NOTES ON CHAPTER 1 OF HTT

§1. A *category* has a set (or class) of objects and morphisms between objects, and an *associative* composition law on morphisms. Intuitively, a *higher category* is supposed to contain "morphisms between morphisms." So, for instance, just as there are 1-morphisms between objects, there should be 2-morphisms between 1-morphisms, 3-morphisms between 2-morphisms, and so on.

Example. A 2-category can be thought of as a category *enriched* over the monoidal category **Cat** of categories. (The monoidal structure is given by the product.)

In this definition, a 2-category consists of:

(1) A class of objects.
(2) For each pair of objects x, y, a *category* (rather than a set) $\hom(x, y)$ of morphisms between x, y. We can think of the *objects* of this category as being the 1-morphisms, and the morphisms of $\hom(x, y)$ as the 2-morphisms (between 1-morphisms $x \to y$).

There are many situations where this is natural. For instance, the category **Cat** itself is more naturally thought of as a 2-category: the objects are categories, and for categories $\mathfrak{X}, \mathfrak{Y}$, we define $\hom(\mathfrak{X}, \mathfrak{Y})$ to be the *category* whose objects are functors $\mathfrak{X} \to \mathfrak{Y}$ and whose morphisms are natural transformations.

A concrete example is in the 2-category of stacks over a base scheme S. These can be defined as categories fibered in groupoids over the big étale site of S, where 1-morphisms are functors and 2-morphisms are natural transformations. This is really a 2-category, because the construction of a "fibered product" of stacks needs to be the 2-fibered product.

We could define a 3-category now as a "category enriched over 2-categories," for instance, but this *fails*. The idea is that, in the previous example, we required composition of morphisms to be strictly associative, or an equality of compositions. In many situations in category theory, equality of compositions is the wrong notion. For instance, for a category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ to be a *monoidal category*, one requires not the equality of functors

$$X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z,$$

but rather a natural isomorphism

$$X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z,$$

satisfying coherence conditions. In the same way, we do not want a strictly associative composition law.

Here is the type of phenomenon we want higher categories to model.

Example. Let X be a topological space. We construct a "higher category" πX (with quotes since we have not defined this) as follows: the objects are the points of X. The 1-morphisms between $p, q \in X$ are the paths running from p to q. The 2-morphisms between two paths γ_1, γ_2 are the homotopies between them. And so on: we consider "homotopies between homotopies."

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Note that since we can reverse homotopies (a homotopy between γ_1, γ_2 yields a homotopy between γ_2, γ_1), the morphisms of this higher category are all *invertible.* In fact, in the models of higher category theory, the ∞-categories that are ∞-groupoids correspond to spaces in this sense.

Principle. ∞-groupoids should be homotopy types of spaces.

In the example πX, note that composition (defined by piecing together homotopies), is *not* strictly associative. If we require in higher category theory that composition be strictly associative, we might then run into problems. It turns out that the naive model for a 3-category (as a category enriched in 2-categories) cannot model πS^3.

In this seminar, we are studying primarily $(\infty, 1)$-categories: these are categories where n-morphisms for $n > 1$ are required to be invertible. It has been said that the study of such objects is actually *simpler* than the study of 2-categories (where 2-morphisms are not required to be invertible).

§2. There are many possible models for $(\infty, 1)$-categories. One bit of intuition is that all the higher-categorical data can be packaged not into a *set* $\hom(x, y)$ of morphisms $x \to y$, but into something more complicated. Indeed, recall that a 2-category is one where mapping sets $\hom(x, y)$ are replaced by *1-categories* instead of sets. Thus we should think of ∞-categories as ones where $\hom(x, y)$ is always itself an ∞-category. We are interested in $(\infty, 1)$-categories, so all morphisms in $\hom(x, y)$ should be invertible.

Principle. For an $(\infty, 1)$-category, $\hom(x, y)$ (for x, y objects) should be a ∞-groupoid, which is the same thing as a space, rather than simply a set.

We might thus suggest that an $(\infty, 1)$-category as a category enriched over the category of spaces (or simplicial sets). As a result, we get:

Model 1. An $(\infty, 1)$-category is a category enriched over the category of topological spaces.

We know that simplicial sets form a good model of topological spaces, up to homotopy. As a result, we might also suggest:

Model 2. An $(\infty, 1)$-category is a category enriched over the category of simplicial sets (i.e. a simplicially enriched category). Usually the ones of interest will be those where $\hom(x, y)$ is a Kan complex for each x, y.

Note that these models of higher categories require a strictly associative law of composition.

This model of ∞-categories makes it clear that *spaces* (or simplicial sets) should play a similar role in higher category theory as sets do in ordinary category theory. For instance, recall:

Example. Let $\mathcal{C}$ be a category. Then a *product* of two objects $X, Y \in \mathcal{C}$ is an object $Z \in \mathcal{C}$ such that there is a natural isomorphism of functors

$$\hom(\cdot, Z) \simeq \hom(\cdot, X) \times \hom(\cdot, Y).$$

That is, the ordinary notion of a cartesian product of sets *determines* what a categorical product should be in any category.

