}^{\mathrm{et}}_{/X}$ be the affine, *étale* site of X.

Definition 2.5. An *even periodic enhancement* or *even periodic refinement* $\mathfrak{X}$ of X is a sheaf $\mathcal{O}^{\mathrm{top}}$ of even periodic E_∞-rings on the site $\mathrm{Aff}^{\mathrm{et}}_{/X}$, lifting the above diagram of homology theories on $\mathrm{Aff}^{\mathrm{et}}_{/X}$.

In other words, for an étale map $\operatorname{Spec} R \to X$, the E_∞-ring $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)$ defines an even periodic cohomology theory, with formal group given by the classifying map $\operatorname{Spec} R \to X \to M_{FG}$: it yields the Landweber-exact (co)homology theory associated to this formal group. The 'sheaf' condition is actually redundant here, because by construction, the homotopy groups of $\mathcal{O}^{\mathrm{top}}$ already form a sheaf on the affine étale site. Note that the phrase 'sheaf of spectra' refers to a functor from the category $\mathrm{Aff}^{\mathrm{et}}_{/X}$ into the ∞-category of spectra (for example, realized via a functor into some model category) and does not refer to the homotopy category.

Even periodic enhancements are examples of (even periodic) derived stacks. Our notion of a derived stack is a special case of the notion of a non-connective spectral Deligne–Mumford stack in Lurie's DAG series (see [**36**, 8.5 and 8.42] for his definition). We prefer to spell this special case out (informally) for the convenience of the reader.

Definition 2.6. A *derived stack* $\mathfrak{X}$ will be for us a Deligne–Mumford stack X together with a sheaf of E_∞-ring spectra $\mathcal{O}^{\mathrm{top}} = \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ on $\mathrm{Aff}^{\mathrm{et}}_{/X}$ and an isomorphism $\pi_0\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}} \cong \mathcal{O}_X$. Here $\pi_i\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ is the sheaf $U \mapsto \pi_i(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}(U))$ on $\mathrm{Aff}^{\mathrm{et}}_{/X}$. Furthermore, one requires $\pi_i\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ to be quasi-coherent as an $\mathcal{O}_X$-module.

The derived stack $\mathfrak{X}$ is called *even periodic* if $\omega = \pi_2\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ is a line bundle such that multiplication induces isomorphisms

$$\pi_{2k}\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}} \otimes \pi_{2l}\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}} \cong \pi_{2(k+l)}\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}, \quad k, l \in \mathbb{Z},$$

and we have

$$\pi_i\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}} = 0 \quad \text{for } i \text{ odd}.$$

Next, we want to define morphisms of derived stacks. If $f\colon Y \to X$ is a map of Deligne–Mumford stacks and $\mathcal{F}$ a sheaf of spectra on $\mathrm{Aff}^{\mathrm{et}}_{/X}$, then we can define a sheaf of spectra $f^{-1}\mathcal{F}$ on $\mathrm{Aff}^{\mathrm{et}}_{/Y}$ as the sheafification of the presheaf given by

$$(f^{-1}_{\mathrm{pre}}\mathcal{F})(U \longrightarrow Y) = \operatorname{hocolim}_{U \to V \to X,\ V \to X \text{ étale}} \mathcal{F}(V).$$

As this homotopy colimit is filtered, in a 2-categorical sense, it follows that if $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ is a sheaf of E_∞-ring spectra, then $f^{-1}_{\mathrm{pre}}\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ is a presheaf of E_∞-ring spectra on $\mathrm{Aff}^{\mathrm{et}}_{/Y}$ (see [**40**, 3.2.3.2]) and thus $f^{-1}\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ a sheaf of E_∞-ring spectra [**36**, 1.15]. Furthermore, $f^{-1}\pi_*(\mathcal{F}) \to \pi_*(f^{-1}\mathcal{F})$ is an isomorphism.

Definition 2.7. Let $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ and $\mathfrak{Y} = (Y, \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}})$ be derived stacks. Then a morphism $f\colon \mathfrak{Y} \to \mathfrak{X}$ of derived stacks consists of a morphism $f_0\colon Y \to X$ of the underlying Deligne–Mumford stacks and a morphism $\alpha\colon f_0^{-1}\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}} \to \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}}$ of E_∞-ring spectra such that $\pi_0\alpha$ coincides with the morphism $f_0^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ defined by f_0.

Given such a morphism $f\colon \mathfrak{Y} \to \mathfrak{X}$ of derived stacks and an $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$-module $\mathcal{F}$, we define $f^*\mathcal{F}$ as $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}} \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}}$.

In an evident manner, an even periodic enhancement of X defines even periodic enhancements of each stack étale over X. Given an even periodic enhancement, it follows that one *can* evaluate the sheaf $\mathcal{O}^{\mathrm{top}}$ on any stack Y étale over X. Namely, one defines

$$\mathcal{O}^{\mathrm{top}}(Y) = \operatorname{holim}_{\operatorname{Spec} R \to Y} \mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R),$$

as $\operatorname{Spec} R \to Y$ ranges over all the étale morphisms from affine schemes. Then $\mathcal{O}^{\mathrm{top}}(Y)$ is naturally an E_∞-ring. Such spectra will generally fail (if Y is not affine) to be Landweber-exact or even periodic, and may exhibit intricate torsion phenomena. For example, we can consider $\mathcal{O}^{\mathrm{top}}(X)$ itself, which we can think of as the ring of 'functions'. Below, we will write $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$ for this.

REMARK 2.8. It is also fruitful to consider derived stacks as representing some type of moduli problem for (possibly non-connective) structured ring spectra. This point of view was used by Lurie to give a construction of the even periodic enhancement of the moduli stack of elliptic curves in [**34**], producing the spectrum of TMF.

2.3. *Quasi-coherent sheaves*

In this subsection, we will review the basics of quasi-coherent sheaves on derived stacks. Fix one such $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$. Given an E_∞-ring A, we write $\operatorname{Mod}(A)$ for the stable ∞-category of A-modules.

DEFINITION 2.9. The ∞-category $\operatorname{QCoh}(\mathfrak{X})$ of quasi-coherent sheaves on $\mathfrak{X}$ is the homotopy limit

$$\operatorname{QCoh}(\mathfrak{X}) \stackrel{\mathrm{def}}{=} \operatorname{holim}_{(\operatorname{Spec} R \to X) \in \operatorname{Aff}^{\mathrm{et}}_{/X}} \operatorname{Mod}(\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)).$$

In other words, a quasi-coherent sheaf on $\mathfrak{X}$ assigns to every étale map $\operatorname{Spec} R \to X$ a module M_R over $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)$, together with equivalences

$$M_R \otimes_{\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)} \mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R') \simeq M_{R'},$$

for each (2-)commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R \\ & \searrow \quad \swarrow & \\ & X & \end{array} \tag{1}$$

and appropriate compatibility data between these equivalences.

We note that one is constructing a homotopy limit of presentable, stable ∞-categories under colimit-preserving, exact functors. It follows that the homotopy limit is itself a presentable, stable ∞-category where homotopy colimits are computed 'pointwise' (see [**33**, Proposition 5.5.3.13]).

Using the derived version of flat descent theory [**38**], which states that the assignment $A \mapsto \operatorname{Mod}(A)$ for an E_∞-ring A is a sheaf of ∞-categories in the flat topology on affine (derived) schemes, it follows that one can give an alternative definition. Suppose first that X has affine diagonal. Choose an étale *surjection* $\operatorname{Spec} R \to X$. Then $\operatorname{QCoh}(\mathfrak{X})$ is the homotopy limit of the cosimplicial diagram of ∞-categories

$$\operatorname{Mod}(\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)) \rightrightarrows \operatorname{Mod}(\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R \times_X \operatorname{Spec} R)) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots .$$

If the diagonal of X is not affine, then one should use an étale hypercover rather than a Cech cover.

Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathfrak{X}$. Then, for each k, the assignment

$$(\operatorname{Spec} R \longrightarrow X) \in \operatorname{Aff}^{\mathrm{et}}_{/X} \longmapsto \pi_k(\mathcal{F}(\operatorname{Spec} R)),$$

defines a quasi-coherent sheaf $\pi_k \mathcal{F}$ on the *ordinary* stack X: that is, it assigns an R-module (in the classical sense) to each étale map $\operatorname{Spec} R \to X$, together with appropriate equivalences and compatibility data. We note that no further sheafification is required since we are working with affine schemes. Given a 2-commuting diagram (1), the map $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R) \to \mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R')$ is flat (even étale) on homotopy groups, and it follows that one has canonical *isomorphisms*

$$\pi_k(\mathcal{F}(\operatorname{Spec} R)) \otimes_R R' \simeq \pi_k(\mathcal{F}(\operatorname{Spec} R'))$$

and thus $\pi_k\mathcal{F}$ is quasi-coherent. In the even periodic case, only π_0 and π_1 are necessary for bookkeeping, because

$$\pi_{n+2k}\mathcal{F} \simeq \pi_n\mathcal{F} \otimes \omega^k,$$

where $\omega = \pi_2\mathcal{O}^{\mathrm{top}}$.

EXAMPLE 2.10. Let T be a spectrum. Then one has a quasi-coherent sheaf $\mathcal{O}^{\mathrm{top}} \otimes T \in \mathrm{QCoh}(\mathfrak{X})$, given by

$$(\operatorname{Spec} R \longrightarrow X) \longmapsto \mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R) \otimes T.$$

In fact, the category $\mathrm{QCoh}(\mathfrak{X})$ (like any presentable, stable ∞-category) is canonically *tensored* over spectra in this way.

Suppose that $\mathfrak{X}$ is an even periodic refinement of a flat map $X \to M_{FG}$. Then the homotopy groups $\pi_0(\mathcal{O}^{\mathrm{top}} \otimes T), \pi_1(\mathcal{O}^{\mathrm{top}} \otimes T)$ are given by the pullback of the $\mathbb{Z}/2$-graded sheaf $\mathcal{F}_*(T)$ on M_{FG} to X via the given map $X \to M_{FG}$, since we have assumed that the diagram $\mathcal{O}^{\mathrm{top}}$ of E_∞-rings lifts the diagram of Landweber-exact homology theories.

These homotopy groups $\pi_k\mathcal{F}$ are important for several reasons; one is that the homotopy groups of the *global sections*

$$\Gamma(\mathfrak{X}, \mathcal{F}) = \operatorname{holim}_{(\operatorname{Spec} R \to X) \in \mathrm{Aff}^{\mathrm{et}}_{/X}} \mathcal{F}(\operatorname{Spec} R)$$

of $\mathcal{F}$ are the abutment of a *descent spectral sequence*

$$H^i(X, \pi_j\mathcal{F}) \implies \pi_{j-i}\Gamma(\mathfrak{X},\mathcal{F}).$$

We will abbreviate the descent spectral sequence to DSS.

Let $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$ be the E_∞-ring of global sections of the structure sheaf. Then the global sections functor on $\mathrm{QCoh}(\mathfrak{X})$ takes values in $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$-modules. Indeed, one has a functor of 'tensoring up'

$$\mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})) \longrightarrow \mathrm{QCoh}(\mathfrak{X}),$$

which sends a $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$-module M to the quasi-coherent sheaf

$$(\operatorname{Spec} R \longrightarrow X) \longmapsto \mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R) \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})} M.$$

The global sections functor is the right adjoint to 'tensoring up'. The relation between these two ∞-categories given by this adjoint pair is the main subject of this paper.

In the next two subsections, we will discuss several important examples of even periodic refinements.

2.4. *Affine schemes*

We begin with the following basic observation.

PROPOSITION 2.11. *Let A be a Landweber exact, even periodic E_∞-ring. Then the affine scheme* $\operatorname{Spec}\pi_0 A$, *together with the natural map* $\operatorname{Spec}\pi_0 A \to M_{FG}$, *has a canonical even periodic enhancement, and its category of quasi-coherent sheaves is equivalent to* $\mathrm{Mod}(A)$.

Proof. It suffices to show that for every étale $\pi_0 A$-algebra A'_0, there exists an even periodic E_∞-A-algebra A' with the property that $\pi_0 A' \simeq A'_0$, and that this construction can be done functorially in A'_0. This follows from [40, Section 8.4], reviewed below, which implies that the ∞-category of such A-algebras is equivalent to the discrete category of étale $\pi_0 A$-algebras. □

The basic result about E_∞-rings needed for the above is the following derived version of the 'topological invariance of the étale site', a proof of which appears in [40, Section 8.4].

THEOREM 2.12. *Let R be an E_∞-ring, and consider the ∞-category $\mathrm{CAlg}_{R/}$ of E_∞-rings under R. Let $\mathrm{CAlg}^{\mathrm{et}}_{R/} \subset \mathrm{CAlg}_{R/}$ be the subcategory of étale R-algebras: that is, those R-algebras R' with the properties that:*

(1) *$\pi_0 R'$ is étale over $\pi_0 R$;*
(2) *$\pi_* R' \simeq \pi_* R \otimes_{\pi_0 R} \pi_0 R'$.*

Then we have an equivalence of ∞-categories

$$\mathrm{CAlg}^{\mathrm{et}}_{R/} \overset{\pi_0}{\simeq} \mathrm{Ring}^{\mathrm{et}}_{\pi_0 R/},$$

with the (discrete) category of étale $\pi_0 R$-algebras.

In other words, if $A \to B$ is an étale morphism in CAlg, then for any $B' \in \mathrm{CAlg}$, we have a homotopy cartesian square of spaces

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{CAlg}}(B, B') & \longrightarrow & \mathrm{Hom}_{\mathrm{CAlg}}(A, B') \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Ring}}(\pi_0 B, \pi_0 B') & \longrightarrow & \mathrm{Hom}_{\mathrm{Ring}}(\pi_0 A, \pi_0 B') \end{array} \tag{2}$$

where both horizontal arrows are given by precomposition. It will be useful to have the following slight generalization (and corollary) of Theorem 2.12.

COROLLARY 2.13. *Let $\mathcal{C}$ be an ∞-category, and $F\colon \mathcal{C} \to \mathrm{CAlg}$ be a functor to E_∞-rings. Consider the composite $\overline{F}\colon \mathcal{C} \to \mathrm{CAlg} \overset{\pi_0}{\to} \mathrm{Ring}$, and consider an extension $\overline{G} \in \mathrm{Fun}(\mathcal{C} \times \Delta^1, \mathrm{Ring})$, of $\overline{F}$, in the sense that the restriction of $\overline{G}$ to the first vertex is identified with $\overline{F}$. Suppose that for each $x \in \mathcal{C}$, the morphism $\overline{G}(x)$ is étale. Then there is a unique extension $G \in \mathrm{Fun}(\mathcal{C} \times \Delta^1, \mathrm{CAlg})$, of both F and $\overline{G}$.*

Proof. Let $\mathrm{Fun}^{\mathrm{et}}(\Delta^1, \mathrm{CAlg})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{CAlg})$ spanned by those morphisms of E_∞-rings that are étale. Define $\mathrm{Fun}^{\mathrm{et}}(\Delta^1, \mathrm{Ring})$ similarly. Then Theorem 2.12 gives us that the natural functor

$$\mathrm{Fun}^{\mathrm{et}}(\Delta^1, \mathrm{CAlg}) \longrightarrow \mathrm{CAlg} \times_{\mathrm{Ring}} \mathrm{Fun}^{\mathrm{et}}(\Delta^1, \mathrm{Ring}), \quad (A \longmapsto B) \longmapsto \{A, \pi_0 A, \pi_0 A \to \pi_0 B\}$$

is an equivalence of ∞-categories.

Indeed, the existence part of Theorem 2.12 gives essential surjectivity. To see full faithfulness, consider two objects $A \to B, A' \to B'$ in $\mathrm{Fun}^{\mathrm{et}}(\Delta^1, \mathrm{CAlg})$. Then

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Fun}^{\mathrm{et}}(\Delta^1, \mathrm{CAlg})}((A \longrightarrow B), (A' \longrightarrow B')) \\ &\quad \simeq \mathrm{Hom}_{\mathrm{CAlg}}(A, A') \times_{\mathrm{Hom}_{\mathrm{CAlg}}(A, B')} \mathrm{Hom}_{\mathrm{CAlg}}(B, B') \\ &\quad \simeq \mathrm{Hom}_{\mathrm{CAlg}}(A, A') \times_{\mathrm{Hom}_{\mathrm{Ring}}(\pi_0 A, \pi_0 B')} \mathrm{Hom}_{\mathrm{Ring}}(\pi_0 B, \pi_0 B'), \end{aligned}$$

where the last equivalence holds because (2) is homotopy cartesian. This shows that our functor is fully faithful.

Finally, we find that

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Fun}^{\mathrm{et}}(\Delta^1, \mathrm{CAlg})) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{CAlg}) \times_{\mathrm{Fun}(\mathcal{C}, \mathrm{Ring})} \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}^{\mathrm{et}}(\Delta^1, \mathrm{Ring}))$$

is an equivalence of ∞-categories, which is equivalent to the desired statement. □

In other words, the 'topological invariance of the étale site' can be done functorially.

EXAMPLE 2.14. Let G be a group acting on an E_∞-ring R. Suppose we have an étale extension T_0 of $\pi_0 R$. Suppose that T_0 is given a G-action in such a way that

$$\pi_0 R \longrightarrow T_0$$

is G-equivariant. Then the étale extension $R \to T$ constructed in Theorem 2.12 canonically has a G-action in view of Corollary 2.13.

2.5. *Group actions*

In this subsection, we give the most basic non-affine example of an even periodic refinement.

Let R be an even periodic, Landweber-exact E_∞-ring, and let G be a finite group acting on R (in the ∞-category of E_∞-rings). Then we get a map from even periodicity, $\operatorname{Spec} \pi_0 R \to M_{FG}$, which has canonically the structure of a G-equivariant map: that is, G acts compatibly on the formal group. Consequently, we get a map of stacks,

$$(\operatorname{Spec} \pi_0 R)/G \longrightarrow M_{FG}.$$

For example, take R to be complex K-theory KU. Then there is a $\mathbb{Z}/2$-action on KU coming from complex conjugation of vector bundles, which can be made into a $\mathbb{Z}/2$-action in E_∞-rings. At the level of formal groups, one has

$$\operatorname{Spf} KU^0(\mathbb{CP}^\infty) \simeq \widehat{\mathbb{G}_m},$$

that is, the formal group of KU is the formal multiplicative group, which is classified by a flat map

$$\operatorname{Spec} \mathbb{Z} \longrightarrow M_{FG},$$

so that KU is Landweber-exact. The $\mathbb{Z}/2$-action on KU corresponds to the involution of $\widehat{\mathbb{G}_m}$ given by $x \mapsto x^{-1}$. In particular, one obtains a map

$$B\mathbb{Z}/2 \longrightarrow M_{FG}.$$

This map takes a $\mathbb{Z}/2$-torsor over a scheme $\operatorname{Spec} R$ and outputs the formal completion of the associated *one-dimensional torus* over $\operatorname{Spec} R$, not necessarily split, in such a way that the $\mathbb{Z}/2$-action on a torsor maps to the $\mathbb{Z}/2$-action on the torus given by inversion. Since $\mathrm{Aut}(\mathbb{G}_m) \simeq \mathbb{Z}/2$, the stack $B\mathbb{Z}/2$ classifies precisely one-dimensional tori.

The next result shows that we can obtain an even periodic refinement of stacks such as $B\mathbb{Z}/2$.

PROPOSITION 2.15. *If R and G are as above and R is Landweber exact, then there is a canonical even periodic refinement of* $(\operatorname{Spec} \pi_0 R)/G \to M_{FG}$.

See also [**32**] for the example of KU-theory.

Proof. Consider an étale map $\operatorname{Spec} T \to (\operatorname{Spec} \pi_0 R)/G$, from which we form the pullback

$$\begin{array}{ccc} \operatorname{Spec} T' & \longrightarrow & \operatorname{Spec} \pi_0 R \\ \downarrow & & \downarrow \\ \operatorname{Spec} T & \longrightarrow & (\operatorname{Spec} \pi_0 R)/G. \end{array}$$

Since $\operatorname{Spec} T'$ is étale over $\operatorname{Spec} \pi_0 R$, we have defined an even periodic, Landweber-exact E_∞-ring $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} T')$, which is étale over R. Since the group G acts on R, it follows (Corollary 2.13) that it acts on $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} T')$ in a compatible manner; we set

$$\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} T) \stackrel{\mathrm{def}}{=} \mathcal{O}^{\mathrm{top}}(\operatorname{Spec} T')^{hG}.$$

Since G acts freely on $\operatorname{Spec} T'$ (that is, the map $\operatorname{Spec} T' \to \operatorname{Spec} T$ is a G-torsor), it follows that there is no higher cohomology for the G-action on $\pi_* \mathcal{O}^{\mathrm{top}}(\operatorname{Spec} T')$. This is a consequence of the fact that $T \to T'$ is a G-Galois extension of commutative rings: that is, after the faithfully flat base-change $T \to T'$, we have an equivalence of T'-modules with G-action, $T' \otimes_T T' \simeq \prod_G T'$. Moreover, coinduced representations of G have no higher cohomology. The homotopy fixed-point spectral sequence thus degenerates and we get

$$\pi_* \mathcal{O}^{\mathrm{top}}(\operatorname{Spec} T) \simeq (\pi_*(\operatorname{Spec} T'))^G,$$

which implies that $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} T')$ is the desired even periodic, Landweber exact E_∞-ring.

This procedure thus gives, for any étale map $\operatorname{Spec} T \to (\operatorname{Spec} \pi_0 R)/G$, an even periodic, Landweber-exact E_∞-ring $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} T)$, and this is the structure sheaf for the even periodic refinement of $(\operatorname{Spec} \pi_0 R)/G$ desired. □

This result has a converse. If $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$ is an even periodic refinement of $X \to M_{FG}$, and if X is the quotient $(\operatorname{Spec} R)/G$ for a finite group G acting on an affine scheme $\operatorname{Spec} R$, then $\mathfrak{X}$ arises in this way from the G-action on the E_∞-ring $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R)$.

Let R be an E_∞-ring with a G-action as above, let $\mathfrak{X}$ be the associated even periodic refinement of $\operatorname{Spec} \pi_0 R$ and let $\mathfrak{Y}$ be the associated even periodic refinement of $(\operatorname{Spec} \pi_0 R)/G$. The next result describes quasi-coherent sheaves on $\mathfrak{Y}$ in terms of $\mathfrak{X}$. Note first that since G acts on R, it acts on the stable ∞-category $\operatorname{Mod}(R)$, in symmetric monoidal ∞-categories.

Proposition 2.16. *One has equivalences of symmetric monoidal ∞-categories*

$$\operatorname{QCoh}(\mathfrak{Y}) \simeq \operatorname{QCoh}(\mathfrak{X})^{hG} \simeq \operatorname{Mod}(R)^{hG}.$$

Proof. This is a formal descent-theoretic statement: in an appropriate ∞-category of derived stacks, $\mathfrak{Y}$ is the homotopy quotient $(\mathfrak{X})_{hG}$ and the construction QCoh is defined so as to send homotopy colimits to homotopy limits (of stable ∞-categories). Let us prove it directly in our setup.

