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P R E F A C E

The 4th Aarhus conference on algebraic topology in connection with the I.C.M. was held August 1.-7. 1982 at the Mathematics Institute, Aarhus University. The conference was supported by the Danish Natural Science Research Council, the Aarhus University Research Fund and the Danish Mathematical Society.

The conference was structured with plenary talks in the morning together with special sessions in the afternoon in three parallel running tracks. The special sessions were divided according to subject into four categories:

- Algebraic K-theory and L-theory
- Geometry of manifolds
- Homotopy theory
- Transformation groups.

Titles of all talks given at the conference are listed below.

These Proceedings contain papers which were presented at the conference, and some related papers. All papers have been refereed and we take this opportunity to thank the many referees. We would like to thank the City of Aarhus for inviting the participants of the conference and companions to a soupé in the Town Hall. Thanks also go to the Aarhus Congress bureau for arranging the accomodations for the participants. Especially we would like to thank Kirsten Boddum and Sonja Eld who handled the administrative and secretarial duties.

Aarhus, October 1983

Ib Madsen, Bob Oliver

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A NORM THEOREM FOR K_2 OF GLOBAL FIELDS

Anthony Bak

A.S. Merkurjev and A.A. Suslin have provided recently in [7] and [11] an important tool for studying problems envolving the functor K_2 applied to division rings. The tool is a reduced norm homomorphism $N_D : K_2(D) \longrightarrow K_2(K)$ where K is any commutative field and D is an arbitrary finite, central, K-division algebra. One promising area of application for the new tool is that of arithmetic. In [7], the authors take already a step in this direction by proving the following theorem [7, 17.4] : If K is a global field and if the index $\sqrt{[D:K]}$ of D/K is square free then the $\mathrm{coker}(N_D) \simeq (\mathbf{Z}/2\mathbf{Z})^{|\Sigma|}$ where Σ is the set of all real primes of K which do not split D. Furthermore, they conjecture that the restriction imposed on the index of D/K is unnecessary. The purpose of this note is to verify their conjecture.

The main theorem of this note will establish a result slightly sharper than that conjectured by Merkurjev and Suslin. The sharpened result is suggested by the Hasse-Schilling norm theorem (cf. [10, 33.15]). We recall this theorem next. If K is a global field, we shall let v denote any noncomplex prime of K. We let K_v denote the completion of K at v and we let $D_v = D \otimes_K K_v$. We let $\mu(K)$ (resp. $\mu(K_v)$) denote the group of all roots of unity of K (resp. K_v) and we set $m = |\mu(K)|$ and $m_v = |\mu(K_v)|$. We let $\left(\frac{\cdot}{v}\right) : K_v^{\bullet} \times K_v^{\bullet} \longrightarrow \mu(K_v)$ denote the m_v'th power norm residue symbol on K_v. Finally, we let Σ_D denote the set of all real primes v of K which do not split D and if Σ is any finite (possibly empty) set of noncomplex primes of K, we let $\Sigma(K) = \{c \mid c \in K^{\bullet},\ c \in (K_v^{\bullet})^{m_v} \text{ for all } v \in \Sigma\}$. The Hasse-Schilling norm theorem says the following: <u>If</u> K <u>is a global field and if</u> $\mathrm{Nrd}_D : D^{\bullet} \longrightarrow K^{\bullet}$ <u>denotes the usual reduced norm homomorphism on</u> $D^{\bullet}$ <u>then the sequence below is exact</u> $D^{\bullet} \xrightarrow{\mathrm{Nrd}_D} K^{\bullet} \longrightarrow K^{\bullet}/\Sigma_D(K)^{\bullet} \longrightarrow 1$. The following theorem is the analogon for K_2 of the theorem above.

THEOREM 1. <u>If</u> K <u>is a global field then the sequence below is exact</u>

$$K_2(D) \xrightarrow{N_D} K_2(K) \xrightarrow{\amalg\left(\frac{\cdot}{v}\right)} \coprod_{v \in \Sigma} (\pm 1) \rightarrow 1 .$$

It is worth mentioning in connection with the arithmetic applications of N_D that the theorem above is sufficient to resolve the ambiguity of (± 1) appearing in certain cases of the solution [2], [3], [4] to the congruence subgroup and metaplectic problems for classical groups of K-rank > 1. The resolution takes the form conjectured in these papers.

We prepare now for the proof of Theorem 1. It will be assumed that the reader is familiar with the definition of the functor K_2 and with Matsumoto's presentation of K_2 of a field in terms of symbols. A good reference for these materials is Milnor [8].

If K is a global field and if Σ is a finite set of noncomplex primes of K, we define the group $K_2(\Sigma(K)) = K^{\bullet} \otimes \Sigma(K)^{\bullet}/\langle(1-a) \otimes a \mid a \in \Sigma(K)^{\bullet},\ (1-a) \neq 0\rangle$. By Matsumoto's theorem (cf. [8, §11]), it is clear that there is a canonical homomorphism $K_2(\Sigma(K)) \longrightarrow K_2(K)$ which is an isomorphism whenever Σ is the empty set. The next result will be required in the proof of Theorem 1.

THEOREM 2. _If_ K _is a global field and if_ Σ _is a finite set of non-complex primes of_ K _then the zero-sequence_

$$K_2(\Sigma(K)) \longrightarrow K_2(K) \xrightarrow{\amalg \left(\frac{}{v}\right)} \coprod_{v \in \Sigma} \mu(K_v) \longrightarrow 1$$

is exact, except possibly at $K_2(K)$; _here, its homology has order at most_ 2. _Moreover, if_ $8 \nmid m_v$ _for all_ $v \in \Sigma$ _then the sequence is exact._

I do not know if the condition that $8 \nmid m_v$ for all $v \in \Sigma$ is necessary in order that the sequence above be exact.

PROOF. By the Moore reciprocity law (cf. [6]), there is an exact sequence

$$\begin{array}{lccccccc}
 & & & \coprod_v \zeta_v & \longmapsto & \prod_v \zeta_v^{\frac{m_v}{m}} & & \\
(*) & K_2(K) & \xrightarrow{\lambda} & \coprod_v \mu(K_v) & \xrightarrow{\partial} & \mu(K) & \longrightarrow & 1 \\
 & (a,b) & \longmapsto & \coprod_v \left(\frac{a,b}{v}\right) & & & &
\end{array}$$

and by a generalization [4, 3.2] of the Moore reciprocity law, there is an exact sequence

$$(**)\qquad \begin{array}{ccccccc} & & \coprod_{v\notin\Sigma}\zeta_v & \longmapsto & \prod_{v\notin\Sigma}\zeta_v^{\frac{m_v}{m}} & & \\ K_2(\Sigma(K)) & \xrightarrow{\lambda_\Sigma} & \coprod_{v\notin\Sigma}\mu(K_v) & \xrightarrow{\partial_\Sigma} & \mu(K) & \longrightarrow & 1\ . \\ (a,b) & \longmapsto & \coprod_{v\notin\Sigma}\left(\frac{a,b}{v}\right) & & & & \end{array}$$

Consider the following commutative diagram

$$\begin{array}{ccccccc} K_2(K) & \xrightarrow{\lambda} & \coprod_{v}\mu(K_v) & \xrightarrow{\partial} & \mu(K) & \longrightarrow & 1 \\ \uparrow & & \uparrow & & \uparrow & & \\ K_2(\Sigma(K)) & \xrightarrow{\lambda_\Sigma} & \coprod_{v\notin\Sigma}\mu(K_v) & \xrightarrow{\partial_\Sigma} & \mu(K) & \longrightarrow & 1\ . \end{array}$$

The diagram induces a homomorphism $\ker(\lambda_\Sigma) \longrightarrow \ker(\lambda)$. We shall show that

$$\left|\ker(\lambda)/\mathrm{image}(\ker \lambda_\Sigma)\right| \quad \leq \quad \begin{cases} 2 & \text{in general} \\ 1 & \text{if } 8 \nmid m_v \text{ for all } v\in\Sigma\ . \end{cases}$$

Once this has been done, the theorem will follow by chasing the commutative diagram above.

By a result of Tate [12, (33)], $\bigcap_n (K_2(K))^n$ is a subgroup of $\ker(\lambda)$ such that $\left|\ker(\lambda)/\bigcap_n (K_2(K))^n\right| \leq 2$. Let k denote the least common multiple of all m_v such that $v\in\Sigma$. If $a,b \in K^{\bullet}$ then it follows from the definition of $K_2(\Sigma(K))$ that the symbol (a,b^k) of $K_2(K)$ lies in $\mathrm{image}(K_2(\Sigma(K))$. Thus, $\bigcap_n (K_2(K))^n \subseteq (K_2(K))^k \subseteq \mathrm{image}(K_2(\Sigma(K))$. This establishes the first assertion of the theorem. By another result of Tate [12, (33)] (cf. also [5, Théorème 9 and Corollaire]), $\ker(\lambda) \subseteq (K_2(K))^n$ providing $8 \nmid n$. If $8 \nmid m_v$ for all $v\in\Sigma$ then clearly $8 \nmid k$. Thus, $\ker(\lambda) \subseteq (K_2(K))^k \subseteq \mathrm{image}(K_2(\Sigma(K))$. This establishes the second assertion of the theorem.

The next result will also be required in the proof of Theorem 1. Let $\mathrm{Nrd}_D : D^{\bullet} \longrightarrow K^{\bullet}$ denote the usual reduced norm homomorphism on $D^{\bullet}$. Let $K_2(\mathrm{Nrd}_D(D\ )) = K^{\bullet} \otimes \mathrm{Nrd}_D(D^{\bullet})/\langle(1-a) \otimes a \mid a \in \mathrm{Nrd}_D(D^{\bullet}),\ 1-a \neq 0\rangle$. By

Matsumoto's theorem, it is clear that there is a canonical homomorphism $K_2(\mathrm{Nrd}_D(D^\bullet)) \longrightarrow K_2(K)$ which is an isomorphism whenever $\mathrm{Nrd}_D(D^\bullet) = K^\bullet$. If K is a local or a global field then by a theorem [9, 2.2] of Rehmann and Stuhler, there is a homomorphism $\psi_D : K_2(\mathrm{Nrd}_D(D^\bullet)) \longrightarrow K_2(D)$, $(a, \mathrm{Nrd}_D(\beta)) \longmapsto (a,\beta)$.

PROPOSITION. _If_ K _is a local or a global field then the diagram below commutes_

$$\begin{tikzcd} & K_2(D) \arrow[dr, "N_D"] & \\ K_2(\mathrm{Nrd}_D(D^\bullet)) \arrow[ur, "\psi_D"] \arrow[rr] & & K_2(K) . \end{tikzcd}$$

PROOF. If E is any field extension of K which splits D, let $D_E = D \otimes_K E$. For a unique natural number n, there is an E-isomorphism $D_E \longrightarrow \mathbb{M}_n(E)$ where $\mathbb{M}_n(E)$ denotes the full matrix ring of $n \times n$-matrices with coeeficients in E. The induced isomorphism $K_2(D_E) \longrightarrow K_2(\mathbb{M}_n(E))$ is independent of the choice of E-isomorphism above. Furthermore, there is a unique Morita isomorphism (cf. [10, 16.18 and §37]) $K_2(\mathbb{M}_n(E)) \longrightarrow K_2(E)$. The composite of the two isomorphisms above will be denoted by $K_2(D_E) \xrightarrow{\simeq} K_2(E)$. By [4, 2.5], the diagram below commutes

$$\begin{tikzcd} & K_2(D) \arrow[r] & K_2(D_E) \arrow[d, "\simeq"] \\ K_2(\mathrm{Nrd}_D(D^\bullet)) \arrow[ur, "\psi_D"] \arrow[r] & K_2(K) \arrow[r] & K_2(E) . \end{tikzcd}$$

By Suslin [11, 5.7], the diagram below commutes

$$\begin{tikzcd} K_2(D) \arrow[r] \arrow[d, "N_D"'] & K_2(D_E) \arrow[d, "\simeq"] \\ K_2(K) \arrow[r] & K_2(E) . \end{tikzcd}$$

Moreover, by Merkurjev-Suslin [7, §7] (cf. also [11, 3.6]), there is an E such that the homomorphism $K_2(K) \longrightarrow K_2(E)$ is injective. The proposition follows now, by chasing the following commutative diagram

$$\begin{tikzcd}
 & K_2(D) \arrow[r] \arrow[d, "N_D"'] & K_2(D_E) \arrow[d, "\simeq"] \\
K_2(\mathrm{Nrd}_D(D)) \arrow[r] \arrow[ur, "\psi_D"] & K_2(K) \arrow[r] & K_2(E) .
\end{tikzcd}$$

COROLLARY. If K is a global field with no real primes or if K is a nonarchimedean local field then $\mathrm{Nrd}_D(D^{\bullet}) = K^{\bullet}$ and the homomorphism $\psi_D : K_2(K) \longrightarrow K_2(D)$ splits the homomorphism $N_D : K_2(D) \longrightarrow K_2(K)$.

PROOF. If K is a global field with no real primes then by the Hasse-Schilling norm theorem cited above, $\mathrm{Nrd}_D(D^{\bullet}) = K'$ and if K is a nonarchimedean local field then by a simple norm theorem (cf. [10, 33.4]), $\mathrm{Nrd}_D(D^{\bullet}) = K^{\bullet}$. The remaining assertion of the corollary follows now from the proposition.

PROOF OF THEOREM 1. It is worth mentioning at the outset that because of the corollary above, one can restrict, if he likes, his attention to the case that K is a number field with at least one real prime. Recall that $\Sigma_D = \{v \mid v$ is a real prime of K, v does not split $D\}$. By the Hasse-Schilling norm theorem, $\Sigma_D(K) = \mathrm{Nrd}_D(D^{\bullet})$. Let $M = \mathrm{image}(K_2(\Sigma_D(K)) \longrightarrow K_2(K))$ and $N = \mathrm{image}(N_D : K_2(D) \longrightarrow K_2(K))$. By Theorem 2, it suffices to show that $M = N$. By the proposition above, $M \subset N$. If λ is defined as in (*) above then the proof of Theorem 2 shows that $\ker(\lambda) \subset M$. Thus, it is enough to show that $\lambda(M) \supset \lambda(N)$.

Consider the following commutative diagram

$$\begin{tikzcd}
K_2(D) \arrow[r] \arrow[d, "N_D"'] & \prod_v K_2(D_v) \arrow[d, "\prod_v N_{D_v}"] \\
K_2(K) \arrow[r] \arrow[dr, "\lambda"'] & \prod_v K_2(K_v) \arrow[d, "\prod \left(\frac{\cdot}{v}\right)"] \\
 & \prod_v \mu(K_v)
\end{tikzcd}$$

By Alperin-Dennis [1] (cf. also [7, (17.1.1)]), the composite map $\left(\frac{\cdot}{v}\right)N_{D_v}$ is trivial for each $v \in \Sigma$. Thus, $\lambda(N) \subset \coprod_{v \notin \Sigma} \mu(K_v) \left(\subset \prod_{v \notin \Sigma} \mu(K_v)\right)$.
By the exact sequence (*), it follows that $\lambda(N) \subset \ker\left(\coprod_{v \notin \Sigma} \mu(K_v) \longrightarrow \mu(K)\right)$
and by the exact sequence (**), it follows that $\ker\left(\coprod_{v \notin \Sigma} \mu(K_v) \longrightarrow \mu(K)\right) = \lambda(M)$.

QUESTION. <u>The corollary above and Theorems 1 and 2 suggest the following question: If K is a global field and Σ is a finite set of nonarchimedean primes of K such that $8 \nmid m_v$ for all $v \in \Sigma$ then is the canonical homomorphism $K_2(\Sigma(K)) \longrightarrow K_2(K)$ injective? If this were the case then it would follow that the Rehmann-Stuhler map ψ_D always induces a splitting to the Merkurjev-Suslin reduced norm N_D.</u>

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THE NON-EXISTENCE OF TENSORPRODUCTS FOR FREE GROUP ACTIONS ON SPHERES

Steffen Bentzen

§0. Introduction.

This paper gives two examples in the theory of finite group actions on spheres. The first example is a counterexample to the existence of tensorproducts of free actions. The second example shows that the finiteness obstruction of the group $Q(8a,b,c)$ cannot be detected on the subgroups $Q(8a,b)$, $Q(8a,c)$ and $Q(8b,c)$.

Suppose we are given a finite group G that allows a free representation, i.e. a complex representation T such that $T(g)$ never has 1 as eigenvalue for $g \in G\setminus\{1\}$. This representation will induce a free action of G on the sphere S^{2d-1} where d is the complex dimension of T. Such an action (or rather the orbit space S^{2d-1}/G) is called an orthogonal space form. For orthogonal space forms there are standard procedures for constructing new ones from old ones: direct sum and tensorproduct are the basic constructions. The direct sum operation can be generalized to arbitrary non-linear actions by taking the join of the given actions. The tensorproduct cannot be generalized.

Consider two finite groups G_1 and G_2 of coprime orders and let T_1 and T_2 be free linear representations of them. The exterior tensorproduct $T_1 \otimes T_2$ is easily seen to be free so we get an orthogonal space form in dimension $2d_1d_2-1$ where d_i is the dimension of T_i. We shall give an example which shows that a similar construction is not possible for free non-linear actions. We give an example of free actions of G_1, G_2 on S^{2d_1-1},

S^{2d_2-1} such that $G_1 \times G_2$ has no free action on $S^{2d_1d_2-1}$.

More precisely, let a,b,c be pairwise prime integers and let Q(8) be the quaternion group of order 8. We consider the group Q(8a,b,c). This is the semidirect product of $\mathbb{Z}/abc$ with Q(8) such that Q(8) acts on $\mathbb{Z}/abc$ via its commutator factor group, $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, in such a way that $\mathbb{Z}/a$ is the (+,-)-eigenspace, $\mathbb{Z}/b$ is the (-,+)-eigenspace and $\mathbb{Z}/c$ is the (-,-)-eigenspace (Milnors notation). We write Q(8a,b) for Q(8a,b,1). Obviously the cyclic group $\mathbb{Z}/r$ has a free linear action on S^1. We shall prove:

Theorem A. There exist distinct primes p,q,r such that Q(8p,q) acts freely on the sphere S^{3+8k} (k>0), but such that $Q(8p,q)\times \mathbb{Z}/r$ has no free action on a finite complex of the homotopy type of the sphere S^{3+8k}. Actually (p,q,r) = (3,313,7) is such an example.

Our second example is connected with the question of free actions of the group Q(8p,q,r) on spheres, or more generally, on finite complexes of the homotopy type of the sphere S^{3+8k}. The existence of such an action is detected by the vanishing of a certain element (the finiteness obstruction $\sigma_4(G)$) of $Cl(G)/S(G)$ - G = Q(8p,q,r). Here Cl(G) is the projective class group and S(G) is the Swan group of G. The finiteness obstruction of groups of the type Q(8a,b) is fairly well understood, cf. [BM], [B1] . Recall from [HM] that the existence of a semifree action on the Euclidian space ($\mathbb{R}^n$,0) can be detected on the subgroups of type Q(8a,b). One could hope that similarly the finiteness obstruction of the group Q(8a,b,c) could be detected on the subgroups Q(8a,b), Q(8a,c) and Q(8b,c). We give an example that shows that this is not the case.

<u>Theorem B.</u> There exist distinct primes p,q,r such that the finiteness obstruction $\sigma_4(Q(8p,q,r)) \neq 0$, but such that the finiteness obstructions of $Q(8p,q)$, $Q(8p,r)$ and $Q(8q,r)$ all vanish.

Both theorems are proved via calculations of the finiteness obstruction. We briefly recall its definition.

For groups with periodic cohomology, the Tate cohomology groups $\hat{H}^{nd}(G, \mathbb{Z})$ are isomorphic to $\mathbb{Z}/|G|$ for all multiples of the period d. Each generator e of $\hat{H}^{nd}(G, \mathbb{Z})$ can be realized by a periodic resolution P_* of $\mathbb{Z}$ by finitely generated projective $\mathbb{Z}G$-modules and the resolution can be taken to be free exactly if the element

$$\sigma_{nd}(e) := \sum_{i=1}^{nd-1} (-1)^i [P_i]$$

of $Cl(G)$ is zero. Topologically the vanishing of some $\sigma_{nd}(e)$ is equivalent to the existence of a free simplicial action of G on a finite simplicial complex, homotopy equivalent to the sphere S^{nd-1}.

The finiteness obstruction of different generators are related by

$$\sigma_{nd}(s \cdot e) = \sigma_{nd}(e) + S(s)$$

where $S: (\mathbb{Z}/|G|)^{\times} \to Cl(G)$ is the Swan map sending $s \in (\mathbb{Z}/|G|)^{\times}$ to the projective ideal of $\mathbb{Z}G$ generated by s and the norm element $\sum_{g \in G} g$ of $\mathbb{Z}G$.

Hence there exists a unique element $\sigma_{nd}(G) \in Cl(G)/S(G)$, and the vanishing of this element is equivalent to the existence of a free action of G on a finite complex of the homotopy type of the sphere S^{nd-1}.

§1. Proof of Theorem A.

We calculate the finiteness obstruction using the same technique as in [B1]. In particular we use Fröhlich's description of the class group: There is an exact sequence

$$(1.1)\qquad Z(\mathbb{Q}G)^{\times}_{+} \xrightarrow{\Phi} J(Z(\mathbb{Q}G))_{+}/\mathrm{Nrd}(U(\mathbb{Z}G)) \xrightarrow{\partial} \mathrm{Cl}(G) \to 0$$

Here Z denotes the center, J denotes the idele group, Nrd is the reduced norm and $U(\mathbb{Z}G)$ is the group of unit ideles:

$$U(\mathbb{Z}G) = \prod_p (\mathbb{Z}_pG)^{\times}$$

Using the formalism of character homomorphisms, the above sequence becomes

$$(1.2)\qquad \mathrm{Hom}^+_{\Omega}(R(G),\bar{\mathbb{Q}}) \xrightarrow{\Phi} \mathrm{Hom}^+_{\Omega}(R(G),J(\bar{\mathbb{Q}}))/\mathrm{Det}(U(\mathbb{Z}G)) \xrightarrow{\partial} \mathrm{Cl}(G) \to 0$$

where Ω is the full Galois group of $\mathbb{Q}$.

Similarly there is an exact sequence

$$(1.3)\qquad \mathrm{Hom}^+_{\Omega}(R(G),\mathcal{O}^{\times}_F) \to \mathrm{Hom}_{\Omega}(R(G),\prod_{\ell \mid |G|}\mathcal{O}^{\times}_{F_\ell})/\mathrm{Det}(\prod_{\ell \mid |G|}(\mathbb{Z}_\ell G)^{\times}) \to D(G) \to 0$$

describing the kernel group. Here F is any number field containing all character values. The middle term in (1.3) is denoted $\hat{D}(G)$ and it decomposes into a product

$$(1.4)\qquad \hat{D}(G) = \prod_{\ell \mid |G|} \mathrm{Hom}_{\Omega}(R(G),\mathcal{O}^{\times}_{F_\ell})/\mathrm{Det}((\mathbb{Z}_\ell G)^{\times})$$

The component at ℓ is called the local kernel group at ℓ and is denoted $\hat{D}_\ell(G)$.

For details on the above we refer to [F1] and the appendix of [F2].

We consider the group $\pi = Q(8p,q)\times \mathbb{Z}/r$ where p,q,r are distinct primes. This group has cohomology of period 4 and the restriction maps induce an isomorphism

$$\hat{H}^4(\pi,\mathbb{Z}) \simeq \hat{H}^4(\mathbb{Z}/p,\mathbb{Z})\times\hat{H}^4(\mathbb{Z}/q,\mathbb{Z})\times\hat{H}^4(\mathbb{Z}/r,\mathbb{Z})\times\hat{H}^4(Q(8),\mathbb{Z})$$

We take the almost linear k-invariant e. This is the generator of $\hat{H}^4(\pi, \mathbb{Z})$ which on the Sylow parts restricts to $c_2(\chi_\ell+\chi_\ell^{-1})$ for $\ell=p,q$, to $c_2(\chi_\ell+\chi_\ell)$ for $\ell=r$ and to $c_2(\Gamma)$ on $Q(8)$. Here, c_2 denotes the second Chern class, χ_ℓ denotes a faithfull character on $\mathbb{Z}/\ell$ ($\ell=p,q,r$) and Γ is the unique 2-dimensional irreducible representation of $Q(8)$.

__Remark:__ Comparing with the formula for the Chern class of the tensorproduct of free representations we see that this generator is what should be considered the tensorproduct of the generators $e|Q(8p,q)$ and $c_1(\chi_r)$ of $\hat{H}^4(Q(8p,q), \mathbb{Z})$ resp. $\hat{H}^2(\mathbb{Z}/r, \mathbb{Z})$.

We prove Theorem A in 5 steps. First we use idempotent endomorphisms of $Q(8p,q)\times\mathbb{Z}/r$ to decompose the sequence (1.2) into a top component plus several lower components. Then we calculate the top component of (1.2) and then we detect the top component of the middle term of (1.2) on the cyclic subgroup $\mathbb{Z}/2pqr$. Wall has specified an element $\hat{\tau}_4(e)$ of $J(Z(\mathbb{Q}\pi))_+/Nrd(U(\mathbb{Z}\pi))$ that maps to $\sigma_4(e)$ via ∂. As the fourth step we use restriction to $\mathbb{Z}/2pqr$ to calculate the top component of $\hat{\tau}_4(e)$. The fifth and final step is then to prove that p,q,r can be chosen such that $Q(8p,q)$ has a free action on S^{3+8k} but the top component of $\partial\hat{\tau}_4(e)$ is not equal to the top component of any Swan module $S(s)$. Therefore π cannot act freely on a finite complex of the homotopy type of the sphere S^{3+8k}. Since the calculations are very similar to the calculations of [BM] and [B1] we shall be brief.

__Step 1:__ The group $\pi = Q(8p,q)\times\mathbb{Z}/r$ allows three idempotent endomorphisms: E_p, E_q and E_r. The endomorphism E_ℓ ($\ell=p,q,r$) is defined as mapping the Sylow-ℓ-part to 1 and is the identity on the rest. They induce commuting idempotents, also denoted E_ℓ, on the virtual character ring and

$$\begin{aligned}Id = {} & E_pE_qE_r + E_pE_q(1-E_r) + E_p(1-E_q)E_r + (1-E_p)E_qE_r \\ & + E_p(1-E_q)(1-E_r) + (1-E_p)E_q(1-E_r) + (1-E_p)(1-E_q)E_r \\ & + (1-E_p)(1-E_q)(1-E_r)\end{aligned}$$

is a decomposition af the identity on $R(\pi)$ into a sum of orthogonal idempotents. This decomposition will induce a decomposition of the sequence (1.2) into 8 sequences. The sequence corresponding to $E_{top} = (1-E_p)(1-E_q)(1-E_r)$ is called the top component.

Step 2: We start by decomosing $\mathbb{Q}\pi$:

$$\begin{aligned}& \mathbb{Q}[Q(8p,q)\times \mathbb{Z}/r] = \mathbb{Q}[Q(8p,q)] \oplus \mathbb{Q}(\zeta_r)[Q(8p,q)] \\ & = \mathbb{Q}_{++} \oplus \mathbb{Q}_{-+} \oplus \mathbb{Q}_{+-} \oplus \mathbb{Q}_{--} \oplus \mathbb{H}_{\mathbb{Q}} \\ & \oplus M_2(\mathbb{Q}(\eta_p))_+ \oplus M_2(\mathbb{Q}(\eta_p))_- \oplus \mathbb{H}_{\mathbb{Q}(\eta_{4p})} \\ & \oplus M_2(\mathbb{Q}(\eta_q))_+ \oplus M_2(\mathbb{Q}(\eta_q))_- \oplus \mathbb{H}_{\mathbb{Q}(\eta_{4q})} \\ & \oplus M_4(\mathbb{Q}(\eta_p,\eta_q)) \oplus A(\mathbb{Q}(\zeta_{pq})/\mathbb{Q}(\eta_p,\eta_q)) \\ & \oplus \mathbb{Q}(\zeta_r)_{++} \oplus \mathbb{Q}(\zeta_r)_{-+} \oplus \mathbb{Q}(\zeta_r)_{+-} \oplus \mathbb{Q}(\zeta_r)_{--} \oplus \mathbb{H}_{\mathbb{Q}(\zeta_r)} \\ & \oplus M_2(\mathbb{Q}(\eta_p,\zeta_r))_+ \oplus M_2(\mathbb{Q}(\eta_p,\zeta_r)_- \oplus \mathbb{H}_{\mathbb{Q}(\eta_{4p},\zeta_r)} \\ & \oplus M_2(\mathbb{Q}(\eta_q,\zeta_r))_+ \oplus M_2(\mathbb{Q}(\eta_q,\zeta_r)_- \oplus \mathbb{H}_{\mathbb{Q}(\eta_{4q},\zeta_r)} \\ & \oplus M_4(\mathbb{Q}(\eta_p,\eta_q,\zeta_r)) \oplus A(\mathbb{Q}(\zeta_{pqr})/\mathbb{Q}(\eta_p,\eta_q,\zeta_r))\end{aligned}$$

Here $A(L/K)$ denotes some simple algebra with center K and split by L.

Corresponding to this decomposition we choose representatives for $Irr(Q(8p,q)\times \mathbb{Z}/r)/\Omega$:

$$\begin{aligned}& x_{++},\ x_{-+},\ x_{+-},\ x_{--},\ x_o \\ & x_{p,+},\ x_{p,-},\ x_{p,o} \\ & x_{q,+},\ x_{q,-},\ x_{q,o} \\ & x_{pq,+},\ x_{pq,-} \\ & x_{r,++}, x_{r,-+},\ x_{r,+-},\ x_{r,--},\ x_{r,o}\end{aligned}$$

$x_{pr,+}, x_{pr,-}, x_{pr,o}$

$x_{qr,+}, x_{qr,-}, xqr,o$

$x_{pqr,+}, x_{pqr,-}$

It is easy to see that E_{top} is zero on these characters, except for $x_{pqr,+}$ and $x_{pqr,-}$ and on these characters E_{top} is :

$$\begin{aligned}E_{top}(x_{pqr,+}) = x_{pqr,+} &- (x_{pq,+} + x_{qr,+} + x_{qr,-} + x_{pr,+} + x_{pr,-}) \\ &+ (x_{q,+} + x_{q,-} + x_{p,+} + x_{p,-} + x_{r,++} + x_{r,+-} \\ &\quad + x_{r,-+} + x_{r,--}) \\ &- (x_{++} + x_{+-} + x_{-+} + x_{--})\end{aligned}$$

$$\begin{aligned}E_{top}(x_{pqr,-}) = x_{pqr,-} &- (x_{pq,-} + x_{qr,o} + x^{\sigma}_{qr,o} + x_{pr,o} + x^{\sigma}_{pr,o}) \\ &+ (x_{q,o} + x^{\sigma}_{q,o} + 2x_{r,o} + x_{p,o} + x^{\sigma}_{p,o} + 2x^{\sigma}_{r,o}) \\ &- (2x_o + 2x^{\sigma}_o)\end{aligned}$$

where σ is the generator of $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$. Hence $E_{top}(R(\pi)/\Omega) = \mathbb{Z}\cdot x_{pqr,+} \oplus \mathbb{Z}\cdot x_{pqr,-}$ and $E_{top}(Z(\mathbb{Q}\pi)^{\times}_{+}) = \mathbb{Q}(\eta_p,\eta_q,\zeta_r)^{\times} \oplus \mathbb{Q}(\eta_p,\eta_q,\zeta_r)^{\times}$.

We turn to the calculation of the top component of $\hat{D}(\pi)$. There are four prime divisors of $|\pi|$: p,q,r and 2.

<u>The case $\ell = p$:</u> In the decomposition of $\mathbb{Z}_p\pi$ we are only interested in the top component. By the above this is the two blocks

$$B_1 = (\mathbb{Z}_p \otimes \mathbb{Z}[\mathbb{Z}/p][\zeta_q,\zeta_r])^t[X,Y \mid X^2 = Y^2 = 1]$$

$$B_2 = (\mathbb{Z}_p \otimes \mathbb{Z}[\mathbb{Z}/p][\zeta_q,\zeta_r])^t[X,Y \mid X^2 = Y^2 = -1]$$

where X and Y are generators of Q(8).

<u>Lemma 1.1.</u> The 2-primary part of

$$(\mathbb{Z}_p \otimes \mathbb{Z}[\eta_p,\eta_q,\zeta_r])^{\times} \oplus 2\cdot(\mathbb{Z}_p \otimes \mathbb{Z}[\eta_p,\eta_q,\zeta_r])^{\times}/\mathrm{Nrd}(B_1^{\times})$$

is equal to

$$(\mathbb{F}_p \otimes \mathbb{Z}[\eta_p,\eta_q,\zeta_r])^{\times}_{(2)}$$

<u>Proof:</u> We assume that $\mathbb{Q}(\zeta_q,\zeta_r)/\mathbb{Q}(\eta_q,\zeta_r)$ doesn't split at p. The block B_1 is contained in

$$\mathbb{Q}_p[\mathbb{Z}/p][\zeta_q,\zeta_r])^t[X,Y|\ X^2 = Y^2 = 1]$$

$$= M_2(\mathbb{Q}_p(\eta_q,\zeta_r)) \oplus M_2(\mathbb{Q}_p(\eta_q,\zeta_r)) \oplus M_4(\mathbb{Q}(\eta_p,\eta_q,\zeta_r))$$

and it is easy to see that the three components of the reduced norm are related by

$$(1.5) \qquad N_1(x)N_2(x) \equiv N_3(x) \mod p$$

for $x \in B_1$. Here p is the prime of $\mathbb{Q}_p(\eta_p,\eta_q,\zeta_r)$. Let A_p be a generator of $\mathbb{Z}/p$ and consider the ideal $I = (A_p-1)$ of B_1. We have an exact sequence

$$\begin{array}{ccccccc}
1 \longrightarrow & (1+I)^{\times} & \longrightarrow & B_1^{\times} & \longrightarrow & \mathbb{Z}_p[\zeta_q,\zeta_r]^t[X,Y|X^2=Y^2=1]^{\times} & \to 1 \\
& \downarrow \mathrm{Nrd} & & \downarrow \mathrm{Nrd} & & \downarrow \mathrm{Nrd} & \\
1 \to & \mathbb{Z}_p[\eta_p,\eta_q,\zeta_r]^{\times} & \to & \mathbb{Z}_p[\eta_p,\eta_q,\zeta_r]^{\times} \oplus 2\,\mathbb{Z}_p[\eta_q,\zeta_r]^{\times} & \to & 2\,\mathbb{Z}_p[\eta_q,\zeta_r]^{\times} & \to 1
\end{array}$$

Using the relation (1.5) we see that $\mathrm{Nrd}((1+I)\ ) \subseteq U_1(\mathbb{Z}_p[\eta_p,\eta_q,\zeta_r])$. Since $\mathbb{Z}_p[\zeta_q,\zeta_r]^t[X,Y|\ X^2 = Y^2 = 1]$ is a maximal order in $\mathbb{Q}_p(\zeta_q,\zeta_r)^t[X,Y|\ X^2 = Y^2 = 1]$ the right hand Nrd is onto. An application of the snake lemma and the relation (1.5) completes the proof.//

Similarly we can prove

<u>Lemma 1.2.</u> The 2-primary part of

$$(\ (\mathbb{Z}_p \otimes \mathbb{Z}[\eta_p,\eta_q,\zeta_r]^{\times}) \oplus 2(\mathbb{Z}_p \otimes \mathbb{Z}[\eta_q,\zeta_r]^{\times})\ )/\mathrm{Nrd}(B_2^{\times})$$

is equal to

$$(\mathbb{F}_p \otimes \mathbb{Z}[\eta_p,\eta_q,\zeta_r])^{\times}_{(2)} .$$

<u>Lemma 1.3.</u> The top component of the map

$$\Phi_p:\ \mathcal{O}^{\times}_{Z(\mathbb{Q}_p\pi)} \longrightarrow \hat{D}_p(\pi)_{(2)}$$

from (1.3) is the reduction map

$$\mathbb{Z}_p[\eta_p,\eta_q,\zeta_r]^\times \oplus \mathbb{Z}_p[\eta_p,\eta_q,\zeta_r]^\times$$
$$\longrightarrow (\mathbb{F}_p \otimes \mathbb{Z}[\eta_p,\eta_q,\zeta_r])^\times_{(2)} \oplus (\mathbb{F}_p \otimes \mathbb{Z}[\eta_p,\eta_q,\zeta_r])^\times_{(2)} .$$

Similar calculations show that

$$E_{top}(\hat{D}_q(\pi))_{(2)} = 2\cdot(\mathbb{F}_q \otimes \mathbb{Z}[\eta_p,\eta_q,\zeta_r])^\times_{(2)}$$
$$E_{top}(\hat{D}_r(\pi))_{(2)} = 2\cdot(\mathbb{F}_r \otimes \mathbb{Z}[\eta_p,\eta_q,\zeta_r])^\times_{(2)}$$

while

$$E_{top}(\hat{D}_2(\pi))_{(2)} = 1 .$$

This determines the 2-primary part of the top component of the sequence

$$\mathcal{O}^\times_{Z(\mathbb{Q}\pi)_+} \longrightarrow \hat{D}(\pi) \longrightarrow D(\pi) \longrightarrow 0.$$

<u>Proposition 1.4.</u> The top component of

$$\mathcal{O}^\times_{Z(\mathbb{Q}\pi)_+} \longrightarrow \hat{D}(\pi)_{(2)}$$

is the reduction map

$$2\cdot \mathbb{Z}[\eta_p,\eta_q,\zeta_r]^\times \longrightarrow 2\cdot\prod_{\ell=p,q,r}(\mathbb{F}_\ell \otimes \mathbb{Z}[\eta_p,\eta_q,\zeta_r])^\times_{(2)}$$

This completes step 2.

<u>Step 3:</u> We consider the cyclic subgroup $C = \mathbb{Z}/2pqr$ of π. The top component of $\mathbb{Q}C$ is equal to $\mathbb{Q}(\zeta_{pqr})_+ \oplus \mathbb{Q}(\zeta_{pqr})_-$. The top component of $R(C)/\Omega$ is therefore equal to $\mathbb{Z}\cdot\omega_{pqr,+} \oplus \mathbb{Z}\cdot\omega_{pqr,-}$ where $\omega_{pqr,+}$ $(\omega_{pqr,-})$ is a character on C with kernel equal to $\mathbb{Z}/2$ (resp. 1). The 2-primary part of the top component of $\hat{D}(C)$ is easily calculated:

<u>Proposition 1.5.</u> The top component of $\prod_{\ell=p,q,r} \hat{D}_{\ell}(C)_{(2)}$ is equal to $2\cdot \prod_{\ell=p,q,r} (\mathbb{F}_{\ell}\otimes \mathbb{Z}[\zeta_{pqr}])^{\times}_{(2)}$ and the map

$$E_{top}(\mathcal{O}^{\times}_{\mathbb{Q}C}) \longrightarrow E_{top}(\prod_{\ell} \hat{D}_{\ell}(C)_{(2)})$$

is the reduction map

$$2(\mathbb{Z}[\zeta_{pqr}]^{\times}) \longrightarrow 2(\prod_{\ell}(\mathbb{F}_{\ell}\otimes \mathbb{Z}[\zeta_{pqr}])^{\times}_{(2)})$$

<u>Proposition 1.6.</u> The restriction map

$$E_{top}(J(Z(\mathbb{Q}\pi))_{+}/\mathrm{Nrd}(U(\mathbb{Z}\pi)))_{(2,f)} \to E_{top}(J(\mathbb{Q}C)/\mathrm{Nrd}(U(\mathbb{Z}C)))$$

is injective. Here (2,f) denotes the 2-primary part plus the free part.

<u>Proof:</u> This is proved just as the corresponding statement in [BM] or [B1]. We have a commutative diagram where the rows are split exact sequences:

$$\begin{array}{ccccccccc}
0 & \to & \hat{D}(\pi)_{(2)} & \to & (J(Z(\mathbb{Q}\pi))_{+}/\mathrm{Nrd}(U(\mathbb{Z}\pi)))_{(2,f)} & \to & I(\pi) & \to & 0 \\
 & & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} & & \\
0 & \to & \hat{D}(C)_{(2)} & \to & (J(\mathbb{Q}C)/\mathrm{Nrd}(U(\mathbb{Z}C)))_{(2,f)} & \to & I(C) & \to & 0
\end{array}$$

It is easy to see that induction of characters commutes with the endomorphisms E_{ℓ} ($\ell=p,q,r$). Since Res is induced by the induction map on the character ring, the above diagram stays commutative when applying E_{top}. But the character $\omega_{pqr,\pm}$ of C induces to the character $\chi_{pqr,\pm}$ of π. Hence the left hand Res is just inclusion of residue fields and the right hand Res is extension of ideals. Therefore the middle Res is injective.//

This completes step 3.

<u>Step 4:</u> By Proposition 1.4, the top component of $D(\pi)_{(2)}$ has two parts, one corresponding to the character $\chi_{pqr,+}$ (the plus part) and one corresponding to $\chi_{pqr,-}$ (the minus part). The finiteness obstruction $\sigma_4(e)$ is known to be an element of

$D(\pi)$ of order at most 2. We calculate the plus part of $E_{top}(\sigma_4(e))$.

Lemma 1.7. The plus part of $E_{top}(\sigma_4(e))$ is the image under ∂ of the element

$$((2-\eta_q)(1-\zeta_r)^2,\ (2-\eta_p)(1-\zeta_r)^2,\ (2-\eta_p)(2-\eta_q))$$

of $(\mathbb{F}_p\otimes\mathbb{Z}[\eta_p,\eta_q,\zeta_r])^\times_{(2)}\oplus(\mathbb{F}_q\otimes\mathbb{Z}[\eta_p,\eta_q,\zeta_r])^\times_{(2)}\oplus(\mathbb{F}_r\otimes\mathbb{Z}[\eta_p,\eta_q,\zeta_r])^\times_{(2)}$

Proof: The restriction of e to C is seen to be equal to the Chern class of the representation $\psi = \chi_p\chi_q\chi_r\chi_2 + \chi_p^{-1}\chi_q^{-1}\chi_r\chi_2$ where χ_ℓ is a faithfull character on $\mathbb{Z}/\ell$ ($\ell = p,q,r,2$). Therefore $\sigma_4(e|C) = 0$. The idelic Reidemeister torsion, $\hat{\tau}_4(e|C)$, defined by Wall (cf. [W]) is therefore the usual Reidemeister torsion $\tau(c_2(\psi))$ of a lens space. This torsion is known. cf. [Mi]. Except for the $\mathbb{Q}_+$-component (which is $(4p^2q^2r^2)^{-1}$) it is the image under the isomorphism

$$\begin{aligned}\mathbb{Q}C \cong\ & \mathbb{Q}_+ \oplus \mathbb{Q}_- \oplus \mathbb{Q}(\zeta_p)_+ \oplus \mathbb{Q}(\zeta_p)_- \oplus \mathbb{Q}(\zeta_q)_+ \oplus \mathbb{Q}(\zeta_q)_- \\ & \oplus \mathbb{Q}(\zeta_r)_+ \oplus \mathbb{Q}(\zeta_r)_- \oplus \mathbb{Q}(\zeta_{pq})_+ \oplus \mathbb{Q}(\zeta_{pq})_- \oplus \mathbb{Q}(\zeta_{pr})_+ \oplus \mathbb{Q}(\zeta_{pr})_- \\ & \oplus \mathbb{Q}(\zeta_{qr})_+ \oplus \mathbb{Q}(\zeta_{qr})_- \oplus \mathbb{Q}(\zeta_{pqr})_+ \oplus \mathbb{Q}(\zeta_{pqr})_-\end{aligned}$$

of the element $(1-t)(1-t^x)$ where t generates C and x is determined by

$$x \equiv \begin{cases} -1 & \text{mod } pq \\ 1 & \text{mod } 2r \end{cases}$$

The action of E_{top} on $(\mathbb{Q}C)^\times$ can easily be calculated: E_{top} is trivial on all blocks except $\mathbb{Q}(\zeta_{pqr})_+^\times$ and $\mathbb{Q}(\zeta_{pqr})_-^\times$ where

$$E_{top}(\omega_{pqr,+}) = \omega_{pqr,+}(\omega_{pq,+}\omega_{pr,+}\omega_{qr,+})^{-1}(\omega_{p,+}\omega_{q,+}\omega_{r,+})\omega_{1,+}^{-1}$$

$$E_{top}(\omega_{pqr,-}) = \omega_{pqr,-}(\omega_{pq,-}\omega_{pr,-}\omega_{qr,-})^{-1}(\omega_{p,-}\omega_{q,-}\omega_{r,-})\omega_{1,-}^{-1}$$

Therefore the plus part of the top component of $\tau_4(c_2(\psi))$ is equal to $n/d \in \mathbb{Q}(\zeta_{pqr})_+$ where

$$n = (1-\zeta_p\zeta_q\zeta_r - \zeta_p^{-1}\zeta_q^{-1}\zeta_r + \zeta_r^2)(2-\eta_p)(2-\eta_q)(1-\zeta_r)^2 4p^2q^2r^2$$

$$d = (1-\zeta_p\zeta_q - \zeta_p^{-1}\zeta_q^{-1} + 1)(1-\zeta_p\zeta_r - \zeta_p^{-1}\zeta_r + \zeta_r^2)(1-\zeta_q\zeta_r - \zeta_q^{-1}\zeta_r + \zeta_r^2)$$

Since the factor $(2-\eta_p)(2-\eta_q)(1-\zeta_r)^2 4p^2q^2r^2$ lies in $E_{top}(Z(\mathbb{Q}\pi))$ we can ignore it. Lemma 1.7 now follows from Proposition 1.5 and 1.6. //

This completes step 4.

Step 5:

Lemma 1.8: Assume that $p \equiv 3 \bmod 4$, $q \equiv 1 \bmod 8$, $r \equiv 3 \bmod 4$ and $(\frac{r}{q}) = -1$, $(\frac{p}{q}) = 1$. Then $\sigma_4(e) \notin S(\pi)$.

Proof: First of all we note that the Swan group $S(\pi)$ is a 2-group. This is because π is a 2-hyperelementary group. The Swan module $S(s)$ ($s \in (\mathbb{Z}/8pqr)^{\times}$) is represented in $\hat{D}(\pi)$ by the character homomorphism that sends x_1 to s and all other irreducible characters to 1. Hence the top component of $S(s)$ has a trivial minus part and a plus part equal to

$$(s,s,s) \in \prod_{\ell=p,q,r} (\mathbb{F}_\ell \otimes \mathbb{Z}[\eta_p,\eta_q,\zeta_r])^{\times}_{(2)}$$

Assume that $\sigma_4(e) = S(s)$. By [BM] we have $\sigma_4(e|Q(8p,q)) = 0$ so $S(s)$ must restrict to zero under $S(\pi) \to S(Q(8p,q))$. By Proposition 3 of [BM] this implies that $(\frac{s}{q}) = 1$. We take norms to the subfield $\mathbb{Q}(\sqrt{q})$ of $\mathbb{Q}(\eta_p,\eta_q,\zeta_r)$. The norm of (s,s,s) is

$$\left((\tfrac{s}{p})^{(q-1)(r-1)/4} , (\tfrac{s}{q})^{(p-1)(r-1)/4} , (\tfrac{s}{r})^{(p-1)(q-1)/8} \right)$$
$$= (\ 1\ ,\ 1\ ,\ 1\ ) \in \prod_{\ell} (\mathbb{F}_\ell \otimes \mathcal{O})^{\times}_{(2)}$$

where $\mathcal{O}$ is the ring of integers of $\mathbb{Q}(\sqrt{q})$. The norm of $((2-\eta_q)(1-\zeta_r)^2, (2-\eta_p)(1-\zeta_r)^2, (2-\eta_p)(2-\eta_q))$ is $(1,-1,1)$. To prove Lemma 1.8 we must show that there is no unit of $\mathcal{O}$ that reduces to $(1,-1,1)$. Let ε_q be the fundamental unit of $\mathcal{O}$. The group $\prod(\mathbb{F}_\ell \otimes \mathcal{O})^{\times}$ is equal to

$$\mathbb{F}_p^{\times} \oplus \mathbb{F}_p^{\times} \oplus \mathbb{F}_q^{\times} \oplus \mathbb{F}_r(\sqrt{q})^{\times}.$$

Let ε_q map to (a_1,a_2,b,c). Using that $N(\varepsilon_q) = -1$ we see that a_1 has odd order and a_2 has order equal to 2 times an odd number.

Furthermore, the order of b is 4 while the order of c must be divisible by 8. Then it is easy to see that no power of ε_q can reduce to $(1,-1,1)$ or $(-1,1,-1)$. Since $\mathcal{O}^{\times} = \langle -1, \varepsilon_q \rangle$ this proves Lemma 1.8.//

Now we can easily prove Theorem A. First we choose $p \equiv 3 \bmod 4$, $q \equiv 1 \bmod 8$ such that the order of p modulo q is odd and the order of q modulo p is maximal odd. By Theorem C of [B2], the group $Q(8p,q)$ acts freely on the sphere S^{3+8k}. Then we choose $r \equiv 3 \bmod 4$ such that $(\frac{r}{q}) = -1$. By Lemma 1.8 , $Q(8p,q) \times \mathbb{Z}/r$ does not act freely on any finite complex having the homotopy type of the sphere S^{3+8k}. This proves Theorem A.

§2. Proof of Theorem B.

We consider the group $\pi = Q(8p,q,r)$. The proof of Theorem B is divided into five steps similar to the proof of Theorem A. The first step is again a decomposition of the sequence (1.2). The second step is the calculation of the top component of (1.2). The third step is to detect the top component of (1.2) on the cyclic subgroup $C = \mathbb{Z}/2pqr$. The fourth step is to calculate the top component of $\hat{\tau}_4(e)$ for e the almost linear generator of $\hat{H}^4(Q(8p,q,r), \mathbb{Z})$. The fifth and final step is to choose p,q and r such that $\sigma_4(e) \notin S(Q(8p,q,r))$ but the finiteness obstruction of $Q(8p,q)$, $Q(8p,r)$ and $Q(8q,r)$ all vanish.

Since the calculations are very similar to the calculations in §1 we shall be very brief.

Step 1: This is quite similar to step 1 of §1. The group allows three idempotent endomorphisms E_p, E_q and E_r. These induce a decomposition of the character ring into 8 parts and we concentrate on the top component, i.e. the component corresponding to the idempotent $(1-E_p)(1-E_q)(1-E_r)$.

<u>Step 2:</u> The rational group algebra splits like

$$\begin{aligned}\mathbb{Q}\pi = {} & \mathbb{Q}_{++} \oplus \mathbb{Q}_{+-} \oplus \mathbb{Q}_{-+} \oplus \mathbb{Q}_{--} \oplus \mathbb{H}_{\mathbb{Q}} \oplus M_2(\mathbb{Q}(\eta_p)) \oplus M_2(\mathbb{Q}(\eta_p)) \\ & \oplus \mathbb{H}_{\mathbb{Q}(\eta_{4p})} \oplus M_2(\mathbb{Q}(\eta_q)) \oplus M_2(\mathbb{Q}(\eta_q)) \oplus \mathbb{H}_{\mathbb{Q}(\eta_{4q})} \oplus M_2(\mathbb{Q}(\eta_r)) \\ & \oplus M_2(\mathbb{Q}(\eta_r)) \oplus \mathbb{H}_{\mathbb{Q}(\eta_{4r})} \oplus M_4(\mathbb{Q}(\eta_p,\eta_q)) \oplus A(\mathbb{Q}(\zeta_{pq})/\mathbb{Q}(\eta_p,\eta_q)) \\ & \oplus M_4(\mathbb{Q}(\eta_p,\eta_r)) \oplus A(\mathbb{Q}(\zeta_{pr})/\mathbb{Q}(\eta_p,\eta_r)) \oplus M_4(\mathbb{Q}(\eta_q,\eta_r)) \\ & \oplus A(\mathbb{Q}(\zeta_{qr})/\mathbb{Q}(\eta_q,\eta_r)) \oplus M_4(\mathbb{Q}(\alpha)) \oplus A(\mathbb{Q}(\zeta_{pqr})/\mathbb{Q}(\alpha))\end{aligned}$$

Here $\alpha = \zeta_p\zeta_q\zeta_r + \zeta_p^{-1}\zeta_q\zeta_r^{-1} + \zeta_p\zeta_q^{-1}\zeta_r^{-1} + \zeta_p^{-1}\zeta_q^{-1}\zeta_r$ and $A(L/K)$ denotes some simple algebra with center K and split by L. We denote the characters corresponding to $M_4(\mathbb{Q}(\alpha))$, $A(\mathbb{Q}(\zeta_{pqr})/\mathbb{Q}(\alpha))$ by $x_{pqr,+}$, $x_{pqr,-}$. The top component of $R(\pi)/\Omega$ is $\mathbb{Z}\cdot x_{pqr,+} \oplus \mathbb{Z}\cdot x_{pqr,-}$ and $E_{top}(\mathcal{O}^{\times}_{Z(\mathbb{Q}\pi)}) = \mathcal{O}^{\times} \oplus \mathcal{O}^{\times}$ where $\mathcal{O}$ is the ring of integers of $\mathbb{Q}(\alpha)$. Using the same technique as in §1 we get

<u>Lemma 2.1.</u> The top component of the map

$$\Phi_\ell : \mathcal{O}^{\times}_{Z(\mathbb{Q}\pi)} \to \hat{D}_\ell(\pi)_{(2)}$$

from 1.3 is the reduction map

$$\mathcal{O}^{\times} \oplus \mathcal{O}^{\times} \to (\mathbb{F}_\ell \otimes \mathcal{O})^{\times}_{(2)} \oplus (\mathbb{F}_\ell \otimes \mathcal{O})^{\times}_{(2)}$$

Therefore we get

<u>Proposition 2.2.</u> The top component of

$$\Phi : \mathcal{O}^{\times}_{Z(\mathbb{Q}\pi)} \to \hat{D}(\pi)_{(2)}$$

is the reduction map

$$\mathcal{O}^{\times} \oplus \mathcal{O}^{\times} \to \prod_{\ell=p,q,r} (\mathbb{F}_\ell \otimes \mathcal{O})^{\times}_{(2)} \oplus \prod_{\ell=p,q,r} (\mathbb{F}_\ell \otimes \mathcal{O})^{\times}_{(2)} .$$

<u>Step 3:</u> This is similar to step 3 of §1.

<u>Proposition 2.3.</u> The restriction map

$$E_{top}(J(Z(\mathbb{Q}\pi))_+/\mathrm{Nrd}(U(\mathbb{Z}\pi)))_{(2,f)} \to E_{top}(J(\mathbb{Q}C)/\mathrm{Nrd}(U(\mathbb{Z}C)))$$

is injective and induced by inclusion maps of ideal groups and residue fields.

Step 4. The restriction map induce an isomorphism

$$\hat{H}^4(\pi, \mathbb{Z}) \simeq \prod_{\ell} \hat{H}^4(\mathbb{Z}/\ell, \mathbb{Z}) \oplus \hat{H}^4(Q(8), \mathbb{Z})$$

We let e be the generator of $\hat{H}^4(\pi, \mathbb{Z})$ that maps to $c_2(\chi_\ell + \chi_\ell^{-1})$ on the Sylow-ℓ-parts for $\ell = p,q,r$ and to $c_2(\Gamma)$ on $Q(8)$. The restriction to C of this generator is equal to $c_2(\chi_p\chi_q\chi_r\chi_2 + \chi_p^{-1}\chi_q^{-1}\chi_r^{-1}\chi_2)$. The idelic Reidemeister torsion of $e|C$ is the Reidemeister torsion of the lens space corresponding to this representation and this can be calculated using [Mi]. From this the top component can be calculated: It has a plus part and a minus part, and the plus part is

$$(2-\eta_{pqr})(2-\eta_p)(2-\eta_q)(2-\eta_r)4p^2q^2r^2/(2-\eta_{pq})(2-\eta_{pr})(2-\eta_{qr}) \in \mathbb{Q}(\zeta_{pqr}).$$

From Proposition 2.3 we get

Proposition 2.4. The plus part of $E_{top}(\sigma_4(e))$ is the image under ∂ of the 2-primary part of the element

$$(\ (2-\eta_q)(2-\eta_r)\ ,\ (2-\eta_p)(2-\eta_r)\ ,\ (2-\eta_p)(2-\eta_q)\ )$$

of $(\mathbb{F}_p \otimes \mathcal{O})^\times \oplus (\mathbb{F}_q \otimes \mathcal{O})^\times \oplus (\mathbb{F}_r \otimes \mathcal{O})^\times$.

Step 5: We choose $p \equiv 3 \mod 4$; $q,r \equiv 1 \mod 8$ and $(\frac{r}{q}) = -1$. By Proposition 5 of [BM] the finiteness obstructions of $Q(8p,q)$, $Q(8p,r)$ and $Q(8q,r)$ are all zero. To finish the proof of Theorem B we must prove that $\sigma_4(e) \notin S(\pi)$.

Lemma 2.5. The unit index of $\mathbb{Q}(\alpha)$ is equal to 1, so $\mathcal{O}^\times = \mathcal{O}_o^\times$ where $\mathcal{O}_o$ is the ring of integers of $\mathbb{Q}(\eta_p,\eta_q,\eta_r)$.

Proof: The maximal real subfield of $\mathbb{Q}(\alpha)$ is $\mathbb{Q}(\eta_p,\eta_q,\eta_r)$. Since $\mathbb{Q}(\zeta_{pqr})/\mathbb{Q}(\alpha)$ is unramified at p,q and r and $\mathbb{Q}(\zeta_{pqr})/\mathbb{Q}(\eta_p,\eta_q,\eta_r)$ has some ramification at p,q and r, the extension $\mathbb{Q}(\alpha)/\mathbb{Q}(\eta_p,\eta_q,\eta_r)$ must have some ramification at p,q and r. By Satz 15 of [Ha] (and the remark following it) the unit index must be 1. //

Assume now that $\sigma_4(e) = S(s)$. Then $E_{top}(\sigma_4(e)) = E_{top}(S(s))$. Let x be the element of $\hat{D}(\pi)_{(2)}$ that has a trivial minus part and a plus part equal to $(s,s,s) \in \prod_{\ell}(\mathbb{F}_\ell \otimes\ )^{\times}_{(2)}$. The top component of the Swan module S(s) is equal to $\partial(x)$. If $E_{top}(\sigma_4(e)) = E_{top}(S(s))$ there would by Lemma 2.3 exist a unit of $\mathcal{O}_o$ reducing to

$$\left((2-\eta_q)(2-\eta_r)s^{-1},\ (2-\eta_p)(2-\eta_r)s^{-1},\ (2-\eta_p)(2-\eta_q)s^{-1} \right)_{(2)}$$

in $\prod_{\ell}(\mathbb{F}_\ell \otimes \mathcal{O}_o)^{\times}_{(2)}$. We take norms from $\mathbb{Q}(\eta_p,\eta_q,\eta_r)$ to $\mathbb{Q}$ and get a unit of $\mathbb{Z}$ reducing to

$$\left((\tfrac{q}{p})^{(r-1)/2}(\tfrac{r}{p})^{(q-1)/2}(\tfrac{s}{p})^{(q-1)(r-1)/4},\ (\tfrac{p}{q})^{(r-1)/2}(\tfrac{r}{q})^{(p-1)/2}(\tfrac{s}{q})^{(r-1)(p-1)/4},\ (\tfrac{p}{r})^{(q-1)/2}(\tfrac{q}{r})^{(p-1)/2}(\tfrac{s}{r})^{(p-1)(q-1)/4} \right)$$

$$= \left(1\ ,\ (\tfrac{r}{q})\ ,\ (\tfrac{q}{r}) \right) = (1\ ,\ -1\ ,\ -1\ )$$

in $\mathbb{F}_p^{\times} \oplus \mathbb{F}_q^{\times} \oplus \mathbb{F}_r^{\times}$. But this is cearly impossible.

This completes the proof of Theorem B.

Remark: This last example shows that the finiteness obstruction of Q(8a,b,c) presents new problems when compared to the finiteness obstruction of Q(8a,b). In [B1] we saw that the plus part of $\sigma_4(e)$ could always be "made rational" for groups of type Q(8a,b). The above example shows that this is not possible for groups of type Q(8a,b,c).

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THE RATIONAL HOMOTOPY TYPE OF $\Omega Wh^{Diff}(*)$.

Marcel Bökstedt

Let $Wh^{Diff}(*)$ be the double delooping of the smooth stable concordancespace of a point. In [11], Waldhausen defined a map $f: G/O \to \Omega Wh^{Diff}(*)$, using the framework of algebraic K-theory of topological spaces.

Theorem 1. The map f is a rational homotopy equivalence.

Such a theorem was originally conjectured by Hatcher, using a different definition of a map $G/O \to \Omega Wh^{Diff}(*)$.

There is a map $Wh^{Diff}(*) \to K(Z)$ which is known to be a rational homotopy equivalence [10]. The space $K(Z)$ has recently been studied using etale homotopy theory [3], [5], [8]. We are going to prove Theorem 1 by comparing geometrical information related to the map $f: G/O \to \Omega Wh^{Diff}(*)$ with input from etale homotopy theory. It then turns out that this procedure yields information about $K(Z)$.

Let $JK(Z)^{\wedge}_{2}$ be the fibre of the composite

$$Z \times BO^{\wedge}_{2} \xrightarrow{\psi^3 - id} BSpin^{\wedge}_{2} \to BSU^{\wedge}_{2}$$

where ψ^3 is the Adams operation, and $X^{\wedge}_{2}$ the Sullivan-Bousfield-Kan completion of X at the prime 2.

Theorem 2. $\Omega K(Z)^{\wedge}_{2}$ splits as a product up to homotopy $\Omega JK(Z)^{\wedge}_{2} \times ?$.

In particular, we get new descriptions of the Borel classes completed at 2.

The author is indepted to F. Waldhausen for explaining the algebraic K-theory of spaces to him in a very large number of discussions.

§1. The main argument.

We begin be recalling the construction of the map $G/O \to \Omega Wh^{Diff}(*)$. We need some preliminary recollections and definitions.

Let $A(*)$ be Waldhausens algebraic K-theory of a point [10], [11], [12].

Fact 1.1. There are maps $QS^0 \to A(*) \to K(Z)$ such that the composition is the standard map, induced by the inclusion of the permutation matrices:

$$QS^0 = \lim_{\substack{\to \\ n}} (B\Sigma_n)^+ \quad \lim_{\substack{\to \\ n}} (BGL_n Z)^+ = K(Z).$$

Fact 1.2. There is a splitting $A(*) \simeq QS^0 \times Wh^{Diff}(*) \times \mu(*)$ [11], [12]. The inclusion of the first factor in this splitting is the map of 1.1.

Fact 1.3. Let SG be the monoid of stable, orientation preserving homotopy equivalences of the sphere. There is a map $BSG \to A(*)$ with the following three properties:

(i) $BSG \to A(*) \to K(Z)$ is the trivial map.

(ii) There is a nullhomotopy of the map [12]

$$BSO \to BSG \to A(*) \to Wh^{Diff}(*)$$

(iii) The composite map $BSG \to A(*) \to QS^0$ is homotopic to a different composite $BSG \to SG \to QS^0$. In this second composite, the map $\eta\colon BSG \to SG$ is defined as "multiplication by the Hopf map". Precisely, BSG is an infinite loop-space in a standard way. In particular, we can write $BSG = \Omega^2 X$ for some X. The Hopf map induces $\eta\colon BSG = \Omega^2 X \to \Omega^3 X = SG$. The map $SG \to QS^0$ is the inclusion of the 1-component [1].

Fact 1.4. $\mu(*) \simeq *$. At the time of writing, there is no published proof of this very important theorem. However, there are at least two independent unpublished proofs, due to Waldhausen [13] respectively Igusa [6].

The nullhomotopy of 1.3(ii) gives rise to a map $G/O \to \Omega Wh^{Diff}(*)$. Because of 1.1, 1.2 and 1.3(i), the composites

$$r_1\colon BSG \to A(*) \to QS^0 \times Wh^{Diff}(*) \times \mu(*) \to Wh^{Diff}(*) \times \mu(*) \to A(*) \to K(Z)$$

$$r_2\colon BSG \to A(*) \to QS^0 \times Wh^{Diff}(*) \times \mu(*) \to \qquad QS^0 \qquad \to A(*) \to K(Z)$$

sum up to a nullhomotopic map $BSG \to K(Z)$.

According to 1.3(iii), r_2 is homotopic to the composite

$$r_3\colon BSG \xrightarrow{\eta} SG \to QS^0 \to K(Z).$$

Since η is 2-torsion, this also implies $r_1 \simeq r_3$. By the vanishing of $\mu(*)$, even the composite

$$r\colon BSG \to A(*) \to Wh^{Diff}(*) \to K(Z)$$

is homotopic to r_3. Note that we are using the deep fact 1.4 in an essential way at this point

In other words, we have the following strictly commutative diagram:

$$\begin{tikzcd}
BSO \arrow[r] \arrow[d, equal] & BSG \arrow[d, "\eta"] \arrow[dr, "r_3"] & \\
BSO \arrow[r, "s"] & SG \arrow[r] & K(Z)
\end{tikzcd}$$

and a homotopy F from r_3 to r.

Composing F with the nullhomotopy of 1.3(ii), we obtain a strictly commutative diagram:

1.5.

$$\begin{tikzcd}
G/O \arrow[r] \arrow[d] & BSO \arrow[r] \arrow[d, "="] & BSG \arrow[d, "\eta"] \\
\text{Fiber}(s) \arrow[r] \arrow[d] & BSO \arrow[r, "s"] \arrow[d] & SG \arrow[d] \\
\Omega K(Z) \arrow[r] & (EK(\mathbb{Z}) \arrow[r] & K(Z)
\end{tikzcd}$$

The composite of the two maps in the left column is homotopic to the map $f\colon G/O \to \Omega K(Z)$ induced by the null-homotopy of 1.3(ii).

Up to homotopy, the map $s\colon BSO \to SG$ actually factors as $BSO \to BO \xrightarrow{\eta} SO \to SG$, where $\eta\colon BO \to SO$ is multiplication by the generator $\eta \in \pi_1(BO)$, and $BSO \to BO$ is the inclusion.

Recall that by the solution to the Adams conjecture [7],[9], the map $SO \to SG$ factors over a ceratin torsion space SJ. The p-part SJ_p of this space is homotopy equivalent to the fiber of the map

$$BO_p^\wedge \xrightarrow{\psi^\ell - id} BSpin_p^\wedge$$

where ℓ generates the multiplicative group of $Z_p^\wedge$. Choose such a number ℓ.

Let t denote the composite $BSO_p^\wedge \to SO_p^\wedge \to SJ_p$.

We obtain the following diagram which commutes up to homotopy:

$$\begin{tikzcd}
BSO^\wedge_p \arrow[r, "t"] \arrow[d, equal] & SJ_p \arrow[d] \arrow[dr] & \\
BSO^\wedge_p \arrow[r, "s"] & SG_p \arrow[r] & K(Z)^\wedge_p
\end{tikzcd}$$

Combining this with 1.5, we can summarize our information in the following diagram, commutative up to homotopy as usual:

1.6.
$$\begin{tikzcd}
(G/O)^\wedge_p \arrow[r] \arrow[dr, "f"'] & \text{Fiber}(s)^\wedge_p \arrow[d, "g"] & \text{Fiber}(t) \arrow[l] \arrow[dl, "h"] \\
& \Omega K(Z)^\wedge_p &
\end{tikzcd}$$

Definition 1.7. $JK(Z)^\wedge_p$ is the fiber of the map

$$Z \times BO^\wedge_p \xrightarrow{\psi^\ell - id} BSpin^\wedge_p \xrightarrow{c} BSU^\wedge_p \ .$$

The map $c\colon BSpin^\wedge_p \to BSU^\wedge_p$ is induced by complexification of vectorbundles.

There is a diagram of fibrations up to homotopy:

1.8.
$$\begin{tikzcd}
BSO^\wedge_p \arrow[r] \arrow[d, "\eta"'] & BSO^\wedge_p \arrow[r] \arrow[d, "t"'] & * \arrow[r] \arrow[d] & BBSO^\wedge_p \arrow[d, "\eta"'] \\
Spin^\wedge_p \arrow[r] & SJ^\wedge_p \arrow[r] \arrow[d, "i"'] & BO^\wedge_p \arrow[r] \arrow[d] & BSpin^\wedge_p \arrow[d, "c"'] \\
& JK(Z)^\wedge_p \arrow[r] & BO^\wedge_p \arrow[r] & BSU^\wedge_p
\end{tikzcd}$$

Here we have used that $\eta\colon BBSO^\wedge_p \to BSpin^\wedge_p$ is homotopic to the inclusion of the fiber of $c\colon BSpin^\wedge_p \to BSU^\wedge_p$.

In lemma 2.1 we will show that here exists a map $\phi\colon K(Z)^\wedge_p \to JK^\wedge_p$ such that the composite $SJ^\wedge_p \to (QS^0)^\wedge_p \to K(Z)^\wedge_p \to JK(Z)^\wedge_p$ is homotopic to the map $i\colon SJ^\wedge_p \to JK(Z)^\wedge_p$ occuring in diagram 1.8. We claim that the map $(\Omega\phi)h$ is a homotopy equivalence for $p = 2$, proving Theorem 2.

Consider the following diagram:

1.9.
$$\begin{array}{ccccc}
\mathrm{Fiber}(t)^\wedge_2 & \longrightarrow & BSO^\wedge_2 & \xrightarrow{t} & SJ^\wedge_2 \\
{\scriptstyle h}\downarrow & & \downarrow & & \downarrow \\
\Omega K(Z)^\wedge_2 & \longrightarrow & PK(Z)^\wedge_2 & \longrightarrow & K(Z)^\wedge_2 \\
{\scriptstyle \Omega\phi}\downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} \\
\Omega JK(Z)^\wedge_2 & \longrightarrow & PJK(Z)^\wedge_2 & \longrightarrow & JK(Z)^\wedge_2
\end{array}$$

The rows of 1.9 are fibrations, and the vertical maps define maps of fibrations. We want to prove that the composite of the maps in the left column is a homotopy equivalence. This is equivalent to showing that the induced map between the fibres of the composites of the two other columns is a homotopy equivalence. According to the assumtion on ϕ, the composite of the right column is homotopic to $i\colon SJ^\wedge_2 \to JK(Z)^\wedge_2$.

Using 1.8, we see that both fibres in question are homotopy equivalent to $BSO^\wedge_2$. Furthermore, under these homotopy equivalences, the map of fibres correspond to a map $s\colon BSO^\wedge_2 \to BSO^\wedge_2$, such that $t \simeq ta$. Theorem 2 now follows from this condition and the following

<u>Lemma 1.10</u>. Let t be the composite $BSO^\wedge_n \overset{\eta}{\to} SO^\wedge_2 \to SJ^\wedge_2$. Given any map $a\colon BSO^\wedge_2 \to BSO^\wedge_2$ so that $t \simeq ta$, the map a is a homotopy equivalence.

<u>Proof</u>. It suffices to show that a induces an isomorphism on homology with coefficients in $Z/2$. To do so, we first determine the image of $t^*\colon H^*(SJ;Z/2) \to H^*(BSO;Z/2)$.

By a spectral sequence argument [14], $H^*(SJ;Z/2) \to H^*(SO;Z/2)$ surjects, so that $\mathrm{im}(t^*) = \mathrm{im}(\eta^*)$.

Recall that $H^*(BSO;Z/2) = Z/2[w_2,w_3,\ldots]$. One can show that $\mathrm{im}(\eta^*)$ is the subalgebra $P \subset H^*(BSO;Z/2)$ generated by the Newton-polynomials σ_i in $w_1 = 0, w_2, w_3, \ldots,$ for instance by looking at the Serre spectral sequence of the fibration $BU \to BSO \to \mathrm{Spin}$.