Similarly, let $\mathfrak{C}$ be a topologically (or simplicially) enriched category. Given objects $x, y \in \mathfrak{C}$, we say that $z \in \mathfrak{C}$ is a *product* for x, y if there is a natural *weak equivalence* of space-valued functors

$$\hom(\cdot, z) \to \hom(\cdot, x) \times \hom(\cdot, y). \tag{1}$$

Note that we want to think of the mapping spaces as not just spaces, but homotopy types. As a result, we should relax the notion of isomorphism: let us say that a morphism $f : x \to y$ in a topologically (or simplicially) enriched category is an *equivalence* if for each z, the map

$$\hom(z, x) \to \hom(z, y)$$

is a weak equivalence of spaces. We generally want our notions to be invariant under equivalence, which explains why we demanded that (1) be a weak equivalence rather than an isomorphism.

Example. More generally, let $\mathcal{I}$ be an ordinary category, and let $F : \mathcal{I} \to \mathfrak{C}$ be a functor to the topologically enriched category $\mathfrak{C}$. We can say that a *limit* of F is an element representing, up to weak equivalence, the functor $\mathrm{holim} F$.

Another possible model for $(\infty, 1)$-categories would be to use categories enriched over the category of chain complexes, or DG-categories; here the mapping "space" is replaced by a mapping chain complex. By the Dold-Kan correspondence, we can think of DG-categories (at least if we throw away half the data) as simplicially enriched categories.

§3. Lurie's HTT and HA primarily use a different model of ∞-categories. Many basic constructions, such as forming over-categories and under-categories, are poorly behaved for topologically enriched categories.

The model that we plan to use in this seminar is different, and is based on the theory of simplicial sets. Recall that simplicial sets can be used to describe ordinary category theory:

Definition 1. Given an ordinary category $\mathcal{C}$, the **nerve** $N\mathcal{C}$ is the following simplicial set: the n-simplices $(N\mathcal{C})_n$ consist of composable n-tuples

$$X_0 \to \cdots \to X_n,$$

and the face and degeneracy maps are defined in the natural way (e.g. composing two maps, dropping an end object).

Alternatively, note that if $[n]$ is the partially ordered set on $\{0, 1, \ldots, n\}$, then $[n]$ can be viewed as a category, and in fact as n varies one obtains a cosimplicial object in **Cat**. The nerve $N\mathcal{C}$ has n-simplices given by $\hom_{\mathbf{Cat}}([n], \mathcal{C})$.

As a result, the nerve is a functor from **Cat** to **SSet**. In fact, the nerve alone contains all the information needed to reconstruct the category up to *isomorphism.* The 0-simplices are the objects, the 1-simplices are the morphisms, and the 2-simplices give the composition law. The higher simplices are not relevant. In fact, one easily sees from this discussion:

Proposition 1. *The nerve functor* $N : \mathbf{Cat} \to \mathbf{SSet}$ *is fully faithful.*

Moreover, categorical properties can be translated to homotopical properties of the nerve.

Example. Suppose given two functors $F, G : \mathcal{C} \to \mathcal{D}$ and a natural transformation $F \to G$. The claim is that this is equivalent to a simplicial homotopy

$$N\mathcal{C} \times \Delta[1] \to N\mathcal{D}$$

between NF, NG. Indeed, such a natural transformation is equivalent to a functor $\mathcal{C} \times [1] \to \mathcal{D}$ restricting to F, G (where $[1]$ is the category associated to the partially ordered set $[1] = \{0, 1\}$). Applying the nerve functor (which preserves products, and sends $[1]$ to $\Delta[1]$) and using the full faithfulness of N gives the claim.

In particular, it follows that any functor which is a left or right adjoint induces a homotopy equivalence on the nerves. As a result, $N\mathcal{C}$ is *contractible* if $\mathcal{C}$ admits an initial or final object, because then there is an adjunction $[0] \rightleftarrows \mathcal{C}$ (or in the other direction).

Example. $N\mathcal{C}$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid. We will say more about this below.

Example. A more subtle example of homotopical information on the nerves describing categorical information is the following: if $F : \mathcal{C} \to \mathcal{D}$ is a functor, then F is *cofinal*[1] if and only if the category $\mathcal{C}/d$ for $d \in \mathcal{D}$ (described as follows: an object is an element $c \in \mathcal{C}$ together with a morphism $F(c) \to d$, and a morphism is a map in $\mathcal{C}$ making the usual diagram commute) has a connected, nonempty nerve for all d. Nonetheless, one notes that only π_0 was used in this criterion: if one stipulates that the categories $\mathcal{C}/d$ have *contractible* nerves for all d, then one gets the stronger notion of *homotopy* cofinality, which is more relevant to higher category theory.

There is a special property that characterizes the nerve of a category. Throughout, we let Λ^i_n denote the ith horn on $\Delta[n]$.