We have an étale cover $\operatorname{Spec} \pi_0 R \to (\operatorname{Spec} \pi_0 R)/G$, and therefore

$$\begin{aligned} \operatorname{QCoh}(\mathfrak{Y}) \simeq \operatorname{holim} \big(& \operatorname{Mod}(\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} \pi_0 R)) \\ & \rightrightarrows \operatorname{Mod}(\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} \pi_0 R \times_{(\operatorname{Spec} \pi_0 R)/G} \operatorname{Spec} \pi_0 R)) \substack{\to \\ \to \\ \to} \cdots \big). \end{aligned}$$

Since $\operatorname{Spec} \pi_0 R \to (\operatorname{Spec} \pi_0 R)/G$ is a G-torsor, the iterated fiber products that appear in the above construction are precisely

$$G \times G \times \cdots \times \operatorname{Spec} \pi_0 R,$$

the rings in question are $\prod_{G^n} R$, and the above cosimplicial diagram is the usual cobar construction for homotopy fixed points: the construction $R \mapsto \operatorname{Mod}(R)$ sends products in R to products of ∞-categories. □

In other words, to give a quasi-coherent sheaf on $\mathfrak{Y}$ is equivalent to giving an R-module M, together with a G-action on M intertwining the G-action on R.

Example 2.17. The most basic example of all this comes from the $\mathbb{Z}/2$-action on KU described above. By Proposition 2.15, it endows the stack $B\mathbb{Z}/2$ of one-dimensional tori with an even periodic refinement. The global sections of the structure sheaf give $KU^{h\mathbb{Z}/2} \simeq KO$.

The ∞-category of quasi-coherent sheaves on the derived version of $B\mathbb{Z}/2$ is precisely $\mathrm{Mod}(KU)^{h\mathbb{Z}/2}$, where the $\mathbb{Z}/2$-action is by complex conjugation on KU-modules: it takes a KU-module M and 'twists' the KU-action by Ψ^{-1}. We will show later (as is well known) that this is equivalent to the ∞-category $\mathrm{Mod}(KO)$.

Example 2.18 (Classical Galois descent). In classical commutative algebra, recall that if $R \to R'$ is a morphism of rings which is a G-torsor for a finite group G (or rather, $\operatorname{Spec} R' \to \operatorname{Spec} R$ is a G-torsor), then we have an equivalence between the category of (ordinary) R-modules and the homotopy fixed points of the G-action on the category of R'-modules.

Namely, given an R-module M, we can form the tensor product $M \otimes_R R'$, which acquires a G-action with G acting on the second factor. Conversely, given an R'-module M' with a compatible G-action, the G-fixed points M'^G define an R-module, which is the inverse of the previous functor.

This equivalence persists at the level of derived ∞-categories, with homotopy fixed points replacing fixed points.

2.6. *Coarse moduli spaces*

It is crucial for our purposes to give criteria for when an algebraic (Artin) stack has finite cohomological dimension. In a quasi-compact and separated setting, every scheme and even every algebraic space has finite cohomological dimension. The best approximation of an algebraic stack by an algebraic space is the coarse moduli space. Later in this subsection, we will define the notion of tameness, which allows us to conclude that an algebraic stack already has finite cohomological dimension, by relating it to its coarse moduli space. Throughout the subsection, we choose implicitly a base scheme S.

Recall first that a *coarse moduli space* of an algebraic stack X is an algebraic space Y together with a map $f\colon X \to Y$ which

(1) is initial among all maps from X to algebraic spaces and
(2) induces a bijection $\pi_0 X(\operatorname{Spec} k) \to \pi_0 Y(\operatorname{Spec} k)$ for every algebraically closed field k, where π_0 denotes the set of isomorphism classes.

The following was first proved by Keel and Mori [**29**] and reformulated by Conrad in [**11**].

Theorem 2.19. *Let X be an algebraic stack locally of finite presentation over a base scheme S with finite inertia stack $\pi\colon X \times_{X \times_S X} X \to X$. Then X has a coarse moduli space $f\colon X \to Y$. The algebraic space Y is separated if X is separated, and the map f is proper and quasi-finite. Moreover, the formation of coarse moduli spaces commutes with flat base change.*

For example, every locally noetherian, separated Deligne–Mumford stack has finite inertia. Indeed, by [**31**, Lemme 4.2], we know that the diagonal $X \to X \times_S X$ of every Deligne–Mumford stack is quasi-finite. Since X is separated, the diagonal is proper; hence, the diagonal is finite.

In the following, we will always assume implicitly that our algebraic stacks are locally of finite presentation over a base scheme S and have finite inertia.

A very convenient class of algebraic stacks is given by the so-called *tame stacks* as studied in [**1**].

Definition 2.20. An algebraic stack X is called *tame* if the map $f\colon X \to Y$ to its coarse moduli space induces an exact functor $f_*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$.

The question remains of how to decide whether a stack is tame. This was completely answered in [**1**]. We begin with a few preliminary definitions and propositions. In the following, we ignore the notation from the introduction and, for an ordinary stack X, we write $\mathrm{QCoh}(X)$ for the ordinary (abelian) category of quasi-coherent sheaves on X. If G is a group scheme over X, then we let $\mathrm{QCoh}^G(X)$ be the ordinary category of G-representations in $\mathrm{QCoh}(X)$.

Definition 2.21. A group scheme $G \to S$ is *linearly reductive* if the functor $\mathrm{QCoh}^G(S) \to \mathrm{QCoh}(S)$, $F \mapsto F^G$, sending an equivariant sheaf to its fixed points, is exact. Note that this is equivalent to the tameness of the stack quotient S/G as $\mathrm{QCoh}^G(S) \simeq \mathrm{QCoh}(S/G)$.

Recall that the datum of an affine group scheme G over $\mathrm{Spec}\ R$ is equivalent to that of a commutative Hopf algebra Γ over R. The group scheme G is commutative if and only if Γ is cocommutative. For example, given a (discrete) abelian group G', we can form the group algebra $R[G']$. The corresponding group scheme is called *diagonalizable*. Examples include μ_n (with $G' = \mathbb{Z}/n\mathbb{Z}$).

Theorem 2.22 ([**1**, Theorem 2.19]). *Let $G \to S$ be a finite, flat group scheme of finite presentation. Then G is linearly reductive if and only if fpqc-locally, we can write G as a semidirect product $\Delta \rtimes H$, where Δ is diagonalizable and H is a constant of an order prime to all residue characteristics of S.*

Theorem 2.23 ([**1**, Theorem 3.2]). *An algebraic stack X is tame if and only if for every geometric point $\mathrm{Spec}\ k \to X$ and object $\xi \in X(\mathrm{Spec}\ k)$, the automorphism group scheme $\underline{\mathrm{Aut}}_k(\xi)$ is linearly reductive over $\mathrm{Spec}\ k$. Here, $\underline{\mathrm{Aut}}_k(\xi)$ is defined to be the scheme equivalent to the pullback of the inertia stack $X \times_{X\times_S X} X$ along the map $\mathrm{Spec}\ k \to X$ classifying ξ and the group structure is induced by the diagonal $X \to X \times_S X$.*

Remark 2.24. Let $q : Y \to \mathrm{Spec}\ k$ be a k-scheme and $q^*\xi \in X(Y)$ be the pullback of ξ. Then $\underline{\mathrm{Aut}}_k(\xi)(Y)$ is isomorphic to the automorphism group of $q^*\xi$ in the groupoid $X(Y)$.

Indeed, a morphism $Y \to \mathrm{Spec}\, k \times_{X\times_S X} X$ over $\mathrm{Spec}\, k$ consists of the choice of $\eta \in X(Y)$ together with two isomorphisms $f : q^*\xi \to \eta$ and $g : q^*\xi \to \eta$ in $X(Y)$ agreeing in $S(Y)$. A morphism between (η, f, g) and (η', f', g') consists of $i : \eta \to \eta'$ such that $f' = if$ and $g' = ig$. Thus, every morphism $Y \to \mathrm{Spec}\, k \times_{X\times_S X} X$ over $\mathrm{Spec}\, k$ is isomorphic to a unique $(q^*\xi, f : q^*\xi \to q^*\xi, \mathrm{id} : q^*\xi \to q^*\xi)$.

Suppose that X is Deligne–Mumford, so that the automorphism group schemes of geometric points are étale and thus constant. Then X is tame if and only if the orders of the automorphism groups (at geometric points) are invertible on X.

Example 2.25. Let X be a Deligne–Mumford stack over a field of characteristic zero. Then X is tame.

Example 2.26. If U is a scheme over a base scheme S and G is a finite group acting on U such that $|G|$ is invertible on S, then the quotient stack U/G is tame. For example, one can take the moduli stack of elliptic curves over $\mathbb{Z}[\frac{1}{6}]$. Indeed, $M_{\mathrm{ell}}[\frac{1}{6}] \simeq M_{\mathrm{ell}}(3)[\frac{1}{6}]/\mathrm{GL}_2(\mathbb{F}_3)$, where $M_{\mathrm{ell}}(3)$ is the moduli *scheme* of elliptic curves with level 3 structures.

Recall for the next proposition that a stack X is called *quasi-compact* if for every collection $\{f_i : U_i \to X\}_{i\in I}$ of open immersions such that $\coprod_{i\in I} f_i : U_i \to X$ is surjective, there exists a finite subset $J \subset I$ such that $\coprod_{j\in J} f_j : U_j \to X$ is still surjective. We will see in the next subsection that X is quasi-compact if and only if its coarse moduli space is.

PROPOSITION 2.27. *Let X be a tame algebraic stack that is quasi-compact and separated. Then there is a natural number n such that $H^i(X;\mathcal{F}) = 0$ for all $i > n$ and all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$.*

Proof. In the case of X an algebraic space, this is [**51**, 072B].

In the general case, denote by $f\colon X \to Y$ the map to the coarse moduli space and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then we have a Leray spectral sequence

$$H^p(Y; R^q f_*(\mathcal{F})) \Rightarrow H^{p+q}(X;\mathcal{F})$$

with $R^q f_*(\mathcal{F}) = 0$ for $q > 0$. The result follows as Y is a quasi-compact and separated algebraic space and thus has finite cohomological dimension. □

Now we want to introduce a relative version of tameness. For this, we will no longer assume our stacks to have finite inertia.

DEFINITION 2.28. Let $f\colon X \to Y$ be a morphism of stacks. We call f *tame* if for every geometric point $\xi\colon \operatorname{Spec} k \to X$, the kernel of the induced map $\underline{\mathrm{Aut}}_k(\xi) \to \underline{\mathrm{Aut}}_k(f(\xi))$ is finite and linearly reductive over $\operatorname{Spec} k$.

If X is Deligne–Mumford, then this is equivalent to assuming that for every $\xi \in X(k)$ for an algebraically closed field k, the kernel of the map $\mathrm{Aut}_{X(k)}(\xi) \to \mathrm{Aut}_{Y(k)}(f(\xi))$ has order coprime to the characteristic of k. Indeed, the automorphism group scheme of ξ is a discrete group (scheme).

Recall for the next proposition that a map $X \to Y$ of stacks is *quasi-compact* if for every map $\operatorname{Spec} A \to Y$, the stack $X \times_Y \operatorname{Spec} A$ is quasi-compact.

PROPOSITION 2.29. *Let $f\colon X \to Y$ be a quasi-compact, separated and tame morphism of stacks, where we assume X to be algebraic, but for Y only that the diagonal $Y \to Y \times_S Y$ is representable by an algebraic stack. Then* $\operatorname{Spec} A \times_Y X$ *has finite cohomological dimension for every map $q\colon \operatorname{Spec} A \to Y$.*

We do not assume that Y is an Artin stack because the moduli stack of formal groups M_{FG} is not an Artin stack. But it has still representable (even affine) diagonal.

Proof. Let $q\colon \operatorname{Spec} A \to Y$ be a morphism. We have to show that $Z = X \times_Y \operatorname{Spec} A$ has finite cohomological dimension. First note that $X \times_Y \operatorname{Spec} A \cong (X \times_S \operatorname{Spec} A) \times_{Y\times_S Y} Y$ is an algebraic stack. Let now $\xi\colon \operatorname{Spec} R \to Z$ be an R-valued point. This corresponds to a point $\xi_X\colon \operatorname{Spec} R \to X$, a point $\xi_A\colon \operatorname{Spec} R \to \operatorname{Spec} A$, and an isomorphism $\phi\colon f(\xi_X) \to q(\xi_A)$. An automorphism of this is an automorphism ψ of ξ_X such that $\phi \circ f(\psi) = \phi$. This is equivalent to ψ being in the kernel of $\underline{\mathrm{Aut}}(\xi)(R) \to \underline{\mathrm{Aut}}(f(\xi))(R)$. In particular, it follows that Z has quasi-finite inertia and all automorphism group schemes of geometric points of Z are kernels of $\underline{\mathrm{Aut}}_k(\xi) \to \underline{\mathrm{Aut}}_k(f(\xi))$ for geometric points ξ of X. Since Z is separated, Z has actually finite inertia. Thus, Z is tame. By Proposition 2.27, it follows that Z has also finite cohomological dimension. □

2.7. *The Zariski topology*

In this subsection, we will define and investigate the Zariski site of an algebraic stack. This is important because certain properties only allow Zariski-descent and not étale or fpqc-descent as we will see in Subsection 3.3.

DEFINITION 2.30. Let X be an algebraic stack. The *Zariski site* of X is given by all open immersions into X and open immersions between them, where a covering is a jointly surjective map. Recall here that a map $Y \to X$ is an open immersion if it is representable and for every map $Z \to X$ from a scheme, the map $Y \times_X Z \to Z$ is an open immersion.

We define an algebraic stack X to be *quasi-compact* if every Zariski cover of X has a finite subcover.

The Zariski site of an algebraic stack X is actually always equivalent to the site of open subsets of the *underlying space* $|X|$ of X. We will first define the space $|X|$, following [**31**] and then prove this equivalence.

The points of $|X|$ are equivalence classes of objects in the groupoids $X(\operatorname{Spec} k)$ for k a field. Two such objects $x_1 \in X(\operatorname{Spec} k_1)$ and $x_2 \in X(\operatorname{Spec} k_2)$ are equivalent if there is a common field extension K of k_1 and k_2 such that $(x_1)_K$ and $(x_2)_K$ are isomorphic in $X(\operatorname{Spec} K)$. The open subsets of $|X|$ are those of the form $|U|$ for an open substack U of X. Recall that substack means in particular that $U(Z)$ is a full subcategory of $X(Z)$ for every scheme Z. The construction $X \mapsto |X|$ is functorial (see [**31**, Section 5] for details).

If X satisfies the conditions of Theorem 2.19, then the map $X \to Y$ in its coarse moduli space induces a homeomorphism $|X| \to |Y|$. Indeed, it is clearly a continuous bijection and it is also closed as $X \to Y$ is proper.

LEMMA 2.31. *Let X be an algebraic stack and $U \to X$ be a presentation, that is, a smooth surjective map from an algebraic space. Then $|X|$ is the coequalizer of the maps* $\mathrm{pr}_1, \mathrm{pr}_2 \colon |U| \times_{|X|} |U| \to |U|$.

Proof. By [**31**, Remarque 5.3], the map $|U| \to |X|$ is surjective. By [**21**, Corollary 14.34; **31**, Proposition 5.6], this map is also open. By the definition of the pullback, the images of two points

$$y_1, y_2 \in U(\operatorname{Spec} k)$$

are isomorphic in $X(\operatorname{Spec} k)$ if and only if there are isomorphic

$$z_1, z_2 \in (U \times_X U)(\operatorname{Spec} k)$$

with $\mathrm{pr}_1(z_1) = y_1$ and $\mathrm{pr}_2(z_2) = y_2$. Thus, the images of y_1 and y_2 in $|X|$ are equal if and only if there is a point z in $|U \times_X U|$ with $|\mathrm{pr}_1|(z) = y_1$ and $|\mathrm{pr}_2|(z) = y_2$. Thus, $|X|$ is the coequalizer of the two maps

$$|\mathrm{pr}_1|, |\mathrm{pr}_2| \colon |U \times_X U| \longrightarrow |U|.$$

As the map

$$|U \times_X U| \longrightarrow |U| \times_{|X|} |U|$$

is surjective by [**31**, Proposition 5.4(iv)], the result follows. □

Proposition 2.32. *The functor $U \mapsto |U|$ defines an order-preserving bijection between open substacks of X and open subsets of $|X|$.*

Proof. Let $W \subset |X|$ be an open subset. Choose a presentation $f\colon Y \to X$ with Y an algebraic space and f smooth and surjective. The preimage $|f|^{-1}(W)$ is open in $|Y|$. By [**51**, 03BZ], there is a unique open algebraic subspace V of $|Y|$ with $|V| = |f|^{-1}W$. If

$$\mathrm{pr}_1, \mathrm{pr}_2\colon Y \times_X Y \longrightarrow Y$$

denote the two projections, we get likewise an open algebraic subspace V' of $Y \times_X Y$ corresponding to

$$(|\,\mathrm{pr}_1\,| \circ |f|)^{-1}(W) = (|\,\mathrm{pr}_2\,| \circ |f|)^{-1}(W).$$

By stackifying the groupoid defined by

$$(\mathrm{pr}_1)|_{V'}, (\mathrm{pr}_2)|_{V'}\colon V' \longrightarrow V,$$

we get an open substack U of X. By the last lemma, we have $|U| = W$.

On the other hand, if U is an open substack of X, then the open substacks associated with $|f|^{-1}|U|$ and $|f \circ \mathrm{pr}_1\,|^{-1}|U|$ agree with $U \times_X Y$ and $U \times_X Y \times_X Y$ by [**51**, 03BZ]. As U equals the substack of X associated with

$$\mathrm{pr}_1, \mathrm{pr}_2\colon U \times_X Y \times_X Y \longrightarrow U \times_X Y,$$

the proposition follows. □

Corollary 2.33. *The Zariski topology on an algebraic stack X is equivalent to the site of open subsets of $|X|$. In particular, X is quasi-compact if and only if $|X|$ is.*

Proof. Note first that every open immersion into X is equivalent over X to an open substack by considering its image. Thus, we only have to show that $\{U_i \to X\}$ is a covering by open substacks if and only if $\{|U_i| \to |X|\}$ is an open covering by open subsets. This follows directly from the fact that one can test the surjectivity of a map between algebraic stacks on the underlying topological spaces by [**31**, Proposition 5.4(ii)]. □

As a last point, we want to discuss non-vanishing loci.

Proposition 2.34. *Let X be an algebraic stack and $\mathcal{L}$ be a line bundle on X. Let $f \in \Gamma(X, \mathcal{L})$. Then we have the following.*

(a) *Let $x_1\colon \mathrm{Spec}\, k_1 \to X$ and $x_2\colon \mathrm{Spec}\, k_2 \to X$ be two morphisms for k_1 and k_2 fields. If x_1 and x_2 define the same point in $|X|$, then $(x_1)^*f = 0$ if and only if $(x_2)^*f = 0$.*

(b) *The locus of points in $|X|$ where f does not vanish is open.*

Proof. As field extensions are always faithfully flat, part (a) follows easily.

For part (b), consider a presentation $q\colon Y \to X$ such that $q^*\mathcal{L}$ is trivial and Y is a scheme. As discussed in the proof of Lemma 2.31, the map $|q|\colon |Y| \to |X|$ is open and surjective. Furthermore, $x^*(q^*f) = 0$ if and only if $(q \circ x)^*f = 0$ for $x\colon \mathrm{Spec}\, k \to Y$ a point. Thus, we can assume that $\mathcal{L} = \mathcal{O}_X$ and that X is a scheme. The result is well known in this case. □

Definition 2.35. Let X be an algebraic stack and $\mathcal{L}$ be a line bundle on X. Let $f \in \Gamma(X, \mathcal{L})$. Then we define the non-vanishing locus $D(f)$ of f to be the open substack of X corresponding to the non-vanishing locus of f on $|X|$ by Proposition 2.32.

The following property will later be freely used.

Proposition 2.36. *Let $q\colon Y \to X$ be a map of algebraic stacks. Let furthermore, $\mathcal{L}$ be a line bundle on X and $f \in \Gamma(X, \mathcal{L})$. Then there is a natural equivalence $D(q^*f) \simeq D(f) \times_X Y$ where $q^*f \in \Gamma(Y, q^*\mathcal{L})$ is the pullback.*

Proof. The map $\mathrm{pr}_2\colon D(f) \times_X Y \to Y$ is an open immersion. The image of $|\,\mathrm{pr}_2\,|$ in $|Y|$ agrees with those points y in $|Y|$ such that $|q|(y) \in |D(f)|$. This agrees with $|D(q^*f)|$. Thus, $D(q^*f)$ agrees with the image of $D(f) \times_X Y \to Y$ in Y. □

3. *Abstract affineness results*

The aim of this section is to give criteria when the global sections functor establishes an equivalence between quasi-coherent sheaves and modules over the ring of global sections for a derived stack. We call such derived stacks *0-affine* (following [**18**]).

Definition 3.1. A derived stack $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$ is called *0-affine* if the global sections functor

$$\Gamma\colon \mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}))$$

is an equivalence of symmetric monoidal ∞-categories.

In particular, Γ is a symmetric monoidal functor (and not only a lax symmetric monoidal functor) if $(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$ is 0-affine. In the next subsection, we will show that a derived stack $\mathfrak{X}$ is 0-affine if and only if

(1) the functor of taking global sections Γ commutes with homotopy colimits in $\mathrm{QCoh}(\mathfrak{X})$ and
(2) the functor Γ is conservative, that is, whenever $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ is such that $\Gamma(\mathfrak{X}, \mathcal{F})$ is contractible, then $\mathcal{F}$ is itself contractible.

In the following two subsections, we show that 0-affineness descends under certain (topological) finiteness conditions fpqc-locally and under certain ampleness conditions Zariski-locally. In the last subsection, we will show that 0-affineness ascends under affine morphisms and open immersions.

The theorems in this section are abstract in the sense that they are not special to chromatic homotopy theory. The main known techniques to guarantee the finiteness assumptions of these abstract theorems, though, will use powerful nilpotence technology in chromatic homotopy theory. This will be the topic of the following section.

One of the main issues can already be illustrated by the following example: If 2 is not inverted, then the functor

$$E \mapsto E^{h\mathbb{Z}/2}, \quad \mathrm{Fun}(B\mathbb{Z}/2, \mathrm{Sp}) \longrightarrow \mathrm{Sp},$$

from spectra with a $\mathbb{Z}/2$-action to spectra, fails to commute with homotopy colimits, or equivalently fails to send wedges to wedges (Example 3.2). The homotopy groups of $E^{h\mathbb{Z}/2}$ are the abutment of a homotopy fixed-point spectral sequence each of whose terms (the group cohomology of π_*E) sends wedges in E to direct sums. However, the potential infiniteness of the spectral sequence (in particular, the infinitude of the filtration) does not allow us to conclude that the abutment sends wedges to wedges. In chromatic homotopy theory, however, it is possible to show that the analogous filtrations are finite under certain conditions, as we will see in the next section.