We claim that if $a^*: H^*(BSO;Z/2) \to H^*(BSO;Z/2)$ is any map of algebras, fixing the subalgebra P and commuting with the standard Bockstein map, then a^* is an isomorphism.

But this is an easy consequence of the two relations $\sigma_{2i+1} = w_{2i+1} + \text{decomposable};\quad Sq^1 w_{2i} = w_{2i+1}$.

Finally, we will show that Theorem 2 implies Theorem 1. Recall from [10] that the maps $Wh^{\mathrm{Diff}}(*) \to A(*)$ and $A(*) \to K(Z)$ induce rational isomorphisms on π_i for $i > 1$. Furthermore, by Borel [2],

$$\pi_i \Omega K(Z) \otimes Q \simeq \pi_i G/O \otimes Q \simeq \begin{cases} Q & \text{if } i = 4j,\ j \geq 1 \\ 0 & \text{else} \end{cases}$$

so that it is enough to show that the map $\pi_i(G/O) \to \pi_i(\Omega K(Z))$ is injective after tensoring with Q. Since $\pi_i(G/O)$ is finitely generated, this is equivalent to the statement that $\pi_i(G/O^\wedge_2[1/2]) \to \pi_i(\Omega K(Z)^\wedge_2[1/2])$ is injective [9].

Now consider the diagram, whose rows are fibrations

$$\begin{array}{ccccc}
G/O^\wedge_2 & \longrightarrow & BSO^\wedge_2 & \longrightarrow & BSG^\wedge_2 \\
\downarrow & & \| & & \downarrow{\scriptstyle \eta} \\
\mathrm{fiber}(s)^\wedge_2 & \longrightarrow & BSO^\wedge_2 & \xrightarrow{s} & SG^\wedge_2 \\
\uparrow & & \| & & \uparrow \\
\Omega JK(Z)_2 \simeq \mathrm{fiber}(t)^\wedge_2 & \longrightarrow & BSO^\wedge_2 & \xrightarrow{t} & SJ^\wedge_2
\end{array}$$

The maps $G/O^\wedge_2 \to \Omega K(Z)^\wedge_2$ respectively $\Omega JK(Z)^\wedge_2 \to \Omega K(Z)^\wedge_2$ factor up to homotopy over a map $\mathrm{fiber}(s)^\wedge_2 \to \Omega K(Z)^\wedge_2$. The spaces in the right column of the diagram above are all torsion spaces, so that the maps in the left column induce maps which become equivalences after inverting the prime 2. The statement above now follows trivially from Theorem 2, saying that $\pi_i \Omega JK(Z)^\wedge_2 \to \pi_i \Omega K(Z)^\wedge_2$ is a split injection.

§2. Etale homotopy theory.

In this paragraph we will use the etale homotopy of simplicial schemes as developed by Friedlander to prove.

Lemma 2.1. There is a map $K(Z) \to JK(Z)^\wedge$ such that the composite

$$SJ_p \to (QS^0)^\wedge_p \to K(Z)^\wedge_p \to JK(Z)^\wedge_p$$

is homotopic to the map of 1.8.

Following ideas by Quillen, Friedlander defines a homotopy class (ℓ prime $\neq p$) $(BGL(Z/\ell)^+) \to BU^\wedge_p$. The way we prove lemma 2.1 is to consider the following diagram (where $K(Z)_1$ denotes the 1-component of $K(Z)$):

$$\begin{array}{ccc} K(Z)_1 & \longrightarrow & BO^\wedge_p \\ \downarrow & & \downarrow \\ (BGL(Z/\ell)^+)_p & \longrightarrow & BU^\wedge_p \end{array} \qquad (2.2.)$$

We will show that this diagram commutes up to homotopy. This homotopy defines a map $K(Z)_1 \to PB$, where PB is the homotopy pullback of the two maps in 2.2 with abutment in $BU^\wedge_p$. The map of 2.1 is the lifting of this map to a certain covering space of PB, homotopy equivalent to $JK(Z)^\wedge_p$.

We note that for $\ell = p$, there is no such homotopy commutative diagram, since $K(Z/\ell)^\wedge_\ell$ is contractible but the map

$QS^0 \to K(Z) \to BU^{\wedge}_{\ell}$ is non-trivial for all ℓ.

Recall that to a locally noetherian simplicial scheme $X.$ we can associate its etale topological type $(X.)_{et}$. The etale topological type is a functor to the category of pro-bi-simplicial sets, and $H^*((X.)_{et})$ is the etale cohomology of X. As an example of this construction, consider the case where $G \to \operatorname{Spec} R$ is a groupscheme over the spectrum of the Noetherian ring R. The classifying simplicial scheme BG is defined to be $\underbrace{G \underset{\operatorname{Spec} R}{\times} G \underset{\operatorname{Spec} R}{\times} G \cdots \underset{\operatorname{Spec} R}{\times} G}_{n}$ in degree n.

The face and degeneracy maps are defined using the product $\mu\colon G \underset{\operatorname{Spec} R}{\times} G \to G$ and the unit $e\colon \operatorname{Spec} R \to G$ in the usual way.

We consider the special case of this construction where $G = GL_n(R)$. More precisely, let $GL_n(R)$ be the scheme over $\operatorname{Spec} R$ defined by

$$GL_n(R) = \operatorname{Spec} A$$

$$A = R[x_{ij},d]/\det(x_{ij})d = 1 \qquad i,j = 1,\dots,n.$$

The groupstructure on $GL_n(R)$ is given as the dual of $\tilde{\mu}\colon A \to A \otimes A$ where $\mu(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$; $\tilde{\mu}(d) = d \otimes d$.

Let $GL_n(R)^{\delta}$ be the group of sections (in the category of schemes over $\operatorname{Spec}(R)$) of the structure map $GL_n(R) \to \operatorname{Spec} R$. We can identify this group in a canonical way with the group of invertible n-by-n matrices with entries in R.

There is an adjunction map $ev\colon GL_n(R)^{\delta} \times \operatorname{Spec}(R) \to GL_n(R)$, such that for $g \in GL_n(R)^{\delta}$ the map $ev(g,-)$ is a map of schemes over $\operatorname{Spec} R$. It induces $ev\colon GL_n(R)^{\delta} \times (\operatorname{Spec} R)_{et} \to GL_n(R)$, and even $ev\colon BGL_n(R) \times (\operatorname{Spec} R)_{et} \to BGL_n(R)_{et}$. In the last formula, one schould interprete the left hand side as a pro-bi-simplicial set, which is a product of a simplicial set with a pro-simplicial set.

Let $a\colon R \to S$ be a ring homomorphism. Consider the commutative diagram:

2.3.
$$\begin{tikzcd}
BGL_n(R)^{\delta} \times (\operatorname{Spec} S)_{et} \arrow[rr, "BGL_n(a)^{\delta} \times id"] \arrow[d, "id \times (\operatorname{Spec} a)_{et}"'] & & BGL_n(S)^{\delta} \times (\operatorname{Spec} S)_{et} \arrow[d, "ev"] \\
BGL_n(R)^{\delta} \times (\operatorname{Spec} R)_{et} \arrow[dr, "ev"'] & & BGL_n(S)_{et} \arrow[dl, "BGL_n(a)_{et}"] \\
& BGL_n(R)_{et} &
\end{tikzcd}$$

Let F_ℓ be the primefield with ℓ elements. Let $W(\bar{F}_\ell)$ be the ring of Witt vectors on its algebraic closure $\bar{F}$. We choose (noncanonically) an ambedding $\iota\colon W(\bar{F}_\ell) \to C$, the complex numbers.

It is known that $(\operatorname{Spec} R)_{et}$ is weakly contractible if either R is a separably closed field, or a local ring satisfying Hensel's lemma, with separably closed residue field. In particular it is true for $R = \bar{F}_\ell$, C, or $W(\bar{F}_\ell)$.

Theorem 2.4. (Friedlander [3],[5]). The base change maps $BGL_n(\bar{F}_\ell)_{et} \to BGL_n(W(\bar{F}_\ell))_{et} \leftarrow BGL_n(C)_{et}$ induce isomorphisms on homology groups with coefficients in Z/p, $p \neq \ell$.

The Friedlander map $K(Z/\ell)^{\wedge}_{p} \to BU^{\wedge}_{p}$ mentioned in the beginning of this paragraph, is defined by constructing a map $BGL_n(Z/\ell)^{\delta} \to BGL_n(\bar{F}_\ell)_{et}$, and then using 2.4 to identify the homotopy limit of the latter pro-object with $BU^{\wedge}_{\ell}$, at least up to homotopy.

This map is given by identifying $BGL(Z/\ell)^{\delta}$ with the prosimplicial set $B(GL(Z/\ell) \times \operatorname{Spec} \bar{F}_\ell)_{et}$, and then including the groupscheme $GL_n(Z/\ell)^{\delta} \times \operatorname{Spec} \bar{F}_\ell$ in $GL_n(\bar{F}_\ell)$ as the kernel of the Frobenius map. Note that this last inclusion is exactly the composite $(ev) \circ (GL_n(a)^{\delta} \times id)$ related to the map $a: Z/\ell \to \bar{F}_{\ell}$.

Lemma 2.5. The following diagram commutes

$$\begin{tikzcd}
BGL_n(Z)^{\delta} \times (\mathrm{Spec}\ \bar{F}_{\ell})_{et} \arrow[r] \arrow[d] & BGL_n(Z)^{\delta} \times \mathrm{Spec}\ W(\bar{F}_{\ell})^{\delta} \arrow[dd] & BGL_n(Z)^{\delta} \times (\mathrm{Spec}\ C)_{et} \arrow[l] \arrow[dd] \\
BGL_n(Z/\ell)^{\delta} \times (\mathrm{Spec}\ \bar{F}_{\ell})_{et} \arrow[d] & & \\
(BGL_n(\bar{F}_{\ell}))_{et} \arrow[r] & (BGL_n(W(\bar{F}_{\ell}))_{et} \arrow[r] & (BGL_n(C))_{et}
\end{tikzcd}$$

The proof of the lemma consists of repeated application of 2.3.

Let $(Z/p)_{\infty}$ be the Bousfield-Kan completion at p. Let $\underleftarrow{\mathrm{holim}}$ denote the Bousfield-Kan homotopy limit. After applying the composite functor $\underleftarrow{\mathrm{holim}}\ (Z/p)_{\infty}$ the maps of 2.4 become homotopy equivalences of simply connected simplicial sets. The map $BGL_n(Z/\ell)^{\delta} \to BGL_n(Z/\ell)^{\delta} \times (\mathrm{Spec}\ \bar{F}\ )_{et} \to (GL_n(\bar{F}_{\ell}))_{et}$ determines a homotopy class $(BGL_n(Z/\ell)^{\delta})^{+} \to \underleftarrow{\mathrm{holim}}(Z/p)_{\infty}(BGL_n(C))_{et}$ (where "+" denotes the plusconstruction of Quillen). Because of 2.5, the composite of the natural map $(BGL_n(Z)^{\delta})^{+} \to (BGL_n(Z/\ell)^{\delta})^{+}$ with this map is homotopic to the composite $(BGL_n(Z)^{\delta})^{+} \to (BGL_n(Z)^{\delta})^{+} \times \underleftarrow{\mathrm{holim}}(Z/p)_{\infty}(\mathrm{Spec}\ C)_{et} \to$
$\to \underleftarrow{\mathrm{holim}}(Z/p)_{\infty}(BGL_n(C))_{et}$.

Recall that for any topological space X there is a simplicial set Sing(X). If X is a scheme of finite type over C, there is a map $\mathrm{Sing}(X) \to (X)_{et}$, inducing a homotopy equivalence $(Z/p)_{\infty}\mathrm{Sing}(X) \to \underleftarrow{\mathrm{holim}}(Z/p)_{\infty}(X)_{et}$. Applying this for the simplicial scheme $X = BGL_n(C)$, we get a map $(BGL_n(Z/\ell)^{\delta})^{+} \to (BU_n)^{\wedge}_{p}$.

Since the map $\mathrm{Sing}(X) \to (X)_{et}$ is natural in X, we have the following commutative diagram

$$\begin{tikzcd} BGL_n(Z)^{\delta} \times \mathrm{Sing}(C) \arrow[r] \arrow[d] & BGL_n(Z)^{\delta} \times (\mathrm{Spec}\, C)_{et} \arrow[d] \\ \mathrm{Sing}(BGL_n(C)) \arrow[r] & (BGL_n(C))_{et} \end{tikzcd}$$

It follows that the composite $(BGL_n(Z)^{\delta})^+ \to (BGL_n(Z/\ell)^{\delta})^+ \to BGL_n(C)^{\wedge}_p$ is homotopic to the standard map. Passing to the limit in n, we obtain the diagram promised in the beginning of the section, that is, we can find a map $K(Z)_1 \to PB$. We want to determine PB.

<u>Theorem 2.6</u>. (Friedlander). The map $(BGL(Z/\ell)^+)^{\wedge}_p \to BU^{\wedge}_p$ defined above is homotopic to the inclusion of the fiber of the map $\psi^{\ell} - \mathrm{id}\colon BU^{\wedge}_p \to BU^{\wedge}_p$.

<u>Proof of 2.1</u>. We can identify the homotopy type of PB by the following diagram of fibrations up to homotopy:

$$\begin{tikzcd} PB \arrow[r] \arrow[d] & BO^{\wedge}_p \arrow[r, "(\psi^{\ell} - \mathrm{id})"] \arrow[d] & BU^{\wedge}_p \arrow[d] \\ (BGL(Z/\ell)^+)^{\wedge}_p \arrow[r] & BU^{\wedge}_p \arrow[r, "\psi^{\ell} - \mathrm{id}"] & BU^{\wedge}_p \end{tikzcd}$$

From definition 1.7, and the long exact sequence of homotopy groups of the upper row, we see that each component in $JK(Z)^{\wedge}_p$ is the covering space of PB corresponding to a certain group $Z/2 \otimes Z^{\wedge}_p$. In particular, we can lift the map defined by 2.2 to $JK(Z)^{\wedge}_p$.

From the construction it follows that the obstruction to finding a homotopy between this lifting and the map of 1.8 is an element of $K^1(SG) = 0$.

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ON NOVIKOV's CONJECTURE FOR COCOMPACT

DISCRETE SUBGROUPS OF A LIE GROUP

by

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I. Introduction and Statement of Results.

Let G be a linear analytic group; i.e. a connected Lie group which admits a faithful representation into $GL_n(\mathbb{R})$ for some integer n. If K is a maximal compact subgroup of G, then $X = K\backslash G$ is diffeomorphic to the Euclidean space $\mathbb{R}^n$. Suppose that $\Gamma \subset G$ is a torsion-free descrete subgroup such that the double coset space

$$M^n = X/\Gamma = K\backslash G/\Gamma$$

is a closed $K(\Gamma,1)$ manifold (i.e., an aspherical manifold of the homotopy type of $K(\Gamma,1)$). If G is semi-simple, then X/Γ is a locally symmetric space with non-positive sectional curvature and the so-called Novikov conjecture was verified in [10][2]. In this paper, we shall combine [2][4][5] to prove that the main result of [2] is valid for G any linear analytic group.

Let us now state our results precisely. Let $\mathbb{D}^i$ denote the closed i-dim disc and let M^n be a closed topological manifold. Recall the surgery exact sequence of [16][9] $(n+i>4)$:

$$\xrightarrow{\theta} L^s_{n+i+1}(\pi_1 M^n, w_1(M^n)) \longrightarrow S(M^n\times\mathbb{D}^i,\partial)$$

$$(1) \qquad \longrightarrow [M^n\times\mathbb{D}^i,\partial;\ G/Top,*] \longrightarrow$$

$$\xrightarrow{\theta} L^s_{n+i}(\pi_1 M^n, w_1(M^n)) \longrightarrow$$

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We have the following theorem.

Theorem A. Let $M^n = X/\Gamma$ be given as above. Then, the surgery map.

$$\theta\colon [M^n \times \mathbb{D}^i, \partial; G/Top, *] \longrightarrow L^s_{n+i}(\pi_1 M^n, w_1(M^n))$$

of (1) is a split monomorphism provided $n + i > 4$.

As we observed in [2], the following corollary follows from Theorem A.

Corollary B. Let $M^n = X/\Gamma$ be given as above. Let $g\colon N^n \to M^n$ be a homotopy equivalence where N^n is a closed manifold. Then,

$$g \times id\colon N^n \times \mathbb{R}^3 \longrightarrow M^n \times \mathbb{R}^3$$

is homotopic to a homeomorphism.

Let us now recall the so-called Novikov's conjecture. Let X^n be a closed oriented manifold and $x \in H^i(\pi, \mathbb{Q})$ a rational cohomology class of $K(\pi,1)$. given a homomorphism $\pi_1 X \to \pi$, we have a natural map

$$(2) \qquad f\colon X^n \longrightarrow K(\pi,1) .$$

Let $L_*(X^n) \in H^{4*}(X^n, \mathbb{Q})$ be a total L-genus of X^n. Consider the number

$$(3) \qquad L(x)(X^n) = \langle L_*(X^n) \cup f^*(x), [X^n] \rangle \in \mathbb{Q} .$$

It is called the higher signature of X^n associated to x.

Novikov's conjecture: For every group π and each element $x \in H^*(\pi, \mathbb{Q})$, the numbers $L(x)(\ )$ are homotopy invariants.

Again, it follows from [2] that the following corollary is an immediate consequence of Theorem A.

Corollary C. If Γ is a torsion-free cocompact discrete subgroup of a linear analytic group G with a finite number of components, then Novikov's conjecture is true.

It has been claimed by Kasparov that Novikov's conjecture is true for any discrete subgroup of $GL(n,\mathbb{C})$ [8]. At times it was also claimed by Miščenko [10] [11] that the conjecture is valid if π is the fundamental group of (merely) a non-positively curved complete manifold M^n. As pointed out in the critique [6], Miščenko did not prove the full conjecture if M^n is not closed. If either one of them were correct, Corollary C would be a consequence of the claim but not the stronger statement Theorem A or its geometric consequence Corollary B.

Let us examine our $M^n = X/\Gamma$ more closely. Consider the Levi decomposition

$$1 \longrightarrow S \longrightarrow G \xrightarrow{p} L \longrightarrow 1$$

of G where S (a solvable group) is the radical of G and L is a semi-simple group [7] [13]. Let K be a maximal subgroup such that $K_S = S \cap K$, $K_L = p(K)$ are maximal compact subgroups of S and L respectively. Similarly, we have $\Gamma_S = \Gamma \cap S$, $\Gamma_L = p(\Gamma)$ (which may have torsion elements) are discrete subgroups of S and L respectively [13].

We have the following facts:

(a) $S_o = K_S \backslash S/\Gamma_S$ is a compact solvamanifold.

(b) If we set $\hat{M}^n = K\backslash G/\Gamma_S$ $\hat{M}^\ell_o = K_L\backslash L$, then we have a fibration

(4) $$S_o \dashrightarrow \hat{M}^n \longrightarrow \hat{M}^\ell_o$$

and we may give $\hat{M}^\ell_o$ a non-positively curved global symmetric space structure. Therefore $\hat{M}^\ell_o$ is diffeomorphic to $\mathbb{R}^\ell$ and (4) is topologically a trivial fibration.

(c) Γ_S is a normal subgroup of Γ and $\Gamma_L = \Gamma/\Gamma_S$ acts on $\hat{M}^n$ and $\hat{M}^\ell_o$ such that $M^n = \hat{M}^n/\Gamma_L$ and such that we have a Siefert fibration [1]

(5) $$S_o \longrightarrow M^n \longrightarrow \hat{M}^\ell_o/\Gamma_L .$$

(d) It follows from [4] [5] that if $f: (N^{t+j+i}, \partial) \longrightarrow (S_o \times T^j \times \mathbb{D}^i, \partial)$ ($t = \dim S_o$) is a homotopy equivalence which is a homeomorphism between the boundaries and $t + j + i > 4$, then f is homotopic to a homeomorphism relative to the boundaries.

(e) $\hat{M}_o^\ell$ has a compactification $\mathbb{D}^\ell$ such that the action of Γ_L on $\hat{M}_o^\ell$ extends to $\mathbb{D}^\ell$. This action satisfies Condition (*) of [2] . (Strictly speaking, it only satisfies (*) after we restrict the action Γ_L to a torsion-free subgroup but this is not an essential point and we shall make it more precise shortly.)

What we realize is that these are the essential properties of the cocompact discrete subgroup Γ of G which make the proof work. Naturally, we should formalize these properties and prove a somewhat stronger version of Theorem A .

Let

(6) $$S_o \longrightarrow M^n \longrightarrow M_o^\ell$$

be a Seifert fibration [1] over an orbifold M_o^ℓ . (See [15] for the definition of an orbifold.) Assume that M^n is an aspherical manifold. We say that M^n satisfies (#) if (6) verifies the following conditions:

(#1) The inclusion $i: S_o \subset M^n$ of the regular fiber induces a map

(7) $$\Gamma_S = \pi_1(S_o) \longrightarrow \Gamma = \pi_1(M^n)$$

such that Γ_S is a normal subgroup of Γ , and $\Gamma_L = \Gamma/\Gamma_S$ is finitely presented.

(#2) The regular covering space of M^n corresponding the inclusion $\Gamma_S \subset \Gamma$ induces a fibre bundle

(8) $$S_o \longrightarrow \hat{M}^n \longrightarrow \hat{M}_o^\ell$$

such that $\Gamma_L = \Gamma/\Gamma_S$ acts on $\hat{M}^n$ properly discontinuously as a group of bundle automorphisms with $M^n = \hat{M}^n/\Gamma_L$, $M_o^\ell = \hat{M}_o^\ell/\Gamma_L$. Moreover, the quotient of $\hat{M}^n$ gives arise the Seifert fibration (6) as constructed in [1] .

(#3) If $f: (N^{t+j+i}, \partial) \longrightarrow (S_o \times T^j \times \mathbb{D}^i, \partial)$ ($t = \dim S_o$) is a homotopy equivalence which is a homeomorphism between the boundaries and $t + j + i > 4$, then f is homotopic to a homeomorphism relative to the boundaries.

(#4) $\hat{M}_o^\ell$ is an open simply connected manifold with a compactification $\bar{M}_o^\ell = \mathbb{D}^\ell$ such that the action Γ_L on $\hat{M}_o^\ell$ extends to $\mathbb{D}^\ell$. Any equivariant homotopy equivalence $h: \hat{M}^{\ell}_{o} \times [0,1] \longrightarrow \hat{M}^{\ell}_{o}$ with $h(x,0) = 0$ (for all $x \in \hat{M}^{\ell}_{o}$) extends to an equivariant homotopy $\bar{h}: \bar{M}^{\ell}_{o} \times [0,1] \longrightarrow \bar{M}^{\ell}_{o}$ with $\bar{h}(y,t) = y$ if either $t = 0$ or $y \in \partial\mathbb{D}^{\ell}$. Furthermore, let $d(\ ,\ )$ be a metric on $\mathbb{D}^{\ell}$, K be any compact subset of $\bar{M}^{\ell}_{o}$ and $\varepsilon > o$. Then there exists a number $\delta > o$ such that the following statement is true for any $\gamma \in \Gamma_L$. If there exist points $x \in K$ and $y \in \partial\mathbb{D}^{\ell}$ such that $d(\gamma x, y) < \delta$, then the diameter of γK (with respect to $d(\ ,\ )$) is less than ε.

Clearly, Theorem A is a special case of

<u>Theorem</u> A'. <u>Let M^n be a closed (triangulable)</u>[(3)] <u>aspherical manifold satisfying</u> $\#$). <u>Then the surgery map</u>

$$\theta: [M^n \times \mathbb{D}^i, \partial; G/Top, *] \longrightarrow L^s_{n+i}(\pi_1 M^n, w_1(M^n))$$

<u>of</u> (1) <u>is a split monomorphism provided</u> $n + i > 4$.

A special case of the above theorem was proved in [14].

2. <u>Proof of Theorem</u> A'.

As we observed in [2], it suffices to show that the surgery map

$$[M^n \times \mathbb{D}^i \times I, \partial; G/Top, *] \xrightarrow{\theta} L^s_{n+i+1}(\pi_1 M^n, w_1(M^n))$$

is split monomorphic for $i \geq 1$, $n + i \geq 5$ (where $I = [0,1]$).

Let $x \in L^s_{n+j+1}(\pi_1 M^n, w_1(M^n))$ be represented by a surgery problem

$$(9) \qquad (W^{n+i+1}; \partial_+ W^{n+i+1}, \partial_- W^{n+i+1}, \partial_o W^{n+i+1}) \xrightarrow{f} (M^n \times \mathbb{D}^i \times I; M^n \times \mathbb{D}^i \times 1, M^n \times \mathbb{D}^i \times 0, M^n \times \partial\mathbb{D}^i \times I)$$

satisfying the following conditions:

(3) We do not have to assume that M^n is triangulable, but it makes the argument simpler.

$$(9')\quad \begin{array}{ll} \text{(i)} & f_-:\partial_- W^{n+i+1} \longrightarrow M^n \times \mathbb{D}^i \times 0 \ \text{ is a homeomorphism;} \\ \text{(ii)} & f_+:\partial_+ W^{n+i+1} \longrightarrow M^n \times \mathbb{D}^i \times 1 \ \text{ is a simple homotopy equivalence;} \\ \text{(iii)} & f_o:\partial_o W^{n+i+1} \longrightarrow M^n \times \partial\mathbb{D}^i \times I \ \text{ is a homeomorphism.} \end{array}$$

Write $U^{n+i} = \partial_+ W^{n+i+1}$ and $\partial_\pm U^{n+i} = f_+^{-1}(M^n \times \partial_\pm \mathbb{D}^i \times 1)$ where $\partial_+\mathbb{D}^i$ and $\partial_-\mathbb{D}^i$ denote the upper and the lower hemispheres of $S^{i-1} = \partial\mathbb{D}^i$, respectively.

Identify $\partial_- U^{n+i}$ with $M^n \times \partial_-\mathbb{D}^i \times 1$ via f_o. Applying the s-cobordism theorem to U^{n+i} with respect to $\partial_- U^{n+i}(\mathrm{rel}.\partial)$, we may identify U^{n+i} as $M^n \times \mathbb{D}^i$ and f_+ may be interpreted as a simple homotopy equivalence

$$(10)\qquad h: M^n \times \mathbb{D}^i \longrightarrow M^n \times \mathbb{D}^i$$

such that $h|M^n \times \partial_-\mathbb{D}^i = \mathrm{id}$ and $h|M^n \times \partial_+\mathbb{D}^i$ is a homeomorphism.

Let us now consider

$$(11)\qquad A^{2n+i} = \tilde{M}^n \times (\hat{M}^n \times \mathbb{D}^i)$$

where $\tilde{M}^n$ denotes the universal cover of M^n. Let $\partial_\pm A^{2n+i}$ be the part of ∂A^{2n+i} corresponding to $\hat{M}^n \times \partial_\pm \mathbb{D}^i$ (of the second factor of A^{2n+i}) respectively. Let Γ act on A^{2n+i} diagonally such that the action of the second factor is factored through $\Gamma_L = \Gamma/\Gamma_S$. We have a fibration

$$(12)\qquad \hat{M}^n \times \mathbb{D}^i \longrightarrow E^{2n+i} = A^{2n+i}/\Gamma \longrightarrow M^n = \tilde{M}^n/\Gamma$$

the map $\mathrm{id} \times h$ which $\hat{h}$ is the lift of h to $\hat{M} \times \mathbb{D}^i$ with $\hat{h}|\hat{M}^n \times \partial_-\mathbb{D}^i = \mathrm{id}$, induces a bundle map (again denoted by h)

$$(13)\qquad \begin{array}{ccccc} \hat{M}^n \times \mathbb{D}^i & \longrightarrow & E^{2n+i} & \xrightarrow{q} & M^n \\ & & \Big\downarrow h & & \Big\| \\ \hat{M}^n \times \mathbb{D}^i & \longrightarrow & E^{2n+i} & \xrightarrow{q} & M^n \end{array}$$

such that $h|\partial_- E^{2n+i} = \mathrm{id}$ and $h|\partial_+ E^{2n+i}$ is a homeomorphism (where $\partial_\pm E^{2n+i}$ are the parts of ∂E^{2n+i} corresponding to $\partial_\pm A^{2n+i}$).

The compactification of $\hat{M}_o^\ell$ to $\mathbb{D}^\ell$ induces a compactification of $\hat{M}^n \times \mathbb{D}^\ell$ to $S^{\ell-1} * (S_o \times \mathbb{D}^i)$ as follows: View $\hat{M}_o^\ell$ as the interior of $\mathbb{D}^\ell$ and

$S^{\ell-1}$ as the boundary of $\mathbb{D}^\ell$. Project $\hat{M}^n \times \mathbb{D}^i$ to $\hat{M}_o^\ell$ and shrink the size of $y \times (S_o \times \mathbb{D}^\ell)$ (where $y \in \hat{M}_o^\ell$) as y moves toward $\partial\mathbb{D}^\ell$ (and becomes a point as it gets to $\partial\mathbb{D}^\ell = S^{\ell-1}$). It follows from $\#$) that the fibration (12) extends to a fibration

(14) $$S^{\ell-1} * (S_o \times \mathbb{D}^i) \longrightarrow \bar{E}^{2n+i} \xrightarrow{\bar{q}} M^n$$

and the bundle map h extends to

(15) $$\begin{array}{ccccc} S^{\ell-1} * (S_o \times \mathbb{D}^i) & \longrightarrow & \bar{E}^{2n+i} & \longrightarrow & M^n \\ & & \downarrow \bar{h} & & \| \\ S^{\ell-1} * (S_o \times \mathbb{D}^i) & \longrightarrow & \bar{E}^{2n+i} & \longrightarrow & M^n . \end{array}$$

Corresponding to $S^{\ell-1} * (S_o \times \partial_\pm \mathbb{D}^i) \subsetneq S^{\ell-1} * (S_o \times \mathbb{D}^i)$, there are two sub-bundles of (14) :

(16) $$S^{\ell-1} * (S_o \times \partial_\pm \mathbb{D}^i) \longrightarrow \partial_\pm \bar{E}^{2n+i} \xrightarrow{\bar{q}} M^n .$$

Each of these fibration is left invariant by $\bar{h}$ such that $\bar{h}|\partial_-\bar{E}^{2n+i} = \mathrm{id}$ and $\bar{h}|\partial_+\bar{E}^{2n+i}$ is a homeomorphism.

We need the following lemma.

Lemma 2.1 *Let*

$$f : (S^{\ell-1} * (S_o \times \mathbb{D}^i)) \times \Delta^j \longrightarrow (S^{\ell-1} * (S_o \times \mathbb{D}^i)) \times \Delta^j$$

be a homotopy equivalence such that $f|\partial = (S^{\ell-1} * (S_o \times \partial\mathbb{D}^i)) \times \Delta^j \cup (S^{\ell-1} * (S_o \times \mathbb{D}^i) \times \partial\Delta^j$ *is a homeomorphism with* $f^{-1}(\partial) = \partial$. *If* $\ell + \dim S_o + i + j \geq 5$, *then* f *is homotopic to a homeomorphism rel* $f|\partial$.

Proof. This lemma essentially follows from the main theorem of [4] (cf. [5]). Let us consider the open regular neighborhood N of $S^{\ell-1} \times \Delta^j$ in $(S^{\ell-1} * (S_o \times \mathbb{D}^i)) \times \Delta^j$. It has an open cone bundle over $S^{\ell-1} \times \Delta^j$. In fact, it is a product bundle and we may write N as

$$(17) \qquad (S_o \times \mathbb{D}^i) \times [1,0] \times (S^{\ell-1} \times \Delta^j)/\sim$$

with $(S_o \times \mathbb{D}^i) \times 0$ as the cone vertex c .

Set

$$N_t = (S_o \times \mathbb{D}^i) \times t \times (S^{\ell-1} \times \Delta^j)/\sim$$

$$(18) \qquad N_{[t,o]} = (S_o \times \mathbb{D}^i) \times [t,o] \times (S^{\ell-1} \times \Delta^j)/\sim$$

$$N_{(t,o]} = (S_o \times \mathbb{D}^i) \times (t,o] \times (S^{\ell-1} \times \Delta^j)/N$$

for $o < t < 1$. It follows from [4] [5] that f is homotopic to f_1 rel ∂ such that $f_1^{-1}(N_{[\frac{1}{2},o]}) = N_{[\frac{1}{2},o]}$ and $f_1|N_{[\frac{1}{2},o]}$ is a homeomorphism. Therefore, f_1 induces a homotopy equivalence

$$(19) \qquad g : L = ((S^{\ell-1} * (S_o \times \mathbb{D}^i)) \times \Delta^j - N_{(\frac{1}{2},o]}) \longrightarrow L$$

such that $g|\partial L$ is a homeomorphism. It follows from (#3) that g is homotopic to a homeomorphism rel ∂ . Combining these two homotopies, we produce a homotopy of f as required.

<u>Lemma</u> 2.2 <u>There is a homotopy</u> $\bar{h}_t$ $(o \le t \le 1)$ of $\bar{h}$ <u>such that</u> $\bar{h}_o = \bar{h}$, $\bar{h}_1$ is <u>a homeomorphism</u>, $\bar{h}_t|\partial\bar{E}$ <u>is a homeomorphism</u>, $\bar{h}_t|\partial_-\bar{E} = \mathrm{id}$ <u>and</u> $\bar{h}_t^{-1}(\partial\bar{E}) = \partial\bar{E}$. <u>Furthermore</u>, $\bar{h}_t(\bar{q}^{-1}(\Delta^j)) \subset \bar{q}^{-1}(\Delta^j)$ <u>for each</u> $t \in [0,1]$ <u>and every closed simplex</u> Δ^j <u>of a fixed triangulation of</u> M^n .

<u>Proof</u>. Since $\bar{E}$ is a fiber bundle over M^n , it may be viewed as a block bundle with $\bar{q}^{-1}(\Delta^i)$ as blocks where Δ^i are simplices of a trianglation of M^n . We shall prove our lemma by an induction on the skeleton of M^n .

Consider the induced map

$$(20) \qquad \bar{h}|\bar{q}^{-1}(\Delta^o) : \bar{q}^{-1}(\Delta^o) \longrightarrow \bar{q}^{-1}(\Delta^o) .$$

If we trivialize $\bar{q}^{-1}(\Delta^o)$ as

(21) $$S^{\ell-1} * (S_o \times \mathbb{D}^i) \times \Delta^o ,$$

then Lemma 2.1 applies and we have a homotopy of $\bar{h}|\bar{q}^{-1}(\Delta^o)$ rel ∂ such that it becomes a homeomorphism at the end of the homotopy. Extending this homotopy blockwise, we may assume that $\bar{h}|\bar{q}^{-1}(\Delta^o)$ is a homeomorphism to begin with. Inductively, we may assume that $\bar{h}$ is a homeomorphism over the blocks over the $(j-1)$-skeleton of M^n $(j \geq 1)$ and $\bar{h}$ preseves the blocks. Let Δ^j be a j-dim simplex of M^n and let us consider the induced map

(22) $$\bar{h}|\bar{q}^{-1}(\Delta^j) : \bar{q}^{-1}(\Delta^j) \longrightarrow \bar{q}^{-1}(\Delta^j) .$$

Again, trivialize $\bar{q}^{-1}(\Delta^j)$ as

(23) $$S^{\ell-1} * (S_o \times \mathbb{D}^i) \times \Delta^j$$

and then apply Lemma 2.1 . By another blockwise homotopy, we finish the induction step. Note that different trivialization of $\bar{q}^{-1}(\Delta^j)$ changes the map of Lemma 2.1 by a conjugation α of the domain and the range where

(24) $$\alpha : S^{\ell-1} * (S_o \times \mathbb{D}^i) \times \Delta^j \longrightarrow S^{\ell-1} * (S_o \times \mathbb{D}^i) \times \Delta^j$$

is a block homeomorphism induced from an element of $\pi_1 M^n$ as gotten from (#2) and (#4) . The change makes no difference for our argument and the lemma is proved.

Let us now continue our proof of Theorem A' . Consider the covering $\hat{W}^{n+i+1}$ of W^{n+i+1} corresponding to Γ_S and set $V^{2n+i+1} = \tilde{M}^n \times \hat{W}^{n+i+1}/\Gamma$.
We have the following fibration

(25) $$\hat{W}^{n+i+1} \longrightarrow V^{2n+i+1} \xrightarrow{q'} M^n$$

and f induces a bundle map

(26) $$\begin{array}{ccccc} \hat{W}^{n+i+1} & \longrightarrow & V^{2n+i+1} & \longrightarrow & M^n \\ \downarrow & & \downarrow g & & \| \\ \tilde{M}^n \times \mathbb{D}^i \times I & \longrightarrow & E^{2n+i} \times I & \longrightarrow & M^n \end{array}$$

Note that we may identify the parts of V^{2n+i+1} corresponding to $M^n \times \partial_- W^{n+i+1}$ and $M^n \times \partial_+ W^{n+i+1}$ as $E^{2n+i} \times 0$ and $E^{2n+i} \times 1$, respectively, such that the maps g_- induced by f_- is the identity and the map g_+ induced by f_+ is h. By Lemma 2.2, we may assume that $h = g_+$ is a homeomorphism. These facts together with $(9',\mathrm{iii})$ implies $g|\partial V^{2n+i+1}$ is a homeomorphism.

Let $M^n \times \mathbb{D}^i \times j \subset E^{2n+i} \times j$ $(j = 0,1)$ by the submanifolds consisting of the quotient points corresponding to $(x,\hat{x},y,j)$ for $x \in \tilde{M}^n$, $y \in \mathbb{D}^i$ where $\hat{x}$ is the image of x in $\hat{M}$. We have $g_-^{-1}(M^n \times \mathbb{D}^i \times 0)$ and $g_+^{-1}(M^n \times \mathbb{D}^i \times 1)$ in ∂V^{2n+i+1}. Applying transversality to g (rel. ∂) with respect to the submanifold $M^n \times \mathbb{D}^i \times I \subset E^{2n+i} \times I$ consisting of the quotient points corresponding to points of the form $(x,\hat{x},y)$ for $x \in \tilde{M}^n$ and $y \in \mathbb{D}^i \times I$, we obtain a submanifold N^{n+i+1} connecting $g_-^{-1}(M^n \times \mathbb{D}^i \times 0)$ and $g_+^{-1}(M^n \times \mathbb{D}^i \times 1)$. This gives a degree 1 map from N^{n+i+1} to $M^n \times \mathbb{D}^i \times I$ which is a homeomorphism when restricted to ∂N^{n+i+1} and determines an element of $[M^n \times \mathbb{D}^i \times I, \partial;\ G/Top, *]$. As we argued in [2, pp.205-206], this is the splitting of θ. This completes the proof of Theorem A'.

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An Introduction to Maps
Between Surgery Obstruction Groups

by

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In order to apply surgery theory we need methods for determining whether a given surgery obstruction $\sigma(f : N \to M, \hat{f}) \in L_n(\mathbb{Z}G, \omega)$ is trivial. Notice that it is more important to have invariants which detect L-groups than to be able to compute the L-group themselves.

One approach is to use numerical invariants such as Arf invariants, multisignatures, or the new "semi-invariants" [M1], [Da], [H-Mad], [P]. Another approach is to use transfer maps. For example,

(i) Dress [D] has shown that when G is a finite group, $L_n(\mathbb{Z}G)$ is detected under the transfer by using all subgroups of G which are hyper-elementary.

(ii) Wall [W9] has shown that when M is closed and G is finite, then image $(\sigma : [M, G/TOP] \to L_n(\mathbb{Z}G, \omega))$ is detected by $L_n(\mathbb{Z}G_2, \omega)$, where G_2 is the 2-Sylow subgroup of G.

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(iii) If G is a finite 2-group, then $L_n^p(\mathbb{Z}G)$ is detected under transfer and projection by using subquotients of G which are dihedral, quaternionc, semi-dihedral, and cyclic (see [T-W]).

The goal of this paper is to give a systematic procedure (when G is a finite 2-group) for computing transfer maps and the "twisted" transfer maps arising from codimension 1 surgery theory. Recall that if H is any subgroup of a finite 2-group G, then there exists a sequence of subgroups $H = H_0 \subset H_1 \subset H_2 \subset \dots \subset H_e = G$ such that H_i is an index 2 subgroup of H_{i+1} foı $i = 0, \dots, e-1$. Thus we might as well assume that H is an index 2 subgroup.

Suppose H is an index 2 subgroup of an arbitrary group G. Then we get the "push forward" exact sequence

$$\dots \to L_n(\mathbb{Z}H,\omega) \xrightarrow{f_!} L_n(\mathbb{Z}G,\omega) \to L_n(f_!) \to \dots$$

(see [R1],§2), and the transfer exact sequence

$$\dots \to L_n(\mathbb{Z}G,\omega) \xrightarrow{f^!} L_n(\mathbb{Z}H,\omega) \to L_n(f^!) \to \dots$$

(see [R1], §7.6).

One can view $S = \mathbb{Z}G$ as a <u>twisted quadratic extension</u> of $R = \mathbb{Z}H$. More precisely, suppose we choose $t \in G - H$. We let $a = t^2 \in H$, and we let $\rho : R \to R$ be conjugation by t. Then,

$$S = R_\rho[\sqrt{a}] = R_\rho[t]/(t^2 - a),$$

where t is viewed as an indeterminate over R such that $tx = \rho(x)t$ for all $x \in R$.

Let γ denote the Galois automorphism of S over R given by

$$\gamma : S \to S; \; x + yt \to x - yt \qquad (x,y \in R).$$

We want to extend the classical results in [L] chap. 7 and [M-H] appendix 2 where $f : R \to S$ is a quadratic extension of <u>fields</u>.

Recall that Wall [W2] defined groups $L_n(R,\alpha,u)$ for any ring with anti-structure, i.e. α is an anti-automorphism, u is a unit, $\alpha^2(r) = uru^{-1}$ for all $r \in R$, and $\alpha(u) = u^{-1}$. For example, $L_n(\mathbb{Z}G,\omega) = L_n(\mathbb{Z}G,\alpha_\omega,1)$, where $\alpha_\omega(\Sigma\, n_g g) = \Sigma\, n_g \omega(g) g^{-1}$.

Suppose we have a map of rings with anti-structure

$$f : (R,\alpha_0,u) \to (S,\alpha,u)$$

where S is a twisted quatratic extension $R_\rho[\sqrt{a}]$ with Galois automorphism γ. Then we also have the following γ-conjugate map

$$^{\gamma}f : (R,\alpha_0,u) \to (S,\gamma\alpha,u).$$

Moreover, we can "twist" (α,u) to get $(\widetilde{\alpha,u}) = (\tilde{\alpha},\tilde{u})$, where $\tilde{\alpha}(s) = \sqrt{a}\;\gamma\alpha(s)\sqrt{a}^{\,-1}$ for all $s \in S$ and $\tilde{u} = \sqrt{a}\;\gamma\alpha(\sqrt{a}^{\,-1})u$. This yields a map

$$\tilde{f} : (R,\tilde{\alpha}_0,\tilde{u}) \to (S,\tilde{\alpha},\tilde{u}),$$

where $\tilde{\alpha}_0$ is the restriction of $\tilde{\alpha}$.

If we twist $(\gamma\alpha,u)$ we get that $(\widetilde{\gamma\alpha,u}) = (\gamma\tilde{\alpha}, -\tilde{u})$, and we get a map

$$\widetilde{{}^{\gamma}f} : (R,\tilde{\alpha}_0, -\tilde{u}) \to (S,\gamma\tilde{\alpha}, -\tilde{u}).$$

Then we get the following amazing isomorphisms.

$$\Gamma_! : L_{n-1}(\tilde{f}_!) \cong L_n(f_!) \qquad \Gamma_! : L_{n-1}(\widetilde{{}^{\gamma}f_!}) \cong L_n({}^{\gamma}f_!)$$

$$\Gamma^! : L_n(f^!) \cong L_{n+1}(\tilde{f}^!) \qquad \Gamma^! : L_n({}^{\gamma}f^!) \cong L_{n+1}(\widetilde{{}^{\gamma}f^!})$$

The maps $\Gamma_!$ and $\Gamma^!$ are defined using an algebraic version of integration along the fibre for line bundles. In the case of group rings the isomorphism $\Gamma_!$ is implicit in [W1] chap. 12.C, [C-S1] and explicit in [H]. The general case is due to Ranicki (after some prodding by us). (See [R1], §7.6 and the appendix by Ranicki in [H-T-W]).

By combining $\Gamma_!$ for f, $\Gamma^!$ for ${}^{\gamma}f$, and scaling isomorphisms

$$\sigma^{\sqrt{a}} : L_n(S,\alpha,u) \to L_n(S,\gamma\tilde{\alpha}, -\tilde{u}) = L_n(S,\widetilde{\gamma\alpha,u}) \quad \text{(see 2.5.5)}$$

Ranicki constructed the following commutative braid of exact sequences

(0.1) Twisting Diagram for $f : (R,\alpha_0,u) \to (S,\alpha,u)$

$$\begin{tikzcd}[column sep=small]
\begin{array}{c} L_n(S,\tilde\alpha,\tilde u) \\ = \\ L_n(S,\gamma\alpha,u) \end{array} \arrow[rr, bend left, "\gamma_f^!"] \arrow[dr] & & L_n(R,\alpha_0,u) \arrow[rr, bend left, "f_!"] \arrow[dr] & & \begin{array}{c} L_n(S,\alpha,u) \\ = \\ L_n(S,\gamma\tilde\alpha,-\tilde u) \end{array} \arrow[rr, bend left, "\widetilde{\gamma_f^!}"] \arrow[dr] & & \begin{array}{c} L_{n-2}(R,\tilde\alpha_0,\tilde u) \\ = \\ L_n(R,\tilde\alpha_0,-\tilde u) \end{array} \\
& \begin{array}{c} L_n(\widetilde{f_!}) \\ = \\ L_{n+1}(f_!) \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c} L_n(\gamma_f^!) \\ = \\ L_{n+1}(\widetilde{\gamma_f^!}) \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c} L_{n-1}(\widetilde{f_!}) \\ = \\ L_n(f_!) \end{array} \arrow[ur] \arrow[dr] & \\
\begin{array}{c} L_{n+1}(S,\alpha,u) \\ = \\ L_{n+1}(S,\gamma\tilde\alpha,-\tilde u) \end{array} \arrow[ur] \arrow[rr, bend right, "\widetilde{\gamma_f^!}"'] & & \begin{array}{c} L_{n-1}(R,\tilde\alpha_0,\tilde u) \\ = \\ L_{n+1}(R,\tilde\alpha_0,-\tilde u) \end{array} \arrow[ur] \arrow[rr, bend right, "\widetilde{f_!}"'] & & \begin{array}{c} L_{n-1}(S,\tilde\alpha,\tilde u) \\ = \\ L_{n-1}(S,\gamma\alpha,u) \end{array} \arrow[ur] \arrow[rr, bend right, "\gamma_f^!"'] & & L_{n-1}(R,\alpha_0,u)
\end{tikzcd}$$

Thus the problem of computing $f_!$ and $\gamma_f^!$ is intimately related to the problem of computing the "twisted" maps $\widetilde{f_!}$ and $\widetilde{\gamma_f^!}$.

Examples:

1. Suppose $L_n = L_n^p$, n is even, plus R and S are semi-simple rings. Recall that L_{odd}^p is trivial for semi-simple rings (see [R2]). Thus, all of the groups along the bottom of (0.1) are trivial and the groups along the top form the following exact octagon (see also [War] and [Le]).

(0.2) Semi-simple 8-Fold Way

$$\begin{tikzcd}
 & L_n(R,\alpha_0,u) \arrow[r, "f_!"] & L_n(S,\alpha,u) \arrow[dr, "\widetilde{\gamma_f!_\sigma\sqrt{a}}"] & \\
L_n(S,\tilde{\alpha},\tilde{u}) \arrow[ur, "\gamma_f!_\sigma\sqrt{a}"] & & & L_n(R,\tilde{\alpha}_0,-\tilde{u}) \arrow[d, "\widetilde{f_!}"] \\
L_n(R,\tilde{\alpha}_0,\tilde{u}) \arrow[u, "\widetilde{f_!}"] & & & L_n(S,\tilde{\alpha},-\tilde{u}) \arrow[dl, "\gamma_f!_\sigma\sqrt{a}"] \\
 & L_n(S,\alpha,-u) \arrow[ul, "\widetilde{\gamma_f!_\sigma\sqrt{a}}"] & L_n(R,\alpha_0,-u) \arrow[l, "f_!"] &
\end{tikzcd}$$

(a) Suppose we have

$$f : (F,\mathrm{id},1) \to (K,\mathrm{id},1)$$

where $F \to K$ is a quadratic extension of fields. If $n = 0$, then $L_n(F,\mathrm{id},-1) \simeq L_n(K,\mathrm{id},-1) \simeq (0)$; and we get the following exact sequence

(0.3) $0 \to L_0(K,\gamma) \to L_0(F) \to L_0(K) \to L_0(F) \to L_0(K,\gamma) \to 0$

which extends the exact sequences of Lam [L], chap. 7 and Milnor-Husemoller [M-H], appendix 2.

(b) Suppose we have

$$f : (K,\mathrm{id},1) \to (D,\alpha,1)$$

where D is a quaternionic division algebra and K is a maximal subfield of D. Since $\alpha|_K = \mathrm{id}$, α must be an involution of type O. If we let $L_0(D;O) = L_0(D,\alpha,1)$, $L_0(D;S_p) = L_0(D,\alpha,-1)$, and $L_0(K,\rho) = L_0(K,\tilde{\alpha}_0,\pm 1)$, then we get the following exact sequence

$$(0.4)\quad \begin{array}{l} 0 \to L_0(D;S_p) \to L_0(K,\rho) \to L_0(D;O) \to L_0(K) \to L_0(D;O) \\ \hphantom{0 \to L_0(D;S_p) \to L_0(K,\rho) \to L_0(D;O) \to L_0(K) \to {}} \downarrow \\ \hphantom{0 \to L_0(D;S_p) \to L_0(K,\rho) \to L_0(D;O) \to L_0(K) \to {}} L_0(K,\rho) \\ \hphantom{0 \to L_0(D;S_p) \to L_0(K,\rho) \to L_0(D;O) \to L_0(K) \to {}} \downarrow \\ \hphantom{0 \to L_0(D;S_p) \to L_0(K,\rho) \to L_0(D;O) \to L_0(K) \to {}} L_0(D;S_p) \\ \hphantom{0 \to L_0(D;S_p) \to L_0(K,\rho) \to L_0(D;O) \to L_0(K) \to {}} \downarrow \\ \hphantom{0 \to L_0(D;S_p) \to L_0(K,\rho) \to L_0(D;O) \to L_0(K) \to {}} 0 \end{array}$$

(c) Suppose D is a division ring with center F. Let K be a quadratic extension of F such that $D\otimes_F K$ is still a division ring. Then for any (anti) involution α_0 on D, we get another example

$$(D,\alpha_0,1) \to (D\otimes_F K,\alpha_0\otimes \mathrm{id},1)$$

(d) (trivial quadratic extension) Suppose we have

$$(d : (R,\alpha_0,u) \to (R \times R,\ \alpha_0 \times \alpha_0,u \times u)$$

where d is the diagonal map. Then $L_n(S,\tilde{\alpha},\pm\tilde{u})$ is trivial and (0.2) breaks into the short exact sequences of the form

$$1 \to L_n(R,\alpha_0,u) \xrightarrow{d_!} L_n(R,\alpha_0,u) \times L_n(R,\alpha_0,u) \to L_n(R,\alpha_0,u) \to 1$$

2. Codimension 1 Surgery (see [B-L], [Me], [W1] Chap. 12C, [C-S1], [C-S2], [H], and [R1] Chap. 7)

Suppose we have

$$f : (\mathbb{Z}H,\alpha_\omega,1) \to (\mathbb{Z}G,\alpha_\omega,1)$$

where H is an index 2 subgroup of G. Also, suppose X^n is a closed manifold with $(\pi_1 X,\omega_1 X) = (G,\omega)$. Let Y^{n-1} be a connected submanifold such that $\omega_1(\nu(Y \to X))$ induces the map $\psi : G \to G/H = \{\pm 1\}$. Then by combining results of Wall

[Wl], chap. 12C, Cappell-Shaneson [C-S1], and Hambleton [H], we get the following commutative diagram with exact rows

$$\begin{tikzcd}
 & & & L_n(\mathbb{Z}H,\omega) \arrow[d, "f_!"] & \\
\cdots \arrow[r] & S(X) \arrow[r] \arrow[ddd, "\tilde{\sigma}_*"] & {[X,G/TOP]} \arrow[r, "\sigma_*"] \arrow[d] & L_n(\mathbb{Z}G,\omega) \arrow[dd] \arrow[ddr, "\widetilde{\gamma_{f^!}\sigma^t}"] & \\
 & & {[Y,G/TOP]} \arrow[d] & & \\
 & & L_{n-1}(\mathbb{Z}G,\omega\psi) \arrow[d, "\wr\,\sigma^t"] & L_n(f_!) & L_n(\mathbb{Z}H,\tilde{\alpha}_\omega,-\tilde{1}) \arrow[d, equal] \\
 & L_{n-1}(\mathbb{Z}H,\tilde{\alpha}_\omega,\tilde{1}) \arrow[r, "\widetilde{f_!}"] & L_{n-1}(\mathbb{Z}G,\tilde{\alpha}_\omega,\tilde{1}) \arrow[r] & L_{n-1}(\widetilde{f_!}) \arrow[r] \arrow[u, "\wr\,\Gamma_!"'] & L_{n-2}(\mathbb{Z}H,\tilde{\alpha}_\omega,\tilde{1})
\end{tikzcd}$$

where $t \in G - H$ and $L_n = L_n^s$.

Assume $\pi_1 Y \doteq \pi_1 X$ and $n \geq 5$. If $f : M \to X$ represents an element in $S(X)$, then $\tilde{\sigma}_*(f)$ is trivial if and only if f is homotopic to a map f_1 such that $f_1^{-1}(Y) \to Y$ and $f_1^{-1}(X - Y) \to X - Y$ are simple homotopy equivalences.

Cappell-Shaneson [C-S1], [C-S2] and Hambleton [H] have observed that since

$$\text{image}(\sigma_* : [M,G/TOP] \to L_n(\mathbb{Z}G,\omega)) \subset \ker(\widetilde{\gamma_{f^!}\sigma^t})$$

$\widetilde{\gamma_{f^!}\sigma^t}(x)$ can be viewed as the <u>primary obstruction</u> to an element $x \in L_n(\mathbb{Z}G,\omega)$ arising from surgery on <u>closed</u> manifolds.

In this paper we compute the twisting diagram (0.1) where

$$f : (\mathbb{Z}H,\alpha_\omega,1) \to (\mathbb{Z}G,\alpha_\omega,1),$$

G is a finite 2-group, and $L_n = L_n^p$. Our motivation is that we have used these results to compute $\sigma_*^p : [M,G/TOP] \to L_n^p(\mathbb{Z}G,\omega)$ (see [H], [T-W], and [H-T-W]).

Roughly speaking, we proceed as follows

(i) We show that the Twisting Diagram (0.1) for f decomposes into a sum of diagrams indexed by the irreducible $\mathbb{Q}$-representations of G.

(ii) We use quadratic Morita theory to construct an isomorphism between each component diagram and the twisting diagrams associated to integral versions of Examples 1. (a),(b),(c),(d) i.e. maximal orders in division rings.

(iii) By using classical results on quadratic forms over division rings and localization sequences we are able to finish the calculation.

In Part I we carry out this program for the groups along the top and bottom of the twisting diagram (0.1). In Part II we compute the actual diagrams.

This paper is a preliminary version of [H-T-W] where we give details, compute the twisting diagrams for other L-groups in addition to L_n^p, and give geometric applications.

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PART I: Computation of the L^p-groups

Let G be a finite 2-group and H an index 2 subgroup.

In the oriented case, the groups $L_n^p(\mathbb{Z}G)$ have been computed by Bak-Kolster [K1], [K2], [B-K], Pardon [P], and Carlson-Milgram [C-M]. These results are nicely summarized by Theorem A in [H-M] where they give a decomposition of $L_n^p(\mathbb{Z}G)$ indexed by the irreducible $\mathbb{Q}$-representation of G. Besides extending their computations to the unoriented L-groups, $L_n^p(\mathbb{Z}G,\omega)$ and the codimension 1 surgery groups $L_n^p(\mathbb{Z}G,\tilde{\alpha}_\omega,\tilde{1})$, we also have to overcome the following problem. All of the above computations were based upon choosing a maximal involution invariant order, M_G which contains $\mathbb{Z}G$. Unfortunately, it is not always true that $M_G \cap \mathbb{Q}H$ is a maximal involution invariant order in $\mathbb{Q}H$. (Bruce Magurn has observed that this can happen even when G is the dihedral group of order 8). Thus it is not clear that the above computations and decompositions are functorial and we have had to modify their method somewhat. We have attempted to keep Part I fairly self-contained, but we would like to emphasize that Part I is based upon the work of the above authors and Wall's fundamental sequence of papers [W1] - [W8]. In Section (2.5) we try to clarify certain questions involving quadratic Morita theory.

§1. Basic Definitions and Overview

(1.1) Intermediate L-group

The use of arithmetic squares forces us to use L-groups other than L_n^p . Thus we shall start by recalling the relationships between the various L-groups.

A <u>ring with antistructure</u> (R,α,u) is an associative ring R, an anti-automorphism $\alpha : R \to R$ and a unit $u \in R$ such that $\alpha^2(a) = uau^{-1}$ for all $a \in R$, and such that $\alpha(u) = u^{-1}$. For any right R-module M, $D^\alpha(M) = \mathrm{Hom}_R(M,R)$ is the right R-module where

$(f \cdot r)(m) = \alpha(r) \cdot f(m)$ where $f \in D^\alpha M$, $m \in M$, and $r \in R$.

Since inner automorphisms act trivially on K-theory, D^α induces involutions on $K_i(R)$ and $\tilde{K}_i(R) = \mathrm{coker}(K_i(\mathbb{Z}) \to K_i(R))$ which we also denote by α.

If Y is an α-invariant subgroup of $\tilde{K}_i(R)$, $i=0$ or 1, then $L_n^Y(R,\alpha,u)$ denotes the <u>standard L-group</u> defined in [Ca], [R3] §9, and [R1] p. 688.

If $R = \mathbb{Z}G$ and $\alpha_\omega : \Sigma\, n_g g \to \Sigma\, n_g \omega(g) g^{-1}$, then we get the following <u>geometric L-groups</u>,

$L_n^s(\mathbb{Z}G,\omega) = L_n^{\{\pi^{ab}\}\subset\tilde{K}_1}(\mathbb{Z}G,\alpha_\omega,1)$ (see [W1])

$L_n^h(\mathbb{Z}G,\omega) = L_n^{\tilde{K}_1}(\mathbb{Z}G,\alpha_\omega,1)$ (see [Sh])

$L_n^p(\mathbb{Z}G,\omega) = L_n^{\tilde{K}_0}(\mathbb{Z}G,\alpha_\omega,1)$ (see [Ma], [P-R])

If $Y_2 \subset Y_1 \subset \tilde{K}_i(R)$, $i = 0$ or 1 are both α-invariant subgroups, then we get the following Rothenberg exact sequence (see [R3] 9.1)

(1.1.1) $\cdots \to L_n^{Y_2}(R,\alpha,u) \to L_n^{Y_1}(R,\alpha,u) \to H_\alpha^n(Y_1/Y_2) \to \cdots$

where $H_\alpha^n(Y_1/Y_2)$ is the Tate cohomology group $\hat{H}^n(\mathbb{Z}/2, Y_1/Y_2)$ associated to the action of $\mathbb{Z}/2$ on Y_1/Y_2 via α.

Also,

(1.1.2) $$L_n^{\tilde{K}_1(R)}(R,\alpha,u) = L_n^{0\subset\tilde{K}_0(R)}(R,\alpha,u)$$

If $Y = \tilde{K}_0$, then

(1.1.3)

$$L_n^Y(R_1 \times R_2, \alpha_1 \times \alpha_2, u_1 \times u_2) \simeq L_n^Y(R_1,\alpha_1,u_1) \times L_n^Y(R_2,\alpha_2,u_2),$$

and

(1.1.4) $L_n^Y(R_1,\alpha_1,u_1) \simeq L_n^Y(R_2,\alpha_2,u_2)$ whenever (R_1,α_1,u_1) and (R_2,α_2,u_2) are quadratic Morita equivalent (see Section 2 for definition)

Since $\tilde{K}_i(R_1 \times R_2) \nsimeq \tilde{K}_i(R_1) \times \tilde{K}_i(R_2)$ and since $\tilde{K}_i$ is not a Morita invariant, (1.3) and (1.4) are false for most Y : e.g. $Y = 0 \subset \tilde{K}_0$. This problem is overcome by introducing the following <u>variant L-groups</u>.

If X is an α-invariant subgroup of $K_i(R)$, then we get L-groups, $L_n^X(R,\alpha,u)$. (See [W3] for $i = 1$ and [B-W] for any $i \geq 0$).

If $\text{image}K_0(\mathbb{Z}) \subset X \subset K_0(R)$, then

(1.1.5) $$L_n^X(R,\alpha,u) \simeq L_n^{\tilde{X}}(R,\alpha,u),$$

where $\tilde{X}$ = image of X in $\tilde{K}_0(R)$.

If $\text{image}K_1(\mathbb{Z}) \subset X \subset K_1(R)$, then we get an exact sequence

$$(1.1.6) \quad \cdots \to L_n^X(R,\alpha,u) \to L_n^{\tilde{X}}(R,\alpha,u) \to H_\alpha^n(\text{image}K_0(\mathbb{Z})) \to \cdots,$$

where $\tilde{X}$ = image of X in $\tilde{K}_1(R)$.

Again we get Rothenberg sequences as in (1.1.1), and

$$(1.1.7) \qquad L_n^{K_i(R)}(R,\alpha,u) = L_n^{0\subset K_{i-1}(R)}(R,\alpha,u).$$

Furthermore,

(1.1.8)

$$L_n^{Y_1\times Y_2}(R_1 \times R_2,\alpha_1 \times \alpha_2,u_1 \times u_2) \simeq L_n^{Y_1}(R_1,\alpha_1,u_1) \times L_n^{Y_2}(R_2,\alpha_2,u_2)$$

and

(1.1.9) $L_n^Y(R_1,\alpha_1,u_1) \simeq L_n^{\phi(Y)}(R_2,\alpha_2,u_2)$ whenever (R_1,α_1,u_1) and (R_2,α_2,u_2) are quadratic Morita equivalent. ($\phi : K_i(R_1) \xrightarrow{\simeq} K_i(R_2)$ is the isomorphism induced by the Morita equivalence)

(1.1.10) Convention: Henceforth $L_n(R,\alpha,u)$ will denote $L_n^{K_1}(R,\alpha,u) \xrightarrow{\simeq} L^{0\subset K_0}(R,\alpha,u)$.

Our first goal is to compute $L_n^p(\mathbb{Z}G,\alpha,u)$ for G any finite 2-group and (α,u) any anti-structure.

First consider the following long exact sequence

(1.1.11)

$$\cdots \to L_n^p(\mathbb{Z}G,\alpha,u) \to L_n^p(\hat{\mathbb{Z}}_2G,\alpha,u) \xrightarrow{\Psi} L_n^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u) \to \cdots$$

(1.2) <u>Computation of $L_n^p(\hat{\mathbb{Z}}_2G,\alpha,u)$</u>

(1.2.1) <u>Theorem</u>: For any finite 2-group and any anti-structure (α,u) on $\hat{\mathbb{Z}}_2G$, we get

$$L_n^p(\hat{\mathbb{Z}}_2G,\alpha,u) \cong L_n^p(\mathbb{Z}/2,\mathrm{id},1) = \begin{matrix} \mathbb{Z}/2 & \text{if} & n \equiv 0(2) \\ 0 & \text{if} & n \equiv 1(2) \end{matrix}$$

Theorem 1.2.1 follows from the following two results.

(1.2.2) <u>Reduction Theorem</u>: If R is a complete local ring then for any 2-sided ideal I,

(i) $K_0(R) \cong K_0(R/I)$, and

(ii) $L_n^p(R,\alpha,u) \cong L_n^p(R/I,\alpha,u)$, (assuming $\alpha(I) = I$)

<u>Proof</u>: See [W5], [B].

(1.2.3) <u>Lemma</u>: If G is a finite p-group, then $\ker\ (\mathbb{Z}/p)G \to \mathbb{Z}/p$ is nilpotent.

<u>Proof</u>: See [SE], p. 57.

Notice (1.2.3) implies that kernel $(\hat{\mathbb{Z}}_pG \to (\mathbb{Z}/p)G \to \mathbb{Z}/p)$ is complete.

(1.3) <u>Computation of $L_n^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u)$</u>

It is well known that

$$\mathbb{Q}G = \pi A_\phi, \qquad \text{(see [S1] or [Y])}$$

where the product is taken over the set of isomorphism classes of irreducible $\mathbb{Q}$-representations. Each A_ϕ is a simple ring, and $A_\phi = a_\phi \mathbb{Q}G$ for some central idempotent a_ϕ which can not be expresses as a sum of nontrivial central idempotents. Since α^2 is an inner automorphism $\alpha(a_\phi)$ is either a_ϕ or $a_{\alpha(\phi)}$ where $\alpha(\phi)$ is another irreducible $\mathbb{Q}$-representation. In fact $a_\phi \in \mathbb{Z}[\frac{1}{2}]G$ (see [Y], p. 4) and $\Lambda_\phi = a_\phi(\mathbb{Z}[\frac{1}{2}]G)$ is a $\mathbb{Z}[\frac{1}{2}]$-maximal order in A_ϕ (see [Re], p. 379). Restriction gives a decomposition of rings with anti-structure.

(1.3.1) $\quad (\mathbb{Z}[\frac{1}{2}]G,\alpha,u) =$

$$\prod_{\phi=\alpha(\phi)}(\Lambda_\phi,\alpha_\phi,u_\phi) \times \prod_{\phi\neq\alpha(\phi)}(\Lambda_\phi \times \Lambda_{\alpha(\phi)},\alpha_{\phi\times\alpha(\phi)},u_\phi \times u_{\alpha(\phi)}) .$$

The $\Lambda_\phi \times \Lambda_{\alpha(\phi)}$ are called type GL factors and make no contribution to any Wall group.

We prove the following result in Section 2.

(1.3.2) <u>Decomposition Theorem</u>: For any finite 2-group G and any anti-structure on $\mathbb{Z}G$, we get the following canonical isomorphisms.

$$L_n^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u) \cong L_n(\mathbb{Z}[\tfrac{1}{2}]G \to \hat{\mathbb{Q}}_2G,\alpha,u)$$

$$\cong \prod_{\substack{\phi \\ \phi=\alpha(\phi)}} L_n(\Lambda_\phi \to \hat{\Lambda}_{\phi(2)},\alpha_\phi,u_\phi)$$

(Recall that L_n denotes $L_n^{K_1} \cong L_n^{O\subset K_0}$)

Consider the following $\mathbb{Z}[\frac{1}{2}]$-algebras, where ζ_j is a primitive 2^j-th root of 1 and $-$ denotes complex conjugation.

(1.3.3)

1) $\Gamma_N = \mathbb{Z}[\frac{1}{2}][\zeta_{N+1}]$

2) $R_N = \mathbb{Z}[\frac{1}{2}][\zeta_{N+2} + \overline{\zeta}_{N+2}]$

3) $F_N = \mathbb{Z}[\frac{1}{2}][\zeta_{N+2} - \overline{\zeta}_{N+2}]$

4) $H_N = (\frac{-1,-1}{\mathbb{Z}}) \otimes R_{N-2}$, where

$$(\frac{-1,-1}{\mathbb{Z}}) = \{\mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k \mid ij = -ji = k,\ i^2 = j^2 = -1\}$$

<u>Remark</u>: Each of these $\mathbb{Z}[\frac{1}{2}]$-algebras is a free $\mathbb{Z}[\frac{1}{2}]$-module of rank 2^N.

Now, consider the following anti-structures.

(1.3.4)

1) On Γ_N, $(\mathrm{Id},1)$, $(-,1)$, $(\tau,1)$, $(\overline{\tau},1)$

Id is the identity; $-$ is complex conjugation; $\tau(\zeta_{N+1}) = -\zeta_{N+1}$ or, equivalently, Γ_{N-1} is the fixed field of τ; $\overline{\tau}$ has fixed field F_{N-1}.

2) On R_N, $(\mathrm{Id},1)$, $(\tau,1)$

3) On F_N, $(\mathrm{Id},1)$, $(-,1)$

4) On H_N, $(\alpha_1,1)$, $(\hat{\alpha},1)$ where $\alpha_1(i) = \hat{\alpha}(i) = i$

$$\alpha_1(j) = \hat{\alpha}(j) = j$$

and $\alpha_1|_{R_{N-2}} = \mathrm{Id}$, $\hat{\alpha}|_{R_{N-2}} = \tau$.

In Section 2 we also prove the following result.

(1.3.5) Identification Theorem: If G is a finite 2-group, and (α,u) is any anti-structure on $\mathbb{Z}G$; then for any irreducible $\mathbb{Q}$-representation ϕ with $\alpha(\phi) = \phi$, we get that

$$L_n(\Lambda_\phi \to \hat{\Lambda}_{\phi(2)},\alpha_\phi,u_\phi) \cong L_n(\Delta_\phi \to \hat{\Delta}_{\phi(2)},\beta_\phi,v_\phi)$$

where $\Lambda_\phi = a_\phi\mathbb{Z}[\frac{1}{2}]G$ and $(\Delta_\phi,\beta_\phi,\pm v_\phi)$ is one of the rings with anti-structure in list (1.3.4). Recall that $L_n(\Delta_\phi,\beta_\phi,v_\phi) \cong L_{n+2}(\Delta_\phi,\beta_\phi,-v_\phi)$.

In Section 3 we compute $L_n(\Delta \to \Delta_{(2)},\beta,v)$ for all of the rings with anti-structure in List (1.3.4) and tabulate the results in Table 1.

Theoretically we could then calculate $L_n^p(\mathbb{Z}G,\alpha,u)$, but we restrict the anti-structure slightly at the start of Appendix I in order to easily identify $(\Delta_\phi,\beta_\phi,v_\phi)$ on List (1.3.4) (see Appendix I, part 1). In part 2 we settle the remaining questions involved in using 1.1.11.

§2. Proofs of the Decomposition Theorem (1.3.2) and of the Identification Theorem (1.3.5)

(2.1) Excision in Arithmetic Squares

Suppose S is a multiplicative subset of a ring A. Then $S^{-1}A$ is the localization of R away from S, $\hat{A} = \lim\limits_{s\in S} A/sA$ is the S-adic completion of A, and

(2.1.1)
$$\begin{array}{ccc} A & \longrightarrow & \hat{A} \\ \downarrow & & \downarrow \\ S^{-1}A & \to & S^{-1}\hat{A} \end{array}$$

is the arithmetic square associated to (A,S).

(2.1.2) <u>K-theory Excision Theorem</u>: For any integer i,

$$K_i(A \to \hat{A}) \cong K_i(S^{-1}A \to S^{-1}\hat{A})$$

(2.1.3) <u>Corollary</u>: For any finite 2-group G and any integer i,

$$K_i(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G) \cong K_i(\mathbb{Z}[\tfrac{1}{2}]G \to \hat{\mathbb{Q}}_2G)$$

$$\cong \prod_{\substack{\phi \\ \text{irred. } \mathbb{Q}\text{-rep.}}} K_i(\Lambda_\phi \to \hat{\Lambda}_{\phi(2)})$$

(same notation as in (1.3))

(2.1.4) <u>L-theory Excision Theorem</u>: (See [R1]. Also [B], [B-W], [C-M], [P], and [W7].)