Proposition 2. *A simplicial set X is the nerve of a category if and only if it satisfies the following condition: whenever $0 < i < n$, any map of simplicial sets*

$$\Lambda^i_n \to X$$

extends uniquely to a map $\Delta[n] \to X$ (i.e. to an n-simplex of X).

Proof. Let $X = N\mathcal{C}$ for a category $\mathcal{C}$. One argument is as follows: we need to show that

$$\hom(\Delta[n], X) \to \hom(\Lambda^i_n, X)$$

is an isomorphism for $0 < i < n$. However, we will see that N is a *right adjoint*; the left adjoint sends a simplicial set to its *fundamental category* $\tau_1 X$. It will be checked below that $\tau_1\Lambda^i_n \to \tau_1\Delta[n]$ is an isomorphism for $0 < i < n$, which implies the result.

Alternatively, we can reason directly. Consider the case $n = 2$. Then a map $\Lambda^i_n \to X$ is given by a diagram

(2)
$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] & & Z. \end{tikzcd}$$

Here $X, Y, Z \in \mathcal{C}$ and f, g are morphisms in $\mathcal{C}$. An extension of such a map to a map $\Delta[2] \to X$ would be given by a commutative diagram

(3)
$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z. \end{tikzcd}$$

However, there is precisely one way of completing a partial diagram as in (2) to a diagram as in (3): we take $h = g \circ f$. For $n = 3$, one can argue similarly. For $n > 3$, we use the fact that a nerve of a category is a 2-coskeleton (i.e. to give a map from a simplicial set into it is the same as giving a map from the 2-skeleton).

Conversely, suppose X is a simplicial set satisfying the relevant condition. We want to construct a category from X. We can do this by taking the objects to be the set of zero-simplices X_0, and the morphisms to be the edges (so the morphisms from x to y are the edges starting at x and ending at y). To compose morphisms f, g, we fit them into a diagram as in (2). This extends uniquely to a diagram as in (3), and we deem h the composition $g \circ f$.

[1] I.e., colimits over $\mathcal{D}$ are the same as colimits over $\mathcal{C}$.

One can check without too much trouble that this is an associative, unital composition law (by using 3-simplices): for instance, the identity at x is the degenerate 1-simplex at x. $\square$

The argument shows more: the simplicial set X has the unique extension property for maps $\Lambda^i_n \to X$ for all $i, 0 \leq i \leq n$, if and only if X is the nerve of a *groupoid.*

Definition 2. An **∞-category** (or **quasicategory**, or **weak Kan complex**), is a simplicial set X such that any map of simplicial sets $\Lambda^i_n \to X$ for $0 < i < n$ extends to a map $\Delta[n] \to X$.

Let X be an ∞-category. We want to think of X as still having objects and morphisms:

(1) The 0-simplices are the objects.
(2) The 1-simplices are morphisms. In fact, an edge $e \in X_1$ has a *source* and a *target* (given by the two boundary maps applied to e). If e is an edge from x to y (where $x, y \in X_0$), we can think of e as a *morphism* from x to y.
(3) The higher simplices measure commutativity data, in some sense.

Suppose given a 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

in a ∞-category $\mathfrak{C}$. Then we can think of the edge h as one candidate for the "composition" $g \circ f$. Note that h is generally not uniquely determined, and composition cannot be thought of as a function in this sense. However, h is at least determined up to homotopy in some sense, by:

Proposition 3 (Joyal)**.** *A simplicial set X is an ∞-category if and only if $X^{\Delta[2]} \to X^{\Lambda^1_2}$ is a trivial Kan fibration.*

In particular, Proposition 3 means that the fiber over an element of $X^{\Lambda^1_2}$—that is, the specification of two composable morphisms—is a contractible Kan complex. This is close to uniqueness. This result is proved in chapter 2. The idea is quite similar to the fact from the homotopy theory of simplicial sets: if $A \to B$ is an anodyne extension (i.e. a trivial cofibration), and X a Kan complex, then $X^B \to X^A$ is a trivial Kan fibration.

Example (∞-groupoids)**.** Earlier, we said that a homotopy type should correspond to an ∞-groupoid. We can now easily describe how to get an ∞-category from a space X: namely, we just take its singular complex $\mathrm{Sing} X$. Since this is a Kan complex, it is an ∞-category.

We also observed that the unique horn filler property characterized the nerves of groupoids (i.e. X was the nerve of a groupoid if and only if $\hom(\Delta[n], X) \to \hom(\Lambda^i_n, X)$ was an isomorphism for all $0 \leq i \leq n$). One might suggest that an ∞-groupoid should then be a simplicial set such that $\hom(\Delta[n], X) \to \hom(\Lambda^i_n, X)$ is always a *surjection.* Indeed, this is just the definition of a Kan complex.