EXAMPLE 3.2. We provide a simple illustration of the fact that the functor

$$E \longrightarrow E^{h\mathbb{Z}/2}, \quad \mathrm{Fun}(B\mathbb{Z}/2, \mathrm{Sp}) \longrightarrow \mathrm{Sp}$$

fails to commute with wedges. Consider the spectrum $X = \bigvee_{n \in \mathbb{Z}} H\mathbb{Z}/2[n]$, and give it the trivial $\mathbb{Z}/2$-action. Then we claim that

$$X^{h\mathbb{Z}/2} = \mathrm{F}(B\mathbb{Z}/2, X) \not\simeq \bigvee_{n \in \mathbb{Z}} \mathrm{F}(B\mathbb{Z}/2, H\mathbb{Z}/2[n]).$$

In fact, this follows from the fact that $\pi_* \mathrm{F}(B\mathbb{Z}/2, X)$ is an uncountable abelian group. We can write

$$\mathrm{F}(B\mathbb{Z}/2, X) = \mathrm{F}(\mathbb{RP}^\infty, X) \simeq \varprojlim_m \mathrm{F}(\mathbb{RP}^m, X),$$

and $\pi_* \mathrm{F}(\mathbb{RP}^m, X) \simeq H^*(\mathbb{RP}^m; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} \pi_*(X)$. As $m \to \infty$, the Milnor exact sequence shows that $\pi_* \mathrm{F}(B\mathbb{Z}/2, X) \simeq \varprojlim_m H^*(\mathbb{RP}^m; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} \pi_*(X)$ is actually uncountable. If we regard X as a ring spectrum such that $\pi_*(X) \simeq \mathbb{F}_2[u^{\pm 1}]$ with $|u| = 1$, then $\pi_*(X^{B\mathbb{Z}/2}) \simeq \mathbb{F}_2[u^{\pm 1}][\![v]\!]$ with $|v| = 0$, while $\pi_*(\bigvee_{n \in \mathbb{Z}} \mathrm{F}(B\mathbb{Z}/2, H\mathbb{Z}/2[n]))$ gives only the polynomial subring $\mathbb{F}_2[u^{\pm 1}][v]$.

3.1. *Schwede–Shipley theory*

Let $\mathcal{A}$ be an abelian category with all colimits. If $\mathcal{A} = \mathrm{Mod}(R)$ is the category of (discrete) modules over a (not necessarily commutative) ring R, then $\mathcal{A}$ has a *compact*, *projective generator*: that is, R itself. More precisely, the functor $\mathrm{Hom}_{\mathcal{A}}(R, \cdot) \colon \mathcal{A} \to \mathbf{Ab}$ (which assigns to a module its underlying abelian group) commutes with all colimits, and is conservative.

It is a basic principle that module categories are characterized precisely by this property: that is, an abelian category is equivalent to a category of modules precisely when it has a compact, projective generator. This point of view explains the classical Morita theorem that describes equivalences between categories of modules: they arise from compact, projective generators.

In the derived setting, the objects of study are not abelian categories, but presentable, stable ∞-categories, and the question one asks is when such an ∞-category is the ∞-category of modules over an A_∞-ring. An answer in the language of stable model categories was given by Schwede and Shipley in [48]; a reformulation of the statement in terms of ∞-categories is in [40, Theorem 8.1.2.1].

THEOREM 3.3 (Schwede–Shipley and Lurie). *A presentable, stable ∞-category $\mathcal{C}$ is equivalent to the ∞-category of modules over an A_∞-ring if and only if it has a compact generator $X \in \mathcal{C}$: that is, X is such that $\mathrm{Hom}_{\mathcal{C}}(X, \cdot) \colon \mathcal{C} \to \mathrm{Sp}$ commutes with filtered homotopy colimits and sends non-zero objects to non-contractible spectra.*

EXAMPLE 3.4 (Beĭlinson [10]). The derived category of quasi-coherent sheaves on projective space $\mathbb{P}^n$ is equivalent to the derived category of modules over the (discrete) ring $\mathrm{End}_{\mathrm{QCoh}(\mathbb{P}^n)}(\mathcal{O}_{\mathbb{P}^n} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(n))$. Namely, Beĭlinson shows that $\mathcal{O}_{\mathbb{P}^n} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(n)$ is a compact generator for the derived category of coherent sheaves on $\mathbb{P}^n$.

We will also need a version in the symmetric monoidal case. When R is an E_∞-ring, the ∞-category $\mathrm{Mod}(R)$ is symmetric monoidal, and it has the property that the unit object (that is, R itself) is a compact generator. This is essentially the distinguishing feature of such module categories according to the next result.

THEOREM 3.5 ([40, Proposition 8.1.2.7]). *Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a presentable stable, symmetric monoidal ∞-category where the tensor product preserves homotopy colimits. The endomorphism ring $R = \mathrm{End}(\mathbf{1})$ has a canonical structure of an E_∞-ring. If the unit object $\mathbf{1} \in \mathcal{C}$ is a* *compact generator, one has a symmetric monoidal equivalence*

$$\mathcal{C} \simeq \mathrm{Mod}(R), \quad X \longmapsto \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X),$$

between $\mathcal{C}$ *and the category of* R*-modules.*

Example 3.6. We asserted in Example 1.2 that the derived category of quasi-coherent sheaves on the stack $B\mathbb{G}_a$ (over the base field $\mathbb{Q}$), or equivalently the derived category of $\mathbb{G}_a$-representations, was equivalent to $\mathrm{Mod}(\mathbb{Q}[x_{-1}])$ via an adjunction of symmetric monoidal ∞-categories. This is equivalent to the assertion that the structure sheaf itself, which corresponds to the trivial one-dimensional representation of $\mathbb{G}_a$, is a compact generator, and this in turn is closely related to the unipotence of $\mathbb{G}_a$.

Our strategy will be to apply the Schwede–Shipley theorem to the ∞-category of quasi-coherent sheaves on a derived stack. More precisely, we will use the following corollary.

Corollary 3.7. *A derived stack* $\mathfrak{X} = (X, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$ *is* 0*-affine if and only if the global sections functor*

$$\Gamma\colon \mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}))$$

commutes with homotopy colimits and is conservative. Here, conservative means that for $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$ *a quasi-coherent* $\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$*-module,* $\Gamma(\mathcal{F}) = 0$ *already implies* $\mathcal{F} = 0$.

Proof. The global sections functor Γ is corepresented by $\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$. Thus, Γ commutes with filtered homotopy colimits and is conservative if and only if $\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$ is a compact generator of $\mathrm{QCoh}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$. By Theorem 3.5, the result follows. Note here that if Γ is an equivalence, then it commutes automatically with all homotopy colimits. □

3.2. *fpqc-descent for* 0*-affineness*

In this section, we describe a basic technique for showing that certain homotopy limits (given by global sections functors) commute with homotopy colimits. The strategy is to first verify that this holds after smashing with something that generates the original category as a thick tensor-ideal; then it is possible to apply 'descent'. We note that the idea of descent via thick tensor-ideals has been explored further in [**7**, **42**].

Let us first recall the definition of a thick tensor-ideal.

Definition 3.8. Given an E_∞-ring R, a *thick tensor-ideal* of $\mathrm{Mod}(R)$ is a full subcategory $\mathcal{C} \subset \mathrm{Mod}(R)$ containing the zero object, such that:

(1) the fiber and cofiber of every morphism $M \to N$ in $\mathcal{C}$ is in $\mathcal{C}$ again;
(2) if $X \oplus Y$ is in $\mathcal{C}$, then $X \in \mathcal{C}$ and $Y \in \mathcal{C}$;
(3) if $X \in \mathcal{C}$ and $Y \in \mathrm{Mod}(R)$ is arbitrary, then $X \otimes_R Y \in \mathcal{C}$.

We say that an R-module M *generates* $\mathcal{C}$ as a thick tensor-ideal if $\mathcal{C}$ is the smallest thick tensor-ideal of $\mathrm{Mod}(R)$ containing M.

Proposition 3.9. *Let* $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$ *be a derived stack whose underlying stack* X *is a quasi-compact, separated Deligne–Mumford stack. Suppose that we have the following.*

(1) *There is a flat, affine morphism* $q\colon Y \to X$ *from an algebraic stack of finite cohomological dimension.*

(2) *There is a* $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$*-module* M *that generates* $\mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}))$ *as a thick tensor-ideal such that we have an isomorphism*

$$\pi_*(\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M) \cong q_* q^* \pi_*(\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$$

of $\pi_*\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$*-modules.*

Then the global sections functor Γ *from quasi-coherent* $\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$*-modules to* $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$*-modules commutes with homotopy colimits.*

Proof. We start by showing that the functor

$$\mathcal{F} \longmapsto \Gamma(\mathfrak{X}, \mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M), \quad \mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})),$$

commutes with homotopy colimits in $\mathcal{F}$. Since the functor is an exact functor between stable ∞-categories, it suffices to show that it commutes with arbitrary direct sums (that is, wedges).

We will prove that the E^2-term of the DSS

$$H^i(X, \pi_j(\mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M)) \implies \pi_{j-i}\Gamma(\mathfrak{X}, \mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M)$$

is concentrated in finitely many rows. In fact, by the isomorphism

$$\pi_*(\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M) \cong q_* q^* \pi_*(\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$$

we know that $\pi_*(\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M)$ is flat as a $\pi_*\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$-module, since q is flat and affine. Note further that, by the projection formula, the morphism

$$\pi_k\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}} \otimes_{\mathcal{O}_X} q_* q^* \mathcal{O}_X \longrightarrow q_* q^* \pi_k \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$$

is an isomorphism as q is affine. Thus, we have the following isomorphisms of $\mathcal{O}_X$-modules:

$$\begin{aligned}
\pi_j(\mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M) &\cong (\pi_*\mathcal{F} \otimes_{\pi_*\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}} \pi_*(\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M))_j \\
&\cong (\pi_*\mathcal{F} \otimes_{\pi_*\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}} q_* q^* \pi_*(\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}))_j \\
&\cong (\pi_*\mathcal{F} \otimes_{\pi_*\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}} \pi_*\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}} \otimes_{\mathcal{O}_X} q_* q^* \mathcal{O}_X)_j \\
&\cong \pi_j\mathcal{F} \otimes_{\mathcal{O}_X} q_* q^* \mathcal{O}_X.
\end{aligned}$$

As q is affine, the projection formula allows us to rewrite this as

$$q_*(q^*\pi_j\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y) \cong q_* q^* \pi_j \mathcal{F}.$$

Thus, we have a Leray spectral sequence

$$H^l(Y, (R^m q_*) q^* \pi_j \mathcal{F}) \Rightarrow H^{l+m}(X, \pi_j(\mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M)).$$

By (1), the E^2-term of this spectral sequence is concentrated in finitely many columns (bounded by the cohomological dimension of Y) and in the 0-row; hence, we see that $H^i(X, \pi_j(\mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M))$ is zero for large i.

Since the E^2-page of the spectral sequence for $\pi_*\Gamma(\mathfrak{X}, \mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M)$ commutes with direct sums (as X is quasi-compact and separated; see Lemma 3.10), it follows thus that these homotopy groups themselves commute with direct sums in $\mathcal{F}$. Indeed, for a collection $(\mathcal{F}_i)_{i\in I}$ of quasi-coherent $\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$-modules, the natural map

$$\bigoplus_{i\in I} \Gamma(\mathfrak{X}, \mathcal{F}_i) \longrightarrow \Gamma\left(\mathfrak{X}, \bigoplus_{i\in I} \mathcal{F}_i\right)$$

induces an isomorphism on the E^2-terms of the corresponding DSSs, and thus on the E^∞-terms and because of the finiteness of the filtration also on the abutment.

Let us consider now the collection $\mathcal{C}$ of all $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$-modules T such that the functor

$$\mathcal{F} \longmapsto \Gamma(\mathfrak{X}, \mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} T), \quad \mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{Sp},$$

commutes with homotopy colimits. As we have just seen, $M \in \mathcal{C}$. Since the composition of homotopy colimit-preserving functors is homotopy colimit-preserving, it follows that $\mathcal{C}$ is an *ideal*: If $T \in \mathcal{C}$ and T' is any $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$-module, then $T \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} T' \in \mathcal{C}$. Moreover, $\mathcal{C}$ is a stable subcategory of Sp, and $\mathcal{C}$ is closed under retracts. (A retract of a functor that preserves homotopy colimits preserves homotopy colimits.) As M generates $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$-modules as a thick tensor-ideal, we see that $\mathcal{C}$ consists of all of $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$-modules; in particular, $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}) \in \mathcal{C}$ and Γ commutes with homotopy colimits. □

In the last proof, we used the following algebraic lemma stating that cohomology commutes with filtered colimits.

LEMMA 3.10. *Let X be a quasi-compact, separated stack. Then the cohomology group functors $H^i(X, \cdot)$ on the category of quasi-coherent sheaves on X commute with filtered colimits.*

Proof. Choose an affine, flat cover $\mathrm{Spec}\, A \to X$. The iterated fiber products $\mathrm{Spec}\, A \times_X \mathrm{Spec}\, A, \ldots$ are all affine schemes, so the cohomology of $\mathcal{F}$ is the cohomology of the cochain complex associated to the cosimplicial abelian group

$$\mathcal{F}(\mathrm{Spec}\, A) \rightrightarrows \mathcal{F}(\mathrm{Spec}\, A \times_X \mathrm{Spec}\, A) \substack{\rightarrow \\ \rightarrow \\ \rightarrow} \cdots,$$

that is, the Cech construction. But this clearly commutes with filtered colimits in $\mathcal{F}$. □

Lemma 3.10 is analogous to the following fact: the homotopy fixed point functor

$$E \longmapsto E^{h\mathbb{Z}/2}, \quad \mathrm{Fun}(B\mathbb{Z}/2, \mathrm{Sp}) \longrightarrow \mathrm{Sp}$$

does commute with filtered homotopy colimits if we restrict to the subcategory of $\mathrm{Fun}(B\mathbb{Z}/2, \mathrm{Sp})$ whose underlying spectra have *bounded-above* (by some fixed value) homotopy groups.

The arguments for the conservativity of Γ are related but different. We will present an algebraic and a topological analog of the last proposition for this purpose. The latter will turn out to be more powerful, yet is also more subtle regarding its input. But first we state a little lemma.

LEMMA 3.11. *Let $\mathfrak{X} = (X, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$ be a derived stack and assume that*

$$\Gamma\colon \mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}))$$

commutes with homotopy colimits. Then the natural map

$$\Gamma(\mathcal{F}) \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} N \longrightarrow \Gamma(\mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} N)$$

is an equivalence for every $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$-module N and every quasi-coherent $\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$-module $\mathcal{F}$.

Proof. This is by definition true for $N = \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$. As the class of $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$-modules for which it is true is closed under homotopy colimits, it is true for every $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$-module N. □

In particular, we see that the left adjoint to Γ (that is, 'tensoring up') is fully faithful if Γ commutes with homotopy colimits.

Proposition 3.12. *Let $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$ be a derived stack whose underlying stack X is a quasi-compact, separated Deligne–Mumford stack. Suppose that we have the following.*

(1) *There is a faithfully flat, affine morphism $q\colon Y \to X$ from a quasi-affine scheme of cohomological dimension at most 1.*

(2) *There is a $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$-module M such that we have an isomorphism*

$$\pi_*(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})} M) \cong q_* q^* \pi_*(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}).$$

(3) *The global sections functor $\Gamma\colon \mathrm{QCoh}(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}) \to \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}))$ commutes with homotopy colimits.*

Then the global sections functor Γ is conservative.

By Proposition 3.9, the last condition is satisfied if M generates $\mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}))$ as a thick tensor-ideal.

Proof. Let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ and assume $\Gamma(\mathcal{F}) = 0$. We have to show that $\pi_j \mathcal{F} = 0$ for every $j \in \mathbb{Z}$.

By the last lemma, $\Gamma(\mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})} M) \simeq \Gamma(\mathcal{F}) \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})} M = 0$. As in the last proof, we have

$$\pi_j(\mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})} M) \cong q_* q^* \pi_j \mathcal{F}.$$

Thus, the DSS for $\mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})} M$ has E^2-term isomorphic to

$$H^i(X; q_* q^* \pi_j \mathcal{F}) \cong H^i(Y; q^* \pi_j \mathcal{F}).$$

As Y has cohomological dimension at most 1, this spectral sequence degenerates at E^2. Since it converges to 0, it follows that $H^0(Y; q^* \pi_j \mathcal{F}) = 0$. As Y is quasi-affine, this implies $q^* \pi_j \mathcal{F} = 0$ by [**21**, Proposition 13.80] and thus $\pi_j \mathcal{F} = 0$ as q is faithfully flat. □

For the next proposition, we need the following definition.

Definition 3.13. We call a morphism $f\colon \mathfrak{Y} \to \mathfrak{X}$ of derived stacks *quasi-compact* or *separated* if the underlying map of classical algebraic stacks is. We call it (*faithfully*) *flat* if the map $f_0\colon Y \to X$ of the underlying stacks is and the map $f^* \pi_k \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}} \to \pi_k \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}}$ is an isomorphism for every $k \in \mathbb{Z}$.

Lemma 3.14. *Let $f\colon \mathfrak{Y} \to \mathfrak{X}$ be a flat map of derived stacks. Then*

$$\pi_* f^* \mathcal{F} \cong f^* \pi_* \mathcal{F}$$

for every quasi-coherent $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$-module $\mathcal{F}$.

Proof. Recall that $f^* \mathcal{F}$ is defined to be $f^{-1} \mathcal{F} \otimes_{f^{-1} \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}} \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}}$. As f is flat, $\pi_0 \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}} \cong \mathcal{O}_Y$ is a flat module over $\pi_0 f^{-1} \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}} \cong f^{-1} \mathcal{O}_X$. By the Künneth spectral sequence, it follows that

$$\pi_*(f^{-1} \mathcal{F} \otimes_{f^{-1} \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}} \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}}) \cong f^{-1}(\pi_* \mathcal{F}) \otimes_{f^{-1} \mathcal{O}_X} \mathcal{O}_Y = f^* \pi_* \mathcal{F}.$$

□

Lemma 3.15. *Let $f\colon \mathfrak{Y} \to \mathfrak{X}$ be a faithfully flat map of derived stacks. Then the functor*

$$f^*\colon \mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{QCoh}(\mathfrak{Y})$$

is faithful.

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$-module such that $f^*\mathcal{F}$ is equivalent to the 0-object. We need to show that $\pi_*\mathcal{F} = 0$. By the last lemma, we know that $f^*\pi_*\mathcal{F} \cong \pi_* f^*\mathcal{F} = 0$. Since f is faithfully flat, the result follows. □

PROPOSITION 3.16. *Let $\mathfrak{X} = (X, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$ be a derived stack whose underlying stack X is quasi-compact, separated, and Deligne–Mumford. Suppose that given a faithfully flat, quasi-compact and separated morphism $q\colon \mathfrak{Y} \to \mathfrak{X}$ from a derived stack $\mathfrak{Y} = (Y, \mathcal{O}_{\mathfrak{Y}}^{\mathrm{top}})$.*
Assume the following.

(1) *The global sections functor $\Gamma\colon \mathrm{QCoh}(\mathfrak{Y}) \to \mathrm{Mod}(\Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}^{\mathrm{top}}))$ is conservative.*
(2) *There is a $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$-module M such that we have an equivalence*

$$\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M \longrightarrow q_* \mathcal{O}_{\mathfrak{Y}}^{\mathrm{top}}$$

of $\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$-modules.
(3) *The global sections functor $\Gamma\colon \mathrm{QCoh}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}) \to \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}))$ for $\mathfrak{X}$ commutes with homotopy colimits.*

Then the global sections functor Γ for $\mathfrak{X}$ is conservative.

Proof. Let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$ and assume $\Gamma(\mathcal{F}) = 0$. We have to show that $\mathcal{F} = 0$ or, equivalently, $q^*\mathcal{F} = 0$ by the last lemma.

By assumption, we have

$$\mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M \simeq \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}} \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M \simeq \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}} q_* \mathcal{O}_{\mathfrak{Y}}^{\mathrm{top}}.$$

By the projection formula (see [**39**, Remark 1.3.14]), this is equivalent to

$$q_*(q^*\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{Y}}^{\mathrm{top}}} \mathcal{O}_{\mathfrak{Y}}^{\mathrm{top}}) \simeq q_* q^* \mathcal{F}.$$

Thus, we get

$$\begin{aligned} \Gamma(q^*\mathcal{F}) &\simeq \Gamma(q_* q^* \mathcal{F}) \\ &\simeq \Gamma(\mathcal{F} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M) \\ &\simeq \Gamma(\mathcal{F}) \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})} M \\ &\simeq 0. \end{aligned}$$

Since the global sections functor on $(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}^{\mathrm{top}})$ is conservative, it follows that $q^*\mathcal{F} = 0$. □

To apply the last proposition, we need as input a good supply of derived stacks with conservative global sections functor. In particular, this will turn out to be the case when the underlying stack X is a quasi-affine scheme. In the next subsection, we will show that this criterion is *Zariski-local* in certain cases.

3.3. *Zariski-descent for* 0*-affineness*

This subsection is concerned with understanding to what extent 0-affineness can be checked Zariski-locally. In this paper, only the quasi-affine case of Corollary 3.25 will be used, but the other criteria are still useful in other situations.

Recall that for 0-affineness it is sufficient that the global sections functor commutes with homotopy colimits and is conservative. The former property can (nearly) always be checked Zariski-locally.

PROPOSITION 3.17. *Let $\mathfrak{X} = (X, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$ be a derived stack with underlying separated and quasi-compact Deligne–Mumford stack X. For $\{U_i \to X\}_{i \in I}$, a finite Zariski covering by open* *substacks, we get induced derived stacks* $\mathfrak{U}_i = (U_i, \mathcal{O}^{\mathrm{top}}_{\mathfrak{U}_i})$. *Assume that the global sections functors*

$$\Gamma\colon \mathrm{QCoh}(\mathfrak{U}_i) \longrightarrow \mathrm{Mod}(\Gamma(\mathfrak{U}_i, \mathcal{O}^{\mathrm{top}}_{\mathfrak{U}_i}))$$

for all $\mathfrak{U}_i$ *commute with homotopy colimits. Then the global sections functor*

$$\Gamma\colon \mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}))$$

for $\mathfrak{X}$ *commutes with homotopy colimits as well.*

The idea is that the global sections over $\mathfrak{X}$ are obtained as a *finite* homotopy limit of the sections over the U_i and their intersections. It is important here that a Zariski cover is used.