We assume that A in (2.1.1) is equipped with an anti-structure (α,u) such that $\alpha|_S$ is the identity. Localization and completion then induce anti-structures on the other rings in (2.1.1). Let $X \subset K_i(S^{-1}A)$ and $Y \subset K_i(\hat{A})$ be α-invariant subgroups. Let C = kernel of $K_i(A) \to K_i(S^{-1}A)/X \oplus K_i(\hat{A})/Y$, and let I = image of $X \oplus Y \to K_i(S^{-1}A) \oplus K_i(\hat{A}) \to K_i(S^{-1}\hat{A})$. Then,

$$L_n^{C\to Y}(A \to \hat{A}) \cong L_n^{X\to I}(S^{-1}A \to S^{-1}\hat{A}).$$

(2.1.5) <u>Corollary</u>: For any finite 2-group G and any anti-structure (α,u) on $\mathbb{Z}G$, letting X and Y be trivial we get

$$L_n^{C_i(G)}(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u) \cong L_n^{K_{i+1}}(\mathbb{Z}[\tfrac{1}{2}]G \to \hat{\mathbb{Q}}_2G,\alpha,u)$$

$$\cong \prod_{\substack{\phi \\ \alpha(\phi)=\phi}} L_n^{K_{i+1}}(\Lambda_\phi \to \hat{\Lambda}_{\phi(2)},\alpha_\phi,u_\phi)$$

where $C_i(G) = \ker K_i(\mathbb{Z}G) \to K_i(\mathbb{Z}[\frac{1}{2}]G) \oplus K_i(\hat{\mathbb{Q}}_2G)$ and the rest of the notation is the same as in (1.3).

If $i = 1$, then the $L_n^{C_i(G)}$-groups are the L_n'-groups which were computed by Wall [W8].

Notice that if we can show that $L_n^{C_0(G)}(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G) \cong L_n^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G)$, then (2.1.5) would imply the Decomposition Theorem (1.3.2).

(2.2) <u>Representation Theory for Finite 2-groups</u>

<u>Definition</u>: A finite 2-group π is <u>special</u> if it has no noncyclic, normal abelian subgroups.

(2.2.1) <u>Proposition</u>: A group π is special if and only if it is one of the following groups

(i) cyclic, $C_N = \langle x \mid x^{2^N} = 1\rangle$

(ii) dihedral, $D_N = \langle x,y \mid x^{2^{N-1}} = y^2 = 1,\ yxy = x^{-1}\rangle$, $N > 3$

(iii) semi-dihedral, $SD_N = \langle x,y \mid x^{2^{N-1}} = y^2 = 1,\ yxy^{-1} = x^{2^{N-2}-1}\rangle$, $N > 3$

(iv) quaternionic, $Q_N = \langle x,y \mid x^{2^{N-1}} = 1,\ y^2 = x^{2^{N-2}},\ yxy^{-1} = x^{-1}\rangle$, $N \geq 3$.

Each special group π has a unique faithful, irreducible, $\mathbb{Q}$-representation $\psi(\pi)$.

For any irreducible $\mathbb{Q}$-representation of a group G $\rho : G \to GL(V_\rho)$, we let

$$D_\rho = \mathrm{End}_{\mathbb{Q}G}(V_\rho).$$

Schur's lemma implies that D_ρ is a division ring.

(2.2.2) Theorem: For any irreducible $\mathbb{Q}$-representation ϕ on a finite 2-group G, there exists a subgroup H with normal subgroup N such that

(i) H/N is special

(ii) If we pull $\psi(H/N)$ back to H and then induce up to G, we get ϕ.

(iii) $D_\phi \simeq D_\psi$, where $\psi = \psi(H/N)$.

Proof: (See [F])

(2.2.3) Table

π	$D_{\psi(\pi)}$
C_N	$\Gamma_{N-1} \otimes \mathbb{Q}$
D_N	$R_{N-3} \otimes \mathbb{Q}$
SD_N	$F_{N-3} \otimes \mathbb{Q}$
Q_N	$H_{N-1} \otimes \mathbb{Q}$

Thus the rings from list (1.3.3) are $\mathbb{Z}[\frac{1}{2}]$-maximal orders in the division rings $D_{\psi(\pi)}$. Notice that the centers of these division rings are precisely the fields which are subfields of $\mathbb{Q}(\zeta_j)$ for some j, namely fields of the form $\mathbb{Q}(\zeta_i)$, $\mathbb{Q}(\zeta_i + \overline{\zeta}_i)$, and $\mathbb{Q}(\zeta_i - \overline{\zeta}_i)$. (Recall ζ_j is a primitive 2^j-th root of 1.)

(2.2.4) Weber's Theorem: Suppose K is a subfield of $\mathbb{Q}(\zeta_j)$ for some j. Let $\mathcal{O}$ be the ring of algebraic integers in

K, and let $R = \mathcal{O}[\frac{1}{2}]$. Then

(i) K/Q is unramified over all odd primes. Over 2, it is totally ramified, and the unique dyadic prime d is principal.

(ii) The ideal class group $\Gamma(K) \simeq \tilde{K}_0(\mathcal{O}) = \tilde{K}_0(R)$ has odd order

(iii) The narrow ideal class group

$$\Gamma^*(K) \simeq \frac{(\text{group of ideals})}{\left(\begin{array}{l}\text{principal ideals } (x) \\ \text{such that } x > 0 \\ \text{for all real places}\end{array}\right)}$$

also has odd order

<u>Proof</u>: For (ii), see Theorem 10.4 in [Was]. Class field theory implies that if $K = \mathbb{Q}(\zeta_i + \overline{\zeta}_i)$, and $\Gamma^*(K)$ does not have odd order; then K has a quadratic entension E/K which is unramified at all finite primes. But, then $E \otimes_K \mathbb{Q}(\zeta_i)$ would be an unramified, quadratic extension of $\mathbb{Q}(\zeta_i)$. Thus (ii) for $K = \mathbb{Q}(\zeta_i)$ implies (iii) for $K = \mathbb{Q}(\zeta_i + \overline{\zeta}_i)$.

(2.2.5) <u>Corollary</u>: For any N, $\tilde{K}_0(\Gamma_N)$, $\tilde{K}_0(R_N)$, $\tilde{K}_0(F_N)$, and $\tilde{K}_0(H_N)$ have odd order.

<u>Proof</u>: Notice that if $R = \Gamma_N$, R_N, or F_N, then $R = \mathcal{O}[\frac{1}{2}]$ where $\mathcal{O}$ = ring of algebraic rings in a subfield of $\mathbb{Q}(\zeta_i)$ for some i. Since H_N is a maximal order in the division algebra $H_N \otimes \mathbb{Q}$, (36.3) in [Re] implies that

$$\tilde{K}_0(H_N) \simeq \Gamma^*(\mathbb{Q}(\zeta_N + \overline{\zeta}_N)).$$

(2.3) <u>(Linear) - Morita Theory</u>: (see [Bass 1] and [Re] for details).

<u>Definition</u>: A <u>Morita equivalence</u> between two rings A and B is a 4-tuple (M,N,u,τ) where M and N are bimodules ${}_BM_A$ and ${}_AN_B$; $u : M\otimes_A N \to B$ and $\tau : N\otimes_B M \to A$ are bimodule isomorphisms such that

$$\tau(n \otimes m) \cdot n' = n \cdot u(m \otimes n'),$$

and

$$u(m \otimes n) \cdot m' = m \cdot \tau(n \otimes m')$$

for all $n, n' \in N$, and all $m, m' \in M$.

For any ring A, we let P_A denote the category of finitely generated projective right R-modules.

(2.3.1) <u>Theorem</u>: Assume (M,N,u,τ) is a Morita equivalence between A and B. Then, we get an equivalence of categories

$$P_A \underset{\otimes_B M}{\overset{\otimes_A N}{\rightleftarrows}} P_B$$

and an isomorphism

$$K_i(A) \overset{\sim}{\leftrightarrow} K_i(B).$$

Furthermore, center(A) = B - A - bimodule endomorphisms of M = center(B).

<u>Examples</u>

(2.3.2) <u>Derived Morita equivalence</u>

Suppose $M \in \mathcal{O}b\ (P_A)$ and A is a direct summand of M^n for some $n > 0$, i.e. M is a <u>progenerator</u>. Then

A and $B = \mathrm{End}_A(M)$ are Morita equivalent via $(M, N = \mathrm{Hom}_A(M,A), u, \tau)$ where $u(m \otimes n) \cdot m' = m \cdot n(m')$ and τ is the evaluation map.

If $\phi : G \to GL_n(\mathbb{Q})$ is a $\mathbb{Q}$-irreducible representation of a finite group G, then we let V_ϕ denote the simple module of the simple component $A_\phi \subset \mathbb{Q}G$. Thus A_ϕ and the division ring $D_\phi = \mathrm{End}_{A_\phi}(V_\phi)$ are Morita equivalent. Furthermore,

$$K_i(\mathbb{Q}G) \simeq \pi_\phi K_i(D_\phi).$$

(2.3.3) If R is a commutative ring, then a R-algebra Λ is <u>Azumaya</u> if there is a R-algebra B and a progenerator M of P_R such that $\Lambda \otimes_R B \cong \mathrm{End}_R(M)$ as R-algebras. (See [K-O].) If Λ is an Azumaya R-algebra, then Λ is central i.e. center $\Lambda = R$. Assume R is a Dedekind domain with field of fractions K. Then, whenever Λ is an Azumaya R-algebra, Λ is also a R-maximal order in $\Lambda \otimes_R K$. Conversely, if Λ is a R-maximal order in a simple K-algebra A with center R, then Λ is Azumaya if and only if $\hat{A}_\rho \simeq M_n(\hat{K}_\rho)$ for all finite prime ideals in R. (See [Rog].)

Suppose $\phi : G \to GL(V_\phi)$ is a irreducible $\mathbb{Q}$-representation of a finite group of order m. Then $\Lambda_\phi = a_\phi \cdot \mathbb{Z}[\frac{1}{m}]G$ is an Azumaya R_ϕ-algebra where $R_\phi = \mathrm{center}(\Lambda_\phi)$. (See [F], Corollaire 1 of Prop. 8.1.)

<u>Definition</u>: For any commutative ring R, $Br(R)$ is the set of Morita equivalence classes of Azumaya R-algebras. It becomes an abelian group under tensor product over R.

Suppose R is a Dedekind domain with quotient field K a finite extension of $\mathbb{Q}$ or $\hat{\mathbb{Q}}_p$ for some prime p.

(2.3.4) Theorem: Let $\Lambda_j \subset A_j$ for $j = 1,2$ be R-maximal orders in simple K-algebras A_j, $j = 1,2$. Then Λ_1 and Λ_2 are Morita equivalent if and only if A_1 and A_2 are Morita equivalent.

Proof: See [Re], Theorem (21.6).

(2.3.5) Corollary: The map $Br(R) \to Br(K)$ is a momomorphism.

(2.3.6) Theorem: Suppose that G is a finite 2-group. Then, for any i,

$$K_i(\mathbb{Z}G \to \hat{\mathbb{Z}}_2 G) \to \prod_{\phi} K_i(\Delta_\phi \to \hat{\Delta}_{\phi(2)})$$

where ϕ runs over the irreducible rational representations of G and Δ_ϕ is one of the rings on list (1.3.3).

Proof: First apply corollary (2.1.3). The result follows from (2.3.4) after consulting paragraph two of (2.3.2); (2.2.2) (iii); and Table 2.2.3.

(2.4) Proof of the Decomposition Theorem (1.3.2)

(2.4.1) Theorem (Swan): If G is a finite group, then $\tilde{K}_0(\mathbb{Z}G)$ is a finite group.

Then (2.3.6), (2.2.5), and (1.2.2 (i)) imply that $C_0(\mathbb{Z}G) \to \tilde{K}_0(\mathbb{Z}G)$ becomes an isomorphism when we localize at 2. A Rothenberg sequence argument then implies that

$$L_n^{C_0(\mathbb{Z}G)}(\mathbb{Z}G \to \hat{\mathbb{Z}}_2 G) \to L_n^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2 G) .$$

Thus (1.3.2) is a special case of (2.1.5).

(2.5) Quadratic Morita Theory (Compare with [F-Mc] and [F-W])

Definition: A quadratic Morita equivalence between two rings with anti-structure (A,α,u) and (B,β,v) is given by a Morita equivalence (M,N,μ,τ) plus a B-A-bimodule isomorphism $h : M \to N$ where we make N into a B-A bimodule using α and β. We also require that

$$\alpha\tau(h(m_1 u) \otimes m_2) = \tau(h(m_2) \otimes m_1 v) \qquad \text{for all } m_1, m_2 \in M$$

(2.5.1) Theorem: A quadratic Morita equivalence between (A,α,u) and (B,β,v) induces isomorphisms

$$\phi : K_i(A) \xrightarrow{\simeq} K_i(B), \quad \text{(equivariant with respect to } \alpha_* \text{ and } \beta_*)$$

$$H_\alpha^n(K_i(A)) \xrightarrow{\simeq} H_\beta^n(K_i(B)),$$

and

$$L_n^X(A,\alpha,u) \xrightarrow{\simeq} L_n^{\phi(X)}(B,\beta,v) \quad \text{where } X \text{ is any } \alpha_*\text{-invariant subgroup of } K_i(A).$$

(2.5.2) Derived Quadratic Morita Equivalence Theorem: Suppose (A,α,u) is a ring with anti-structure and (M,N,μ,τ) is a (linear) Morita equivalence between A and B. Let R = center A = center B. Assume $h : M \to N$ is a right A-module isomorphism, where we use α to make N into a right A-module. Then,

(i) B admits a unique anti-automorphism β such that h becomes a B-A-bimodule isomorphism when we use β to make N a left S-module. $(\beta | R = \alpha | R)$; and

(ii) there exists a unique unit $v \in B$ such that (M,N,μ,τ,h) is a quadratic Morita equivalence between (A,α,u) and (B,β,v).

(2.5.3) Corollary: Suppose (A,α,u) is a ring with anti-structure where A is a simple algebra over a field K. Let V = simple right A-module and let D = the division algebra $\mathrm{End}_A(V)$. Then (A,α,u) is quadratic Morita equivalent to (D,β,v) for some anti-structure (β,v).

Proof: Since V and $V^{\alpha} = \mathrm{Hom}_A(V,A)$ are both simple right A-modules, there exists a right A-module isomorphism $h : V \to V^{\alpha}$.

(2.5.4) Corollary: Suppose (α,u) is an anti-structure on $\mathbb{Q}G$ for some finite group G. Then,

$$L_n(\mathbb{Q}G,\alpha,u) \doteq \prod_{\substack{\phi \\ \phi=\alpha(\phi)}} L_n(D_\phi,\beta_\phi,v_\phi).$$

(2.5.5) Definition: If (R,α,u) is a ring with anti-structure and w is a unit in R, then the scaling of (α,u) by w is the new anti-structure

$$(\alpha,u)^w = (\beta,v)$$

where $\beta(r) = w\alpha(r)w^{-1}$ for all $r \in R$, and $v = w\alpha(w^{-1})u$.

For any R-module M, there exists an isomorphism

$$D^{\alpha}M \to D^{\beta}M;\ f \to (f^w : x \to wf(x)).$$

Thus we get an isomorphism

$$\sigma^w : L_n(R,\alpha,u) \to L_n(R,(\alpha,u)^w)$$

Alternatively, σ^w can be gotten by applying (2.5.2) with

$M_R = R$ and $h : R \to \mathrm{Hom}_R(R,R) = R$ the map that sends r to rw^{-1}.

(2.5.6) Definition: Suppose R is a commutative ring with involution α_0. Then $\underline{Br(R,\alpha_0)}$ is the set of quadratic Morita equivalence classes of rings with anti-structure (Λ,α,u), where Λ is a Azumaya R-algebra and $\alpha|_R = \alpha_0$. $Br(R,\alpha_0)$ is an abelian group under tensor product.

Warning: We shall see in (2.5.9) that the quadratic analogue of (2.3.4) is not true in general.

Let $Br_0(R,\alpha_0)$ be the kernel of the forgetful map $Br(R,\alpha_0) \to Br(R)$.

Assume that R is a Dedekind domain with quotient field K. Let I = the group of R-fractional ideals in K, and let $g : K^* \to I$ be the map that sends $x \in K^*$ to the ideal (x). By sending elements and fractional ideals to their images under the map $\alpha_0 : K \to K$ we get an action of $\mathbb{Z}/2$ on K^* and I.

Warning: The map $I \to K_0(R)$ which sends a fractional ideal $\mathfrak{a}$ to the underlying module $[\mathfrak{a}]$ is not equivariant. Indeed, $[\alpha_0(\mathfrak{a}^{-1})] \simeq [\mathfrak{a}]^{\alpha_0} = \mathrm{Hom}_R(\mathfrak{a},R)$ made into a right R-module via α_0.

(2.5.7) Theorem: There exists an isomorphism

$$\Psi : Br_0(R,\alpha_0) \to \hat{H}^0(\mathbb{Z}/2, K^* \to I) \cong \frac{\{(x,\mathfrak{a}) \in K^* \oplus I \mid \alpha_0(x) = x^{-1}, (x)\mathfrak{a} = \alpha_0(\mathfrak{a})\}}{\{(y\alpha_0(y^{-1}),\ (y)\beta\alpha_0(\beta) \mid (y,\beta) \in K^* \oplus I\}}$$

The map Ψ is defined as follows. Suppose (Λ,α,u) represents an element in $Br_0(R,\alpha_0)$. Choose M so that $\Lambda = End_R(M)$. Let $V = M\otimes_R K$ and $A = \Lambda\otimes_R K$. as in (2.5.3) we can choose a right A-module isomorphism $h : V \to V^\alpha$ which yields an anti-structure (β,v) on $End_A(V)$. Notice that $K = End_A(V)$ and $\beta = \alpha_0$. Let $ad(h) : V \times V \to K$ be the adjoint of $h : V \to V^\alpha = Hom_K(V,K)$. Then $\Psi(\Lambda,\alpha,u)$ is represented by $(v,\mathfrak{a})$ where $\mathfrak{a}$ is the fractional ideal generated by $h(M \times M)$.

The map Ψ has the following interpretation. Assume h is choosen so that $\mathfrak{a} \subset R$. Then the (linear) Morita equivalence derived from M and the pairing $h : M \times M \to \mathfrak{a}$ determines an equivalence of categories $Sesq(\Lambda,\alpha,u) \to Sesq(R,\alpha_0,v)$. But, nonsingular forms are sent to $\mathfrak{a}$-valued <u>modular</u> forms.

The following result was suggested to us by Karoubi.

(2.5.8) <u>Proposition</u>: Any ring with anti-structure (A,α,u) is quadratic Morita equivalent to $(M_2(A),\beta,1)$ where

$$\beta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha(d) & \alpha(b)u \\ u^{-1}\alpha(c) & u^{-1}\alpha(a)u \end{pmatrix}$$

<u>Proof</u>: Let $\{e_1,e_2\}$ be the standard basis for $M = A \oplus A$ and $\{e_1^*,e_2^*\}$ be the dual basis for $N = M^*$. Then we let $h : M \to N$ be given by $h(e_1) = ue_2^*$ and $h(e_2) = e_1^*$ and we apply (2.5.2) to the derived Morita equivalence.

This implies that $Br_0(R,\alpha_0)$ is isomorphic to $B_0(R,\mathbb{Z}/2)$ in the sense of Frohlich-Wall [F-W]. Thus (2.5.7) is at least implicit in [F-W].

(2.5.9) Sample Calculations:

(i) $R = K$

If $\operatorname{char}K \neq 2$, then $Br_0(K, id)$ has two elements which are represented by $(K, \alpha_0, 1)$ and $(K, \alpha_0, -1)$.

$Br_0(K, \alpha_0) = (1)$ when $\alpha_0 \neq id$ or $\operatorname{char}K = 2$.

(ii) R = finite extension of $\hat{\mathbb{Z}}_p$.

$$Br_0(R, id) \xrightarrow{\sim} Br_o(K, id) \simeq \{\pm 1\}$$

If $\alpha_0 \neq id$, we let r be the fixed subring of α_0 . Then, $Br_0(R, \alpha_0) = (1)$, when R is inert over r, and $Br_0(R, \alpha_0)$ has order 2 when R is ramified over r. Notice that $Br(R, \alpha_0) \to Br(K, \alpha_0)$ is not an injection.

In cases (iii) and (iv) we let $R = \mathcal{O}_{(\Sigma)}$ where $\mathcal{O}$ is a ring of algebraic integers and Σ is a set of prime ideals in $\mathcal{O}$, e.g. the center of the rings in List (1.3.3).

(iii) $Br_0(R, id) = {}_2R^* \oplus \Gamma/\Gamma^2$

where $\Gamma = \operatorname{coker}(K^* \to I)$, and ${}_2R^* = \{x \in R^* \mid x^2 = 1\}$.

The isomorphism comes from the following braid

$$\begin{tikzcd}
\hat{H}^1(R^*) \arrow[rr, bend left, "\sim"] \arrow[dr] & & \hat{H}^1(\hat{K}^*) \arrow[dr] & & \\
& \hat{H}^0(K^* \to I) \arrow[ur] \arrow[dr] & & 0 \arrow[dr] & \\
\hat{H}^0(I) \arrow[ur] \arrow[rr, bend right] & & \hat{H}^0(\Gamma) \arrow[ur] \arrow[rr, bend right] & & \hat{H}^0(R^*)
\end{tikzcd}$$

which is induced by the following exact sequence of $\mathbb{Z}$ $\mathbb{Z}/2$-modules

$$1 \to R^* \to K^* \to I^* \to \Gamma \to 1$$

Remark: The localization sequence (see [R1], §4.2) implies that

$$L_3^p(R,id,1) \simeq \operatorname{coker}(L_0^p(K,id,1) \to \bigoplus_\rho L_0^p(\hat{R}_\rho \to \hat{K}_\rho,id,1)) .$$

Similarly, if (Λ,α,u) represents an element $a \in \Gamma/\Gamma^2 \subset Br_0(R,id)$ and $A = \Lambda \otimes_R K$, then $L_3^p(\Lambda,\alpha,u) \simeq$ $\operatorname{coker}(L_0^p(A,id,1) \to \bigoplus_\rho L_0^p(\hat{\Lambda}_\rho \to \hat{A}_\rho,id,1))$, where $L_0^p(A,id,1) \simeq$ $L_0^p(K,id,1)$ and where $L_0^p(\hat{\Lambda}_\rho \to \hat{A}_\rho,id,1) \not\simeq L_0^R(\hat{R}_\rho \to \hat{K}_\rho,id,1)$. But it is not true in general that $L_3^p(R,id,1) \simeq L_3^p(\Lambda,\alpha,u)$. For example if a is nontrivial and $\frac{1}{2} \in R$, then

$$\text{order } L_3^p(R,id,1) = 2 \times \text{order } L_3^p(\Lambda,\alpha,u).$$

(iv) Assume $\alpha_0 \neq id$.

Case 1: K is unramified over the fixed field of α_0 and Σ = the set of all prime ideals in $\mathcal{O}$. Then $Br_0(R,\alpha_0)$ has order 2, but the map $Br_0(R,\alpha_0) \to Br_0(K,\alpha_0) \oplus \prod_\rho Br_0(\hat{R}_\rho,\alpha_0)$ is trivial. Furthermore, if (Λ,α,u) represents the nontrivial element in $Br_0(R,\alpha_0)$. Then

$$L^p(R,\alpha_0,1) \not\simeq L_0^p(\Lambda,\alpha,u).$$

Case 2: Otherwise,

$$\mathrm{Br}_0(R,\alpha_0) \simeq \oplus\, \mathrm{Br}_0(\hat{R}_\rho,\alpha_0),$$

where we can sum all finite primes ρ in Σ which are ramified over the fixed field of α_0 .

These results are proven by using the isomorphism

$$\hat{H}^0(\mathbb{Z}/2;K^* \to I) \simeq \hat{H}^0(\mathbb{Z}/2;\ \prod_{\rho\notin\Sigma} \hat{R}^*_\rho \times \prod_{\rho\in\Sigma} \hat{K}^*_\rho \times \prod_{\substack{v\\ \text{arch}}} \hat{K}^*_v \to \mathcal{C}(K)),$$

where $\mathcal{C}(K)$ is the idele class group of K.

<u>Remark</u>: If $R = \mathcal{O}$, then case (iv) is related to Connor's book [C]. In fact,

$$\hat{H}^0(\mathbb{Z}/2;K^* \to I) \simeq \mathrm{Gen}(K/K^{\alpha_0})$$

(see chap. I in [C]), and if $\Psi(\Lambda,\alpha,u) = [(x,\mathfrak{a})]$, then $L^p_0(\Lambda,\alpha,u) \simeq H_x(\mathfrak{a})$, where $H_x(\mathfrak{a})$ is the Witt group of x-symmetric, $\mathfrak{a}$-modular forms studied in chap. IV of [C].

(2.6) <u>Proof of the Identification Theorem (1.3.5)</u>

Theorem (1.3.5) will follow from (2.5.1) (or rather its relative version) if we can prove that $(\Lambda_\phi,\alpha_\phi,u_\phi)$ is quadratic Morita equivalent to $(\Delta_\phi,\beta_\phi,\pm 1)$ where $(\Delta_\phi,\beta_\phi,1)$ is one of the rings with anti-structure in List (1.3.4)

From the proof of Theorem (2.3.6), we know that Λ_ϕ is linearly Morita equivalent to Γ_N, F_N, R_N, or H_N for some N. Let R denote the center of Λ_ϕ. Then $(\Lambda_\phi,\alpha_\phi,u_\phi) \in \mathrm{Br}(R,\alpha_0)$ for some α_0.

The proof divides into three cases.

1) D_ϕ is commutative, $\alpha_0 = \mathrm{Id}$.

Then $(\Lambda_\phi,\alpha_\phi,u_\phi) \in \mathrm{Br}_0(R,\mathrm{Id})$. From (2.5.9) (iii) and (2.2.4) (ii), $\mathrm{Br}_0(R,\mathrm{Id}) \simeq \mathbb{Z}/2\mathbb{Z}$ and from (2.5.9) (i) we see that (R,Id,1) and (R,Id,-1) are the two elements.

2) D_ϕ is non-commutative, $\alpha_0 = \mathrm{Id}$.

The calculation in 1) shows that $(H_N,\alpha_1,1)$ and $(H_N,\alpha_1,-1)$ are the two distinct elements in Br(R,Id) which map to $[H_N] \in$ Br(R). From (2.3.6) we know that Λ_ϕ is linearly Morita equivalent to H_N, so done.

3) $\alpha_0 \neq \mathrm{Id}$.

First notice that R with each non-trivial involution occurs on List (1.3.4). From (2.2.4) (i) and (2.5.9) (iv) case 2, we see $Br_0(R,\alpha_0) \simeq (1)$. Using (2.3.6) we are finished.

Remark: Notice that if $R = \Gamma_N$, F_N, or R_N, then, for any α_0,

$$Br(R,\alpha_0) \to Br(K,\alpha_0) \quad \text{is one to one.}$$

§3. Localization Sequence

The goal of this section is to compute $L_n(\Delta \to \hat{\Delta}_2,\beta,1)$ where Δ is any of the rings from List (1.3.3) and β is any (anti)-involution on Δ. Recall that L_n denotes $L_n^{O\subset K_0}$. Henceforth, we shall suppress writing the 1 in $(\beta,1)$.

The results are summarized in Table 1.

(3.1) General Background

Suppose K is an algebraic number field with ring of algebraic integers $\mathcal{O}$. Let $R = \mathcal{O}[\frac{1}{2}]$; D = central, simple, K-division algebra; Δ = R-maximal order in D; and β any (anti)-involution of Δ. We assume $\widetilde{K_0}(\Delta)$ has odd order.

Consider the following arithmetic square

$$\begin{array}{ccc} \Delta & \to & D \\ \downarrow & & \downarrow \\ \hat{\Delta} & \to & \hat{D} \end{array} .$$

Then,

(3.1.1)

$L_n(\Delta \to D,\beta) \simeq L_n(\hat{\Delta} \to \hat{D},\beta)$ (by L-theory Excision Theorem (2.1.4))

$\simeq \oplus\, L_n(\hat{\Delta}_\rho \to \hat{D}_\rho,\beta)$ (by 4.1.2 and 4.1.5 in [R1]),

where we sum over all maximal ideals ρ in R such that $\beta(\rho) = \rho$.

(3.1.2) Local Quadratic Morita Theorem: Suppose ρ is a maximal ideal in R such that $\beta(\rho) = \rho$ and such that $\hat{D}_\rho \simeq M_k(\hat{K}_\rho)$ for some k.

If $\beta|_R = \mathrm{id}$, we assume that $(\Delta,\beta,1) \in Br(R,\mathrm{id})$ maps to the trivial element in $Br(\bar{K},\mathrm{id}) \simeq \{\pm 1\}$, where $\bar{K}$ is the algebraic closure of K.

If $\beta|_R \neq \mathrm{id}$, we assume that ρ is unramified over the fixed field for $\beta|_K$.

Then,

$$L_n(\hat{\Delta}_\rho \to \hat{D}_\rho,\beta) \simeq L_n(\hat{R}_\rho \to \hat{K}_\rho,\beta)$$

Proof: Apply (2.5.1) and (2.5.9) (ii).

(3.1.3) Divissage Theorem: Suppose ρ is a maximal ideal in R such that $\beta(\rho) = \rho$. If $\beta|_R \neq \mathrm{id}$, we also assume that ρ is unramified over the fixed field for $\beta|_K$. Then, since $\frac{1}{2} \in \hat{R}_\rho$, we get

$$L_n(\hat{R}_\rho \to \hat{K}_\rho,\beta) \simeq L_n^p(k_\rho,\beta),$$

where k_ρ is the residue field R/ρ.

<u>Proof</u>: See 4.2.1 in [R1].

If (Δ,β) satisfies the assumptions in (3.1.2) and (3.1.3), then we get the following <u>localization diagram</u> with exact rows and columns

(3.1.4)

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
L_{i+1}(\Delta\to\hat{\Delta}_2,\beta) \arrow[r] \arrow[d] & L_i(D,\beta) \arrow[r, "\psi"] \arrow[d, equal] & L_i(\hat{\Delta}_2,\beta)\oplus\oplus L_i^p(k_\rho,\beta) \arrow[r] \arrow[d] & L_i(\Delta\to\hat{\Delta}_2,\beta) \arrow[r] \arrow[d] & \cdots \\
L_i(\Delta,\beta) \arrow[r] \arrow[d] & L_i(D,\beta) \arrow[r] & \oplus L_i^p(k_\rho,\beta) \arrow[r] \arrow[d, "0\text{-map}"] & L_{i-1}(\Delta,\beta) \arrow[r] \arrow[d] & \cdots \\
L_i(\hat{\Delta}_2,\beta) \arrow[d] & & L_{i-1}(\hat{\Delta}_2,\beta) \arrow[r, equal] & L_{i-1}(\hat{\Delta}_2,\beta) & \\
\vdots & & & &
\end{tikzcd}$$

Since $\frac{1}{2} \in \hat{\Delta}_2$, we get that $\hat{\Delta}_2 \simeq \hat{D}_2$. If $\mathcal{O}$ contains a unique dyadic prime, then $\hat{D}_2$ is a simple ring. Recall that in $\oplus L_i^p(k_\rho,\beta)$ we are summing over the set of maximal ideals in R such that $\beta(\rho) = \rho$. Notice that this is the same as summing over the β-invariant, N.D. maximal ideals in $\mathcal{O}$ (where N.D. stands for nondyadic).

We shall compute $L_*(\Delta \to \hat{\Delta}_2,\beta)$ by computing the map ψ. Notice that the domain and range of ψ is expressed in terms of L-groups of semi-simple rings.

(3.1.5) Semi-simple Theorem: If A is a semi-simple ring, then $L^p_{2i+1}(A,\beta) = 0$ for any involution β.

Proof: See [R2].

(3.1.6) Reduction Theorem: $L_i(\hat{R}_\rho,\beta) = L_i(k_\rho,\beta)$, where $k_\rho = R_\rho/\rho$.

For any abelian group G, ${}_2G = \{g \in G \mid g^2 = 1\}$.

(3.2) Type 0-Commutative Case: (Γ_N,id), (R_N,id), (F_N,id),

We assume $\beta = \mathrm{id}$ which we suppress writing.

Then for any field K with $\mathrm{char}K \neq 2$, $L^p_0(K) = W(K)$, the classical Witt ring of symmetric bilinear pairings over K (see [L], [M-H], [O'M], and [W4], p. 135). Multiplication in $W(K)$ comes from the tensor product of pairings. Let $I(K) = \text{kernel } r : W(K) \to \mathbb{Z}/2$ where r is the rank map.

The group $L^p_2(K) \simeq (1)$ because any skew-symmetric non-singular pairing b has a symplectic basis, i.e. b is hyperbolic (see [M-H], 3.5).

The Rothenberg sequence plus (3.1.5) then imply that

$$L_n(K) \simeq 0,0,{}_2K^*,I(K) \quad \text{for } n \equiv 3,2,1,0(4)$$

(3.2.1) Examples

(i) If k is a finite field with $\mathrm{char}\,k \neq 2$, then

$$\mathrm{disc}: I(k) \xrightarrow{\simeq} k^*/k^{*2} \quad \text{has order 2.}$$

(ii) If $\hat{K}_\rho/\hat{\mathbb{Q}}_p$ with $[\hat{K}_\rho,\hat{\mathbb{Q}}_p] = \ell$, then

$$\mathrm{disc} : I(\hat{K}_\rho)/I^2(\hat{K}_\rho) \xrightarrow{\sim} \hat{K}^*_\rho/\hat{K}^{*2}_\rho ;$$

Hasse-Witt: $I^2(\hat{K}_\rho) \xrightarrow{\sim} {}_2\mathrm{Br}(\hat{K}_2) \simeq \{\pm 1\}$. If ρ is N.D., then the map $L_0(\hat{K}_\rho) \to L_0(\hat{R}_\rho \to \hat{K}_\rho) \xrightarrow{\sim} L_0^p(k_\rho)$, sends $I^i(\hat{K}_\rho)$ onto $I^{i-1}(k_\rho)$ (see [M-H], IV, 1.4). Thus $I^1(\hat{K}_\rho) \to k^*_\rho/k^{*2}_\rho$ can be identified with the Hasse-Witt invariant. We also get the following exact sequence

$$\begin{array}{ccccccccc} 1 & \to & \hat{\mathcal{O}}^*_\rho/\hat{\mathcal{O}}^{*2}_\rho & \to & \hat{K}^*_\rho/\hat{K}^{*2}_\rho & \xrightarrow{v} & \mathbb{Z}/2 & \to & 1 \\ & & & & \wr\uparrow \mathrm{disc} & & \wr\uparrow r & & \\ & & & & I(\hat{K}_\rho)/I^2(\hat{K}_\rho) & \to & L_0(k_\rho)/I(k_\rho), & & \end{array}$$

where $\hat{\mathcal{O}}_\rho$ is the integral closure of $\hat{\mathbb{Z}}_p$ in $\hat{K}_\rho$.

For any ρ, $\hat{\mathcal{O}}^*_\rho \simeq \mu(\hat{K}_\rho) \times \hat{\mathbb{Z}}^\ell_\rho$ (see [S2], XIV, §4), where $\mu(\hat{K}_\rho)$ = roots of unity, Thus

$$\hat{\mathcal{O}}^*_\rho/\hat{\mathcal{O}}^{*2}_\rho \simeq \begin{cases} \mathbb{Z}/2 & \text{if } \rho \text{ is N.D.} \\ \mathbb{Z}/2^{\ell+1} & \text{if } \rho \text{ is dyadic} \end{cases}$$

(iii) $I(\mathbb{C}) \simeq (0)$ and $\mathrm{sig} : I(\mathbb{R}) \xrightarrow{\sim} 2\mathbb{Z}$.

(iv) If $K/\mathbb{Q}$ with $[K,\mathbb{Q}] = r_1 + 2r_2$ where r_1 is the number of embeddings of K into $\mathbb{R}$, then

$$\mathrm{disc} : I(K)/I^2(K) \xrightarrow{\sim} K^*/K^{*2},$$

Hasse-Witt: $I^2(K)/I^3(K) \xrightarrow{\sim} {}_2\mathrm{Br}(K)$, and

$\mathrm{sig} : I^3(K) \xrightarrow{\sim} \bigoplus_v I^3(\hat{K}_v) \xrightarrow{\sim} (8\mathbb{Z})^{r_1}$, where v varies over the real embeddings of K.

Suppose K is a field on 2.2.3 , $\mathcal{O}$ is the ring of algebraic integers in K, and $R = \mathcal{O}[\frac{1}{2}]$. Since $\mathcal{O}$ has a unique prime over 2, $\hat{K}_2$ is a field and the Localization sequence (3.1.4) implies that $L_3(R \to \hat{R}_2)$ and $L_2(R \to \hat{R}_2)$ are trivial. We also get the following commutative diagram with exact rows and columns.

(3.2.2)

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & & \\
 & & I^2(K) \arrow[r, "\psi_2"] \arrow[d] & I^2(\hat{K}_2) \oplus & \bigoplus I(k_\rho) \arrow[d, "\rho_{ND}"] & & \\
0 \arrow[r] & L_1(R \to \hat{R}_2) \arrow[r] & L_0(K) \arrow[r, "\psi"] \arrow[d] & L_0(\hat{K}_2) \oplus & \bigoplus\limits_{\substack{\rho \\ N.D.}} L_0^p(k_\rho) \arrow[r] \arrow[d] & L_0(R \to \hat{R}_2) \arrow[r] & 0 \\
 & & K^*/K^{*2} \arrow[r, "\psi_1"] \arrow[d] & \hat{K}_2^*/\hat{K}_2^{*2} \oplus & \bigoplus\limits_{\rho} \mathbb{Z}/2 \arrow[d, "N.D."] & & \\
 & & 0 & & 0 & &
\end{tikzcd}$$

The snake lemma then yields the following exact sequence

(3.2.3)

$$0 \to \ker\psi_2 \to L_1(R \to \hat{R}_2) \to \ker\psi_1 \xrightarrow{\partial} \operatorname{coker}\psi_2 \to L_0(R \to \hat{R}_2) \to \operatorname{coker}\psi_1 \to 0$$

<u>Computation of ψ_1</u>:

If I = group of $\mathcal{O}$-fractional ideals in K and Γ = ideal class group, then we get the following exact sequence

$$1 \to \mathcal{O}^* \to K^* \to I \to \Gamma \to 1$$

Since Γ has odd order by Weber's Theorem (2.2.4), we get the following short exact sequence

$$1 \to \mathcal{O}^*/\mathcal{O}^{*2} \to K^*/K^{*2} \xrightarrow{i} I/I^2 \to 1$$

Since any $\mathcal{O}$-fractional ideal can be expressed uniquely as a product of prime ideals, we can identify I with the free abelian group generated by the maximal ideals in $\mathcal{O}$. Thus,

$$I/I^2 \cong \bigoplus_{\rho \subset \mathcal{O}} \mathbb{Z}/2\,.$$

Consider the following commutative braid of exact sequences

(3.2.4)

$$\begin{tikzcd}
 & & & \hat{\mathcal{O}}_2^*/\hat{\mathcal{O}}_2^{*2} \arrow[dr] \arrow[rr, bend left] & & \operatorname{coker} \psi_1 \\
 & \mathcal{O}^*/\mathcal{O}^{*2} \arrow[urr, "\tilde{\psi}_1"] \arrow[dr] & & & \hat{K}_2^*/\hat{K}_2^{*2} \oplus \bigoplus\limits_{N.D.} \mathbb{Z}/2 \arrow[ur] \arrow[dd, "v \oplus id"] & \\
\ker \psi_1 \arrow[ur] \arrow[rr, bend right] & & K^*/K^{*2} \arrow[urr, "\psi_1"] \arrow[drr, bend right, "i"'] & & & \\
 & & & & \mathbb{Z}/2 \oplus \bigoplus\limits_{N.D.} \mathbb{Z}/2 &
\end{tikzcd}$$

(3.2.6) <u>Lemma</u>: Assume $L/\hat{\mathbb{Q}}_p$ is a finite extension and p is odd. Then for any element $x \in L^*$, $L(\sqrt{x})/L$ is unramified if and only if $v_\rho(x)$ is even, where $v_\rho : L^* \to \mathbb{Z}$ is the valuation map.

<u>Proof</u>: Recall that the extension $L(\sqrt{x})/L$ is determined by $\bar{x}$, the image of x in L^*/L^{*2}. Since $v_\rho(x)$ is even, $\bar{x} \in A^*/A^{*2}$; where A is the integral closure of $\hat{\mathbb{Z}}_p$ in L.

Since p is odd, $A^*/A^{*2} \cong \ell^*/\ell^{*2} \simeq \mathbb{Z}/2$, where ℓ = residue field. Thus we get that either $\bar{x} = 1$ and $L\sqrt{x}$ is a product of two fields i.e. split or $\ell(\sqrt{x})/\ell$ is quadratic and $L(\sqrt{x})/L$ is inert.

(3.2.7) <u>Corollary</u>: $\mathrm{Kernel}(\tilde{\psi}_1) = \mathrm{Kernel}(\psi_1) = (1)$

<u>Proof</u>: Suppose $x \in \mathcal{O}^*$ represents a nontrivial element $\bar{x}$ in the kernel of $\tilde{\psi}_1$. Since $\tilde{\psi}_1(x) = 1$, $K\sqrt{x}/K$ is split over the unique dyadic prime in K. Since $x \in \mathcal{O}^*$, $v_\rho(x) = 0$ for all prime ideals in $\mathcal{O}$, and (3.2.6) implies that $K\sqrt{x}/K$ is split at all N.D. primes. Global class field theory implies that $\mathrm{Gal}(K\sqrt{x}/K) \simeq \mathbb{Z}/2$ is a quotient group of $\Gamma^*(K)$ the narrow class group. But this is impossible by Weber's Theorem (2.2.4).

Let $[K,\mathbb{Q}] = r_1 + 2r_2$, where r_1 is the number of embeddings of K into $\mathbb{R}$. Then,

$$\mathcal{O}^* = \mu(K) \oplus \mathbb{Z}^{r_1+r_2-1} \qquad \text{(Dirichlet Unit Theorem)}$$

and

$$\hat{\mathcal{O}}_2^* = \mu(\hat{K}_2) \oplus \hat{\mathbb{Z}}_2^{r_1+2r_2} \qquad \text{(see [Se], XIV, §4, Prop. 10)}$$

Thus $\operatorname{coker} \psi_1 \simeq \operatorname{coker} \tilde{\psi}_1 = (\mathbb{Z}/2)^{r_2+1}$.

<u>Computation of ψ_2</u>

Recall the reciprocity sequence (see [C-F]).

$$1 \to \mathrm{Br}(K) \to \bigoplus_{\substack{\text{all}\\ \rho}} \mathrm{Br}(\hat{K}_\rho) \oplus \bigoplus_{\substack{\text{real}\\ v}} \mathrm{Br}(\hat{K}_v) \xrightarrow{R} \mathbb{Q}/\mathbb{Z} \to 1$$

where $\mathcal{R}$ restricted to $Br(\hat{K}_\rho)$ is an isomorphism for any ρ and $\mathcal{R}$ restricted to $Br(\hat{K}_v)$ maps isomorphically onto $2\mathbb{Z}/\mathbb{Z}$. Thus we get the following commutative diagram.

$$\begin{array}{ccccccc}
1 \to & I^2(K)/I^3(K) & \overset{\tilde{\psi}_2}{\to} & I^2(\hat{K}_2) \oplus \oplus I(k_\rho) \oplus \underset{v}{\oplus} Br(\hat{K}_v) & \to & \mathbb{Z}/2 & \to 1 \\
& \uparrow\wr & & \wr\| & & \| & \\
1 \to & {}_2Br(K) & \longrightarrow & \underset{\substack{\text{all}\\ \rho}}{\oplus}\ {}_2Br(\hat{K}_\rho) \oplus \underset{v}{\oplus} Br(\hat{K}_v) & \longrightarrow & \mathbb{Z}/2 & \to 1
\end{array}$$

<u>Case 1</u>: $r_2 = 0$ i.e. $R = R_N$ with $N = r_1$.

Then,

$$\text{coker } \psi_2 = 0, \quad L_0(R \to \hat{R}_2) \simeq \mathbb{Z}/2,$$

and we get the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & I^3(K) & \longrightarrow & \ker \psi_2 & \to & \ker \psi_2/I^3(K) & \to 1 & \\
 & & \downarrow\wr & & \downarrow & & \downarrow & & \\
1 & \to & \underset{v}{\oplus} I^3(\hat{K}_v) & \to & \underset{v}{\oplus} I^2(\hat{K}_v) & \to & \underset{v}{\oplus} I^2(\hat{K}_v)/I^3(\hat{K}_v) & \simeq \underset{v}{\oplus} Br(\hat{K}_v) & \to 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{Z}/2 & = & \mathbb{Z}/2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Thus $\ker \psi_2 \simeq L_1(R \to \hat{R}_2) \simeq \mathbb{Z}^{r_1}$.

<u>Case 2</u>: $r_1 = 0$ i.e. $R = \Gamma_N$ or F_N, with $N = 2r_2$.

Then, $\ker \psi_2 \simeq L_1(R \to \hat{R}_2) \simeq 0$, and we get the following diagram

$$\begin{array}{ccccccccc}
 & & \mathbb{Z}/2 & & & & & & \\
 & & \wr\| & & & & & & \\
1 & \longrightarrow & I^2(\hat{K}_2) & \longrightarrow & L_0(\hat{K}_2) & \longrightarrow & \hat{K}_2^*/\hat{K}_2^{*2} & \to & 1 \\
 & & \downarrow\wr & & \downarrow & & \downarrow & & \\
1 & \to & \operatorname{coker}\psi_2 & \to & L_0(R \to \hat{R}_2) & \to & \operatorname{coker}\psi_1 & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

<u>Case 2(a)</u>: $R = \Gamma_{2r_2}$

Then Theorem 2.29 in [L] implies the top sequence splits. Since $K_2^*/K_2^{*2} \to \operatorname{coker}\psi_1$ splits, we can conclude the bottom sequence also splits. Thus,

$$L_0(\Gamma_{2r_2} \to \hat{\Gamma}_{2r_{2(2)}}) \simeq \mathbb{Z}/2^{r_2+2}.$$

<u>Case 2(b)</u>: $R \simeq F_{2r_2}$.

Then Theorem 2.29 in [L] implies the top sequence does not split. Thus,

$$L_0(F_{2r_2} \to \hat{F}_{2r_{2(2)}}) \simeq \mathbb{Z}/2^{r_2} \oplus \mathbb{Z}/4.$$

(3.3) <u>Type U-Commutative Case</u>: $(\Gamma_N,-)$, (Γ_N,τ), $(\Gamma_N,\bar{\tau})$, (R_N,τ), $(F_N,-)$.

If β is nontrivial, $Br_0(K,\beta) \simeq (1)$. Thus $(K,\beta,1)$ and $(K,\beta,-1)$ are quadratic Morita equivalent and $L_i(K,\beta) \xrightarrow{\sim} L_{i+2}(K,\beta)$ for any i. For any field K, $L_0^p(K,\beta) = W_\beta(K)$ the classical Witt ring of hermitian pairings over K (see [C] and [W], p. 135). Again, let $I_\beta(K) = \text{kernel } r : W_\beta(K) \to \mathbb{Z}/2$, where r is the rank map.

The Rothenberg sequence plus (3.1.5) imply that $L_n(K,\beta) \simeq 0, I_\beta(K)$ for $n = 1,0(2)$.

(3.3.1) Examples

(i) If k is a finite field, then $I_\beta(k) \simeq (1)$.

(ii) If $\hat{K}_\rho$ is a finite extension of $\hat{\mathbb{Q}}_p$, then disc: $I(\hat{K}_\rho) \xrightarrow{\simeq} \hat{F}^*_{\rho_0}/N_{\hat{K}_\rho/\hat{F}_{\rho_0}} \hat{K}^*_\rho$, where $\hat{F}_{\rho_0}$ is the fixed field for β. Local class field theory ([S2]) implies that $\hat{F}^*_{\rho_0}/N_{\hat{K}_\rho/\hat{F}_{\rho_0}} \hat{K}^*_\rho \simeq \mathrm{Gal}(\hat{K}_\rho/\hat{F}_{\rho_0}) \simeq \mathbb{Z}/2$.

If $\hat{K}_\rho/\hat{F}_{\rho_0}$ is unramified, then the Divissage Theorem (3.1.3) implies that

$$L_{2i}(\hat{K}_\rho,\beta) \xleftarrow{\simeq} L_{2i}(\hat{\mathcal{O}}_\rho \to \hat{K}_\rho,\beta) \xleftarrow{\simeq} L^p_{2i}(k_\rho,\beta) \xleftarrow{\simeq} \mathbb{Z}/2,$$

where $\hat{\mathcal{O}}_\rho$ is the integral closure of $\hat{\mathbb{Z}}_p$ in $\hat{K}_\rho$

(iii) The signature map yields an isomorphism

$$\mathrm{sig} : I\_(\mathbb{C}) \to 2\mathbb{Z}$$

(iv) If $K/\mathbb{Q}$ with $[K,\mathbb{Q}] = r_1 + 2r_2$, where r_1 is the number of embeddings of K into $\mathbb{R}$; then

$$\mathrm{disc}: I_\beta(K)/I^2_\beta(K) \to F^*/N_{K/F}K^*,$$

where F is the fixed field for β. If $[F,\mathbb{Q}] = s_1 + s_2$ where s_1 is the number of embeddings of K into $\mathbb{R}$, then

$$\mathrm{sig} : I^2_\beta(K) \xrightarrow{\simeq} \bigoplus_v I^2_\beta(K_v) \xrightarrow{\simeq} (4\mathbb{Z})^{s_1 - \frac{r_1}{2}},$$

where we sum over conjugate pairs of embeddings $v : K \to \mathbb{C}$ such that $v(F) \subset \mathbb{R}$, but $v(K) \not\subset \mathbb{R}$, i.e. the ramified archimedian places for K/F.

Suppose $K \subseteq \mathbb{Q}(\zeta_j)$ for some j, $\mathcal{O}$ is the ring of integers in K, and $R = \mathcal{O}[\frac{1}{2}]$. Then we get the following commutative diagram with exact rows and columns. (Recall that K/F is unramified over N.D. primes.)

(3.3.2)
$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & & \\
 & I^2_\beta(K) \arrow[r, equal] \arrow[d] & I^2_\beta(K) \arrow[d] & & & \\
1 \arrow[r] & L_{2i+1}(R\to\hat{R}_2,\beta) \arrow[r] \arrow[d] & L_{2i}(K,\beta) \arrow[r, "\psi"] \arrow[d] & L_{2i}(\hat{K}_2,\beta) \oplus \bigoplus\limits_{\substack{\beta(\rho)=\rho \\ \text{N.D.}}} L^p_{2i}(k_\rho,\beta) \arrow[r] \arrow[d, "\wr"] & L_{2i}(R\to\hat{R}_2,\beta) \arrow[r] \arrow[d, "\wr"] & 1 \\
1 \arrow[r] & \ker\psi_1 \arrow[r] \arrow[d] & F^*/NK^* \arrow[r, "\psi_1"] \arrow[d] & \hat{F}^*_2/N\hat{K}^*_2 \oplus \bigoplus \hat{F}^*_{\rho_0}/N\hat{K}^*_\rho \arrow[r] & \operatorname{coker}\psi_1 \arrow[r] & 1 \\
 & 1 & 1 & & &
\end{tikzcd}$$

Global Class Field Theory (see [C-F]) yields the following short exact sequence

(3.3.3)
$$1 \to F^*/NK^* \to \bigoplus_{\beta(\rho)=\rho} \hat{F}^*_{\rho_0}/N\hat{K}^*_\rho \oplus \bigoplus_v \hat{F}^*_{v_0}/NK^*_v \xrightarrow{\mathcal{R}} \mathbb{Z}/2 \to 0$$

where v varies over the ramified archimedian places for K/F. Furthermore, $\mathcal{R}$ becomes an isomorphism when restricted to $\hat{F}^*_{\rho_0}/N\hat{K}^*_\rho$ for any ρ (N.D. or dyadic) or $\hat{F}^*_{v_0}/N\hat{K}^*_v$ for any v.

<u>Type UI</u>: $r_1 = 0$ and $s_2 = 0$, i.e. K is totally nonreal and F is totally real. $((\Gamma_{2r_2},-)$ or $(F_{2r_2},-))$

Then ψ_1 is onto, and

$$L_{2i}(R \to \hat{R}_2,\beta) \simeq \operatorname{coker}\psi_1 \simeq (0).$$

We also get the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & I^2_\beta(K) \arrow[r] \arrow[d, "\wr"] & L_{2i+1}(R \to \hat{R}_2,\beta) \arrow[r] \arrow[d] & \ker\psi_1 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \bigoplus\limits_{\substack{\text{arch.}\\ \text{unram.}}} I^2_\beta(\hat{K}_v) \arrow[r] & \bigoplus\limits_{\substack{\text{arch.}\\ \text{unram.}}} L_{2i}(\hat{K}_v,\beta) \arrow[r] \arrow[d] & \bigoplus F^*_{v_0}/NK^*_v \arrow[r] \arrow[d] & 1 \\
 & & \mathbb{Z}/2 \arrow[r, "\sim"'] \arrow[d] & \mathbb{Z}/2 \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}$$

Thus $L_{2i+1}(R \to \hat{R}_2,\beta) \simeq \mathbb{Z}^{r_2}$.

<u>Type UII</u>: Otherwise, $((\Gamma_N,\tau),\ (\Gamma_N,\bar{\tau}),$ or $(R_N,\tau))$. Then $L_{2i+1}(R \to \hat{R}_2,\beta) = 0$ and $L_{2i}(R \to \hat{R}_2,\beta) \simeq \mathbb{Z}/2$.

(3.4) <u>Type O - Noncommutative</u>: $(H_N,\alpha_1,1)$

Let $D = H_N \otimes \mathbb{Q}$. Then D is a quaternionic division ring over $K = \mathbb{Q}(\zeta_N + \bar{\zeta}_N)$. If ρ is a N.D. prime, then $\hat{D}_\rho \simeq M_2(\hat{K}_\rho)$. Furthermore, for any real embedding v of K, $\hat{D}_v$ is a division ring. If $N = 2$, then $\hat{D}_2$ is a division ring; but if $N > 2$, then $\hat{D}_2 = M_2(\hat{K}_2)$.

The (anti) involution α_1 is such that $\alpha_1|K = \mathrm{id}$ and $(D,\alpha_1,1) \in \mathrm{Br}(K,\mathrm{id})$ maps to the trivial element in $\mathrm{Br}(\bar{K},\mathrm{id}) \simeq \{\pm 1\}$.

Then $L_0^p(D,\alpha_1)$ is the classical Witt group of Hermitian pairings over (D,α_1), i.e. what Wall calls Type O_D; and $L_2^p(D,\alpha_1)$ is the classical Witt group of skew-Hermitian pairings over (D,α_1), i.e. what Wall calls type Sp_D. For background see [W4], p. 135 and [K].

Examples:

(i) If $N=2$, then $L_n(\hat{D}_2,\alpha_1) \simeq 0,0,0,\ \hat{\mathbb{Q}}_2^*/\hat{\mathbb{Q}}_2^{*2}$ for $n \equiv 3,2,1,0(4)$.

(ii) For any real embedding v of K,
$L_n(\hat{D}_v,\alpha_1) \simeq 0\ ,2\mathbb{Z},0,0$ for $n \equiv 3,2,1,0(4)$.

(iii) For any i, $L_{2i+1}(D,\alpha_1) = 0$ (apply the Semi-simple Theorem (3.1.5) and the Rothenberg sequence).

We also get $L_2(D,\alpha_1) \xrightarrow{\simeq} \bigoplus_v L_2(\hat{D}_v,\alpha_1) \simeq (2\mathbb{Z})^{2^{N-2}}$.

The discriminate yields an onto map

disc: $L_0(D,\alpha_1) \to K^+/K^{*2}$, where

$K^+ = \{x \in K^* \mid v(x) \in \mathbb{R}^+$ for all real embeddings $v\}$.

Let $I_2(D) = \ker \mathrm{disc}$, and let $I_3(D)$ be the kernel of the onto map

$$I_2(D) \to \oplus\, I^2(\hat{K}_\rho) \simeq \oplus\, \mathbb{Z}/2$$

where we sum over all finite primes ρ (dyadic or N.D.) such $\hat{D}_\rho \simeq M_2(\hat{K}_\rho)$. Then

$$I_3(D) \simeq \begin{matrix} \mathbb{Z}/2^{2^{N-2}-2} & \text{if } N > 2 \\ 0 & \text{if } N = 2. \end{matrix}$$

From (3.1.4) we get the following exact sequence

(3.4.1)

$$\cdots \to L_{i+1}(H_N \to \hat{H}_{N(2)}, \alpha_1) \to L_i(D,\alpha_1) \xrightarrow{\psi} L_i(\hat{D}_2,\alpha_1) \oplus \bigoplus_{N.D.} L_i^p(k_\rho) \to \cdots$$

When $i = 2$ we get,

$$L_3(H_N \to \hat{H}_{N(2)},\alpha_1) \simeq L_2(D,\alpha_1) \simeq (2\mathbb{Z})^{2^{N-2}},$$

and

$$L_2(H_N \to \hat{H}_{N(2)},\alpha_1) \simeq (0).$$

<u>Case 1</u>: $(N = 2)$ Then, when $i = 0$, (3.4.1) yields the following commutative diagram with exact rows and columns

(3.4.2)

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & & \\
 & & I_2(D) \arrow[r, "\sim"'] \arrow[d] & 0 \oplus \bigoplus_{N.D.} I^2(\hat{\mathbb{Q}}_p) \arrow[d] & & \\
1 \arrow[r] & L_1(H_2 \to \hat{H}_{2(2)},\alpha_1) \arrow[r] \arrow[d, "\wr"] & L_0(D,\alpha_1) \arrow[r, "\psi"] \arrow[d, "\wr"] & L_0(\hat{D}_2,\alpha_1) \oplus \bigoplus_{N.D.} L_0^p(\mathbb{F}_p) \arrow[r] \arrow[d] & L_0(H_2 \to \hat{H}_{(2)},\alpha_1) \arrow[r] \arrow[d, "\wr"] & 1 \\
1 \arrow[r] & \ker \overline{\Psi}_1 \arrow[r] & \mathbb{Q}^+/\mathbb{Q}^{*2} \arrow[r, "\overline{\Psi}_1"] \arrow[d] & \hat{\mathbb{Q}}_2^*/\hat{\mathbb{Q}}_2^{*2} \oplus \bigoplus_{N.D.} \mathbb{Z}/2 \arrow[r] & \operatorname{coker} \overline{\Psi}_1 \arrow[r] & 1 \\
 & & 1 & & &
\end{tikzcd}$$

<u>Case 2</u>: $(N > 2)$ Then $\hat{D}_2 \simeq M_2(\hat{K}_2)$, and $L_i(\hat{D}_2,\alpha_1) \simeq L_i(\hat{K}_2)$. When $i = 0$, (3.4.1) yields the following commutative diagram with exact rows and columns

(3.4.3)

$$\begin{tikzcd}[column sep=small]
 & & & 1 \arrow[d] & 1 \arrow[d] & & \\
 & {}_2K^* \arrow[d, phantom, "\wr\|"] & & I_2(D) \arrow[r] \arrow[d] & I^2(\hat{K}_2) \oplus \bigoplus\limits_{N.D.} I^2(\hat{K}_\rho) \arrow[d] & & \\
0 \arrow[r] & L_1(\hat{K}_2) \arrow[r] & L_1(H_N \to \hat{H}_{N(2)}, \alpha_1) \arrow[r] & L_0(D, \alpha_1) \arrow[r] \arrow[d] & L_0(\hat{K}_2, \alpha_1) \oplus \bigoplus\limits_{N.D.} L_0^p(k_\rho) \arrow[r] \arrow[d] & L_0(H_N \to \hat{H}_{N(2)}, \alpha_1) \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\
 & & & K^+/K^{*2} \arrow[r, "\overline{\psi}_1"] \arrow[d] & K_2^*/K_2^{*2} \oplus \bigoplus\limits_{N.D.} \mathbb{Z}/2 \arrow[r] \arrow[d] & \operatorname{coker} \overline{\psi}_1 \arrow[r] & 0 \\
 & & & 1 & 1 & &
\end{tikzcd}$$

Consider the following commutative braid of exact sequences (compare with (3.2.4)).

(3.4.4)

$$\begin{tikzcd}[column sep=small]
 & & \bigoplus\limits_{v} \hat{K}_v^*/\hat{K}_v^{*2} \simeq \mathbb{Z}/2^{2^{N-2}} \arrow[dr] & & \\
 & K^*/K^{*2} \arrow[ur] \arrow[dr, "\psi_1"] & & \operatorname{coker} \overline{\psi}_1 \arrow[dr] & \\
K^+/K^{*2} \arrow[ur] \arrow[rr, bend right, "\overline{\psi}_1"'] & & \hat{K}_2^*/\hat{K}_2^{*2} \oplus \oplus \mathbb{Z}/2 \arrow[ur] \arrow[rr, bend right] & & \mathbb{Z}/2 \quad (r_2 = 0)
\end{tikzcd}$$

Since (3.2.7) implies that ψ_1 is injective, we get that $\overline{\psi}_1$ is also injective. Also, in both Case 1 and Case 2, we get $L_0(H_2 \to \hat{H}_{(2)}, \alpha) \simeq \operatorname{coker} \overline{\psi}_1 \simeq \mathbb{Z}/2^{2^{N-2}+1}$. In Case 1, we get that $L_1(H_2 \to \hat{H}_{2(2)}, \alpha) \simeq 0$. In Case 2, we get the following short exact sequence

$$\begin{tikzcd}[column sep=small]
(3.4.6) \quad 1 \arrow[r] & {}_2K^* \arrow[r] & L_1(H_N \to \hat{H}_{H(2)}, \alpha) \arrow[r] & I_3(K) \arrow[r] \arrow[d, phantom, "\wr\|"] & 1 \quad (N > 2). \\
 & & & \mathbb{Z}/2^{2^{N-2}-2} &
\end{tikzcd}$$

In Part II, (4.5.6) we show that a twisting braid argument implies that this sequence splits.

(3.5) <u>Type U - Noncommutative Case</u> $(H_N,\hat{\alpha})$, $N > 2$

Again, let $D = H_N \otimes Q$ with center K. Since $\hat{\alpha}|K \neq \mathrm{id}$, $Br_0(K,\hat{\alpha}) \simeq (1)$ (see (2.5.9) (i)) and $L_i(D,\hat{\alpha}) = L_{i+2}(D,\hat{\alpha})$. Furthermore, $L^p_{2i}(D,\hat{\alpha})$ is the classical Witt group of Hermitian pairings over $(D,\hat{\alpha})$, i.e. what Wall calls Type U_D. For background see [W4], p. 135.

For any i, $L_{2i+1}(D,\hat{\alpha}) = 0$ (apply the Semi-simple Theorem (3.1.1) and the Rothenberg sequence). The discriminate map yields an isomorphism

$$\mathrm{disc}\colon L_{2i}(D,\hat{\alpha}) \xrightarrow{\sim} F^+/F^+ \cap N_{K/F}K^*,$$

where F = fixed field for $\hat{\alpha}|K$ and

$$F^+ = \{x \in F^* | w(x) > 0 \text{ for all real embeddings } w \text{ of } F\}.$$

(3.5.1) <u>Lemma</u>: $\phi : F^+/F^+ \cap N_{K/F}K^* \to F^*/N_{K/F}K^*$ is an isomorphism.

<u>Proof</u>: Clearly ϕ is injective and the cokernel of ϕ is isomorphic to the cokernel of

$$K^* \xrightarrow{N_{K/F}} F^* \to F^*/F^+.$$

Consider the following commutative diagram

$$\begin{array}{ccc}
K^* & \xrightarrow{N_{K/F}} & F^* \\
\downarrow & & \downarrow \\
K^*/K^+ & & F^*/F^+ \\
\downarrow{s_K} & & \downarrow{s_F} \\
\bigoplus_{v \text{ real}} K_v^*/K_v^+ & \xrightarrow{N} & \bigoplus_{w \text{ real}} F_w^*/F_w^+
\end{array}$$

The Weak Approximation Theorem (see [C-F]) implies that s_K and s_F are isomorphisms. Since K and F are both totally real, N is onto.

We then get the following localization sequence

$$\begin{array}{ccccccccc}
0 \to L_{2i+1}(\Delta \to \hat{\Delta}_2, \hat{\alpha}) & \to & L_{2i}(D) & \xrightarrow{\psi} & L_{2i}(\hat{K}_2, \hat{\alpha}) \oplus \bigoplus_{\substack{\hat{\alpha}(\rho)=\rho \\ \text{N.D.}}} L^p_{2i}(k_\rho, \hat{\alpha}) & \to & L_{2i}(\Delta \to \hat{\Delta}_2, \hat{\alpha}) \to 0 \\
\wr\| & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
0 \longrightarrow \ker \psi_1 & \longrightarrow & F^*/NK^* & \xrightarrow{\psi_1} & \hat{F}_2^*/N\hat{K}_2^* \oplus \bigoplus \hat{F}^*_{\rho_0}/N\hat{K}^*_\rho & \longrightarrow & \operatorname{coker} \psi_1 \longrightarrow 0
\end{array}$$

Since both F and K are totally real, (3.3.3) implies that

$$L_{2i+1}(\Delta \to \hat{\Delta}_2, \hat{\alpha}) = 0, \quad \text{and} \quad L_{2i}(\Delta \to \hat{\Delta}_2, \alpha) \simeq \mathbb{Z}/2.$$

(3.6) <u>Summary</u>: Let $\Delta = \Gamma_N$, F_N, R_N, or H_N, R = center of Δ, K = the quotient field for R, and $\overline{K}$ = the algebraic closure of K. Suppose (α,u) is an anti-structure on Δ. Let F be the quotient field for r, where r is the fixed ring for $\alpha|R$.

<u>Definition</u>: Assume $\alpha|_R = \text{id}$. Then

$$(\alpha,u) \text{ has type } \begin{cases} 0 & \text{if } (\alpha,u) \text{ maps to the trivial element in } Br(\overline{K},\text{id}), \\ S_p & \text{otherwise} \end{cases}$$

Assume $\alpha|_R \neq \mathrm{id}$. Then

$$(\alpha,u) \text{ has type } \begin{cases} \mathrm{UI} & \text{if } K \text{ is \underline{fake} i.e. has no real places and } F \text{ is totally real,} \\ \mathrm{UII} & \text{otherwise} \end{cases}$$

By combining (2.6) with the computations in this chapter we get the following result.

(3.6.1) Theorem: Assume (α,u) is any anti-structure on $\Delta = \Gamma_N, F_N, R_N,$ or H_N.

If $\alpha|_R = \mathrm{id}$, then $L_i(\Delta \to \hat{\Delta}_2,\alpha,u)$ is determined by Δ and the type of (α,u). Furthermore, if (α,u) has type 0 and (α',u') has type S_p, then $L_i(\Delta \to \hat{\Delta}_2,\alpha,u) \cong L_{i+2}(\Delta \to \hat{\Delta}_2,\alpha',u')$.

If $\alpha|_R \neq \mathrm{id}$, then $L_i(\Delta \to \hat{\Delta}_2,\alpha,u)$ is determined by just the type of (α,u). Thus, there exist the following isomorphisms.

UI: $$L_i(\Gamma_N,-,1) \cong L_i(F_N,-,1)$$

UII: $L_i(\Gamma_N,\tau,1) \cong L_i(\Gamma_N,\overline{\tau},1) \cong L_i(R_N,\tau,1) \cong L_i(H_N,\alpha,1)$.

Furthermore, $L_i(\Delta,\alpha,u) \cong L_{i+2}(\Delta,\alpha,u)$.

PART II: Maps Between L-groups

§4. Basic definitions for transfers and twisted quadratic extensions

(4.1) Transfer maps in Algebraic K-Theory

Suppose $f : R \to S$ is any ring homomorphism. Then we get the "push forward"map

$$f_! : K_n(R) \to K_n(S);\ M \to M \otimes_R S.$$

If the map f makes S into a finitely generated, projective right R-module map, then restriction of scalers induces a transfer map

$$f^! : K_n(S) \to K_n(R).$$

If S is a progenerator as a right R-module, then $f^!$ also has the following alternative description. Let

$$T(f) : S \to \mathrm{End}_R(S)$$

be the map given by left-multiplication. Then the Morita equivalence derived from S viewed as a right R-module yields an isomorphism $\phi : K_n R \to K_n \mathrm{End}_R(S)$ such that the following diagram commutes

(4.1.1)
$$\begin{tikzcd}
K_n(S) \arrow[r, "f^!"] \arrow[dr, "T(f)_!"'] & K_n(R) \arrow[d, "\phi"] \\
& K_n(\mathrm{End}_R(S))
\end{tikzcd}$$

If ${}_R S_S$ is isomorphic to ${}_R\mathrm{Hom}(S,R)_S$, then we also get that the following diagram commutes

(4.1.2)
$$\begin{tikzcd}
K_n(R) \arrow[rr, "f_!"] \arrow[dr, "\phi"'] & & K_n(S) \\
& K_n(\mathrm{End}_R(S)) \arrow[ur, "T(f)^!"'] &
\end{tikzcd}$$

(4.1.3) Examples:

(i) If $\Delta : R \to R \times R$ is the diagonal map, then $T(\Delta)$ can be identified with the map $R \times R \to M_2(R)$ which sends (r_1, r_2) to

$$\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

(ii) Suppose $f : K \to D$ is the inclusion map of a maximal subfield in a division ring where $F = \text{center}(D)$ and $m^2 = [D,F]$. Then $T(f)$ can be identified with the map $D \to D\otimes_F K \simeq M_m(K)$.

(4.2) <u>Relative (linear) Morita Theory</u>

Suppose M is a progenerator for P_R. Let $R_1 = \text{End}_R(M)$ and $S_1 = \text{End}_S(M\otimes_R S)$. Then we get the following commutative diagrams

(4.2.1)

$$\begin{array}{ccc} K_n(R) & \xrightarrow{f_!} & K_n(S) \\ \wr\Big\downarrow \phi_M & & \wr\Big\downarrow \phi_{M\otimes_R S} \\ K_n(R_1) & \xrightarrow{f_{1!}} & K_n(S_1) \end{array} \qquad \begin{array}{ccc} K_n(S) & \xrightarrow{f^!} & K_n(R) \\ \wr\Big\downarrow \phi_{M\otimes_R S} & & \wr\Big\downarrow \phi_M \\ K_n(S_1) & \xrightarrow{f_1^!} & K_n(R_1) \end{array}$$

where $f_1 : R_1 \to S_1$ is given by tensoring with 1_S, and the maps ϕ_M and $\phi_{M\otimes_R S}$ come from derived Morita equivalences.