§4. Given a higher category $\mathfrak{C}$, thought of here as a topologically enriched category, we can consider its *homotopy category.* This is what happens when we throw away all the higher-categorical data—if $x, y \in \mathfrak{C}$, then instead of the mapping space $\hom_{\mathfrak{C}}(x, y)$, one just applies π_0 and takes connected components. We get a (1-)category $h\mathfrak{C}$ whose objects are the same as that of $\mathfrak{C}$, but such that $\hom_{h\mathfrak{C}}(x, y) = \pi_0(x, y)$. This is analogous to the construction of an ordinary (**Ab**-enriched) category from a DG-category, by applying H^0.

We shall see that in the setting of quasi-categories, there is a simple construction of the homotopy category.

Given a simplicial set X, we can consider its *fundamental groupoid* (or edge-path groupoid) $\pi_{\leq 1}X$. This can be described as follows:

(1) The objects are the elements of X_0.
(2) A morphism from x to y (with $x, y \in X_0$) is given by a "composable" sequence of edges and their inverses from x to y.
(3) We declare that if $\sigma \in X_2$ is a 2-simplex, then ...

In other words, to construct the 1-groupoid $\pi_{\leq 1}X$, we think of the edges (or 1-simplices) in X as morphisms, formally invert them, and add relations corresponding to the 2-simplices. We will need a slight variant of this:

Definition 3. Given a simplicial set X, we define $\tau_1 X$ to be the category with objects the elements of X_0, generated by the edges of X and subject to the usual relations on 2-simplices: given a 2-simplex with boundary e_0, e_1, e_2, we should require $e_2 = e_1 \circ e_2$. Alternatively, τ_1 can be described as the unique colimit-preserving functor from simplicial sets to categories sending $\Delta[n]$ to $[n]$, the partially ordered set $\{0, 1, \ldots, n\}$ (considered as a category). In this way, it is the left adjoint to the nerve functor N from categories to simplicial sets.

In particular, $\pi_1 X$ is the localization of $\tau_1 X$ at all its morphisms.

For an ∞-category X, there is a simple way of presenting the homotopy category $\tau_1 X$, also denoted hX. Namely, one considers the objects to be the 0-simplices. We say that two 1-simplices e, e' from the same source x to the same target y are *homotopic* if there is a 2-simplex connecting them,

$$\begin{tikzcd} & y \arrow[dr, "1"] & \\ x \arrow[ur, "e"] \arrow[rr, "e'"] & & y. \end{tikzcd}$$

Here 1 is the degenerate edge at y. It is shown in HTT that this is an *equivalence relation* (this is an exercise in the definitions and some simplex-chasing). To describe the homotopy category hX, one considers $\hom_{hX}(x, y)$ to be the equivalence classes of edges from x to y. Given edges e from x to y and e' from y to z, their composition can be obtained by finding a filler in the horn

$$\begin{tikzcd} & y \arrow[dr, "e'"] & \\ x \arrow[ur, "e"] \arrow[rr, dashed] & & z, \end{tikzcd}$$

and then taking the boundary. It is shown in HTT that this operation of composition is well-defined (up to homotopy).

§5. We have described several models for higher categories. They have been proved to be equivalent to each other. First, it is intuitively clear that topologically enriched and simplicially enriched categories should represent the same thing, because up to homotopy, topological spaces and simplicial sets represent the same thing. However, comparing simplicially enriched categories with ∞-categories is more subtle.

The main result, due to Lurie, makes precise that these two models are equivalent. There are two main functors: a *homotopy coherent nerve* from simplicially enriched categories to simplicial sets, and its left adjoint. By abstract nonsense, to define such a functor, it is enough to construct a cosimplicial object in the category of simplicial categories.

Definition 4. Given a partially ordered set P, we construct the simplicial category $\mathfrak{C}[P]$ as follows.

(1) The objects are the elements of P.
(2) Given $a, b \in P$, the mapping space $\hom(a, b)$ is the nerve of the partially ordered set $P^{[a,b]}$ of subsets $S \subset [a, b]$ containing both a, b. (Here $[a, b]$ is the interval of elements between a, b.)
(3) Composition can be described as follows. There is a map $P^{[a,b]} \times P^{[b,c]} \to P^{[a,c]}$ given by the union of subsets. As a result, there is a map $N(P^{[a,b]}) \times N(P^{[b,c]}) \to N(P^{[a,c]})$; we take this to be the composition law.

The construction of $\mathfrak{C}[P]$ is functorial in P. Given a morphism $f : P \to P'$ of partially ordered sets, there is a natural map $P^{[a,b]} \to P^{[f(a),f(b)]}$ for each $a, b \in P$ (by sending a subset to its image). This gives a morphism of simplicial categories $\mathfrak{C}[P] \to \mathfrak{C}[P']$.

Consider the case $P = [n]$. Note that then the mapping spaces $\hom(a, b)$ are all contractible, and are in fact cubes. To see this, we need to just check that if Q is the partially ordered set of subsets of a finite set, then NQ is contractible: this follows for instance because Q has a largest element (and any category with a final object has a contractible nerve). In other words, $\mathfrak{C}[P]$ is a "thickening" of the category $[n]$.