Proof. Observe that the pushforward along an open immersion of derived stacks commutes with homotopy colimits by [**37**, Example 2.5.6 and Proposition 2.5.12]. It follows easily that the global sections functor for every open substack of each $\mathfrak{U}_i$ also commutes with homotopy colimits. Observe furthermore, that restriction of a quasi-coherent sheaf to an open substack commutes with arbitrary homotopy colimits. Thus, the functor

$$\mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{Sp}, \quad \mathcal{F} \longmapsto \mathcal{F}(U)$$

commutes with arbitrary homotopy colimits for any substack U of some U_i.

Let $\mathcal{F}$ be an $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$-module. By Proposition A.18 and Remark A.19, the canonical map

$$\Gamma(\mathfrak{X}, \mathcal{F}) \longrightarrow \mathrm{holim}_{\mathcal{P}_I{}^{\mathrm{op}}} \mathcal{F}(\mathfrak{C}^{U,c})$$

is an equivalence. Here, $\mathcal{P}_I$ denotes the (finite) poset of non-empty subsets of I and $\mathfrak{C}^{U,c}(S) = \bigcap_{i\in S} U_i$ for a subset $S \subset I$.

As before, it is enough to show that the global sections functor

$$\Gamma\colon \mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}))$$

commutes with direct sums. So let $(\mathcal{F}_j)_{j\in J}$ be a family of quasi-coherent $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$-modules. Consider the commutative diagram

$$\begin{array}{ccc} \bigoplus_{j\in J} \Gamma(\mathcal{F}_j) & \longrightarrow & \Gamma(\bigoplus_{j\in J} \mathcal{F}_j) \\ \downarrow & & \downarrow \\ \bigoplus_{j\in J} \mathrm{holim}_{\emptyset\neq S\subset I} \mathcal{F}_j(\mathfrak{C}^{U,c}(S)) & \longrightarrow & \mathrm{holim}_{\emptyset\neq S\subset I} \bigoplus_{j\in J} \mathcal{F}_j(\mathfrak{C}^{U,c}(S)). \end{array}$$

As just discussed, the vertical arrows are equivalences. Moreover, the lower horizontal arrow is an equivalence since finite homotopy limits commute with arbitrary homotopy colimits (in a stable ∞-category). Thus, the upper horizontal arrow is an equivalence as well. □

Now we turn to the conservativeness of Γ. This will depend on the notion of an ample line bundle, which in turn, depends on the notion of non-vanishing loci as in Definition 2.35.

DEFINITION 3.18. Let X be a quasi-compact and separated Deligne–Mumford stack with coarse moduli space $f\colon X \to Y$. We call then a line bundle $\mathcal{L}$ on X *ample* if Y is a scheme and the non-vanishing loci $D(x)$ of sections $x \in \Gamma(X, \mathcal{L}^{\otimes k})$ form a basis of the *Zariski* topology of X. This agrees with the usual definition if X is a scheme by [**21**, Proposition 13.47].

Let $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ be a derived stack. Let $\mathcal{L}$ be a locally free $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$-module of rank 1. We say that $\mathcal{L}$ is *ample* if the non-vanishing loci $D(\overline{x})$ of the reductions $\overline{x} \in \Gamma(\pi_k(\mathcal{L}^{\otimes l}))$ of elements $x \in \pi_k\Gamma(\mathfrak{X}, \mathcal{L}^{\otimes l})$ form a basis of the Zariski topology of X.

PROPOSITION 3.19. *Let X be a quasi-compact and separated Deligne–Mumford stack and $\mathcal{L}$ be a line bundle on X. Then the following are equivalent.*

(1) *The line bundle $\mathcal{L}$ is ample.*

(2) *There are finitely many sections $x_i \in \Gamma(X, \mathcal{L}^{\otimes k_i})$ with $k_i \geqslant 1$ such that the $D(x_i)$ have affine coarse moduli space and cover X.*

(3) *The affine non-vanishing loci $D(x)$ of sections $x \in \Gamma(X, \mathcal{L}^{\otimes k})$ form a basis of the Zariski topology.*

Proof. This is true if X is a scheme by [**21**, Propositions 13.47 and 13.49]. As the Zariski topologies of X and its coarse moduli space Y agree, we have only to show in the cases (2) and (3) that the coarse moduli space Y is a scheme. This follows from [**11**, Theorem 3.1]. □

EXAMPLE 3.20. The line bundle $\omega \cong \pi_2\mathcal{O}^{\mathrm{top}}$ on $M_{\overline{\mathrm{ell}}}$ is ample. Indeed, the non-vanishing loci of $c_4 \in \Gamma(\omega^{\otimes 4})$ and $\Delta \in \Gamma(\omega^{\otimes 12})$ have affine coarse moduli space and cover $M_{\overline{\mathrm{ell}}}$. More details about $(M_{\overline{\mathrm{ell}}}, \mathcal{O}^{\mathrm{top}})$ will be given in Section 7.

Let $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ be an even periodic derived stack, $f \in \pi_k\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ and $\overline{f} \in \Gamma(X, \pi_k\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ be its reduction. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$-module. By the theory of [**40**, Section 8.2.4], we can consider the localization

$$\Gamma(\mathfrak{X}, \mathcal{F}) \longrightarrow \Gamma(\mathfrak{X}, \mathcal{F})[1/f].$$

This has the following universal property: let M be a $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$-module such that f operates invertible on π_*M. Then the induced map

$$\mathrm{Map}(\Gamma(\mathfrak{X}, \mathcal{F})[1/f], M) \longrightarrow \mathrm{Map}(\Gamma(\mathfrak{X}, \mathcal{F}), M)$$

is an equivalence.

Now assume that the global sections functor $\Gamma\colon \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}))$ commutes with homotopy colimits. Then the presheaf

$$\mathcal{F}[1/f]\colon U \longmapsto \mathcal{F}(U) \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})} \Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})[1/f] \simeq \mathcal{F}(U)[1/f]$$

is already a sheaf by Lemma 3.11. As $\mathcal{F} \to \mathcal{F}[1/f]$ is an equivalence étale locally on $D(\overline{f})$, we can conclude thus that $\mathcal{F}(D(\overline{f})) \simeq \mathcal{F}(D(\overline{f}))[1/f]$. In particular, there is thus a canonical map $\Gamma(\mathcal{F})[1/f] \to \mathcal{F}(D(\overline{f}))$.

LEMMA 3.21. *Let $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ be an even periodic, quasi-compact, and separated derived stack. Assume that the global sections functor commutes with homotopy colimits. Let $f \in \pi_k\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$-module. Then the canonical map*

$$\Gamma(\mathfrak{X}, \mathcal{F})[1/f] \longrightarrow \mathcal{F}(D(\overline{f}))$$

is an equivalence, where $\overline{f} \in \Gamma(\pi_k\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ is the reduction of f.

In other words, restricting to a basic open affine gives the corresponding localization at the level of sections.

Proof. First assume that $\mathfrak{X}$ is an affine derived scheme with $\pi_2\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ trivial. In particular, we can assume f to be in $\pi_0\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$. By the definition of a derived scheme, we know that $\Gamma(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})[1/f] \simeq \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}(D(\overline{f}))$. Now the result follows by the quasi-coherence of $\mathcal{F}$.

Now consider the general case. Let $p\colon \mathfrak{U} \to \mathfrak{X}$ be an affine étale cover with p affine and such that $\pi_2\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$ is trivial on $\mathfrak{U}$. Define $p_n\colon \mathfrak{U}_n = \mathfrak{U}^{\times_{\mathfrak{X}} n} \to \mathfrak{X}$. We have a commutative diagram

$$\begin{array}{ccc} \Gamma(\mathcal{F})[1/f] & \longrightarrow & \mathcal{F}(D(\overline{f})) \\ \downarrow & & \downarrow \\ \operatorname{holim}(\mathcal{F}(\mathfrak{U}_n)[1/p_n^* f]) & \longrightarrow \operatorname{holim}\mathcal{F}(D(p_n^*\overline{f})) \simeq & \operatorname{holim}\mathcal{F}(\mathfrak{U}_n \times_{\mathfrak{X}} D(\overline{f})). \end{array}$$

The vertical maps are equivalences since $\mathcal{F}[1/f]$ and $\mathcal{F}$ are sheaves. The lower horizontal map is an equivalence by the affine case. Thus, the result follows. □

Proposition 3.22. *Let $\mathfrak{X} = (X, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$ be an even periodic, quasi-compact, separated derived stack. Assume that the global sections functor commutes with homotopy colimits and that $\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$ is ample. Let $\{U_i \to X\}_{i\in I} \to X$ be a Zariski covering and $\mathfrak{U}_i = (U_i, \mathcal{O}^{\mathrm{top}})$ be the induced open derived substacks. Assume that the global sections functor for every $\mathfrak{U}_i$ is conservative. Then the global sections functor*

$$\Gamma : \mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}))$$

for $\mathfrak{X}$ is conservative.

In other words: under the assumptions, the conservativity of the global sections functor is a Zariski-local property.

Proof. By shrinking the U_i, we can assume that $U_i = D(\overline{x_i}) \subset X$, where $\overline{x_i} \in \Gamma(\pi_{k_i}\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$ is the reduction of an element $x_i \in \pi_{k_i}\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$. By Proposition 3.28, this preserves the property that the global sections functor is conservative.

Now let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$-module. By the last lemma, we know that

$$\mathcal{F}(D(\overline{x_i})) \simeq \Gamma(\mathcal{F})[1/x_i] = 0.$$

As the global sections functor is conservative on each $D(\overline{x_i})$, we have $\mathcal{F}|_{D(\overline{x_i})} = 0$ for every $i \in I$. Since $\mathcal{F}$ is a sheaf, it follows that $\mathcal{F} = 0$. □

Next, we want to prove an algebraic criterion for ampleness of the structure sheaf $\mathcal{O}_{\mathfrak{X}}^{\mathrm{top}}$. We need first a simple lemma.

Lemma 3.23. *Let X be a quasi-compact and separated Deligne–Mumford stack, $\mathcal{L}$ be a line bundle on X and $x \in H^0(X; \mathcal{L})$. Then $H^i(D(x); \mathcal{L}^{\otimes *}) \cong H^i(X; \mathcal{L}^{\otimes *})[1/x]$ for $\mathcal{L}^{\otimes *} \cong \bigoplus_{n\in\mathbb{Z}} \mathcal{L}^{\otimes n}$.*

Proof. Let $j\colon D(x) \to X$ be the inclusion of the non-vanishing locus. Define the quasi-coherent graded $\mathcal{O}_X$-module $\mathcal{L}^{\otimes *}[1/x]$ as the colimit over

$$\mathcal{L}^{\otimes *} \xrightarrow{r_x} \mathcal{L}^{\otimes *} \xrightarrow{r_x} \cdots,$$

where r_x denotes multiplication by x. Then the map $\mathcal{L}^{\otimes *} \to j_*j^*\mathcal{L}^{\otimes *}$ factors over $\mathcal{L}^{\otimes *}[1/x]$. The map $\mathcal{L}^{\otimes *}[1/x] \to j_*j^*\mathcal{L}^{\otimes *}$ is an isomorphism as it is on affine schemes with $\mathcal{L}$ trivial. As j is affine, we have an isomorphism

$$H^*(X; j_*j^*\mathcal{L}^{\otimes *}) \cong H^*(D(x); j^*\mathcal{L}^{\otimes *}).$$

It remains to show that cohomology commutes with localization at x on X. This follows from the fact that cohomology commutes with filtered colimits, by Lemma 3.10. □

PROPOSITION 3.24. *Let $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ be an even periodic, quasi-compact and separated derived Deligne–Mumford stack. Assume that $\pi_k\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ is ample for some $k \in \mathbb{Z}$ and that the length of the differentials in the DSS for $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ is bounded. Then $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ itself is ample.*

Proof. Consider some $x \in H^0(X, \pi_k(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}))$ such that the non-vanishing locus $D(x)$ has affine coarse moduli space. We want to show that some power of x is a permanent cycle in the DSS for $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$. Note that k has to be even.

Let $d_{i_1}(x)$ be the first non-zero differential of x. Consider the morphism $\mathfrak{X} \to \mathfrak{X}_{\mathbb{Q}} = (X_{\mathbb{Q}}, (\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})_{\mathbb{Q}})$ and the image $x_{\mathbb{Q}}$ of x in $H^0(X_{\mathbb{Q}}, \pi_k(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})_{\mathbb{Q}})$. We show first that $d_{i_1}x_{\mathbb{Q}}$ is annihilated by a power of $x_{\mathbb{Q}}$. Denote by $j\colon D(x_{\mathbb{Q}}) \to X_{\mathbb{Q}}$ the inclusion. Then we have a map of spectral sequences $DSS((\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})_{\mathbb{Q}}) \to DSS(j_*j^*(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})_{\mathbb{Q}})$. As j_* is affine, the E^2-term of the latter is isomorphic to $H^*(D(x_{\mathbb{Q}}); j^*\pi_*(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})_{\mathbb{Q}})$. As shown in the last lemma, this is isomorphic to $H^*(X; \pi_*(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})_{\mathbb{Q}})[1/x_{\mathbb{Q}}]$ since $\pi_{2k}(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})_{\mathbb{Q}} \cong (\pi_2\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})^{\otimes k}_{\mathbb{Q}}$. On the other hand, $H^i(D(x_{\mathbb{Q}}); j^*\pi_*(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})_{\mathbb{Q}}) = 0$ for $i > 0$ as the coarse moduli space of $D(x_{\mathbb{Q}})$ is affine and $D(x_{\mathbb{Q}})$ is tame by Example 2.25 as it is rational. Thus, indeed, $d_{i_1}(x_{\mathbb{Q}})$ has to be annihilated by a power of $x_{\mathbb{Q}}$.

For $m \in \mathbb{N}$, we have $d_{i_1}x^m = mx^{m-1}d_{i_1}(x)$. Hence, $d_{i_1}x^m_{\mathbb{Q}} = 0$ for some m and thus $d_{i_1}x^m$ is l-torsion for some l. It follows that $d_{i_1}(x^{lm}) = lx^{m(l-1)}d_{i_1}(x^m) = 0$. The argument can be repeated for the first non-trivial differential of x^{lm}, etc. and comes to an end somewhere as the length of the differentials is bounded. Thus, there is some power x^K of x that is a permanent cycle and $D(x^K) = D(x)$.

As the $D(x)$ with affine coarse moduli space form the basis of the Zariski topology by Proposition 3.19, $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ is ample. □

The existence of an upper bound for the length of differentials in the DSS is actually closely related to the cocontinuity of Γ. One situation where these conditions are trivially fulfilled is that of bounded cohomological dimension, which is satisfied for a derived *scheme*.

COROLLARY 3.25. *Let $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ be an even periodic, quasi-compact and separated derived* scheme *X. Assume that $\pi_k\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ is an ample $\mathcal{O}_X$-module for some $k \in \mathbb{Z}$. Then the global sections functor*

$$\Gamma\colon \mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}))$$

is an equivalence. In particular, this is true for X quasi-affine as every line bundle is ample on a quasi-affine scheme.

Proof. There is an $n \geqslant 0$ such that $H^i(X; \pi_l\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}) = 0$ for $i > n$ and all $l \in \mathbb{Z}$. The global sections functor Γ commutes with homotopy colimits as the DSS for $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ is concentrated in finitely many rows. Likewise it follows that the length of differentials in the DSS for $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ is bounded. By the last proposition, it follows that $\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$ is ample. Furthermore, it is certainly Zariski-locally true that the global sections functor is conservative (as it is true on every affine scheme). Thus, we can apply Proposition 3.22 to see that Γ is conservative. This implies 0-affineness by Corollary 3.7. □

REMARK 3.26. In the quasi-affine case, Corollary 3.25 was already proved in [**37**, Propositions 2.4.4 and 2.4.8] for connective spectral Deligne–Mumford stacks. From this, the even periodic case can be easily recovered by taking the connective cover. Any (possibly non-connective) spectral Deligne–Mumford stack has a connective cover it maps to (just as any E_∞-ring R receives a map from its connective cover $\tau_{\geqslant 0}R$), and any quasi-coherent sheaf can be pushed forward to the connective cover. The global sections are the same in either case. In this way, one can always reduce to the case in which the derived stack is connective.

Our original formulation of our main results was restricted to the case of cohomological dimension 1. We are indebted to Jacob Lurie for explaining to us his (only slightly different) argument for Corollary 3.25, which consequently yielded a stronger formulation of Theorem 4.1.

Example 3.27. Consider the compactified moduli stack of elliptic curves $M_{\overline{\mathrm{ell}}}(n)$ with level n structure. By work of Goerss–Hopkins and Hill–Lawson [**23**], this can be refined to an even periodic derived stack $(M_{\overline{\mathrm{ell}}}(n), \mathcal{O}^{\mathrm{top}})$ with $\pi_2\mathcal{O}^{\mathrm{top}}$ ample. Thus it follows directly from Corollary 3.25 that for $n \geqslant 3$, when $M_{\overline{\mathrm{ell}}}(n)$ is a scheme, $(M_{\overline{\mathrm{ell}}}(n), \mathcal{O}^{\mathrm{top}})$ is 0-affine. As explained in Section 7, this is actually true for all n.

3.4. *Ascent for 0-affineness*

In this subsection, we note a couple of additional easy criteria for 0-affineness.

Proposition 3.28. *Suppose that $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$ is a 0-affine derived stack. Let $U \subset X$ be an open substack and let $\mathfrak{U} = (U, \mathcal{O}^{\mathrm{top}}_{\mathfrak{U}})$ be the induced derived stack. Then $\mathfrak{U}$ is also 0-affine.*

Proof. Given a quasi-coherent sheaf $\mathcal{F}$ on $\mathfrak{U}$, if its global sections are zero, then we show $\mathcal{F} = 0$ as follows: form the pushforward $j_*\mathcal{F}$ along the inclusion $j\colon U \hookrightarrow X$. By assumption, $\Gamma(j_*\mathcal{F}) \simeq \Gamma(\mathcal{F}) \simeq 0$, so that $j_*\mathcal{F} \simeq 0$ by the 0-affineness of $\mathfrak{X}$. Since $j^*j_*\mathcal{F} \simeq \mathcal{F}$, it follows that $\mathcal{F} \simeq 0$.

Furthermore, the pushforward functor j_* commutes with homotopy colimits by [**37**, Example 2.5.6 and Proposition 2.5.12]. Thus, $\Gamma(\mathfrak{U}, -) \simeq \Gamma(\mathfrak{X}, -) \circ j_*$ also commutes with homotopy colimits. □

Proposition 3.29. *Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of derived stacks for $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ and $\mathfrak{Y} = (Y, \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}})$ such that the underlying morphism $Y \to X$ is affine. If $\mathfrak{X}$ is 0-affine, then $\mathfrak{Y}$ is 0-affine.*

Proof. The conservativeness of $\Gamma(\mathfrak{Y}, -)$ follows again, because the pushforward along $\mathfrak{Y} \to \mathfrak{X}$ is a conservative functor. Indeed, we can choose an étale cover $\{U_i \to X\}_{i \in I}$ by affine schemes. Thus, $\{U_i \times_X Y \to Y\}_{i \in I}$ is an étale cover of Y. If $f_*\mathcal{F} = 0$ for a quasi-coherent sheaf $\mathcal{F}$ on $\mathfrak{Y}$, then $f_*\mathcal{F}(U_i) = \mathcal{F}(U_i \times_X Y) = 0$ for every $i \in I$. Thus, $\mathcal{F} = 0$.

Furthermore, the pushforward functor f_* commutes with homotopy colimits by [**37**, Example 2.5.6 and Proposition 2.5.12]. Thus, $\Gamma(\mathfrak{Y}, -) \simeq \Gamma(\mathfrak{X}, -) \circ f_*$ also commutes with homotopy colimits. □

4. *Affineness results in chromatic homotopy theory*

The main result of this section is the following theorem.

Theorem 4.1. *Let X be a noetherian and separated Deligne–Mumford stack, equipped with a flat map $X \to M_{FG}$ that is quasi-affine. Let $\mathfrak{X}$ be an even periodic refinement of X. Then the derived stack $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$ is 0-affine.*

Recall here that a map $X \to M_{FG}$ is *quasi-affine* if for every map $\operatorname{Spec} A \to M_{FG}$ the pullback $\operatorname{Spec} A \times_{M_{FG}} X$ is quasi-affine, that is, a quasi-compact open subscheme of an affine scheme.

Remark 4.2. The condition that X is separated is not very restrictive. Recall that X is separated if its diagonal is universally closed. This implies for (locally noetherian) Deligne–Mumford stacks only that the diagonal is finite, but not that it is a closed immersion. The Deligne–Mumford stacks most commonly considered by algebraic geometers, like the (compactified) moduli stack of elliptic curves or PEL-Shimura stacks, are separated.

We will first show Theorem 4.1 locally at every prime p using Propositions 3.9 and 3.16. This relies crucially on the fact that Morava E-theory E_n generates the E_n-local stable homotopy category as a thick tensor-ideal, which in turn follows from the nilpotence technology to be reviewed in the next subsection. We will then glue the p-local results together to deduce an integral statement.

4.1. *Nilpotence technology*

The power of the use of formal groups in stable homotopy theory is especially illustrated by the nilpotence and periodicity theorems of [**14**, **25**], and their cousin, the Hopkins–Ravenel smash product theorem.

Theorem 4.3 (Nilpotence theorem [**14**]).

(1) *Let R be a (not necessarily structured) ring spectrum and let $\alpha \in \pi_* R$. Suppose that α maps to zero in $MU_*(R)$. Then α is nilpotent.*

(2) *Let $f\colon \Sigma^k X \to X$ be a self-map of a finite spectrum. Then f is nilpotent if and only if $MU_*(f)$ is nilpotent.*

In particular, the association $X \mapsto \mathcal{F}_*(X)$, from spectra to quasi-coherent sheaves on M_{FG}, is sufficient to detect all maps except up to nilpotence, at least for finite spectra. Conversely, many of the 'periodicities' visible in the geometry of M_{FG} can be realized topologically, via the *periodicity theorem* of [**25**].

Next, recall that a subcategory of the ∞-category $\mathrm{Sp}^{\omega}_{(p)}$ of finite p-local spectra is called *thick* if it is stable and closed under retracts.

Theorem 4.4 (Thick subcategory theorem [**25**]). *The thick subcategories of $\mathrm{Sp}^{\omega}_{(p)}$ are in natural bijection with the reduced, closed substacks of $(M_{FG})_{(p)}$ of finite presentation. In particular, if a thick subcategory $\mathcal{C}$ contains a spectrum with non-trivial rational homology, then $\mathcal{C} = \mathrm{Sp}^{\omega}_{(p)}$.*

The next result is the strongest finiteness theorem that we need. Let E_n be the nth *Morava E-theory*; then E_n is a Landweber-exact, even periodic E_∞-ring with $\pi_0 E_n \simeq W(\mathbb{F}_{p^n})[[v_1, \ldots, v_{n-1}]]$. Given a spectrum, the functor of E_n-localization can be thought of as (after localizing at p) restriction to the open substack of $(M_{FG})_{(p)}$ parameterizing formal groups of height at most n.