(4.2.2) <u>Examples</u>:

Suppose H is an index 2 subgroup of a finite 2-group G. Then t, the nontrivial element in G/H acts on $\{a_\rho\}$ = the set of primitive central idempotents in H. Furthermore, the map $\mathbb{Q}H \to \mathbb{Q}G$ decomposes as a product of maps

<u>Case 1</u>: $a_\rho\mathbb{Q}H \to a_\rho\mathbb{Q}G$ (for $t(\rho) = \rho$), and

<u>Case 2</u>: $a_\rho\mathbb{Q}H \times a_{t(\rho)}\mathbb{Q}H \to (a_\rho + a_{t(\rho)}) \cdot \mathbb{Q}G$ (for $t(\rho) \neq \rho$)

For any ρ we let V_ρ be the simple module for $a_\rho\mathbb{Q}H$. In Case 1 we let M be V_ρ, and in Case 2 we let M be $V_\rho \times V_{t(\rho)}$. In both cases we get that $f_1 : R_1 \to S_1$ is either one of the following maps or T applied to one of the following maps

(a) $F \subset K$, where F and K are subfields of $\mathbb{Q}(\zeta_N)$ for some N; and K is a quadratic extension of F.

(b) $K \to D_N = \left(\dfrac{-1,-1}{\mathbb{Q}(\zeta_N + \overline{\zeta}_N)}\right)$, where K is either $\mathbb{Q}(\zeta_N)$ or $\mathbb{Q}(\zeta_{N+1} - \overline{\zeta}_{N+1})$.

(c) $D_N \to D_{N+1}$

or

(d) $\Delta : A \to A \times A$, where A is either a subfield of $\mathbb{Q}(\zeta_N)$ or $\left(\dfrac{-1,-1}{\mathbb{Q}(\zeta_N + \overline{\zeta}_N)}\right)$ for some N.

(Compare with Example 1 in the Introduction.)

Thus the problem of computing

$$f_! : K_n(\mathbb{Q}H) \to K_n(\mathbb{Q}G) \quad \text{and} \quad f^! : K_n(\mathbb{Q}G) \to K_n(\mathbb{Q}H)$$

can be reduced to the problem of computing the push forward and transfer maps associated to the maps in (a), (b), (c), and (d).

(4.3) <u>Transfer maps in L-Theory</u>

Suppose $f : (R,\alpha_0,u) \to (S,\alpha,u)$ is a map of rings with anti-structure. Then we get the "push forward" map

$$f_! : L_n(R,\alpha_0,u) \to L_n(S,\alpha,u)$$

(4.3.1) <u>Definition</u>: <u>A trace for f</u> is a map $X : S \to R$ such that

(i) X is a right R-linear map where we use f to make S a right R-module.

(ii) $X(\alpha(s)) = \alpha_0 X(s)$ for all $s \in S$.

(iii) if $\lambda^X : S \times S \to R$ sends (s_1, s_2) to $X(\alpha(s_1)s_2)$, then $\mathrm{ad}(\lambda^X) : S \to \mathrm{Hom}_R(S,R)$ is onto.

and

(iv) S is a finitely-generated projective right R-module.

Notice that a choice of trace X for f (assuming one exists) determines a functor

$\mathrm{Sesq}(S,\alpha,u) \to \mathrm{Sesq}(R,\alpha_0,u)$; $(b : N \times N \to S) \to (X \cdot b : N \times N \to R)$,

and a transfer map

$$f^X : L_n(S,\alpha,u) \to L_n(R,\alpha_0,u)$$

(4.3.2) <u>Example</u>: Suppose $f : (\mathbb{Z}H, \alpha_\omega, 1) \to (\mathbb{Z}G, \alpha_\omega, 1)$ is induced by an inclusion of groups $H \subset G$. The $\mathbb{Z}$-linear map $X : \mathbb{Z}G \to \mathbb{Z}H$ such that

$$X(g) = \begin{matrix} g & \text{if } g \in H \\ 0 & \text{if } g \in G - H \end{matrix}$$

is a trace. Furthermore, the induced transfer map is the same as the geometric transfer defined using covering spaces.

Consider the map $T(f) : S \to \mathrm{End}_R(S)$. By the Derived Quadratic Morita Equivalence Theorem (2.5.2) we get that $\mathrm{ad}(\lambda^X)$ determines an anti-structure (β, v) on $\mathrm{End}_R(S)$, such that (R,α_0,u) and $(\mathrm{End}_R(S),\beta,v)$ are quadratic Morita equivalent.

(4.3.3) Proposition: We get a map of rings with anti-structure

$$T(f) : (S,\alpha,u) \to (\mathrm{End}_R(S),\beta,v),$$

and the following diagram commutes

$$\begin{tikzcd} L_n(S,\alpha,u) \arrow[r, "f^X"] \arrow[dr, "T(f)_!"'] & L_n(R,\alpha_0,u) \arrow[d, "\wr"] \\ & L_n(\mathrm{End}_R(S),\beta,v) \end{tikzcd}$$

(4.4) Twisted quadratic extensions

Recall that in the Introduction we considered the notion of a twisted quadratic extension.

$$f: R \to R_\rho[\sqrt{a}] = S, \quad \text{with Galois automorphism } \gamma.$$

Notice that the examples in (4.2.2) can all be viewed as twisted quadratic extensions. We are particularly interested in the following examples where we pass to $\mathbb{Z}[\frac{1}{2}]$-maximal orders.

(4.4.1) List: (see (1.3.5) for notation)

$f : R \to R_\rho[\sqrt{a}]$	ρ	$t = \sqrt{a}$
$R_{N-1} \to R_N$	Id	$\zeta_{N+2} + \bar{\zeta}_{N+2}$
$\Gamma_{N-1} \to \Gamma_N$	Id	ζ_{N+1}
$F_{N-1} \to \Gamma_N$	Id	i
$R_{N-1} \to \Gamma_N$	Id	i
$R_{N-1} \to F_N$	Id	$\zeta_{N+2} - \bar{\zeta}_{N+2}$
$f_{\pm} : \Gamma_{N-1} \to H_N$, where	-	j
$f_+(i) = i$ and $f_-(i) = k$		
$f : F_{N-1} \to H_N$, where	-	j
$f(\zeta_{N+1} - \bar{\zeta}_{N+1}) = k(1 - \zeta_N)$		
$H_{N-1} \to H_N$	Id	$\zeta_N + \bar{\zeta}_N$
$d : \Delta \to \Delta \times \Delta$, diagonal	Id	$(1,-1)$
map, where $\Delta = \Gamma_N, R_N, F_N$, or H_N		

(4.4.2) Proposition: Assume $\frac{1}{2} \in R$, a is a unit in R, and

$$f : R \to R_\rho[\sqrt{a}] = S$$

is a twisted quadratic extension with Galois automorphism γ. Then

(i) $$T(f) : S \to \mathrm{End}_R(S)$$

is also a twisted quadratic extension. More precisely, there exists a ring isomorphism $G : S_\gamma[\sqrt{1}] \to \mathrm{End}_R(S)$ such that the following diagram commutes

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "T(f)"'] & S_\gamma[\sqrt{1}] \arrow[d, "G"] \\ & \mathrm{End}_R(S). \end{tikzcd}$$

For any $s_1 + s_2\sqrt{1} \in S_\gamma(\sqrt{1})$, $G(s_1 + s_2\sqrt{1})$ is the endomorphism of S (as a right R-module) which sends $z \in S$ to $s_1 z + s_2\gamma(z)$, and

(ii) ${}_RS_S$ is isomorphic to ${}_R\mathrm{Hom}(S,R)_S$.

(4.4.3) <u>Theorem</u>: If H is an index 2 subgroup of a finite 2-group, then $\mathbb{Z}[\frac{1}{2}]H \to \mathbb{Z}[\frac{1}{2}]G$ can be expressed as a product of maps such that each component map is either in List (4.4.1) or it is T of a map in List (4.4.1) (up to Morita equivalence).

Thus the problem of computing the K-theory push forward and transfer maps for $\mathbb{Z}[\frac{1}{2}]H \to \mathbb{Z}[\frac{1}{2}]G$ is reduced to the analogous problem for the maps in (4.4.1).

(4.5) <u>L-Theory for twisted quadratic extensions</u>

Suppose we have a map of rings with anti-structure

$$f : (R,\alpha_0,u) \to (S,\alpha,u)$$

where $f : R \to R_\rho[\sqrt{a}] = S$ is a twisted quadratic extension with Galois automorphism γ.

Then a trace for f is given by

$$X : R_\rho[\sqrt{a}] \to R;\ X(x + yt) = x, \quad \text{for all } x,y \in R.$$

Since our X is fixed, we also denote f^X by $f^!$.

As in the Introduction we get a twisting diagram for $f : (R,\alpha_0,u) \to (S,\alpha,u)$. Notice that the twisting diagram for $\tilde{f} : (R,\tilde{\alpha}_0,\tilde{u}) \to (S,\tilde{\alpha},\tilde{u})$ is the same as the twisting diagram for f (up to reindexing).

If $\frac{1}{2} \in R$ and a is a unit, then (4.3.3) and (4.4.2) imply that

$$T(f) : (S,\alpha,u) \to (\mathrm{End}_R(S),\beta,v)$$

is a map of rings with anti-structure where $\mathrm{End}_R(S) \simeq S_\gamma[\sqrt{1}]$ and $T(f)$ is a twisted quadratic extension.

(4.5.1) Proposition: The twisting diagram for $T(f)$ is isomorphic to the twisting diagram for ${}^\gamma f$ (up to reindexing).

Suppose we equip one of the twisted quadratic extensions in (4.4.1) with anti-structure $f: (\alpha_0,u) \to (\alpha,u)$. Then as in (3.6.1) one can show that the twisting diagram for f is determined by the rings, type (α_0,u), and type (α,u).

(4.5.2) List: Twisted quadratic extensions with anti-structure

	$f: R \to R_\rho[\sqrt{a}]$	Type (f)	Type $(\tilde{f})$	Type $({}^{\gamma}f)$	Type $(\widetilde{{}^{\gamma}f})$
(1)	$\Gamma_{N-1} \to \Gamma_N$	$0 \to 0$	$S_p \to UII$	$0 \to UII$	$0 \to 0$
(2)		$UI \to UI$	$UI \to UII$	$UI \to UII$	$UI \to UI$
(3)	$R_{N-1} \to R_N$	$0 \to 0$	$S_p \to UII$	$0 \to UII$	$0 \to 0$
(4)	$R_{N-1} \to \Gamma_N$	$0 \to 0$	$S_p \to UI$	$0 \to UI$	$0 \to 0$
(5)		$UII \to UII$	$UII \to UII$	$UII \to UII$	$UII \to UII$
(6)	$R_{N-1} \to F_N$	$0 \to 0$	$S_p \to UI$	$0 \to UI$	$0 \to 0$
(7)	$F_{N-1} \to \Gamma_N$	$0 \to 0$	$S_p \to UII$	$0 \to UII$	$0 \to 0$
(8)		$UI \to UI$	$UI \to UII$	$UI \to UII$	$UI \to UI$
(9)	$\Gamma_{N-1} \to H_N$	$0 \to 0$	$UI \to 0$	$0 \to 0$	$UI \to 0$
(10)		$UII \to UII$	$UII \to UII$	$UII \to UII$	$UII \to UII$
(11)	$F_{N-1} \to H_N$	$0 \to 0$	$UI \to 0$	$0 \to 0$	$UI \to 0$
(12)	$H_{N-1} \to H_N$	$0 \to 0$	$S_p \to UII$	$0 \to UII$	$0 \to 0$
(13)	$\Delta \xrightarrow{d} \Delta \times \Delta$	$(\alpha_0,1) \to (\alpha_0,1) \times (\alpha_0,1)$	$(\alpha_0,-1) \to GL$	$(\alpha_0,1) \to GL$	$(\alpha_0,-1) \to (\alpha_0,-1) \times (\alpha_0,-1)$

In fact we get isomorphisms of twisting diagrams between Cases (2) and (8), and also between Cases (5) and T of (10).

(4.5.3) Theorem: Suppose we have a map of rings with anti-structure $f : (\mathbb{Z}[\frac{1}{2}]H, \alpha_0, u) \to (\mathbb{Z}[\frac{1}{2}]G, \alpha, u)$ where G is a finite 2-group and H is an index 2 subgroup. Then the L^p-twist diagram for f decomposes into a direct sum of diagrams such that each component diagram is isomorphic (up to reindexing) to the L^p-twist diagram for one of the twisted quadratic extensions with anti-structure in List (4.5.2).

(4.5.4) Definition: For any ring with anti-structure (S,α,u) we let

$$(S,\alpha,u)_n = L_n^{0\subset K_0}(S \to \hat{S}_2,\alpha,u)$$

If $f : (R,\alpha_0,u) \to (R_\rho[\sqrt{a}],\alpha,u)$ is a twisted quadratic extension of rings with anti-structure; then we get a "push forward" exact sequence

$$\cdots \to (R,\alpha_0,u)_n \xrightarrow{f_!} (S,\alpha,u)_n \to (f_!)_n \to \cdots,$$

a transfer exact sequence

$$\cdots \to (S,\alpha,u)_n \xrightarrow{f^!} (R,\alpha_0,u)_n \to (f^!)_n \to \cdots,$$

and a

(4.5.5) Relative Twist Diagram

$$\begin{tikzcd}
(S,\gamma\alpha,u)_3 \arrow[d, phantom, sloped, "\cong"] \\
(S,\widetilde{\alpha,u})_3 \arrow[dr] \arrow[rr, bend left] & & (R,\alpha,u)_3 \arrow[dr] \arrow[rr, bend left] & & (S,\alpha,u)_3 \arrow[dr] \arrow[rr, bend left] & & (R,\widetilde{\alpha,u})_1 \arrow[dr] \arrow[rr, bend left] & & {} \\
 & (f_!)_0 \arrow[ur] \arrow[dr] & & ({}^\gamma f^!)_3 \arrow[ur] \arrow[dr] & & (f_!)_3 \arrow[ur] \arrow[dr] & & ({}^\gamma f^!)_2 \arrow[ur] \arrow[dr] & \\
(S,\alpha,u)_0 \arrow[ur] \arrow[rr, bend right] & & (R,\widetilde{\alpha,u})_2 \arrow[ur] \arrow[rr, bend right] & & (S,\widetilde{\alpha,u})_2 \arrow[ur] \arrow[rr, bend right] \arrow[d, equal] & & (R,\alpha,u)_2 \arrow[ur] \arrow[rr, bend right] & & {} \\
 & & & & (S,\gamma\alpha,u)_2
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[dr] \arrow[rr, bend left] & & (S,\widetilde{\alpha,u})_1 \arrow[dr] \arrow[rr, bend left] & & (R,\alpha,u)_1 \arrow[dr] \arrow[rr, bend left] & & (S,\alpha,u)_1 \arrow[dr] \arrow[rr, bend left] & & (R,\widetilde{\alpha,u})_3 \arrow[dr] & \\
 & ({}^\gamma f^!)_2 \arrow[ur] \arrow[dr] & & (f_!)_2 \arrow[ur] \arrow[dr] & & ({}^\gamma f^!)_1 \arrow[ur] \arrow[dr] & & (f_!)_1 \arrow[ur] \arrow[dr] & & ({}^\gamma f^!)_0 \\
{} \arrow[ur] \arrow[rr, bend right] & & (S,\alpha,u)_2 \arrow[ur] \arrow[rr, bend right] & & (R,\widetilde{\alpha,u})_0 \arrow[ur] \arrow[rr, bend right] & & (S,\widetilde{\alpha,u})_0 \arrow[ur] \arrow[rr, bend right] \arrow[d, equal] & & (R,\alpha,u)_0 \arrow[ur] & \\
 & & & & & & (S,\gamma\alpha,u)_0
\end{tikzcd}$$

Furthermore, we get a relative version of (4.5.3).

At the end of the paper there are tables giving the relative push forward and transfer exact sequences for all cases in (4.5.2) except cases (10) and (13). The twist diagram for (13) is easy: the one for (10) is T of the one for (5). In particular, the push forward map for $\Gamma_{N-1} \to H_N$, type UII $\to$ type UII is read off Table 3 not Table 2 !

Each relative twist diagram from (4.5.2) is determined by the groups along the top and bottom rows of the diagram except in cases (5) and (10). These are determined by using

$f_!: (R_{N-1},\tau)_0 \to (\Gamma_N,\tau)_0$ is trivial, and

$\gamma_f^!: (H_N,\hat{\alpha})_0 \to (\Gamma_{N-1},\tau)_0$ is trivial.

Both these facts can be derived from the other diagrams.

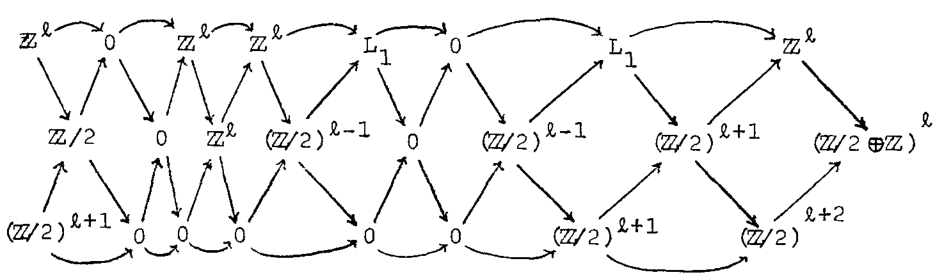
Recall from (3.4.6) the short exact sequence

$$1 \to {}_2K^* \to L_1(H_N \to \hat{H}_{N(2)},\alpha) \to \mathbb{Z}/2^{2^{N-2}-2} \to 1.$$

We write out the twisting diagram below to show that this sequence splits.

(4.5.6) $(f_+ \colon \Gamma_{N-1} \to H_N)$, Type $0 \to 0$, $L_1 = L_1(H_N \to \hat{H}_{N(2)},\alpha)$

$\ell = 2^{N-2}$



APPENDIX I: Computing $L^p_r(\mathbb{Z}G,\alpha,u)$

To compute $L^p_r(\mathbb{Z}G,\alpha,u)$ we shall use the sequence

$$\ldots \to L^p_r(\mathbb{Z}G,\alpha,u) \to L^p_r(\hat{\mathbb{Z}}_2G,\alpha,u) \xrightarrow{\Psi_r} L^p_r(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u) \quad \ldots$$

Since $L^p_r(\hat{\mathbb{Z}}_2G,\alpha,u) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & r \text{ even} \\ 0 & r \text{ odd} \end{cases}$ (1.2.1), most of the work

comes in computing $L^p_r(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u)$.

All of the antistructures encountered in surgery theoretic applications have the following description. We are given a homomorphism $\omega: G \to \pm 1$; an automorphism $\theta: G \to G$; and an element $b \varepsilon G$. We require $\omega \circ \theta = \omega$; $\theta\circ\theta(g) = bgb^{-1}$ for all $g \varepsilon G$; $\omega(b) = 1$; and $\theta(b) = b$. We define two associated antistructures (α,u) by

$$\alpha(g) = \omega(g)\theta(g^{-1}) \quad \text{for all } g \varepsilon G: \quad u = \pm b.$$

We call such an antistructure a geometric antistructure.

Given any anti-automorphism $\alpha: \mathbb{Z}G \to \mathbb{Z}G$ which takes G to $\pm G$, there are θ and ω so that $\alpha(g) = \omega(g)\theta(g^{-1})$ for all $g \in G$. No integral group ring is known to have units of finite order other than $\pm G$, so it is conceivable that all anti-automorphisms have the above form. One can produce units which are not of the form $\pm b$ (scale by some strange unit in the group ring).

Any geometric antistructure can arise in the codimension 1 surgery diagram. The small group, H, is our G and the G is

$$\pi = G * \mathbb{Z} \,/\, tgt^{-1} = \theta(g);\ t^2 = b$$

where t generates $\mathbb{Z}$. There are two extensions of ω to π and the correct choice yields α for $\tilde{\alpha}_\omega$ and u for $\tilde{1}$.

In Part 1 we compute $L^p_r(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u)$ for any geometric antistructure. In Part 2 we compute Ψ_{2r} and settle the extension questions which arise.

Part 1: Compute $L^p_r(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u)$

The goal of this section is to explain how to use Table 1 to compute $L^p_r(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u)$ for any geometric antistructure given the characters of the irreducible rational representations. Henceforth, χ denotes such a character.

The subtables of Table 1 are labeled by a type, U or O: we assign a type (GL, U, O, or S_p) to each χ. The columns of these subtables are labeled by a symbol Γ_N, F_N, R_N, or H_N or by a symbol UI_N or UII.

In steps 1 and 2 below we show how to determine Type χ. In steps 2 and 3 we show how to assign a symbol $E\chi = \Gamma_N$, F_N, R_N, or H_N or a symbol $U\chi = UI_N$ or UII.

Step 1: Initial crucial remarks.

The type of χ really depends on χ and (α,u) but as the antistructure is fixed during one of these calculations we suppress it.

We first determine if χ has type GL or not:

Type χ is GL iff $\chi(g) \neq \omega(g)\chi(\theta(g^{-1}))$ for some $g\in G$.

Define a character χ^α by $\chi^\alpha(g) = \omega(g)\chi(\theta(g^{-1}))$ for all $g\in G$.

If χ has type GL, it makes no contribution to any L theory. If χ does not have type GL, we let $L_r(\chi)$ denote the contribution of χ to $L^p_r(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u)$. In the remaining steps we assume that the type of χ is not GL.

Step 2: Type and initial symbol calculations.

Compute the two numbers

$$T_\chi = \frac{1}{|G|} \sum_{g\in G}\omega(g)\chi(g\theta(g)u) \quad ; \quad S_\chi = \frac{1}{|G|} \sum_{g\in G} \chi(g^2)$$

From T_χ and S_χ we find the type of χ; partial information about $E\chi$; and define a number m_χ for later use. Explicitly

T_χ	Type
positive	0
zero	U
negative	S_p

S_χ	$E\chi$	m_χ
positive	R_N	1
zero	Γ_N or F_N	1
negative	H_N	2

where N is defined as follows: $2^N = m_\chi \delta_\chi$ and we can always find δ_χ from

$$\delta_\chi = \frac{1}{|G|} \sum_{g \in G} (\chi(g))^2$$

<u>However</u>, if $T_\chi \neq 0$, $\delta_\chi = |T_\chi|$; if $S_\chi \neq 0$, $\delta_\chi = |S_\chi|$

If $T_\chi \neq 0$ and $S_\chi \neq 0$, go directly to step 4.

If $T_\chi = 0$ and $S_\chi \neq 0$, we have $U\chi = UII$: go to step 4.

<u>Step 3</u>: Unresolved issues and a pairing.

If $S_\chi = 0$, we must determine a symbol. If the type of χ is U, we use AI.1.1 below to decide if $U\chi = UI_N$ or UII: if the type of χ is 0 or S_p, we use AI.1.2 below to decide if $E\chi = \Gamma_N$ or F_N.

We will determine these symbols by using a pairing

$$\Lambda : \overline{QG} \times \overline{QG} \to \mathbb{Q}$$

where $\overline{QG}$ is the rational vector space based on the conjugacy classes of G, and

$$\Lambda(C_1, C_2) = \sum_{\substack{g \in C_1 \\ h \in C_2}} \chi(gh)$$

We shall need some related pairings which we proceed to define.

For each N there is an operation, λ_N, on $\overline{QG}$ which sends a conjugacy class, C, to $C^{-5^{2^{N-1}}}$. Define $T_N(C_1, C_2) = \Lambda(C_1, \lambda_N(C_2))$.

There is an operation, α, on $\overline{QG}$ which sends a conjugacy class, C, to $\omega(C)\theta(C^{-1}) \in \overline{QG}$. Define $A(C_1, C_2) = \Lambda(C_1, \alpha(C_2))$.

These pairings are used in the following results.

(AI.1.1) Assume that $S_\chi = T_\chi = 0$ and let $2^N = m_\chi \delta_\chi$. Then $U\chi = UI_N$ or UII:

$U\chi = UI_N$ iff $T_1(C,C) = A(C,C)$ for every conjugacy class C of G.

(AI.1.2) Assume $S_\chi = 0$ and let $2^N = m_\chi \delta_\chi$. Then $E\chi = \Gamma_N$ or F_N:

$E\chi = \Gamma_N$ iff $T_1(C,C) = T_N(C,C)$ for every conjugacy class C of G.

Remark: Of course the symbol $E\chi$ is just the name for a $\mathbb{Z}[\frac{1}{2}]$-maximal order in the division algebra associated to χ (see section 2.2) and hence $E\chi$ is independent of the antistructure. We could use S_χ, N, and AI.1.2 to find $E\chi$ for any χ we wanted. Working through the steps as outlined above only computes $E\chi$ if it is needed to read Table 1.

Step 4: Find the contribution of χ to $L^p_r(\mathbb{Z}G \to \hat{\mathbb{Z}}_2 G,\alpha,u)$

If χ has type U, we use subtable U: $L_r(\chi)$ is found on column $U\chi$ on the row "odd" if r is odd or on the row "even" if r is even.

If χ has type O or S_p, we use subtable O: $L_r(\chi)$ is found in column $E\chi$ on the row $\hbar = 3, 2, 1$, or 0: $\hbar \equiv r \pmod 4$ if Type χ is O; $\hbar \equiv r+2 \pmod 4$ if Type χ is S_p.

Part 2: Compute $L^p_r(\mathbb{Z}G, \alpha,u)$

We have reduced this problem to understanding a pair of exact sequences

$$0 \to L^p_{2r+1}(\mathbb{Z}G \to \hat{\mathbb{Z}}_2 G,\alpha,u) \xrightarrow{\partial_{2r+1}} L^p_{2r}(\mathbb{Z}G,\alpha,u) \xrightarrow{\kappa_{2r}} L^p_{2r}(\hat{\mathbb{Z}}_2 G,\alpha,u) \xrightarrow{\Psi_{2r}}$$

$$L^p_{2r}(\mathbb{Z}G \to \hat{\mathbb{Z}}_2 G,\alpha,u) \xrightarrow{\partial_{2r}} L^p_{2r-1}(\mathbb{Z}G,\alpha,u) \to 0$$

(for r = 0, 1). Some terminology will be useful.

A representation (or its character χ) is called cyclic if it can be obtained by pulling back the faithful irreducible rational representation of C_N along some epimorphism $\gamma: G \to C_N$, $N \geq 0$. A representation (or its character) is called dihedral if it can be obtained by pulling back the faithful irreducible rational representation of D_N along some epimorphism $\gamma: G \to D_N$, $N \geq 3$.

The epimorphism γ determines χ but not vice versa. The kernel of χ determines the kernel of γ but χ only determines γ up to an automorphism of the quoient.

Next we determine Ψ_{2r}. Since $L^p_{2r}(\hat{\mathbb{Z}}_2 G,\alpha,u) \simeq \mathbb{Z}/2\mathbb{Z}$, Ψ_{2r} is either trivial or one to one.

<u>Theorem AI.2.1</u>: Ψ_{2r} is determined by:

Ψ_0 is one to one iff there is a type 0 cyclic representation;

Ψ_2 is one to one iff there is a type S_p cyclic representation.

It is easy to describe the right-hand extension.

<u>Theorem AI.2.2</u>: ∂_{2r} is a split epimorphism.

The left-hand extension is more difficult to describe, since even if κ_{2r} is onto, two different things can happen.

Associated to a dihedral representation $\gamma: G \to D_N$ there are two other maps; $\gamma_1: G \to \pm 1 = D_N/C_{N-1}$ and $\gamma_2: \ker \gamma_1 \to C_{N-1} \twoheadrightarrow \pm 1$. A dihedral representation is <u>twisted</u> (with respect to the geometric antistructure θ,ω,b) iff γ_1 is θ invariant and $\gamma_2(y^{-1}\theta(y)) = -1$ for any (and hence every) $y \in G\text{-}\ker \gamma_1$.

A cyclic representation is <u>twisted</u> iff the composite $G \xrightarrow{\gamma} C_N \twoheadrightarrow \pm 1$ sends b to -1.

We have

<u>Theorem AI.2.3</u>: If κ_0 is onto, then it is split unless there is a type UI twisted cyclic or a type 0 twisted dihedral representation. Then κ_0 is not split.

If κ_2 is onto, then it is split unless there is a type UI twisted cyclic or a type S_p twisted dihedral representation. Then κ_2 is not split.

If κ_{2r} is not split, then any $x \in L^p_{2r}(\mathbb{Z} G,\alpha,u)$ with $\kappa_{2r}(x) = -1$ has infinite order.

To apply the above results it is desirable to be able to find cyclic and dihedral representations. A return to the theory behind the results in Part I yields the critera below.

A character χ is cyclic iff $E\chi = \Gamma_N$ or R_0 and $\chi(e) = 2^N$ or 1. A character χ is dihedral iff $E\chi = R_N$ and $\chi(e) = 2^{N+2}$. A cyclic character χ is twisted iff $(b^{2^N}) = -\chi(e)$; a dihedral character χ is twisted iff $\chi((g^{-1}\theta(g))^{2^{N+1}}) = -\chi(e)$ for at least one $g \in G$.

Another way to give such representations is to give the epimorphism γ directly. In this case there is a quicker way to find the type than by using step 2 of Part 1.

In the cyclic case, extend $\gamma\colon G \to C_N$ to $\hat{\gamma}\colon \pm G \to C_N$ by defining $\hat{\gamma}(-g) = -\gamma(g)$. Recall that α induces a map from G to $\pm G$.

χ has type UI iff $\hat{\gamma}(g^{-1}) = \hat{\gamma}(\alpha(g))$ for all $g \in G$

χ has type 0 or S_p iff $\hat{\gamma}(g) = \hat{\gamma}(\alpha(g))$ for all $g \in G$

the type is 0 if $\hat{\gamma}(u) = 1$; the type is S_p if $\hat{\gamma}(u) = -1$.

Any twisted dihedral representation has type 0 or S_p. For any $g \in G$, define $\tau_g = \omega(g)\chi(g\theta(g)u)$: τ_g is either 0, $\chi(e)$, or $-\chi(e)$. We have type 0 if there exists a $g \in G$ with $\tau_g = \chi(e)$: we have type S_p if there exists a $g \in G$ with $\tau_g = -\chi(e)$.

Any cyclic character with $\chi(e) = 1$ is called <u>linear</u>: any cyclic character with $\chi(e) = 2$ is called <u>quadratic</u>. A linear cyclic character is either the trivial character or a cyclic character with $\gamma\colon G \to C_1$. The quadratic characters are the cyclic characters with $\gamma\colon G \to C_2$. Notice that the linear characters are in one to one correspondence with $H^1(G; \mathbb{Z}/2\mathbb{Z})$.

<u>Examples</u>:

1) $(\mathbb{Z}G, \alpha, e)$ $\alpha(g) = g^{-1}$: any non-linear cyclic representation is type U; all the linear ones are type 0. Therefore Ψ_2 is trivial: Ψ_0 is one to one. There are no twisted cyclic or twisted dihedral representations, so κ_2 is split.

2) $(\mathbb{Z}G, \alpha_\omega, e)$ $\alpha(g) = \omega(g)g^{-1}$; ω non-trivial: any non-quadratic cyclic

representation has type GL or U. A quadratic representation $\gamma\colon G \to C_2$ has either type GL or O. The type is O iff the composite $G \xrightarrow{\gamma} \mathbb{Z}/4\mathbb{Z} \to \pm 1$ is ω. If we consider $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$, there are type O quadratic representations iff $\omega^2 = 0 \in H^2(G; \mathbb{Z}/2\mathbb{Z})$. Therefore Ψ_2 is always trivial: Ψ_0 is trivial iff $\omega^2 \neq 0$. There are no twisted cyclic or twisted dihedral representations, so κ_{2r} is split whenever it is onto.

3) $(\mathbb{Z}C_N, \alpha, x)$ $\alpha(g) = g^{-1}$; $x \in C_N$ a generator: the non-linear representations have type U. The trivial representation has type O; the other linear representation has type S_p. Therefore both Ψ_0 and Ψ_2 are one to one.

4) $(\mathbb{Z}G, \alpha, u)$: $G = C_N \times \mathbb{Z}/2\mathbb{Z}$ generated by $x \in C_N$ and $t \in \mathbb{Z}/2\mathbb{Z}$; $\omega(x) = 1 = -\omega(t)$; $\theta(x) = x$; $\theta(t) = tx^{2^{N-1}}$; $u = x$. There is a type UI twisted cyclic representation and no type O or S_p cyclic representations. Hence both κ_0 and κ_2 are onto but neither is split.

APPENDIX II: Computing push forward maps and transfers

We wish to describe how to compute the push forward and transfer maps associated to an index 2 inclusion of groups, say $H \subset G$. We will assume that we have a map of rings with antistructure and that the antistructures are geometric, but we begin by describing the "simple pieces" of the map $\mathbb{Q}H \to \mathbb{Q}G$.

To do this requires some notation. If χ_0 is the character of an irreducible rational representation of H, define

$$\chi_0^t(h) = \chi_0(tht^{-1}) \qquad \text{for all } h \in H;\ t \in G-H \text{ is a fixed element.}$$

If χ is the character of an irreducible rational representation of G, define

$$\chi^\psi(g) = \psi(g)\chi(g) \qquad \text{for all } g \in G;\ \text{where } \psi\colon G \to \pm 1 \text{ has kernel } H.$$

Recall (1.3) that $\mathbb{Q}H$ is a product of simple rings indexed by the characters, χ_0, of the irreducible rational representations of H: $\mathbb{Q}G$ has a similar description. The map $\mathbb{Q}H \to \mathbb{Q}G$ is a product of the following three sorts of maps. In the three descriptions below, χ_0 is a constituent of χ restricted to H:

Case I: $\chi_0^t = \chi_0$; $\chi \neq \chi^\psi$: $A_{\chi_0} \to A_\chi \times A_{\chi^\psi}$

Case II: $\chi_0^t \neq \chi_0$; $\chi = \chi^\psi$: $A_{\chi_0} \times A_{\chi_0^t} \to A_\chi$

Case III: $\chi_0^t = \chi_0$; $\chi = \chi^\psi$: $A_{\chi_0} \to A_\chi$

When we add the antistructures to the picture, we need to refine this decomposition further into types. We proceed to describe the various cases which occur. Recall $\chi^\alpha(g) = \omega(g)\chi\,(\theta(g^{-1}))$ (Appendix I, step 1).

The easiest to describe is the GL type. Here, two pieces of the same sort (I, II, III) are interchanged by the antistructure. A GL type makes no contribution to the L theory and so can be ignored.

In case I there are two types in addition to the GL type discussed above. These further types are denoted IX_0GL and $IX_0\Delta$. In IX_0GL, $\chi_0^\alpha = \chi_0$ and $\chi^\alpha = \chi^\psi$: in $IX_0\Delta$, $\chi_0^\alpha = \chi_0$ and $\chi^\alpha = \chi$.

There are similar types in case II: denoted IIGLX and IIAX. We have type IIGLX if $\chi_0^\alpha = \chi_0^t$ and $\chi^\alpha = \chi$: we have type IIAX if $\chi_0^\alpha = \chi_0$ and $\chi^\alpha = \chi$.

In case III the type is either GL or $\chi_0^\alpha = \chi_0$ and $\chi^\alpha = \chi$. This time we divide into type III2 and III3. To describe these two types compute $d = 2^N$ and $m = m_\chi$ for χ. (This was probably done in computing the L group, but, if not, step 2 in Appendix I will do it.) Compute the corresponding numbers d_0 and m_0 for χ_0. Finally, decide if χ and χ_0 both have type UII or not.

Assume either that $m_0 = m$ or that not both χ and χ_0 have type UII:

we have type III2 iff $2d_0 = d$
we have type III3 iff $d_0 = 2d$

Assume that $m_0 = m$ and that both χ and χ_0 have type UII:

we have type III2 iff $m_0 = 2$

we have type III3 iff $m = 2$

Some further definitions will be useful. To describe the maps which come up in cases I and II, define the following kinds of maps:

a Δ-map is a map $A \to B_0 \times B_1$ so that the two composites $A \to B_0 \times B_1 \to B_i$ are isomorphisms;

an A-map is a map $A_0 \times A_1 \to B$ so that the two composites $A_i \to A_0 \times A_1 \to B$ are isomorphisms.

In case III, we introduce the notion of subtype:

if Type χ_0 = Type χ is 0, the subtype is 0;

if Type χ_0 = Type χ is S_p, the subtype is S_p;

if Type χ_0 = Type χ is U, the subtype is U:

if Type $\chi_0 \neq$ Type χ we have a mixed subtype.

There are four cases of mixed subtype denoted

$$0 \to U \qquad U \to 0$$
$$S_p \to U \qquad U \to S_p$$

<u>Part 1</u>: <u>Relative push forward maps</u>

Our goal is to describe

(AII.1.1) $\ldots \to L_r^p(\mathbb{Z}H \to \hat{\mathbb{Z}}_2H,\alpha,u) \xrightarrow{i_!} L_r^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u) \to L_r^p(i_!) \to \ldots$

This sequence decomposes into a product of exact sequences where the product is taken over the types in the decomposition of the map $\mathbb{Q}G \to \mathbb{Q}H$. Since GL types make no contribution we need only describe what happens in the remaining cases. We begin with cases I and II: in the four cases below we list the contribution of the type to AII.1.1.

IX_0GL: $\ldots \to L_r(\chi_0) \to 0 \to L_r(IX_0GL_!) \to \ldots$

IX_0 : $\ldots \to L_r(\chi_0) \to L_r(\chi) \times L_r(\chi^\psi) \to L_r(LX_0\Delta_!) \to \ldots$

where $i_!$ is a Δ-map

$IIGLX$: $\ldots \to 0 \to L_r(\chi) \to L_r(IIGLX_!) \to \ldots$

$IIAX$: $\ldots \to L_r(\chi_0) \times L_r(\chi_0^t) \to L_r(\chi) \to L_r(IIAX_!) \to \ldots$

where $i_!$ is an A-map.

In case III we either use Table 2 or Table 3. We must decide which subtable to use; which row of that subtable; and which columns to use. An integer $\mathfrak{k}$ mod 4 (or mod 2 on the $U \to U$ subtable) determines a sequence of three groups on each row: this sequence will be isomorphic to the contribution of this factor of the map to sequence AII.1.1.

If the type is III2 we use Table 2. If the subtype is mixed it is Type $\chi_0 \to$ Type χ.

(A.II.1.2)

subtype	subtable	row	$\mathfrak{k}$
0	$0 \to 0$	$E\chi_0 \subset E\chi$	$\mathfrak{k} \equiv r \pmod 4$
S_p	$0 \to 0$	$E\chi_0 \subset E\chi$	$\mathfrak{k} \equiv r+2 \pmod 4$
U	$U \to U$	$U\chi_0 \to U\chi$	$\mathfrak{k} \equiv r \pmod 2$
$0 \to U$	$0 \to U\chi$	$E\chi_0 \subset E\chi$	$\mathfrak{k} \equiv r \pmod 4$
$S_p \to U$	$0 \to U\chi$	$E\chi_0 \subset E\chi$	$\mathfrak{k} \equiv r+2 \pmod 4$
$U \to 0$	$U\chi \to 0$	—	$\mathfrak{k} \equiv r \pmod 4$
$U \to S_p$	$U\chi \to 0$	—	$\mathfrak{k} \equiv r+2 \pmod 4$

<u>Remarks</u>: The — in the row column means that the subtable in question has only 1 row. In the $0 \to U$ (or $S_p \to U$) case we may need to go back to steps 2 and 3 in Appendix I to compute $E\chi$. Note that we do not need $E\chi$ if we are using subtable $0 \to UII$, $E\chi_0$ suffices.

If the type is III3 we use Table 3. If the subtype is mixed, it is Type $\chi \to$ Type χ_0.

(AII.1.3)

subtype	subtable	row	$\mathfrak{k}$
0	$0 \to 0$	$E\chi \subset E\chi_0$	$\mathfrak{k} \equiv r \pmod 4$
S_p	$0 \to 0$	$E\chi \subset E\chi_0$	$\mathfrak{k} \equiv r+2 \pmod 4$
U	$U \to U$	$U\chi \to U\chi_0$	$\mathfrak{k} \equiv r \pmod 2$
$0 \to U$	$0 \to U\chi$	$E\chi \subset E\chi_0$	$\mathfrak{k} \equiv r \pmod 4$
$S_p \to U$	$0 \to U\chi$	$E\chi \subset E\chi_0$	$\mathfrak{k} \equiv r+2 \pmod 4$
$U \to 0$	$U\chi \to 0$	—	$\mathfrak{k} \equiv r \pmod 4$
$U \to S_p$	$U\chi \to 0$	—	$\mathfrak{k} \equiv r+2 \pmod 4$

Remark: The only visable difference between AII.1.2 and AII.1.3 is that in the row column we have interchanged the role of χ_0 and χ. A closer study shows that on the $0 \to UI_N$ subtable we need EX to use Table 2 but that on Table 3, this subtable has only one row.

Part 2: Relative transfer maps

This time our goal is to describe

$$\text{(AII.2.1)} \quad \ldots \to L_r^p(\mathbb{Z}G \to \hat{\mathbb{Z}}_2G,\alpha,u) \xrightarrow{i^!} L_r^p(\mathbb{Z}H \to \hat{\mathbb{Z}}_2H,\alpha,u) \to L_r^p(i^!) \to \ldots$$

As in part 1 of Appendix II, we get that AII.2.1 is a sum of exact sequences. We describe the contribution from each of the non-GL types.

IX_0GL: $\ldots \to 0 \to L_r(\chi_0) \to L_r(IX_0GL^!) \to \ldots$

$IX_0\Delta$: $\ldots \to L_r(\chi) \times L_r(\chi^\psi) \to L_r(\chi_0) \to L_r(IX_0\Delta^!) \to \ldots$

where $i^!$ is an A-map

$IIGLX$: $\ldots \to L_r(\chi) \to 0 \to L_r(IIGLX^!) \to \ldots$

$IIAX$: $\ldots \to L_r(\chi) \to L_r(\chi_0) \times L_r(\chi_0^t) \to L_r(IIAX^!) \to \ldots$

where $i^!$ is a Δ-map.

If the type is III2, we use Table 3: if the subtype is mixed it is Type $\chi_0 \to$ Type χ. The subtable-row-$\mathcal{k}$ data is read off chart AII.1.2.

If the type is III3, we use Table 2: if the subtype is mixed it is Type $\chi \to$ Type χ_0. The subtable-row-$\mathcal{k}$ data is read off chart AII.1.3.

Part 3: Push forward and transfer maps

We want to describe

$$\text{(AII.3.1)} \quad \ldots \to L_r^p(\mathbb{Z}H,\alpha,u) \xrightarrow{i_!} L_r^p(\mathbb{Z}G,\alpha,u) \to L_r^p(i_!) \to \ldots$$

and

$$\text{(AII.3.2)} \quad \ldots \to L_r^p(\mathbb{Z}G,\alpha,u) \xrightarrow{i^!} L_r^p(\mathbb{Z}H,\alpha,u) \to L_r^p(i^!) \to \ldots$$

The map $L_r^p(\hat{\mathbb{Z}}_2H,\alpha,u) \to L_r^p(\hat{\mathbb{Z}}_2G,\alpha,u)$ is an isomorphism, so $L_r^p(i_!)$ is isomorphic to the relative group computed in part 1 of Appendix II. The map $L_r^p(\hat{\mathbb{Z}}_2G,\alpha,u) \to L_r^p(\hat{\mathbb{Z}}_2H,\alpha,u)$ is always the zero map so we have not yet computed $L_r^p(i^!)$. We leave this for [H-T-W].

The maps in AII.3.1 and AII.3.2 are almost completely determined by the corresponding maps in the relative sequences, AII.1.1 and AII.2.1. In some cases the fate of elements which map non-zero into the 2-adic terms is ambiguous. In one case we can give a complete description.

Define the notion of a <u>twisted quaternionic representation</u> by replacing D_N, $N \geq 3$ with Q_N, $N \geq 4$ everywhere. We say that $(\mathbb{Z}G,\alpha,u)$ satisfies condition $\underline{ARF_0}$ iff there are <u>no</u> UI twisted cyclic; 0 twisted dihedral; 0 twisted quaternionic; or 0 cyclic representations: $(\mathbb{Z}G,\alpha,u)$ satisfies condition $\underline{ARF_2}$ iff there are <u>no</u> UI twisted cyclic; S_p twisted dihedral; S_p twisted quaternionic; or S_p cyclic representations.

If $(\mathbb{Z}G,\alpha,u)$ satisfies condition ARF_{2r}, we can define an element $A_{2r} \in L^p_{2r}(\mathbb{Z}G,\alpha,u)$ such that A_{2r} has order 2; $\kappa_{2r}(A_{2r}) = -1$; and the following theorem holds.

<u>Theorem AII.3.3</u>: Let $i:(\mathbb{Z}H,\alpha,u) \to (\mathbb{Z}G,\alpha,u)$ be the usual map. If $(\mathbb{Z}H,\alpha,u)$ satisfies condition ARF_{2r} then so does $(\mathbb{Z}G,\alpha,u)$. Moreover

$$i_!(A_{2r}) = A_{2r} \qquad ; \qquad i^!(A_{2r}) = 0.$$

The antistructures which arise in ordinary surgery theory (the ones with $\alpha = \alpha_\omega$ and $u = e$) never have any twisted representations. Hence they satisfy condition ARF_{2r} iff κ_{2r} is onto.

Our proofs of these results must wait for [H-T-W], but perhaps a word is in order as to how they go.

The first step is to use representation theory to show that all problems can be resolved by studying a short list of groups (e.g. Theorem 2.2.2).

To do the necessary calculations for these groups involves the explicit calculations in Section 3 and the work of C. T. C. Wall [W4-W8]. Finally, whenever the going gets tough, we resort to a twisting diagram (e.g. 4.5.6). Twisting diagrams seem to be a new tool of some power in the long history of these sorts of calculations.

TABLE 1

Type O	Γ_N	F_N	R_N	H_N
3	0	0	0	$\mathbb{Z}^{2^{N-2}}$
2	0	0	0	0
1	0	0	$\mathbb{Z}^{2^N}$	$\mathbb{Z}/2^{2^{N-2}-1}$
0	$\mathbb{Z}/2^{2^{N-1}+2}$	$\mathbb{Z}/2^{2^{N-1}} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/2^{2^{N-2}+1}$

Type U	$U\,I_N$	$U\,\underline{II}$
odd	$\mathbb{Z}^{2^{N-1}}$	0
even	0	$\mathbb{Z}/2$

TABLE 2

$$\cdots \to \underset{\underset{R_k}{\parallel}}{L^p_k(R\to\hat{R}_2,\alpha_0,1)} \longrightarrow \underset{\underset{S_k}{\parallel}}{L^p_k(S\to\hat{S}_2,\alpha,1)} \longrightarrow \underset{\underset{\underset{P_k}{\parallel}}{L^p_k(R\to S,\alpha,1)}}{L^p_k(f_!)} \to \cdots$$

$R\subset S$

$0\to 0$	R_3	S_3	P_3	R_2	S_2	P_2	R_1	S_1	P_1	R_0	S_0	P_0
$R_{N-1}\subset R_N$	0	0	0	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^N}$	$\mathbb{Z}^{2^{N-1}}\oplus\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
$H_{N-1}\subset H_N$	$\mathbb{Z}^{2^{N-3}}$	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-3}}$	0	0	0	$(\mathbb{Z}/2)^{2^{N-3}-1}$	$(\mathbb{Z}/2)^{2^{N-2}-1}$	$(\mathbb{Z}/2)^{2^{N-3}}$	$(\mathbb{Z}/2)^{2^{N-3}+1}$	$(\mathbb{Z}/2)^{2^{N-2}+1}$	$(\mathbb{Z}/2)^{2^{N-3}}$
$\Gamma_{N-1}, F_{N-1}\subset\Gamma_N$	0	0	0	0	0	0	0	0	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^{2^{N-2}+2}$ $(\mathbb{Z}/2)^{2^{N-2}}\oplus\mathbb{Z}/4$	$(\mathbb{Z}/2)^{2^{N-1}+2}$	$(\mathbb{Z}/2)^{2^{N-2}+1}$
$R_{N-1}\subset\Gamma_N, F_N$	0	0	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-1}}$	0	0	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^{2^{N-1}+2}$ $(\mathbb{Z}/2)^{2^{N-1}}\oplus\mathbb{Z}/4$	$(\mathbb{Z}/2)^{2^{N-1}+1}$ $(\mathbb{Z}/2)^{2^{N-1}-1}\oplus\mathbb{Z}/4$
$\Gamma_{N-1}, F_{N-1}\subset H_N$	0	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-2}}$	0	0	0	0	$(\mathbb{Z}/2)^{2^{N-2}-1}$	$(\mathbb{Z}/2)^{2^{N-2}+1}$	$(\mathbb{Z}/2)^{2^{N-2}+2}$ $(\mathbb{Z}/2)^{2^{N-2}}\oplus\mathbb{Z}/4$	$(\mathbb{Z}/2)^{2^{N-2}+1}$	$\mathbb{Z}/2$

$0\longrightarrow UI_N$

$R_{N-1}\subset\Gamma_N, F_N$	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-1}}$	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-1}}$	$(\mathbb{Z}/2)^{2^{N-1}+1}$ $(\mathbb{Z}/2)^{2^{N-1}-1}\oplus\mathbb{Z}/4$	$\mathbb{Z}/2$	0	0

$UI_{N-1}\to 0$

Fake $\subset H_N$	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}/2$	0	0	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-1}}$	$(\mathbb{Z}/2)^{2^{N-2}-1}$	0	0	$(\mathbb{Z}/2)^{2^{N-2}+1}$	$(\mathbb{Z}/2)^{2^{N-2}+1}$

$0\to UII$

$R_{N-1}\subset R_N$	0	0	0	0	$\mathbb{Z}/2$	$\mathbb{Z}^{2^{N-1}}\oplus\mathbb{Z}/2$	$\mathbb{Z}^{2^{N-1}}$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
$H_{N-1}\subset H_N$	$\mathbb{Z}^{2^{N-3}}$	0	0	0	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^{2^{N-3}}$	$(\mathbb{Z}/2)^{2^{N-3}-1}$	0	$(\mathbb{Z}/2)^{2^{N-3}}$	$(\mathbb{Z}/2)^{2^{N-3}+1}$	$\mathbb{Z}/2$	$\mathbb{Z}^{2^{N-3}}$
$\Gamma_{N-1}, F_{N-1}\subset\Gamma_N$	0	0	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	$(\mathbb{Z}/2)^{2^{N-2}+1}$	$(\mathbb{Z}/2)^{2^{N-2}+2}$ $(\mathbb{Z}/2)^{2^{N-2}}\oplus\mathbb{Z}/4$	$\mathbb{Z}/2$	0

$U\to U$	R_{odd}	S_{odd}	P_{odd}	R_{even}	S_{even}	P_{even}
$UI_{N-1}\to UI_N$	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-2}}\oplus\mathbb{Z}/2$	0	0	0
$UI_{N-1}\to UII$	$\mathbb{Z}^{2^{N-2}}$	0	0	0	$\mathbb{Z}/2$	$\mathbb{Z}^{2^{N-2}}\oplus\mathbb{Z}/2$
$UII\to UII$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

TABLE 3

$$\cdots \to \underset{\overset{\shortparallel}{S_k}}{L^p_k(S \to \hat{S}_2, \alpha, 1)} \longrightarrow \underset{\overset{\shortparallel}{R_k}}{L^p_k(R \to \hat{R}_2, \alpha_0, 1)} \longrightarrow \underset{\overset{\shortparallel}{Tr_k}}{L^p_k(f^!)} \to \cdots$$

$R \subset S$

$0 \to 0$	S_3	R_3	Tr_3	S_2	R_2	Tr_2	S_1	R_1	Tr_1	S_0	R_0	Tr_0
$R_{N-1} \subset R_N$	0	0	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^N}$	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H_{N-1} \subset H_N$	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-3}}$	0	0	0	$\mathbb{Z}/2^{2^{N-3}}$	$\mathbb{Z}/2^{2^{N-2}-1}$	$\mathbb{Z}/2^{2^{N-3}-1}$	$\mathbb{Z}/2^{2^{N-3}+1}$	$\mathbb{Z}/2^{2^{N-2}+1}$	$\mathbb{Z}/2^{2^{N-3}+1}$	$\mathbb{Z}^{2^{N-3}} \oplus \mathbb{Z}/2$
Γ_{N-1} F_{N-1} $\subset \Gamma_N$	0	0	0	0	0	0	0	0	$\mathbb{Z}/2^{2^{N-2}+1}$	$\mathbb{Z}/2^{2^{N-1}+2}$	$\mathbb{Z}/2^{2^{N-2}+2}$ $\mathbb{Z}/2^{2^{N-2}} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2$
$R_{N-1} \subset$ Γ_N F_N	0	0	0	0	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-1}} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2^{2^{N-1}+2}$ $\mathbb{Z}/2^{2^{N-1}} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2$	0
Γ_{N-1} F_{N-1} $\subset H_N$	$\mathbb{Z}^{2^{N-2}}$	0	0	0	0	$\mathbb{Z}/2^{2^{N-2}-1}$	$\mathbb{Z}/2^{2^{N-2}-1}$	0	$\mathbb{Z}/2^{2^{N-2}-1}$	$\mathbb{Z}/2^{2^{N-2}+1}$	$\mathbb{Z}/2^{2^{N-2}+2}$ $\mathbb{Z}/2^{2^{N-2}} \oplus \mathbb{Z}/4$	$(\mathbb{Z}/2 \oplus \mathbb{Z})^{2^{N-2}}$
$0 \to UI_N$												
$R_{N-1} \subset Fake$	$\mathbb{Z}^{2^{N-1}}$	0	0	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-1}}$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}^{2^{N-1}} \oplus \mathbb{Z}/2$
$UI_{N-1} \to 0$												
$Fake \subset H_N$	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}/2^{2^{N-2}-1}$	0	0	$\mathbb{Z}/2^{2^{N-2}-1}$	$\mathbb{Z}/2^{2^{N-2}-1}$	$\mathbb{Z}^{2^{N-2}}$	$(\mathbb{Z}/2 \oplus \mathbb{Z})^{2^{N-2}}$	$\mathbb{Z}/2^{2^{N-2}+1}$	0	0
$0 \to UII$												
$R_{N-1} \subset R_N$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	0	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H_{N-1} \subset H_N$	0	$\mathbb{Z}^{2^{N-3}}$	$\mathbb{Z}^{2^{N-3}} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	0	$\mathbb{Z}/2^{2^{N-3}-1}$	$\mathbb{Z}/2^{2^{N-3}}$	$\mathbb{Z}/2$	$\mathbb{Z}/2^{2^{N-3}+1}$	$\mathbb{Z}/2^{2^{N-3}+1}$
Γ_{N-1} F_{N-1} $\subset \Gamma_N$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	0	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2^{2^{N-2}+2}$ $\mathbb{Z}/2^{2^{N-2}} \oplus \mathbb{Z}/4$	$\mathbb{Z}/2^{2^{N-2}+1}$

$U \to U$	S_{odd}	R_{odd}	Tr_{odd}	S_{even}	R_{even}	Tr_{even}
$UI_{N-1} \to UI_N$	$\mathbb{Z}^{2^{N-1}}$	$\mathbb{Z}^{2^{N-2}}$	0	0	0	$\mathbb{Z}^{2^{N-2}}$
$UI_{N-1} \to UII$	0	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}^{2^{N-2}}$	$\mathbb{Z}/2$	0	0
$UII \to UII$	0	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0

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UNITARY K-HOMOLOGY AND THE LICHTENBAUM-QUILLEN CONJECTURE ON THE ALGEBRAIC K-THEORY OF SCHEMES

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§1: Introduction - Statement of Results

Let X be a regular scheme, quasiprojective over a noetherian ring, S, which has finite Krull dimension. Let ℓ be a prime which is invertible in S. Assume also that the residue fields of X have uniformly bounded étale cohomological dimension for ℓ-torsion sheaves. Such schemes are quite commonplace and include the spectrum of a local or global field or the localisation away from ℓ of the algebraic integers in such a field (see §5.4 for further examples).

For such schemes, X, the Lichtenbaum-Quillen conjecture may be formulated succinctly as the assertion that the map from algebraic K-theory to étale K-theory, constructed in [FI,II],

$$\rho : K_i(X;Z/\ell^\nu) \to K_i^{et}(X;Z/\ell^\nu) \tag{1.1}$$

is an isomorphism for all i greater than some integer depending on X (and possibly ℓ, ν). The point about étale K-theory is that it is constructed to be computable in terms of étale cohomology. It has an Atiyah-Hirzebruch type of spectral sequence (see §5.13) which collapses if the étale cohomological dimension of X is less than 2ℓ. In [D-F-S-T] it is shown that (1.1) is eventually onto in many cases (for example, X a regular quasiprojective scheme over an algebraically closed field of characteristic not ℓ). In fact this result is much improved in [Th2] where it is shown that

$$K_i^{et}(X;Z/\ell^\nu) \cong K_i(X;Z/\ell^\nu)[1/\beta_\nu] \tag{1.2}$$

where $\beta_\nu \in K_{2\ell^{\nu-1}(\ell-1)}(S;Z/\ell^\nu)$ is a so called Bott element (see [D-F-S-T]). Actually the primes ℓ = 2 and 3 are somewhat exceptional in the statement

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of (1.2) in that S is required to have 16-th or 9-th roots of unity respectively in these cases. In addition (1.2) has the property that it identifies ρ with the localisation map

$$K_i(X;Z/\ell^\nu) \to K_i(X;Z/\ell^\nu)[1/\beta_\nu] \tag{1.3}$$

from algebraic K-theory to "Bott periodic" K-theory, introduced in [Sn1,§IV]. See also [Sn3]. With these significant reductions to the problem we are reduced to studying the injectivity of (1.3).

The first remark to make is that the case of a general scheme, X, may readily be reduced (by Jouanolou's device, see §5.5) to the case X = Spec A. That is, we may study the K-theory of commutative rings.

If ℓ is a prime and ν is an integer choose any prime t such that $\nu_\ell(t-1) \geq \nu$ and define $j_*(\_;Z/\ell^\nu)$ to be the fibre homology theory of

$$\psi^t - 1 : KU_*(\_;Z/\ell^\nu) \to KU_*(\_;Z/\ell^\nu)$$

(see §2.12). Let h_* denote either $j_*(\_;Z/\ell^\nu)$ or $KU_*(\_;Z/\ell^\nu)$.

Let $\mathcal{K}_*(A;Z/\ell^\nu)$ denote $K_*(A;Z/\ell^\nu)[1/\beta_\nu]$.

<u>3.24: Theorem</u>. Let ℓ be an odd prime invertible in A. There exists an integer s (depending on ℓ and ν, if $\nu = 1$, s = 3) such that the following diagram commutes for all $i \geq s$.

$$\begin{tikzcd} K_i(A;Z/\ell^\nu) \arrow[rr, "\rho"] \arrow[dr, "H"'] & & \mathcal{K}_i(A;Z/\ell^\nu) \arrow[dl, "I"] \\ & h_i(BGLA^+) & \end{tikzcd}$$

Here H is the Hurewicz map.

If A*, the units of A, have the form $Z/\ell^a \times H$ with $\nu \leq a \leq \infty$ we have a diagram which is the "converse" of §3.24. If $a = \infty$ set $h_* = KU_*(\_;Z/\ell^\nu)$ and if $a < \infty$ set $h_* = j_*(\_;Z/\ell^\nu)$ where $\nu_\ell(t-1) = a$ in the above definition of $j_*(\_;Z/\ell^\nu)$.

Let $BSLA^+ = \text{fibre}(\det : BGLA^+ \to BA^*)$.

<u>3.4: Theorem</u>. Let ℓ be any prime (assume $a \geq 2$ if $\ell = 2$). There is a commutative diagram for all $i \geq 2$

$$\begin{array}{ccc}
\pi_i(BSLA^+ \times BH; Z/\ell^\nu) & \xrightarrow{H} & h_i(BSLA^+ \times BH) \\
\downarrow & & \downarrow J \\
K_i(A;Z/\ell^\nu) & \xrightarrow[\rho]{} & K_i(A;Z/\ell^\nu)
\end{array}$$

When A is such that both §3.4 and §3.24 apply, we obtain

3.25: Corollary. I is injective in §3.24, when A is a $Z[1/\ell]$-algebra as in §3.4, for h_* as in §3.4.

For such A, at odd primes, we obtain from §3.25 a commutative diagram of the following form.

4.1: Theorem. Let ℓ be an odd prime and let A be as in §§3.4/3.24 the

diagram

$$\begin{tikzcd}
K_2(A;Z/\ell) \arrow[rr, "\rho"] \arrow[dr, "\hat{H}"'] & & K_2(A;Z/\ell) \arrow[dl, "I"] \\
& h_0(BGLA^+) &
\end{tikzcd}$$

commutes if $\hat{H}(x) = H(x) - H(x)^\ell$.

Roughly speaking in §5 we show that if $\hat{H}(x) = 0$ implies $H(x) = 0$ then the Lichtenbaum-Quillen conjecture follows. Precisely we explain in §5 that to study the injectivity of

$$\rho : K_{n+1}(A;Z/\ell) \to K_{n+1}(A;Z/\ell) \tag{1.4}$$

it suffices, by work of Dayton-Weibel, to study

$$\rho : K_2(\dot{\Delta}^n_A;Z/\ell) \to K_2(\dot{\Delta}^n_A;Z/\ell)$$

for a suitable ring, $\dot{\Delta}^n_A$. If $n \geq 2$ $\dot{\Delta}^n_A$ is the ring of "zero forms over A on $\dot{\Delta}^n$" and the definition is to be found in §5.6. In §5.11 we show that (1.4) is an isomorphism - i.e. that the Lichtenbaum-Quillen conjecture is true in dimension n+1 for A - if ρ is onto in dimensions n+1 and n+2 and if

$$\{x \in h_0(BGL\dot{\Delta}^n_A) \mid x = x^\ell\} = \underline{0}\,. \tag{1.5}$$

In §§5.16,5.17 we give many examples in which the Lichtenbaum-Quillen conjecture is implied by (virtually equivalent to) the verification of the J-theoretic/KU-theoretic condition (1.5).

Invaluable in this work have been the following:

(i) The work of Eric Friedlander, who reformulated the conjecture in terms of his étale K-theory [FI,II] and later, with Bill Dwyer [D-F] extended Soulé's work [So] on algebraic integers in terms of étale K-theory.

(ii) The work of Chuck Weibel [We] which enables one to work with Karoubi-Villamayor K-theory and to use his work with Barry Dayton [D-WI,II] to reduce to dimension two.

(iii) The work of Bob Thomason - the sine qua non - [Th1;Th2] which permits one to replace étale K-theory by "Bott periodic" algebraic K-theory of [Sn1,§IV] and hence to construct the crucial diagrams.

To all of them I would like to express my gratitude for conversations, correspondence and for keeping me informed of their work.

This paper provides complete proofs of the results announced in [Sn 5].

§2: KU_*-Hurewicz homomorphisms

Throughout this section X will denote a connected space which is a two-fold loop space ($X = \Omega^2 Y$). Eventually our applications will be to the case when $X = BGLA^+$, where A is a ring with unit. In this example X is an infinite loop space (see [Ma2], for example).

Let ℓ be any prime and let ν be an integer ($1 \leq \nu \leq \infty$). In this section we study the injectivity of the KU-theory Hurewicz map

$$H_K : \pi_2(X;Z/\ell^\nu) \to KU_0(X;Z/\ell^\nu). \tag{2.1}$$

In (2.1), as elsewhere, we abide by the convention that KU_* is considered (by virtue of Bott periodicity) to be Z/2-graded. Also in (2.1), when $1 \leq \nu < \infty$, $\pi_2(X;Z/\ell^\nu)$ denotes the (abelian) group of based homotopy classes of maps of the Moore space, $S^1 \cup_{\ell^\nu} e^2 = P^2(\nu)$, to X. When $\nu = \infty$ $\pi_2(X;Z/\ell^\infty)$ denotes $\varinjlim_\nu \pi_2(X;Z/\ell^\nu)$.

We will also have need of corresponding results about the analogous J-theory Hurewicz map. These are given, together with the definition of J-theory, at the end of the section.

For background on homotopy with coefficients the reader is referred to [N;B]. For KU-homology with coefficients and Hurewicz maps I suggest [A,Part III] as a background source.

<u>2.2: Theorem</u>. With the notation established above the kernel of (2.1), when $1 \leq \nu < \infty$, is contained in the image of

$$\{y \in \pi_2(X) \mid \ell^N y = 0 \text{ for some } N\}$$

under reduction mod ℓ^ν, $\pi_2(X) \to \pi_2(X;Z/\ell^\nu)$.

Taking the limit over the coefficient exact sequences associated to $Z \overset{\ell^\nu}{\rightarrowtail} Z \twoheadrightarrow Z/\ell^\nu$ we obtain the following result.

<u>2.3: Corollary</u>. When $\nu = \infty$, H_K is injective in (2.1).

The proof of Theorem 2.2 will consist of a reduction to the following special cases, in which Theorem 2.2 asserts that H_K is <u>injective</u>.

<u>2.4: Lemma</u>. Theorem 2.2 is true for the Eilenberg-MacLane spaces $X = K(Z,2) = \mathbb{C}P^\infty$ or $X = K(Z/\ell^a,1) = BZ/\ell^a$.

<u>Proof</u>: The case $X = \mathbb{C}P^\infty$ is well-known, following from the classical Hurewicz theorem [Sp] and the fact that the KU-theory Atiyah-Hirzebruch spectral sequence for $\mathbb{C}P^\infty$ collapses.

We reduce the case $X = BZ/\ell^a$ to that of $\mathbb{C}P^\infty$ using the canonical map $\pi_a : BZ/\ell^a \to \mathbb{C}P^\infty = BS^1$ which is induced by the inclusion of Z/ℓ^a into S^1.

We have the following commutative diagram of Hurewicz maps in which the horizontal maps are induced by the natural inclusions $j_a : Z/\ell^a \to Z/\ell^{a+1}$.

$$\begin{tikzcd}
\pi_2(BZ/\ell^a;Z/\ell^\nu) \arrow[rr, "(j_a)_\#"] \arrow[dd, "H_K"'] \arrow[dr, "(\pi_a)_\#"] & & \pi_2(BZ/\ell^{a+1};Z/\ell^\nu) \arrow[dd, "H_K"] \arrow[dl, "(\pi_{a+1})_\#"'] \\
& \pi_2(\mathbb{C}P^\infty;Z/\ell^\nu) & \\
KU_0(BZ/\ell^a;Z/\ell^\nu) \arrow[rr, "(j_a)_*" near end] \arrow[dr, "(\pi_a)_*"'] & & KU_0(BZ/\ell^{a+1};Z/\ell^\nu) \arrow[dl, "(\pi_{a+1})_*"] \\
& KU_0(\mathbb{C}P^\infty;Z/\ell^\nu) \arrow[from=uu, crossing over, "H_K" near start] &
\end{tikzcd} \tag{2.5}$$

By [At;Ho] there is a natural identification of $KU_0(BG;Z/\ell^\nu)$ with $\mathrm{Hom}(R(G),Z/\ell^\nu)$ when G is a finite ℓ-group, R(G) being the complex representation ring of G. Since j_a^* is surjective on representation rings, $(j_a)_*$ is one-one in (2.5). Also, from the coefficient exact sequence

$$\pi_2(K(Z/\ell^a,1);Z/\ell^\nu) \cong Z/\ell^b$$

where $b = \min(a,\nu)$ and $(j_a)_\#$ is one-one. From the mod ℓ^ν homotopy sequence of the fibring $BZ/\ell^a \xrightarrow{\pi_a} \mathbb{C}P^\infty \xrightarrow{\ell^a} \mathbb{C}P^\infty$ it follows that $(\pi_a)_\#$ is an isomorphism when $a \geq \nu$. The result now follows from diagram (2.5).

2.6: Corollary. Theorem 2.2 is true in the following cases:

(a) $X = K(B,2)$ with B torsion free or

(b) $X = K(C,1)$ with C a torsion abelian group or

(c) $X = K(D,1)$ an arbitrary abelian group.

Proof: It suffices to assume B, C and D are finitely generated since we may write each group as the direct limit over its finitely generated subgroups and we may identify H_K with the direct limit of the Hurewicz homomorphisms for those subgroups.

In (b) if $C = \prod_{i=1}^{T} C_i$ is a product of prime power order cyclic groups $\pi_2(BC;Z/\ell^\nu) = \bigoplus_j \pi_2(BC_j;Z/\ell^\nu)$ where C_j runs through the ℓ-primary factors of C. This is because $BC = \prod_{i=1}^{T} BC_i$. However, for any homology theory, h_*, the natural map $h_*(X_1) \oplus h_*(X_2) \to h_*(X_1 \times X_2)$ is injective. Hence the result follows from the following commutative diagram.

$$\begin{array}{ccc}
\bigoplus_j \pi_2(BC_j;Z/\ell^\nu) & \xrightarrow{\oplus_j H_K} & \bigoplus_j KU_0(BC_j;Z/\ell^\nu) \\
\downarrow \cong & & \downarrow \\
\pi_2(\prod_j BC_j;Z/\ell^\nu) & \xrightarrow{H_K} & KU_0(\prod_j BC_j;Z/\ell^\nu) \\
\downarrow \cong & & \downarrow \cong \\
\pi_2(BC;Z/\ell^\nu) & \xrightarrow{H_K} & KU_0(BC;Z/\ell^\nu)
\end{array}$$

A similar argument deals with (a) when B is finitely generated. Finally for (c) the result follows in a similar manner by observing that, if D is finitely generated, $\pi_2(BD;Z/\ell^\nu)$ depends only on the ℓ-Sylow subgroup of D, which is a direct summand, and on its maximal free abelian summand.

2.7: Proof of Theorem 2.2. We construct a map

$$a = (a_1, a_2) : X \to K(\pi_1 X, 1) \times K(B,2) \tag{2.8}$$

and study the following commutative diagram.

$$\begin{tikzcd}
\pi_2(X;Z/\ell^\nu) \arrow[rr] \arrow[d, "a_\#"] & & KU_0(X;Z/\ell^\nu) \arrow[dd, "a_*"] \\
\pi_2(K_1;Z/\ell^\nu) \oplus \pi_2(K_2;Z/\ell^\nu) \arrow[d, "H_K \oplus H_K"] & & \\
KU_0(K_1;Z/\ell^\nu) \oplus KU_0(K_2;Z/\ell^\nu) \arrow[rr, tail] & & KU_0(K_1 \times K_2;Z/\ell^\nu)
\end{tikzcd} \tag{2.9}$$

In (2.9) $K_1 = K(\pi_1 X,1)$, $K_2 = K(B,2)$ and the lower horizontal map is the canonical injection.

We choose a_1 in (2.8) to be the first k-invariant of X. It is in the choice of a_2 that we use the condition, $X = \Omega^2 Y$.

Firstly we may write $\pi_2(X) = B_1 \times B_2$ where B_1 is a maximal divisible subgroup of $\pi_2(X)$. Set $B = B_2/\text{Tors } B_2$, the quotient of B_2 by its torsion subgroup.

Next consider the following commutative diagram of homomorphisms.

$$\begin{tikzcd}
H_2(X) \arrow[r, "\cong"] & H_4(\Sigma^2 X) \arrow[d, "\phi"] & \pi_4(\Sigma^2 X) \arrow[l, "H'"'] \arrow[dl] \cong \pi_2(\Omega^2\Sigma^2 X) \arrow[d, "e_\#"] \\
 & B & \pi_2(X) = B_1 \times B_2 \arrow[l, two heads, "\pi"']
\end{tikzcd} \tag{2.10}$$

In (2.10) the singular homology groups are taken with integer coefficients, H' is the Hurewicz homomorphism, e is the evaluation map

$$\Omega^2\Sigma^2 X = \Omega^2\Sigma^2\Omega^2 Y \to \Omega^2 Y = X$$

and π is the natural projection onto B. The homomorphism, ϕ, exists to make the triangle commute because [W,p.555] H' is onto with a kernel which is a $Z/2$-vector space and B is torsion free. We choose the map, a_2, to represent an element of $H^2(X;B)$ which maps to $\phi \in \mathrm{Hom}(H_2(X),B)$ under the surjection in the universal coefficient theorem [Sp;W].