The construction of $\mathfrak{C}[n]$ (the one that will be of primary interest to us) is a little elaborate, but the idea is as follows. There are $n+1$ objects marked $0, 1, \ldots, n$. Given $i \in [0, n)$, there is one element p_i of $\hom_{\mathfrak{C}[n]}(i, i+1)$. The various composites are not equal, but they are homotopic to each other.

Definition 5. Given a simplicial set X, we can define the simplicial category $\mathfrak{C}[X]$ as the colimit of $\mathfrak{C}[n]$ over the simplex category of X (i.e. we extend the cosimplicial simplicial category $\{\mathfrak{C}[n]\}$ to a colimit-preserving functor from simplicial sets to simplicial categories). Its right adjoint N is called the **homotopy coherent nerve**. We can write

$$N\mathfrak{D} = \hom(\mathfrak{C}[n], \mathfrak{D})$$

for a simplicial category $\mathfrak{D}$.

We shall use the same notation as for the ordinary nerve.

The homotopy coherent nerve is thus a way of obtaining a simplicial set from a simplicial category, which is better than (say) taking the nerve of the zero-simplices.

Example. The simplicial categories $\mathfrak{C}[0], \mathfrak{C}[1]$ are just the ordinary categories $[0], [1]$ (regarded as constant simplicial categories). Consequently, if $\mathfrak{D}$ is a simplicial category, then the zero-simplices of $N\mathfrak{D}$ are just the objects of $\mathfrak{D}$. The 1-simplices of $N\mathfrak{D}$ are just the elements of $\hom_{\mathfrak{D}}(x, y)_0$ (i.e. the zero simplices).

Example. The simplicial category $\mathfrak{C}[2]$ is more interesting. By definition, there are three objects $0, 1, 2$. We have $\hom_{\mathfrak{C}[2]}(0, 1) = *$, $\hom_{\mathfrak{C}[2]}(1, 2) = *$ (the point). However, $\hom_{\mathfrak{C}[2]}(0, 2) = \Delta[1]$.

Let $\mathfrak{D}$ be a simplicial category. To give a functor $\mathfrak{C}[2] \to \mathfrak{D}$ is to give three objects $x, y, z \in \mathfrak{D}$, maps $f : x \to y$ (so $f \in \hom(x, y)_0$), $g : y \to z$ (so $g \in \hom(y, z)_0$), $h : x \to z$ (so $h \in \hom(x, z)_0$), and a homotopy H (i.e. an element of $\hom(x, z)_1$) between h and $g \circ f$. Note that the homotopy H is part of the data.

More generally, we can view a functor $\mathfrak{C}[n] \to \mathfrak{D}$ as a diagram of morphisms in $\mathfrak{D}$, which is not necessarily commutative, but which is commutative up to homotopies (which are themselves part of the data). The homotopies are themselves required to be coherent.

We have defined an adjunction between simplicial sets and simplicial categories. We note, moreover, that at least nice simplicial categories are taken (by N) to ∞-categories.

Proposition 4. *If $\mathfrak{D}$ is a simplicial category whose* hom-*spaces are Kan complexes (i.e. $\mathfrak{D}$ is fibrant), then the coherent nerve $N\mathfrak{D}$ is an ∞-category.*

Proof. We will give the idea; the proof is in HTT. The idea is that we need to show that any map $\Lambda^i_n \to N\mathfrak{D}$ extends to a map $\Delta[n] \to N\mathfrak{D}$ for $0 < i < n$. By adjointness, it is equivalent to consider a map of simplicial categories (where $\mathfrak{C}[]$ is the functor above from simplicial sets to simplicial categories)

$$\mathfrak{C}[\Lambda^i_n] \to \mathfrak{D}$$

and to extend it to

$$\mathfrak{C}[n] \to \mathfrak{D}.$$

However, the point is that the hom-spaces are almost the same for these two categories; there is only one places (between 0 and n) where they differ, and there it is an anodyne extension. As a result, the Kan complex hypothesis allows one to make the extension. □

We can now state in what way simplicial categories (at least, fibrant simplicial categories) and ∞-categories model the same mathematics.

(1) There is a model structure on simplicial categories (due to Bergner) where the fibrant objects are the fibrant simplicial categories, and where the weak equivalences are the equivalences of simplicial categories (i.e. those that induce equivalences on the homotopy categories, considered as enriched categories).
(2) There is a model structure on simplicial sets (due to Joyal) where the cofibrations are the monomorphisms, the fibrant objects are the ∞-categories, and the weak equivalences are those that induce an equivalence of simplicially enriched categories after applying $\mathfrak{C}$.

Lurie shows that $\mathfrak{C}$ and the homotopy coherent nerve N form a Quillen equivalence between these two model categories.

§6. We now need to describe various operations that one can perform on ∞-categories. Ideally, these will reduce to ordinary 1-categorical operations when one applies them to $N\mathcal{C}$ for an ordinary category $\mathcal{C}$.