Theorem 4.5 (Smash product theorem [**45**]). *The E_n-localization functor $L_n\colon \mathrm{Sp} \to \mathrm{Sp}$ commutes with homotopy colimits, so that for any spectrum X, the natural map*

$$X \otimes L_n S^0 \longrightarrow L_n X$$

is an equivalence.

Note here that E_n has the same Bousfield class as the Johnson–Wilson theory $E(n)$ by [**27**, Proposition 5.3].

Theorem 4.5 is essentially a statement about certain homotopy colimits and limits commuting with each other, and is crucial to describing the structure of the ∞-category $L_n\mathrm{Sp}$ of E_n-local spectra. A general reference for this category is [**27**]. The smash product theorem implies that $L_n\mathrm{Sp}$ is a full subcategory of Sp which is closed under homotopy limits *and colimits*. The ∞-category $L_n\mathrm{Sp}$ has much better finiteness properties than the category of spectra (or even the category of p-local spectra), as we will explain next.

We will need various slightly stronger versions of Theorem 4.5 (which are ultimately the ingredients used to prove it) for our purposes. We can recover the E_n-local sphere L_nS^0 via the totalization of the classical *cobar construction*

$$E_n \rightrightarrows E_n \otimes E_n \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \cdots ,$$

whose associated Tot spectral sequence is the E_n-local ANSS. A strong form of the smash product theorem implies that this spectral sequence (drawn with the Adams indexing $(s, t-s)$) degenerates at a finite stage with a *horizontal* vanishing line. More generally, this is true for any E_n-local spectrum replacing L_nS^0 by the following result.

Theorem 4.6 ([**27**, Proposition 6.5]). *There is a uniform bound $N = N(n)$ such that given any E_n-local spectrum X, the ANSS for X satisfies $E_N^{s,t} = 0$ for $s > N$.*

We can formulate a closely related statement in the language of *pro-spectra*. The ∞-category Pro(Sp) is the ∞-category of pro-objects in Sp in the sense, for example, of [**33**, Chapter 5]: a pro-spectrum is a formal filtered homotopy inverse limit of spectra.

Given a cosimplicial diagram $F\colon \Delta \to \mathrm{Sp}$, we can form the homotopy inverse limit $\mathrm{Tot}F$ in spectra, but we can also do it in pro-spectra. This amounts to considering the Tot tower

$$\cdots \longrightarrow \mathrm{Tot}^2F \longrightarrow \mathrm{Tot}^1F \longrightarrow \mathrm{Tot}^0F,$$

as a pro-object. There is a fully faithful inclusion of ∞-categories $\mathrm{Sp} \to \mathrm{Pro(Sp)}$. Pro-spectra in the image of Sp are called *constant*. Another reformulation of the smash product theorem is the following.

Theorem 4.7 (Hopkins–Ravenel [**45**, Chapter 8]). *The pro-spectrum associated to the cobar construction on E_n is constant with value L_nS^0.*

The proof of this is explained in Lectures 30 and 31 of Lurie's course on chromatic homotopy theory [**35**]. As explained in [**35**, Lecture 30], this is closely related to (and in fact follows from) the horizontal vanishing line in the L_n-local Adams–Novikov spectral sequence (Theorem 4.6), using a delicate criterion for constancy of pro-spectra due to Bousfield. In particular, Theorem 4.6 is actually used to prove Theorem 4.7.

Theorem 4.7 states that there is an equivalence of pro-objects between the Tot-tower for the cobar construction and the constant pro-object with value L_nS^0. In particular, the natural maps

$$L_nS^0 \longrightarrow \mathrm{Tot}^m((E_n^{\otimes(\bullet+1)}))$$

have, for large enough m, sections up to homotopy. For further discussion, see also [**35**, Lecture 30] and, for connections between these ideas and descent theory, [**42**, Sections 3 and 4].

In particular, we get the following corollary, which is the piece of the nilpotence technology that we shall use.

Corollary 4.8. *The spectrum E_n generates $L_n\mathrm{Sp}$ as a thick tensor-ideal.*

Proof. Let $\mathcal{C} \subset L_n\mathrm{Sp}$ be a thick tensor-ideal containing E_n. This implies that the partial totalizations Tot^m of the cobar construction on E_n belong to $\mathcal{C}$. Since $\mathcal{C}$ is closed under retracts, it follows that $L_nS^0 \in \mathcal{C}$ and thus $\mathcal{C} = L_n\mathrm{Sp}$. □

A similar result is discussed in [**26**, Theorem 5.3], where it is shown that every spectrum in $L_n\mathrm{Sp}$ is '$E(n)$-nilpotent', or equivalently belongs to the thick tensor-ideal generated by $E(n)$.

REMARK 4.9. Similarly, the nilpotence theorem is closely related to the statement that for any connective spectrum X, the (MU-based) Adams–Novikov spectral sequence for X has a *vanishing curve* of slope tending to zero as $t - s \to \infty$ at E^∞. See [**24**] for further discussion of this. However, the MU-based cobar construction (whose homotopy inverse limit is S^0) is definitely non-constant, because of the existence of non-trivial MU-acyclic spectra (for instance, the Brown–Comenentz dual I of the sphere; see [**25**, Appendix B]).

4.2. *The p-local case*

Let X be a noetherian separated Deligne–Mumford stack over $\mathbb{Z}_{(p)}$ with a flat map to M_{FG}. In this section, everything is implicitly localized at p: for instance, M_{FG} really means $M_{FG} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}_{(p)}$.

We want to find in this case instances of the abstract theorems of the last section. We begin by choosing an $n \in \mathbb{N}$ such that the flat map $X \to M_{FG}$ factors through the open substack $M_{FG}^{\leqslant n} \subset M_{FG}$ of formal groups of height at most n. We can do this because we have a descending sequence of closed substacks

$$M_{FG} \supset M_{FG}^{\geqslant 1} \supset M_{FG}^{\geqslant 2} \supset \cdots,$$

where each $M_{FG}^{\geqslant n+1}$ is cut out by the vanishing of a regular element on $M_{FG}^{\geqslant n}$. Since X is noetherian, $X \times_{M_{FG}} M_{FG}^{\geqslant m}$ must be empty for $m \gg 0$.

Observe the following (well-known) lemma.

LEMMA 4.10. *Let $f\colon \mathrm{Spec}\,R \to M_{FG}$ be a flat map, which factors over $M_{FG}^{\leqslant n}$, and E_R be the corresponding Landweber exact spectrum. Then E_R is E_n-local.*

Proof. Let T be a spectrum. The MU_*-comodule $(MU_*T, (MU \otimes MU)_*(T))$ defines a $\mathbb{Z}/2$-graded quasi-coherent sheaf $\mathcal{F}_*$ on M_{FG} (see Subsection 2.1 for a discussion). Then $\pi_*(E_R \otimes T) \cong f^*\mathcal{F}_*(\mathrm{Spec}\ R)$. Denote by q the canonical map $\mathrm{Spec}\ \pi_0E_n \to M_{FG}^{\leqslant n} \subset M_{FG}$. So, likewise, we have an isomorphism $\pi_*(E_n \otimes T) \cong q^*\mathcal{F}_*(\mathrm{Spec}\ R)$. Now assume that T is E_n-acyclic. Thus, $q^*\mathcal{F}_* = 0$ and hence, since $\mathrm{Spec}\ \pi_0E_n \to M_{FG}^{\leqslant n}$ is faithfully flat, $\mathcal{F}_*|_{M_{FG}^{\leqslant n}} = 0$. This implies $f^*\mathcal{F}_* = 0$ and thus $E_R \otimes T = 0$. Thus

$$[T, E_R] = 0$$

for every E_n-acyclic spectrum T, since E_R (as a ring spectrum) is local with respect to itself. □

Now let $\mathfrak{X} = (X, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$ be an even periodic refinement of $X \to M_{FG}$. By the lemma, for every étale map $\mathrm{Spec}\,R \to X$, the E_∞-ring $\mathcal{O}^{\mathrm{top}}(\mathrm{Spec}\,R)$ is E_n-local. Since E_n-local spectra are closed under homotopy limits, thus for every étale map $Y \to X$, the E_∞-ring $\mathcal{O}^{\mathrm{top}}(Y)$ is E_n-local. We see that the whole argument takes place in the E_n-local category.

PROPOSITION 4.11. *Let X be a noetherian and separated Deligne–Mumford stack over a p-local ring, equipped with a flat and tame map $X \to M_{FG}$. Let $\mathfrak{X} = (X, \mathcal{O}_{\mathfrak{X}}^{\mathrm{top}})$ be an even* *periodic refinement of X. Then the functor of taking global sections*

$$\Gamma\colon \mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}))$$

commutes with homotopy colimits.

Proof. Let $Y = X \times_{M_{FG}} \operatorname{Spec} \pi_0 E_n$. This has finite cohomological dimension by Proposition 2.29 as the map $X \to M_{FG}$ is tame. It is also quasi-compact and separated as X, $\operatorname{Spec} \pi_0 E_n$ and the diagonal of M_{FG} are. We want to apply Proposition 3.9 with $M = \Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}) \otimes E_n$. It follows from Corollary 4.8 that M generates $\mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}))$ as a thick tensor-ideal. We have to show that

$$\pi_*(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}} \otimes_{\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})} M) \cong \pi_*(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}} \otimes E_n) \cong q_* q^* \pi_*(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$$

for $q\colon Y \to X$ the projection map. This follows from Proposition 2.4, that for flat maps $\operatorname{Spec} R \to M_{FG}$ and $\operatorname{Spec} R' \to M_{FG}$ the smash product of the two Landweber exact spectra E_R and $E_{R'}$ can be computed as

$$\pi_{2k}(E_R \otimes E_{R'}) \cong \omega^{\otimes k}(\operatorname{Spec} R \times_{M_{FG}} \operatorname{Spec} R').$$

Specialized to our situation, we get that for every flat map $\operatorname{Spec} R \to X$ we have the following natural isomorphisms:

$$\pi_{2k}(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}(\operatorname{Spec} R) \otimes E_n) \cong \pi_{2k}(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})(\operatorname{Spec} R \times_{M_{FG}} \operatorname{Spec} \pi_0 E_n) \cong q_* q^* \pi_{2k}(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}). \qquad \square$$

Set again $Y = X \times_{M_{FG}} \operatorname{Spec} \pi_0 E_n$. We want to define an even periodic refinement of this. There are two equivalent ways of doing this.

(1) Let $U \to X$ be an affine étale cover and $U_\bullet$ be the corresponding Cech simplicial object. We can define even periodic refinements on $U_k \times_{M_{FG}} \pi_0 E_n$ by considering $\operatorname{Spec} \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}(U_k) \otimes E_n$. Note here that E_n has the structure of an E_∞-ring spectrum by the Goerss–Hopkins–Miller theorem. Then we can define $\mathfrak{Y} := \operatorname{hocolim} \operatorname{Spec}(\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}(U_k) \otimes E_n)$.

In other words, one notes that to realize Y as a derived stack, one needs to construct an appropriate *diagram* of even periodic E_∞-rings, corresponding to any given presentation of Y as an ordinary stack. Given any étale map $\operatorname{Spec} R \to X$, we can realize $\operatorname{Spec} R \times_{M_{FG}} \operatorname{Spec} \pi_0 E_n$ via the E_∞-ring $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R) \otimes E_n$. This constructs a diagram of E_∞-rings which is enough (by a descent procedure) to produce the sheaf of E_∞-rings on the étale site of Y (in an analogous way to Proposition 2.15).

(2) We can define $\mathfrak{Y}$ as the 'relative Spec' of the sheaf of algebras $\mathcal{O}^{\mathrm{top}} \otimes E_n$, using essentially the previous construction.

Thus, we get an even periodic stack $\mathfrak{Y} = (Y, \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}})$ with a faithfully flat, separated and quasi-compact map $q : \mathfrak{Y} \to \mathfrak{X}$ such that

$$q_* \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}} \simeq \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}} \otimes E_n.$$

We now get the following theorem.

THEOREM 4.12. *We use the same notation and assumptions from Proposition 4.11. Assume furthermore that Y is quasi-affine. Then $\Gamma\colon \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}))$ is conservative and thus an equivalence.*

Proof. In light of Proposition 4.11, this is a direct application of Proposition 3.16 and Corollary 3.25 as the underlying stack Y of $\mathfrak{Y}$ is quasi-affine. $\square$

REMARK 4.13. Note that the condition that $\pi_k \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}}$ is ample is not more general as $\pi_k \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}} \cong \pi_0 \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}}$ for k even and 0 else (since E_n is strongly even periodic).

4.3. *The integral version*

In this section, we complete the proof of Theorem 4.1. To start with, we extend the proof of the first step when localized at p, as done in the previous section, to an integral statement. Once again, we have a slightly stronger statement.

THEOREM 4.14. *Let X be a noetherian and separated Deligne–Mumford stack, equipped with a flat map $X \to M_{FG}$. Let $\mathfrak{X}$ be an even periodic refinement of X. Then, if $X \to M_{FG}$ is tame, the global sections functor*

$$\Gamma\colon \mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}))$$

commutes with homotopy colimits.

Proof. In order to prove this, we will use the p-local version proved earlier, for each prime, together with an arithmetic square to fit everything together integrally. There is an obstacle in that the 'arithmetic square' is infinite in nature. To deal with this, we use the following lemma. □

LEMMA 4.15. *Let X be a quasi-compact and separated Deligne–Mumford stack over some quasi-compact scheme S. Then there is an N such that $X[1/N]$ has bounded cohomological dimension.*

Proof. We want first to show that the order of automorphism groups of points in X is bounded. As X is separated, the inertia stack $X \times_{X \times_S X} X = \mathcal{I}_X$ is finite over X. Since X is quasi-compact, there is an étale covering $q\colon \operatorname{Spec} A \to X$ for some ring A. The pullback $q^*\mathcal{I}_X \to \operatorname{Spec} A$ corresponds to an A-module generated by m elements for some m. If $x\colon \operatorname{Spec} k \to X$ is a geometric point, then the pullback $\operatorname{Spec} k \times_X \operatorname{Spec} A$ is equivalent to a disjoint union of $\operatorname{Spec} k$. Thus, $x^*\mathcal{I}_X \to \operatorname{Spec} k$ has also rank at most m, that is, the stabilizer of x has at most m elements.

Let $N = m!$. Then all stabilizers have invertible order on $X[1/N]$. Thus, $X[1/N]$ is tame, which implies the result by Proposition 2.27. □

It follows from this that there exists an integer $N \in \mathbb{N}$ such that, after tensoring with the localization $S^0[N^{-1}]$, the functor

$$\mathcal{F} \longmapsto \Gamma(\mathfrak{X}, \mathcal{F} \otimes S^0[N^{-1}]), \quad \mathrm{QCoh}(\mathfrak{X}) \longrightarrow \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})),$$

commutes with homotopy colimits. In fact, the spectral sequence to compute the homotopy groups of $\Gamma(\mathfrak{X}, \mathcal{F} \otimes S^0[N^{-1}])$ starts from the cohomology of $\pi_*\mathcal{F}$ on the open substack $\mathfrak{X}[N^{-1}]$, since cohomology commutes with localization, and is consequently concentrated in finitely many rows at E^2.

In view of Propositions 4.11 and 2.29, we can thus apply the following lemma to conclude the proof of the theorem.

LEMMA 4.16. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between cocomplete stable ∞-categories. Suppose that:*

(1) *$F(\cdot \otimes S^0_{(p)})$ commutes with homotopy colimits for every prime number p;*
(2) *there exists an integer N such that $F(\cdot \otimes S^0[1/N])$ commutes with homotopy colimits.*

Then F commutes with homotopy colimits.

Proof. Consider the collection $\mathcal{I}$ of spectra T such that $F(\cdot \otimes T)$ commutes with homotopy colimits. It is an *ideal* in spectra: that is, if X is a spectrum and $Y \in \mathcal{I}$, then $X \otimes Y \in \mathcal{I}$. By hypothesis, this ideal contains $S^0[1/N]$ for some N and each $S^0_{(p)}$ for each prime number p. We want to show that it contains S^0.

To do this, we use an inductive argument. Let $N \in \mathbb{Z}_{>0}$ be chosen *minimal* such that $S^0[1/N] \in \mathcal{I}$. We want to show that $N = 1$. Observe that if $(m, p) = 1$, then there is an *arithmetic square*, that is, a homotopy pullback diagram

$$\begin{array}{ccc} S^0[1/m] & \longrightarrow & S^0[1/mp] \\ \downarrow & & \downarrow \\ S^0_{(p)}[1/m] & \longrightarrow & S^0_{\mathbb{Q}}. \end{array}$$

It follows that if $N > 1$, then $N = pm$ for $(p, m) = 1$ (N is squarefree by minimality), and then the above arithmetic square implies that $S^0[1/m] \in \mathcal{I}$, a contradiction. Thus $N = 1$ and we are done.

This completes the proof of Theorem 4.14. □

Proof of Theorem 4.1. Let us now complete the proof of the main theorem. By Corollary 3.7, it suffices now to show that if $\Gamma(\mathfrak{X}, \mathcal{F}) = 0$ for some quasi-coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$, then $\mathcal{F} = 0$.

So assume $\Gamma(\mathfrak{X}, \mathcal{F}) = 0$. Then $\Gamma(\mathfrak{X}_{(p)}, \mathcal{F}_{(p)}) \simeq \Gamma(\mathfrak{X}, \mathcal{F})_{(p)} = 0$ for every prime p. Indeed, since Γ commutes with homotopy colimits, it commutes with localization at p. By Theorem 4.12, it follows that $\mathcal{F}_{(p)} = 0$ for every prime p. Thus, $\mathcal{F} = 0$. □

5. *Applications to Galois theory*

Let R be an E_∞-ring. Recall that an E_∞-R-algebra R' is said to be *étale* if $\pi_0 R \to \pi_0 R'$ is an étale morphism of commutative rings, and the natural map $\pi_0 R' \otimes_{\pi_0 R} \pi_* R \to \pi_* R'$ is an isomorphism. The theory of étale extensions in this sense is entirely algebraic: the ∞-category of étale R-algebras is equivalent to the (ordinary) category of étale $\pi_0 R$-algebras.

This definition excludes useful examples such as the map $KO \to KU$, which behaves in many respects as an étale morphism in commutative algebra, albeit not on the level of homotopy groups. Since $\pi_0 KO \simeq \mathbb{Z}$, there are no finite étale extensions of KO. Nonetheless, $KO \simeq KU^{h\mathbb{Z}/2}$, and, as we have shown (for example, in view of Theorem 4.1), there is a good theory of $\mathbb{Z}/2$-'descent' from KU to KO. Rognes's notion of a faithful Galois extension (Definition 5.1) is a generalization of the above notion of étaleness (or at least the Galois version) that encompasses examples such as $KO \to KU$.

In this section, we analyze the Galois theory, in this sense, for E_∞-rings such as KO and TMF which arise as 'rings of functions' $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$ of 0-affine derived stacks. Our main result (Theorem 5.8) is that a Galois cover (in the algebraic sense) of the underlying stack yields a faithful Galois extension of $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}})$. This provides examples of Galois extensions of (localizations of) TMF via level structures, for instance.

5.1. *Galois extensions*

Let B be an E_∞-ring with the action of a finite group G and let $A = B^{hG}$ be the homotopy fixed points. We recall the following definition of Rognes.

Definition 5.1 ([**47**]). The map $A \to B$ is said to be a *G-Galois extension* if the map of E_∞-A-algebras

$$B \otimes_A B \longrightarrow \prod_{g \in G} B,$$

which informally is given by $b_1 \otimes b_2 \mapsto \{b_1 \cdot g(b_2)\}_{g \in G}$, is an equivalence. A Galois extension is said to be *faithful* if the Bousfield classes of A and B (for A-modules) are equivalent: that is, if an A-module smashes to zero with B, then it itself is zero.

This is inspired by the notion of a Galois extension of (discrete) commutative rings, which can be defined in the same way, but where faithfulness is automatic. Equivalently, a map $R \to S$ of commutative rings is G-Galois if $\operatorname{Spec} S \to \operatorname{Spec} R$ is an étale G-torsor in the sense of algebraic geometry.

Faithful Galois extensions (which are the only type of Galois extensions we shall consider) are very well-behaved. The map $A \to B$ exhibits B as a *perfect* (that is, compact or dualizable) A-module, and for any E_∞ A-algebra A', the map of rings $A' \to B \otimes_A A'$ is again faithful and G-Galois. Moreover, one can develop [**42**] a version of Grothendieck's étale fundamental group formalism in this setting.

We start by noting a few examples and properties of faithful Galois extensions.

Example 5.2. Suppose that A is an E_∞-ring, and suppose that B_0 is a G-Galois extension of the ring $\pi_0 A$. Then there exists a unique E_∞-ring B étale over A with $\pi_0 B \simeq B_0$, and a G-action on B in the ∞-category of A-algebras such that the natural map $A \to B^{hG}$ is an equivalence (by Theorem 2.12).

Example 5.3 ([**6**, Proposition 3.6; **42**, Proposition 6.28]). Suppose that A is an even periodic E_∞-ring such that $\pi_0 A$ is a field. Then G-Galois extensions of A are equivalent to G-Galois extensions of $\pi_0 A$: that is, they are étale. The main ingredient is the Künneth isomorphism for A-modules.

Example 5.4. A simple example of a Galois extension that is not étale is as follows: let A be an E_∞-ring with $\pi_*(A) \simeq \mathbb{Z}[1/2, t^{\pm 1}]$, where $|t| = 2$. Consider a $\mathbb{Z}/2$-action on A that sends $t \mapsto -t$. In this case, the map $A^{h\mathbb{Z}/2} \to A$ is a $\mathbb{Z}/2$-Galois extension realizing on homotopy the map $\mathbb{Z}[1/2, u^{\pm 1}] \to \mathbb{Z}[1/2, t^{\pm 1}]$, $u \mapsto t^2$, as we will now show.

First observe that the map

$$\begin{aligned} \Phi\colon \mathbb{Z}[1/2, t^{\pm 1}] \otimes \mathbb{Z}[1/2, t^{\pm 1}] &\longrightarrow \mathbb{Z}[1/2, t^{\pm 1}] \times \mathbb{Z}[1/2, t^{\pm 1}], \\ x \otimes y &\longmapsto (x \cdot y, x \cdot g(y)) \end{aligned}$$

is surjective, where g generates $\mathbb{Z}/2$. As this map is $\mathbb{Z}[1/2, t^{\pm 1}]$-linear, this follows from the fact that $\Phi(\frac{1}{2} \otimes 1 + \frac{1}{2} t^{-1} \otimes t) = (1, 0)$ and $\Phi(\frac{1}{2} \otimes 1 - \frac{1}{2} t^{-1} \otimes t) = (0, 1)$. By a graded version of [**22**, Theorem 1.6], we see that $\mathbb{Z}[1/2, u^{\pm 1}] \to \mathbb{Z}[1/2, t^{\pm 1}]$, $u \mapsto t^2$ is a $\mathbb{Z}/2$-Galois extension in the graded sense. As $\mathbb{Z}[1/2, t^{\pm 1}]$ is free over $\mathbb{Z}[1/2, u^{\pm 1}]$, this implies that $A^{h\mathbb{Z}/2} \to A$ is a $\mathbb{Z}/2$-Galois.