To complete the proof it remains to determine $\ker a_\#$ in (2.9) since $H_K \oplus H_K$ is injective by Corollary 2.6.

From the universal coefficient theorem we obtain an exact sequence

$$B_2 \otimes Z/\ell^\nu \rightarrowtail \pi_2(X;Z/\ell^\nu) \twoheadrightarrow {}_{\ell^\nu}\pi_1(X) \qquad (2.11)$$

where ${}_nD = \{x \in D \mid nx = 0\}$.

An element of $\ker a_\#$ must be in $B_2 \otimes Z/\ell^\nu$ since $\pi_2(K_1;Z/\ell^\nu) \cong {}_{\ell^\nu}\pi_1(X)$ and under this isomorphism $(a_1)_\#$ is identified with the epimorphism in (2.11). However, $(a_2)_\#$ maps $B_2 \otimes Z/\ell^\nu$ to $B \otimes Z/\ell^\nu$ by the canonical projection whose kernel is the image of $(\mathrm{Tors}\ B_2) \otimes Z/\ell^\nu$. To see this last assertion, which completes the proof, we observe that $e_\#$ in (2.10) is split by the double suspension map

$$\Sigma^2 : \pi_2(X) \to \pi_4(\Sigma^2 X)$$

and by definition for $b \in \pi_2(X)$

$$\begin{aligned}(a_2)_\#(b) &= \phi(H'(\Sigma^2 b)) \\ &= \pi(e_\#(\Sigma^2(b))) \\ &\equiv \pi(b) \pmod{\ell^\nu}.\end{aligned}$$

<u>2.12</u>: We conclude this section by recording the results we will need concerning the J-theory Hurewicz homomorphism, H_J.

For our purpose $1 \leq \nu \leq a < \infty$ will be integers and t will be a prime such that $a = \nu_\ell(t-1)$, the ℓ-adic valuation of $t-1$. We define $j_*(Y;Z/\ell^\nu)$ to be the homology theory in the fibre sequence

$$\cdots \to j_\alpha(Y;Z/\ell^\nu) \xrightarrow{\mu} KU_\alpha(Y;Z/\ell^\nu) \xrightarrow{\psi^t - 1} KU_\alpha(Y;Z/\ell^\nu) \to \cdots \qquad (2.13)$$

where ψ^t is the Adams operation [At2]. Note that $j_*(Y;Z/\ell^\nu)$ is a $Z/2$-graded theory.

Setting $Y = P^2(\nu) = S^1 \cup_{\ell^\nu} e^2$ in (2.13) we see that $\mu : j_0(P^2(\nu);Z/\ell^\nu) \to KU_0(P^2(\nu);Z/\ell^\nu) \cong Z/\ell^\nu$ is an isomorphism. If $x \in j_0(P^2(\nu);Z/\ell^\nu)$ is a generator then H_J is defined by sending the class of $f : P^2(\nu) \to X$ to $f_*(x) \in j_0(X;Z/\ell^\nu)$ while $H_K(f) = f_*\mu(x)$. There results a commutative diagram.

$$\begin{tikzcd} \pi_2(X;Z/\ell^\nu) \arrow[rr, "H_J"] \arrow[dr, "H_K"'] & & j_0(X;Z/\ell^\nu) \arrow[dl, "\mu"] \\ & KU_0(X;Z/\ell^\nu) & \end{tikzcd} \tag{2.14}$$

From (2.14) we see at once

2.15: Proposition. With the above notation Theorem 2.2 remains true when H_K is replaced by H_J.

§3: Bott periodic algebraic K-theory and the J-theory Hurewicz diagrams

In this section I give two diagrams which involve the localisation map

$$\rho : K_i(A;Z/\ell^\nu) \to \mathcal{K}_i(A;Z/\ell^\nu)$$

and the Hurewicz maps, H_K or H_J, of §2.

The first of these diagrams, which was established in [Sn2,§IV.3] and improved in [Sn4], is the one which led to this entire programme. In order to state the result we will need some preliminary conventions.

3.1: Let ℓ be any prime and let ν be a positive integer (if $\ell = 2$ we assume $\nu \geq 2$). Suppose that A is a commutative ring whose units have a chosen (and thereafter fixed) decomposition $A^* \cong Z/\ell^a \times H$ $(\nu \leq a \leq \infty)$. For example, if A is a ring of algebraic integers in a number field, the limit of such, a field or if $a = \infty$ then A^* decomposes in this manner. In the result I am about to state it is probable that one can weaken this condition to require only a central copy of Z/ℓ^a in A^*.

Let $\beta \in \pi_2(BZ/\ell^a;Z/\ell^\nu) \cong Z/\ell^\nu$ be a generator. The inclusion of $A^* = GL_1A$ into GLA induces

$$\gamma : \pi_2(BZ/\ell^a;Z/\ell^\nu) \to K_2(A;Z/\ell^\nu) = \pi_2(BGLA^+;Z/\ell^\nu)$$

and we will write β also for the image of β under γ. We may define Bott periodic K-theory (following [Sn1,ChIV;Sn2]) by inverting (left) multiplication by β

$$K_i(A;Z/\ell^\nu) = K_i(A;Z/\ell^\nu)[1/\beta]. \tag{3.2}$$

If the factor, Z/ℓ^a, in A^* is generated by the root of unity, ξ_{ℓ^a}, in a $Z[\xi_{\ell^a}]$-algebra or a $Z[1/\ell,\xi_{\ell^a}]$-algebra then (3.2) coincides with the definition of

$$K_i(A;Z/\ell^\nu) = K_i(A;Z/\ell^\nu)[1/\beta_\nu] \tag{3.3}$$

which applies to $Z[1/\ell]$-algebras (see below) for which we invert β_ν with $|\beta_\nu| = 2\ell^{\nu-1}(\ell-1)$. The equivalence of (3.2) and (3.3) is explained in [D-F-S-T].

Next we establish our conventions about the identity of h_*, the homology which we use when we are considering a ring whose units have the above form. When $a < \infty$ choose a prime t with $\nu_\ell(t-1) = a$ and set $h_*(\_) = j_*(\_;Z/\ell^\nu)$, the J-theory which appears in (2.13) for this choice of t. When $a = \infty$ we set $h_*(\_) = KU_*(\_;Z/\ell^\nu)$. Note that we have suppressed in our notation the dependence of h_* upon ν, since the context will make this dependence clear.

Let $BSLA^+$ denote the fibre of the determinant map $BGLA^+ \to BA^*$.

3.4: Theorem. [Sn2,§IV.3;Sn4] Let A be a commutative ring with A^*, ℓ, ν and h_* as in §3.1.

There is a commutative diagram for $i \geq 2$

$$\begin{array}{ccc}
\pi_i(BSLA^+ \times BH;Z/\ell^\nu) & \xrightarrow{H} & h_i(BSLA^+ \times BH) \\
\Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle J} \\
K_i(A;Z/\ell^\nu) & \xrightarrow[\rho]{} & K_i(A;Z/\ell^\nu)
\end{array}$$

Here H denotes the appropriate Hurewicz map, H_K or H_J, as in §2.

3.5: Theorem 3.4 in full generality is proved by the use of localised stable homotopy to describe $h_*(BSLA^+ \times BH)$. However there exists a very simple case of Theorem 3.4 which I will describe. Let $\overline{\mathbb{F}}_q$ denote the algebraic closure of $\mathbb{F}_q$, the finite field with q elements and assume

$(q,\ell) = 1$. Let A be an $\overline{\mathbb{F}}_q$-algebra then we have an associative pairing

$$m : (BGLA^+) \wedge (BGL\overline{\mathbb{F}}_q^+) \to BGLA^+ .$$

Since, at the prime ℓ, $BU \simeq BGL\overline{\mathbb{F}}_q^+$ by [Q2], we have an isomorphism

$$KU_i(BGLA^+;Z/\ell^\nu) \cong \varinjlim_n \pi_{i+2n}((BGLA^+) \wedge (BGL\overline{\mathbb{F}}_q^+);Z/\ell^\nu)$$

where the limit is taken over successive multiplications by $\beta \in \pi_2(BGL\overline{\mathbb{F}}_q;Z/\ell^\nu)$,

using the pairing $(BGL\overline{\mathbb{F}}_q^+) \wedge (BGL\overline{\mathbb{F}}_q^+) \to BGL\overline{\mathbb{F}}_q^+$. The map, m, therefore, induces a map, J, (which is clearly onto) from $KU_i(BGLA^+;Z/\ell^\nu)$ to $\varinjlim_n \pi_{i+2n}(BGLA^+;Z/\ell^\nu) = K_i(A;Z/\ell^\nu)$. In this example ρ clearly equals $H_K J$.

3.6: Throughout the remainder of this section ℓ will denote an odd prime. I will now give a different description of $K_*(A:Z/\ell^\nu)$, suggested to me during a conversation with Bill Dwyer, and I will use this description to obtain a diagram which is the "converse" of §3.4.

For $1 \leq n \leq \infty$ let Σ_n denote the symmetric group and let $\Sigma_n \int (Z/\ell)$ denote the wreath product given by the semi-direct product of Σ_n with the n-fold product of the cyclic group of order ℓ. Let $\xi_\ell = \exp(2\pi i/\ell)$ and let GL_nA denote the general linear group of $n\times n$ invertible matrices with entries in A.

Consider the following diagram of homomorphisms.

$$\begin{tikzcd}
\Sigma_n \int (Z/\ell) \arrow[rr, "d_1"] \arrow[d, "i"'] & & GL_n Z[\xi_\ell] \arrow[d, "\tau"] \\
\Sigma_{n\ell} \arrow[d, "d_2"'] & & GL_{n(\ell-1)} Z \arrow[d, "s"] \\
GL_{n\ell} Z \arrow[dr, "\varepsilon"'] & & GL_{n\ell} Z \arrow[dl, "\varepsilon"] \\
& GL_{n\ell} Z[1/\ell] &
\end{tikzcd} \tag{3.7}$$

In (3.7) i is induced by sending a generator of Z/ℓ to the ℓ-cycle, $(1,\ldots,\ell)$; d_1 and d_2 are induced by sending $\sigma \in \Sigma_n$ to the corresponding permutation matrix and a generator of Z/ℓ to ξ_ℓ; τ is induced by considering $Z[\xi_\ell]$ as the free abelian group on $1,\xi,\xi^2,\ldots,\xi^{\ell-2}$; s is the stabilisation map sending M to $\begin{pmatrix} M & \\ & I_\ell \end{pmatrix}$ and ε is induced by the localisation map.

3.8: Lemma. (3.7) commutes up to inner automorphism.

Proof: Firstly s differs by an inner (permutation) automorphism from s', the stabilisation map which inserts the 1's at the $k\ell$-th diagonal place $(1 \leq k \leq n)$ rather than at the $(n(\ell-1)+k)$-th place.

Choose $1,\xi_\ell,\ldots,\xi_\ell^{\ell-2}$ as a basis for $Z[\xi_\ell]$ as a Z-module then, if $g \in Z/\ell$ is the chosen generator and $\sigma \in \Sigma_n$,

$$\varepsilon s'\tau d_1(\sigma(g^{j_1},g^{j_2},\ldots,g^{j_n})) = d_2(\sigma(T^{j_1},T^{j_2},\ldots,T^{j_n})) \qquad (3.9)$$

where

$$T = \begin{pmatrix} 0 & & & & & & -1 & 0 \\ 1 & 0 & & & & & -1 & 0 \\ & 1 & 0 & & & & & \\ & & & \cdot & & & \cdot & \cdot \\ & & & & \cdot & & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot \\ & & & & & 0 & -1 & \\ & & & & & 1 & -1 & 0 \\ & & & & & & 0 & 1 \end{pmatrix} \in GL_\ell Z$$

With respect to the basis of Q^ℓ given by

$$\begin{aligned} \underline{v}_1 &= (1,1,\ldots,1), \\ \underline{v}_2 &= (1,-1,0,\ldots), \\ &\vdots \\ \underline{v}_\ell &= (0,0,\ldots,0,1,-1) \end{aligned}$$

the ℓ-cycle also has matrix T. Since $\det(\underline{v}_1|\underline{v}_2|\ldots|\underline{v}_\ell) = (-1)^{\ell-1}\ell$ the $\{\underline{v}_i\}$ form a basis of $(Z[1/\ell])^\ell$. Hence (3.9) implies that $\varepsilon d_2 i$ equals $\varepsilon s'\tau d_1$ up to an inner automorphism.

Since inner automorphisms induce the identity map on BG, up to homotopy, we obtain the following result.

3.10: Corollary. There is a homotopy commutative diagram of natural maps induced by (3.7).

$$\begin{array}{ccc} B\Sigma_\infty \int (Z/\ell)^+ & \xrightarrow{d_1} & BGLZ[1/\ell,\xi_\ell]^+ \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \tau} \\ B\Sigma_\infty{}^+ & \xrightarrow{d_2} & BGLZ[1/\ell]^+ \end{array}$$

Recall [H-S] that there are homotopy equivalences (of infinite loop spaces [Ma2])

(3.11) $$B\Sigma_\infty^+ \simeq Q_0 S^0, \quad B\Sigma_\infty \int (Z/\ell)^+ \simeq Q_0((BZ/\ell)_+)$$

where $QX = \lim\limits_{\vec{n}} \Omega^n\Sigma^n X$, $Q_0 X$ is the base-point component and $Y_+ = Y \cup (pt)$, the disjoint union of Y with a base-point.

Let $b_1 \in \pi_2^S(BZ/\ell;Z/\ell) \cong Z/\ell$ (ℓ is an odd prime) be a generator. Since, by (3.11) $\pi_2^S((BZ/\ell)_+;Z/\ell) \cong \pi_2(B\Sigma_\infty\int(Z/\ell)^+;Z/\ell)$, we may also denote by b_1 the image of this element in $\pi_2(B\Sigma_\infty\int(Z/\ell)^+;Z/\ell)$. Using the product induced by that on BZ/ℓ we may form

$$b_1^{\ell-1} \in \pi^S_{2(\ell-1)}((BZ/\ell)_+;Z/\ell) \cong \pi_{2(\ell-1)}(B\Sigma_\infty\int(Z/\ell)^+;Z/\ell).$$

If $e_i \in H_i(BZ/\ell;Z/\ell)$ is a generator the Hurewicz images of b_1 and $b_1^{\ell-1}$ are e_2 and $e_{2(\ell-1)}$ respectively in $H_*(B\Sigma_\infty\int(Z/\ell)^+;Z/\ell) \cong H_*(B\Sigma_\infty\int Z/\ell;Z/\ell)$. Thus (see [K-P], for example)

$$i(b_1^{\ell-1}) \in \pi_{2(\ell-1)}(B\Sigma_\infty^+;Z/\ell) \cong \pi^S_{2(\ell-1)}(S^0;Z/\ell) \cong Z/\ell$$

has non-zero Hurewicz image and is the generator of the image of the mod ℓ J-homomorphism [A,2]. The Bockstein of $i(b_1^{\ell-1})$ generates the (integral) ℓ-primary image of J in $\pi^S_{2\ell-3}(S^0)$.

Define $\beta_1 \in K_{2(\ell-1)}(Z[1/\ell];Z/\ell)$ by

$$\beta_1 = (d_2)_\#(i_\#(b_1^{\ell-1})) \in \pi_{2(\ell-1)}(BGLZ[1/\ell]^+;Z/\ell). \qquad (3.12)$$

Let $j : Z[1/\ell] \to Z[1/\ell,\xi_\ell]$ denote the natural map.

3.13: Lemma. With the notation introduced above

$$j_\#(\beta_1) = (\ell-1)((d_1)_\#(b_1))^{\ell-1} \in K_{2(\ell-1)}(Z[1/\ell,\xi_\ell];Z/\ell).$$

Proof: By [B,§2.8] $j_{\#}\tau_{\#}(x) = \sum_g g_{\#}(x)$ where g runs through $\mathrm{Gal}(Q(\xi_\ell)/Q) \cong (Z/\ell)^*$. Hence

$$\begin{aligned} j_*(\beta_1) &= j_{\#}(d_2)_{\#} i_{\#}(b_1^{\ell-1}) \\ &= j_{\#}\tau_{\#}((d_1)_{\#}(b_1))^{\ell-1}, \text{ by §3.10,} \\ &= \sum_g g_{\#}((d_1)_{\#}(b_1))^{\ell-1} \\ &= (\ell-1)((d_1)_{\#}(b_1))^{\ell-1} \end{aligned}$$

since $((d_1)_{\#}(b_1))^{\ell-1}$ is Galois invariant.

3.14: Bott elements. In [D-F-S-T] the Bott element $\beta_\nu \in K_{2\ell^{\nu-1}(\ell-1)}(Z[1/\ell];Z/\ell^\nu)$ is characterised as follows. β_1 is required to satisfy $j_{\#}(\beta_1) = b_1^{\ell-1}$. Hence §3.13 implies that (up to units) a choice for β_1 may be defined explicitly by (3.12). For $\nu \geq 2$, β_ν may be chosen to be any element whose mod ℓ reduction is $\beta_1^{\ell^{\nu-1}}$.

The map, d_2, of §3.10 is the "discrete model" (via the equivalences of (3.11)) for the base-point component of the infinite loop map [Ma2]

$$d : QS^0 \to Z \times BGLZ[1/\ell]^+$$

determined by sending the non-base-point of S^0 into the 1-component. The infinite loop space QS^0 corresponds to the suspension spectrum $\Sigma^\infty S^0$ [A;Ma1;Ma2] while $Z \times BGLZ[1/\ell]^+$ corresponds to the K-theory spectrum $\underline{KZ[1/\ell]}$. The infinite loop map, d, corresponds to the unit of the K-theory spectrum - that is, to the S-map

$$D : \Sigma^\infty S^0 \to \underline{KZ[1/\ell]}. \tag{3.15}$$

In the stable homotopy category [A]

$$\pi_i(\Sigma^\infty S^0;Z/\ell) \cong \pi_i^S(S^0;Z/\ell)$$

and

$$\pi_i(\underline{KZ[1/\ell]};Z/\ell) \cong K_i(Z[1/\ell];Z/\ell).$$

Thus the foregoing discussion has shown that β_1 may be taken to be the image under D of (3.15) of an element in the image of the J-homomorphism, $i_{\#}(b_1^{\ell-1})$. We now proceed to give a similar representation of $\beta_\nu \in \pi_{2\ell^{\nu-1}(\ell-1)}(\underline{KZ[1/\ell]};Z/\ell^\nu)$, for which purpose we must recall how, in stable homotopy, $i_{\#}(b_1^{\ell-1})$ is constructed. Write $a_1 = i_{\#}(b_1^{\ell-1})$ and $P^n(s) = S^{n-1} \cup_{\ell^s} e^n$.

For sufficiently large q we have a commutative diagram [A2,IV§12].

$$\begin{tikzcd} P^{q+2(\ell-1)}(1) \arrow[r, "A_1"] \arrow[dr, "a_1"] & P^q(1) \arrow[d, "j"] \\ S^{q+2\ell-3} \arrow[u, "i"] \arrow[r, "\alpha_1"'] & S^q \end{tikzcd} \tag{3.16}$$

In (3.16) i, j are the canonical inclusion and collapse respectively. The map, A_1, is a K-theory (and hence J-theory) isomorphism and consequently [A2,IV§12.3] the e-invariant of α_1 is $1/\ell$. In (3.16)

$$j_* : \widetilde{KU}_q(P^q(1);Z/\ell) \to \widetilde{KU}_q(S^q;Z/\ell)$$

is an isomorphism.

Suppose in (3.16) q = 2s (for convenience of exposition). A representative of $a_1^{\ell^{\nu-1}}$ is formed as a composite of the form

$$\begin{tikzcd} P^{2\ell^{\nu-1}(s+\ell-1)}(1) \arrow[r, "\chi"] \arrow[dr, "a_1^{\ell^{\nu-1}}"'] & \overset{\ell^{\nu-1}}{\wedge} P^{2s+2(\ell-1)}(1) \arrow[d, "\overset{\ell^{\nu-1}}{\wedge} a_1"] \\ & S^{2s\ell^{\nu-1}} \end{tikzcd}$$

where χ is the Moore space pairing of [B,§1.4]. In particular χ_* sends a generator of

$$KU_0(P^{2\ell^{\nu-1}(s+\ell-1)}(1);Z/\ell)$$

to the $\ell^{\nu-1}$-fold tensor product of the generator of $KU_0(P^{2s+2(\ell-1)}(1);Z/\ell)$. Hence $a_1^{\ell^{\nu-1}}$ is an isomorphism on $KU_0(\_;Z/\ell)$. If s is large enough we may fill in the following homotopy commutative diagram ($f = s\ell^{\nu-1}$)

$$
\begin{tikzcd}
P^{2f+2\ell^{\nu-1}(\ell-1)}(\nu) \arrow[r, "A_\nu"] \arrow[dr, "a_\nu"] & P^{2f}(\nu) \arrow[dd, "j"] \\
P^{2f+2\ell^{\nu-1}(\ell-1)}(1) \arrow[u, "i_1"] \arrow[dr, "a_1^{\ell^{\nu-1}}"'] & \\
S^{2f-1+2\ell^{\nu-1}(\ell-1)} \arrow[u, "i"] \arrow[r, "\alpha_\nu"'] \arrow[uu, bend left, "i"] & S^{2f}
\end{tikzcd}
\tag{3.17}
$$

In (3.17) $(i_1)_*$ is a $KU_0(\_;Z/\ell)$ isomorphism and therefore so is $(A_\nu)_*$. By construction $a_1^{\ell^{\nu-1}}$ is the mod ℓ reduction of $a_\nu \in \pi^S_{2\ell^{\nu-1}(\ell-1)}(S^0;Z/\ell^\nu)$.

Therefore, since D is a map of ring spectra [A;Ma2] we may choose β_ν to be given, for all $\nu \geq 1$, by

(3.18) $$\beta_\nu = D_\#(a_\nu) \in \pi_{2\ell^{\nu-1}(\ell-1)}(\underline{KZ[1/\ell]};Z/\ell^\nu).$$

<u>3.19</u>: Let A be a commutative $Z[1/\ell]$-algebra. Let $\underline{P}(\nu)$ denote the Moore spectrum $\Sigma^{\infty-2}P^2(\nu)$ and let $\underline{S}$ denote $\Sigma^\infty S^0$. In addition write $\underline{K}$ for the algebraic K-theory spectrum of A. Suppose that $g : \underline{P}(\nu) \to \underline{K}$ is a map of spectra which represents

$$[g] \in \pi_i(\underline{K};Z/\ell^\nu) \cong K_i(A;Z/\ell^\nu).$$

We have the following commutative diagram of spectra

$$
\begin{tikzcd}
\underline{P}(\nu) \arrow[r, "\chi"] \arrow[dd, "A'_\nu"'] & \underline{P}(\nu) \wedge \underline{P}(\nu) \arrow[r, "\beta_\nu \wedge g"] \arrow[d, "A_\nu \wedge 1"] \arrow[dr, "a_\nu \wedge g"] & \underline{KZ[1/\ell]} \wedge \underline{K} \arrow[r, "\mathrm{mult}"] & \underline{K} \\
 & \underline{P}(\nu) \wedge \underline{P}(\nu) \arrow[r, "j \wedge g"] \arrow[d, "j \wedge 1"] & \underline{S} \wedge \underline{K} \arrow[u, "D \wedge 1"] \arrow[ur, "\cong"'] & \\
\underline{P}(\nu) \arrow[r, "="] & \underline{S} \wedge \underline{P}(\nu) \arrow[ur, "1 \wedge g"'] & &
\end{tikzcd}
\tag{3.20}
$$

In (3.20) χ is the Moore space pairing so that the top row represents the product $\beta_\nu[g] \in K_{i+2\ell^{\nu-1}(\ell-1)}(A;Z/\ell^\nu)$. The middle of (3.20) commutes by (3.17) while the right hand triangle commutes because $\underline{K}$ is a

$\underline{KZ[1/\ell]}$-module spectrum. The left hand rectangle defines A'_ν. In fact A'_ν equals A_ν, by commutativity of χ and its S-dual, $j \wedge 1$. However, all we need to know is that A'_ν is one of Adams maps between Moore spaces - that is, it induces a KU-theory isomorphism - which is obvious. When $\nu = 1$ this actually shows $A'_1 = A_1$ as they both induce the same $KU_*(\_;Z/\ell)$-map, which happens in this case to characterise, A_1.

Hence we have shown that

$$A'^*_\nu[g] = \beta_\nu[g] \in K_{i+2\ell^{\nu-1}(\ell-1)}(A;Z/\ell^\nu). \tag{3.21}$$

From (3.21) we can give the following description of $K_*(A;Z/\ell^\nu)$.

$\underline{\text{3.22: Theorem}}$. Let ℓ be an odd prime and suppose there exists a map of Moore spaces $\hat{A}_\nu : P^{s+2\ell^{\nu-1}(\ell-1)}(\nu) \to P^s(\nu)$ such that the stable homotopy class of $\hat{A}_\nu$ is A'_ν of (3.20). (When $\nu = 1$ we may take $s = 3$ [CMN].) Let $d = 2\ell^{\nu-1}(\ell-1)$ and suppose $i \geq s$ then $K_i(A;Z/\ell^\nu) = K_i(A;Z/\ell\ )[1/\beta_\nu]$, for a $Z[1/\ell]$-algebra, A, is isomorphic to the direct limit of

$$K_i(A;Z/\ell^\nu) \xrightarrow{(\Sigma^{i-s}\hat{A}_\nu)^*} K_{i+d}(A;Z/\ell^\nu) \xrightarrow{(\Sigma^{i+d-s}\hat{A}_\nu)^*} \cdots .$$

$\underline{\text{Proof}}$: By (3.21), $K_*(A;Z/\ell^\nu)$ may be defined as the limit over iterations of $(A'_\nu)^*$ on the mod ℓ^ν homotopy of the K-theory spectrum of A. However the isomorphism

$$[P^i(\nu),BGLA^+] \cong \pi_i(BGLA^+;Z/\ell^\nu) \cong \pi_i(\underline{K};Z/\ell^\nu)$$

identifies $(\hat{A}'_\nu)^*$ with $(\Sigma^{i-s}\hat{A}_\nu)^*$, provided that $i \geq s$.

$\underline{3.23}$: Let h_* denote either $KU_*(\_;Z/\ell^\nu)$ or a J-theory, $j_*(\_;Z/\ell^\nu)$, which is given by the fibre of ψ^t-1, as in (2.13), with $\nu_\ell(t-1) \geq \nu$. For $n \geq 2$, $P^n(\nu) = S^{n-1} \cup_{\ell^\nu} e^n$ and $h_i(P^n(\nu)) \cong Z/\ell^\nu$ for each i (mod 2). Since $\hat{A}_\nu$ induces KU_*-isomorphisms it induces an isomorphism

$$(\hat{A}_\nu)_* : h_*(P^{s+2\ell^{\nu-1}(\ell-1)}(\nu)) \xrightarrow{\cong} h_*(P^s(\nu)).$$

Hence, up to multiplication by units, we have

$$H((\Sigma^{i-s}\hat{A}_{\nu})_*(x)) = H(x) \in h_i(BGLA^+)$$

where $x \in \pi_i(BGLA^+;Z/\ell^{\nu})$ and H is the h_*-theory Hurewicz map. Consequently we obtain the following result in which $I = \varinjlim H$.

3.24: Theorem. Let ℓ be an odd prime and let A be a $Z[1/\ell]$-algebra. Let s be as in §3.22 (so s = 3 when ν = 1). Then there is a commutative diagram as follows, provided that $i \geq s$.

$$\begin{tikzcd} K_i(A;Z/\ell^{\nu}) \arrow[rr, "\rho"] \arrow[dr, "H"'] & & K_i(A;Z/\ell^{\nu}) \arrow[dl, "I"] \\ & h_i(BGLA^+) & \end{tikzcd}$$

3.25: Corollary. Let A be a commutative ring in which ℓ is invertible. Suppose that A^*, h_*, ℓ, ν and H are as in §§3.1, 3.4 and 3.24. Then (a) I is injective in §3.24; (b) ker ρ = ker H in $K_i(A;Z/\ell^{\nu})$ with $i \geq s$ (≥ 3)

Proof: To prove (b) we observe in §3.4 that $\pi_i(BSLA^+ \times BH;Z/\ell^{\nu})$ equals $K_i(A;Z/\ell^{\nu})$ if $i \geq 3$ since $BGLA^+ \simeq BSLA^+ \times BH \times BZ/\ell^{\nu}$. Hence §3.4 and §3.24 combine to show ker ρ = ker H.

To prove (a) we observe that the groups $\{\rho(K_{i+wd}(A;Z/\ell^{\nu})\,|\,w \geq 0,$ $d = 2\ell^{\nu-1}(\ell-1)\}$ generate $K_i(A;Z/\ell^{\nu})$. If $x \in K_{i+wd}(A;Z/\ell^{\nu})$ represents an element in ker I then $0 = I\rho(x) = H(x)$ so, by (b), $\rho(x) = 0$ and x represents the zero class.

§4: The J-theory Hurewicz diagram in dimension two

In this section ℓ will be an odd prime. We will construct a diagram of the following form, analogous to that of §3.24. Let $h_*(BGLA^+)$ be as in §3.24, it is an algebra under a product induced by direct sum of matrices.

4.1: Theorem. Let ℓ be an odd prime and let A be a commutative $Z[1/\ell]$-algebra satisfying the conditions of §3.25. Let H denote the h_*-theory Hurewicz map and set $\hat{H}(x) = H(x) - H(x)^{\ell}$. Then there is a commutative diagram

$$\begin{tikzcd} K_2(A;Z/\ell) \arrow[rr, "\rho"] \arrow[dr, "\hat{H}"'] & & K_2(A;Z/\ell) \arrow[dl, "I"] \\ & h_0(BGLA^+) & \end{tikzcd}$$

4.2: This result will be proved in a series of steps, sketched in [Sn 5, §2.12]

Recall first that if $P^n(1) = S^{n-1} \underset{\ell}{\cup} e^n$ then $\widetilde{KU}_\alpha(P^n(1);Z/\ell) \cong Z/\ell$ for each α(mod 2) with generators u_n, v_n having $\deg(u_n) \equiv n$(mod 2) and $B(u_n) = v_n$, where B is the Bockstein. Also the mod ℓ Atiyah-Hirzebruch homology spectral sequences collapses for $P^n(1)$ and therefore for $\Omega P^3(1)$ since

$$\Sigma\Omega P^3(1) = \Sigma\Omega\Sigma P^2(1) \simeq \bigvee_{k=1}^{\infty} \Sigma(\bigwedge_1^k P^2(1))$$

and
$$P^a(1) \wedge P^b(1) \simeq P^{a+b}(1) \vee P^{a+b-1}(1) . \qquad [B]$$

Therefore we have an isomorphism

$$KU_*(\Omega P^3(1);Z/\ell) \cong \{\text{free algebra on } u_2, v_2\}$$

since $H_*(\Omega P^3(1);Z/\ell)$ is the free algebra on two generators which represent u_2 and v_2.

Since ψ^t acts like the identity on u_2 and v_2 (as $\nu_\ell(t-1) \geq 1$) the exact sequence of §2.13 yields

$$j_*(\Omega P^3(1);Z/\ell) \cong E(w) \otimes KU_*(\Omega P^3(1);Z/\ell)$$

where w generates $j_1(pt;Z/\ell)$.

4.3: Lemma. Let $A_1 : P^{2\ell+1}(1) \to P^3(1)$ denote Adams' map, as in §3.16. Let $\tilde{A}_1 : P^{2\ell}(1) \to \Omega P^3(1)$ be the adjoint of A_1. Then in $j_0(\Omega P^3(1);Z/\ell)$ or $KU_0(\Omega P^3(1);Z/\ell)$

$$(\tilde{A}_1)_*(u_{2\ell}) = u_2 - u_2^\ell + c$$

where c is a primitive which is a linear combination of commutators.

Proof: By the discussion of §4.2 it suffices to treat the case of mod K-theory. As pointed out in [N] $(\tilde{A}_1)_*(u_{2\ell})$ will be primitive. Furthermore in the Atiyah-Hirzebruch spectral sequence it will be represented as follows. Let $\bar{u}, \bar{v} \in \tilde{H}_*(P^2(1);Z/\ell)$ represent u_2 and v_2 respectively in the spectral sequence. The only primitives in $H_{2\ell}(\Omega P^3(1);Z/\ell)$ are of the form $\lambda\bar{u}^\ell + \bar{c}$ where $\bar{c}$ is a linear combination of iterated commutators of $\bar{u}$ and $\bar{v}$. In addition $\lambda \not\equiv 0 \pmod{\ell}$ since the composite

$$\alpha : P^{2\ell}(1) \xrightarrow{\tilde{A}_1} \Omega\Sigma P^2(1) \xrightarrow{\Sigma} QP^2(1) \qquad (QX = \varinjlim_n \Omega^n\Sigma^n X)$$

represents a non-zero "image of J" element in $\pi^S_{2\ell}(P^2(1);Z/\ell)$ so that

$$0 \neq \alpha_*(\bar{u}_{2\ell}) = \Sigma_*(\lambda\bar{u}^\ell + \bar{c}) = \lambda\Sigma_*(\bar{u})^\ell .$$

By choosing $\tilde{A}_1$ correctly we may assume $\lambda \equiv -1 \pmod{\ell}$. Now consider the suspension of A_1 from $P^{2\ell+1}(1)$ to $\Sigma\Omega P^3(1)$, which is a wedge of Moore spaces. Since $\pi^S_j(P^2(1);Z/\ell) = 0$ for $2 < j < 2\ell-1$ an inspection of the dimensions of the wedge summands in $\Sigma\Omega P^3(1)$ shows that

$$(\tilde{A}_1)_*(u_{2\ell}) = \mu u_2 - u_2^\ell + c$$

where c is a linear combination of commutators. Since $(A_1)_*$ is an isomorphism on K-theory we see $\mu \not\equiv 0 \pmod{\ell}$. In fact $\mu = 1$, for consider the composite

$$C : P^{2\ell}(1) \xrightarrow{\tilde{A}_1} \Omega\Sigma P^2(1) \xrightarrow{\Omega\Sigma g} \Omega\Sigma BZ/\ell \xrightarrow{D} BZ/\ell .$$

Here g generates $\pi_2(BZ/\ell;Z/\ell)$ and D is the retraction (an H-map) coming from the H-space structure of BZ/ℓ. Hence in $KU_0(BZ/\ell;Z/\ell)$

$$\begin{aligned} C_*(u_{2\ell}) &= D_*(\Omega\Sigma g)(\mu u_2 - u_2^\ell + c) \\ &= D_*(\Omega\Sigma g)_*(\mu u_2 - u_2^\ell) \\ &= \mu g_*(u_2) - g_*(u_2)^\ell \\ &= (\mu - 1)g_*(u_2) \end{aligned}$$

since $g_*(u_2) = g_*(u_2)^\ell$. But $\pi_{2\ell}(BZ/\ell;Z/\ell) = 0$ so $C_*(u_{2\ell}) = 0$ and $\mu \equiv 1 \pmod{\ell}$.

<u>4.4</u>: Define $\sigma : K_2(A;Z/\ell) \to K_{2\ell}(A;Z/\ell)$ by sending $f : P^2(1) \to BGLA^+$ to the composite

$$\sigma(f) : P^{2\ell}(1) \xrightarrow{\tilde{A}_1} \Omega\Sigma P^2(1) \xrightarrow{\Omega\Sigma f} \Omega\Sigma BGLA^+ \xrightarrow{D} BGLA^+$$

where $\tilde{A}_1$ is as in §4.3 and D comes from the direct sum loop space structure on $BGLA^+$.

<u>4.5: Proof of Theorem 4.1</u>. Recall that for A as in §3.25 we have Bott periodicity so that we may identify $K_n(A;Z/\ell)$ and $K_{n+2}(A;Z/\ell)$.

As in the proof of §4.3 we see that $H(\sigma(f)) = \hat{H}(f)$ so from Theorem

3.24 we obtain an equation $I\rho'(x) = \hat{H}(x)$ for $x \in K_2(A;Z/\ell)$. It remains to show that $\rho'(x) = \rho(x)$. For this we use §3.25 which accounts for the accumulation of conditions on A. Since $I : \mathcal{K}_{2m}(A;Z/\ell) \to h_{2m}(BGLA^+)$ is injective then, by Bott periodicity, $I : \mathcal{K}_2(A;Z/\ell) \to h_2(BGLA^+) \cong h_0(BGLA^+)$ is also injective.

Furthermore, we have an injection (induced by multiplication by v)

$$v_\# : K_i(A;Z/\ell) \rightarrowtail K_{i+1}(A[v,v^{-1}];Z/\ell)$$

and similarly for $\mathcal{K}_*(\_;Z/\ell)$. Hence $\rho'(x) = \rho(x)$ if and only if $Iv_\#(\rho(x)) = Iv_\#(\rho'(x))$. However

$$\begin{aligned} Iv_\#\rho'(x) &= v_\# I\rho'(x) \\ &= v_\#(H(x) - H(x)^\ell) \\ &= v_\# H(x) \quad (v_\# \text{ annihilates decomposables}) \\ &= Hv_\#(x) \\ &= I\rho v_\#(x) \quad (\text{as } \deg v_\#(x) \geq 3) \\ &= Iv_\#\rho(x) \text{ , as required.} \end{aligned}$$

§5: Concerning injectivity of the localisation map, ρ

5.1: Throughout this section A will be a regular, commutative ring in which the odd prime, ℓ, is invertible. In addition we will require, as in §3.1, that the units of A have the form $A^* = Z/\ell^a \times H$, $1 \leq a \leq \infty$. Also we will assume that $a \geq 2$ when $\ell = 3$.

In addition X will be a scheme over a noetherian ring S, which has finite Krull dimension. The prime, ℓ, is invertible in S. Also all of the residue fields of X have uniformly bounded êtale cohomological dimension for ℓ-torsion sheaves. If $\ell = 3$, S contains a 9-th root of unity.

Under the above conditions on X we have Thomason's result.

5.2: Theorem. If X is as above ρ, of §1, induces an isomorphism for all $\nu \geq 1$

$$\rho : \mathcal{K}_*(X;Z/\ell^\nu) \to K_*^{et}(X;Z/\ell^\nu).$$

Here $\mathcal{K}_*(X;Z/\ell^\nu) = K_*(X;Z/\ell^\nu)[1/\beta_\nu]$ where β_ν is the Bott element of §3.18. Hence the effect of Theorem 5.2 is to identify Friedlander's map between K_* and K_*^{et} with the localisation map

$$\rho : K_i(X;Z/\ell^\nu) \to \mathcal{K}_i(X;Z/\ell^\nu). \tag{5.3}$$

When $X = \operatorname{Spec} A$ we write $K_*(X;Z/\ell^\nu) = K_*(A;Z/\ell^\nu)$ and similarly for $\mathcal{K}_*$ and K_*^{et}.

5.4: Examples of schemes satisfying the conditions of §5.1 are quite common and include the following.

(a) regular quasiprojective varieties over algebraically closed or finite fields of characteristic not ℓ;

(b) localisations away from ℓ of the algebraic integers, $\mathcal{O}$, in a local or global field, k [D-F-S-T] (char k $\neq \ell$);

(c) the local or global fields, k, in (b);

(d) the integral closure of $\mathcal{O}[\frac{1}{\ell}, \xi_{\ell^\infty}]$ in $k(\xi_{\ell^\infty})$ where $\mathcal{O}$, k are as in (b) and ξ_{ℓ^∞} indicates the adjunction of all ℓ-primary roots of unity;

(e) $k(\xi_{\ell^\infty})$ as in (d);

(f) separably closed fields or strictly local Hensel rings containing $1/\ell$.

5.5: Jouanolou's device [J;Q;FI] Let X be as in §5.1.

There exists an algebraic fibring $v : \hat{X} \to X$ with affine fibres, $\mathbb{A}^N$, in which $\hat{X} = \operatorname{Spec} A$. When $X = \mathbf{P}^m_S$, A^* contains S^* as a direct factor. For $X \subset \mathbf{P}^m_S$ $\hat{X}$ is formed by pulling back the bundle, $\hat{\mathbf{P}}^m_S$. Hence, if $a = \infty$ or if $S^* = Z/\ell^a \times H$ as in §5.1 and if $\pi : X \to S$ has a section then A^* has Z/ℓ^a as a split factor, in the manner of §5.1. In addition, if X satisfies the conditions of §5.1 so does $\hat{X} = \operatorname{Spec} A$.

The above discussion permits us to restrict our study of (5.3) to the cases $X = \operatorname{Spec} A$ where X and A satisfy the conditions of §5.1.

5.6: Dayton-Weibel Theory. We assume, as in §5.1, that ℓ is invertible in A and that A is regular.

IN [We,§3] a Karoubi-Villamayor type of K-theory (with coefficients $\wedge$) is defined. It is denoted $KV_*(A;\wedge)$ where the index ranges through the integers in general but only through non-negative integers for the A which we study. For such A there is a natural isomorphism

$$KV_i(A;\wedge) \cong K_i(A;\wedge) \quad \text{for } \wedge = Z \text{ or } Z/\ell^\nu,$$

where $i \geq 2$ if $\wedge = Z/\ell^\nu$.

For $n \geq 2$ let

$$\dot{\Delta}^n_A = A[x_0,\dots,x_n]/((\Sigma x_i - 1)x_0x_1\cdot\dots\cdot x_n)$$

then, from [D-WI, §§2.4, 4.8 and 5.2;D-WII], there is a natural isomorphism ($\wedge = Z$ or Z/ℓ^ν).

(5.7) $$K_2(\dot{\Delta}^n_A;\wedge) \cong K_2(A;\wedge) \oplus K_{n+1}(A;\wedge)\ .$$

This decomposition commutes with multiplication by β_ν and induces

(5.8) $$\mathcal{K}_2(\dot{\Delta}^n_A;Z/\ell^\nu) \cong \mathcal{K}_2(A;Z/\ell^\nu) \oplus \mathcal{K}_{n+1}(A;Z/\ell^\nu)\ .$$

From the localisation sequence [Q;G-Q]

(5.9)
$$K_2(A[x,x^{-1}];\wedge) \cong K_2(A;\wedge) \oplus K_1(A;\wedge)$$
$$K_2(A[x,y,x^{-1},y^{-1}];\wedge) = K_2(A;\wedge) \oplus (K_1(A;\wedge))^2 \oplus K_0(A;\wedge)$$

and similarly for $\mathcal{K}_*$.

Set $\dot{\Delta}^1_A = A$, $\dot{\Delta}^0_A = A[x,x^{-1}]$ and $\dot{\Delta}^{-1}_A = A[x,y,x^{-1},y^{-1}]$.

<u>5.10: Theorem</u>. Let A and X = Spec A satisfy the conditions of §5.1. Suppose that h_* be, as in §3.1, either $KU_*(\_;Z/\ell)$ or $j_*(\_;Z/\ell)$. Suppose further that the group $\{z \in h_0(BGL(\dot{\Delta}^n_A)^+) \mid z = z^\ell\}$ is zero. Then $\ker(\rho : K_{n+1}(A;Z/\ell) \to \mathcal{K}_{n+1}(A;Z/\ell))$ is contained in the image under reduction mod ℓ of

$$\{w \in K_{n+1}(A) \mid \ell^M w = 0 \text{ for some } M\}.$$

<u>Proof</u>: When n = 1 our assumptions ensure that $\ker \rho = \ker \hat{H} = \ker H$, by §4.1, and the result follows from §§2.2,2.15. When $n \neq 1$ the result follows by application of the argument to $\dot{\Delta}^n_A$.

<u>5.11: Theorem</u>. Let ℓ be an odd prime and suppose A and X = Spec A satisfy the conditions of §5.1. Suppose that $\{z \in h_0(BGL(\dot{\Delta}^n_A)^+) \mid z = z^\ell\}$ is zero, as in §5.10. Assume further that

$$\rho : K_{N+2}(A;Z/\ell^\nu) \to \mathcal{K}_{N+2}(A;Z/\ell^\nu) \cong K^{et}_{N+2}(A;Z/\ell^\nu)$$

is onto for all $\nu \geq 1$. Then

$$\rho : K_{N+1}(A;Z/\ell) \to K^{et}_{N+1}(A;Z/\ell)$$

is injective.

<u>Proof</u>: Write $(K/\ell^\nu)_j$ for $K_j(A;Z/\ell^\nu)$ and similarly $(K^{et}/\ell^\nu)_j$ for $K^{et}_j(A;Z/\ell^\nu)$.

Consider the following commutative diagram.

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & (K/\ell)_{N+2} & \xrightarrow{i} & (K/\ell^{M+1})_{N+2} & \xrightarrow{\lambda} & (K/\ell^{M})_{N+2} & \xrightarrow{\sigma} & (K/\ell)_{N+1} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle\rho_1} & & \downarrow{\scriptstyle\rho_2} & & \downarrow{\scriptstyle\rho_3} & & \downarrow{\scriptstyle\rho_4} & \\
\cdots \longrightarrow & (K^{et}/\ell)_{N+2} & \xrightarrow{i^{et}} & (K^{et}/\ell^{M+1})_{N+2} & \xrightarrow{\lambda^{et}} & (K^{et}/\ell^{M})_{N+2} & \xrightarrow{\sigma^{et}} & (K^{et}/\ell)_{N+1} & \longrightarrow \cdots
\end{array} \tag{5.12}$$

In (5.12) ρ_1, ρ_2, ρ_3 and ρ_4 are examples of the natural map. Suppose $z \in \ker \rho_4$. By Theorem 5.10 there exists $y \in K_{N+1}(A)$ such that $\ell^M y = 0$. Hence there exists $u_M \in (K/\ell^M)_{N+2}$ such that $\sigma(u_M) = z$.

By a diagram chase we will reduce to the case $M = 1$ and finally we will show $z = 0$.

There exists $w \in (K^{et}/\ell^{M+1})_{N+2}$ such that $\lambda^{et}(w) = \rho_3(u_M)$ since $\sigma^{et}\rho_3(u_M) = \rho_4\sigma(u_M) = \rho_4(z) = 0$. By surjectivity of ρ_2 there is $v \in (K/\ell^{M+1})_{N+2}$ with $\rho_2(v) = w$ so that

$$\rho_3(\lambda(v) - u_M) = \lambda^{et}\rho_2(v) - w = 0.$$

Hence, by §5.10, there exists

$$r \in K_{N+2}(A) \text{ with } \ell^T r = 0$$

for some T and $u_{M-1} \in (K/\ell^{M-1})_{N+2}$ such that

$$\lambda(v) - u_M$$

is the sum of the images of u_{M-1} and r.

Since the image of r lies in $\mathrm{im}(\lambda)$, $\sigma(u_{M-1}) = \sigma(\lambda(v) - u_M) = -\sigma(u_M)$. By induction in this manner we are reduced to the case when $M = 1$ in (5.12) and $z = \sigma(u_1)$. Repeating the induction step once more shows $z = 0$, as required.

<u>5.13</u>: Let X be a scheme over S satisfying the conditions of §5.1. Then there exists a spectral sequence [Th1;Th2;FI]

$$E_2^{p,q} = \left\{\begin{array}{ll} H^p_{et}(X;Z/\ell^{\nu}(i)) & \text{if } q = 2i \\ 0 & \text{if } q = 2i-1 \end{array}\right\} \Rightarrow K^{et}_{q-p}(X;Z/\ell^{\nu}).$$

Here $E_2^{p,q}$ is étale cohomology with Tate-twisted coefficients. From the construction of this spectral sequence [FI,§1.5] - from $KU^*(\_;Z/\ell^{\nu})$ Atiyah-Hirzebruch spectral sequences by a delicate limit process - it is clear that the first differential in this spectral sequence is

$$d_{2\ell-1} : E_2^{p,q} \to E_2^{p+2\ell-1,q-2\ell} .$$

For $p > 2 \dim X$, $E_2^{p,q} = 0$ so this spectral sequence often collapses.

This is true, for example, in §5.4(b)-(f) and in the case of smooth, connected curves over an algebraically closed field, as in §5.4(a). In all these cases ($X = \operatorname{Spec} A$ in §5.4(b)-(f))

$$(5.14)\qquad \begin{aligned} K_{2i}^{et}(X;Z/\ell) &= H_{et}^{0}(X;Z/\ell(i)) \oplus H_{et}^{2}(X;Z/\ell(i+1)) \\ K_{2i-1}^{et}(X;Z/\ell) &= H_{et}^{1}(X;Z/\ell(i)) . \end{aligned}$$

Specific examples of (5.14) may be found in [Sn5;Th2].

I will conclude by stating what Theorem 5.11 boils down to in the explicit examples of §5.4.

<u>5.15</u>: Let ξ_ℓ be an ℓ-th root of unity and for those rings A in §5.4(b)-(f) which do not already contain ξ_ℓ let $A(\xi_\ell)$ denote A with ξ_ℓ adjoined.

Hence $A(\xi_\ell) = A$ if A already possesses this root of unity. Let h_* denote $j_*(\_;Z/\ell)$ or $KU_*(\_;Z/\ell)$, chosen for the ring $A(\xi_\ell)$ according to the conventions established in §3.1.

For examples, §5.4(b)-(f), it is known that $K_i(A;Z/\ell^\nu) \to K_i^{et}(A;Z/\ell^\nu)$ is an isomorphism for $i = 1,2$.

<u>5.16: Theorem</u>. Let ℓ be an odd prime and for $n \geq 2$ let $\dot{\Delta}^n_{A(\xi_\ell)}$ be as in §5.6. If

$$\{x \in h_0(BGL(\dot{\Delta}^n_{A(\xi_\ell)})^+) \mid x = x^\ell\} = 0$$

then

$$\rho : K_{n+1}(A;Z/\ell) \to K_{n+1}^{et}(A;Z/\ell)$$

is an isomorphism.

<u>Proof</u>: If ρ is an isomorphism for $A(\xi_\ell)$ then it is also an isomorphism for A, by a transfer argument. Hence, in order to apply Theorem 5.11 it suffices to know that ρ is onto in dimensions greater than one with coefficients mod ℓ^ν. In example §5.4(b) this follows from [D-F], in

example 5.4(d) it follows from (b) by taking limits. In example 5.4(c) it follows from the localisation sequences [Q;G-Q;Sou] for K-theory and étale cohomology and in §5.4(e) by limits of such. In §5.4(f) $K_*^{et}(A;Z/\ell^\nu) = Z/\ell^\nu[\beta]$ where $\beta \in K_2(A;Z/\ell^\nu)$, which completes the proof.

5.17: Theorem. Let ℓ be an odd prime.

If X is a connected, smooth curve over an algebraically closed field. Let A denote the ring provided by Jouanolou's device, in §5.5. Suppose for $n \geq -1$

$$\{x \in KU_0(BGL\dot{\Delta}^n_A;Z/\ell) \mid x = x^\ell\} = 0$$

then

$$K_{n+1}(A;Z/\ell) = \begin{cases} (Z/\ell)^{2g} & \text{if } n \text{ even} \\ (Z/\ell)^2 & \text{if } n = 2m-1 \end{cases} .$$

where g is the genus of X.

Proof: Thomason has shown (see [F II,§3.10]) that ρ is onto in all dimensions with modℓ^ν coefficients for X as above.Hence the result follows as in §5.16 .

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CANADA

THE CANONICAL INVOLUTION ON THE ALGEBRAIC K-THEORY OF SPACES

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It is well known, e.g. [4], [1] that in studying the homotopy type of the stable concordance spaces of manifolds an important technical tool is provided by a certain canonical involution (up to homotopy) on these spaces.

On the other hand much of the information we have about concordance spaces comes from their relation with Waldhausen's functor $A(X)$, the algebraic K-theory of spaces, [8], [9].

There is an involution on $A(X)$ which can be defined as to be rather obviously compatible with the involution on concordance spaces,[9] .

One way to obtain numerical information about $A(X)$ is to compare it with its 'linearization', the algebraic K-theory of (group) rings.

We will show that there is an involution on $A(X)$ linearizing to the standard involution on the K-theory of group rings which is given by associating to a matrix its conjugate transpose inverse.

In the context of the categories used to define $A(X)$ this corresponds to the transition from a simplicial G-set to its (equivariant) Spanier-Whitehead dual.

At this point we are confronted with the problem of having defined two involutions on $A(X)$ which could possibly disagree.

It is proved in this paper that this is not the case:

Theorem: There is an involution up to homotopy on $A(X)$ such that the natural transformations $A(X) \to Wh^{CAT}(X)$ to the (CAT-) Whitehead space, and $A(X) \to K(\mathbb{Z}[\pi_1 X])$ to the K-theory of group rings, are compatible with the involutions, where the involutions on $Wh^{CAT}(X)$ and on $K(\mathbb{Z}[\pi_1 X])$ are the usual ones.

This result has been anticipated in various papers, e.g. [2] [3] [5]. There is no published proof, however.

In particular, the theorem establishes the conjecture in [5 ,p.135] that the rational isomorphism of $\pi_i Wh^{DIFF}(M)$ with $K_i(\mathbb{Z}[G])$ is compatible with the involutions, where M is an orientable spherical space form with fundamental group G.

To prove the theorem we use three approaches to $A(X)$, starting respectively from a certain category of manifolds, $h\mathcal{P}(X)$, from the category of retractive spaces over X and their homotopy equivalences, $h\mathcal{R}(X)$, or, finally, from the category $h\mathcal{U}(G)$ of simplicial sets with an action of the loop group of X.

The category $h\mathcal{P}(X)$ carries a natural involution leading to the standard involution on $Wh^{CAT}(X)$, cf. [9].

We construct categories $h\mathcal{DR}(X)$, $h\mathcal{DU}(G)$, and a commutative diagram

$$\begin{tikzcd} h\mathcal{P}(X) \arrow[r] & h\mathcal{R}(X) \arrow[r] & h\mathcal{U}(G) \\ h\mathcal{P}(X) \arrow[r] \arrow[u, "="] & h\mathcal{DR}(X) \arrow[r] \arrow[u] & h\mathcal{DU}(G) \arrow[u] \end{tikzcd} .$$

The maps in the upper row are homotopy equivalences by [8] [9] .

It is shown (in § 3) that the vertical arrows are homotopy equivalences (after suitable stabilization). Hence the maps in the lower row are also homotopy equivalences (after stabilization).

It is proved that there is an involution on $h\mathcal{DR}(X)$, and on $h\mathcal{DU}(G)$ and that the maps in the lower row of the diagram are compatible with the involutions.

Using the definition of $A(X)$ in terms of the category $h\mathcal{DU}(G)$ it is proved that the natural transformation $A(X) \to K(\mathbb{Z}[\pi_1 X])$ commutes with the involutions.

This will establish the theorem.

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§ 1.

Following [9], we will describe a geometrical model of $A(X)$, the algebraic K-theory of X, and show that this model admits a weak involution T, in the sense that the restriction of T^2 to any compactum is homotopic to the restriction of the identity.

We briefly review the definitions. Let X be a compact manifold of dimension d with boundary ∂X. Let I denote the interval $[a,b]$. A partition is a triple (M, F, N), where M is a compact codimension zero submanifold of $X \times I$, N is the closure of the complement of M, and

$F = M \cap N$. F is to be standard in a neighborhood of $\partial X \times I$, i.e. there exists a number $t \in I$ such that F equals $X \times t$ in this neighborhood.

Let $P(X)$ denote the simplicial set in which a p-simplex is a (CAT-) locally trivial p-parameter family of partitions. A partial ordering on the set of partitions is defined by letting $(M,F,N) < (M',F',N')$ if firstly M is contained in M', and secondly the maps

$$F' \rightarrow M' - (M-F) \leftarrow F$$

are homotopy equivalences. This defines a simplical partially ordered set, and hence a simplicial category which will be denoted $hP(X)$.

We have a particular partition given by attaching k trivial m-handles to $X \times [a,a']$ in such a way that the complementary (d-m)-handles are trivially attached to $X \times [b,b']$, a', $b' \varepsilon [a,b]$. Let $hP_k^{m,n}(X)$ be the connected component of $hP(X)$ containing this particular partition.

An anti-involution on $hP(X)$ is defined by the contravariant functor

$$T' : hP(X) \rightarrow hP(X), \quad (M,F,N) \mapsto (N^*,F^*,M^*),$$

where M^* (resp. N^*) is the image of M (resp. N) under the map $r : X \times I \rightarrow X \times I$, which reflects I at its mid-point. It restricts to a contravariant functor

$$T' : hP_k^{m,n}(X) \rightarrow hP_k^{n,m}(X) .$$

In [8] , it is proved that the categories $hP_k^{m,n}(X)$ approximate $A(X)$. To fit these approximations together one needs a stabilization process. There are two ways to stabilize a partition (M,F,N), namely taking the lower (resp. upper) part to its product with an interval. These disagree because of the condition of standard behaviour near the boundary. We have to consider a technical modification of the various spaces of partitions.

We fix some standard choices. Let $X' \subset \operatorname{Int} X$ be a submanifold of X such that $Cl(X-X')$ is a collar on ∂X. Similarly, let J denote an interval containing two subintervals J', J'' such that $J' \subset \operatorname{Int} J$, $J'' \subset \operatorname{Int} J'$; further let $[a',b']$ be a symmetric subinterval of I.

Let $\underline{P}(X)$ be the simplicial subset of $P(X)$ of those partitions satisfying that

$$F \subset X \times [a',b'] \;; \quad F \cap (X-X') \times I = (X-X') \times a' .$$

The inclusion $\underline{P}(X) \subset P(X)$ (resp. $h\underline{P}(X) \subset hP(X)$) is a homotopy equivalence.

Define the <u>lower stabilization</u> as the map

$$\sigma_1 : h\underline{P}(X) \rightarrow h\underline{P}(X \times J)$$

which takes the lower part of a partition (M,F,N) to

$$MxJ' \cup Xx[a,a']\,xJ \subset XxIxJ \quad .$$

The upper part of the partition is mapped by σ_1 not to a product but to a "fibrewise suspension" of N considered as a space over X.

The <u>upper stabilization</u> is the map

$$\sigma_u : h\underline{P}(X) \to h\underline{P}(X \times J)$$

defined by

$$M \mapsto MxJ' \cup X'x[a,b']\,x\,Cl(J'-J'') \cup Xx[a,a']\,xJ \subset XxIxJ \quad .$$

The involution T' does not restrict to a map $\underline{P}(X) \to \underline{P}(X)$. So a slight modification of T' is necessary, e.g. first shrink the lower part of the partition a little, then take the complement, reflect in the I-direction, and finally remove a certain standard part near the boundary to obtain a partition in $\underline{P}(X)$ again.

<u>Lemma</u> 1.1. This defines a map $T : h\underline{P}(X) \to h\underline{P}(X)$ such that the following diagram commutes up to homotopy

$$\begin{array}{ccc} h\underline{P}(X) & \to & hP(X) \\ T \downarrow & & \downarrow T' \\ h\underline{P}(X) & \to & hP(X) \end{array} \quad .$$

In particular $T^2 \simeq id$. #

We have the following relations:

(i) $\sigma_1 \sigma_u = \sigma_u \sigma_1$, (ii) $\sigma_u T \simeq T \sigma_1$, (iii) $T \sigma_u \simeq \sigma_1 T$.

Using a mapping cylinder argument, T can be defined as a map

$$|\varinjlim_{m,n} h\underline{P}(XxJ^{m+n-d})| \to |\varinjlim_{m,n} h\underline{P}(XxJ^{m+n-d})| \quad ,$$

where the maps in the direct system are given by σ_1, and σ_u , respectively.

<u>Lemma</u> 1.2. T is a weak involution in the sense that the restriction of T^2 to any compactum is homotopic to the restriction of the identity; in particular it induces an involution on π_*. #

In [9] it is proved that a connected component of A(X) can be obtained by performing the +-construction on the space

$$|\varinjlim_{k,m,n} h\underline{P}_k^{m,n}(XxJ^{n+m-d})| \quad .$$

We thus obtain a weak involution on A(X).

Further the homotopy fibre of the inclusion $P_k^{m,n}(X) \to hP_k^{m,n}(X)$ is identified in a dimension range with the h-cobordism space of X.

From the description given there it is clear that the involution on $P^m_k(X)$ induces the canonical involution on the fibre, given by turning an h-cobordism upside down.

§ 2.

In this section we will construct an involution on A(X) in a different way and show that it agrees up to homotopy with the involution defined above. The construction starts from the approach to A(X) which uses as building blocks spaces containing X as a retract, cf. [8]. We will need some modifications, however.

Let $R(X)$ denote the category of retractive spaces over X, i.e. an object is a diagram $Y \underset{s}{\overset{r}{\rightleftarrows}} X$ of simplicial sets such that $rs=id_X$. A morphism is a map $Y \to Y'$ commuting with the retractions and with the sections. An h-equivalence is a morphism in $R(X)$ which is a weak homotopy equivalence. Let $hR(X)$ denote the subcategory of h-equivalences; $R^f(X)$ (resp. $R^{hf}(X)$) is the subcategory of those objects (Y,r,s) satisfying that Y/s(X) is finite (resp. finite up to homotopy).

There is an external pairing ("fibre-wise smash-product")

$$
\begin{aligned}
{}_X\wedge_{X'} : \quad R(X) \times R(X') &\to R(X\times X') \\
(Y,Y') &\mapsto Y\times Y' \cup_{X\times Y' \cup Y\times X'} X\times X'
\end{aligned}
$$

Notation: We will write $Y \wedge Y'$ instead of the more accurate $Y_X\wedge_{X'}Y'$ if the context is clear enough.

A special case of this pairing is the fibre-wise suspension over X, defined as $\Sigma^n_X(Y) = S^n \,{}_*\wedge_X\, Y$.

Let $hR^n_k(X)$ denote the connected component of $hR(X)$ containing the object $X \cup_{\underbrace{\partial D^n \cup \ldots \cup \partial D^n}_{k}} \underbrace{D^n \cup \ldots \cup D^n}_{k} \rightleftarrows X$, with trivial attaching maps.

In [8] the following definition of the algebraic K-theory of X is given.

Definition: $\quad A(X) = \mathbb{Z} \times |\varinjlim_{m,k} hR^m_k(X)|^+ .$

The limit in the m-variable is given by Σ_X .

To get an involution on this functor, we translate the concept of Spanier-Whitehead duality into the framework of retractive spaces.

Let ξ denote an (orientable) spherical fibration over X (having a section) such that the fibre is $\simeq S^d$. Let $Th(\xi)$ denote any space in $\mathcal{R}(X\times X)$ satisfying that $\mathrm{Th}(\xi) \rightleftarrows X^2$ is in the same component of $h\mathcal{R}(X\times X)$ as $X\times X\cup_X \xi \rightleftarrows X\times X$, and further that $\mathrm{Th}(\xi) \to X\times X$ is a fibration. Such a space will be called a Thom space of ξ. Note that $\mathrm{Th}(\xi)/X^2$ is essentially the Thom space of ξ in the usual sense.

Let (Y,r,s) (resp. (Y',r',s')) denote an object of $h\mathcal{R}^{hf}(X)$, satisfying that s_* (resp. s'_*) is an isomorphism on π_i $(i=0,1)$.
An n-duality map is a map in $\mathcal{R}(X\times X)$

$$u : Y \wedge Y' \to \mathrm{Th}_{n-d}(\xi) \overset{\mathrm{def}}{=} \Sigma^{n-d}_{X^2}(\mathrm{Th}(\xi))$$

satisfying that it induces an isomorphism

$$\alpha_u : H_*(Y',X;\ \mathbb{Z}[\pi_1 X]) \to H^{n-*}(Y,X;\ \mathbb{Z}[\pi_1 X])$$

given by $z \mapsto u^*(t)/z$, i.e. slant product with a Thom class t of ξ. By abuse of language we will call Y' an n-dual of Y if there exists such a duality map.

Define a category $\mathcal{DR}^n_{\mathrm{Th}(\xi)}(X)$ in which an object is given by a triple (Y,Y',u), where Y (resp. Y') is an object of $\mathcal{R}^{hf}(X)$, subject to the technical condition that the inclusion of X in Y (resp. Y') induces an isomorphism on π_0 and π_1; and $u\colon Y\wedge Y' \to \mathrm{Th}_{n-d}(\xi)$ is an n-duality map.
A morphism $(Y,Y',u) \to (Z,Z',v)$ is a pair of morphisms in $\mathcal{R}^{hf}(X)$

$$f\colon Y \to Z,\quad f'\colon Z' \to Y'$$

such that the diagram

$$\begin{tikzcd} Y \wedge Z' \arrow[r, "f\wedge id"] \arrow[d, "id\wedge f'"'] & Z\wedge Z' \arrow[d, "v"] \\ Y \wedge Y' \arrow[r, "u"'] & \mathrm{Th}_{n-d}(\xi) \end{tikzcd}$$

commutes.

A morphism (f,f') in $\mathcal{DR}^n_{\mathrm{Th}(\xi)}(X)$ is called an h-equivalence if f and f' are h-equivalences. The subcategory of h-equivalences will be called $h\mathcal{DR}^n_{\mathrm{Th}(\xi)}(X)$. Let $h\mathcal{DR}_{\mathrm{Th}(\xi)} = \coprod_n h\mathcal{DR}^n_{\mathrm{Th}(\xi)}$.

The category $(h)\mathcal{DR}_{\mathrm{Th}(\xi)}(X)$ does not essentially depend on the particular choice of the space $\mathrm{Th}(\xi)$. Namely, suppose $\mathrm{Th}'(\xi)$ is another model for the Thom space of ξ in the sense defined above. The conditions on such a space imply that there is a fibre homotopy equivalence $\mathrm{Th}(\xi) \to \mathrm{Th}'(\xi)$.

Lemma 2.1. A fibre homotopy equivalence $\mathrm{Th}(\xi) \to \mathrm{Th}'(\xi)$ induces a homotopy equivalence $h\mathcal{DR}_{\mathrm{Th}(\xi)}(X) \to h\mathcal{DR}_{\mathrm{Th}'(\xi)}(X)$. #

In view of this lemma we shall use the simplified notation $\mathcal{DR}_\xi(X)$ for any of the categories $\mathcal{DR}_{Th(\xi)}(X)$.

There are two suspension functors on the category $\mathcal{DR}_\xi(X)$, $\Sigma_l: \mathcal{DR}_\xi(X) \to \mathcal{DR}_\xi(X)$ (resp. $\Sigma_r: \mathcal{DR}_\xi(X) \to \mathcal{DR}_\xi(X)$) given by $(Y,Y',u) \mapsto (\Sigma_X(Y),Y',\Sigma(u))$ (resp. $(Y,Y',u) \mapsto (Y,\Sigma_X Y', \Sigma(u))$).

There is a canonical anti-involution on the category $\mathcal{DR}_\xi(X)$. It is given by

$$\begin{aligned} \tau \quad : \quad h\mathcal{DR}_\xi(X) \quad &\to \quad h\mathcal{DR}_\xi(X)^{op} \\ (Y,Y',u) \quad &\mapsto \quad (Y',Y,\bar{u}), \end{aligned}$$

where $\bar{u}: Y'\wedge Y \xrightarrow{\approx} Y\wedge Y' \xrightarrow[u]{} Th_{n-d}(\xi)$. In view of later application we allow for a slight generalization. Assume that ξ comes equipped with a $\mathbb{Z}/2$-action. In this case $\bar{u}$ is defined to be the composite $Y'\wedge Y \to Y\wedge Y' \to Th_{n-d}(\xi) \to Th_{n-d}(\xi)$, where the last arrow is (induced from) the twist on ξ.

Stably the category $\mathcal{DR}_\xi(X)$ does not depend on the spherical fibration ξ . This follows from

Lemma 2.2. There is a homotopy equivalence

$$\varinjlim_{\Sigma_l} h\mathcal{DR}_\varepsilon(X) \quad \to \quad \varinjlim_{\Sigma_l} h\mathcal{DR}_\xi(X) \ ,$$

where $\varepsilon = X\times S^o$ is the trivial fibration. (Similarly with Σ_l replaced by Σ_r .)

Proof: Using a suitable version of the Thom isomorphism one defines a functor $h\mathcal{DR}_\varepsilon(X) \to h\mathcal{DR}_\xi(X)$ by
$(Y,Y',u) \mapsto (\Delta^*(\xi\wedge Y),Y', \bar{u}: \Delta^*(\xi\wedge Y)\wedge Y' \to \Delta_1^*(\xi_X\wedge_{X^2} Th(\varepsilon)) \cong Th(\xi))$, where Δ^* (resp. Δ_1^*) denotes the pullback along the diagonal $X \to X^2$ (resp. $\Delta_1 = \Delta \times id : X^2 \to X^3$). Choosing an inverse for ξ gives a functor in the other direction. It is easy to check that the composite gives an iterated suspension on $h\mathcal{DR}_\varepsilon(X)$ (resp. $h\mathcal{DR}_\xi(X)$). This proves the lemma. #

Proposition 2.3. The forgetful functor

$$\begin{aligned} \delta : \varinjlim_{\Sigma_l,\Sigma_r} h\mathcal{DR}^n_\varepsilon(X) \quad &\to \quad \varinjlim_{\Sigma} hR^{hf}(X) \\ (Y,Y',u) \quad &\mapsto \quad Y \end{aligned}$$

is a homotopy equivalence.

The proof will be given after prop. 3.6. #

By imposing a condition on the homotopy type like in the definition of the categories $hR^m_k(X)$ one defines categories $h\mathcal{DR}^{m,n}_k(X)$. (We drop the spherical fibration from the notation of these categories.)

The map δ restricts to a homotopy equivalence

$$\delta \; : \; \varinjlim_{m,n} \; h\mathcal{DR}_k^{m,n}(X) \; \rightarrow \; \varinjlim_{m} \; h\mathcal{R}_k^{m}(X) \; .$$

Corollary 2.4. $\quad A(X) \;\simeq\; \mathbb{Z} \times \left|\varinjlim_{m,n,k} \; h\mathcal{DR}_k^{m,n}(X)\right|^{+}$ #

The involution is compatible with the suspension functors, i.e. $\tau\Sigma_1 = \Sigma_r\tau$, $\tau\Sigma_r = \Sigma_1\tau$. Hence it defines a contravariant functor

$$\tau \; : \; \varinjlim_{m,n} \; h\mathcal{DR}_k^{m,n}(X) \; \rightarrow \; \varinjlim_{m,n} \; h\mathcal{DR}_k^{m,n}(X)$$

In view of the corollary it therefore defines an involution on $A(X)$.

We next relate the involution defined above in the context of manifolds to this one. To compare the categories $h\underline{P}(X)$ and $h\mathcal{DR}_\xi(X)$ we have to make $\mathcal{DR}_\xi(X)$ into a simplicial category $\mathcal{DR}_\xi(X)$. . This is done in a straightforward way:

Objects of $\mathcal{DR}_\xi(X)_p$ are locally trivial p-parameter families Y, Y' of objects of $\mathcal{R}(X)$, together with a p-parameter family of duality maps

$$\begin{tikzcd}
Y \wedge Y' \arrow[r] \arrow[dr] & Th_{n-d}(\xi) \times \Delta^p \arrow[d] \\
& \Delta^p
\end{tikzcd}$$

Lemma 2.5. The inclusion $h\mathcal{DR}_\xi(X) \rightarrow h\mathcal{DR}_\xi(X)$. is a homotopy equivalence. #

Let X be d-dimensional compact (orientable) manifold. To a partition $(M,F,N) \in h\underline{P}(X)$ there can be associated a duality map. Let $M' = M-F$, $N' = N-F$. The inclusion

$$i: \; M' \times N' \; \rightarrow \; (X \times [a,b]\,)^2 \; - \; \text{diagonal}$$

induces a map over $X \times X$

$$j: \; M' \wedge N' \; \rightarrow \; ((X\times[a,b])^2 - \Delta\,) \cup_{X^2\times[a,b]\times b\, \cup\, X^2\times a\times[a,b]} X^2$$

(Δ = diagonal). Let Z denote the latter space. It is clearly homotopy equivalent to $(X \times [a,b])^2 - \Delta$. Slant product with a Thom class t in $H^{d+1}((X \times [a,b])^2, (X \times [a,b])^2 - \Delta) \approx H^d(Z, X^2)$ gives a map

$$\alpha_t: \; H_*(N',X) \; \rightarrow \; H^{d-*}(M',X) \; .$$

Lemma 2.6. This map is an isomorphism.