Example. Given a simplicial set K and an ∞-category X, we define the **functor category** to be the simplicial set X^K (parametrizing maps $K \to X$). One can show that X^K is an ∞-category. This construction is compatible with the nerves, as well: if X, K are the nerves of ordinary categories, then the usual functor category has nerve X^K.

Example. As another example, we can define the **opposite ∞-category** of an ∞-category X. This is a special case of the operation of forming the *opposite simplicial set* of a simplicial set. This is done via the involution of the simplex category that sends a finite linearly ordered set to its opposite (i.e. reverses the order).

Next, we need to describe the construction of over-categories and under-categories, which will be useful in the sequel. We will start with a simpler construction.

Definition 6. Given categories $\mathcal{C}, \mathcal{C}'$, we define the **join** $\mathcal{C} \star C'$ as the following category.

(1) The objects of $\mathcal{C} \star \mathcal{C}'$ are the disjoint union of the objects of $\mathcal{C}$ and the objects of $\mathcal{C}'$.
(2) If $x, y \in \mathcal{C}$, then $\hom_{\mathcal{C}\star\mathcal{C}'}(x, y) = \hom_{\mathcal{C}}(x, y)$. If $x \in \mathcal{C}, y \in \mathcal{C}'$, then $\hom_{\mathcal{C}\star\mathcal{C}'}(x, y) = *$. If $x, y \in \mathcal{C}'$, then $\hom_{\mathcal{C}\star\mathcal{C}'}(x, y) = \hom_{\mathcal{C}'}(x, y)$. Finally, if $x \in \mathcal{C}', y \in \mathcal{C}$, then $\hom_{\mathcal{C}\star\mathcal{C}'}(x, y) = \emptyset$.

In other words, we have taken the two categories $\mathcal{C}, \mathcal{C}'$, and drawn a unique morphism from every object in $\mathcal{C}$ to every object in $\mathcal{C}'$, and added nothing else.

The join of two categories will be a useful construction. For instance, it will let us form over and under-categories. We shall first show that the join operation descends to simplicial sets.

Definition 7. Let X, X' be simplicial sets. We define the **join** $X \star X'$ as follows. For a finite ordered nonempty set J, we write

$$(X \star X')(J) = \bigsqcup_{J = I \sqcup I'} X(I) \times X'(I'),$$

where for $I = \emptyset$ we set $X(I) = *$ (and similarly for $X'(\emptyset)$).

One way to think about the join is that one extends a simplicial set to a presheaf on the *monoidal* category of all finite ordered sets (not necessarily nonempty) but setting $X(\emptyset) = *$, and then using the process of *Day convolution* to go from two such presheaves to a new one.

Note that the join of X, X' contains a copy of X, and a copy of X', as simplicial subsets. If X is fixed, the operation $X' \mapsto X \star X'$ does *not* commute with colimits in the category of simplicial sets. However, it *does* commute with colimits if the target is taken in the category of simplicial sets under X. This is clear from the description.

Proposition 5. *Given categories $\mathcal{C}, \mathcal{C}'$, the nerve $N(\mathcal{C} \star \mathcal{C}')$ is the join $N\mathcal{C} \star N\mathcal{C}'$ (where the join of simplicial sets is defined as above).*

Proof. Indeed, an n-simplex of $N(\mathcal{C} \star \mathcal{C}')$ consists of an $n+1$-tuple of composable maps $X_0 \to X_1 \to \cdots \to X_n$, for the $X_i \in \mathcal{C}$. This means that there is i such that $X_0, \dots, X_i \in \mathcal{C}$, and $X_{i+1}, \dots, X_n \in \mathcal{C}'$; to give an n-simplex of $N(\mathcal{C} \star \mathcal{C}')$ is thus equivalent to giving an i-simplex of $N\mathcal{C}$ and a $n-(i+1)$-simplex of $N\mathcal{C}'$. This is precisely the definition of the join. ∎

Consequently, we can treat the join of categories on the level of simplicial sets. As a result, we might define the *join* of ∞-categories as the join in the category of simplicial sets. We can do this, in view of:

Proposition 6. *The join $X \star X'$ of two ∞-categories X, X' is an ∞-category.*

Proof. Suppose $0 < i < n$, and suppose given a map $\Lambda^i_n \to X \star X'$. There are various possibilities:

(1) The n vertices of Λ^i_n are taken into X. Then the map $\Lambda^i_n \to X \star X'$ factors through $\Lambda^i_n \to X \to X \star X'$, and this map extends to a map $\Delta[n] \to X$. Consequently we can extend it to $\Delta[n] \to X \to X \star X'$.
(2) The n vertices of Λ^i_n are taken into X'. In this case, we can use the same argument to extend to $X' \subset X \star X'$, and thus to $X \star X'$.
(3) It is possible the first k vertices are taken into X and the last k' vertices are taken into X'. However, then we have maps $\Delta[k] \to X$ and $\Delta[k'] \to X'$, which we amalgamate by taking the join to get a map $\Delta[n] \to X \star X'$. (Note that $\Delta[m] \star \Delta[m'] = \Delta[m+m'+1]$.)