Example 5.5. While the notion of a faithful Galois extension generalizes that of an étale Galois extension (see [**5**]), the notions coincide on *connective* E_∞-rings A. We prove this here if $\pi_0(A)$ is *noetherian*. In fact, let A be as in the previous sentence, and let B be a faithful G-Galois extension. For any morphism $\pi_0 A \to k$, for k a field, we get a map of E_∞-rings

$$A \longrightarrow \tau_{\leqslant 0} A \simeq H\pi_0 A \longrightarrow Hk,$$

and the base-change $B \otimes_A Hk$ is therefore a faithful G-Galois extension of Hk, which, thanks to the Künneth isomorphism, is necessarily discrete (and the Eilenberg–MacLane spectrum associated to a product of copies of finite separable extensions of k).

It follows that B, which is a perfect A-module, is actually connective, and indeed *flat* in the sense of [**40**, Section 8.2.2]: In [**40**], it is shown that an A-module M is flat if and only if, for every discrete A-module (that is, $\pi_0 A$-module) N, the A-module $M \otimes_A N$ is discrete. However, it suffices to show that $M \otimes_A H\pi_0 A$ is a discrete, flat $H\pi_0 A$-module. Now we can appeal to a classical fact from commutative algebra (see the discussion in [**51**, Tag 0656] for the local case to which one reduces) given a commutative noetherian ring R, and a perfect complex $P^\bullet$ of R-modules, then $P^\bullet$ is quasi-isomorphic to a projective R-module concentrated in dimension 0 if and only if the same holds (over k) for $P^\bullet \otimes_R k$ for every residue field k of R.

It follows that $\pi_0(B)$ is flat over $\pi_0(A)$ and is unramified in the sense of classical commutative algebra: therefore, $\pi_0(A) \to \pi_0(B)$ is étale. Since $A \to B$ is flat, we are done. See also [**42**, Theorem 6.16].

Our goal is to show that even periodic refinements provide a rich source of Galois extensions which are not étale.

Example 5.6. The map $KO \to KU$ is a $\mathbb{Z}/2$-Galois extension, as shown in [**47**, Chapter 5] using the following result, a proof of which appears in [**41**].

Theorem 5.7 (Wood). *There is an equivalence of spectra $KO \otimes \Sigma^{-2}\mathbb{CP}^2 \simeq KU$.*

Our next result is a generalization of this, which states that Galois coverings of an associated stack can be used to manufacture Galois extensions of ring spectra. For example, the $\mathbb{Z}/2$-Galois extension $KO \to KU$ arises in this way from the $\mathbb{Z}/2$-torsor $\operatorname{Spec}\mathbb{Z} \to B\mathbb{Z}/2$.

Theorem 5.8. *Let G be a finite group acting on a Deligne–Mumford stack X, with $Y = X/G$ the stack quotient. Consider a flat map $Y \to M_{FG}$. Let $\mathfrak{Y}$ be a 0-affine even periodic refinement of Y and $\mathfrak{X}$ be the induced refinement of X. Then $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ is a faithful G-Galois extension of $\Gamma(\mathfrak{Y}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}}) = \Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})^{hG}$. In particular, the Tate spectrum of G acting on $\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ is contractible.*

Proof. Choose an étale map $\operatorname{Spec} R \to Y$. Then the map

$$\mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}}(\operatorname{Spec} R) \longrightarrow \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}(\operatorname{Spec} R \times_Y X) \tag{3}$$

is a G-Galois extension: in fact, it is so even on homotopy groups, in the sense of Example 5.2. In particular, the map

$$\mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}(\operatorname{Spec} R \times_Y X) \otimes_{\mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}}(\operatorname{Spec} R)} \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}(\operatorname{Spec} R \times_Y X) \to \prod_{g \in G} \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}(\operatorname{Spec} R \times_Y X)$$

is an equivalence. In other words, if $f \colon \mathfrak{X} \to \mathfrak{Y}$ is the projection, then the map

$$f_* \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}} \otimes_{\mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}}} f_* \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}} \longrightarrow \prod_G f_* \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}$$

is an equivalence. Now, using the fact that $\Gamma(\mathfrak{Y}, \cdot)$ is a symmetric monoidal functor (by 0-affineness), we find that the map

$$\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}) \otimes_{\Gamma(\mathfrak{Y}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}})} \Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}}) \longrightarrow \prod_{g \in G} \Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$$

is an equivalence, as desired.

The claim about faithfulness follows from the following commutative square of ∞-categories:

$$\begin{array}{ccc} \mathrm{Mod}(\Gamma(\mathfrak{Y}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}})) & \longrightarrow & \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{QCoh}(\mathfrak{Y}) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathfrak{X}) \end{array}$$

where the lower horizontal functor (pullback) has trivial kernel, since $Y \to X$ is faithfully flat (see Lemma 3.15). This shows that the Galois extension is faithful, and implies that the Tate spectrum vanishes [**47**, Proposition 6.3.3]. □

5.2. *Tate spectra*

In this section, we give a strengthening of the earlier result on vanishing of Tate spectra, which will apply in certain non-Galois cases as well.

We begin by reviewing the Tate spectrum in more detail. Let X be a spectrum with the action of a finite group G. Recall that there is a *norm map*

$$X_{hG} \longrightarrow X^{hG},$$

from homotopy coinvariants to homotopy invariants, whose cofiber is defined to be the *Tate spectrum* X_{tG}. If X has a 'free G-action' in that it is freely induced from an ordinary spectrum Y, then the Tate spectrum is contractible. The Tate spectrum commutes with *finite* homotopy colimits and limits in the ∞-category $\mathrm{Fun}(BG, \mathrm{Sp})$ of spectra with a G-action, so it vanishes identically on the thick subcategory of $\mathrm{Fun}(BG, \mathrm{Sp})$ generated by the spectra with free G-action.

EXAMPLE 5.9. Suppose that $X \in \mathrm{Fun}(BG, \mathrm{Sp})$ has the property that the functor

$$Y \longmapsto (Y \otimes X)^{hG}, \quad \mathrm{Fun}(BG, \mathrm{Sp}) \longrightarrow \mathrm{Sp},$$

commutes with homotopy colimits. (Equivalently, X has the property that the functor $Y \mapsto (Y \otimes X)_{tG}$ commutes with homotopy colimits.) Here Y is a spectrum with a G-action, and $Y \otimes X$ is given the 'diagonal G-action': that is, at the level of functors, the smash product is computed pointwise. Then, the Tate construction X_{tG} is contractible.

To see this, observe first that if $Y = \bigsqcup_G Z$ is free, then

$$Y \otimes X \simeq \bigsqcup_G Z \otimes X,$$

so that the Tate construction $(Y \otimes X)_{tG}$ is contractible. Since we can write the sphere S^0 with trivial G-action as a geometric realization (via the bar construction) of objects in $\mathrm{Fun}(BG, \mathrm{Sp})$ with free G-action, it follows that $(S^0 \otimes X)_{tG} \simeq (X)_{tG}$ is contractible too.

We can now prove our main result on the vanishing of Tate spectra.

THEOREM 5.10. *Let X be a noetherian and separated Deligne–Mumford stack equipped with a flat map $X \to M_{FG}$, which is tame. Let $Y \to X$ be a G-torsor for a finite group G.*

Let $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}}_{\mathfrak{X}})$ be an even periodic refinement, and let $\mathfrak{Y} = (Y, \mathcal{O}^{\mathrm{top}}_{\mathfrak{Y}})$ be the induced even periodic refinement of $Y \to X \to M_{FG}$, which acquires a G-action. Let $q : \mathfrak{Y} \to \mathfrak{X}$ be the induced morphism. Then, for any $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, we have

$$(\Gamma(\mathfrak{Y}, q^* \mathcal{F}))_{tG} \simeq 0.$$

Proof. By Galois descent, we obtain an equivalence of ∞-categories

$$\mathrm{QCoh}(\mathfrak{X}) \simeq \mathrm{QCoh}(\mathfrak{Y})^{hG},$$

where the G-action on $\mathfrak{Y}$ induces a G-action on the ∞-category of quasi-coherent sheaves. This is true locally in view of étale descent as in [**37**, **38**], and then follows globally by sheafification. Moreover, for any $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, we get $\Gamma(\mathfrak{X}, \mathcal{F}) \simeq \Gamma(\mathfrak{Y}, q^*\mathcal{F})^{hG}$.

As a result, given a spectrum T with a G-action and given any quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, we can form a twisted pullback $\mathcal{F} \otimes' T \in \mathrm{QCoh}(\mathfrak{Y})^{hG} \simeq \mathrm{QCoh}(\mathfrak{X})$, which intertwines the G-action on T. At the level of global sections, we have

$$\Gamma(\mathfrak{Y}, q^*(\mathcal{F} \otimes' T)) \simeq \Gamma(\mathfrak{Y}, q^*\mathcal{F}) \otimes T \in \mathrm{Fun}(BG, \mathrm{Sp}),$$

that is, using the diagonal G-action on each tensor factor. We note that $\Gamma\colon \mathrm{QCoh}(\mathfrak{Y}) \to \mathrm{Mod}(\Gamma(\mathfrak{Y}, \mathcal{O}^{\mathrm{top}}))$ and $\Gamma\colon \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{Mod}(\Gamma(\mathfrak{X}, \mathcal{O}^{\mathrm{top}}))$ commute with homotopy colimits by Theorem 4.14.

Now, it follows from Galois descent again that we have natural equivalences

$$\Gamma(\mathfrak{X}, \mathcal{F} \otimes' T) \simeq (\Gamma(\mathfrak{Y}, q^*\mathcal{F}) \otimes T)^{hG}, \quad T \in \mathrm{Fun}(BG, \mathrm{Sp}).$$

Since Γ commutes with homotopy colimits on $\mathrm{QCoh}(\mathfrak{X})$, and since the construction $\otimes'$ preserves homotopy colimits, it follows by Example 5.9 that the Tate construction $(\Gamma(\mathfrak{Y}, q^*\mathcal{F}))_{tG}$ is contractible. □

6. *Some examples*

In this section, we discuss a few basic examples of even periodic refinements and discuss applications of the results of this paper. The main example that motivated us, that of TMF, will be discussed in more detail in the next section.

6.1. *Non-examples*

As a non-example, consider the $\mathbb{Z}/2$-action on KU-theory where $\mathbb{Z}/2$ acts trivially. The induced map $B\mathbb{Z}/2 \to M_{FG}$ is the 'constant' map $B\mathbb{Z}/2 \to \operatorname{Spec}\mathbb{Z} \to M_{FG}$, where $\operatorname{Spec}\mathbb{Z} \to M_{FG}$ classifies the multiplicative formal group. In particular, it is a flat morphism. Since KU-theory is an E_∞-ring, and it is possible to make $\mathbb{Z}/2$ act trivially on KU, this gives a derived version of $B\mathbb{Z}/2$, whose global sections are given by $K^{h\mathbb{Z}/2}$.

In this case,

$$K^{h\mathbb{Z}/2} \simeq \mathrm{F}(B\mathbb{Z}/2, K),$$

whose homotopy groups are computed, by the classical Atiyah–Segal completion theorem [**4**], to be the completion of the representation ring of $\mathbb{Z}/2$ in even dimensions and zero in odd dimensions.

The homotopy fixed-point spectral sequence (equivalently, the Atiyah–Hirzebruch spectral sequence for $K^*(\mathbb{RP}^\infty)$) has no room for differentials and degenerates at E^2, with an infinite 'checkerboard' of non-zero terms, and thus without a horizontal vanishing line. It follows that Theorem 4.1 and many of the results in this paper definitely fail for a derived stack arising from a flat morphism $X \to M_{FG}$ which is not representable. For example, the associated pro-object is not constant, as there are elements in E^∞ of arbitrarily high filtration.

Even if $X \to M_{FG}$ is representable, Theorem 4.1 may fail for more mundane reasons. For instance, let us work over $\mathbb{Q}$, so that $M_{FG} \simeq B\mathbb{G}_m$ and any map to M_{FG} is flat: to give a formal group over a $\mathbb{Q}$-algebra is equivalent to giving its cotangent space ω, a line bundle. Given a scheme X over $\mathbb{Q}$ and a line bundle ω on X, we can produce a sheaf $\mathcal{O}^{\mathrm{top}}$ of E_∞-rings on X via

$$\mathcal{O}^{\mathrm{top}} \stackrel{\mathrm{def}}{=} \mathrm{Sym}^*(\Sigma^2\omega)[\Sigma^2\omega^{-1}],$$

where the notation means that over an open affine $U \simeq \operatorname{Spec} R \subset X$ over which ω is trivial, $\mathcal{O}^{\mathrm{top}}(\operatorname{Spec} R) \simeq R[x, x^{-1}]$ is the free E_∞ R-algebra on a generator x in degree two, with x inverted. The gluing data comes from the gluing data on ω. In particular, the choice of (X, ω) determines a canonical choice (not necessarily unique) of even periodic refinement $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$.

In this case, $\mathcal{O}^{\mathrm{top}}$ is a sheaf of $\mathcal{O}_X$-algebras, so given any coherent sheaf $\mathcal{F}_0$ on X, we can produce a quasi-coherent sheaf $\mathcal{F} = \mathcal{O}^{\mathrm{top}} \otimes_{\mathcal{O}_X} \mathcal{F}_0$ on $\mathfrak{X}$. If $\mathcal{F}_0$ is such that $H^i(X, \mathcal{F}_0 \otimes \omega^j) = 0$ for all i, j, then $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ has no global sections. To be concrete, we can take $X = \mathbb{P}^1_{\mathbb{Q}}$, $\omega = \mathcal{O}_X$ and $\mathcal{F}_0 = \mathcal{O}[-1]$.

6.2. *Finite group actions*: *KO-theory and EO_n spectra*

Let R be a Landweber-exact, E_∞-ring with the action of a finite group G. This induces an action of G on the formal group of R compatible with the action on $\operatorname{Spec} \pi_0 R$: as we have seen, we get a map

$$(\operatorname{Spec} \pi_0 R)/G \longrightarrow M_{FG}.$$

This map is affine (equivalently, representable) precisely when, for every field-valued point $x\colon \operatorname{Spec} k \to \operatorname{Spec} \pi_0 R$, the stabilizer $G_x \subset G$ of x acts faithfully on the pullback of the formal group to $\operatorname{Spec} k$. Under these hypotheses, it follows that

$$R^{hG} \longrightarrow R$$

is a faithful G-Galois extension, and Galois descent goes into effect.

We discuss two basic examples of this.

Example 6.1 (KO-theory, again). As discussed in Example 2.17, we have a map

$$B\mathbb{Z}/2 \longrightarrow M_{FG},$$

sending a one-dimensional torus (equivalently, $\mathbb{Z}/2$-torsor) to its formal completion. It is flat and affine. The $\mathbb{Z}/2$-action on KU-theory by complex conjugation enables the construction of a derived stack $\mathfrak{B}\mathbb{Z}/2 = (B\mathbb{Z}/2, \mathcal{O}^{\mathrm{top}})$, which is an even periodic refinement of the above map, such that $\Gamma(\mathfrak{B}\mathbb{Z}/2, \mathcal{O}^{\mathrm{top}}) \simeq KO$.

As a result, we recover the equivalence of ∞-categories

$$\mathrm{Mod}(KO) \simeq \mathrm{QCoh}(\mathfrak{B}\mathbb{Z}/2) \simeq \mathrm{Mod}(KU)^{h\mathbb{Z}/2},$$

which we could have also seen by Galois descent.

Example 6.2 (EO_n). Let E_n be the Morava E-theory with coefficient ring $W(\mathbb{F}_{p^n})[[v_1, \ldots, v_{n-1}]]$. By the Goerss–Hopkins–Miller theorem [20], E_n is an E_∞-ring with an action of the *extended Morava stabilizer group* $\mathbb{G}$: that is, the semidirect product of the automorphism group of the Honda formal group with $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. For a discussion, see [47, Section 5.4.1].

Given a finite subgroup $H \subset \mathbb{G}$, it follows from the above discussion that we can construct a derived stack $(\operatorname{Spec} \pi_0 E_n/H, \mathcal{O}^{\mathrm{top}})$, and that $(E_n)^{hH} \to E_n$ is a faithful H-Galois extension. This is proved $K(n)$-locally in [47]. Especially interesting is the case where H is a *maximal* finite subgroup, where $(E_n)^{hH}$ is denoted by EO_n (with implicit dependence on H).

6.3. *Open subsets*

Let R be a Landweber-exact, even periodic E_∞-ring. Then any open subset of $\mathrm{Spec}\,\pi_0 R$ yields a derived stack, which by Proposition 3.28 is 0-affine. This includes the case where we are localized at a prime p, so that R is E_n-local for some n. For $m < n$, the conclusion is that

$$\mathrm{Mod}(L_m R) \simeq \mathrm{QCoh}(\mathfrak{X})$$

for $\mathfrak{X}$ an even periodic refinement of $\mathrm{Spec}\,\pi_0 R \times_{M_{FG}} M_{FG}^{\leqslant m}$.

Although elementary, this construction has some uses because the associated rings of functions are definitely far from being even periodic. For instance, in [**43**, Theorem C], it is shown that the Picard groups of $L_m R$ can be unexpectedly large, even when $R = E_n$ (although the algebraic Picard group is trivial).

6.4. *The affine line*

In this subsection, we note an important example. Let $\mathbb{Z}_{\geqslant 0}$ be the (discrete) topological, commutative monoid of non-negative integers. Since

$$\Sigma^\infty_+ \colon \mathcal{S} \longrightarrow \mathrm{Sp}$$

is a symmetric monoidal functor, it carries E_∞-monoids in spaces to E_∞-ring spectra. In particular, we get an E_∞-ring $\Sigma^\infty_+ \mathbb{Z}_{\geqslant 0}$, which we can think of as the 'group algebra' on $\mathbb{Z}_{\geqslant 0}$. Given an even periodic E_∞-ring R, the smash product $R[\mathbb{Z}_{\geqslant 0}] \stackrel{\mathrm{def}}{=} R \otimes \Sigma^\infty_+ \mathbb{Z}_{\geqslant 0}$ is still even periodic, with $\pi_0 R[\mathbb{Z}_{\geqslant 0}] = (\pi_0 R) \otimes_{\mathbb{Z}} \mathbb{Z}[x]$, and the map

$$\mathrm{Spec}\,(\pi_0 R)[x] \longrightarrow M_{FG},$$

associated to $R[\mathbb{Z}_{\geqslant 0}]$ is the one obtained from $\mathrm{Spec}\,\pi_0 R \to M_{FG}$ obtained by taking the product with the constant map $\mathrm{Spec}\,\mathbb{Z}[x] \to \mathrm{Spec}\,\mathbb{Z}$. If R is Landweber-exact, so is $R[\mathbb{Z}_{\geqslant 0}]$.

It follows from this that if $\mathfrak{X} = (X, \mathcal{O}^{\mathrm{top}})$ is an even periodic refinement of a flat map $X \to M_{FG}$, then we get a natural choice of even periodic refinement $\mathbb{A}^1_{\mathfrak{X}}$ of $\mathbb{A}^1_X \to M_{FG}$ (together with a map $\mathbb{A}^1_{\mathfrak{X}} \to \mathfrak{X}$). By Proposition 3.29, if $\mathfrak{X}$ is 0-affine, so is $\mathbb{A}^1_{\mathfrak{X}}$.

7. *Applications to topological modular forms*

In this section, we discuss the primary example that motivated this work, and apply our results in this case.

Let $M_{\overline{\mathrm{ell}}}$ be the moduli stack of stable, 1-pointed genus 1 curves (that is, the Deligne–Mumford compactification of the moduli stack M_{ell} of elliptic curves). A map $\mathrm{Spec}\,R \to M_{\overline{\mathrm{ell}}}$ is equivalent to a flat family of proper curves $p\colon C \to \mathrm{Spec}\,R$ together with a section (or marked point) $e\colon \mathrm{Spec}\,R \to C$ contained in the smooth locus of p, such that each geometric fiber is irreducible of arithmetic genus 1 with at worst nodal singularities. Then $M_{\overline{\mathrm{ell}}}$ is a Deligne–Mumford stack of finite type over $\mathbb{Z}$. See [**13**] for more details.

Given such a curve $C \to \mathrm{Spec}\,R$, one has a canonical group scheme structure on the smooth locus C°, with the marked point as the identity, and taking the formal completion gives a morphism of stacks

$$M_{\overline{\mathrm{ell}}} \longrightarrow M_{FG},$$

which one can check is flat using the Landweber-exact functor theorem (see [**15**, Chapter 4.4]). In this case, one has the fundamental theorem.

THEOREM 7.1 (Goerss–Hopkins–Miller and Lurie). *The stack $M_{\overline{\mathrm{ell}}}$ (together with the map $M_{\overline{\mathrm{ell}}} \to M_{FG}$) admits an even periodic refinement $\mathfrak{M}_{\overline{\mathrm{ell}}}$.*

A construction of $\mathfrak{M}_{\overline{\mathrm{ell}}}$ is detailed in [8], and another is sketched in [34]. In other words, the Goerss–Hopkins–Miller–Lurie theorem states that given a stable 1-pointed genus 1 curve $C \to \operatorname{Spec} R$, such that the classifying map $\operatorname{Spec} R \to M_{\overline{\mathrm{ell}}}$ is étale. (This requires $\operatorname{Spec} R$ to be regular, and the Kodaira–Spencer map at each point of the base to be an isomorphism.) One can build an E_∞-ring spectrum from the associated formal group; moreover, one can do this functorially in the elliptic curve.

Using this derived stack, one defines the spectra of *topological modular forms*:

$$\mathrm{Tmf} = \Gamma(\mathfrak{M}_{\overline{\mathrm{ell}}}, \mathcal{O}^{\mathrm{top}}), \quad \mathrm{TMF} = \Gamma(\mathfrak{M}_{\mathrm{ell}}, \mathcal{O}^{\mathrm{top}}),$$

where $\mathfrak{M}_{\mathrm{ell}} \subset \mathfrak{M}_{\overline{\mathrm{ell}}}$ is the open derived substack corresponding to smooth elliptic curves. These will provide examples of the results in this paper.

When 6 is inverted, the moduli stack $M_{\overline{\mathrm{ell}}}$ is the weighted projective stack $\mathbb{P}(4,6)$, and the homotopy limits necessary to describe Tmf take a simple form. However, the stack $M_{\overline{\mathrm{ell}}}$ is quite complicated at the primes 2 and 3 (that is, there are elliptic curves with relatively large automorphism groups), which contributes to significant torsion at those primes in $\pi_*\mathrm{Tmf}$; moreover, it makes working with Tmf-modules trickier, and it is not a priori clear how well the homotopy limit that builds Tmf behaves. The results of this paper show that the homotopy limit behaves well.