Proof: Let $Y = X \times [a,b]$. We have a commutative diagram

$$\begin{array}{ccccc} H^d(Z, X^2) & \otimes & H_*(N', X) \to & H^{d-*}(M',X) \approx H^{d-*}(M, X) \\ & \approx\downarrow & & \downarrow\approx \\ H^{d+1}(Y^2, Y^2-\Delta) & \otimes & H_*(Y-M, Y-(Y-X)) \to & H^{d+1-*}(Y-X, M) \end{array}$$

where the vertical isomorphisms come from the exact sequence of the triple $(Y^2, Y^2-\Delta, X^2)$ (resp. $(Y-X, M, X)$). The upper row is the pairing inducing α_t, the lower row is the pairing inducing the usual Alexander duality isomorphism. Hence the upper row gives a non-singular pairing, too. (One has to be a little careful since the assumptions of the duality theorem are not quite satisfied here, e.g. Y-X is not compact, and, more seriously, M is not contained in the interior of Y. But in this special situation at hand this does not affect the result because the intersection of M with the boundary of Y is homotopy equivalent to X.) #

Let $\alpha(X)$ denote the tangent bundle of X; $\xi := S(\alpha \oplus \varepsilon^1)$ is the sphere bundle associated to the sum of α with a trivial line bundle. There is a $\mathbb{Z}/2$ - action on ξ given by reflection in ε^1. It is easy to verify that the space Z defined above is a Thom space of ξ in the sense defined before, except for the technical condition that $Z \to X^2$ be a fibration which is not satisfied here. Replacing Z by another space Z' satisfying this condition we thus obtain a map

$$\varphi : h\underline{P}(X) \to h\mathcal{DR}_\xi(X). \qquad (M,F,N) \mapsto (M',N',j)$$

$$(\text{resp. } \varphi : h\underline{P}^{m,n}_k(X) \to h\mathcal{DR}^{m,n}_{k,\xi}(X). \quad , n = d-m) .$$

(We have to add the technical condition that $\pi_1 X = \pi_1 M = \pi_1 N$ at this point.)

It is easy to see that the diagram

$$\begin{array}{ccc} h\underline{P}(X) & \xrightarrow{\varphi} & h\mathcal{DR}_\xi(X). \\ T\downarrow & & \downarrow\tau \\ h\underline{P}(X) & \xrightarrow[\varphi]{} & h\mathcal{DR}_\xi(X). \end{array}$$

commutes up to homotopy.

One can obtain a map in the limit

$$\varphi : \varinjlim_{k,m,n} h\underline{P}^{m,n}_k(X \times J^{n+m-d}) \to \varinjlim_{k,m,n} h\mathcal{DR}^{m,n}_{k,\xi}(X \times J^{n+m-d}).$$

(by suitably modifying the definition of the stabilization maps on the right hand side).

This map is a homotopy equivalence since its composition with the forgetful map of prop. 2.3. is a homotopy equivalence by the argument of [9, prop. 5.4.] .

Standard mapping cylinder arguments now show

Proposition 2.7. The map φ is compatible with the involutions, up to weak homotopy, i.e. the restrictions of $\tau\varphi$ (resp. φ T) to any compactum are homotopic. #

Corollary 2.8. The involutions T and τ agree up to homotopy. #

§ 3.

Let R be a ring with an anti-involution $\bar{\ }: R \to R$. There is an involution on $Gl_k(R)$ given by $A \mapsto (\bar{A}^t)^{-1}$ which induces the usual involution on K(R), the algebraic K-theory of R.

More generally, if R is a simplicial ring, its algebraic K-theory is defined, cf. [8]. It can be regarded as a "linearized" version of the algebraic K-theory of spaces.

In this section we construct an involution on A(X) which "linearizes" to an involution on the K-theory of simplicial (group) rings. It is shown that this involution on A(X) agrees with those defined in the first two paragraphs, and further that in the case of an ordinary (group) ring, considered as a simplicial ring in a trivial way, it agrees with the involution defined in the beginning of this paragraph.

Let G be a simplicial group. $\mathcal{U}(G)$ (resp. $\mathcal{U}(G^{op})$) is the category of pointed simplicial sets with right (resp. left) G-action, $\mathcal{U}^f(G)$ is the subcategory of those G-sets which are free (in the pointed sense) and finitely generated over G. An h-equivalence is a G-map which is a homotopy equivalence of the underlying simplicial sets, $h\mathcal{U}^f(G) \subset \mathcal{U}^f(G)$ is the subcategory of h-equivalences.

Let M (resp. M') denote an object of $\mathcal{U}^f(G)$ (resp. $\mathcal{U}^f(G^{op})$). An n-duality map is a pointed (right) $(G \times G^{op})$-map

$$u: \quad M' \wedge M \quad \to \quad S^n \wedge G_+$$

satisfying that it induces an isomorphism of $\mathbb{Z}[\pi_0 G]$-modules

$$\alpha_u: \quad H_q^{G^{op}}(M') \quad \to \quad H_G^{n-q}(M)$$

by $$z \quad \mapsto \quad u^*(t)/z, \qquad 0 \leqq q \leqq n,$$

t being a generator of $H^n_G(S^n \wedge G_+) \approx \mathbb{Z}[\pi_0 G]$.

By definition, $H^G_*(M) = H_*(M \times^G E, * \times^G E; \mathbb{Z}[\pi_0 G])$, $H^*_G(M) = H^*(M \times^G E, * \times^G E, \mathbb{Z}[\pi_0 G])$, where E is a universal G-bundle.

Example 3.1. The map $\mu\colon (S^k \wedge G_+) \wedge (S^{n-k} \wedge G_+) \overset{\approx}{\to} S^n \wedge G_+ \wedge G_+ \to S^n \wedge G_+$ induced from the multiplication $G \times G \to G$ is an n-duality map. #

Let $F^n_G(M)$ denote the simplicial set of pointed (right) G-equivariant maps from M to $S^n \wedge G_+$. G acts freely (pointed) from the left on this function space.

The evaluation map

$$F^n_G(M) \wedge M \quad \to \quad S^n \wedge G_+$$

induces a map

$$\alpha_e : H^{G^{op}}_q(F^n_G(M)) \to H^{n-q}_G(M).$$

Let $M \in \mathcal{U}^f(G)$ be of (G-)dimension k.

Lemma 3.2. The map α_e is an isomorphism in the range $0 \leq q \leq 2(n-k) - 1$.

Proof: By induction. The assertion is trivially true in the case $M = *$. For $M = G \coprod *$ we have $F^n_G(M) = \mathrm{Map}_*(S^0, S^n \wedge G_+) = S^n \wedge G_+$, and the evaluation map is the multiplication $G_+ \wedge S^n \wedge G_+ \overset{\approx}{\to} S^n \wedge G_+ \wedge G_+ \to S^n \wedge G_+$. So α_e is an isomorphism by example 3.1.. M has a G-skeleton filtration $* \subset M_0 \subset M_1 \subset \dots \subset M_k = M$, such that we have cofibration sequences

$$M_{i-1} \rightarrowtail M_i \twoheadrightarrow \bigvee_\alpha S^i \wedge G_+, \quad \alpha \in \text{some finite}$$

index set. The general case then follows by a five lemma argument and the fact that the canonical map $S^{n-i} \wedge G_+ \to \Omega^i(S^n \wedge G_+)$ is $2(n-i)-1$-connected. #

Corollary 3.3. Let M be as before, M' a left G-set satisfying that $H^{G^{op}}_i(M') = 0$, $i > n$. Suppose that $n \geq 2k+1$. If there exists a pointed left G-map $\varphi\colon M' \to F^n_G(M)$ inducing an isomorphism on G-homology up to dimension n, then M' is an n-dual of M. #

Corollary 3.4. Let M be an object of $\mathcal{U}^f(G)$. There exists an n-dual of M, if n is large enough.

Proof: Suppose M is k-dimensional as a G-set. Let $n \geq 2k+1$. Consider the inclusion of the (n+1)-skeleton $i\colon sk_{n+1} F^n_G(M) \to F^n_G(M)$. Attaching finitely many G-cells to the (n+1)-skeleton of $F^n_G(M)$ kills the homology in dimension n+1 without introducing any new homology.

Let M' be the space obtained by attaching these cells. By construction i extends to a map $\bar{i} : M' \to F^n_G(M)$, satisfying the assumptions of cor. 3.3. #

Define a category $\mathcal{DU}(G)$ in which an object is a triple (M,M',u), where M (resp. M') is an object of $\mathcal{U}(G)$ (resp. $\mathcal{U}(G^{op})$), $u: M' \wedge M \to S^n \wedge G_+$ a duality map. We add the technical condition that $\pi_i M = \pi_i M' = 0$, $i=0,1$. A morphism $(M,M',u) \to (N,N',v)$ is a pair of morphisms $M \xrightarrow{f} N$, $N' \xrightarrow{f'} M'$, such that the diagram

$$\begin{array}{ccc} N' \wedge M & \xrightarrow{id \wedge f} & N' \wedge N \\ {\scriptstyle f' \wedge id}\downarrow & & \downarrow{\scriptstyle v} \\ M' \wedge M & \xrightarrow[u]{} & S^n \wedge G_+ \end{array}$$

commutes.

A morphism (f,f') is called an h-equivalence if both f and f' are homotopy equivalences.

There are two suspension functors Σ_l, Σ_r on the category $\mathcal{DU}(G)$ given by suspending M (resp. M'). Further there is a canonical involution given by $\tau: (M,M',u) \to (\bar{M}',\bar{M}, \iota \circ u)$, where 'bar' means that the action is changed via $\iota: G \to G$, $g \mapsto g^{-1}$.

Similarly as in the case of $\mathcal{DR}(X)$ certain subcategories $\mathcal{DU}^{m,n}_k(G)$ of $\mathcal{DU}(G)$ may be defined. The involution can be defined as a (contravariant) functor

$$\tau : \varinjlim_{\Sigma_l,\Sigma_r} h\mathcal{DU}^{m,n}_k(G) \to \varinjlim_{\Sigma_l,\Sigma_r} h\mathcal{DU}^{m,n}_k(G)$$

Proposition 3.5. The forgetful functor

$$\varepsilon : \varinjlim_{\Sigma_l,\Sigma_r} h\mathcal{DU}^n(G) \to \varinjlim_{\Sigma} h\mathcal{U}^f(G)$$

$$(M,M',u) \mapsto M$$

is a homotopy equivalence.

Proof: We will use Quillen's theorem A, cf.[7]. We have to show that for any object N in $\varinjlim_{\Sigma} h\mathcal{U}^f(G)$ the right fibre N/ε is contractible. To prove this it suffices to show that for every finite diagram $\mathcal{D}: \mathcal{I} \to N/\varepsilon$ there exists an initial object. Let the diagram $\mathcal{D}$ be represented by $(M_i, M'_i, u_i: M'_i \wedge M_i \to S^n \wedge G_+;\ a_i: N \to M_i)_{i \in \mathcal{I}}$. Let $n > 2 \max_i \{\dim M_i, \dim N\} + 1$. Further assume that $\dim M'_i \leq n$. Consider the composite

$$M_i^! \xrightarrow{\hat{u}_i} F_G^n(M_i) \xrightarrow{a_i^*} F_G^n(N) \quad .$$

Since the maps $\hat{u}_i$ are (induced from) duality maps, the composite $a_i^*\hat{u}_i$ is an isomorphism on G-homology up to dimension $2(n - \max(\dim M_i)) - 1 > n$. Since all the G-sets $M_i^!$ are finite and by assumption their dimension is smaller than n, the image of all the maps $a_i^*\hat{u}_i$ is contained in the n-skeleton of $F_G^n(N)$, since we are using simplicial maps throughout. Attaching some G-cells to the (n+1)-skeleton of $F_G^n(N)$ gives a space $\bar{N}$ with the properties (i) $sk_n F_G^n(N) \subset \bar{N}$, (ii) $\bar{N}$ maps to $F_G^n(N)$, (iii) this map is the natural inclusion on $sk_n F_G^n(N)$ and it induces an isomorphism on homology up to dimension n+1, (iv) $H_i^{G^{op}}(\bar{N}) = 0$, $i \geqq n+1$.
We have thus obtained a factorization $M_i^! \to \bar{N} \to F_G^n(N)$ of the maps $a_i^*\hat{u}_i$. This implies that $M_i^! \to \bar{N}$ induces an isomorphism on G-homology at least up to dimension n, by lemma 3.2.. Since both spaces have no homology in dimensions greater than n, and by the Hurewicz theorem, the map $M_i^! \to \bar{N}$ is therefore an h-equivalence.

The inclusion $\bigcup_i a_i^*\hat{u}_i(M_i^!) \hookrightarrow \bar{N}$ factors

$$\bigcup_i a_i^*\hat{u}_i(M_i^!) \to N' \xrightarrow{\simeq} \bar{N} ,$$

where N' is obtained from the finite G-set $\bigcup_i a_i^*\hat{u}_i(M_i^!)$ by attaching of finitely many G-cells, hence is itself finite. The map

$$b_i : \quad M_i^! \to \bigcup_i a_i^*\hat{u}_i(M_i^!) \to N'$$

is an h-equivalence since its composition with $N' \xrightarrow{\simeq} \bar{N}$ is one.

By construction N' is an n-dual of N. Let v denote the corresponding duality map. The object (N,N',v) of $h\mathcal{DU}^f(G)$ maps to the diagram $(M_i, M_i^!, u_i; a_i)$ by (a_i, b_i). Hence ((N,N',v); id: N → N) is a cone point for the diagram $\mathcal{D}$.

This proves that every finite diagram in N/ε is nullhomotopic, as was to be shown. #

<u>Remark</u> (i) The proposition remains true if the finiteness condition on the objects is relaxed to admit objects that are finite up to homotopy. (ii) By restriction to a connected component one obtains another homotopy equivalence

$$\varepsilon : \varinjlim_{m,n} h\mathcal{DU}_k^{m,n}(G) \to \varinjlim_m h\mathcal{U}_k^m(G) \qquad \#$$

Let G denote a loop group of X, [6]. Let E be a universal G-bundle. There is an adjoint functor pair (cf.[8].)

$$\begin{array}{llll} \Phi_X: & hR(X) & \to & hU(G) \\ & (Y,r,s) & \mapsto & Y\ x_X E/E \\ \Psi_G: & hU(G) & \to & hR(X) \\ & M & \mapsto & M\ x^G E \rightleftarrows *\ x^G E\ . \end{array}$$

Let $\varepsilon = X \times S^o$ be a trivial spherical fibration over X. The space $\Psi_{G^{op}\,xG}(G_+)$ can be used as a Thom space of ε in the sense defined above. (Here G_+ is considered as an object of $U(G^{op}xG)$).

Define a functor

$$\begin{array}{llll} D\Phi & : & hDR_\varepsilon(X) & \to \ hDU(G) \\ & & (Y,Y',u) & \mapsto \ (\Phi_X(Y),\Phi_X(Y'),u'), \end{array}$$

where u' is the composite

$$\Phi_X(Y')\wedge\Phi_X(Y) \to \Phi_{X^2}(\Sigma^n_{X^2}(\Psi_{G^{op}\,xG}(G_+))) = \Phi_{X^2}\Psi_{G^{op}\,xG}(S^n\wedge G_+)\to S^n\wedge G_+ \ .$$

Similarly,

$$D\Psi \ : \ hDU(G) \ \to \ h\,DR_\varepsilon(X)$$

$$(u\colon M'\wedge M \to S^n\wedge G_+) \ \mapsto \ (\Psi_G(M')\wedge\Psi_G(M) \to \Psi_{G^{op}\,xG}(S^n\wedge G_+)\approx \Sigma^n_{X^2}(\Psi_{G^{op}\,xG}(G_+)= Th_n(\varepsilon)).$$

Proposition 3.6. $D\Phi$ and $D\Psi$ are mutually inverse homotopy equivalences.

Proof: We first remark that $D\Phi$ and $D\Psi$ are not adjoint. Let $f\colon hDR(X) \to hDR(X)$ be given by $(Y,Y',u) \mapsto (\Psi\Phi(Y),Y',\bar u)$,
$\bar u\colon \Psi\Phi(Y)\wedge Y' \to \Psi\Phi(Y\wedge Y') \xrightarrow{\Psi\Phi(u)} \Psi\Phi\Psi(S^n\wedge G_+) \to \Psi(S^n\wedge G_+)$.
Similarly, f' is the corresponding endofunctor of $hDR(X)$ defined by a condition on Y'. There is a natural transformation from the identity to f, and another one from f' to the identity. Therefore $D\Psi\cdot D\Phi = f'f$ is homotopic to the identity; similarly with the other composition. #

Remark: $D\Phi$ (resp. $D\Psi$) restricts to a functor $hDR^{hf}_\varepsilon(X) \to hDU^{hf}(G)$ (resp. $hDU^{hf}(G) \to hDR^{hf}_\varepsilon(X)$). Both functors commute for obvious reasons with the canonical involution.

Proof of prop. 2.3.: There is a commutative diagram

$$\begin{array}{ccc} hDR^{hf}(X) & \xrightarrow{D\Phi} & hDU^{hf}(G) \\ \delta\downarrow & & \downarrow\varepsilon \\ hR^{hf}(X) & \xrightarrow{\Phi} & hU^{hf}(G) \end{array} \quad .$$

Φ is a homotopy equivalence because it has an adjoint, $D\Phi$ is a homotopy equivalence by prop. 3.6.; ε becomes a homotopy equivalence after passing to the limit by prop. 3.5., therefore so does δ. #

We now consider the analogous construction for a simplicial ring R. Its algebraic K-theory K(R) can be defined from the category of free simplicial R-modules in a way formally quite similar to the construction of A(X) from the category of free simplicial G-sets.

In particular, if $R = \mathbb{Z}[G]$, G a simplicial group, the concept of duality can be defined, and K(R) can be constructed from a larger category of R-modules by including duality data. This leads to an involution on K(R), which will be denoted τ'.

There is a linearization map $A(X) \to K(\mathbb{Z}[G(X)])$ from the K-theory of X to the K-theory of the simplicial loop group of X. It is given by associating to a free pointed simplicial G-set M the simplicial $\mathbb{Z}[G]$-module $\tilde{\mathbb{Z}}[M]$, the underlying simplicial abelian group of which is generated by the non-basepoint elements of M.

By its construction the linearization map is compatible with the involution.

The connected component map $\pi_0 : \mathbb{Z}[G(X)] \to \mathbb{Z}[\pi_1 X]$ induces a map $K(\mathbb{Z}[G(X)]) \to K(\mathbb{Z}[\pi_1 X])$. Let τ'' denote the involution on the latter space defined in the introduction to § 3.

Proposition 3.7.: There is a commutative diagram

$$\begin{array}{ccccc} A(X) & \to & K(\mathbb{Z}[G(X)]) & \xrightarrow{\pi_0} & K(\mathbb{Z}[\pi_1 X]) \\ \downarrow \tau & & \downarrow \tau' & & \downarrow \tau'' \\ A(X) & \to & K(\mathbb{Z}[G(X)]) & \xrightarrow{\pi_0} & K(\mathbb{Z}[\pi_1 X]) \end{array} .$$

Proof: Let $R = \mathbb{Z}[\pi_1 X]$. Let iso $F_k(R)$ denote the category of free (right) R-modules of rank k and their isomorphisms. The K-theory of R (meaning that of its free modules) may be defined in the following way:

$$K(R) = \mathbb{Z} \times |\varinjlim_k \text{ iso } F_k(R)|^+ .$$

There is a natural map

$$\begin{array}{ccc} \text{iso } F_k(R) & \to & \text{iso } F_k(R) \\ A & \mapsto & \mathrm{Hom}_R(A, R) \end{array} .$$

This map restricts to the natural involution

$$Gl_k(R) \to Gl_k(R), \qquad \alpha \to (\bar{\alpha}^t)^{-1} .$$

The map $A(X) \to K(R)$ may be described by first mapping a duality $u\colon M'\wedge M \to S^n\wedge G_+$ in $\mathcal{DU}_k^{1,m}(G)$ to the induced pairing

$$\bar{u}\colon (\tilde{\mathbb{Z}}[M']\otimes_{\mathbb{Z}[G]}\mathbb{Z}[\pi_0 G])\otimes_{\mathbb{Z}[\pi_0 G]}(\tilde{\mathbb{Z}}[M]\otimes_{\mathbb{Z}[G]}\mathbb{Z}[\pi_0 G]) \to \tilde{\mathbb{Z}}[S^n\wedge(\pi_0 G)_+]$$

and then mapping to the free $\mathbb{Z}[\pi_0 G]$-module of rank k

$$\pi_\ell(\tilde{\mathbb{Z}}[M]\otimes_{\mathbb{Z}[G]}\mathbb{Z}[\pi_0 G]) = H_\ell^G(M).$$

By definition of a duality map in $\mathcal{DU}(G)$, $\bar{u}$ is nonsingular, i.e. it gives an identification of

$$\pi_m(\tilde{\mathbb{Z}}[M']\otimes_{\mathbb{Z}[G]}\mathbb{Z}[\pi_0 G]) \quad \text{with} \quad \mathrm{Hom}_{\mathbb{Z}[\pi_0 G]}(H_\ell^G(M),\ \mathbb{Z}[\pi_0 G]),$$

considered as a right $\mathbb{Z}[\pi_0 G]$-module.

This proves the proposition. #

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ALGEBRAIC K-THEORY OF SPACES, LOCALIZATION, AND THE CHROMATIC FILTRATION OF STABLE HOMOTOPY.

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This paper represents a first step in applying localization techniques to the computation of the algebraic K-theory of spaces, and in particular to the task of reducing that computation to the computation of the algebraic K-theory of rings.

In order not to obscure the essential points by great generality we shall restrict ourselves to the special case of the space $A(*)$, the algebraic K-theory of a point. What we would like to do is to reduce the computation of $A(*)$ to that of $K(Z)$, the algebraic K-theory of the ring of integers, and in particular to compute $\text{fibre}(\, A(*) \to K(Z)\,)$, the homotopy fibre of the natural map.

That task is not easy. For, as will be explained in an appendix, it follows from the Lichtenbaum-Quillen conjecture (which is regarded as rather respectable among experts in the algebraic K-theory of rings) that $\text{fibre}(\, A(*) \to K(Z)\,)$ must in some way or other account for all of that formidable object, the cokernel of J .

Here is an outline of what is done in this paper. The space $A(*)$ may be constructed according to a certain recipe out of the category of pointed spaces of finite (homotopy) type; alternatively one could use spectra of finite type for the purpose (these matters are explained in section 1 below). The recipe is fairly general and can be applied in the same way to other categories of spaces or spectra. In particular if p is a prime, the recipe can be applied to the category of p-local spectra of finite type.

Let us denote the result of this construction by $A(*,p)$. Let $Z_{(p)}$ denote the ring of integers localized at p . There is a natural map

$$A(*,p) \longrightarrow K(Z_{(p)})$$

and we shall show that its homotopy fibre may be identified to the p-local part of $\text{fibre}(\, A(*) \to K(Z)\,)$. In this sense the task of computing the latter has been broken up into its p-local parts now.

In contradistinction to what one might expect by analogy with the algebraic K-theory of the ring Z, it is possible here to continue fracturing by localization methods. This is where the *chromatic filtration* comes in (there is one such for each prime p). By definition, the chromatic filtration is a particular sequence of localization functors in stable homotopy. The characteristic feature of these localization functors, as opposed to localization functors in general, is that they may be defined in terms of *acyclic spaces of finite type* (these matters are explained in section 2 below). The existence of the sequence is still conjectural beyond the first few terms; the relevant conjectures are due to Bousfield and Ravenel.

As will be explained (in section 3) the existence of the chromatic filtration implies the existence of a *localization tower* (whose maps are induced by localization functors)

$$A(*,p) = A(*,p,\infty) \longrightarrow \ldots \longrightarrow A(*,p,2) \longrightarrow A(*,p,1) \longrightarrow A(*,p,0) .$$

The bottom term $A(*,p,0)$ turns out to be the same (up to homotopy) as $K(Q)$, the algebraic K-theory of the ring of rational numbers; the next term $A(*,p,1)$ is in some sense the algebraic K-theory of the *non-connective* J (image-of-J-theory at the prime p). The layers of the tower (the homotopy fibres of the maps of consecutive terms) represent the contributions of what in Ravenel's terminology are the *monochromatic* phenomena in stable homotopy theory.

There is a second tower associated to the chromatic filtration, an *integral* (or *connective*) analogue of the former tower,

$$A(*,p) = \tilde{A}(*,p,\infty) \longrightarrow \ldots \longrightarrow \tilde{A}(*,p,2) \longrightarrow \tilde{A}(*,p,1) \longrightarrow \tilde{A}(*,p,0) .$$

The bottom term $\tilde{A}(*,p,0)$ here is $K(Z_{(p)})$, the algebraic K-theory of the ring of p-local integers, and the next term $\tilde{A}(*,p,1)$ is the algebraic K-theory of the *connective* J. The construction of the spaces $\tilde{A}(*,p,n)$ is very much like that of the algebraic K-theory of rings in the framework of the plus construction. This means that a certain amount of explicit computation is possible in low degrees. There does not however seem to exist a direct description of the layers in the tower. This suggests to try reducing to the former tower in order to obtain information about the layers.

There is a natural transformation $\tilde{A}(*,p,n) \to A(*,p,n)$. Modulo certain technical assumptions we can give an explicit description of the fibre, by localization methods again. For $n = 0$ the map is the natural map $K(Z_{(p)}) \to K(Q)$, and in that case our description of the fibre reduces to a case of Quillen's localization theorem.

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1. Review of algebraic K-theory.

We recall the definition of $A(*)$ from the category viewpoint [14], [5], [16]. Let C be the category of pointed spaces of finite type, that is, pointed spaces having the homotopy type of a finite CW complex (as a technical point, C is not a 'small' category, but we can replace it by one). Then $A(*)$ is defined as the loop space of the CW complex

$$\bigcup_{m,n} w_m S_n C \times \Delta^m \times \Delta^n \ / \sim ,$$

the geometric realization of the bisimplicial set $[m],[n] \mapsto w_m S_n C$, where $w_m S_n C$ is the set of commutative diagrams in C,

$$\begin{array}{ccccccccc}
* = Y^0_{0,0} & \rightarrowtail & Y^0_{0,1} & \rightarrowtail & Y^0_{0,2} & \rightarrowtail & \cdots & \rightarrowtail & Y^0_{0,n} \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & & & \downarrow\wr \\
* = Y^1_{0,0} & \rightarrowtail & Y^1_{0,1} & \rightarrowtail & Y^1_{0,2} & \rightarrowtail & \cdots & \rightarrowtail & Y^1_{0,n} \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & & & \downarrow\wr \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & & & \downarrow\wr \\
* = Y^m_{0,0} & \rightarrowtail & Y^m_{0,1} & \rightarrowtail & Y^m_{0,2} & \rightarrowtail & \cdots & \rightarrowtail & Y^m_{0,n}
\end{array}$$

in which the horizontal arrows $\rightarrowtail$ denote cofibrations, and the vertical arrows $\xrightarrow{\sim}$ denote (weak) homotopy equivalences.

The face and degeneracy maps in the vertical direction are given by omission and reduplication of data. This may conveniently be summarized by saying that the bisimplicial set arises as the nerve of a simplicial category; namely of $[n] \mapsto wS_n C$, the category of the diagrams

$$* = Y_{0,0} \rightarrowtail Y_{0,1} \rightarrowtail Y_{0,2} \rightarrowtail \cdots \rightarrowtail Y_{0,n}$$

and their weak homotopy equivalences.

The face structure in the horizontal direction is slightly more complicated. All but one of the face maps are still given by omission of data, but the face map numbered 0 involves a quotient space construction. It takes the above object to

$$* = Y_{1,1} \rightarrowtail Y_{1,2} \rightarrowtail \dots \rightarrowtail Y_{1,n} \ , \qquad \text{where} \quad Y_{1,k} = Y_{0,k} \,/\, Y_{0,1} \ .$$

(As a technical point, quotient spaces are only well defined up to canonical isomorphism. This need not concern us very much, however. One just rearranges the construction a little by including the choices of quotients $Y_{i,j} = Y_{0,j}/Y_{0,i}$ in the data of the diagrams, cf. [14], [16]).

The construction is formal in the sense that it uses little knowledge about the category C . Indeed, the only thing required (apart from the technical point concerning the existence of an object in C which is both initial and terminal) is the fact that there are singled out two particular kinds of morphisms which are called *cofibrations* and *weak equivalences*, respectively, and which have suitable properties (e.g. cofibrations have quotients, and the weak equivalences satisfy a gluing lemma).

This suggests defining the notion of a *category with cofibrations and weak equivalences*. This is a category C equipped with subcategories co(C) and w(C) , and the data are subject to a short list of plausible axioms (which will not be repeated here, cf. [14], [16]). The definition of the simplicial category $[n] \mapsto wS_nC$ (or wS.C , for short) now carries over word for word. We think of this simplicial category (or rather of the loop space of its geometric realization) as the *algebraic* K-*theory of the category* C or better, to be precise, as the K-*theory of* C *with respect to the chosen notions of cofibration and weak equivalence*.

In practice it turns out that the notion of cofibration is usually fixed once and for all. That is, it just doesn't occur in practice that some category C is considered as a category with cofibrations in more than one way. In particular, for the spaces and spectra in the present paper the term cofibration will always have its usual meaning. By contrast, it is not at all pathological nor even exceptional that some category C is considered as a category with weak equivalences in more than one way. For example if E is a spectrum, and C the category of pointed spaces (resp. of spectra) then the notion of E-equivalence is a perfectly acceptable notion of weak equivalence in C . In fact, the interplay between different notions of weak equivalence arising in this way is one of the things that localization theory is going to be about.

It may be appropriate to say a word about the ever recurring *finite type* condition. One could take it as one of the facts of life that in connection with algebraic K-theory there is always some finiteness condition around, be it explicit or implicit. But one can also give a simple explanation: in the absense of a finiteness condition algebraic K-theory just isn't interesting and therefore is not considered. For as soon as, say, infinite sums are allowed in the category C one can go through a version of the Eilenberg swindle. Namely if the endofunctor $F(A) = A \vee A \vee \ldots$ is defined then one certainly has an isomorphism $Id \vee F \approx F$. On the other hand the sum in C induces a composition law on $|wS.C|$ making it an infinite loop space in the manner of Segal [10] and in particular therefore a group-like H-space (cf. [16] for details). In the homotopy $Id \vee F \simeq F$ one can then cancel F to conclude that the identity map on $|wS.C|$ is null-homotopic.

There is one general computation that is easy to do. This is the determination of K_0 , the class group, in terms of generators and relations. By definition this group is $\pi_0\Omega|wS.C|$ or what is the same thing, the fundamental group of the CW complex $|wS.C|$. There is a well known recipe on how to compute the fundamental group of a reduced CW complex in terms of the cells of dimension 1 and 2. Applying the recipe in the case at hand one obtains that the class group is the abelian group generated by the objects $A \in C$, and subject to two kinds of defining relations,

$$[A^0] = [A^1] \quad \text{if there is a weak equivalence} \quad A^0 \xrightarrow{\sim} A^1 \,, \quad \text{and}$$

$$[A_{01}] + [A_{12}] = [A_{02}] \quad \text{if there is a cofibration sequence} \quad A_{01} \rightarrowtail A_{02} \twoheadrightarrow A_{12} \,.$$

In particular, in the case of the pointed spaces of finite type and their weak homotopy equivalences one obtains the group $\pi_0 A(*) \approx Z$, and the integer represented by a space is just its (reduced) Euler characteristic. Other cases will be considered later.

To conclude this review we shall outline an argument now to justify the fact that the space $A(*)$ may not only be defined in terms of pointed spaces of finite type but also in terms of spectra of finite type. We will need to know about some general results for this.

A functor between categories with cofibrations and weak equivalences, say $F\colon C \to C'$, is called *exact* if it preserves all the relevant structure. In that case it induces a map $wS.F\colon wS.C \to wS.C'$.

A *weak equivalence* between exact functors $C \to C'$ is a natural transformation $F \to F'$ so that for every $A \in C$ the map $F(A) \to F'(A)$ is a weak equivalence in C' . Not very surprisingly there results a homotopy between $wS.F$ and $wS.F'$ in this case. For example the cone functor on the category of pointed spaces is exact, so it induces a self-map on $A(*)$, and it is weakly equivalent to the trivial map, so the self-map is null-homotopic.

A *cofibration sequence* of exact functors $C \to C'$ is a sequence of natural transformations $F' \to F \to F''$, or $F' \rightarrowtail F \twoheadrightarrow F''$ as we shall write, having the property that for every $A \in C$ the map $F'(A) \to F(A)$ is a cofibration in C' , and $F(A) \to F''(A)$ represents the associated quotient map. A basic technical tool about the construction $C \mapsto wS.C$ is the *additivity theorem*. One of several equivalent formulations says if $F' \rightarrowtail F \twoheadrightarrow F''$ is a cofibration sequence of exact functors then there exists a homotopy between $wS.F$ and the sum of the maps $wS.F'$ and $wS.F''$.

To come back to the situation at hand, there is a cofibration sequence of exact functors on the category of pointed spaces,

$$\text{identity} \rightarrowtail \text{cone} \twoheadrightarrow \text{suspension} .$$

In view of the additivity theorem therefore the self-map $\mathrm{Id} \vee \Sigma$ of $A(*)$ is null-homotopic, thus the suspension represents a homotopy inverse for the additive H-space structure on $A(*)$. In particular the suspension induces a homotopy equivalence of $A(*)$ to itself.

Now $C \mapsto wS.C$ is compatible with direct limits, so we obtain that (up to homotopy) $A(*)$ is also definable in terms of the category with cofibrations and weak equivalences $\overline{C}$ say,

$$\overline{C} = \varinjlim C_n ,$$

where each C_n is the category of pointed spaces of finite type, and $C_n \to C_{n+1}$ is the suspension map. $\overline{C}$ is a category of spectra containing the full subcategory of the finite spectra but it is somewhat smaller than $\widetilde{C}$, say, the category of spectra of finite (homotopy) type. We will therefore want to know that the inclusion $wS.\overline{C} \to wS.\widetilde{C}$ is a homotopy equivalence. While this is certainly plausible it is not self-evident, and an argument is required. The argument is provided by the following useful criterion whose applicability in the present situation is straightforward to check.

The criterion gives a sufficient condition for an exact functor $F: C \to D$ to induce a homotopy equivalence $wS.C \to wS.D$. We refer to it as the *approximation theorem*. The idea behind is that the homotopy type of $wS.C$ should only depend on the 'homotopy theory underlying C ' (whatever that may be). The approximation theorem makes this precise in the form of three axioms [16]. The first axiom says, roughly, that the general setup should be as in homotopy theory (in particular this rules out some fancy notions of weak equivalence and asks that mapping cylinder constructions should be available). The second axiom says if $A \to A'$ is a map in C then if $F(A) \to F(A')$ is a weak equivalence in D it follows that $A \to A'$ is a weak equivalence in C (the converse is implied by the exactness of F , of course). The third axiom finally insists that objects of D are 'homotopy equivalent' to objects coming from C , and morphisms too; the precise formulation is that given objects $A \in C$ and $B \in D$, and a map $f: F(A) \to B$ in D , then there exist a cofibration $g: A \to A'$ in C and a weak equivalence $h: F(A') \to B$ in D so that the resulting triangle commutes, i.e. $f = hF(g)$.

2. Review of localization.

The main references are to papers by Adams [1], Bousfield [2], and Ravenel [8].

Let E be a spectrum. A spectrum X is called E-*acyclic* if the E-homology groups $E_*X = \pi_*(E \wedge X)$ are trivial. Likewise a map $X' \to X''$ is called an E-*equivalence* if it induces an isomorphism $E_*X' \to E_*X''$. A spectrum Y is said to be E-*local* if it does not admit any non-trivial map from an E-acyclic spectrum; an equivalent condition is that for every E-equivalence $X' \to X''$ the induced map of sets of homotopy classes $[X'',Y] \to [X',Y]$ is an isomorphism.

By an E-*localization* of a spectrum X is meant any E-local spectrum Y together with an E-equivalence $X \to Y$. It follows from the definitions that the E-localization is unique up to (weak) homotopy equivalence under X. Bousfield has shown that it always exists, in fact that there exists an E-*localization functor* L_E [2].

There is a correspondence between localization functors and *acyclicity types*. For on the one hand the E-localization depends only on the class of the E-acyclic spectra: if E' and E" happen to have the same acyclic spectra then their associated localization functors are the same, by definition. And on the other hand the E-acyclic spectra may be recovered from the localization functor L_E as the 'pre-image of zero'; that is, the E-acyclic spectra are precisely the ones whose E-localization is trivial (up to homotopy). The correspondence allows us to formulate a finite type condition on the localization functor L_E in terms of the associated acyclicity type. The condition is simply that $Cl(L_E)$, the class of the E-acyclic spectra, is in some sense generated by finite spectra.

To make this precise let us say that a class of spectra is *saturated* if it is closed under

- homotopy equivalence and shifting (suspension and de-suspension)
- the formation of (possibly infinite) wedges
- the formation of mapping cones.

For any spectrum E the class of the E-acyclic spectra is saturated. Conversely it is known [2] that any saturated class occurs in this fashion from a suitable E. If M is any collection of spectra let the *saturation of* M mean the smallest saturated class of spectra containing M ; we denote it sat(M). We will say that a localization functor L , resp. the associated acyclicity type $Cl(L)$, is *generated* by a collection of spectra M if $Cl(L) = sat(M)$. And we will say that a localization functor is of *finite type*, or that it is a *finite localization functor*, if it is generated by some collection M any member of which is a finite spectrum. (Note that the number of spectra in M may well be infinite, however).

A finite localization functor has an important property which we refer to as the *convergence property*. It says that for every X the localization $L_E(X)$ may be obtained, up to homotopy, as the direct limit of a sequence of E-equivalences each of which has *finite* homotopy cofibre. In particular if X is finite then $L_E(X)$ is the direct limit (up to homotopy) of a sequence of finite spectra E-equivalent to X .

The proof may first of all be reduced to the assertion that the E-acyclic spectrum $L_E(X)/X$, the (homotopy-)cofibre of $X \to L_E(X)$, is the direct limit (up to homotopy) of a sequence of finite E-acyclic spectra. (For $L_E(X)$ can be reconstructed by attaching $L_E(X)/X$ to X). By hypothesis now $Cl(L_E)$ is generated by some collection M any member of which is finite and therefore certainly has the property asserted of $L_E(X)/X$. Inspection of the individual constructions permitted in generating $sat(M)$ out of M now shows that each member of $sat(M)$ must have the property also; in particular therefore $L_E(X)/X$ does.

The following properties of a spectrum E and of the associated localization functor L_E are particularly desirable. It is known that these four properties are mutually equivalent [8].

- Every direct limit of E-local spectra is E-local,
- L_E commutes with direct limit (up to homotopy),
- $L_E = L_T$ where $T = L_E(S)$, the localization of the sphere spectrum,
- $L_E(X) = T \wedge X$ (up to homotopy), in particular $T = L_E S = L_E L_E S = T \wedge T \wedge S = T \wedge T$.

A spectrum (resp. localization functor) having these properties is called *smashing* [8].

Finite localization functors are smashing. For if L_E is any such then for every X the localization $L_E(X)$ is obtainable from X by repeated attaching of finite E-acyclic spectra (the convergence property). It follows that $L_E(X)$ is the direct limit of the localizations of the finite subspectra of X , thus L_E commutes with direct limit and is therefore smashing.

It has been conjectured by Bousfield [2] and Ravenel [8] that, conversely, all smashing localization functors should be of finite type. Furthermore Ravenel has formulated some spectacular conjectures which assert a complete classification of the smashing localization functors. We shall discuss these conjectures below.

One defines a partial ordering on localization functors by saying that $L' \geqslant L''$ if L' retains at least as much information as L'' does; in other words if every L'-trivial spectrum is also L''-trivial. One knows that, up to homotopy, $L'L'' = L''$ $= L''L'$ in this situation.

If a smashing localization is not trivial it is $\geqslant L_{(0)}$, the rationalization. On the other hand every rationally trivial spectrum decomposes into its p-primary parts. There is therefore no essential loss of generality in restricting attention to localization functors which are $\leqslant L_{(p)}$, the localization at a prime p. The conjectures of Ravenel, below, assert that there is precisely a sequence of smashing (or indeed, finite) localization functors between $L_{(p)}$ and $L_{(0)}$,

$$L_{(p)} = L(p,\infty) > \dots > L(p,2) > L(p,1) > L(p,0) = L_{(0)} ;$$

this (conjectural) sequence is the *chromatic filtration*.

Following Ravenel, but adapting the notion a little, let us say that a spectrum is *disharmonic* (at p, to be precise) if it is trivial with respect to all finite localization functors $< L_{(p)}$. Examples of disharmonic spectra are provided by the bounded-above p-torsion spectra (I am indebted to Bökstedt for pointing out this fact and for contributing the following argument):

Let L be a finite localization functor $< L_{(p)}$. Then L is smashing and it trivializes at least one bounded-below spectrum X not trivialized by $L_{(p)}$. Since X is bounded below the Hurewicz theorem applies, and $X \wedge Z/p$ contains as a summand a (shifted) copy of the Eilenberg-MacLane spectrum Z/p. The triviality of $L(X) = T\wedge X$ thus not only entails that of $T \wedge X \wedge Z/p$ but also that of $T \wedge Z/p$ $= L(Z/p)$. We conclude by a cofibration argument that L trivializes every p-torsion spectrum bounded both above and below, i.e. having only finitely many non-zero homotopy groups. A bounded-above spectrum, finally, is a direct limit of such, so it is trivialized by L, too.

Here is an interesting special case. Let L be a finite localization functor, and $S_L = L(S)$ the localization of the sphere spectrum. Then $S \to S_L$ is a S_L-equivalence since L is smashing. Let $\widetilde{S}_L$ be the connected cover of S_L. Then $S_L/\widetilde{S}_L$ is bounded above and hence disharmonic. It follows that $\widetilde{S}_L \to S_L$ and $S \to \widetilde{S}_L$ are also S_L-equivalences.

To conclude this review we will now describe in more detail the conjectures of Ravenel [8] as far as they are relevant to the present context. The conjectures were motivated by the manifestation of certain algebraic phenomena in the context of the Adams-Novikov spectral sequence associated to the Brown-Peterson spectrum BP. The conjectures seek to say that the algebraic phenomena are there for geometric reasons.

Let $BP_{(p)}$ denote the p-localization of BP ; it is a ring spectrum (in the sense of stable homotopy theory - no coherence conditions asserted) and its homotopy groups form a polynomial ring $Z_{(p)}[v_1,v_2,\ldots,v_n,\ldots]$ where the generator v_n has grading $2p^n-2$; it is convenient to let $v_0 = p$, the prime at hand. The multiplication by v_n gives a (graded) self-map of $BP_{(p)}$, and one defines $BP_{(p)}[v_n^{-1}]$ as the telescope of this self-map; that is, the homotopy direct limit of the sequence

$$BP_{(p)} \xrightarrow{\cdot v_n} BP_{(p)} \xrightarrow{\cdot v_n} \cdots .$$

The spectrum $BP_{(p)}[v_n^{-1}]$ admits the multiplication by v_n as an automorphism, it is thus a periodic spectrum (if $n>0$).

Following Ravenel we let L_n denote the localization functor associated to $BP_{(p)}[v_n^{-1}]$, the prime p being understood.

The *smashing conjecture* [8] asserts that L_n is smashing. This is known to be true for $n \leq p-2$ as well as for $n = 1$ if $p = 2$ [8].

When combined with the *finiteness conjecture* of Bousfield and Ravenel (that smashing localizations are necessarily finite) it asserts that L_n is finite. This is known to be true for L_1 [2] (and of course for L_0). The situation is slightly better with regard to the existence of finite L_n-trivial spectra. Such spectra have been obtained for small values of n in connection with the construction of the so-called periodic families in the stable homotopy of spheres [8], [3].

The *class invariance conjecture* [8] finally asserts that, as far as finite spectra are concerned, there are no acyclicity types beyond those provided by the L_n .

It is known [8] that the functors L_n form a sequence with respect to the partial ordering of the localization functors, namely $L_n > L_{n-1}$. The three conjectures taken together then say that the sequence of the L_n is the aforementioned chromatic filtration.

Independently of the conjectures one knows that all finite spectra X are *harmonic* [8], that is, they are local for the homology theory given by the wedge of all the $BP_{(p)}[v_n^{-1}]$; in particular if X is finite and non-trivial then $L_n(X)$ is non-trivial for sufficiently large n .

On the other hand one also knows many (infinite) X which are *dissonant*, that is, they are trivialized by each of the L_n (if the conjectures are true then "dissonant" is the same as "disharmonic"). For example the p-torsion Eilenberg-MacLane spectra are known to be dissonant [8].

3. The local counterparts of $A(*)$.

Let C denote the category of spectra. Let $L\colon C \to C$ be a localization functor. Associated to L there is a category of weak equivalences wC where, by definition, a map in C is in wC (or is a *w-map*, as we shall say) if the homotopy cofibre is trivialized by L .

A spectrum is *finite up to w-equivalence* if it is in the same connected component, in wC , as some finite spectrum; we denote the subcategory of the w-finite spectra by C_{wf} . Let $C_{(L)}$ denote the category of the L-local spectra, and

$$C_{(L)f} = C_{(L)} \cap C_{wf} .$$

If L' is a second localization functor, coarser than L , we let $C^{L'}$ denote the category of the L'-trivial spectra, and

$$C^{L'}_{(L)} = C_{(L)} \cap C^{L'} .$$

Let the *h-maps*, finally, mean the weak homotopy equivalences.

Localization theorem. Let L and L' be localization functors of finite type, and $L > L'$. There is a homotopy cartesian square

$$\begin{array}{ccc} hS.C^{L'}_{(L)f} & \longrightarrow & hS.C_{(L)f} \\ \downarrow & & \downarrow \\ hS.C^{L'}_{(L')f} & \longrightarrow & hS.C_{(L')f} \end{array}$$

where the term on the lower left is contractible.

In other words, if one considers the K-theories of the L-local and of the L'-local spectra, respectively, then their difference (i.e. the homotopy fibre of the natural map) is explicitly describable, namely it is represented by the K-theory of the category of those L-local spectra which are L'-trivial.

Proof. There is a similar looking result which is valid in a much more general context. In the situation at hand we check that the terms may be re-written in the desired form.

Namely if a category with cofibrations is equipped with two notions of weak equivalence, one finer than the other, then under rather general hypotheses which we will not spell out here, there results a homotopy cartesian square of the associated K-theories [14], [5], [16]. In particular there is such a square in the case of the category C_{hf} of the homotopy-finite spectra, equipped with the two notions of weak equivalence w and w' given by L and L', respectively. It reads

$$\begin{array}{ccc} wS.C_{hf}^{w'} & \longrightarrow & wS.C_{hf} \\ \downarrow & & \downarrow \\ w'S.C_{hf}^{w'} & \longrightarrow & w'S.C_{hf} \end{array} .$$

In order to put this square into the desired form we will need to know of the finiteness of the localization functors, and of the ensuing smashing property (section 2).

Since L is smashing we can replace it, if necessary, by the functor given by smash-product with a L-localization T of the sphere spectrum. The L-localization can thus be an exact functor in the technical sense, so it induces a map in K-theory.

Similarly L' can be replaced, if necessary, by smash-product with T'. But it can also be replaced by smash-product with $T \wedge T'$ (since $L > L'$). It results that we can define a natural transformation from the above square to the square of the theorem: On the upper terms the map is induced by smash-product with T, and on the lower terms it is induced by smash-product with $T \wedge T'$. (We are using here that $hS.C_{(L)f} = wS.C_{(L)f}$ in view of the fact that h-maps and w-maps are the same in $C_{(L)}$; and similarly with the other terms).

To conclude we check that the map of squares is a homotopy equivalence on each term. We treat only the case of the map $wS.C_{hf} \to hS.C_{(L)f}$. The other cases are similar.

The map factors as

$$wS.C_{hf} \longrightarrow wS.C_{wf} \longrightarrow hS.C_{(L)f} ,$$

so it suffices to show that these two maps are homotopy equivalences.

The inclusion $wS.C_{hf} \to wS.C_{wf}$ is a homotopy equivalence because of the approximation theorem (section 1) which applies in view of the convergence property (section 2) of the finite localization functor L.

The localization map $C_{wf} \to C_{(L)f}$ is left inverse to the inclusion $C_{(L)f} \to C_{wf}$ up to a natural transformation which is a w-equivalence. It results that the localization map induces a deformation retraction from $wS.C_{wf}$ to $wS.C_{(L)f} = hS.C_{(L)f}$. This completes the proof of the localization theorem. ∎

Let now P be a set of primes. We denote by $A(*,P)$ the analogue of $A(*)$ constructed from P-local spaces or spectra; that is, $\Omega|hS.C_{(P)f}|$.

Lemma 1. There is a natural map $A(*,P) \to K(Z_{(P)})$ which is an equivalence away from P . More precisely, the homotopy groups of the homotopy fibre are P-torsion, and the first p-torsion, $p \in P$, occurs in dimension $2p-2$.

Proof. The map is given by *linearization* (this involves a definition of the algebraic K-theory of rings analogous to that of the algebraic K-theory of spaces, but in terms of abelian-group-objects, resp. module-objects, cf. [16]). To obtain the numerical statement we have to know that $A(*,P)$ can also be defined in other terms. This is one of the main results about the algebraic K-theory of spaces, the argument is given in [16] for the case where P is the set of all primes, i.e. the case of $A(*)$. It is not difficult to modify the argument so as to apply to the case of general P . The outcome is that $A(*,P)$ may be redefined, up to homotopy, as

$$Z \times \varinjlim BH(V^k S_{(P)})^+$$

where $V^k S_{(P)}$ denotes a wedge of k P-local sphere spectra, $H(..)$ is the simplicial monoid of homotopy equivalences, $BH(..)$ its classifying space, and $(..)^+$ denotes the plus construction of Quillen. Given that, under the translation, the map $A(*,P) \to K(Z_{(P)})$ corresponds to the natural map $BH(V^k S_{(P)}) \to BGl_k(Z_{(P)})$, the asserted numerics now follows easily from the fact that the higher homotopy of $S_{(P)}$ is P-torsion only and the first p-torsion occurs in dimension $2p-3$. ∎

Lemma 2. The map $A(*,(0)) \to K(Q)$ is a homotopy equivalence.

Proof. This is the special case $P = \emptyset$ of the preceding lemma. ∎

Let $F(*,P)$ denote the K-theory of the P-local torsion spaces, or what is the same, the P-torsion spaces.

Lemma 3. There is a homotopy equivalence

$$F(*,P) \simeq \prod_{p \in P}' F(*,p)$$

where $\prod'$ denotes the restricted product, the direct limit of the products indexed by the finite subsets of P .

Proof. Every P-torsion spectrum decomposes, up to homotopy, into its p-primary parts, and only finitely many of these parts are non-trivial because of the finite type condition on the spectrum. This shows that the approximation theorem (section 1) applies to the reconstruction map $\prod'_{p \in P} C^p_f \to C^P_f$ which takes a finite collection of p-primary spectra to the wedge of these spectra. ∎

<u>Lemma</u> 4. There is a diagram of homotopy fibrations

$$\begin{array}{ccccc}
\prod'_{p\in P} F(*,p) & \longrightarrow & A(*,p) & \longrightarrow & A(*,(0)) \\
\downarrow & & \downarrow & & \downarrow \simeq \\
\prod'_{p\in P} K(Z/p) & \longrightarrow & K(Z_{(P)}) & \longrightarrow & K(Q)
\end{array} \quad .$$

In particular the square on the left is homotopy cartesian.

<u>Proof</u>. The upper row is given by the localization theorem applied to the rationalization map $A(*,P) \to A(*,(0))$, together with the rewriting provided by lemma 3. The lower row is the analogous case of Quillen's localization theorem for the map $K(Z_{(P)}) \to K(Q)$. To obtain the map from top to bottom it is necessary to rewrite the lower row suitably, namely as the analogue of the upper row in the framework of abelian-group-objects, cf. [16]. The map on the right is a homotopy equivalence by lemma 2. ∎

<u>Theorem</u>. The square

$$\begin{array}{ccc}
A(*) & \longrightarrow & \prod_p A(*,p) \\
\downarrow & & \downarrow \\
K(Z) & \longrightarrow & \prod_p K(Z_{(p)})
\end{array}$$

is homotopy cartesian, and for every prime p there is a homotopy equivalence

$$\text{fibre}(\ A(*) \to K(Z)\ )_{(p)} \quad \simeq \quad \text{fibre}(\ A(*,p) \to K(Z_{(p)})\ )\ .$$

<u>Proof</u>. By lemma 4 there are homotopy cartesian squares

$$\begin{array}{ccc}
\prod'_p F(*,p) & \longrightarrow & A(*) \\
\downarrow & & \downarrow \\
\prod'_p K(Z/p) & \longrightarrow & K(Z)
\end{array}
\qquad\qquad
\begin{array}{ccc}
F(*,p) & \longrightarrow & A(*,p) \\
\downarrow & & \downarrow \\
K(Z/p) & \longrightarrow & K(Z_{(p)})
\end{array}$$

and the localization at p induces a map from the former to the latter. We take the product of all these maps. Then the square formed by the right hand columns gives the square of the theorem. To show it is homotopy cartesian it suffices to show that the square formed by the left hand columns is homotopy cartesian. That is, we want to show that the map

$$\text{fibre}(\ \prod'_p F(*,p) \to \prod'_p K(Z/p)\ ) \longrightarrow \text{fibre}(\ \prod_p F(*,p) \to \prod_p K(Z/p)\ )$$

is a weak homotopy equivalence; equivalently (by lemma 4 and since the homotopy fibre commutes with products and direct limits, up to homotopy) that the inclusion map

$$\prod_p{}' \text{ fibre}(A(*,p) \to K(Z_{(p)})) \longrightarrow \prod_p \text{ fibre}(A(*,p) \to K(Z_{(p)}))$$

is one. But by lemma 1 the homotopy group π_n fibre($A(*,p) \to K(Z_{(p)})$) is zero for sufficiently large p (depending on n). So the map induces an isomorphism on homotopy groups.

The second part of the theorem follows from the first by taking p-localizations of the vertical fibres and noting that

$$\prod_q \text{ fibre}(A(*,q) \to K(Z_{(q)}))_{(p)} \simeq \text{ fibre}(A(*,p) \to K(Z_{(p)}))$$

in view of lemma 1. ∎

Let us fix a prime p now. Recall from section 2 the localization functors

$$L_{(p)} = L_\infty > \ldots > L_n > \ldots > L_1 > L_0$$

where L_n is associated to $BP_{(p)}[v_n^{-1}]$ (and L_0 is the same as rationalization). Following the conjectures of Bousfield and Ravenel discussed in section 2 we make the

Hypothesis. L_n is a *finite* localization functor.

Let us denote the category of the L_n-local spectra by $C_{(p,n)}$. We define $A(*,p,n)$ to be its K-theory,

$$A(*,p,n) = \Omega|hS.C_{(p,n)f}| ,$$

where as usual the subscript f indicates the finite type condition. Localization induces maps between these spaces, so we obtain a tower of spaces and maps,

$$A(*,p) = A(*,p,\infty) \longrightarrow \ldots \longrightarrow A(*,p,n) \longrightarrow \ldots \longrightarrow A(*,p,0) ,$$

interpolating between $A(*,p)$ and the K-theory of the rational numbers.

Next, let $C^{n-1}_{(p,n)}$ be the subcategory of $C_{(p,n)}$ of the spectra which are L_{n-1}-trivial; this is what Ravenel calls the n-*th monochromatic category* [8]. By the localization theorem its K-theory

$$M(*,p,n) = \Omega|hS.C^{n-1}_{(p,n)f}|$$

represents the n-th layer in the localization tower,

$$M(*,p,n) \simeq \text{fibre}(A(*,p,n) \to A(*,p,n-1)) .$$

The following argument, due to Bökstedt, can be used to prove the non-triviality of $M(*,p,n)$ in certain cases. Suppose h_* is a homology theory coarser than L_n

(that is, L_n-triviality implies h_*-acyclicity). Suppose further that for finite L_{n-1}-trivial X the groups $h_i X$ are finite and periodic of period $2s$, say. Let $c_i X$ denote the order of $h_i X$. Then, as one checks, the rational number given by the alternating product

$$cX = c_0 X \cdot (c_1 X)^{-1} \cdot c_2 X \cdot \ldots \cdot c_{2s-2} X \cdot (c_{2s-1} X)^{-1}$$

is multiplicative for cofibration sequences. It results that c defines a homomorphism from the class group $\pi_0 M(*,p,n)$ to the multiplicative group of rational numbers.

The argument applies in the case of $\pi_0 M(*,p,1)$ and shows that this group is not trivial. For it is known [8] that the localization functor L_1 is definable in terms of p-local complex K-theory, and KU_i applied to a finite torsion spectrum is certainly finite and periodic. It suffices then to note that the number cX is not 1 in the case of the Moore spectrum S/p.

It is likely that a similar argument can be applied to show that $\pi_0 M(*,p,2)$ is not trivial, and more specifically that the Toda spectrum $V(1)$ represents an element of infinite order. (Recall that $V(1)$ is the mapping cone of a certain graded self-map on the Moore spectrum S/p ; the self-map induces multiplication by (a power of) v_1 in BP-homology). Assuming this is so, we can deduce a strange looking consequence. Namely the element $[V(1)]$ in $\pi_0 M(*,p,2)$ projects to zero in $\pi_0 A(*,p,2)$ because the cofibration sequence $\Sigma^k(S/p) \rightarrowtail S/p \twoheadrightarrow V(1)$ (where k is even) implies a relation $[V(1)] = [S/p] - [S/p]$. Therefore $[V(1)]$ must be the image of some element υ_1, say, in $\pi_1 A(*,p,1)$. Thus the periodicity operator $v_1 \in \pi_* BP_{(p)}[v_1^{-1}]$ somehow corresponds to a 'phantom unit' υ_1 in algebraic K-theory.

As to a general attack on the spaces $M(*,p,n)$, the first (and perhaps main) step should be the search for a *devissage theorem*. Its content would be that for the purpose of constructing $M(*,p,n)$ one does not really need *all* of the monochromatic category $C^{n-1}_{(p,n)}$ but only a subcategory of *elementary* objects. A good candidate for the elementary objects would seem to be the spectra in $C^{n-1}_{(p,n)}$ which are periodic of minimal period.

We proceed to the construction of the *integral localization tower* $\widetilde{A}(*,p,n)$.

Recall our standing hypothesis that L_n is a finite localization functor. As a consequence L_n is smashing (section 2), and $S_{(n)}$, the L_n-localization of the sphere spectrum, satisfies $S_{(n)} \wedge S_{(n)} \simeq S_{(n)}$ and is thus a very particular kind of ring spectrum. In particular the associated infinite loop space $QS_{(n)}$ is a ring space.

Let $M_k(QS_{(n)})$ denote the space of $k \times k$ matrices. It is a multiplicative H-space and, if $n \geq 1$, the monoid of connected components is $M_k(Z_{(p)})$. Define

$\hat{GL}_k(QS_{(n)})$ as the union of connected components given by pullback with the inclusion of $GL_k(Z_{(p)})$ in $M_k(Z_{(p)})$.

Lemma. The H-space $\hat{GL}_k(QS_{(n)})$ has a canonical (up to homotopy) classifying space.

Proof. $QS_{(n)}$ may be defined as the space (or better, simplicial set) of maps $S \to S_{(n)}$, and $M_k(QS_{(n)})$ may be identified to the mapping space $Map(V^kS, V^kS_{(n)})$. The latter is homotopy equivalent to $Map(V^kS_{(n)}, V^kS_{(n)})$ which is a monoid by composition of maps; the requisite homotopy equivalences are given by restriction along $V^kS \to V^kS_{(n)}$ on the one hand and by smash product with $S_{(n)}$ on the other, using that $S_{(n)} \wedge S_{(n)} \simeq S_{(n)}$. It results that $\hat{GL}_k(QS_{(n)})$ is homotopy equivalent, as H-space, to a monoid. ∎

We define

$$\tilde{A}(*,p,n) \;=\; Z \times \varinjlim_{k} B\hat{GL}_k(QS_{(n)})^+ \;.$$

The factor Z is the class group of the ring $\pi_0 QS_{(n)}$, it has to be taken care of in this artificial way since the class group is invisible to the plus construction. The case $n = 0$ is exceptional from the present point of view, we can include it by defining $\tilde{A}(*,p,0)$ as $Z \times \varinjlim BGL_k(Z_{(p)})^+$.

By exploiting the plus construction one can arrive at a certain amount of numerics (as in [14], [16]). There is one general result which can be obtained in this way, namely the fact that the map

$$\tilde{A}(*,p,n) \longrightarrow \tilde{A}(*,p,n-1)$$

is an equivalence away from p (this uses that $QS_{(n)} \to QS_{(n-1)}$ is an equivalence away from p , as well as 1-connected). Note this is in sharp distinction from the situation with the other localization tower.

Beyond that it is possible to obtain quantitative results in (very) low dimensions. For example the first homotopy in fibre($\tilde{A}(*,p,1) \to \tilde{A}(*,p,0)$) occurs in dimension $2p-2$ and is cyclic of order p . But it seems unreasonable to expect that one can go much further in this way.

Perhaps the best approach eventually will be to compare the two localization towers. The idea is that in order to obtain information about

$$\text{fibre}(\, \tilde{A}(*,p,n) \to \tilde{A}(*,p,n-1)\,)$$

one should first try to compute with fibre($A(*,p,n) \to A(*,p,n-1)$) as well as the fibres of a natural transformation

$$\tilde{A}(*,p,n) \longrightarrow A(*,p,n) \;.$$

There is no problem in defining a map $\tilde{A}(*,p,n) \to A(*,p,n)$. Briefly, one can also construct $\tilde{A}(*,p,n)$ out of $\coprod_k B\hat{GL}_k(QS_{(n)})$ by *group completion* (with respect to block sum). And $\coprod_k B\hat{GL}_k(QS_{(n)})$ is practically contained in $|hS_1C_{(p,n)f}|$ (there are some technicalities; in particular the category $hS_1C_{(p,n)f}$ should be blown up to a homotopy equivalent simplicial category in order that one can have an honest inclusion, cf. corresponding constructions in [16]). The inclusion of $\coprod_k B\hat{GL}_k(QS_{(n)})$ into $|hS_1C_{(p,n)f}|$, the geometric realization of the category in degree 1, now induces an inclusion of the suspension $\Sigma(\coprod_k B\hat{GL}_k(QS_{(n)}))$ into $|hS.C_{(p,n)f}|$, the geometric realization of the full simplicial category. The adjoint of the latter inclusion then extends, by the group completion principle, to the desired map of $\tilde{A}(*,p,n)$ into the loop space $\Omega|hS.C_{(p,n)f}|$.

We will describe a localization theorem for the map $\tilde{A}(*,p,n) \to A(*,p,n)$ now. We need a further hypothesis. In fact we need the further hypothesis even for formulating the theorem.

The hypothesis is that there exists a category of modules over the ring spectrum $\tilde{S}_{(n)}$, the connected cover of $S_{(n)}$ (for $n \geq 1$). The hypothetical part about it is that the morphisms in the category should be actual maps, not homotopy classes of maps. (There has been done some work on module spectra in this sense by Robinson [9]; recent unpublished work of Schwänzl and Vogt is also relevant). Let the hypothetical category be denoted $\mathrm{Mod}(\tilde{S}_{(n)})$. It will be a category with cofibrations and weak equivalences in the technical sense of section 1. In fact there are two notions of weak equivalence, the h-*maps* and the w-*maps*, where the former are the weak homotopy equivalences and the latter are the maps which become equivalences upon changing the ground ring from $\tilde{S}_{(n)}$ to $S_{(n)}$ (or what amounts to the same, cf. below, the maps which become homotopy equivalences by L_n-localization).

An object of $\mathrm{Mod}(\tilde{S}_{(n)})$ is said to be *finite* if there is a finite filtration (sequence of cofibrations, that is) any quotient of which is free of rank 1, i.e. a perhaps shifted copy of $\tilde{S}_{(n)}$. Somewhat more generally we can also speak of *finiteness up to* h-*equivalence* (resp. w-*equivalence*); we indicate this by the subscript hf (resp. wf). The coarser notion of weak equivalence gives rise to the subcategory $\mathrm{Mod}(\tilde{S}_{(n)})^w$ of the w-trivial modules, or *torsion modules* as we will say.

The desired localization theorem says that the homotopy fibre

$$\mathrm{fibre}(\ \tilde{A}(*,p,n) \to A(*,p,n)\ )$$

is represented by the K-theory of the category of torsion modules over $\tilde{S}_{(n)}$.

The argument of proof is similar to that given in the proof of the localization theorem in the beginning of this section. Namely for general reasons there is a homotopy cartesian square

$$\begin{array}{ccc}
\Omega|hS.\mathrm{Mod}(\widetilde{S}_{(n)})^{w}_{hf}| & \longrightarrow & \Omega|hS.\mathrm{Mod}(\widetilde{S}_{(n)})_{hf}| \\
\downarrow & & \downarrow \\
\Omega|wS.\mathrm{Mod}(\widetilde{S}_{(n)})^{w}_{hf}| & \longrightarrow & \Omega|wS.\mathrm{Mod}(\widetilde{S}_{(n)})_{hf}|
\end{array}$$

in which the lower left term is contractible. The upper left term is the K-theory of the category of torsion modules over $\widetilde{S}_{(n)}$. It only remains to be shown, therefore, that the map on the right may be identified to the map $\widetilde{A}(*,p,n) \to A(*,p,n)$.

The identification of the upper right term with $\widetilde{A}(*,p,n)$ comes from the main result of [16]; cf. the proofs of lemmas 1 and 4 above for similar points.

The identification of the lower right term with $A(*,p,n)$ is similar to the argument at the end of the proof of the localization theorem (the last three paragraphs). Two points deserve mentioning. The first is that one can construct a L_n-localization of a given $\widetilde{S}_{(n)}$-module by (infinitely) repeated attaching of *finite* L_n-acyclic modules; this uses Bökstedt's lemma (section 2) that $S \to \widetilde{S}_{(n)}$ is a $S_{(n)}$-equivalence. It results that there exists a L_n-localization functor on $\mathrm{Mod}(\widetilde{S}_{(n)})$ which is of finite type (in view of its construction) and therefore also has the convergence property (section 2). The second point is that a L_n-local spectrum has a *unique* $S_{(n)}$-module structure which may therefore be suppressed or resurrected according to the need of the moment.

It is a matter of checking the definitions, finally, to see that under these identifications the two maps correspond as desired.

4. Appendix: An implication of the Lichtenbaum-Quillen conjecture.

We give a quick review of the Lichtenbaum-Quillen conjecture, a homotopy theoretic reformulation, and finally the application to obtaining a kind of lower bound on the difference of $A(*)$ and $K(Z)$.

The content of LQC is that for many rings (and schemes) the algebraic K-theory ought to be expressible in terms of etale cohomology and thereby computable. With the advent of the *etale K-theory* of Dwyer and Friedlander [4] a simpler, and more explicit, formulation became possible. The new formulation is that the natural transformation

$$K_*(R,Z/p) \longrightarrow K_*^{et}(R,Z/p)$$

should be an isomorphism for suitable R . Actually this is conjectured only for odd primes p , and for sufficiently high degrees; it is known that some such restriction is necessary, cf. [12].

As usual here $K_*(R,Z/p)$ denotes the *K-theory of* R *with coefficients in* Z/p . We think of it in terms of spectra, namely as the homotopy of $K(R,Z/p)$, the smash product of the K-theory spectrum $K(R)$ and the Moore spectrum S/p .

The necessity of working with finite coefficients comes from the fact that the etale homotopy, and therefore also the etale K-theory, does not behave properly unless one restricts to working with finite coefficients.

We will not define the etale K-theory here. We don't have to, in fact. For Thomason has proved the amazing result that etale K-theory is the same, in many cases, as "Bott periodic" algebraic K-theory [13]. In view of this result LQC translates into the conjecture that the map

$$K_*(R,Z/p) \longrightarrow K_*(R,Z/p)[\beta^{-1}]$$

is an isomorphism (for suitable R , odd p , and in sufficiently high degrees).

As to the *Bott periodic algebraic K-theory*, we find it convenient to use the definition given by Snaith [11]. Namely the Moore spectrum S/p supports a self-map known as the *Adams map*; if p is odd the map is of degree $2p-2$. It induces a graded self-map of $K(R,Z/p)$, and $K(R,Z/p)[\beta^{-1}]$ is now defined as the mapping telescope of the latter, the homotopy direct limit of the sequence

$$K(R,Z/p) \longrightarrow K(R,Z/p) \longrightarrow \cdots$$

in which each map is the map in question.

Actually Snaith's procedure is slightly different in that he defines $K(R,Z/p)$ as the spectrum of maps $S/p \to K(R)$, so the self-map on $K(R,Z/p)$ is given by composition with the Adams map. However the distinction is minor since the Moore spectrum and the Adams map are self-dual with respect to Spanier-Whitehead duality.

At any rate, the definition is equivalent to letting

$$K(R,Z/p)[\beta^{-1}] \;=\; K(R) \wedge S/p[\beta^{-1}]$$

where $S/p[\beta^{-1}]$ is the mapping telescope of the Adams map.

Recall the localization functor L_1 (section 2). It is known [2] that

$$S/p[\beta^{-1}] \;=\; L_1(S/p) \;.$$

Since L_1 is smashing (section 2) we obtain

$$K(R,Z/p)[\beta^{-1}] \;=\; K(R) \wedge S/p \wedge L_1(S) \;=\; L_1(K(R)) \wedge S/p \;.$$

So LQC translates into a conjecture saying that the homotopy cofibre, F say, of the localization map

$$K(R) \longrightarrow L_1(K(R))$$

is annihilated by smash product with S/p (for suitable R and odd p, that is, and in sufficiently high degrees). In view of the cofibration sequence

$$S \xrightarrow{\;\cdot p\;} S \longrightarrow S/p$$

this means that the self-map of F given by multiplication by p is an equivalence (in high degrees), so F may be identified (in high degrees) to the telescope of the self-map; that telescope is $F[p^{-1}]$, the localization away from p.

Replacing $K(R)$ by $K(R)_{(p)}$ now (the localization at p) we conclude that the homotopy cofibre of

$$K(R)_{(p)} \longrightarrow L_1(K(R)_{(p)})$$

is unchanged (in high degrees) by inverting p, that is, by the rationalization functor L_0. Since $L_0 = L_0L_1$ it follows that the homotopy cofibre is trivial (in high degrees).

We have thus translated LQC into a conjecture saying that, for suitable R, and odd p, the localization map

$$K(R)_{(p)} \longrightarrow L_1(K(R)_{(p)})$$

should be an equivalence of sufficiently highly connected covers; in other words that, apart from some bounded piece, the p-local $K(R)_{(p)}$ should already be L_1-local; in still other words that, in terms of the chromatic filtration, $K(R)_{(p)}$ should support first order phenomena only.