∎

Example. We define the join $X \star \Delta[0]$ to be the **right cone** on X, and denote it by $X^{\triangleright}$. Similarly, the **left cone** on X is $\Delta[0] \star X$. In each case, the 0-simplices of the cone are the same as those of X, with an extra "cone point" added. In the right cone, there is exactly one edge from each vertex of X to the cone point. In the left cone, there is exactly one edge from the cone point to each vertex of X.

With the construction of the join, we can construct the ∞-categorical analog of the over-category. Recall the 1-categorical construction. If $\mathcal{C}$ is a category and $X \in \mathcal{C}$, then the *overcategory* $\mathcal{C}_{/X}$ consists of morphisms $Y \to X$, while morphisms consist of commutative diagrams. Note the following characterization of the over-category: to give a functor

$$\mathcal{D} \to \mathcal{C}_{/X},$$

for any category $\mathcal{D}$, is the same as giving a functor $\mathcal{D} \star [0] \to \mathcal{C}$ that takes the added vertex (corresponding to $[0]$) into X. This is easy to see from the definition. In other words,

$$\hom(\mathcal{D}, \mathcal{C}_{/X}) = \hom_X(\mathcal{D} \star [0], \mathcal{C}),$$

where the X in the second hom indicates functors that send the "cone point" into X.

All this is makes sense on the level of simplicial sets. Consequently, if K is a simplicial set and $x \in K_0$ is a vertex, we should define the simplicial set $K_{/x}$ via the mapping property

$$\hom(Y, K_{/x}) = \hom_x(Y^{\triangleright}, K) = \hom_x(Y \star [0], K).$$

We define it more generally:

Definition 8. Let $p : K \to X$ is a morphism of simplicial sets, we define the "**overcategory**" $X_{/p}$ as the simplicial set with the universal property

$$\hom(Y, X_{/p}) = \hom_p(Y \star K, X).$$

Here $\hom_p$ means that we are considering maps $Y \star K \to X$ that take K into X via $p : K \to X$.

We should check that such a simplicial set $X_{/p}$ actually exists. To do this, we set $(X_{/p})_n = \hom_p(\Delta[n] \star K, X)$, as we must. It follows that $X_{/p}$ as constructed will satisfy the universal property for Y a standard simplex. However, it follows then that it must be satisfied for all simplicial sets Y, because the construction $Y \mapsto Y \star K$ preserves colimits when $Y \star K$ is taken in the category of simplicial sets under K.

Because this universal property is precisely analogous to the universal property of ordinary overcategories, and because the nerve functor is compatible with the join, it follows that if $\mathcal{C}$ is an ordinary category and $x \in \mathcal{C}$, then $N(\mathcal{C}_{/x}) = (N(\mathcal{C}))_{/x}$ (where the second "overcategory" is in the simplicial set sense).

Example. We can understand the objects of the "overcategory" as follows. An object, or 0-simplex, of $X_{/p}$ is given by a map $\Delta[0] \to X_{/p}$, or equivalently by a map $\Delta[0] \star K \to X$ (where the restriction $K \to X$ is p). These can be thought of as the "cones on K," at least if X is the nerve of a category. The 1-morphisms are the maps $\Delta[1] \star K \to X$, and consequently correspond to morphisms of cones.

Example. There is a canonical morphism $X_{/p} \to X$ of simplicial sets. Indeed, we can see this on the level of representable functors by Yoneda's lemma. Given $Y \to X_{/p}$, we can get a map $Y \to X$ by restricting the associated map $Y \star K \to X$ to Y. This corresponds to the fact that the overcategory of a category maps functorially to the category.

Proposition 7. *If X is an ∞-category and $p : K \to X$ is any morphism of simplicial sets, then $X_{/p}$ is an ∞-category.*

We do not give the proof; presumably a future speaker will do it. The point is that $X_{/p} \to X$ is an example of a "right fibration."

In a similar vein, we can construct an *undercategory* $X_{p/}$ when $p : K \to X$ is a morphism of simplicial sets. The universal property here is

$$\hom(Y, X_{p/}) = \hom_p(K \star Y, X).$$

§7. Next, we describe limits and colimits in ∞-categories. In a topological (or simplicial) category, we recall that limits and colimits are easy to describe using *homotopy* limits and colimits of spaces (so that the category of spaces is in some sense the universal example). There is a useful description in the simplicial set approach.

Let $\mathcal{C}$ be an ordinary category, and let $x \in \mathcal{C}$ be an object. We can describe what it means for x to be *final* as follows: the projection

$$\mathcal{C}_{/X} \to \mathcal{C}$$

should be an equivalence of categories. We shall use a version of this on the level of nerves:

Proposition 8. *$X \in \mathcal{C}$ is a final object if and only if the map $N(\mathcal{C}_{/X}) \to N(\mathcal{C})$ is a trivial Kan fibration.*

Note that $N(\mathcal{C}_{/X}) = (N\mathcal{C})_{/X}$ can be computed purely simplicially.