In fact, the idea of this paper arose, in part, from the analysis of the homology of connective $\mathrm{tmf} \stackrel{\mathrm{def}}{=} \tau_{\geqslant 0}\mathrm{Tmf}$ by the first author in [41]. There, working over $\mathbb{Z}_{(2)}$ rather than $\mathbb{Z}$, it was shown that there is a 2-local eight cell complex $DA(1)$ such that the homotopy group sheaf π_0 of $\mathcal{O}^{\mathrm{top}} \otimes DA(1) \in \mathrm{QCoh}(\mathfrak{M}_{\overline{\mathrm{ell}}})$ is given by the pushforward of the structure sheaf via an eightfold cover

$$p\colon \mathbb{P}(1,3) \longrightarrow M_{\overline{\mathrm{ell}}},$$

where the weighted projective stack $\mathbb{P}(1,3)$ is the quotient of a scheme by a $\mathbb{G}_m$-action, and in particular is much simpler cohomologically than $M_{\overline{\mathrm{ell}}}$. Using this, it followed that after smashing with $DA(1)$, the category of quasi-coherent sheaves on $\mathfrak{M}_{\overline{\mathrm{ell}}}$ becomes much better behaved. For example, it was possible to conclude that

$$\Gamma(\mathfrak{M}_{\overline{\mathrm{ell}}}, \mathcal{O}^{\mathrm{top}} \otimes DA(1) \otimes T) \simeq \Gamma(\mathfrak{M}_{\overline{\mathrm{ell}}}, \mathcal{O}^{\mathrm{top}} \otimes DA(1)) \otimes T$$

for any spectrum T, because the spectral sequence to compute the homotopy groups of $\Gamma(\mathfrak{M}_{\overline{\mathrm{ell}}}, \mathcal{F} \otimes DA(1))$ is concentrated in the bottom two rows (in dramatic contrast to the spectral sequence for $\Gamma(\mathfrak{M}_{\overline{\mathrm{ell}}}, \mathcal{O}^{\mathrm{top}})$). In general, the global sections functor Γ is exact, so it commutes with *finite* homotopy colimits and limits, but we cannot a priori expect it to commute with arbitrary homotopy colimits.

Applying the thick subcategory theorem of [25], one may replace $DA(1)$ with the sphere spectrum, and thus show that

$$\Gamma(\mathfrak{M}_{\overline{\mathrm{ell}}}, \mathcal{O}^{\mathrm{top}} \otimes T) \simeq \Gamma(\mathfrak{M}_{\overline{\mathrm{ell}}}, \mathcal{O}^{\mathrm{top}}) \otimes T$$

for all $T \in \mathrm{Sp}$. As an application, it is possible to compute the Tmf-homology of infinite spectra such as MU using the DSS. In this paper, we did not have such finite complexes available to work with, but we used the E_n-spectra themselves to prove analogous results in more generality.

We apply our results to the case of TMF (respectively, Tmf) and the derived stacks $\mathfrak{M}_{\mathrm{ell}}$ (respectively, $\mathfrak{M}_{\overline{\mathrm{ell}}}$) that give rise to them; recall that these are even periodic refinements of the moduli stacks of elliptic curves (respectively, possibly nodal elliptic curves). We will study both the ∞-categories of modules and the Galois theory.

7.1. *Modules over TMF*

Our first main result is the following theorem.

Theorem 7.2. (1) *The ∞-category of* TMF*-modules is equivalent* (*via* Γ) *to the ∞-category of quasi-coherent sheaves on* $\mathfrak{M}_{\mathrm{ell}}$.

(2) *The ∞-category of* Tmf*-modules is equivalent* (*via* Γ) *to the ∞-category of quasi-coherent sheaves on the compactified derived stack* $\mathfrak{M}_{\overline{\mathrm{ell}}}$.

Away from the prime 2, the first part of Theorem 7.2 was originally proved in [**44**]. The result was also known to Lurie.

Proof. Indeed, for the first claim, it suffices by Theorem 4.1 to show that the map

$$M_{\mathrm{ell}} \longrightarrow M_{FG}$$

is affine. To see this, observe that the moduli stack of elliptic curves together with a coordinate to order four on the formal group is precisely $\operatorname{Spec}\mathbb{Z}[a_1, a_2, a_3, a_4, a_6][\Delta^{-1}]$: that is, a choice of coordinate to order four is precisely the data needed to put an elliptic curve in a canonical *Weierstrass form.* See [**46**, Proposition 12.2]. The universal elliptic curve with such a coordinate is given by the equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

and the coordinate on the formal group is given by $-x/y$.

Since the moduli stack of formal groups with a coordinate to order four $M_{FG}^{\leqslant 4}$ is affine over M_{FG}, it follows easily that $M_{\mathrm{ell}} \to M_{FG}$ is affine. Indeed, $\operatorname{Spec} L \times_{M_{FG}} M_{FG}^{\leqslant 4} \times_{M_{FG}} M_{\mathrm{ell}}$ is affine as the diagonal of M_{FG} is affine and $\operatorname{Spec} L \times_{M_{FG}} M_{FG}^{\leqslant 4} \to M_{FG}$ is an affine fpqc cover, for L the Lazard ring.

The map $M_{\overline{\mathrm{ell}}} \to M_{FG}$ is not affine, but it is quasi-affine (and even of cohomological dimension 1). The moduli stack of generalized elliptic curves together with a coordinate to order four is precisely $\operatorname{Spec}\mathbb{Z}[a_1, a_2, a_3, a_4, a_6] \setminus V((c_4, \Delta))$, where c_4, Δ are the standard modular forms evaluated on the cubic curve given by $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. Therefore, we can still apply Theorem 4.1 to the derived stack $\mathfrak{M}_{\overline{\mathrm{ell}}}$ and conclude that it is 0-affine, as desired. □

7.2. *Galois theory*

Next, we study the Galois theory of TMF (respectively, Tmf).

The use of level structures provides various covers of the moduli stack of elliptic curves that rigidify the 'stackiness'. These can be realized topologically.

Fix a positive integer n.

Definition 7.3. Let $M_{\mathrm{ell}}(n)$ be the moduli stack (over $\mathbb{Z}[1/n]$) of elliptic curves with a *level n structure:* that is, if S is a scheme where n is invertible, then maps

$$S \longrightarrow M_{\mathrm{ell}}(n)$$

are given by (smooth) elliptic curves $p\colon C \to S, 0\colon S \to C$ together with sections $\phi_1, \phi_2\colon S \to C$ contained in the n-torsion subgroup $C[n] \subset C$, such that, over each geometric fiber $C_{\overline{s}}$, for $s \in S$, the sections ϕ_1, ϕ_2 form a basis for the n-torsion $C_{\overline{s}}[n] \simeq (\mathbb{Z}/n\mathbb{Z})^2$.

Then $M_{\mathrm{ell}}(n)$ is étale over $M_{\mathrm{ell}}[1/n]$, and in fact the natural forgetful map

$$M_{\mathrm{ell}}(n) \longrightarrow M_{\mathrm{ell}}[1/n],$$

is a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-torsor, where the $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ acts on $M_{\mathrm{ell}}(n)$ by matrix multiplication on the level structure. It follows that the composite map

$$M_{\mathrm{ell}}(n) \longrightarrow M_{\mathrm{ell}}[1/n] \longrightarrow M_{FG},$$

is flat, and $M_{\mathrm{ell}}(n)$ is realizable by a derived stack $\mathfrak{M}_{\mathrm{ell}}(n)$ over $\mathfrak{M}_{\mathrm{ell}}$.

DEFINITION 7.4. The global sections $\Gamma(\mathfrak{M}_{\mathrm{ell}}(n)[1/n], \mathcal{O}^{\mathrm{top}})$ are called *TMF of level n* and are denoted by $\mathrm{TMF}(n)$.

It follows in particular that $\mathrm{TMF}(n)$ has a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-action, and that

$$\mathrm{TMF}[1/n] \simeq \mathrm{TMF}(n)^{h\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})}.$$

For $n \geqslant 3$, $M_{\mathrm{ell}}(n)$ is actually an affine scheme, and the resulting spectra $\mathrm{TMF}(n)$ are therefore Landweber-exact, even periodic E_∞-rings.

EXAMPLE 7.5. The moduli stack (over $\mathbb{Z}[1/2]$) of elliptic curves together with a full level 2-structure is given by $\mathrm{Spec}\,\mathbb{Z}[1/2,\lambda][\lambda^{-1},(\lambda-1)^{-1}] \times B\mathbb{Z}/2$, given by putting the elliptic curve in 'Legendre form'

$$y^2 = x(x-1)(x-\lambda), \quad \lambda \neq 0, 1,$$

together with the 2-torsion points $(0,0),(0,1)$. The $B\mathbb{Z}/2$ factor is necessary to account for the automorphism -1.

Since $M_{\mathrm{ell}}(2)[\frac{1}{2}]$ is not affine (in fact, not even a scheme), the spectrum $\mathrm{TMF}(2)$ is not even periodic, but only 4-periodic, with homotopy groups given by

$$\pi_*\mathrm{TMF}(2)[\tfrac{1}{2}] \simeq \mathbb{Z}[\tfrac{1}{2},\lambda,t][\lambda^{-1},(\lambda-1)^{-1},t^{-1}], \quad |\lambda| = 0, |t| = 4.$$

The $S_3 \simeq \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$-Galois descent from $\mathrm{TMF}(2)[\frac{1}{2}]$ to $\mathrm{TMF}[\frac{1}{2}]$ is studied in detail in [**50**].

By taking various partial quotients of $M_{\mathrm{ell}}(n)$ over M_{ell}, one can realize other variants of 'moduli of elliptic curves with level structure'. For instance, let $M_{\mathrm{ell},1}(n)$ be the moduli stack of elliptic curves with a $\Gamma_1(n)$-structure: that is, a choice of an n-torsion point that generates a $\mathbb{Z}/n\mathbb{Z}$-summand in the n-torsion on each fiber. Then $M_{\mathrm{ell},1}(n) \simeq M_{\mathrm{ell}}(n)/H$ where $H \subset GL_2(\mathbb{Z}/n\mathbb{Z})$ consists of matrices of the form $\left[\begin{smallmatrix} 1 & b \\ 0 & c\end{smallmatrix}\right]$. The stack $M_{\mathrm{ell},1}(n)$ is étale (though no longer Galois) over $M_{\mathrm{ell}}[1/n]$ and can consequently be realized by a derived stack, whose E_∞-ring of global sections is denoted by $\mathrm{TMF}_1(n)$. Similarly, one defines $\mathrm{TMF}_0(n)$ from the moduli stack of elliptic curves together with a cyclic degree n subgroup.

Using the 0-affineness of $\mathfrak{M}_{\mathrm{ell}}$, Proposition 3.29 and Theorem 5.8, we find the following theorem.

THEOREM 7.6. *The map* $\mathrm{TMF}[1/n] \to \mathrm{TMF}(n)$ *is a faithful* $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$*-Galois extension. Similarly, the map* $\mathrm{TMF}_0(n) \to \mathrm{TMF}_1(n)$ *is a faithful* $(\mathbb{Z}/n\mathbb{Z})^\times$*-Galois extension. In particular, the Tate spectra of these group actions vanish.*

The vanishing of Tate spectra in the latter case is proved for $n = 5$ via different means in [**9**].

REMARK 7.7. One can show in fact that *all* Galois covers of TMF and its localizations arise from Galois extensions of the associated stack; in particular, TMF over $\mathbb{Z}$ is 'separably closed', that is, has no non-trivial Galois extensions. This is carried out in [**42**, Section 10].

7.3. *Tate spectra and compactified moduli*

Our earlier results showed that $\mathrm{TMF}[1/n] \to \mathrm{TMF}(n)$ is a faithful $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-Galois extension; in particular, the Tate spectrum for the action of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ on the latter is contractible.

In this subsection, we show the analogous Tate spectra for the non-periodic versions of TMF(n) also vanish. The associated extensions are no longer Galois, as the associated covers of stacks are now ramified. However, we will still be able to apply Theorem 5.10.

Recall that it is useful to compactify the moduli stack $M_{\mathrm{ell}}(n)$ by allowing the elliptic curve to degenerate, although we will need to drop irreducibility and allow slightly more complicated degenerations: instead of $\mathbb{P}^1$ with two points glued together (a nodal cubic), we need to allow *Néron n-gons*, which are obtained by gluing n copies of $\mathbb{P}^1$, where 0 in the ith $\mathbb{P}^1$ (for $i \in \mathbb{Z}/n\mathbb{Z}$) is attached to ∞ in the $(i+1)$st. This theory was developed in [**13**]; another helpful reference (which extends the theory to the cusps in characteristics dividing n, which we do not need) is [**12**].

Definition 7.8. Let $M_{\mathrm{ell}}^{(n)}$ be the moduli stack that assigns to a $\mathbb{Z}[1/n]$-scheme S the groupoid of *generalized elliptic curves* [**13**, Chapter II] $p\colon C \to S$, such that each geometric fiber of p is either smooth or an n-gon.

We do not review the definition of a generalized elliptic curve, except to note that it requires more than a curve over the base S together with a section: the group structure (on the smooth locus $C^\circ \subset C$) must be part of the data, rather than a consequence of the definition. In [**13**, Theorem 2.5, Chapter III], it is shown that $M_{\mathrm{ell}}^{(n)}$ is a smooth Deligne–Mumford stack of finite type over $\operatorname{Spec}\mathbb{Z}[1/n]$. Moreover, there is a morphism

$$M_{\mathrm{ell}}^{(n)} \longrightarrow M_{\overline{\mathrm{ell}}}[1/n],$$

which sends a generalized elliptic curve $C \to S$ to the stable elliptic curve $\overline{C} \to S$ obtained by fiberwise *contracting* all irreducible components not containing the identity section. (This process is discussed in [**13**, Section IV.1].)

In particular, $M_{\mathrm{ell}}^{(n)} \to M_{\overline{\mathrm{ell}}}[1/n]$ is an equivalence of stacks away from the 'cusps'. Near the cusps, it fails to be representable: the automorphism group scheme of a Néron n-gon is a semidirect product $\mathbb{Z}/2\mathbb{Z} \ltimes \mu_n$ ([**13**, Section 2, Proposition 1.10]), while the automorphism group scheme of a nodal elliptic curve (that is, a Néron 1-gon) is simply $\mathbb{Z}/2\mathbb{Z}$. However, using $M_{\mathrm{ell}}^{(n)}$, one can construct the compactification of $M_{\mathrm{ell}}(n)$.

Definition 7.9 ([**12**, **13**]). The stack $M_{\overline{\mathrm{ell}}}(n)$ classifies *generalized elliptic curves* $p\colon C \to S$ over a base S with n invertible, such that each geometric fiber of p is either smooth or a Néron n-gon, together with an isomorphism of group schemes $\phi\colon (\mathbb{Z}/n\mathbb{Z})^2 \simeq C^\circ[n]$ (that is, a trivialization of the n-torsion points on the smooth locus).

Similarly, one defines $M_{\overline{\mathrm{ell}},1}(n)$, a compactification of the moduli stack of elliptic curves with $\Gamma_1(n)$-structure, to classify generalized elliptic curves over a base S (of the same form) with an injection of group schemes $\mathbb{Z}/n\mathbb{Z} \to C^\circ[n]$ such that the divisor cut out by the image of $\mathbb{Z}/n\mathbb{Z}$ is ample (that is, intersects each irreducible component in every geometric fiber). One also defines $M_{\overline{\mathrm{ell}},0}(n)$, a compactification of the moduli stack of elliptic curves with $\Gamma_0(n)$-structure, to classify generalized elliptic curves over a base S with a (finite flat) subgroup $G \subset C^\circ[n]$ which is ample and which is étale locally isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Here, the moduli interpretation of $M_{\overline{\mathrm{ell}},1}(n)$ and $M_{\overline{\mathrm{ell}},0}(n)$ is from [**12**]. Note that while Conrad only requires an fppf-local generator, in our situation we have actually an étale-local generator as we assume that n is invertible so that G is étale and étale locally free (see, for example, [**49**, Theorem 34 and discussion below Theorem 33]).

As $M_{\mathrm{ell}}(n)$ is to $M_{\mathrm{ell}}[1/n]$, the stack $M_{\overline{\mathrm{ell}}}(n)$ lives as a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-torsor over $M_{\mathrm{ell}}^{(n)}$. The moduli stack $M_{\overline{\mathrm{ell}}}(n)$ is a smooth Deligne–Mumford stack over $\mathbb{Z}[1/n]$. There is a morphism of stacks

$$M_{\overline{\mathrm{ell}}}(n) \longrightarrow M_{\overline{\mathrm{ell}}}[1/n],$$

which sends a pair $p\colon C \to S, \phi\colon (\mathbb{Z}/n\mathbb{Z})^2 \simeq C^\circ[n]$ as above (over some base S) to the stable elliptic curve over S obtained from C by fiberwise *contracting* all non-identity irreducible components. This map is naturally equivariant for the natural $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-action on the source and the trivial $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-action on the target. This comes from the map $M_{\overline{\mathrm{ell}}}^{(n)} \to M_{\overline{\mathrm{ell}}}[1/n]$.

In [**13**], it is shown that $M_{\overline{\mathrm{ell}}}(n) \to M_{\overline{\mathrm{ell}}}$ is finite and flat. It fails to be étale over the cusps, and the existence of a topological realization is not a direct consequence of the existence of Tmf. However, one has the following theorem.

THEOREM 7.10 (Goerss–Hopkins and Hill–Lawson [**23**]). *The moduli stack $M_{\overline{\mathrm{ell}}}(n)$ has an even periodic refinement $\mathfrak{M}_{\overline{\mathrm{ell}}}(n)$, in such a way that*

$$\mathfrak{M}_{\overline{\mathrm{ell}}}(n) \longrightarrow \mathfrak{M}_{\overline{\mathrm{ell}}},$$

is a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-equivariant morphism of derived stacks. Equivalently, $M_{\overline{\mathrm{ell}}}^{(n)} \to M_{\overline{\mathrm{ell}}} \to M_{FG}$ has an even periodic refinement.

In particular, it is possible to construct E_∞-algebras

$$\mathrm{Tmf}(n) \overset{\mathrm{def}}{=} \Gamma(\mathfrak{M}_{\overline{\mathrm{ell}}}(n), \mathcal{O}^{\mathrm{top}})$$

over Tmf, which acquire $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-actions. Similarly, one defines even periodic refinements of $M_{\overline{\mathrm{ell}},1}(n), M_{\overline{\mathrm{ell}},0}(n)$, and obtains E_∞-rings $\mathrm{Tmf}_0(n), \mathrm{Tmf}_1(n)$, where $\mathrm{Tmf}_1(n)$ has a $(\mathbb{Z}/n\mathbb{Z})^\times$-action with homotopy fixed points given by $\mathrm{Tmf}_0(n)$.

Our first main result in this section is the following theorem.

THEOREM 7.11. *The Tate spectrum of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ on $\mathrm{Tmf}(n)$ is contractible.*

For $n = 2$, this result appears in [**50**].

Proof. This is a consequence of Theorem 5.10, once we show that the map

$$M_{\overline{\mathrm{ell}}}^{(n)} \longrightarrow M_{FG}$$

is quasi-compact, separated and tame.

To see this, it suffices to check at the level of stabilizers. Away from the cusps, there is no issue: the stabilizers of $M_{\overline{\mathrm{ell}}}^{(n)}$ (equivalently, of M_{ell}) inject into those of M_{FG}. At the cusps, we recall that the automorphism group of a Néron n-gon is a semidirect product $(\mathbb{Z}/2) \rtimes \mu_n$, where the $\mathbb{Z}/2$ piece (which acts by inversion) injects into the associated stabilizer for M_{FG}. Since we have inverted n, it follows that the kernels of the maps of stabilizers are invertible and the map is tame. Thus by Theorem 5.10, we are done. □

For $\mathrm{Tmf}_1(n)$, the situation is even better.

THEOREM 7.12. *The map $\mathrm{Tmf}_0(n) \to \mathrm{Tmf}_1(n)$ is a faithful $(\mathbb{Z}/n\mathbb{Z})^\times$-Galois extension.*

Proof. By [**12**, 4.1.1], $M_{\overline{\mathrm{ell}},0}(n)$ is finite over $M_{\overline{\mathrm{ell}}}$. While he states it only for n squarefree, it is also true if n is invertible, as in our setting. By Proposition 3.29, Theorem 5.8 and the 0-affineness of $M_{\overline{\mathrm{ell}}}$, it is enough to show that $M_{\overline{\mathrm{ell}},1}(n) \to M_{\overline{\mathrm{ell}},0}(n)$ is a $(\mathbb{Z}/n\mathbb{Z})^\times$-Galois cover.

More precisely, we claim that we have a cartesian square

$$\begin{array}{ccc} M_{\overline{\mathrm{ell}},1}(n) & \longrightarrow & \operatorname{Spec}\mathbb{Z}[\frac{1}{n}] \\ \downarrow & & \downarrow \\ M_{\overline{\mathrm{ell}},0}(n) & \longrightarrow & \operatorname{Spec}\mathbb{Z}[\frac{1}{n}]/(\mathbb{Z}/n\mathbb{Z})^{\times}. \end{array}$$

As $\mathbb{Z}[1/n] \to \mathbb{Z}[1/n]/(\mathbb{Z}/n\mathbb{Z})^{\times}$ is a $(\mathbb{Z}/n\mathbb{Z})^{\times}$-torsor, this is sufficient.