Before discussing any implications of LQC we must briefly comment on which rings R are supposed to be 'suitable'. Etale homotopy requires all coefficients to be finite, as pointed out before, but it also requires them to be prime to the residue characteristics at hand. As a result the etale K-theory $K_*^{et}(R,Z/p)$ is only defined if p is invertible in R, and there can't possibly be any conjecture about it otherwise.

On the other hand the homotopy theoretical reformulation of LQC makes perfect sense for general R. A standard argument shows that for some R the validity of LQC in this sense is equivalent to its validity for the ring of fractions $R[p^{-1}]$. In particular this is so for Z, the ring of integers. Namely by the theorems of Quillen, the difference of $K(Z)$ and $K(Z[p^{-1}])$ is given by $K(Z/p)$, and that is trivial at p except in degree 0.

By naturality of localization applied to the map $QS^0 \to K(Z)$ now there is a commutative diagram

$$\begin{array}{ccc} QS^0_{(p)} & \longrightarrow & K(Z)_{(p)} \\ \downarrow & & \downarrow \\ L_1(QS^0_{(p)}) & \longrightarrow & L_1(K(Z)_{(p)}) \end{array} .$$

If the right hand vertical map is assumed to be an equivalence it follows that, at p, the map $QS^0 \to K(Z)$ factors through J, the connective cover of $L_1(QS^0)$.

On the other hand the map $QS^0 \to K(Z)$ factors through $QS^0 \to A(*)$ which is known to be a split injection [15], [17]. If one assumes the validity of LQC it thus follows that (at least for odd p and in sufficiently high degrees) the difference between $A(*)$ and $K(Z)$ must in some way or other account for the difference between QS^0 and J.

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COXETER GROUPS AND ASPHERICAL MANIFOLDS

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1. Contractible manifolds and aspherical manifolds.

Suppose that M^n is a compact, contractible n-manifold with boundary. These assumptions imply that the boundary of M has the homology of an (n-1)-sphere; however, they do not imply that it is simply connected. If $n \geq 3$, then for M^n to be homeomorphic to a disk it is obviously necessary that ∂M be simply connected. For $n \geq 5$ this condition is also sufficient (cf. [5], [13], [18], [19]). On the other hand, for $n \geq 4$, there exist examples of such M^n with non-simply connected boundary (cf. [11], [12], [14]). In fact, if $n \geq 6$, then the fundamental group of the boundary can be any group G satisfying $H_1(G) = 0 = H_2(G)$ (cf. [9]).

A non-compact space W is simply connected at ∞ if every neighborhood of ∞ (i.e., every complement of a compact set) contains a simply connected neighborhood of ∞. Suppose W is a locally compact, second countable, Hausdorff space with one end (i.e., it is connected at ∞). Then W can be written as an increasing union of compact sets $W = \bigcup_{i=1}^{\infty} C_i$ where $C_1 \subset C_2 \subset \subset \ldots$, and where each $W - C_i$ is connected. The space W is semi-stable if the inverse sequence

$$\pi_1\ (W-C_1) \leftarrow \pi_1\ (W-C_2) \leftarrow \cdots$$

satisfies the Mittag-Leffler Condition, i.e., if there exists a subsequence of epimorphisms. (This condition is independent of the choice of C_i.) If W is semi-stable, then the isomorphism class of the inverse limit $\pi_1^{\infty}(W) = \varprojlim \pi_1(W-C_i)$ is independent of all choices (including base points). The space W is simply connected at ∞ if and only if it is semi-stable and $\pi_1^{\infty}(W)$ is trivial (cf. [6], [7], [17]).

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Now suppose that W^n is an open contractible n-manifold. If $n \geq 3$, then for W^n to be homeomorphic to $\mathbb{R}^n$ it is obviously necessary that it be simply connected at ∞. For $n \geq 4$ this condition is also sufficient (cf. [5],[20]). If W^n is the interior of a compact manifold M^n, then, clearly, W^n is semi-stable and $\pi_1^\infty(W) \cong \pi_1(\partial M)$. Hence, in view of our previous remarks, there exist examples which are not simply connected at ∞, for any $n \geq 4$. (For $n = 3$, there is a well-known example of Whitehead [22] of an open contractible 3-manifold which is not simply connected at ∞.)

A space is <u>aspherical</u> if its universal cover is contractible.

Aspherical manifolds arise naturally in a variety of geometric contexts. In such contexts the proof that the manifold is aspherical usually consists of a direct identification of its universal cover with Euclidean space. As examples we have: 1) the universal cover of a Riemann surface of genus > 0 is either the plane or the interior of the disk, more generally, 2) if M^n is any complete manifold of non-positive sectional curvature then the exponential map $\exp : T_xM \to M$ (at any point $x \in M$) is a covering projection; hence, the universal cover is diffeomorphic to $T_xM \cong \mathbb{R}^n$, and 3) if G is any Lie group with maximal compact subgroup K and if $\Gamma \subset G$ is any torsion-free discrete subgroup, then the universal cover of the manifold $\Gamma\backslash G/K$ is G/K which is diffeomorphic to Euclidean space. On the basis of such examples some people believed the following well-known conjecture (cf. [7], [8; p. 423]).

CONJECTURE. <u>The universal cover of any closed aspherical manifold is homeomorphic to Euclidean space.</u>

Of course, the issue here is not the existence of exotic contractible manifolds (they exist), but rather the existence of exotic contractible manifolds which simultaneously admit a group of covering transformations with compact quotient. Some positive results (i.e., non-existence results) have been obtained, e.g., in [6], [7], [10].

This paper, which is an expanded version of my lecture, is basically an exposition of some of the results of [3]. We shall discuss a method of [3] of using the theory of Coxeter groups to construct a large number of new examples of closed aspherical manifolds. Although the construction is quite classical, its full potential had not been realized previously. The most striking consequence of the construction is the existence of counterexamples to the above conjecture in each dimension ≥ 4.

In Section 2 of this paper we give some background material on Coxeter groups. In Section 3 we explain the construction in dimension two, where it reduces to the classical theory of groups generated by reflections on simply connected complete Riemann surfaces of constant curvature. The main results are explained in Section 4 where we consider the same construction in higher dimensions. In Section 5 we discuss a modification of the construction which gives many further examples. This modification is used in Section 6 to prove a result (the only new result in this paper) concerning the Novikov Conjecture. Finally, in Section 7 we discuss a conjecture concerning Euler characteristics of even-dimensional closed aspherical manifolds.

2. Coxeter groups.

In this section we review some standard material on Coxeter groups. For the complete details, see [2].

Let Ω be a finite graph (i.e., a 1-dimensional finite simplicial complex), with vertex set V and edge set E and let $m : E \to \mathbb{Z}$ be a function which assigns to each edge an integer ≥ 2. For each pair $(v,w) \in V \times V$ put

$$m(v,w) = \begin{cases} 1 & ; \text{ if } v = w \\ m(\{v,w\}); & \text{ if } \{v,w\} \in E \\ \infty & ; \text{ otherwise.} \end{cases}$$

These data give a presentation of a group:

$$\Gamma = \langle V; (vw)^{m(v,w)} = 1\rangle, \quad (v,w) \in V \times V.$$

Let $(e_v)_{v\in V}$ be the standard basis for the vector space $\mathbb{R}^V$. Define a symmetric bilinear form B on $\mathbb{R}^V$ by

$$B(e_v,e_w) = -\cos(\pi/m(v,w))$$

(where π/∞ is interpreted as 0). For each $v \in V$, let σ_v denote the linear reflection on $\mathbb{R}^V$ defined by $\sigma_v(x) = x - 2B(e_v,x)e_v$ and let $\bar{\Gamma}$ be the subgroup of $GL(\mathbb{R}^V)$ generated by $(\sigma_v)_{v\in V}$. (Note that $\bar{\Gamma}$ leaves the form B invariant.)

Suppose that v,w are distinct elements of V, that P is the plane spanned by e_v and e_w and that $m = m(v,w)$. The restriction of B to P is positive semi-definite and it is positive definite if and only if $m \neq \infty$. Moreover, if m is finite, then $\sigma_v|P$ and $\sigma_w|P$ are the orthogonal reflections through the lines orthogonal to e_v and to e_w, respectively, and these lines make an angle of π/m. Hence, $\sigma_v\sigma_w|P$ is a rotation through and angle of $2\pi/m$ and $\sigma_v|P$ and $\sigma_w|P$ generate a dihedral group of order $2m$. Since $\sigma_v\sigma_w$ fixes $P^\perp$, it follows that $\sigma_v\sigma_w$ has order m. If $m = \infty$, then one can easily show that $\sigma_v\sigma_w$ has order m.

It follows that the map $v \to \sigma_v$ extends to an isomorphism $\Gamma \to \bar{\Gamma}$, called the _canonical representation_ of Γ. The construction of this representation shows that a) the natural map $i : V \to \Gamma$ is an injection, and that b) the order of $i(v)i(w)$ is equal to $m(v,w)$ (rather than just dividing $m(v,w)$). Henceforth, we identify V with $i(V)$. The pair (Γ,V) is a _Coxeter system_ and Γ is a _Coxeter group_. The graph Ω together with the labelling of its edges is the _associated labelled graph_. It follows from property b) above that the correspondence between labelled graphs and isomorphism classes of Coxeter systems is bijective.

There is another way to record the same information as is contained in the associated labelled graph. Let Ω' be the graph with the same vertex set V as Ω but with edge set E' obtained by first deleting the elements of E

labelled 2 and then adding edges for each unordered pair of distinct vertices $\{v,w\}$ not in E (i.e., with $m(v,w) = \infty$). As a notational simplification the edges labelled 3 are usually left unmarked. The graph Ω' together with the labelling of its edges is called the Coxeter diagram of (Γ,V). A Coxeter system is irreducible if its Coxeter diagram is connected.

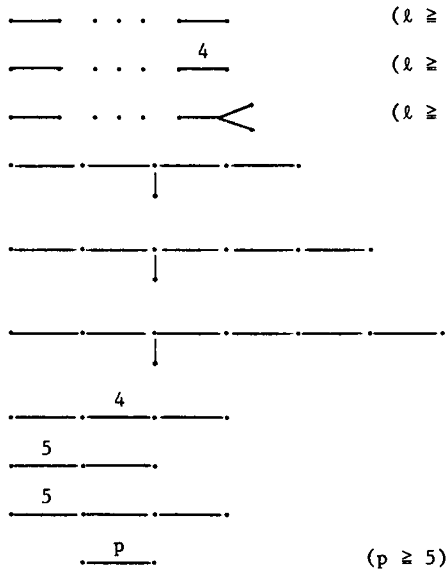
Suppose that (Γ,V) is a Coxeter system. For any subset S of V denote by Γ_S the subgroup generated by S. (It turns out that the pair (Γ_S,S) is also a Coxeter system.) If the Coxeter diagram of (Γ,V) has k components with vertex sets $V_1,\ldots,V_k$, then $\Gamma = \Gamma_{V_1} \times \ldots \times \Gamma_{V_k}$.

Finite Coxeter groups. A Coxeter group Γ is finite if and only if the form B is positive definite. Suppose that this is the case. Let C be the simplicial cone in $\mathbb{R}^V$ defined by the equations: $B(e_v,x) \geq 0$, $v \in V$. Thus, $(e_v)_{v \in V}$ is the set of inward pointing unit normals to the "panels" (i.e., codimension one faces) of C. Moreover, C is a closed fundamental domain for Γ on $\mathbb{R}^V$ in the sense that it intersects each Γ-orbit in exactly one point. (It follows that the orbit space $\mathbb{R}^V/\Gamma$ is homeomorphic to C.)

Still supposing that Γ is finite, we have that order $(vw) = m(v,w) < \infty$ for each pair (v,w) of vertices. Hence, the associated graph Ω is the 1-skeleton of the simplex with vertex set V. On the other hand, it turns out, that each component of the Coxeter diagram Ω' is a tree. The well-known list of Coxeter diagrams of irreducible Coxeter systems of finite Coxeter groups is given below. The list contains one infinite family $I_2(p)$ in dimension 2 (where $I_2(p)$ denotes the dihedral group of order $2p$), three families A_ℓ, B_ℓ, D_ℓ in each dimension ℓ (with a few restrictions in low dimensions), and 6 additional groups.

Many of these groups have other convenient descriptions. For example, A_ℓ is the symmetric group on $\ell + 1$ symbols, while A_3, B_3, H_3 are the full groups of motions of regular solids, namely, the tetrahedron, the octahedron, and the icosahedron, respectively.

<u>Coxeter Diagrams of Irreducible Finite Coxeter Groups</u>



A_ℓ ($\ell \geq 1$ vertices)

B_ℓ ($\ell \geq 2$ vertices)

D_ℓ ($\ell \geq 4$ vertices)

E_6

E_7

E_8

F_4

H_3

H_4

$I_2(p)$ ($p \geq 5$)

3. <u>The construction in dimension two.</u>

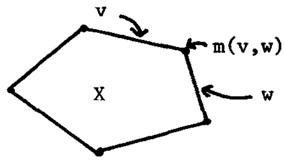
In dimension two all our constructions reduce to well-known classical results. We shall now review these results.

Let X be a polygon. We shall find it convenient to work with the graph Ω which is the dual of ∂X. Thus, if V denotes the vertex set of Ω and E the edge set, then

$$V = \{\text{edges of } \partial X\}$$

$$E = \{\{v,w\} \mid v,w \in V,\ v \neq w,\ v \cap w \neq \emptyset\}.$$

Equivalently, E is the set of vertices of X. Choose a labelling $m : E \to \{2,3,\ldots\}$. Thus, we label the vertices of X by integers ≥ 2.

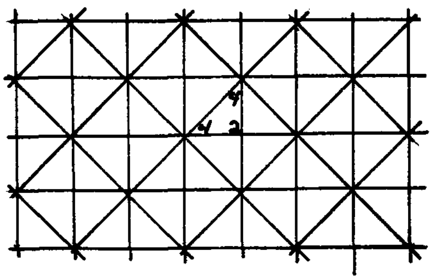


The labelled graph Ω defines a Coxeter system (Γ, V). For each x in X, let $V(x)$ denote the set of v in V such that $x \in v$. Let $\Gamma_{V(x)}$ be the subgroup generated by $V(x)$. (By convention $\Gamma_\emptyset$ is the trivial group.) Thus, if x belongs to the interior of X, then $\Gamma_{V(x)}$ is trivial; if x belongs to the interior of an edge v, then $\Gamma_{V(x)}$ is the cyclic group of order 2 generated by v; and if x is a vertex, then $\Gamma_{V(x)}$ is the dihedral group generated by the edges containing x.

There is an obvious method for constructing a Γ-space $\mathcal{U}$ by pasting together copies of X, one for each element of Γ. To be precise, put $\mathcal{U} = (\Gamma \times X)/\sim$ where the equivalence relation $\sim$ is defined by

$$(g,x) \sim (h,y) \iff x = y \quad \text{and} \quad g^{-1}h \in \Gamma_{V(x)}.$$

Let $[g,x]$ denote the equivalence class of (g,x). There is a natural Γ-action on $\mathcal{U}$ defined by $h[g,x] = [hg,x]$. The isotropy group at $[g,x]$ is clearly $g\Gamma_{V(x)}g^{-1}$. Since each of these isotropy groups is finite, it is easy to see that the action is proper. It is also not difficult to see that $\mathcal{U}$ is a 2-manifold. (At each edge two copies of X fit together. At a vertex labelled m the picture is locally isomorphic to the canonical action of the dihedral group of order $2m$ on $\mathbb{R}^2$.)



The surface $\mathcal{U}$ is simply connected. This can be seen geometrically using the developing map. (See Remark 1 at the end of this section or [21].) It also follows from the results of the next section. There is also a direct argument using covering space theory. (Let $p : \tilde{\mathcal{U}} \to \mathcal{U}$ be the universal cover of $\mathcal{U}$, let $\tilde{X}$ be a component of $p^{-1}(X)$, and let $s : X \to \tilde{X}$ be the inverse of the homeomorphism $p|X$. Lift each involution v in V to an involution $\tilde{v}$ on $\tilde{\mathcal{U}}$ such that the fixed set of $\tilde{v}$ contains the corresponding edge of $\tilde{X}$. This defines a lift of the Γ-action to $\tilde{\mathcal{U}}$. The mapping $s: X \to \tilde{X}$, then extends to a Γ-equivariant section $\mathcal{U} \to \tilde{\mathcal{U}}$. Hence, the covering is trivial and $\mathcal{U}$ is simply connected.)

Since $\mathcal{U}$ is a simply connected surface, it is homeomorphic either to S^2 or to $\mathbb{R}^2$. The case $\mathcal{U} = S^2$ occurs if and only if Γ is finite. By the classification of finite Coxeter groups described in the previous section, this happens if and only if X is a triangle and the set of labels $\{p,q,r\}$ is either $\{2,2,r\}$, $\{2,3,3\}$, $\{2,3,4\}$, or $\{2,3,5\}$ (corresponding, respectively, to the groups $A_1 \times I_2(r), A_3, B_3$, or H_3).

The moral to be drawn from the above discussion is that apart from a few exceptional cases this construction always leads to a contractible 2-manifold $\mathcal{U}$. As we shall see in the next section, virtually the same construction works in any dimension. The surprising fact is that, under a mild restriction, the resulting manifold is also contractible.

At this point we have not yet constructed any closed aspherical manifolds. The problem is that the transformation group Γ does not act freely on $\mathcal{U}$. This can be remedied as follows. Suppose Γ is infinite and let Γ' be any torsion-free subgroup of finite index in Γ. (There are various algorithms for finding such subgroups; however, in general, none of them are very satisfactory. However, as we have seen in the previous section any Coxeter group is a subgroup of some linear group; hence, it follows from Selberg's Lemma (cf. [15]) that any Coxeter group is virtually torsion-free.) Since each Γ-isotropy group is finite, each Γ'-isotropy group is trivial; hence, $\mathcal{U} \to \mathcal{U}/\Gamma'$ is a covering projection. Since $[\Gamma:\Gamma'] < \infty$, $\mathcal{U}/\Gamma'$ is compact; hence, $\mathcal{U}/\Gamma'$ is a closed aspherical surface.

REMARK 1. Since the local picture in $\mathcal{U}$ near a vertex in X labelled m is isomorphic to the canonical action of the dihedral group of order $2m$ on $\mathbb{R}^2$, we should think of the label as specifying an interior angle of π/m at this vertex. Depending on whether the sum of this interior angles is greater than, equal to, or less than $\pi(\text{Card } V - 2)$, X can be realized as a convex polygon in, respectively, S^2, the Euclidean plane $\mathbb{R}^2$, or the hyperbolic plane H^2 with interior angles as specified by the labels. There is then a well-defined homomorphism from Γ onto $\bar{\Gamma}$, the group generated by the orthogonal reflections through the sides of this convex polygon. Using the Γ-actions, we obtain a map $\mathcal{U} \to M^2$, where M^2 denotes the appropriate choice of $S^2, \mathbb{R}^2$, or H^2. This map is easily seen to be a covering projection. Since M^2 is simply connected, this map is a homeomorphism. It follows that $\bar{\Gamma}$ is discrete and isomorphic to Γ.

The classification of finite Coxeter groups in dimension 3 can then be recovered from the facts that 1) any convex polygon in S^2 with non-obtuse interior angles is a spherical triangle and 2) the sum of the interior angles in such a triangle is $> \pi$.

REMARK 2. In higher dimensions the situation with cocompact geometric reflection groups is as follows. In the spherical case, the group Γ is a finite Coxeter group and the fundamental chamber X is a spherical simplex. In the flat case, there is also a complete classification of possible Coxeter groups (cf. [2, p. 199]); moreover, X is a product of Euclidean simplices, one for each irreducible factor of Γ. In the hyperbolic case, the Coxeter group Γ must be irreducible; however, the chamber X need not be a simplex (as we have seen already in dimension 2). If it is a simplex, then there are only a few possibilities: 9 in dimension 3, 5 in dimension 4, and none in higher dimensions (cf. Exercise 15, p. 133 in [2]). In the general hyperbolic case, the situation is as follows.

In dimension 3 there is a rich theory and complete result due to Andreev (cf. [1] or [21]). In dimensions > 3 there are a few isolated examples but no general understanding of the possibilities; while in very high dimensions (something like dimensions > 30) Vinberg has apparently proved that cocompact hyperbolic reflection groups do not exist. In summary, relatively few Coxeter groups have representations as cocompact geometric reflection groups and in these cases, at least in dimensions > 3, there are very few combinatorial types of convex polyhedra which can occur as fundamental chambers. As we shall see in the next section, if we drop our geometric requirements, then the situation reverts to its original simplicity.

4. The construction in dimension n.

Let X be a compact, contractible n-manifold with boundary and let L be a PL-triangulation of its boundary. The simplicial complex L will be used for two purposes. First, a Coxeter system will be constructed by labelling the edges of the 1-skeleton of L. Second, ∂X will be given the structure of the dual cell complex to L.

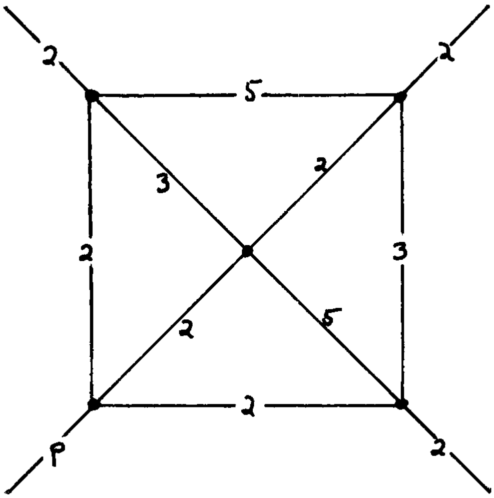
Let V be the vertex set of L, E the edge set, and Ω the 1-skeleton. There are two conditions which we want our labelling $m : E \to \{2,3,\ldots\}$ to satisfy. The first condition is the following:

(*) For each simplex $S \in L$ the subgroup Γ_S, generated by S, is finite.

This means that for any $S \in L$ if we discard the edges labelled 2, then the resulting labelled graph is the Coxeter diagram of a finite Coxeter group. For example, if L is an octahedron we could label its edges as below. In general, for any simplicial complex L if we label every edge by 2, then condition (*) holds, since in this case for each $S \in L$ we will have $\Gamma_S = (\mathbb{Z}/2\mathbb{Z})^S$.

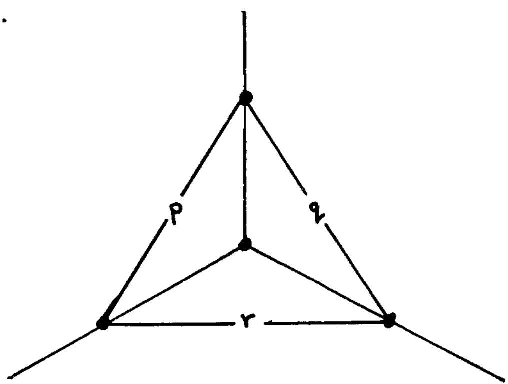
The second condition is the converse to the first:

(**) If S is a subset of V such that Γ_S is finite, then $S \in L$.



L is an octahedron with a vertex at ∞.

Since by construction an unordered pair of vertices $\{v,w\}$ belongs to E if and only if $m(v,w) < \infty$, this condition is vacuous for subsets of cardinality less than 3. Hence, condition (**) means that if Ω contains a subgraph with vertex set S which is isomorphic to the 1-skeleton of a simplex and which is not equal to the 1-skeleton of a simplex in L, then the edge labels must be such that Γ_S is infinite. For example, if L is the suspension of a triangle, then the labels p,q,r on the edges of the triangle must satisfy $p^{-1} + q^{-1} + r^{-1} \leq 1$.



L is the suspension of a triangle with a vertex at ∞.

If any subgraph of Ω which is isomorphic to the 1-skeleton of a simplex is equal to the 1-skeleton of some simplex in L, then condition (**) holds vacuously. For example, the octahedron has this property as does any polygon with more than 3 edges. More generally, if L is any simplicial complex, then its barycentric subdivision has this property (cf. Lemma 11.3 in [3]). Therefore, conditions (*) and (**) are always satisfied if we replace L by its barycentric subdivision and label each edge 2 (or in any other fashion which satisfies (*)). We now assume that we have labelled the edges of L in some fashion so that conditions (*) and (**) hold and we let (Γ, V) denote the resulting Coxeter system.

Next we cellulate ∂X as the dual cell complex. Thus, for example, if L is an octahedron, X will be a cube. For each $v \in V$, let X_v denote the dual cell of $\{v\}$ and for each simplex $S \in L$, let X_S be the dual cell of S. Thus,

$$X_S = \bigcap_{v \in S} X_v$$

(The X_v, which are faces of codimension one in X, are called the <u>panels</u> of X.) Also, for each subset S of V put

$$X_{\sigma(S)} = \bigcup_{v \in S} X_v.$$

If S is actually a simplex of L, then $X_{\sigma(S)}$ is a regular neighborhood of S in the barycentric subdivision of L. Hence,

(D) <u>If</u> $S \in L$, <u>then the union of panels</u> $X_{\sigma(S)}$ <u>is a disk of codimension zero in</u> ∂X.

For each $x \in X$ let $V(x) = \{v \in V | x \in X_v\}$ be the set of panels which contain x and let $\Gamma_{V(x)}$ be the subgroup generated $V(x)$. As before, we define a

Γ-space $\mathcal{U} = (\Gamma \times X)/\sim$, where the equivalence relation $\sim$ is defined exactly as in the previous section. The map $x \to [1,x]$ induces an embedding $X \to \mathcal{U}$ which we regard as an inclusion. Observe that a) X is a fundamental domain for Γ on $\mathcal{U}$ and that b) for each $x \in X$ the isotropy subgroup is $\Gamma_{V(x)}$. We claim that:

(1) Γ acts properly on $\mathcal{U}$.

(2) $\mathcal{U}$ is a manifold and Γ acts locally smoothly.

(3) $\mathcal{U}$ is contractible.

(4) If L $(=\partial X)$ is not simply connected, then $\mathcal{U}$ is not simply connected at ∞.

Basically, (1) is equivalent to condition (*), while (3) is equivalent to condition (**).

<u>Proof of</u> (1) <u>and</u> (2). To say that the cells of a polyhedron intersect in general position means that the dual polyhedron is a simplicial complex. Hence, the panels of X intersect in general position. This means that for each $x \in X$ we can find a neighborhood U_x of x in X of the form $\mathbb{R}^m \times C^{V(x)}$ where $\mathbb{R}^m$ is a neighborhood of x in $X_{V(x)}$ and where $C^{V(x)}$ is the standard simplicial cone in $\mathbb{R}^{V(x)}$. Let $W_x = \Gamma_{V(x)}U_x$ be the corresponding neighborhood of x in $\mathcal{U}$. Recall that an action of a discrete group is proper if and only if (i) the orbit space is Hausdorff, (ii) each isotropy group is finite and (iii) each point has a neighborhood which is invariant under the isotropy subgroup and which is disjoint from all other translates of itself.
Since $\mathcal{U}/\Gamma \cong X$, (i) holds. Condition (*) implies (ii). Also, gW_x is a neighborhood of $[g,x]$ satisfying (iii). Hence, Γ acts properly. Since $\Gamma_{V(x)}$ is a finite Coxeter group, a fundamental chamber for its canonical action on $\mathbb{R}^{V(x)}$ is $C^{V(x)}$. Hence, $\Gamma_{V(x)}C^{V(x)} \cong \mathbb{R}^{V(x)}$ and $W_x = \Gamma_{V(x)}U_x \cong \mathbb{R}^m \times \mathbb{R}^{V(x)}$. This shows that $\mathcal{U}$ is locally Euclidean and that the action is locally linear, proving (2).

<u>Proof of</u> (3) <u>and</u> (4). For any $g \in \Gamma$ let $\ell(g)$ denote its word length with respect to the generating set V. Let V^g denote the set of "reflections" through the panels of gX (i.e., $V^g = gVg^{-1}$.) Put

$$C(g) = \{w \in V^g \mid \ell(wg) < \ell(g)\} \quad \text{and}$$
$$B(g) = \{v \in V \mid \ell(gv) < \ell(g)\} = g^{-1}C(g)g.$$

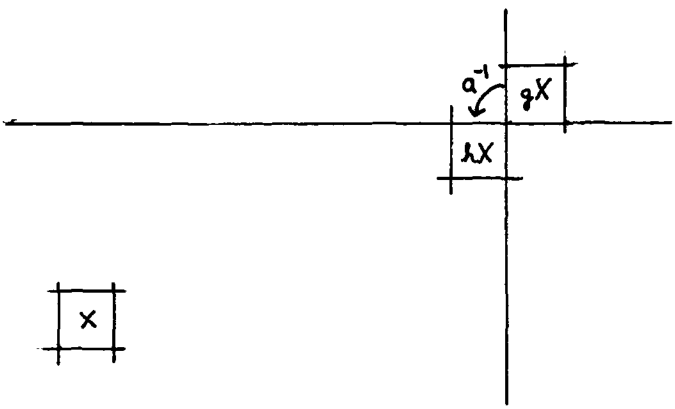
($C(g)$ is the set of reflections across panels of gX such that the reflected image of gX is closer to X than is gX. $B(g)$ is the set of reflections across these same panels after they have been translated back to X.) Also, put

$$\delta(gX) = gX_{\sigma(B(g))},$$

i.e., $\delta(gX)$ is the union of those panels of gX which are indexed by $C(g)$.

LEMMA A (cf. [3, Lemma 7.12] or [16, p. 108]). _For any_ $g \in \Gamma$, _the subgroup_ $\Gamma_{B(g)}$ _is finite_.

Sketch of Proof. Finite Coxeter groups are distinguished from infinite Coxeter groups by the fact that each finite one has a unique element of longest length. It is not hard to see that $\Gamma_{B(g)}$ has such an element. Explicitly, let h be the (unique) element of shortest length in the coset $g\Gamma_{B(g)}$. Then it follows from Exercises 3 and 22, p. 43, in [2] that $a = gh^{-1}$ is the element of longest length in $\Gamma_{B(g)}$. Q.E.D.



Next order the elements of Γ,

$$g_1, g_2, \ldots .$$

so that $\ell(g_{i+1}) \geq \ell(g_i)$. Since 1 is the unique element of length 0, $g_1 = 1$. For each integer $m \geq 1$, put

$$X_m = g_m X, \qquad \delta X_m = \delta(g_m X),$$

$$T_m = \bigcup_{i=1}^{m} X_i .$$

The next lemma asserts that the chambers intersect as one would expect them to. (For a proof see [3, Lemma 8.2].)

LEMMA B. <u>For each integer</u> $m \geq 2$, $X_m \cap T_{m-1} = \delta X_m$.

Thus, T_m is obtained from T_{m-1} by pasting on a copy of X along a certain union of panels.

By Lemma A, for each $g \in \Gamma$ the group $\Gamma_{B(g)}$ is finite. Condition (**) implies that $B(g) \in L$. Statement (D) then implies that the union of panels $X_{\sigma(B(g))}$ is a disk of codimension zero in ∂X for each $g \in \Gamma$. Since $\delta X_m = g_m X_{\sigma(B(g_m))}$, this means that δX_m is a disk of codimension zero in ∂X_m. Hence T_m is the boundary connected sum of m copies of X. Since X is contractible, so is T_m. Since $\mathcal{U} = \bigcup_{m=1}^{\infty} T_m$, $\mathcal{U}$ is contractible, which proves 3).

Since $\mathcal{U}$ is formed by successively pasting on copies of X (which are contractible) to T_m along disks, $\mathcal{U} - T_m$ is homotopy equivalent to ∂T_m. Since ∂T_m is the connected sum of m copies of ∂X, we have (provided $\dim X \geq 3$) that $\pi_1(\partial T_m)$ is the free product of m copies of $\pi_1(\partial X)$.

Moreover, the map $\pi_1(\mathcal{U}-T_{m+1}) \cong \pi_1(\partial T_{m+1}) \to \pi_1(\mathcal{U}-T_m) \cong \pi_1(\partial T_m)$ induced by inclusion can clearly be identified with the projection onto the first m factors of the free product. In particular, this map is onto for each $m \geq 1$. Thus, $\mathcal{U}$ is semi-stable and the inverse limit $\pi_1^\infty(\mathcal{U})$ is the "projective free product" of an infinite number of copies of $\pi_1(\partial X)$. Hence, if $\pi_1(\partial X)$ is not trivial, then this inverse limit is not trivial (or even finitely generated). This proves (4).

REMARK 1. As we have previously remarked, Γ always contains torsion-free subgroups of finite index and any such subgroup Γ' leads to a closed aspherical manifold $\mathcal{U}/\Gamma'$. In view of the fact that ∂X may be non-simply connected whenever $\dim X \geq 4$, statement (4) implies that the conjecture of Section 1 is false in dimensions ≥ 4.

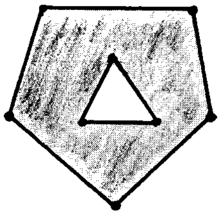
REMARK 2. In the special case where (Γ,V) is obtained by labelling each edge of a graph by 2, there is any easy construction of a torsion-free subgroup Γ'. Let (H,V) be the Coxeter system defined by setting $m(v,w) = 2$ for each pair $\{v,w\}$ of distinct vertices in V. Thus, H is the finite Coxeter group $(\mathbb{Z}/2\mathbb{Z})^V$. There is a natural epimorphism $\varphi : \Gamma \to H$ which is the identity on V. Let Γ' be the kernel of φ. If S is any subset of V such that Γ_S is finite, then $\Gamma_S \cong ((\mathbb{Z}/2\mathbb{Z})^S = H_S$; hence, for any such S, $\varphi|\Gamma_S$ is an isomorphism onto H_S. Since any finite subgroup of Γ is contained in some isotropy group and is consequently conjugate to a subgroup of some Γ_S, we have that Γ' is torsion-free. (Incidentally, Γ' is the commutator subgroup.) If $\mathcal{U} = (\Gamma\times X)/\sim$, then there is an alternative description of the quotient $\mathcal{U}/\Gamma'$. Namely, $\mathcal{U}/\Gamma' \cong (H\times X)/\sim'$ where $(g,x) \sim' (h,x) \Leftrightarrow g^{-1}h \in H_{V(x)}$. Thus, H acts a reflection group on $\mathcal{U}/\Gamma'$ with quotient X.

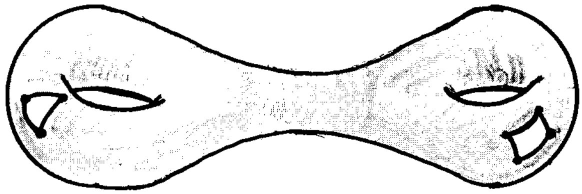
REMARK 3. The construction of this section suggests several questions concerning fundamental groups at ∞. First of all, R. Geoghegan has conjectured that if the universal cover of a finite complex has one end, then it must be semi-stable. After seeing the above construction, H. Sah and R. Schultz both asked if the fundamental group at ∞ of the universal cover of a closed aspherical manifold can ever be non-trivial and finitely generated (i.e., can the universal cover be stable at ∞ without being simply connected at ∞).

REMARK 4. There is some flexibility in the main construction of this section. First of all, for $\mathcal{U}$ to be contractible it is not necessary that each face of X be a cell. All that the proof requires is that X be contractible and that each proper face be acyclic. (For example, we can change a panel of X by taking connected sun with a homology sphere.) Secondly and more interestingly, it is not necessary for the simplicial complex L to be a PL triangulation of a homology sphere. All that is required is that L have the homology of an (n-1)-sphere and that it be a polyhedral homology manifold (i.e., the link of each k-simplex must have the homology of an (n-k-2)-sphere). Any such L can then be "dualized" (i.e., "resolved") to produce a contractible n-manifold X with contractible faces. This X can then be used to produce a contractible $\mathcal{U}$ as before. If one is interested in constructing proper, locally smooth actions of a discrete group generated by "reflections" on a contractible manifold with compact quotient, then there is no further flexibility. That is to say, with the above two provisos, *every cocompact reflection group on a contractible manifold can be constructed as above* (cf. [3]). However, as we shall see in the next section, there is quite a bit more flexibility if we are only interested in producing more examples of aspherical manifolds.

5. *A generalization*.

We begin this section by considering a modification of our construction in dimension 2. Rather than starting with the underlying space of X a 2-disk, let it be any compact surface with nonempty boundary. Take a polygonal subdivision of the boundary and label the vertices by integers ≥ 2. For example, X could be one of the "orbifolds" pictured below with random labels on the vertices. The polygonal subdivision of the boundary together with the labelling





defines a Coxeter system (Γ, V). As before, paste together copies of X to obtain a surface $\mathcal{U}$ $(=(\Gamma \times X)/\sim)$ with Γ-action. The surface $\mathcal{U}$ will no longer be contractible, but it will be aspherical provided X is not a 2-disk. (After all, almost every surface is aspherical.) If Γ' is any closed torsion-free subgroup of finite index in Γ, then $\mathcal{U}/\Gamma'$ will be a closed aspherical surface.

Next let us try to make the same modification in an arbitrary dimension. Let X be a compact aspherical n-manifold with boundary. (The boundary need not be aspherical.) Let L be a triangulation of ∂X. After possibly replacing L by its barycentric subdivision, we can find a labelling of its edges so that the resulting Coxeter system (Γ, V) satisfies conditions (*) and (**). As in Section 4, there results an n-manifold $\mathcal{U}$ with proper Γ-action. The proof of Claim (3) in Section 4 shows that $\mathcal{U}$ is homeomorphic to the infinite boundary connected sum of copies of X. Since X is aspherical, so is $\mathcal{U}$ (the wedge of two aspherical spaces is again aspherical). The fundamental group of $\mathcal{U}$ is an infinite free product of copies of $\pi_1(X)$. If Γ' is a torsion-free subgroup of finite index in Γ, then $\mathcal{U}/\Gamma'$ is a closed aspherical manifold; its fundamental group is, of course, an extension of Γ' by $\pi_1(\mathcal{U})$.

REMARK. One can imagine situations where it would be convenient if one could double a compact aspherical manifold along its boundary and obtain a closed aspherical manifold as the result. However, the doubled manifold is not aspherical unless the boundary is aspherical and its fundamental group injects into the fundamental group of the original manifold. The method described above can be viewed as a fancy method of doubling so that the result will be aspherical.

In [21] Thurston considers the above construction for certain compact aspherical 3-manifolds. (In fact, the discussion in [21] inspired the results of Sections 4 and 5.) Thurston shows that with a few more hypotheses the 3-dimensional orbifold can be given a hyperbolic structure. This allows him to "double" certain hyperbolic 3-manifolds along their boundaries (or actually sub-surfaces of their boundaries). This "orbifold trick" plays an important technical role in the proof of his famous theorem on atoroidal Haken 3-manifolds.

6. An observation concerning the Novikov Conjecture.

A group is geometrically finite if it is the fundamental group of an aspherical finite complex (or equivalently, if it is the fundamental group of an aspherical compact manifold with boundary).

If M^n is a manifold with boundary with fundamental group π, then there is a surgery map $\sigma : [(M,\partial M);(G/TOP,*)] \to L_n(\pi)$. The map σ need not be a homomorphism; however, it is if we replace M by $M \times D^i$, $i > 0$, and ∂M by $\partial(M\times D^i)$.

The Novikov Conjecture for a geometrically finite group π asserts that if M^n is any aspherical compact manifold with fundamental group π, then $\sigma : [M\times I,\partial(M\times I),(G/TOP,*)] \to L_{n+1}(\pi)$ becomes a monomorphism after tensoring with the rationals. A stronger version of this asserts that σ is a monomorphism. (See [4] for a discussion of these conjectures.)

PROPOSITION. *If the Novikov Conjecture (resp. the strong version of the Novikov Conjecture) holds for the fundamental group of every closed aspherical manifold, then it holds for every geometrically finite group.**

*The fact that the construction of the previous section could be used to prove this proposition first came up during a conversation with John Morgan.

Proof. Let $(X,\partial X)$ be an aspherical compact manifold with fundamental group π. Triangulate ∂X, take the barycentric subdivision, and label each edge by 2 to form a Coxeter system (Γ,V). Let $H = (\mathbb{Z}/2\mathbb{Z})^V$, $\Gamma' = \ker(\Gamma \to H)$, $\mathcal{U} = (\Gamma\times X)/\sim$, and $\mathcal{U}' = \mathcal{U}/\Gamma' = (H\times X)/\sim'$, be as in Remark 2 of Section 4. There is a commutative diagram

$$\begin{array}{ccc} [(X,\partial X),\ (G/TOP,*)] & \xrightarrow{\sigma} & L_n(\pi) \\ \downarrow \Lambda^* & & \downarrow i_* \\ [\mathcal{U}',\ G/TOP] & \xrightarrow{\sigma} & L_n(\pi') \end{array}$$

where $\Lambda : \mathcal{U}' \to X/\partial X$ is the map which collapses everything outside X to a point, where $\pi' = \pi_1(\mathcal{U}')$, and where i_* is the map induced by the inclusion $\pi = \pi_1(X) \hookrightarrow \pi_1(\mathcal{U}') = \pi'$. The proposition follows easily from the next claim. (If σ is not a homomorphism, then replace X by $X \times D^4$, ∂X by $\partial(X\times D^4)$, $\mathcal{U}'$ by $\mathcal{U}' \times D^4$ and $[\mathcal{U}',G/TOP]$ by $[(\mathcal{U}'\times D^4,\ \mathcal{U}'\times S^3);\ (G/TOP,*)]$.)

CLAIM. The map $\Lambda^* : [(X,\partial X),(G/TOP,*)] \to [\mathcal{U}',G/TOP]$ is a monomorphism.

Proof of Claim. We first proved the corresponding statement in cohomology. The argument is based on the existence of an "alternation map" (cf. Section 9 in [3]). If α is a singular chain in X, then we can "alternate" it to form the chain

$$A(\alpha) = \sum_{h\in H} (-1)^{\ell(h)} h \cdot \alpha$$

in $\mathcal{U}'$. The map $\alpha \to A(\alpha)$ clearly vanishes on $C_*(\partial X)$; hence, there is a chain map $A : C_*(X,\partial X) \to C_*(\mathcal{U}')$. This induces a homomorphism $A^* : H^*(\mathcal{U}') \to H^*(X,\partial X)$ which is a splitting for $\Lambda^* : H^*(X,\partial X) \to H^*(\mathcal{U}')$. Hence, Λ^* is a split monomorphism on cohomology (with arbitrary coefficients). Since for any space Y, $[Y,G/TOP] \otimes \mathbb{Q} \cong \Sigma H^{4*}(Y;\mathbb{Q})$, this is enough to prove that Λ^* is rationally a monomorphism on $[(X,\partial X),(G/TOP,*)]$. (Hence, the proposition holds for the weak version of the Novikov Conjecture.) According to Sullivan's calculation of the homotopy type of G/TOP, showing that Λ^* is a monomorphism is equivalent to showing that it is a monomorphism on $H^*(\ ,\mathbb{Z}_{(2)})$ and on $KO^*(\ ) \otimes \mathbb{Z}[\frac{1}{2}]$. We have already proved it for ordinary cohomology with arbitrary coefficients. To prove the corresponding statement for an extraordinary cohomology theory we first need to make some small modifications.

Let R be a ring in which $|H|$ is invertible (i.e., in which 2 is invertible). Put

$$\hat{A} = |H|^{-1} \sum_{h \in H} (-1)^{\ell(h)} h \in R[H].$$

For any $R[H]$-module M define a submodule $M^{\mathrm{Alt}} = \{z \in M \mid vz = -z, \forall v \in V\}$. It is easily checked that

(a) $\hat{A}^2 = \hat{A}$

(b) $M^{\mathrm{Alt}} = \mathrm{Image}(\hat{A}: M \to M)$.

For an arbitrary cohomology theory it is not clear how to split Λ^*. However, suppose $\mathcal{H}^*(\ )$ is a cohomology theory with value in R-modules and that 2 is invertible in R (e.g. $\mathcal{H}^*(\ ) = KO^*(\ ) \otimes \mathbb{Z}[\frac{1}{2}]$). Let $\Sigma \subset \mathcal{U}$ be the singular set. By excision,

(c) $\mathcal{H}^*(\mathcal{U}', \Sigma) \cong \sum_{h \in H} \mathcal{H}^*(X, \partial X)$ from which it follows that

(d) $\mathcal{H}^*(\mathcal{U}', \Sigma)^{\mathrm{Alt}} \cong \mathcal{H}^*(X, \partial X)$.

Using (a), (b), and the sequence of the pair $(\mathcal{U}', \Sigma)$, we find that $\mathcal{H}^*(\mathcal{U}')^{\mathrm{Alt}} \cong \mathcal{H}^*(\mathcal{U}', \Sigma)^{\mathrm{Alt}}$. Hence, there is a map

$$\hat{A}^* : \mathcal{H}^*(\mathcal{U}') \to \mathcal{H}^*(\mathcal{U}')^{\mathrm{Alt}} = \mathcal{H}^*(X, \partial X)$$

which splits Λ^*. This proves the claim and consequently, the proposition.

7. The rational Euler characteristic and some conjectures.

Associated to any orbifold there is a rational number, called its "Euler characteristic," which is multiplicative with respect to orbifold coverings (cf. [21]). If the orbifold is cellulated so that each stratum is a subcomplex, then this is defined as the alternating sum of the number of cells in each dimension where each cell is given a weight of the inverse of the order of its isotropy group.

We suppose, as usual, that X^n is a compact n-manifold with boundary, that L is a triangulation of ∂X, that (Γ,V) is a Coxeter system obtained by labelling the edges of L, that condition (*) of Section 4 is satisfied, and that $\mathcal{U} = (\Gamma\times X)/\sim$. For each $S \in L$, let $X_S \subset \partial X$ be the dual cell of S. Let $e(X)$ be the ordinary Euler characteristic of the underlying topological space of X minus that of ∂X. The rational Euler characteristic of X is then defined by

$$(1)\quad \chi(X) = e(X) + \sum_{S\in L} (-1)^{\dim X_S} \frac{1}{|\Gamma_S|}$$

$$= e(X) + (-1)^n \sum_{S\in L} (-1)^{\mathrm{Card}(S)} \frac{1}{|\Gamma_S|}$$

where $|\Gamma_S|$ denotes the order of Γ_S. It is then clear that if Γ' is a torsion-free subgroup of finite index in Γ, then $\chi(\mathcal{U}/\Gamma') = [\Gamma:\Gamma']\chi(X)$. Also, if $\mathcal{U}$ is contractible, then $\chi(X) = \chi(\Gamma)$, where $\chi(\Gamma)$ is the rational Euler characteristic as defined in [16].

For example, if X is a triangle and Γ is the (p,q,r)-triangle group, then

$$\chi(X) = 1 - \frac{3}{2} + \left(\frac{1}{2p} + \frac{1}{2q} + \frac{1}{2r}\right) = \frac{1}{2}((p^{-1}+q^{-1}+r^{-1})-1).$$

If M^2 is a closed surface which is aspherical, then $\chi(M^2) \leq 0$. It follows that if $N^{2k} = M_1^2 \times \dots \times M_k^2$ is a product of closed aspherical surfaces, then $(-1)^k\chi(M^{2k}) \geq 0$.

A well-known conjecture of Hopf is the following:

CONJECTURE 1 (Hopf). *If M^{2k} is a closed manifold of non-positive sectional curvature, then $(-1)^k\chi(M^{2k}) \geq 0$.*

This was proved by Chern for 4-manifolds and by Serre [16] for local symmetric spaces. Recently, H. G. Donnelly and F. Xavier have established it under a hypothesis of pinched negative curvature.

More generally W. Thurston has asked if the following conjecture is true.

CONJECTURE 2 (Thurston). *If M^{2k} is a closed aspherical manifold, then $(-1)^k\chi(M^{2k}) \geq 0$.*

It should be pointed out that this conjecture contradicts another conjecture of Kan-Thurston which asserts that any closed manifold has the same homology as some closed aspherical manifold.

Conjecture 2 implies the following conjecture.

CONJECTURE 3. *Let $X^{2k},\Gamma,\mathcal{U}$ be as above and let $\chi(X)$ denote the rational Euler characteristic. If $\mathcal{U}$ is aspherical, then $(-1)^k\chi(X^{2k}) \geq 0$.*

One might try to construct a 4-dimensional counterexample to the above conjecture as follows. Let X be a 4-cell and L some specific triangulation of S^3. Label the edges in L in some fashion so that conditions (*) and (**) hold and calculate $\chi(X)$ using (1). After making a number of such calculations and having the result invariably come out non-negative, I now believe Conjectures 2 and 3 are true. A more or less random example of such a calculation is included below.

EXAMPLE. Let J be a triangulation of S^2 and let L be the suspension of J. Label each edge in J by 2 and each edge in $L-J$ by 3. This satisfies (*). If J is not the boundary of a tetrahedron and if each circuit of length three in J bounds a triangle then it also satisfies (**). Let a_i denote the number of i-simplices in J. Let us calculate $\chi(X)$ using

$$(1) \quad \chi(X) = 1 + \sum_{S\in L} (-1)^{\mathrm{Card}(S)}|\Gamma_S|^{-1}$$

There are (a_0+2) vertices in L each of weight $\frac{1}{2}$; hence, the vertices contribute $-\frac{1}{2}(a_0+2)$. There are two types of edges: a_1 of type (weight $\frac{1}{4}$) and $2a_0$ of type (weight $\frac{1}{6}$); hence, the edges contribute $\frac{1}{4}a_1 + \frac{1}{3}a_0$. There are two types of triangles: a_2 of type (weight $\frac{1}{8}$) and $2a_1$ of type (weight $\frac{1}{24}$); hence the triangles contribute $-(\frac{1}{8}a_2 + \frac{1}{12}a_3)$. There are $2a_2$ tetrahedron each of type (weight $\frac{1}{192}$); hence, the tetrahedra contribute $\frac{1}{96}a_2$. Thus,

$$\begin{aligned}\chi(X) &= 1 - \tfrac{1}{2}(a_0+2) + (\tfrac{1}{4}a_1 + \tfrac{1}{3}a_0) - (\tfrac{1}{8}a_2 + \tfrac{1}{12}a_1) + \tfrac{1}{96}a_2 \\ &= \tfrac{1}{6}(-a_0 + a_1 - \tfrac{11}{16}a_2).\end{aligned}$$

Since $a_0 - a_1 + a_2 = 2$ and $3a_2 = 2a_1$, we can rewrite this as

$$\chi(X) = \frac{1}{6}\left(\frac{5}{24}a_1 - 2\right).$$

If this is to be ≤ 0, then we must have $a_1 \leq \frac{48}{5} < 10$. Thus, $a_1 \leq 9$. If $a_1 \leq 9$, then the equation $a_0 - \frac{1}{3}a_1 = 2$ implies that either $(a_0, a_1) = (4,6)$ or $(5,9)$. The first case can only happen if J is the boundary of a tetrahedron and the second only if J is the suspension of the boundary of a triangle. In either case (**) does not hold; while in every other case $\chi(X) \geq 0$.

The geometric picture of X is as follows. Let Y be a 3-cell with ∂Y cellulated as the dual polyhedron to J and with all dihedral angles 90°. Combinatorially, X is $Y \times I$; however, the top and bottom faces $Y \times \{0\}$ and $Y \times \{1\}$ meet each (face of Y) $\times$ I at a dihedral angle of 60°.

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ACYCLIC MAPS AND POINCARE SPACES

Ian HAMBLETON and Jean-Claude HAUSMANN

1. The "minus" problem for Poincaré spaces

Recall that a continous map $f : Y \longrightarrow Z$ is called *acyclic* if its homotopy theoretic fiber is an acyclic space, or equivalently if it induces an isomorphism on homology or cohomology with any local coefficients. If the space Y is fixed, the correspondence $f \mapsto \ker\pi_1 f$ produces a bijection between equivalence classes of acyclic maps $f : Y \longrightarrow Z$ and perfect normal subgroups of $\pi_1(Y)$. A representative $Y \longrightarrow Y_P^+$ of the class corresponding to the perfect normal subgroup P of $\pi_1(Y)$ can be obtained by a *Quillen plus construction*, which means that Y_P^+ is obtained by attaching cells of dimension 2 and 3 to Y. For details and other properties of acyclic maps, see [HH].

A space X is called a *Poincaré space (of formal dimension n)* if it is homotopy equivalent to a finite complex and if there exists a class $[X] \in H_n(X;\mathbb{Z})$ so that $- \cap X : H^k(X;B) \longrightarrow H_{n-k}(X;B)$ is an isomorphism for any $\mathbb{Z}\pi_1(X)$-module B. If Y is a Poincaré space and $f : Y \longrightarrow X$ an acyclic map with $\pi_1(X)$ finitely presented, then X is a Poincaré space. The homology condition is obviously satisfied for X and it only remains to prove that X is homotopy equivalent to a finite complex. As $\pi_1(X)$ is finitely presented, the group $\pi_1(X)$ is finitely presented iff $\ker\pi_1 f$ is the normal closure of finitely many elements in $\pi_1(Y)$. Hence a space Y_P^+ ($P=\ker\pi_1 f$) homotopy equivalent to X may be obtained by attaching to Y finitely many 2-cells and then the same number of 3-cells.

Let X be a Poincaré space. For each epimorphism $\varphi : \Gamma \twoheadrightarrow \pi_1(X)$ with Γ finitely presented and $\ker\varphi$ perfect, we consider the problem of finding an acyclic map $f : Y \longrightarrow X$, where Y is a Poincaré space, $\pi_1(Y) = \Gamma$ and $\pi_1 f = \varphi$. In other words : is X obtained by performing a plus construction on a Poincaré space with fundamental group Γ) (the "minus" problem for (X,φ)).

First observe that the existence of such an acyclic map $f : Y \to X$ implies some conditions on X. The following commutative diagram :

$$\begin{tikzcd}
Y \arrow[rr, "\alpha_Y"] \arrow[dd, "f"] & & B\Gamma \arrow[dl, "\iota"] \arrow[dd, "B\varphi"] \\
& B\Gamma^+_{\ker\varphi} \arrow[dr, "B\varphi^+"] & \\
X \arrow[ur, "\alpha_Y^+"] \arrow[rr, "\alpha_X"] & & B\pi_1(X)
\end{tikzcd}$$

shows the existence of a lifting $\alpha_Y^+ : X \to B\Gamma^+_{\ker\varphi}$ of the characteristic map $\alpha_X : X \to B\pi_1(X)$ (see [H-H, Proposition 3.1]). Moreover, recall that for any space Z, the homomorphism $H_2\alpha_Z : H_2(Z;C) \to H_2(B\pi_1(Z);C)$ is surjective for any $\mathbb{Z}\pi_1(Z)$-module C (since $B\pi_1(Z)$ is obtainable from Z by adding cells of dimension ≥ 3). Hence the following commutative diagram :

$$\begin{tikzcd}
H_2(Y;C) \arrow[r, two heads, "H_2\alpha_Y"] \arrow[d, "\simeq", "H_2f"'] & H_2(B\Gamma;C) \arrow[d, "\simeq", "H_2\iota"'] \\
H_2(X;C) \arrow[r, "H_2\alpha_Y^+"] \arrow[dr, two heads, "H_2\alpha_X"'] & H_2(B\Gamma^+_{\ker\varphi};C) \arrow[d, "H_2B\varphi^+"] \\
& H_2(B\pi_1(X);C)
\end{tikzcd}$$

shows that for any $\mathbb{Z}\pi_1(X)$-module C, the homomorphisms $H_2\alpha_Y^+$ and $H_2B\varphi^+$ are both surjective. This, of course, implies non-trivial compatibilities between $H_2(X;C)$ and $H_2(B\Gamma;C) = H_2(\Gamma;C)$.

These first remarks suggest a more natural formulation of the above problem, using the following definition :

<u>(1.1) Definition</u> : Let X be a Poincaré space. Let us consider pairs $(\varphi,\tilde{\alpha})$, where :

1) $\varphi : \Gamma \twoheadrightarrow \pi_1(X)$ is an epimorphism of finitely presented groups with $\ker\varphi$ perfect, and

2) $\tilde{\alpha} : X \longrightarrow B\Gamma^+_{\ker\varphi}$ makes the following diagram commute :

$$\begin{tikzcd} & B\Gamma^+_{\ker\varphi} \arrow[d, "B\varphi^+"] \\ X \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha_X"'] & B\pi_1(X) \end{tikzcd}$$

and $H_2\tilde{\alpha} : H_2(X;C) \longrightarrow H_2(B\Gamma^+_{\ker\varphi};C)$ is surjective for any $\mathbb{Z}\pi_1(X)$-module C.

Such a pair $(\varphi,\tilde{\alpha})$ is *realizable* if there exists an acyclic map $f : Y \longrightarrow X$ with Y a Poincaré space, $\pi_1(Y) = \Gamma, \pi_1 f = \varphi$ and $\alpha^+_Y = \tilde{\alpha}$.

Our problem then becomes : given a Poinçaré space X and a pair $(\varphi,\tilde{\alpha})$ as in (1.1), is this pair realizable ? The answer that we are able to give to this more precise problem is contained in Theorem (1.2) below. Recall that a group G is called *locally perfect* if any finitely generated subgroup of G is contained is a finitely generated perfect subgroup of G.

<u>(1.2) Theorem</u> Let X be a Poincaré space of formal dimension $n \geq 4$.

i) a pair $(\varphi,\tilde{\alpha})$ as in (1.1) determines an element $\sigma(\varphi,\tilde{\alpha})$ in the Wall surgery obstruction group $L_n(\varphi)$. If $(\varphi,\tilde{\alpha})$ is realizable, then $\sigma(\varphi,\tilde{\alpha}) = 0$.

ii) If $\tilde{\alpha}' : X \longrightarrow B\Gamma^+_{\ker\varphi}$ is another lifting of α_X such that the pair $(\varphi,\tilde{\alpha}')$ satisfies to the conditions of (1,1), then $\sigma(\varphi,\tilde{\alpha}) = \sigma(\varphi,\tilde{\alpha}')$.

iii) If in addition $n \geq 5$ and $\ker\varphi$ is locally perfect, then $\sigma(\varphi,\tilde{\alpha}) = 0$ implies that $(\varphi,\tilde{\alpha})$ is realizable.

<u>(1.3) Remarks</u> : a) The Wall group used in (1.2) is the obstruction group for surgery to a homotopy equivalence (sometimes called L^h_n). Recall that the group $L_n(\ )$ fits in the exact sequence :

$$\longrightarrow L_n(\Gamma) \xrightarrow{\varphi} L_n(\pi_1(X)) \longrightarrow L_n(\varphi) \longrightarrow L_{n-1}(\Gamma) \longrightarrow$$

b) The same theory holds for simple Poincaré spaces [Wa, Chapter 2]. using simple acyclic maps (the Whitehead torsion of an acyclic map $f : Y \longrightarrow X$ is well defined in $Wh(\pi_1(X))$;if this torsion vanishes, the acyclic map is called *simple*). The relevant Wall group is then $L_n^s(\varphi)$.

c) The same theory holds for non-orientable Poincaré spaces. The relevant Wall group is then $L_n(\varphi, w_1(X))$, where $w_1(X) : \pi_1(X) \to \mathbb{Z}/2\mathbb{Z}$ is the orientation character for X.

Proof of (1.2) : Write $B\Gamma^+$ for $B\Gamma^+_{\ker\varphi}$. Let us consider the pull-back diagram :

$$\begin{tikzcd} T \arrow[r] \arrow[d, "g"] & B\Gamma \arrow[d, "\iota"] \\ X \arrow[r, "\tilde{\alpha}"] & B\Gamma^+ \end{tikzcd}$$

The fiber of g is the same as the fiber of ι, therefore g is an acyclic map. If F is the homotopy theoretic fiber of $\tilde{\alpha}$ one has the following diagram :

$$\begin{tikzcd} \pi_2(X) \arrow[r] \arrow[d, "\pi_2\tilde{\alpha}"] & \pi_1(F) \arrow[r] \arrow[d, "="] & \pi_1(T) \arrow[r] \arrow[d] & \pi_1(X) \arrow[r] \arrow[d, "\simeq"] & 1 \arrow[d] \\ \pi_2(B\Gamma^+) \arrow[r] & \pi_1(F) \arrow[r] & \Gamma \arrow[r] & \Gamma/\ker\varphi \arrow[r] & 1 \end{tikzcd}$$

Hence $\pi_1(T) = \Gamma$ if $\pi_2\tilde{\alpha}$ is surjective. But this is the case, as can be seen by the following diagram :

$$\begin{tikzcd} \pi_2(X) \arrow[r, "\simeq"] \arrow[d, "\pi_2\tilde{\alpha}"] & H_2(\tilde{X}) \arrow[r, "\simeq"] \arrow[d] & H_2(X;\pi_1(X)) \arrow[d] \\ \pi_2(B\Gamma^+) \arrow[r, "\simeq"] & H_2(\widetilde{B\Gamma}^+) \arrow[r, "\simeq"] & H_2(B\Gamma^+;\pi_1(X)) \end{tikzcd}$$

the right-hand vertical arrow being surjective by Part b) of (1.1).

Let Z be a space . We denote by $\Omega_n^P(Z)$ (*Poincaré bordism group*) the bordism group of maps $f : U \longrightarrow Z$ where U is an oriented Poincaré space of formal dimension n. According to the theory of Quinn ([Qn], see [HV2] for proofs), these groups fit in a natural long exact sequence :

$$H_{n+1}(Z;MSG) \longrightarrow L_n(\pi_1(Z)) \longrightarrow \Omega_n^P(Z) \longrightarrow H_n(Z;MSG) \qquad (n\geq 4)$$

If Z' is a subspace of Z, one defines $\Omega_n^P(Z,Z')$ similarly, using Poincaré pairs, and on gets a corresponding sequence. Specializing to $Z = X, Z' = T$ and using the fact that $T \to X$ is an acyclic map, one gets the following commutative diagram in which the rows and columns are exact :

$$\begin{array}{ccccccc}
H_{n+1}(T;MSG) & \longrightarrow & L_n(\Gamma) & \longrightarrow & \Omega_n^P(T) & \longrightarrow & H_n(T;MSG) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+1}(X;MSG) & \longrightarrow & L_n(\pi_1(X)) & \longrightarrow & \Omega_n^P(X) & \longrightarrow & H_n(X;MSG) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & L_n(\varphi) & \xrightarrow{\simeq} & \Omega_n^P(X,T) & \longrightarrow & 0
\end{array}$$

This permits us to define $\sigma(\varphi,\tilde{\alpha})$ as the image of $[id_X] \in \Omega_n^P(X)$ under the composite map $\Omega_n^P(X) \to \Omega_n^P(X,T) \simeq L_n(\varphi)$.

Now, suppose that $(\varphi,\tilde{\alpha})$ is realizable by an acyclic map $f : Y \to X$ with Y a Poincaré space. Thus, f factors through a map $f : Y \to T$ representing a class in $\Omega_n^P(T)$. As f is acyclic, its mapping cylinder constitutes a Poincaré cobordism from id_X to f. Therefore, the class $[id_X]$ is mapped to zero in $\Omega_n^P(X,T)$ (since f factors through T) and $(\varphi,\tilde{\alpha}) = 0$. This proves part i) of (1.2).

To prove ii), let us consider the pull-back diagram

$$\begin{array}{ccc}
T' & \longrightarrow & B\Gamma \\
\downarrow & & \downarrow \\
X & \xrightarrow{\tilde{\alpha}'} & B\Gamma^+
\end{array}$$

and form again the pull-back diagram

$$\begin{array}{ccc}
\hat{T} & \longrightarrow & T' \\
\downarrow & & \downarrow \\
T & \longrightarrow & X
\end{array}$$

in which all the maps are now acyclic. Then the composed map $\hat{T} \to X$ is also acyclic. Denote by $\hat{\varphi}$: $\hat{\Gamma}=\pi_1(\hat{T}) \twoheadrightarrow \pi_1(X)$ the induced homomorphism. One has a commutative diagram

$$\begin{tikzcd}
 & & \Omega_n^P(X,T) & & \\
\Omega_n^P(X) \arrow[urr] \arrow[rr] \arrow[drr] & & \Omega_n^P(X,\hat{T}) \arrow[u] \arrow[d] & L_n(\hat{\varphi}) \arrow[l, "\simeq"'] \arrow[r] & L_n(\varphi) \arrow[ull, "\simeq"'] \arrow[dll, "\simeq"] \\
 & & \Omega_n^P(X,T') & &
\end{tikzcd}$$

Therefore, $\sigma(\varphi,\tilde{\alpha})$ and $\sigma(\varphi,\tilde{\alpha}')$ are both image of a single element of $L_n(\Phi)$. This proves Part ii) of (1.2).

Let us finally prove part iii) of (1.2). If $\sigma(\varphi,\tilde{\alpha}) = 0$, then there is a map $\beta_0 : Y_0 \longrightarrow T$ representing a class in $\Omega_n^P(T)$ such that $g\circ\beta_0$ is Poincaré cobordant to id_X. To show that (φ,α) is realizable, we shall find a representative $\beta : Y \longrightarrow T$ of the class β_0 such that $\pi_1\beta$ and $\beta_* : H_*(Y;\mathbb{Z}\pi_1(X)) \longrightarrow H_*(T;\mathbb{Z}\pi_1(X))$ are isomorphisms.

By construction of the space T, the group $\ker\varphi$ acts trivially on $\pi_2(T)$ (use [HH, Proposition 5.4] to the maps ι and g). As $\ker\varphi$ is locally perfect, one can construct, as in [H2, proof of Theorem 3.1],a finite complex T_1 and a commutative diagram :

$$\begin{tikzcd}
Y_0 \arrow[r, "\beta_0"] \arrow[dr, "\beta_1"'] & T \arrow[r, "g"] & X \\
 & T_1 \arrow[u, "\gamma"] \arrow[ur, "g_1"'] &
\end{tikzcd}$$

such that g_1 is an acyclic map and $\pi_1\gamma$ is an isomorphism. Thus, T_1 is a finite complex satisfying Poincaré duality with coefficients $\mathbb{Z}\pi_1(X)$ and β_1 can be covered by a map of the Spivak bundles. By surgery with coefficients for Poincaré spaces (the Cappell-Shaneson type of generalization of [Qn, Corollary 1.4]; for proofs, see[HV2]), the map β_1 determines an element $\sigma(\beta_1) \in \Gamma_n(\varphi)$, where $\Gamma_n(\varphi)$ is the Cappell-Shaneson surgery obstruction group $\Gamma_n^h(\mathbb{Z}\Gamma \longrightarrow \mathbb{Z}\pi_1(X))$ defined in [CS]. The existence of the required map $\beta : Y \longrightarrow T$ will be implied by the nullity of $\sigma(\beta_1)$.

As in [H1,§3] , it can be checked (see [HV2]) that the image of $\sigma(\beta_1)$ under the homomorphism $\Gamma_n(\varphi) \longrightarrow L_n(\pi_1(X))$ is the obstruction to $g_1 \circ \beta_1$ being Poincaré cobordant to a homotopy equivalence. The latter is obviously zero since, by construction, $g_1 \circ \beta_1 = g \circ \beta_0$ is Poincaré cobordant to id_X. Since both Γ and $\pi_1(X)$ are finitely presented, $\ker\varphi$ locally perfect is equivalent to $\ker\varphi$ being the normal closure of a finitely generated perfect group. Therefore, the homomorphism $\Gamma_n(\varphi) \longrightarrow L_n(\pi_1(X))$ is an isomorphism [H1, Theorem 1]. Then $\sigma(\beta_1) = 0$ and Part ii) of (1.2) is proved.

2. The invariant $\sigma(\varphi,\tilde{\alpha})$ as part of a total surgery obstruction theory

Let X be a Poincaré space of formal dimension $n\geq 4$. By (1.2) to each pair $(\varphi,\tilde{\alpha})$ as in (1.1), one can associate the element $\sigma(\varphi,\tilde{\alpha}) \in L_n(\varphi)$. This gives a large collection of invariants associated to X. In this context, Theorem 2.1 of [HV1] may be rephrased as follows :

(2.1) Theorem Let X be a Poincaré space of formal dimension $n\geq 5$. Let $(\varphi,\tilde{\alpha})$ be a pair as in (1.1) with $\ker\varphi$ locally perfect. If X has the homotopy type of a topological closed manifold then $\sigma(\varphi,\tilde{\alpha}) = 0$.

Thus, the elements $\sigma(\varphi,\tilde{\alpha})$ occurs as obstruction for X being homotopy equivalent to a closed topological manifold and we can except some relationship between our $\sigma(\varphi,\tilde{\alpha})$'s and the total surgery obstruction of [Ra]. We are indebted to A. Ranicki for pointing out a mistake in our first draft of this section.

Let X be a Poincaré space of formal dimension ≥ 5. According to [Ra], there is an exact sequence :

$$(2.1) \quad \dots \to \mathcal{S}_{m+1}(X) \to H_m(X;\underline{\mathbb{L}}_0) \to L_m(\pi_1(X)) \to \mathcal{S}_m(X) \to H_{m-1}(X;\underline{\mathbb{L}}_0) \to \dots$$

and an element $s(X) \in \mathcal{S}_n(X)$ which vanishes if and only if X is homotopy equivalent to a closed topological manifold. Here the groups are defined for $m\geq 0$ by

$$\mathcal{S}_m(X) = \pi_m(\sigma_* : X_+\wedge\underline{\mathbb{L}}_0 \longrightarrow \underline{\mathbb{L}}_0(\pi_1(X)))$$

where σ_* is the assembly map and $\underline{\mathbb{L}}_0$ is the 1-connective covering of the spectrum $\underline{\mathbb{L}}_0(1)$ (see [Ra, p.285]; we use the notations of [Ra]). Observe that our definition of $\mathcal{S}_m(X)$ slightly differs from the one in [Ra] (we take the whole spectrum $\underline{\mathbb{L}}_0(\pi_1(X))$ instead of its 1-connective covering). This difference only affects the group $\mathcal{S}_0(X)$. Since the assembly map σ_* can be extended to $\bar{\sigma}_* : X_+\wedge\underline{\mathbb{L}}_0(1) \longrightarrow \underline{\mathbb{L}}_0(\pi_1(X))$ we can define : $\bar{\mathcal{S}}_m(X) = \pi_m(\bar{\sigma}_*)$. This gives the exact sequences :

$$\to \bar{\mathcal{S}}_{m+1}(X) \to H_m(X;\underline{\mathbb{L}}_0(1)) \to L_m(\pi_1(X)) \to \bar{\mathcal{S}}_m(X) \to H_{m-1}(X;\underline{\mathbb{L}}_0(1))) \to$$

and

$$(2.2)\qquad \cdots \longrightarrow H_m(X;\mathbb{Z}) \longrightarrow \mathscr{S}_m(X) \xrightarrow{\lambda_m} \overline{\mathscr{S}}_m(X) \longrightarrow H_{m-1}(X;\mathbb{Z}) \longrightarrow \cdots$$

Let us define $\overline{s}(X) = \lambda_n(s(X)) \in \overline{\mathscr{S}}_n(X)$.
If $(\varphi,\tilde{\alpha})$ is any pair for X as in (1.1), consider the pull-back diagram :

$$\begin{tikzcd} T \arrow[r] \arrow[d, "g"'] & B\Gamma \arrow[d] \\ X \arrow[r, "\tilde{\alpha}"] & B\Gamma^+ \end{tikzcd}$$

which gives rise to the following diagram :

$$(2.3)\qquad \begin{tikzcd} H_n(T;\mathbb{L}_0) \arrow[r] \arrow[d] & L_n(\Gamma) \arrow[r] \arrow[d] & \mathscr{S}_n(T) \arrow[r] \arrow[d] & H_{n-1}(T;\mathbb{L}_0) \arrow[d] \\ H_n(X;\mathbb{L}_0) \arrow[r] \arrow[d] & L_n(\pi_1(X)) \arrow[r] \arrow[d] & \mathscr{S}_n(X) \arrow[r] \arrow[d] & H_{n-1}(X;\mathbb{L}_0) \arrow[d] \\ 0 \arrow[r] & L_n(\varphi) \arrow[r, "\simeq"] & \mathscr{S}_n(X,T) \arrow[r] & 0 \end{tikzcd}$$

in which rows and collumns are exact. One has also the corresponding diagram for $\overline{\mathscr{S}}(X)$. Let $\eta_m : \mathscr{S}_m(X) \longrightarrow L_m(\varphi)$ be the composed homomorphism $\mathscr{S}_m(X) \longrightarrow \mathscr{S}_m(X,T) \xleftarrow{\simeq} L_m(\varphi)$. Define $\overline{\eta}_m : \overline{\mathscr{S}}_m(X) \longrightarrow L_m(\varphi)$ accordingly, and notice that $\eta_m = \overline{\eta}_m \circ \lambda_m$.