Proof. To say that a map of simplicial sets is a trivial Kan fibration is to say that it has the right lifting property with respect to the inclusions $\partial\Delta[n] \hookrightarrow \Delta[n]$ for each n. Consider the lifting problem

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & N(\mathcal{C}_{/X}) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & N(\mathcal{C}). \end{tikzcd}$$

The top map is equivalent to a map $\partial\Delta[n] \star \Delta[0] \to N(\mathcal{C})$ taking the last vertex to X. Here $\partial\Delta[n] \star \Delta[0] = \Lambda^{n+1}_{n+1}$. So we have a map $\Lambda^{n+1}_{n+1} \to N(\mathcal{C})$ taking the last vertex to x; the bottom map gives a map $\Delta[n] \to N(\mathcal{C})$. This sits inside $\Delta[n+1]$ as the zeroth face. So the data of the above diagram is equivalent to a map $\partial\Delta[n+1] \to N(\mathcal{C})$ that sends the last vertex into X. A lifting is the same as a map $\Delta[n] \star \Delta[0] = \Delta[n+1] \to N\mathcal{C}$ extending the given map.

In other words, $N(\mathcal{C}_{/X}) \to N\mathcal{C}$ is a trivial Kan fibration if and only if whenever $p : \partial\Delta[n+1] \to N\mathcal{C}$ is a map taking the last vertex to X, then p extends to a map $\Delta[n+1] \to N\mathcal{C}$.

Suppose now $X \in \mathcal{C}$ is final. Then the above condition is true. Indeed, suppose $n = 0$ for example. The data of a map $\partial\Delta[1] \to N\mathcal{C}$ is a pair of objects in $\mathcal{C}$; the latter is required to be X. The fact that the map can be extended to $\Delta[1] \to N\mathcal{C}$ states that any map admits a map to X. When $n = 1$, a map $\partial\Delta[n+1] \to N\mathcal{C}$ sending the last vertex to X is a not necessarily commutative diagram

$$\begin{tikzcd} & B \arrow[dr] & \\ A \arrow[ur] \arrow[rr] & & X. \end{tikzcd}$$

To say that it extends to a map $\Delta[2] \to N\mathcal{C}$ is to say that this diagram automatically commutes, which it does if X is final. When $n > 1$, we can use the coskeletal property of nerves to make the argument.

The above argument can be reversed to show that if the above extension property holds, X must be final. □

Definition 9. If K is a simplicial set and $x \in K$ is a vertex, then we say that x is **final** if the projection

$$K_{/x} \to K$$

is a trivial Kan fibration.

By the arguments above, we see that:

Proposition 9. *$x \in K$ is final if and only if every map $\partial\Delta[n+1] \to X$ taking the last vertex into x extends to a map $\Delta[n+1] \to X$.*

Proposition 10. *In an ∞-category X, an element x is final if and only if x is final in the homotopy category hX (considered as a category enriched over the homotopy category of spaces).*

To obtain hX as a category enriched over the homotopy category, one has to use the equivalence between ∞-categories and topological (or simplicial) categories. In other words, x is final if and only if the mapping *spaces* $\hom(x, y)$ are *contractible* for all x.

Proof. This relies on a simplicial description of the mapping spaces in quasi-categories, which we cannot give here. □

In ordinary category theory, there is a standard result that final objects are unique up to unique isomorphism. Here is the higher categorical result:

Proposition 11. *Let X be an ∞-category. Then the subcategory spanned by the final objects is either empty or a contractible Kan complex.*

The subcategory "spanned" means that one considers all simplices of X whose vertices are final objects.

Proof. Let $K \subset X$ be the subcategory spanned by all these final objects. Suppose K is nonempty. To show that K is a contractible Kan complex, we need to show that any map $\partial\Delta[n] \to K$ extends to a map $\Delta[n] \to X$. When $n = 0$, this will happen as K is nonempty. When $n > 0$, then we can use the characterization of final objects above to see that such an extension exists. □

We can formulate a dual notion of *initial object* in an ∞-category.

With this in mind, we can describe limits and colimits. Let $p : K \to X$ be a morphism, where K is a simplicial set and X an ∞-category. Recall that a *limit* in the 1-categorical sense should be thought of as a universal cone: that is, it is a terminal object in the category of cones. But we know that $X_{/p}$ is such a category of cones.

Definition 10. If $p : K \to X$ is a morphism, we define a **limit** of p to be an terminal object of the overcategory $X_{/p}$. We can similarly define a **colimit** of p to be an initial object of the undercategory.

It is a (nontrivial!) fact that when one makes the correspondence between ∞-categories and topologically enriched categories, then the notions of limit and colimit given here coincides with that mentioned earlier (in terms of homotopy limits and colimits). It is another nontrivial fact that, when model categories are used to present ∞-categories (one way to think of this is to consider *simplicial* model categories, which are automatically simplicially enriched categories), then homotopy limits and colimits in these model categories correspond to ∞-categorical limits.