We first have to define the map $M_{\overline{\mathrm{ell}},0}(n) \to \operatorname{Spec}\mathbb{Z}[1/n]/(\mathbb{Z}/n\mathbb{Z})^{\times}$. The target classifies étale $(\mathbb{Z}/n\mathbb{Z})^{\times}$-torsors over a scheme S with n invertible on S. The datum of an étale $(\mathbb{Z}/n\mathbb{Z})^{\times}$-torsor over S is equivalent to that of an S-group scheme that is étale locally isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Indeed, the latter is given by an étale cover $U \to S$ together with a section of the constant group scheme $\operatorname{Aut}(\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^{\times}$ on $U \times_S U$ satisfying a cocycle condition (by descent). Exactly the same datum defines an étale $(\mathbb{Z}/n\mathbb{Z})^{\times}$-torsor over S. Thus, $\mathbb{Z}[1/n]/(\mathbb{Z}/n\mathbb{Z})^{\times}$ classifies group schemes over S (for n invertible on S) which are étale locally isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Recall that a $\Gamma_0(n)$-level structure consists of a subgroup $G \subset C^{\circ}[n]$ which is étale locally isomorphic to $\mathbb{Z}/n\mathbb{Z}$ and which is an ample divisor. Thus, we get a map $M_{\overline{\mathrm{ell}},0}(n) \to \operatorname{Spec}\mathbb{Z}[\frac{1}{n}]/(\mathbb{Z}/n\mathbb{Z})^{\times}$. The fiber product $M_{\overline{\mathrm{ell}},0}(n) \times_{\operatorname{Spec}\mathbb{Z}[\frac{1}{n}]/(\mathbb{Z}/n\mathbb{Z})^{\times}} \operatorname{Spec}\mathbb{Z}[1/n]$ classifies generalized elliptic curves with a subgroup $G \subset C^{\circ}[n]$, which is an ample divisor together with a chosen isomorphism $G \cong \mathbb{Z}/n\mathbb{Z}$. This is exactly a generalized elliptic curve with a $\Gamma_1(n)$-structure. □

Appendix A. *Homotopy limits and sheaves*

Let $U \cup V = X$ be an open covering of a topological space. Let $\mathcal{F}$ be a sheaf on X with values in an ∞-category. Is then the square

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \cap V) \end{array}$$

(homotopy) cartesian? A priori, we can compute $\mathcal{F}(X)$ only as the homotopy limit over the (infinite) Cech cosimplicial object associated to the covering. Nevertheless, in this appendix, we will give a positive answer to the question for arbitrary finite covers. This is used in Proposition 3.17, where we need to compute $\mathcal{F}(X)$ as a finite homotopy limit. Our strategy is first to compare an ordered and an unordered version of the Cech cosimplicial object for a sheaf on a space. At least in outline, this material is surely known to the experts.

Let X be a topological space and let $(U_i)_{i\in I}$ be open subsets covering X. Let I be finite of cardinality n and totally ordered. For a tuple $\underline{i} = (i_1, \dots, i_k) \in I^k$, denote by $U_{\underline{i}}$ the intersection of $U_{i_1}, \dots, U_{i_k}$. Let $I^k_{\leqslant}$ be the set of weakly increasing k-tuples (that is, $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$).

To these data, we can associate (at least) two simplicial objects:

$$\mathfrak{C}^U_\bullet : k \longmapsto \coprod_{\underline{i}\in I^{k+1}} U_{\underline{i}} \cong \left(\coprod_{i\in I} U_i\right)^{\times_X k+1},$$

$$\mathfrak{C}^{U,\leqslant}_\bullet : k \longmapsto \coprod_{\underline{i}\in I^{k+1}_{\leqslant}} U_{\underline{i}}.$$

The face maps are given by leaving out elements and the degeneracies by repeating elements. There is an obvious simplicial map $e\colon \mathfrak{C}^{U,\leqslant}_\bullet \to \mathfrak{C}^U_\bullet$.

Note that we are given a presheaf $\mathcal{F}$ on X, we can evaluate $\mathcal{F}$ on a disjoint union $\coprod_{\underline{i}\in I^k} U_{\underline{i}}$ by setting $\mathcal{F}(\coprod_{\underline{i}\in I^k} U_{\underline{i}}) = \prod_{\underline{i}\in I^k} \mathcal{F}(U_{\underline{i}})$. We view the disjoint unions here as formal and never identify the disjoint union of more than one open subsets of X with an open subset of X.

The following proof is inspired by [**16**, Proposition 2.7].

Proposition A.13. *Let* Top *be the category of topological spaces. Let $\mathcal{F}$ be a* Top*-valued presheaf on X. Then the canonical map*

$$e^*\colon \operatorname{Tot}\mathcal{F}(\mathfrak{C}^U_\bullet) \longrightarrow \operatorname{Tot}\mathcal{F}(\mathfrak{C}^{U,\leqslant}_\bullet)$$

is a homotopy equivalence.

Proof. For each multi-index $\underline{i} = (i_0, \dots, i_m)$, there is a unique permutation $\sigma_{\underline{i}} \in S_{m+1}$ with

(1) $i_{\sigma_{\underline{i}}(0)} \leqslant i_{\sigma_{\underline{i}}(1)} \leqslant \cdots \leqslant i_{\sigma_{\underline{i}}(m)}$ and
(2) $\sigma_{\underline{i}}^{-1}(k) < \sigma_{\underline{i}}^{-1}(l)$ if $i_k = i_l$ for some $k < l$.

This will allow us to define an inverse map $g\colon \operatorname{Tot}\mathcal{F}(\mathfrak{C}^{U,\leqslant}_\bullet) \to \operatorname{Tot}\mathcal{F}(\mathfrak{C}^U_\bullet)$ to e^*:

Recall that $\operatorname{Tot}\mathcal{F}(\mathfrak{C}^U_\bullet) \subset \prod_{[m]\in\Delta} \mathcal{F}(\mathfrak{C}^U_m)^{\Delta^m}$ consists of all maps of cosimplicial topological spaces $\Delta^\bullet \to \mathcal{F}(\mathfrak{C}^U_\bullet)$. Thus, to give a map into $\operatorname{Tot}\mathcal{F}(\mathfrak{C}^U_\bullet)$ is equivalent to giving maps into all $\mathcal{F}(U_{\underline{i}})^{\Delta^m}$, $\underline{i} : [m] \to I$ a multi-index, compatible with coface and codegeneracy maps. Given a multi-index $\underline{i} : [m] \to I$, we define the map $g_{\underline{i}} : \operatorname{Tot}\mathcal{F}(\mathfrak{C}^{U,\leqslant}_\bullet) \to \mathcal{F}(U_{\underline{i}})^{\Delta^m}$ as the composition

$$\operatorname{Tot}\mathcal{F}(\mathfrak{C}^{U,\leqslant}_\bullet) \xrightarrow{\operatorname{pr}_{\underline{i}\sigma_{\underline{i}}}} \mathcal{F}(U_{\underline{i}\sigma_{\underline{i}}})^{\Delta^m} \xrightarrow{\sigma_{\underline{i}}^*} \mathcal{F}(U_{\underline{i}\sigma_{\underline{i}}})^{\Delta^m} \xrightarrow{=} \mathcal{F}(U_{\underline{i}})^{\Delta^m}.$$

Here, $\sigma_{\underline{i}}$ sends a point $(t_0, \dots, t_m) \in \Delta^m$ to $(t_{\sigma_{\underline{i}}(0)}, t_{\sigma_{\underline{i}}(1)}, \dots, t_{\sigma_{\underline{i}}(m)})$. We will only check compatibility with coface maps. Let $f \in \operatorname{Tot}\mathcal{F}(\mathfrak{C}^{U,\leqslant}_\bullet)$. We want to show that the diagram

$$\begin{array}{ccc} \Delta^{m-1} & \xrightarrow{g_{\underline{i}d^j}(f)} & \mathcal{F}(U_{\underline{i}d^j}) \\ \downarrow{\scriptstyle d^j} & & \downarrow{\scriptstyle d^j_{\underline{i}}} \\ \Delta^m & \xrightarrow{g_{\underline{i}}(f)} & \mathcal{F}(U_{\underline{i}}) \end{array}$$

commutes for $d^j : [m-1] \to [m]$. Here, $d^j_{\underline{i}}$ denotes $\mathcal{F}(\mathfrak{C}^U_{m-1}) \xrightarrow{d^j} \mathcal{F}(\mathfrak{C}^U_m) \xrightarrow{\operatorname{pr}_{\underline{i}}} \mathcal{F}(U_{\underline{i}})$, factoring through $\mathcal{F}(U_{\underline{i}d^j})$.

By definition, this is the outer part of the rectangle

$$\begin{array}{ccccccc} \Delta^{m-1} & \xrightarrow{\tau} & \Delta^{m-1} & \xrightarrow{f_{\underline{i}d^j\tau}} & \mathcal{F}(U_{\underline{i}d^j\tau}) = \mathcal{F}(U_{\underline{i}\sigma d^{\sigma^{-1}(j)}}) & \xrightarrow{=} & \mathcal{F}(U_{\underline{i}d^j}) \\ \downarrow{\scriptstyle d^j} & & \downarrow{\scriptstyle d^{\sigma^{-1}(j)}} & & \downarrow{\scriptstyle d^{\sigma^{-1}(j)}_{\underline{i}\sigma}} & & \downarrow{\scriptstyle d^j_{\underline{i}}} \\ \Delta^m & \xrightarrow{\sigma} & \Delta^m & \xrightarrow{f_{\underline{i}\sigma}} & \mathcal{F}(U_{\underline{i}\sigma}) & \xrightarrow{=} & \mathcal{F}(U_{\underline{i}}). \end{array}$$

Here, $\sigma = \sigma_{\underline{i}}$ and $\tau = \sigma_{\underline{i}d^j}$ for short. We claim that all the small squares commute (and make sense).

One can check that $d^{\sigma^{-1}(j)}\tau^{-1} = \sigma^{-1}d^j$. This gives the commutativity of the first square (note how the permutations become inverted). The equality in the upper right corner of the next square follows from $d^j\tau = \sigma d^{\sigma^{-1}(j)}$. The commutativity of this square follows since $f \in \operatorname{Tot}\mathcal{F}(\mathfrak{C}^{U,\leqslant}_\bullet)$. In the last square, the vertical morphisms are induced by inclusions between the same open subsets and thus have to be equal. The proof for the codegeneracies is similar. Thus, we get a well-defined map

$$g : \operatorname{Tot}\mathcal{F}(\mathfrak{C}^{U,\leqslant}_\bullet) \longrightarrow \operatorname{Tot}\mathcal{F}(\mathfrak{C}^U_\bullet).$$

The composition e^*g equals the identity. We want to show that ge^* is homotopic to the identity. For a permutation $\sigma \in S_{m+1}$ and an element $s \in [0,1]$, define a map $u_{\sigma,s} : \Delta^m \to \Delta^{2m+1}$ by

$$(t_0, \ldots, t_m) \mapsto (st_0, \ldots, st_m, (1-s)t_{\sigma(0)}, \ldots, (1-s)t_{\sigma(m)}).$$

Then for $\sigma = \sigma_{\underline{i}}$, we define maps

$$\operatorname{Tot} \mathcal{F}(\mathfrak{C}^U_\bullet) \times [0,1] \longrightarrow \mathcal{F}(U_{i_0 i_1 \cdots i_m})^{\Delta^m}$$

as the composition

$$\begin{array}{c}
\operatorname{Tot} \mathcal{F}(\mathfrak{C}^U_\bullet) \times [0,1] \\
\big\downarrow \scriptstyle \mathrm{pr}_{i_0 \cdots i_m i_{\sigma(0)} \cdots i_{\sigma(m)}} \times u_{(\sigma,\bullet)} \\
\mathcal{F}(U_{i_0 \cdots i_m i_{\sigma(0)} \cdots i_{\sigma(m)}})^{\Delta^{2m+1}} \times \operatorname{Map}(\Delta^m, \Delta^{2m+1}) \\
\big\downarrow \\
\mathcal{F}(U_{i_0 \cdots i_m i_{\sigma(0)} \cdots i_{\sigma(m)}})^{\Delta^m} \\
\big\downarrow = \\
\mathcal{F}(U_{i_0 i_1 \cdots i_m})^{\Delta^m}.
\end{array}$$

This defines the required homotopy $H : \operatorname{Tot} \mathcal{F}(\mathfrak{C}^U_\bullet) \times [0,1] \to \operatorname{Tot} \mathcal{F}(\mathfrak{C}^U_\bullet)$ between id and ge^*, once we have checked the compatibility with coface and codegeneracies. We will only treat the coface maps again. We set $\tau = \sigma_{\underline{i}d^j}$ again and choose $(f,s) \in \operatorname{Tot} \mathcal{F}(\mathfrak{C}^U_\bullet) \times [0,1]$. Furthermore, for functions $a, b : [m] \to I$ we use the notation $a|b : [2m+1] = [m] \sqcup [m] \to I$ for the sum of a, b. For example, $\underline{i}|\underline{i}\sigma = (i_0, \ldots, i_m, i_{\sigma(0)}, \ldots, i_{\sigma(m)})$. The compatibility follows from the commutative diagram

$$\begin{array}{ccccccc}
\Delta^{m-1} & \xrightarrow{u_{\tau,s}} & \Delta^{2m-1} & \xrightarrow{f_{\underline{i}d^j | \underline{i}d^j \tau}} & \mathcal{F}(U_{\underline{i}d^j | \underline{i}d^j \tau}) & \xrightarrow{=} & \mathcal{F}(U_{\underline{i}d^j}) \\
\big\downarrow{\scriptstyle d^j} & & \big\downarrow{\scriptstyle d^j \sqcup d^{\sigma^{-1}(j)}} & & \big\downarrow{\scriptstyle d^j_{\underline{i}} \sqcup d^{\sigma^{-1}(j)}_{\underline{i}\sigma}} & & \big\downarrow{\scriptstyle d^j_{\underline{i}}} \\
\Delta^m & \xrightarrow{u_{\sigma,s}} & \Delta^{2m+1} & \xrightarrow{f_{\underline{i}|\underline{i}\sigma}} & \mathcal{F}(U_{\underline{i}|\underline{i}\sigma}) & \xrightarrow{=} & \mathcal{F}(U_{\underline{i}}).
\end{array}$$

The commutativity is shown as above. □

Note that we did this proof in a topological and not in a simplicial setting since the symmetric group S_{m+1} acts on the topological m-simplex, but not on the simplicial m-simplex.

As a preparation for the following proof, we note that $\mathfrak{C}^U_\bullet$ and $\mathfrak{C}^{U,\leqslant}_\bullet$ have free degeneracies in the sense of the following definition.

Definition A.14. Let $\mathcal{C}$ be a category with coproducts. A $\mathcal{C}$-valued simplicial object $X_\bullet$ is said to have *free degeneracies* if there exist maps $N_k \to X_k$ from $N_k \in \mathcal{C}$ such that the canonical map

$$\coprod_{\sigma : [k] \twoheadrightarrow [m]} N_m \longrightarrow X_k$$

is an isomorphism for every k.

Equivalently, the restriction of $X_\bullet$ to $(\Delta_{\mathrm{epi}})^{\mathrm{op}}$ is isomorphic to the left Kan extension of $N : \mathbb{N}_0 \to \mathcal{C}$ along $\mathbb{N}_0 \to (\Delta_{\mathrm{epi}})^{\mathrm{op}}$. Here, Δ_{epi} is the subcategory of Δ consisting of order-preserving epimorphism.

Both $\mathfrak{C}_\bullet^{U,\leqslant}$ and $\mathfrak{C}_\bullet^U$ have free degeneracies: In the case of $\mathfrak{C}_\bullet^{U,\leqslant}$, we choose $N_k = \coprod_{i_0<i_1<\cdots<i_k} U_{i_0i_1\cdots i_k}$. In the case of $\mathfrak{C}_\bullet^U$, we choose $N_k = \coprod_{i_0\neq i_1\neq i_2\neq\cdots\neq i_k} U_{i_0i_1\cdots i_k}$. Here, we really do not mean pairwise inequality, but just $i_l \neq i_{l+1}$. This can be refined to the following statement, which we will use for Corollary A.17.

Lemma A.15. *Define a functor* $\mathfrak{C}_\bullet^{U,<} : \Delta^{\mathrm{op}}_{\mathrm{mono},\leqslant n-1} \to \mathrm{Top}$ *by* $\mathfrak{C}_k^{U,<} = \coprod_{i_0<i_1<\cdots<i_k} U_{i_0i_1\cdots i_k}$. *Then the canonical map* $\mathrm{LKan}_F \mathfrak{C}_\bullet^{U,<} \to \mathfrak{C}_\bullet^{U,\leqslant}$ *along the functor* $F : \Delta^{\mathrm{op}}_{\mathrm{mono},\leqslant n-1} \to \Delta^{\mathrm{op}}$ *is an isomorphism.*

Here, $\Delta_{\mathrm{mono},\leqslant n-1}$ denotes the subcategory of Δ consisting of order-preserving monomorphisms between $[0],\ldots,[n-1]$. Note furthermore that a similar statement with Δ_{mono} instead of $\Delta_{\mathrm{mono},\leqslant n-1}$ is also true for $\mathfrak{C}_\bullet^U$.

Proof. By definition

$$(\mathrm{LKan}_F \mathfrak{C}_\bullet^{U,<})([k]) = \mathrm{colim}_{[k]\longrightarrow[l], l\leqslant n-1} \mathfrak{C}_l^{U,<},$$

where a morphism in the index category between $f : [k] \to [l]$ and $g : [k] \to [l']$ consists of an injection $i : [l'] \to [l]$ such that $f = i \circ g$. As every morphism in Δ factors uniquely as an epimorphism followed by a monomorphism, the full subcategory on all $[k] \twoheadrightarrow [l], l \leqslant n-1$ is final. As this category is discrete, the result follows. □

Let $X_\bullet$ be again a simplicial object in a category $\mathcal{C}$ (with coproducts) and $\mathcal{F}$ be a product-preserving functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Top}$. Assume that $X_\bullet$ has free degeneracies with maps $N_k \to X_k$ as above. Then the mth matching object of $\mathcal{F}(X_\bullet)$ is isomorphic to

$$\lim_{[m]\twoheadrightarrow[k], k<m} \prod_{[k]\twoheadrightarrow[l]} \mathcal{F}(N_l) \cong \prod_{[m]\twoheadrightarrow[l], l<m} \mathcal{F}(N_l).$$

Thus, the matching map

$$\mathcal{F}(X_m) \cong \prod_{[m]\twoheadrightarrow[l]} \mathcal{F}(N_l) \longrightarrow \prod_{[m]\twoheadrightarrow[l], l<m} \mathcal{F}(N_l)$$

is a projection and thus a fibration.

Corollary A.16. *Let* $\mathcal{F}$ *be a* Top*-valued presheaf on* X. *Then the canonical map*

$$e^* : \mathrm{holim}_\Delta \mathcal{F}(\mathfrak{C}_\bullet^U) \longrightarrow \mathrm{holim}_\Delta \mathcal{F}(\mathfrak{C}_\bullet^{U,\leqslant})$$

is a homotopy equivalence.

Proof. The relevant cosimplicial objects are Reedy fibrant by the discussion preceding this corollary. Thus, the statement follows from Proposition A.13. □

Corollary A.17. *Let* $\mathcal{F}$ *be a sheaf on* X *with values in a complete* ∞*-category* $\mathcal{C}$. *Then the maps*

$$\mathcal{F}(X) \xrightarrow{\simeq} \mathrm{holim}_\Delta \mathcal{F}(\mathfrak{C}_\bullet^U) \longrightarrow \mathrm{holim}_\Delta \mathcal{F}(\mathfrak{C}_\bullet^{U,\leqslant}),$$

induced by the inclusion $\mathfrak{C}_\bullet^{U,\leqslant} \to \mathfrak{C}_\bullet^U$, *and*

$$\mathrm{holim}_\Delta \mathcal{F}(\mathfrak{C}_\bullet^{U,\leqslant}) \longrightarrow \mathrm{holim}_{\Delta_{\mathrm{mono},\leqslant n-1}} \mathcal{F}(\mathfrak{C}_\bullet^{U,<}),$$

induced by the inclusions $\Delta_{\mathrm{mono},\leqslant n-1} \to \Delta$ *and* $\mathfrak{C}_\bullet^{U,<} \to \mathfrak{C}_\bullet^{U,\leqslant}$, *are equivalences.*

Proof. By the last corollary, the first map is an equivalence if $\mathcal{C}$ is the ∞-category $\mathcal{S}$ of spaces. Indeed, $\mathcal{S}$ is equivalent to the coherent nerve of the simplicial category of Kan simplicial sets. Given an $\mathcal{S}$-valued sheaf $\mathcal{F}$, this can be thus strictified to a presheaf of simplicial sets on X by [**33**, Theorem 4.2.4.4]. Its geometric realization is a presheaf of topological spaces. As geometric realization commutes with homotopy limits and homotopy limits in $\mathcal{S}$ can be computed as homotopy limits in simplicial sets by [**33**, Theorem 4.2.4.1], we can apply the last corollary.

Thus, the first part of this corollary follows also for the ∞-category of presheaves $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ for an arbitrary ∞-category $\mathcal{C}$. As the canonical map $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ is a (homotopy) limit-preserving embedding [**33**, Propositions 5.1.3.1 and 5.1.3.2], the corollary follows for an arbitrary complete ∞-category $\mathcal{C}$.

The second part follows as we can see from (the proof of) Lemma A.15 that $\mathcal{F}(\mathfrak{C}_\bullet^{U,\leqslant})$ is (homotopy) right Kan extended from $\mathcal{F}(\mathfrak{C}_\bullet^{U,<})$, first for the ∞-category of spaces and then for all complete ∞-categories as above, so the homotopy limits agree. □

For example, if $X = U \cup V$, then this implies that $\mathcal{F}(X)$ is the homotopy equalizer of

$$\mathcal{F}(U) \times \mathcal{F}(V) \rightrightarrows \mathcal{F}(U \cap V).$$

This formulation is all we need for this article, but to answer the question posed at the beginning of this appendix, we introduce one further reformulation of this homotopy limit.

Let $\mathcal{P}_I$ be the poset of non-empty subsets of I. Each subset of I with k elements has a unique order-preserving bijection to $[k-1]$. This defines a functor $G : \mathcal{P}_I \to \Delta_{\mathrm{mono}, \leqslant n-1}$, where n is still the cardinality of I. Furthermore, there is a functor

$$\mathfrak{C}^{U,c} : \mathcal{P}_I^{\mathrm{op}} \longrightarrow \mathrm{Top}, \quad S \longmapsto \bigcap_{i \in S} U_i.$$

Proposition A.18. *The map* $\mathrm{LKan}_{G^{\mathrm{op}}} \mathfrak{C}^{U,c} \to \mathfrak{C}^{U,<}$ *is an isomorphism. Thus, for* $\mathcal{F}$ *a sheaf with values in a complete* ∞*-category, the map*

$$\mathcal{F}(X) \longrightarrow \mathrm{holim}_{\mathcal{P}_I} \mathcal{F}(\mathfrak{C}^{U,c})$$

is an equivalence.

Proof. By definition,

$$(\mathrm{LKan}_{G^{\mathrm{op}}} \mathfrak{C}^{U,c})([k]) = \mathrm{colim}_{\emptyset \neq S \subset I, [k] \hookrightarrow [|S|-1]} \bigcap_{i \in S} U_i.$$

The index category has the discrete subcategory of all subsets of I with $k+1$ elements as a final subcategory (note the op). This proves the first part of the proposition. The second part follows from Corollary A.17 and the fact that $\mathcal{F}$ sends this left Kan extension to a right Kan extension. □

In particular, for $X = U \cup V$, this implies that the square

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \cap V) \end{array}$$

is (homotopy) cartesian. People familiar with Goodwillie calculus will note that this special case actually implies the last proposition for arbitrary finite covers.

Remark A.19. Note that we can apply the whole discussion also to a Zariski covering $\{U_i \to X\}$ of an algebraic stack X by using the underlying space of X (see Corollary 2.33).

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