<u>(2.4) Proposition</u> In $L_n(\varphi)$, one has the equalities :

$$\eta_n(s(X)) = \overline{\eta}_n(\overline{s}(X)) = \sigma(\varphi,\tilde{\alpha}).$$

<u>Proof</u> This follows directly from the definitions, since there is a homomorphism $\delta_X : \Omega_n^P(X) \longrightarrow \mathscr{S}_n(X)$ such that the following diagram

$$\begin{tikzcd} L_n(\pi_1(X)) \arrow[r] \arrow[d] & \Omega_n^P(X) \arrow[r, "\delta_X"] \arrow[d] & \mathscr{S}_n(X) \arrow[r, "\lambda_n"] \arrow[d] & \overline{\mathscr{S}}_n(X) \arrow[d] \\ L_n(\varphi) \arrow[r, "\simeq"] & \Omega_n^P(X,T) \arrow[r, "\delta_{X,T}", "\simeq"'] & \mathscr{S}_n(X,T) \arrow[r, "\simeq"] & \overline{\mathscr{S}}_n(X,T) \end{tikzcd}$$

commutes and $\delta_X([id_X]) = s(x)$ [Ra,pp.307-308].

(2.5) Corollary Let X be a Poincaré complex of formal dimension $n \geq 5$, and let $(\varphi, \tilde{\alpha})$ a pair as in (1.1). Suppose that the Spivak bundle for X has a TOP-reduction ξ which defines a surgery obstruction $\sigma(\xi) \in L_n(\pi_1(X))$. Then, $\sigma(\varphi, \tilde{\alpha})$ is the image of $\sigma(\xi)$ under the homomorphism $L_n(\pi_1(X)) \longrightarrow L_n(\varphi)$.

Proof By [Ra,p. 298], the element $\sigma(\xi)$ has image $s(X)$ under the homomorphism $L_n(\pi_1(X)) \longrightarrow \mathcal{S}_n(X)$. The result thus follows from (2.4). Thus, if $\bar{s}(X) = 0$, one has $\sigma(\varphi, \tilde{\alpha}) = 0$ for any pair $(\varphi, \tilde{\alpha})$ as in (1.1). A converse to this fact might be obtained by considering some "test pairs" $(\varphi_X, \tilde{\alpha}_X)$ for X as follows : let $\mathcal{A}_i$, $i=0,1,\ldots$, and $\mathcal{A} = \cup_i \mathcal{A}_i$ be the smallest classes of groups such that :

$\mathcal{A}_0$ contains the trivial group
$G \in \mathcal{A}_i$ iff at least one of the following conditions holds :

(a) there exist groups G_1, G_2 and $G_0 = G_1 \cap G_2$, all in $\mathcal{A}_{i-1}$ such that $G = G_1 *_{G_0} G_2$ and the inclusions $G_0 \subset G_i$ are $\sqrt{\ }$-closed in the sense of [Cl] : if $g \in G_i$ and $g^2 \in G_0$ then $g \in G_0$.

or

(b) $G = G_0 \times \mathbb{Z}$, with $G_0 \in \mathcal{A}_{i-1}$

(2.6) Proposition Let X be a finite complex of dimension n. Then there exists a pair $(\varphi_X : \Gamma_X \longrightarrow \pi_1(X), \tilde{\alpha}_X)$ satisfying 1) and 2) of (1.1) such that :

1) $\Gamma_X \in \mathcal{A}$
2) $B\Gamma_X$ is a finite complex of dimension n
3) $\tilde{\alpha}_X$ is a homotopy equivalence.

The pair $(\varphi_X, \tilde{\alpha}_X)$ is associated to a triangulation of X, according an algorithm as in [B-D-H] or [Ma]. Its construction is given in §4.

Recall that a standard conjecture is that $\tilde{K}_0(G) = 0 = Wh(G)$ for $G \in \mathcal{A}$(1). (or even for G such that BG is a finite complex).

(2.7) Theorem Suppose that $\tilde{K}_0(G) = Wh(G) = 0$ for all $G \in \mathcal{A}$. Then, for X a Poincaré space of formal dimension $n \geq 5$, the following conditions are equivalent :

(1) P. Vogel informs us that he has recently obtained a proof of this conjecture.

1) $\bar{s}(X) = 0$

2) $\sigma(\varphi,\tilde{\alpha}) = 0$ for any pair $(\varphi,\tilde{\alpha})$ for X as in (1.1)

3) $\sigma(\varphi_X,\tilde{\alpha}_X) = 0$ for some pair $(\varphi_X,\tilde{\alpha}_X)$ of (2.6).

Proof : Condition 1) implies Condition 2) by (2.4). The implication from 2) to 3) is straightforward. Therefore it remains to prove that 3) implies 1). As the map $\tilde{\alpha}_X$ is a homotopy equivalence, the diagram for $\mathcal{S}_m(X)$ similar to (2.3) gives the long exact sequence :

$$(2.8) \qquad \ldots \bar{\bar{\mathcal{S}}}_m(B\Gamma_X) \longrightarrow \bar{\bar{\mathcal{S}}}_m(X) \xrightarrow{\bar{\eta}_m} L_m(\varphi_X) \longrightarrow \bar{\bar{\mathcal{S}}}_{m-1}(B\Gamma_X) \longrightarrow \ldots$$

Therefore, it suffices to establish that $\bar{\bar{\mathcal{S}}}_m(B\Gamma_X) = 0$ for $m \geq n$. As $\dim B\Gamma_X = n$, this follows from the following lemma :

(2.9) Lemma Let $G \in \mathcal{A}$ such that $\tilde{K}_0(P) = 0 = Wh(P)$ for any subgroup P of G with $P \in \mathcal{A}$. Then the homomorphism

$$\bar{\sigma}_m : H_m(G;\underline{\mathbb{L}}_0(1)) \longrightarrow L_m(G)$$

induced by the assembly map $\bar{\sigma}_*$ is an isomorphism for $m \geq \dim BG$ and is injective for $m = \dim BG - 1$.

Proof We shall prove Lemma (2.9) for $G \in \mathcal{A}_j$ by induction on j, using the classical idea of S. Cappell [C3] . The class $\mathcal{A}_0$ contains only the trivial group and $H_m(pt;\underline{\mathbb{L}}_0(1))$ is isomorphic to $L_m(1)$ for $m \geq 0$ (this is the main point where we need the spectrum $\underline{\mathbb{L}}_0(1)$ instead of $\underline{\mathbb{L}}_0$). Also $H_{-1}(pt;\underline{\mathbb{L}}_0(1)) = 0$, thus lemma (2.9) is proved for $G \in \mathcal{A}_0$.

If now $G \in \mathcal{A}_j$, then

$$\begin{array}{ccccccc}
H_m(BG_0;\underline{\mathbb{L}}_0(1)) & \to & H_m(BG_1;\underline{\mathbb{L}}_0(1))\oplus H_m(BG_2;\underline{\mathbb{L}}_0(1)) & \to & H_m(BG;\underline{\mathbb{L}}_0(1)) & \to & H_{m-1}(BG_0;\underline{\mathbb{L}}_0(1)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_m(G_0) & \longrightarrow & L_m(G_1) \oplus L_m(G_2) & \longrightarrow & L_m(G) & \longrightarrow & L_{m-1}(G_0)
\end{array}$$

in the first case and

$$\begin{array}{ccccccccc}
0 & \to & H_m(G_0;\underline{\mathbb{L}}_0(1)) & \to & H_m(G;\underline{\mathbb{L}}_0(1)) & \to & H_{m-1}(G_0;\underline{\mathbb{L}}_0(1)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & L_m(G_0) & \longrightarrow & L_m(G) & \longrightarrow & L_{m-1}(G_0) & \longrightarrow & 0
\end{array}$$

in the second case, in which all the rows are exact. The exact sequences involving L-groups are those of [Cl] . As dim BG_1 and dim BG_2 are $\leq$ dim BG and dim $BG_0 \leq$ dim BG-1 (in both cases), the induction step follows from the five lemma.

Using Exact sequences (2.2) and (2.3) together with Lemma (2.9), one obtains the following theorem :

(2.10) Theorem Suppose that $\tilde{K}_0(G) = 0 = Wh(G)$, for all $G \in \mathscr{A}$. Let X be a Poincaré space of formal dimension $n \geq 5$ and let $(\varphi_X, \tilde{\alpha}_X)$ be a pair as in (2.6). Then :

a) $\eta_m : \mathscr{S}_m(X) \longrightarrow L_m(\varphi_X)$ is an isomorphism for $m \geq n+2$

b) One has an exact sequence :

$$0 \longrightarrow \mathscr{S}_{n+1}(X) \xrightarrow{\eta_{n+1}} L_{n+1}(\varphi_X) \longrightarrow \mathbb{Z} \longrightarrow \mathscr{S}_n(X) \xrightarrow{\eta_n} L_n(\varphi_X)$$

Finally, we mention the following proposition which will be of interest in Remarks 4 and 5 below :

(2.11) Proposition Let G be a group as in (2.9) such that BG is a (finite) complex of dimension n . Let X be a space with $\pi_1(X) = G$ and such that the canonical map $X \longrightarrow BG$ induces an isomorphism on integral homology. Then $\mathscr{S}_m(X) = \bar{\mathscr{S}}_m(X) = 0$ for $m>n$, $\mathscr{S}_n(X) \cong \mathbb{Z}$ and $\bar{\mathscr{S}}_n(X) = 0$.

Proof This follows from Lemma (2.9) and from the comparison of the exact sequences (2.1) and (2.1 bis) for X and for BG.

2.12) Remarks 1) If one is interested in Statements (2.9), (2.10) and (2.11) only modulo 2-torsion, one can drop the assumption $\tilde{K}_0(G) = 0 = Wh(G)$ for $G \in \mathscr{A}$ as well as the condition $\sqrt{}$-closed in the definition of the class $\mathscr{A}$ (this would simplify §4). Indeed, the exact sequences of surgery groups used in the proof of (2.9) always exist when all the groups are tensored by $\mathbb{Z}[1/2]$.

2) From Proposition (2.11), it follows that $\mathscr{S}_m(B\mathbb{Z}^n) = 0$ for $m>n$ and $\mathscr{S}_n(B\mathbb{Z}^n) = \mathbb{Z}$. This result is mentioned in [Ra, p.310].

3) The class $\mathscr{A}$ has been chosen minimal in order to obtain (2.6) and (2.7). But Lemma (2.9) is valid for a larger class in which we allow HNN-extension (with the relevant $\sqrt{\ }$-closed condition). As in 2), one is then able to prove for instance that $\mathscr{S}_m(X) = 0$ for m>3 and $\mathscr{S}_3(X) = \mathbb{Z}$ for X belonging to a large class of sufficiently large 3-manifolds (the result is valid mod 2-torsion for all sufficiently large 3-manifolds).

4) We now construct a Poincaré space Y of formal dimension n such that $\sigma(\varphi,\tilde{\alpha}) = 0$ for all pairs $(\varphi,\tilde{\alpha})$ for Y as in (1.1) but which is not homotopy equivalent to a closed topological manifold. We assume that $\tilde{K}_0(G) = 0 = Wh(G)$ for all $G \in \mathscr{A}$ thus it suffices to prove that $\bar{s}(Y) = 0$ by (2.7).

We apply (2.6) to the case $X = S^n$. We thus obtain a group $\Gamma_n \in \mathscr{A}$ such that $B\Gamma_n$ is a finite complex of dimension n and $H_*(B\Gamma_n;\mathbb{Z}) \cong H_*(S^n;\mathbb{Z})$.

The Atiyah-Hirzebruch spectral sequence shows that $H_m(B\Gamma_n;\underline{\mathbb{L}}_0) = L_m(1)$ for $1\le m\le n$ and the homomorphism $H_m(B\Gamma_n;\underline{\mathbb{L}}_0) \longrightarrow L_m(\Gamma_n)$ induced by the assembly map coincides with the inclusion $L_m(1) \rightarrowtail L_m(\Gamma_n)$. Thus, the reduced surgery group $\tilde{L}_n(\Gamma_n) = \operatorname{coker}(L_n(1) \longrightarrow L_n(\Gamma_n))$ is isomorphic to $\mathscr{S}_n(B\Gamma_n) = \mathbb{Z}$ by (2.1) and (2.11).

Let us consider the Poincaré homology sphere bordism group $\Omega_n^{PHS}(B\Gamma_n)$ defined in [H3], whose elements are represented by maps $f : \Sigma \longrightarrow B\Gamma_n$, where Σ is an oriented Poincaré space with the homology of S^n. For $n\ge 6$, the theory of [H3] gives an isomorphism :

$$\Omega_n^{PHS}(B\Gamma_n) \cong \pi_n(S^n) \oplus \tilde{L}_n(\Gamma_n) \cong \mathbb{Z} \oplus \mathbb{Z}$$

so that the class of $f : \Sigma \longrightarrow B\Gamma_n$ corresponds to the pair $(\deg f, \widetilde{f_*(\sigma)})$, where $\sigma \in L_n(\pi_1(\Sigma))$ is the surgery obstruction for any surgery problem with target Σ. As Γ_n is finitely presented and $H_1(\Gamma_n;\mathbb{Z}) = H_2(\Gamma_n;\mathbb{Z}) = 0$, it actually follows from [H3, "proof of the surjectivity of σ_n"] that for any class of $\Omega_n^{PHS}(B\Gamma_n)$ has a representative $f : \Sigma \longrightarrow B\Gamma_n$ with $\pi_1 f$ an isomorphism . Therefore, the pair (1,k) with $k\neq 0$ corresponds to a map $f : Y \longrightarrow B\Gamma_n$ such that :

- f induces an isomorphism on the fundamental groups
- f induces an isomorphism on integral homology (since deg f = 1)
- Y has not the homotopy type of a closed topological manifold (otherwise k would be zero).
- $\bar{s}(Y) = 0$ (since $\mathcal{S}_n(Y) = 0$ by (2.11)).

5) The following is a version of the Novikov Conjecture : if G is a group such that BG is a Poincaré space of formal dimension n, then

a) $\mathcal{S}_m(BG) = 0$ for $m>n$ and $\mathcal{S}_n(BG) = \mathbb{Z}$
b) $s(BG) = 0$

Proposition (2.11) shows that a) is satisfied if $G \in \mathscr{A}$ (modulo the vanishing assumptions on $\tilde{K}_0$ and Wh). On the other hand, the space Y of Remark 4) above has fundamental group $\Gamma_n \in \mathscr{A}$, the same integral homology as $B\Gamma_n$ and thus satisfies a) by (2.11). But $s(Y) \neq 0$. This shows some independence between condition a) and b) and emphasizes the importance of the assumption that BG itself be a Poincaré space in the Novikov conjecture.

3. Homotopy equivalences of closed manifolds

As one might except, the results of §1 and 2 have analogues for homotopy equivalences of closed manifolds. We give here the "simple homotopy" version of this theory, which seems more natural in this framework.

(3.1) Theorem Let $j : M \longrightarrow N$ be a simple homotopy equivalence between closed manifolds of dimension $n \geq 5$. Then any pair $(\varphi, \tilde{\alpha})$ for N as in (1.1) with $\ker\varphi$ locally perfect determines an element $\sigma(j, \varphi, \tilde{\alpha}) \in L^s_{n+1}(\varphi)$ such that the following three conditions are equivalent :

a) there is a commutative diagram :

$$\begin{tikzcd} M_- \arrow[r, "j_-"] \arrow[d, "f_M"'] & N_- \arrow[r, "\alpha_{N_-}"] \arrow[d, "f_N"'] & B\Gamma \arrow[d, "\iota"] \\ M \arrow[r, "j"] & N \arrow[r, "\tilde{\alpha}"] & B\Gamma^+ \end{tikzcd}$$

where M_- and N_- are closed manifolds, f_M and f_N are simple acyclic maps and j_- is a simple homotopy equivalence.

b) any commutative diagram

$$\begin{tikzcd} & N_- \arrow[r, "\alpha_{N_-}"] \arrow[d, "f_N"'] & B\Gamma \arrow[d, "\iota"] \\ M \arrow[r, "j"] & N \arrow[r, "\tilde{\alpha}"] & B\Gamma^+ \end{tikzcd}$$

with N_- a closed manifold and f_N a simple acyclic map can be completed in a diagram as in a).

c) $\sigma(j, \varphi, \tilde{\alpha}) = 0$.

Proof Recall that in the proof of (1.2) we checked that in the pull-back diagram :

$$\begin{tikzcd} T \arrow[r] \arrow[d, "g"'] & B\Gamma \arrow[d] \\ N \arrow[r, "\tilde{\alpha}"] & B\Gamma^+ \end{tikzcd}$$

the map g is acyclic, $\pi_1(T) = \Gamma$ and $\ker\varphi$ acts trivially on $\pi_2(T)$. By [H2, Theorem 3.1], there is a commutative diagram :

$$\begin{tikzcd} N_- \arrow[r] \arrow[dr, "f_N"'] & T \arrow[d, "g"] \\ & N \end{tikzcd}$$

such that f_N is a simple acyclic map and $\pi_1(N_-) = \pi_1(T) = \Gamma$. (This existence of f_N shows that b) implies a).)

For P a closed manifold of dimension n, let $\mathcal{S}_{TOP}(P)$ be the Sullivan-Wall set of topological structures on P [Wa, Chapter 10] According to [Ra, p.277] there is an identification $\mathcal{S}_{TOP}(P) \xrightarrow{\simeq} \mathcal{S}_{n+1}(P)$. Let $h : Q_- \longrightarrow N_-$ represent a class in $\mathcal{S}_{TOP}(N_-)$. Using a simple plus cobordism (W,Q_-,Q) (i.e. $Q_-^+ \simeq W$) one gets a simple homotopy equivalence $h^+ : Q \longrightarrow N$ whose class in $\mathcal{S}_{TOP}(N)$ is well defined. One checks that this correspondance $[h] \longrightarrow [h^+]$ is actually given by the composite :

$\mathcal{S}_{TOP}(N_-) \xrightarrow{\simeq} \mathcal{S}_{n+1}(N_-) \xrightarrow{f_{N*}} \mathcal{S}_{n+1}(N) \xrightarrow{\simeq} \mathcal{S}_{TOP}(N)$. Finally, observe that one has the following commutative diagram :

$$\begin{tikzcd} \mathcal{S}_{n+1}(N_-) \arrow[rr, "f_{N*}"] \arrow[dr, "\simeq"'] & & \mathcal{S}_{n+1}(N) \\ & \mathcal{S}_{n+1}(T) \arrow[ur, "g_*"'] & \end{tikzcd}$$

The map $\mathcal{S}_{n+1}(N_-) \longrightarrow \mathcal{S}_{n+1}(T)$ is an isomorphism by the Ranicki exact sequence [Ra, p.276] indeed the map $N_- \longrightarrow T$ induces an isomorphism on the funcamental groups and on the homology.

These considerations make Theorem (3.1) straightforward if we define $\sigma(j,\varphi,\tilde{\alpha})$ to be the image of $[j] \in \mathcal{S}_{TOP}(N)$ under the composite map $\mathcal{S}_{TOP}(N) \xrightarrow{\simeq} \mathcal{S}_{n+1}(N) \xrightarrow{\eta_{n+1}} L_{n+1}(\varphi)$ (see (2.3) and (2.4)).

If $(\varphi_N,\tilde{\alpha}_N)$ is a pair for N as in (2.6), the homomorphism $\mathcal{S}_{n+1} : \mathcal{S}_{n+1}(N) \longrightarrow L_{n+1}(\varphi_N)$ is injective by (2.10). One thus obtains the analogue of (2.7) :

<u>(3.2) Theorem</u> Let $j : M \longrightarrow N$ as in (3.1). Assume that $\tilde{K}_0(G) = Wh(G) = 0$ for all $G \in \mathcal{A}$. Then, the following conditions are equivalent :

1) j is homotopic to a homeomorphism
2) $\sigma(j,\varphi,\tilde{\alpha}) = 0$ for all pair $(\varphi,\tilde{\alpha})$ for N as in (1.1)
3) $(j,\varphi_N,\tilde{\alpha}_N) = 0$ for some pair $(\varphi_N,\tilde{\alpha}_N)$ for N as in (2.6)

4. Proof of Proposition (2.6)

Our proof makes use of Statements (4.1)-(4.4) below. The proof of (4.1) is given at the end of this section.

(4.1) Lemma Let R_i ($i \in I$) be a familly of groups having a common subgroup B and let R be the amalgamated product $(*_B)_{i\in I} R_i$. Let S be a subgroup of R and let $S_i = S \cap R_i$. Suppose that the following conditions hold :

1) the union of S_i's generates S
2) S_i is $\sqrt{\ }$-closed in R_i for all i
3) if $s_i b \hat{s}_i \in B$ with $s_i, \hat{s}_i \in S_i$ and $b \in B$, then $b \in S_i$.

Then S is $\sqrt{\ }$-closed in R.

(4.2) Examples a) Condition 3) holds trivially if $B \subset S_i$ for all $i \in I$. For instance, if $B = 1$, case of a free product.

b) If B is $\sqrt{\ }$-closed in R_i for all $i \in I$, then B is $\sqrt{\ }$-closed in R (case $S_i = B$).

c) If $J \in I$ and B is $\sqrt{\ }$-closed in R_i for $i \in I \setminus J$, then the subgroup generated by $\cup_{i\in J} R_i$ is $\sqrt{\ }$-closed in R. (Take $S_i = R_i$ for $i \in J$ and $S_i = B$ for $i \notin J$).

(4.3) Lemma If G_1 and G_2 are groups in $\mathscr{A}$, so is $G_1 \times G_2$.

Proof Let $G_1 \in \mathscr{A}_m$ and $G_2 \in \mathscr{A}_n$. The proof is by induction on m+n. The statement is trivial if m+n = 0 and the induction step is easily obtained, using the isomorphisms
$G_1 \times (G_2 *_G G_3) = (G_1 \times G_2) *_{G_1 \times G} (G_1 \times G_3)$ and $G_1 \times (\mathbb{Z} \times G) = (G_1 \times G) \times \mathbb{Z}$.

(4.4) Lemma There exists an acyclic group A in $\mathscr{A}_4$ such that dim BA = 2. (G **acyclic** means that $H_*(BG;\mathbb{Z}) = 0$ where $\mathbb{Z}$ is endowed with the trivial G-action).

Proof : Let $G = \langle a,b \mid a^3 = b^5 \rangle$ (the group of the (3.5)-torus knot; one could take another (p,q)-knot with p and q relatively prime odd integers). The group G belongs to $\mathscr{A}_2$. One has G/[G,G] infinite cyclic generated by $m = a^{-1}b^2$. The commutator group [G,G] is free of rank 8 on $[a^i,b^j]$ for $i = 1,2$ and $1 \le j \le 4$. The center $\zeta(G)$ of G is infinite cyclic on a^3.

(4.4.a) Sublemma The equation $m^k x m^{-k} = x^{-1}$ is possible in G only if $x = 1$. The equation $m^k x m^{-k} = x$ is possible in G iff $x = m^i z$ with $z \in \zeta(G)$.

As the proof of (4.1), our proof of (4.4.a) uses the Serre theory of groups acting on trees. It is also posponed till the end of this section.

The element $u = [a,b]$ generates a $\sqrt{\ }$-closed subgroup U in G. Indeed, U is $\sqrt{\ }$-closed in $[G,G]$ (since u is part of a basis of $[G,G]$) and $[G,G]$ is $\sqrt{\ }$-closed in G (since $G/[G,G]$ has no 2-torsion). On the other hand, the element m generates a subgroup M of G which is also $\sqrt{\ }$-closed. Indeed, suppose that $g^2 = m^k$. As $G/[G, G]$ is infinite cyclic generated by m, one has $k = 2i$ and $g = ym^i$ with $y \in [G,G]$. Then, one has $m^{2i} = g^2 = ym^iym^i = ym^iym^{-i}m^{2i}$ which implies $m^iym^{-i} = y^{-1}$. Thus $y = 1$ by (4.4.a).

Let G_1 and G_2 be two copies of G, with corresponding elements m_1,u_1 and m_2,u_2. By the above, the group $P = G_1 * G_2/\{m_1=u_2\}$ is in the class $\mathscr{A}_3$. By the Mayer-Vietoris sequence for amalgamated products, one checks easily that $H_*(P) = 0$ if $* \neq 0,1$ and $H_1(P) = \mathbb{Z}$, generated by m_2.

Let us consider the subgroup Q of P generated by u_1 and m_2. As $M \cap U = (1)$ in G, Q is free on u_1 and m_2 [Se, Corollary p.14]. and we have $Q \cap G_1 = U_1$ and $Q \cap G_2 = M_2$. We will prove that Q is $\sqrt{\ }$-closed in P, using (4.1) with $R_i = G_i$, $Q = S$, $S_1 = U_1$ and $S_2 = M_2$. It just remains to check Condition 3) of (4.1) which we do by showing that the equations $m^iu^sm^j = u^t$ and $u^im^su^j = m^t$ are possible in G only if $s = t = 1$.

Let us first consider the equation $m^iu^sm^j = u^t$. Passing to $G/[G,G]$ shows that $j = -i$. Thus u^t is the image of u^s under an automorphism of the free group $[G,G]$. This implies that $t = \pm s$. One checks easily that this contradicts (4.4.a).

As for the equation $u^im^su^j = m^t$, one must have $s = t$ for homological reasons. The equation is then equivalent to $m^su^jm^{-s} = u^{-i}$

which drives us back to the former case.

Let $\bar{P}$ be another copy of P. By the above, the group $A = P\times\bar{P}/\{m_2=\bar{u}_1,\ u_1=\bar{m}_2\}$ belongs to $\mathscr{A}_4$. Using the Mayer-Vietoris sequence again, one checks that A is acyclic. Observe that dim BA=2.

<u>(4.5) Remarks on the proof of (4.4)</u> : a) The subgroup $U_1\subset G_1\subset Q\subset A$ generated by u_1 is $\sqrt{\ }$-closed in A. Indeed, U_1 is $\sqrt{\ }$-closed in $Q = U_1\times M_1$ and Q is $\sqrt{\ }$-closed in A by (4.2.b).

b) Acyclic groups can be obtained by the amalgamation of two copies of a free group F of rank 2 over a suitable subgroup 5 (see [BDH , p.11]). Problem : find such a situation where S is $\sqrt{\ }$-closed in F.

<u>(4.6) Proof of Proposition (2.6)</u> Following the procedure of [Ma], we consider for any polyedron L (polyedron = finite simplicial complex) the following condition $\mathscr{M}(L)$:

<u>Condition $\mathscr{M}(L)$</u> : There exists a map $t : (UL,TL) \longrightarrow (CL,L)$ (where CL denotes the cone over L) such that, for each connected subpolyedron M of L, one has :

a) $t|t^{-1}(CM) : t^{-1}(CM) \longrightarrow CM$ and $t|t^{-1}(M) : t^{-1}(M) \longrightarrow M$ are acyclic maps

b) $t^{-1}(CM) = B\Gamma_{CM}$ and $t^{-1}(M) = B\Gamma_M$, where Γ_M and Γ_{CM} are groups in $\mathscr{A}$;moreover, dim $B\Gamma_M$ = dim M and $\dim B\Gamma_{CM} = \dim M + 1$

c) $\ker(\Gamma_M \longrightarrow \pi_1(M))$ is locally perfect

d) If M' is a connected subpolyedron of L containing M, the inclusion $t^{-1}(CM,M)\subset t^{-1}(CM',M')$ induces four homomorphisms

$$\begin{array}{ccc}\Gamma_M & \rightarrowtail & \Gamma_{CM}\\ \downarrow & & \downarrow\\ \Gamma_{M'} & \longrightarrow & \Gamma_{CM'}\end{array}$$

which are all monomorphisms and $\sqrt{\ }$-closed (a monomorphism $\gamma : G \to G'$ is $\sqrt{\ }$-closed if $\gamma(G)$ is $\sqrt{\ }$-closed in G').

We shall prove that Condition $\mathscr{M}(L)$ holds by induction on dim L .

<u>dim L = 0</u> One takes t to be the identity map.

<u>dim L = 1</u> One takes t to be the identity map on TL = L and on the

1-skeleton $UL^{(1)}$ of UL which is $L \cup C(L^{(0)})$. Let A be the acyclic group constructed for (4.4) and $u_1 \in A$ be the element considered in (4.5.a). Then BA can be taken to be a polyedron having a subpolyedron isomorphic to the boundary of a 2-simplex which represent the class u_1. Form the polyedron

$$UL = UL^{(1)} \amalg \left(\coprod_{\sigma} (CA)_{\sigma} \right) / \{\partial\sigma = (u_1)_{\sigma}\}$$

where $(BA)_{\sigma}$ is a copy of BA and σ runs over the set of 2-cells of CL. One easily check Conditions a)-d), using (4.4),(4.5.a), (4.2.b) and (4.2.c) for the latter.

Induction step : one assumes by induction that $\mathcal{M}(L)$ holds if $\dim L \le n-1$. By induction on the number of n-cells of L, it is enough to prove that $\mathcal{M}(L_0)$ implies $\mathcal{M}(L)$ when L is the union of L_0 with one n-simplex σ. As $n \ge 2$, $\partial\sigma$ is connected and one may assume that L_0 is connected.

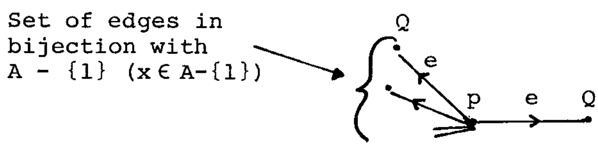
As $\mathcal{M}(L_0)$ holds, $t^{-1}(C\partial\sigma) = U\partial\sigma$ and $t^{-1}(\partial\sigma) = T\partial\sigma$ are subpolyedra of UL_0 and TL_0 respectively. Let TL be $TL_0 \cup U'\partial\sigma$, where $U'\partial\sigma$ is another copy of $U\partial\sigma$ attached to $T\partial\sigma$ and extend t to TL by sending $U'\partial\sigma$ to σ . Then $TL = B\Gamma_L$ where Γ_L is the free product $\Gamma_{L_0} * \Gamma_{C'\partial\sigma}$ with amalgamation over $\Gamma_{\partial\sigma}$ (where $C'\partial\sigma$ is another copy of $C\partial\sigma$). Observe also that $t^{-1}(L \cup CL_0) = B\Gamma_{L \cup CL_0}$, where $\Gamma_{L \cup CL_0}$ is the free product $\Gamma_{C'\partial\sigma} * \Gamma_{CL_0}$ with amalgamation over $\Gamma_{\partial\sigma}$ and that $\Gamma_{C'\partial\sigma} (*_{\Gamma_{\partial\sigma}}) \Gamma_{C_{\partial\sigma}} = \Gamma_{\Sigma(\partial\sigma)}$ is a subgroup of $\Gamma_{L \cup CL_0}$. As in [BDH, Theorem 6.1] one embedds $\Gamma_{\Sigma(\partial\sigma)}$ into the acyclic group $(A \times \Gamma_{\partial\sigma}) * \Gamma_{C\partial\sigma} = \Gamma_{C\Sigma\partial\sigma}$ (amalgamation over $\Gamma_{\partial\sigma}$; A is the acyclic group of (4.4)) by sending $g \longrightarrow g$ if $g \in \Gamma_{C\partial\sigma}$ and $g \longrightarrow aga^{-1}$ if $g \in \Gamma_{C'\partial\sigma}$, where a $A - \{1\}$. Take $UL = TL \cup UL_0 \cup m$ where m is the mapping cylinder of the above embedding and extend t to UL by sending m onto $C\sigma$. One easily check Condition a)-c) of $\mathcal{M}(L)$ (observe that $\Gamma_{C\Sigma\partial\sigma} \in \mathcal{A}$ by (4.4) and (4.3)). For Condition d), one checks that the monomorphisms $\Gamma_Y \longrightarrow \Gamma_X$ corresponding to all the inclusion $Y \longrightarrow X$ of the following diagram :

$$\begin{tikzcd}
C'\partial\sigma \arrow[r, leftrightarrow, "(1)"] \arrow[d, "(2)"] \arrow[rrr, bend left, "(2)"] & \partial\sigma \arrow[r, "(1)"] \arrow[d] & L_0 \arrow[r, "(3)"] \arrow[d, "(1)"] & L \arrow[ddl, "(4)"'] \\
\Sigma\partial\sigma \arrow[r, leftrightarrow, "(2)"] \arrow[dr, "(3)"'] & C\partial\sigma \arrow[r, "(1)"] & CL_0 \arrow[d, "(4)"] & \\
& C\Sigma\partial\sigma \arrow[r, "(4)"] & CL &
\end{tikzcd}$$

are $\sqrt{\ }$-closed. This is done as follows :

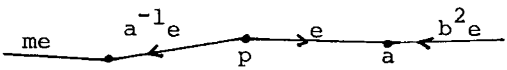
- inclusions (1) are $\sqrt{\ }$-closed because $\mathscr{M}(L_0)$ holds.
- " (2) " " " inclusions (1) are, using (4.2.b) and (4.2.c).
- if inclusion (3) is $\sqrt{\ }$-closed, then inclusions (4) are $\sqrt{\ }$closed, using several times (4.2.b) and (4.2.c). For instance, the inclusion $L \subset CL$ has to be decomposed : $L \subset L \cup C\partial\sigma \subset (C\Sigma\partial\sigma \cup L) \cup_{L \cup C\partial\sigma} (CL_0)$, etc.

It thus remains to prove that Inclusion (3) is $\sqrt{\ }$-closed. To simplify the notation, write Inclusion (3) under the form $G' *_H G \longrightarrow (A\times H) *_H G$ (G' a copy of G). As for the proof of (4.4.a) and (4.1), we shall use the Serre theory of amalgamated product acting on trees [Se, 4 and 5]. Recall that an amalgamated product $R_1 *_B R_2 = R$ acts on a tree T_R characterised by the following properties : there is a fundamental domain which is a segment $\overset{P}{\bullet}\xrightarrow{e}\overset{Q}{\bullet}$ isomorphic to the quotient tree $R \backslash T_R$ with isotropy groups $R_P = R_1, R_Q = R_2$ and $R_e = B$. Applying this to $R = (A\times H) *_H G$ and making the normal closure $\overline{G}$ of G act on T_R, one see that a fundamental domain isomorphic to $\overline{G} \backslash T_R$ is given by the following tree :



The isotropy group are : $R_{xe} = H$ and $R_{xQ} = xGx^{-1}$. Using [Se, §5] one deduces that $\overline{G}$ is the free product of the groups xGx^{-1} ($x \in A$) amalgamated over their common subgroup H. Therefore, the subgroup $G' *_H G$ of R which is the subgroup generated by G and aGa^{-1} is $\sqrt{\ }$-closed in $\overline{G}$ by (4.2.c) (the inclusion $H \subset A\times H$ is $\sqrt{\ }$-closed since $A \in \mathscr{A}$ and groups in $\mathscr{A}$ have no 2-torsion). On the other hand, $\overline{G}$ is $\sqrt{\ }$-closed in R since $R/\overline{G} = A$ has no 2-torsion. Therefore, $G' *_H G$ is $\sqrt{\ }$-closed in R.

Proof of Sublemma (4.4.a) Observe that the first statement is implied by the second since $m^k x m^{-k} = x^{-1}$ implies that $m^{2k} x m^{-2k} = x$. To establish the second statement, observe that the tree T_G has fundamental domain $\overset{P}{\bullet}\xrightarrow{e}\overset{Q}{\bullet}$ with isotropy groups $G_P = \langle a \rangle$, $G_Q = \langle b \rangle$ and $G_e = \zeta(G)$. One has the following situation in T_G :



By [Se, Proposition 25 §6], one deduces that the subgraph drawn above is part of an infinite chain L on which m acts by a translation of amplitude 2. Observe that the orientations of the edges of L imply that m is a generator of the oriented-automomorphisms group of L. Now, if m^k commutes with x, one deduces from [Se, Propositions 25 and 27 §6] that $xL = L$ and thus $xe = m^i e$ for some i. As $G_e = \zeta(G)$, this implies that $xm^{-i} \in \zeta(G)$.

Proof of Lemma (4.1) The Serre tree T_R has here fundamental domain (isomorphic to $R\diagdown T_R$) a cone on the set of vertices $\{P_i\}_{i\in I}$ (the cone vertex is called P; the edge from P_i to P is called e_i) , and the isotropy groups are $R_{P_i} = R_i$, $R_P = R_{e_i} = B$.

Let T_S be the smallest subgraph of T_R such that $\{e_i; i\in I\} \subset \{\text{Edges } T_S\}$ and $ST_S = T_S$. As S is generated by $S_i = S_{P_i}$, T_S is connected by the obvious generalisation of [Se, Lemme 2, p.49] and thus T_S is a subtree of T_R.

Let $g \in R$ such that $g^2 \in S$. As an oriented automorphism of T_R, g has either a fixed vertex or there is an infinite chain L in T on which g acts by a non-trivial translation [Se, Proposition 25 §6]. Suppose that g has a fixed vertex V. Hence $g^2 V = V$ and, as $gT_S \cap T_S \neq \emptyset$, g must fix the whole path joining V to T_S. Therefore one may suppose that $V \in T_S$ which implies that $g = tr_i t^{-1}$ with $r_i \in R_i$ (for some i) and $t \in S$. Thus, $r_i^2 = t^{-1}g^2 t \in S \cap R_i = S_i$. As S_i is $\sqrt{\ }$-closed in R_i, one has $t^{-1}gt \in S_i$ and then $g \in S$.

It then remains to check the case where g translates a chain L. As $g^2 \in S$, one has $L \subset T_S$ (otherwise $gT_S \;\; T_S = \emptyset$). Therefore, by replacing if necessary g by one of its conjugate by an element of S, one may suppose that L contains the edge e_i for some $i \in I$. As $T_S \cap \mathrm{Orbit}_R(P) = \mathrm{Orbit}_S(P)$, there is $h \in S$ such that $b = h^{-1}g \in R_p = B$. One has $g^2 = hbhb \in S$ which means $bhb \in S$. As $L \subset T_S$, the vertex P_i is common to the edges e_i and s_ie_i with $s_i \in S_i$. Observe that the path joining $hb(s_ie_i)$ to P_i contains s_ie_i, and therefore $bhb(s_ie_i) \in T_S$ implies that $bs_ie_i \in T_S$. The latter means $bs_i = \tilde{s}_i\tilde{b}$ for some $\tilde{s}_i \in S_i$ and $\tilde{b} \in B$. This contradicts Condition 3) of (4.1).

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SOME CLOSED 4-MANIFOLDS WITH EXOTIC DIFFERENTIABLE STRUCTURE

Meinem Vater zum 75. Geburtstag gewidmet

M. KRECK

§ 1 Introduction

Recently there has been a phantastic progress in the theory of 4-manifolds. M. Freedman using techniques of A. Casson has solved the topological 4-dimensional Poincaré conjecture and classified all 1-connected closed almost smooth 4-manifolds showing that in the Spin case ($w_2=0$) they are bijectively determined by the intersection form and in the non-Spin case ($w_2 \neq 0$) by the intersection form plus the Kirby-Siebenmann smoothing obstruction $k \in \mathbb{Z}_2$ ([8],[17]). F. Quinn has shown that the same is true for Top 4-manifolds [14].

On the other hand S.K. Donaldson has shown that the only definite form which can be realized as the intersection form of a smooth closed 1-connected 4-manifold is up to sign the standard Euclidean form ([6], [3]).

These results imply that there are some non-compact 4-manifolds which have an exotic differentiable structure. In 1978 Freedman has constructed manifolds which are proper homotopy equivalent to $S^3 \times \mathbb{R}$ but not diffeomorphic ([7], [16]). Combining this with his new results it implies that $S^3 \times \mathbb{R}$ has at least one exotic differentiable structure. The most surprising consequence of both Freedman's and Donaldson's results is that $\mathbb{R}^4$ has at least one exotic differentiable structure ([18]).

In the theory of closed 4-manifolds there are some good candidates for manifolds which might have an exotic differentiable structure, for instance Cappell and Shaneson have constructed a manifold Q^4 which is topologically h - cobordant to $\mathbb{R}P^4$ but not diffeomorphic [5] . But as far as I know it is not known until now whether they are homeomorphic.

The following observation, which is a consequence of Freedman's results, shows that there are many closed smooth 4-manifolds which are homeomorphic but, as we will show, in some cases are not diffeomorphic. Let K be the Kummer surface (compare for instance [19]).

Lemma 1: Let M be a non-orientable closed smooth 4-manifold. Then M # K is homeomorphic to M # 11 $(S^2 \times S^2)$.

Proof: By Freedman's classification of 1-connected topological 4-manifolds there exists an almost differentiable almost parallelizable 1-connected closed 4-manifold $M(E_8)$ whose intersection form is the form corresponding to the Lie group E_8 ([8] ,[17]). Furthermore it implies that $M(E_8)$ # $M(E_8)$ # $3(S^2 \times S^2)$ is homeomorphic to K, as the intersection form of K is $E_8 \oplus E_8 \oplus 3H$, H the hyperbolic form .

Now, as M is non-orientable, we have that ($M(E_8)$ # $M(E_8)$) # M is homeomorphic to ($M(E_8)$ # ($-M(E_8)$)) # M. Thus K # M is homeomorphic to ($M(E_8)$ # ($-M(E_8)$)) # $3(S^2 \times S^2)$ # M.

On the other hand the intersection form of $M(E_8)$ # ($-M(E_8)$) is indefinite and even and thus classified by the signature, which is zero, and the rank [12] . Freedman's Theorem implies that $M(E_8)$ # ($-M(E_8)$) is homeomorphic to $8(S^2 \times S^2)$.

Thus K # M is homeomorphic to $11(S^2 \times S^2)$ # M.

q.e.d.

In § 3, Theorem 1' we will describe some classes of non-orientable closed smooth 4-manifolds M for which M # K is not diffeomorphic to M # $11(S^2 \times S^2)$, in fact they are even not stably diffeomorphic. As a consequence we formulate here the following

Theorem 1: Let π be a finitely generated group and $w_1 \in H^1(\pi;\mathbb{Z}_2)$ a non-trivial element. Then there exists a closed smooth 4-manifold M with $\pi_1(M) = \pi$ and $w_1(M) = w_1$ s.t. $M \# r(S^2 \times S^2)$ has at least one exotic differentiable structure for all $r \geq 11$.

Examples of manifolds M with this property:

$\pi = \mathbb{Z}_2$: a) $M = \mathbb{R}P^4$

b) M = total space of the linear S^2-bundle over $\mathbb{R}P^2$ with w_1 of the S^2-bundle trivial and w_2 non-trivial

We will describe more examples with $\pi = \mathbb{Z}_2$ in §5.

$\pi = \mathbb{Z}$: $M = S^1 \tilde{\times} S^3$, the total space of the non-trivial linear S^3-bundle over S^1.

$\pi = \pi_1(F)$, F a closed surface.

a) F non-orientable and Euler characteristic e(F) even:

$M = F \times S^2$

b) F non-orientable and e(F) odd:

M = total space of the linear S^2-bundle over F with $w_1 = 0$ and $w_2 \neq 0$.

c) F orientable of genus ≥ 1: Let $x \in H^1(F;\mathbb{Z}_2)$ be non-trivial. M = total space of the linear S^2-bundle over F with $w_1 = x$ and $w_2 = 0$.

For arbitrary π we will describe a construction in the proof of Theorem 1' (§3).

I would like to thank Stephan Stolz for many helpful and stimulating discussions when this work was in statu nascendi. Furthermore I would like to thank the Matematisk Institut in Aarhus and the Max-Planck Institut/SFB in Bonn for their support during the periods I could spend there (summer 1981 in Aarhus and from September 1981 to September 1982 in Bonn).

§ 2 Manifolds of type B

In [11] I have developed a modified surgery approach to the classification of closed manifolds. Instead of classifying all manifolds of a given homotopy type I classify - roughly spoken - all s-cobordism classes of n-manifolds with prescribed $\left[\frac{n}{2}\right]$ - skeleton and prescribed normal bundle over it. This can be better formulated in the language of Postnikov decompositions (compare [4]). In our context we only need some basic notations and properties which we collect in this chapter.

<u>Definition:</u> Let $B \longrightarrow BO$ be a k-Postnikov fibration, i.e. a fibration with fibre F such that $\pi_i(F) = \{0\}$ for all $i \geq k$. Let $n = 2k$ or $2k + 1$. We say that a differentiable n-manifold is <u>of type B</u> if there is a Postnikov decomposition $\tilde{\nu}_M$ of the normal Gauß map $\nu_M : M \longrightarrow BO$ over B, that is a k-equivalence $\tilde{\nu}_M : M \longrightarrow B$ s.t.

$$\begin{tikzcd} & B \arrow[d] \\ M \arrow[ur, "\tilde{\nu}_M"] \arrow[r, "\nu_M"'] & BO \end{tikzcd}$$

commutes.

A specific choice of such a Postnikov decomposition $\tilde{\nu}_M$ is called a <u>B-typisation of M.</u> The set of s-cobordism classes of closed smooth n-manifolds of type B is denoted by $Ty_n(B)$ and the set of s-cobordism classes of such manifolds together with a typisation is called $\widetilde{Ty}_n(B)$.

There is a canonical projection $\widetilde{Ty}_n(B) \twoheadrightarrow Ty_n(B)$. The group of homotopy classes of fibre homotopy self equivalences of the fibration $B \longrightarrow BO$ is denoted by Aut(B). It operates on $\widetilde{Ty}_n(B)$ by composition. It is not difficult to show, using the uniqueness of a Postnikov decomposition ([4], Chap. 5.3), that $Ty_n(B)$ is the orbit space of $\widetilde{Ty}_n(B)$ under this action.

<u>Lemma 2([11] , § 3):</u> $Ty_n(B) = \widetilde{Ty}_n(B) \, / \, {}_{Aut(B)}$.

Let Ω_n^B be the cobordism group of B-manifolds in the sense of Lashof (compare [20]). There is a map $\widetilde{Ty}_n(B) \longrightarrow \Omega_n^B$. As before Aut(B) operates on Ω_n^B by composition. This operation is linear and the map $Ty_n(B) \rightarrow \Omega_n^B$ is equivariant. Thus we have an induced map of sets

$$Ty_n(B) \longrightarrow \Omega_n^B \; / \; {}_{Aut(B)} .$$

We will use this map to distinguish some closed smooth 4-manifolds.

<u>Remark:</u> The main result of [11] is that for $n \geq 4$ the map $Ty_n(B) \rightarrow \Omega_n^B/_{Aut(B)}$ is surjective if B has finite $\left[\frac{n}{2}\right]$ - skeleton and elements in the same fibre are distinguished by an invariant in a semi group $l_{n+1}(\pi_1(B),w_1)$ mod some indeterminacy. These l-semi groups replace in my surgery approach Wall's L-groups. It is perhaps interesting in our context to note that for n even the algebraic obstruction reduces to the Euler characteristic if we pass from s-cobordism classes to stable diffeomorphism classes ([11] , §5): $n = 2k$. If M and N are in $Ty_n(B)$, $[M] = [N]$ in $\Omega_n^B \, / \, {}_{Aut(B)}$ and M and N have same Euler characteristic then $\exists \; r \in \mathbb{N}$ s.t. $M \# r(S^k \times S^k) \cong N \# r(S^k \times S^k)$.

§ 3 Reduction of Theorem 1 to stable homotopy theory

If $M \in Ty_4(B)$ and N is a 1-connected smooth closed almost parallelizable 4-manifold then $M \# N \in Ty_4(B)$. Especially for the Kummer surface K we have $M \# K \in Ty_4(B)$ and more generally $M \# K \# r(S^2 \times S^2) \in Ty_4(B)$. On the other hand $M \# (r+11)\cdot(S^2 \times S^2) \in Ty_4(B)$. Thus if $M \# K \# r(S^2 \times S^2)$ and $M \# (r+11)\cdot(S^2 \times S^2)$ are not equivalent in $\Omega_4^B / {}_{\mathrm{Aut}(B)}$ and M is non-orientable, then these manifolds are not diffeomorphic and thus by Lemma 1 there exists an exotic differentiable structure on $M \# (r+11)\cdot(S^2 \times S^2)$.

In general it is not easy to compute Ω_4^B and the action of Aut(B) on it. We have no problems with the action of Aut(B) if $M \in Ty_4(B)$ represents zero in Ω_4^B and $M \# K$ is not zero bordant in Ω_4^B. For the action of Aut(B) on Ω_4^B is linear and thus $M \# K$ and M are different in $\Omega_4^B / {}_{\mathrm{Aut}\, B}$. So, in this case for all $r \geq 11$, $M \# r(S^2 \times S^2)$ has an exotic differentiable structure if M is non orientable.

With other words we have proved the main part of Theorem 1 if for any finitely generated group π and $w_1 \in H^1(\pi\,; \mathbb{Z}_2)$ non-trivial there exists a 2-Postnikov fibration $B \longrightarrow BO$ with $\pi_1(B) = \pi$ and a zero-bordant manifold M of type B with $w_1(M) = w_1$ and if the Kummer surface is not zero bordant in Ω_4^B.

Consider the line bundle L over $K(\pi,1)$ with $w_1(L) = w_1$. Let $p: B(\pi,w_1) \to BO$ be the homotopy fibration of the map $K(\pi,1) \times B\,Spin \xrightarrow{f \times p} BO \times BO \xrightarrow{\oplus} BO$, where $f: K(\pi,1) \longrightarrow BO$ is the classifying map of $3L$ and $p: B\,Spin \longrightarrow BO$ the canonical projection.

<u>Remark:</u> A closed smooth 4-manifold M is of type $B(\pi,w_1)$ if and only if $\pi_1(M) \cong \pi$, $w_1(\nu(M)) \in H^1(M;\mathbb{Z}_2) = \mathrm{Hom}(\pi;\mathbb{Z}_2)$ corresponds under this

isomorphism to w_1 and $w_2(\nu(M)) = w_1^2(\nu(M))$ or equivalently $w_2(\tau(M)) = 0$ (compare [11], § 2).

Proposition 1: The Kummer surface K considered as a $B(\pi, w_1)$-manifold is not zero-bordant in $\Omega_4^{B(\pi, w_1)}$.

Proof: As $f : K(\pi,1) \longrightarrow BO$ factorizes over $\mathbb{R}P^{\infty} = K(\mathbb{Z}_2,1)$, the fibration $B(\pi, w_1) \longrightarrow BO$ factorizes over $B(\mathbb{Z}_2,1)$:

$$B(\pi, w_1) \longrightarrow B(\mathbb{Z}_2,1) \longrightarrow BO.$$

Thus we are finished if K considerd as a $B(\mathbb{Z}_2,1)$-manifold is non-trivial in $\Omega_4^{B(\mathbb{Z}_2,1)}$. We will show in the next chapter that $\Omega_4^{B(\mathbb{Z}_2,1)} = \mathbb{Z}/_{16\cdot\mathbb{Z}}$ and that K represents the non-trivial element of order 2 in this group (Proposition 2).

q.e.d.

Now we have all the material to prove a slightly stronger statement than Theorem 1.

Theorem 1' : a) Let π be a finitely generated group and $w_1 \in H^1(\pi; \mathbb{Z}_2)$ a non-trivial element. Let M be a manifold of type $B(\pi, w_1)$ which with some typisation is zero bordant in $\Omega_4^{B(\pi, w_1)}$. Then $M \# r(S^2 \times S^2)$ has at least one exotic differentiable structure for $r \geqslant 11$. For all π and w_1 such a manifold M exists.
b) For $\pi = \pi_1(F)$, F a closed surface, the manifolds M described in Theorem 1 are of type $B(\pi_1(F),x)$ and represent zero in $\dot{\Omega}_4^{B(\pi_1(F),x)}$,

where $x = w_1(F)$ if F is non-orientable and some prescribed non-trivial element in $H^1(F;Z_2)$ if F is orientable. Thus $M \# r(S^2 \times S^2)$ has an exotic differentiable structure for all $r \geqslant 11$.

Proof : For a) we have shown everything except the existence of a zero-bordant manifold of type $B(\pi, w_1)$. For this one can either refer to the surjectivity of $Ty_4(B(\pi, w_1)) \longrightarrow \Omega_4^{B(\pi, w_1)}/Aut(B)$ mentioned in the Remark at the end of § 2 or one can use the following construction.

Consider a presentation of π: $[x_1, \ldots, x_m; r_1, \ldots, r_n]$. Let X be a geometric realization of this presentation by a 2-complex. Let $X \longrightarrow \mathbb{R}P^4$ be the classifying map of w_1 considered as element of $H^1(X; \mathbb{Z}_2)$. In $\mathbb{R}P^4 \times \mathbb{R}$ one can approximate this map by an embedding $X \hookrightarrow RP^4 \times R$. Let U be a smooth compact regular neighborhood of X. Then $M = \partial U$ is a closed smooth 4-manifold of type $B(\pi, w_1)$ and U is a zero-bordism of it. For by construction we know that $w_2(\tau(U)) = 0$ and $w_1(\nu(U)) = w_1$. Thus $\nu : U \longrightarrow BO$ factors over $B(\pi, w_1)$ by a 2 - equivalence. As $\pi_1(\partial U) \longrightarrow \pi_1(U)$ is an isomorphism the restriction of the lift to U is a typisation.

b) follows from a) as by the Remark before Proposition 1 the manifold M is of type $B(\pi_1(F), x)$ and M bounds the corresponding disk-bundle.

q.e.d.

Remark : In some cases one can use the invariant of ([5],§1) which is very similar to the invariant we will study in §4 to show that $M \# K$ and $M \# 11(S^2 \times S^2)$ are not stably diffeomorphic.

§ 4 Some stable homotopy groups

To finish the proof of Proposition 1 we have to compute $\Omega_4^{B(\mathbb{Z}_2,1)}$.

Proposition 2: $\Omega_4^{B(\mathbb{Z}_2,1)} = \mathbb{Z} / 16 \cdot \mathbb{Z}$ and the Kummer surface K represents the non-trivial element or order two in it.

Proof: By the Pontrjagin-Thom construction $\Omega_4^{B(\mathbb{Z}_2,1)}$ is isomorphic to $\pi_7(M(3L) \wedge M\,\mathrm{Spin})$, where M(3L) is the Thom space of 3L over $\mathbb{R}P^{\infty}$ which is equal to $\mathbb{R}P^{\infty} / \mathbb{R}P^{2}$ and M Spin is the spectrum of Spin-cobordism. This homotopy group can be interpreted as

$$\pi_7(M(3L) \wedge M\,\mathrm{Spin}) = \tilde{\Omega}_7^{\mathrm{Spin}}(M(3L)).$$

The Atiyah-Hirzebruch spectral sequence implies that this group has at most 16 elements. The corresponding line in the E_2-term consists of $H_7(M(3L); \Omega_0^{\mathrm{Spin}}) = \mathbb{Z}_2$; $H_6(M(3L); \Omega_1^{\mathrm{Spin}}) = \mathbb{Z}_2$, $H_5(M(3L); \Omega_2^{\mathrm{Spin}}) = \mathbb{Z}_2$ and of $H_3(M(3L); \Omega_4^{\mathrm{Spin}}) = \mathbb{Z}_2$.

Thus we are finished if we can construct a surjective homomorphism

$$\alpha : \Omega_4^{B(\mathbb{Z}_2,1)} \longrightarrow \mathbb{Z} / 16 \cdot \mathbb{Z}.$$

We will do this as follows. Consider a $B(\mathbb{Z}_2,1)$-manifold M represented by a map $M \longrightarrow \mathbb{R}P^N \times B\,\mathrm{Spin}$, $N \gg 4$. Take the composition $M \longrightarrow \mathbb{R}P^N \times B\,\mathrm{Spin} \longrightarrow \mathbb{R}P^N$ and make it transversal to $\mathbb{R}P^{N-1}$. We denote the inverse image of $\mathbb{R}P^{N-1}$ in M by F. The normal bundle of F is induced from $4L \times \gamma\mathrm{Spin}$ over $\mathbb{R}P^{N-1} \times B\,\mathrm{Spin}$. If we fix a Spin-structure on 4 L we can consider F together with the map $F \longrightarrow \mathbb{R}P^{N-1}$ as an element of the bordism group $\Omega_3^{Spin}(\mathbb{R}P^{\infty})$. This element is a bordism invariant of the $B(\mathbb{Z}_2,1)$-manifold M.

As a consequence of the next Proposition we will show (Corollary 1) that every element of $\Omega_3^{Spin}(\mathbb{R}P^{\infty})$ considered as a 2-fold covering $\hat{F} \longrightarrow F$ bounds an oriented ramified covering $\hat{W} \longrightarrow W$, where W is a Spin-manifold extending the Spin-structure of F. We denote the involution on $\hat{W}$ by τ.

We define

$$\alpha(M) := \operatorname{sign} M_F - \operatorname{sign} \hat{W} - \operatorname{Fix}\tau \circ \operatorname{Fix}\tau \in \mathbb{Z} / 32 \cdot \mathbb{Z},$$

where $M_F = M$-(open tubular neigborhood of F)sign is the signature of the manifold and $\operatorname{Fix}\tau \circ \operatorname{Fix}\tau$ is the self-intersection number of the fixed point set.

We have to show that this invariant is well defined. Let $M = \partial X$ and $Y \subseteq X$ the inverse image of $\mathbb{R}P^{N-1}$ as constructed before. Y is a Spin manifold with $\partial Y = F$. Let $\hat{Y} \longrightarrow Y$ be the (unramified) 2-fold covering.

Now the manifold $M_F \cup (-\hat{Y})$ bounds and thus $\operatorname{sign} M_F - \operatorname{sign} \hat{Y} = 0$. On the other hand $(-\hat{Y}) \cup \hat{W}$ is a ramified covering over the Spin-manifold $(-Y) \cup W$ and by a formula of Hirzebruch we have

$$\operatorname{sign}(-\hat{Y} \cup \hat{W}) = 2 \cdot \operatorname{sign}(-Y \cup W) - \operatorname{Fix}\tau \circ \operatorname{Fix}\tau \quad ([9], \S 4).$$

By Rohlin's Theorem $\operatorname{sign}(-Y \cup W) = 0 \bmod 16$ and thus

$$\operatorname{sign}(-\hat{Y} \cup \hat{W}) = -\operatorname{Fix}\tau \circ \operatorname{Fix}\tau \bmod 32.$$

Combining these formulas we obtain

$$\operatorname{sign} M_F - \operatorname{sign} \hat{W} - \operatorname{Fix}\tau \circ \operatorname{Fix}\tau = 0 \bmod 32.$$

To finish the proof of Proposition 2 we have to show that α maps $\Omega_4^{B(\mathbb{Z}_2,1)}$

surjectively onto $2\cdot\mathbb{Z}/_{32}\cdot\mathbb{Z}$ and that $\alpha(K) = 16 \bmod 32$. The last statement is obvious as $\alpha(K) = \operatorname{sign} K \bmod 32$. For the first statement we consider $\mathbb{R}P^4$ as a $B(\mathbb{Z}_2,1)$-manifold. Then $F = \mathbb{R}P^3$, $\hat{F} = S^3$ and $\mathbb{R}P^4{}_F = D^4$. Let H be the Hopf disk-bundle over S^2 and H^2 the disk-bundle of $H \otimes H$. H^2 is a Spin-manifold. If we choose the $B(\mathbb{Z}_2,1)$-structure on $\mathbb{R}P^4$ appropriately, then $\partial H^2 = F$ as Spin-manifold. The covering $\hat{F} \longrightarrow F$ extends to a ramified covering $H \longrightarrow H^2$, ramified along S^2. Thus

$$\alpha(\mathbb{R}P^4) = -\operatorname{sign} H - S^2 \circ S^2 = -2 \bmod 32$$

q.e.d.

In the proof of Proposition 4 we have used the fact that every oriented 2-fold covering $\hat{F} \longrightarrow F$ over a Spin-manifold F bounds an oriented ramified covering $\hat{W} \longrightarrow W$ where W is a Spin manifold extending the Spin-structure of F. This follows immediately from the following Proposition.

<u>Proposition 3:</u> $\Omega_3^{Spin}(\mathbb{R}P^\infty) \cong \mathbb{Z}/_8\cdot\mathbb{Z}$ generated by $S^3 \longrightarrow \mathbb{R}P^3$ with some Spin-structure on $\mathbb{R}P^3$.

At the end of the proof of Proposition 2 we have constructed a ramified covering bounding $S^3 \longrightarrow \mathbb{R}P^3$. This implies:

<u>Corollary 1:</u> Every element in $\Omega_3^{Spin}(\mathbb{R}P^\infty)$ bounds an oriented ramified covering over a Spin manifold such that the given Spin structure extends.

<u>Proof of Proposition 3:</u> We proceed as in the proof of Proposition 2. The Atiyah-Hirzebruch sprectral sequence implies that $\Omega_3^{\mathrm{Spin}}(\mathbb{R}P^{\infty})$ has at most 8 elements. Again we construct a surjective homomorphism $\Omega_3^{\mathrm{Spin}}(\mathbb{R}P^{\infty}) \to \mathbb{Z}_8$. Let $\hat{F} \to F$ represent an element in $\Omega_3^{\mathrm{Spin}}(\mathbb{R}P^{\infty})$. We can assume that F is connected. Choose a framing α on F compatible with the given Spin structure. Let $\hat{\alpha}$ be the induced framing on $\hat{F}$. Then the invariant is $e(\hat{F},\hat{\alpha}) - 2e(F,\alpha) \in \mathbb{Z}/_{24\cdot\mathbb{Z}}$ where e is the Adams e-invariant.

To show that this is well defined we first observe that the Spin structure on F fixes a framing on $F - \{pt\}$. This implies that any other framing α' on F compatible with the given Spin structure is of the form $(F,\alpha) \# (S^3,\beta)$ for some framing β on S^3. Thus $(\hat{F},\alpha') = (\hat{F},\hat{\alpha}) \# 2(S^3,\beta)$ and this implies that the invariant is independent of the choice of the framing. (The same statement holds in a more general context ([2], § 4)). On the other hand if $\hat{W} \longrightarrow W$ is an oriented covering over a connected Spin-manifold bounding $\hat{F} \longrightarrow F$ then W has a framing compatible with the Spin-structure as $\partial W \neq \emptyset$. Thus the invariant is a bordism invariant.

We compute this invariant in the following example:

Consider SO(3) with the left-invariant framing. The induced framing on SU(2) is again the left-invariant framing. It is well known that SO(3) with this framing represents $2\cdot[SU(2), L]$ and that $[SU(2), L]$ generates $\pi_3^S = \mathbb{Z}/_{24\cdot\mathbb{Z}}$ ([13], § 1). Thus the invariant of the covering $SU(2) \longrightarrow SO(3)$ with the corresponding Spin-structure has value $3 \in \mathbb{Z}/_{24\cdot\mathbb{Z}}$.

q.e.d.

§ 5 Examples with fundamental group $\mathbb{Z}_2$.

In this Chapter we will discuss the case of manifolds M with $\pi_1(M) = \mathbb{Z}_2$ and $w_2(\tau(M)) = 0$. Such a manifold is of type $B(\mathbb{Z}_2,1)$ (Remark before Proposition 1). In Proposition 2 we have shown that $\Omega_4^{B(\mathbb{Z}_2,1)} \cong \mathbb{Z}/16\cdot\mathbb{Z}$ and that K is the element of order 2 in it. Obviously the total space M of the linear S^2-bundle over $\mathbb{R}P^2$ with $w_1 = 0$ and $w_2 \neq 0$ represents the zero in $\Omega_4^{B(\mathbb{Z}_2,1)}$.

As $\mathbb{R}P^4$ has odd Euler characteristic it represents a generator of $\Omega_4^{B(\mathbb{Z}_2,1)}$ if we fix some typisation. We can construct a representative of $k[\mathbb{R}P^4] \in \Omega_4^{B(\mathbb{Z}_2,1)}$ by a manifold M_k of type $B(\mathbb{Z}_2,1)$ as follows. $M_0 := M$ as above and $M_1 := \mathbb{R}P^4$. For $k > 1$ we construct M_k inductively. Fix a $B(\mathbb{Z}_2,1)$ structure on $S^1 \tilde{\times} D^3$ and choose an embedding of $S^1 \tilde{\times} D^3$ into M_{k-1} which is compatible with the $B(\mathbb{Z}_2,1)$ structures and such that $S^1 \times \{0\}$ represents a generator of $\pi_1(M_{k-1})$. Define M_k as $M_{k-1} - S^1 \tilde{\times} \mathring{D}^3 \cup_f \mathbb{R}P^4 - S^1 \tilde{\times} \mathring{D}^3$, where $f : S^1 \tilde{\times} S^2 \longrightarrow S^1 \tilde{\times} S^2$ is the $B(\mathbb{Z}_2,1)$ - structure reversing diffeomorphism obtained by reflection along a section of this fibre bundle. It is easy to check that the typisation of M_{k-1} and of $\mathbb{R}P^4$ extend to a typisation of M_k and that M_k is bordant to $M_{k-1} + \mathbb{R}P^4$.

We will show now that for $k \leq 3$ $M_k \# r(S^2 \times S^2)$ has at least one exotic differentiable structure for $r \geq 11$. For this we compute the action of $\mathrm{Aut}(B(\mathbb{Z}_2,1))$ on $\Omega_4^{B(\mathbb{Z}_2,1)}$. The first step is to show that $B(\mathbb{Z}_2,1) \longrightarrow BO$ is a principal fibration. For this we consider the principal fibration $B \longrightarrow BO$ induced by the map $w_1^2 - w_2 : BO \longrightarrow K(\mathbb{Z}_2,2)$. As $w_1^2\nu(\mathbb{R}P^4) = w_2\nu(\mathbb{R}P^4)$, the normal Gauß map of $\mathbb{R}P^4$ factors over B and the lift is a 2-equivalence. Thus

$$\begin{tikzcd} & B \arrow[d] \\ RP^4 \arrow[ur] \arrow[r] & BO \end{tikzcd}$$

is a 2-Postnikov decomposition. The uniqueness of a Postnikov decomposition implies that B is fibre homotopy equivalent to $B(\mathbb{Z}_2,1)$. Now, if $B(\mathbb{Z}_2,1)$ is a principal fibration with fibre $K(\mathbb{Z}_2,1)$, $\mathrm{Aut}(B(\mathbb{Z}_2,1)) = [BO, K(\mathbb{Z}_2,1)] = Z_2$. The operation on $\Omega_4^{B(\mathbb{Z}_2,1)}$ is given by $-\mathrm{Id}$.

Thus under the map $\mathrm{Ty}(B(\mathbb{Z}_2,1)) \longrightarrow \Omega_4^{B(Z_2,1)}/_{-\mathrm{Id}}$ the manifolds $M_k \# K \# (r-11)(S^2 \times S^2)$ and $M_k \# r(S^2 \times S^2)$ have different images for $k \leq 3$ and $r \geq 11$ and thus, by Lemma 1, $M_k \# r(S^2 \times S^2)$ has at least one exotic differentiable sructure.

Now we show that the manifolds $M_k \# r(S^2 \times S^2)$ and $M_l \# s(S^2 \times S^2)$ are not homeomorphic for $k \neq l \leq 3$. For this we note that the theory of manifolds of type B as described in §2 also works for topological manifolds. In our case we have to replace the space B Spin in the definition of $B(\pi, w_1)$ by B(Spin Top). We denote it by $B_{Top}(\mathbb{Z}_2,1)$. A simple calculation shows that the kernel of $\Omega_4^{B(\mathbb{Z}_2,1)} \longrightarrow \Omega_4^{B_{Top}(\mathbb{Z}_2,1)}$ is $\mathbb{Z}_2$ generated by K. This implies the desired statement. We summarize:

Proposition 4: Let M_k be as constructed above. For $k \leq 3$ and $r \geq 11$ the manifolds $M_k \# r(S^2 \times S^2)$ have an exotic differentiable structure and are in pairs not homeomorphic.

Remark : All our exotic differentiable structures are stable in the sense that the structure remains exotic after an arbitrary connected sum with $S^2 \times S^2$. On the other hand our computation above and the stable classification result mentioned in the Remark in §2 imply that M_4 has no stable exotic structure.

§ 6 Relation to the exotic structures on $\mathbb{R}^4$ and on $S^3 \times \mathbb{R}$

It is natural to ask whether there is a relation between Freedman's exotic structure on $S^3 \times \mathbb{R}$ or the exotic structure on $\mathbb{R}^4$ mentioned in the beginning of the introduction and our exotic structures on closed 4-manifolds. More precisely we ask whether the exotic structures on $S^3 \times \mathbb{R}$ and on $\mathbb{R}^4$ embed into both structures of $M \# r(S^2 \times S^2)$ where M is a manifold as in Theorem 1'. If not this would indicate a strong connection between all these exotic structures. But the answer is, at least stably, that they embed into both structures.

This is obvious for $\mathbb{R}^4$, as the exotic $\mathbb{R}^4$ embeds into K and into $S^2 \times S^2$ ([14],[18]). For $S^3 \times \mathbb{R}$ we will show that a typical exotic structure on $S^3 \times \mathbb{R}$ as constructed by Freedman embeds into $k(S^2 \times S^2)$ for k sufficiently large.

Let me recall how Freedman shows the existence of an exotic structure on $S^3 \times \mathbb{R}$ ([7], [16]). Let H^3 be a smooth homology 3-sphere. He proves that there exists a smooth 4-manifold V^4 which by his recent results is homeomorphic to $S^3 \times \mathbb{R}$ and which has a involution τ such that the fixed point set is H^3 and H^3 separates V into two parts. If the Rohlin μ-invariant of H^3 is nonzero [10] then V^4 is not diffeomorphic to $S^3 \times \mathbb{R}$.

Proposition 5: For every homology sphere H^3 there exists a V^4 homeomorphic to $S^3 \times \mathbb{R}$ as described above, s.t. V^4 embeds into $k(S^2 \times S^2)$ for k sufficiently large.

Proof: Let W be a 1-connected smooth 4-dimensional Spin manifold bounding H^3. There exists an open submanifold V^4 homeomorphic to $S^3 \times \mathbb{R}$ in

$W \cup -W$ containing H^3 such the interchanging of the two sides of $W \cup -W$ restricts to an involution on V with fixed point set H^3. This follows from ([21] , Theorem C) or by a similar argument as in ([16] , § 5). By Wall [22] $W \cup -W$ is stably diffeomorphic to $b(S^2 \times S^2)$ where b is the second Betti number of W.

q.e.d.

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SUR LES IMMERSIONS DE BOY

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On précise les entiers positifs n pour lesquels il existe une immersion générique d'une variété fermée de dimension n-1 dans l'espace euclidien de dimension n ayant un nombre impair de points n-uples. On montre en particulier que le seul n divisible par 4 qui convienne est 4, complétant ainsi les résultats de P. J. Eccles [4].

0. LE PROBLEME DES IMMERSIONS DE BOY

La question que l'on pose est la suivante : pour quels entiers positifs n existe-t-il une immersion générique, d'une variété fermée de dimension n-1 dans l'espace euclidien de dimension n, ayant un nombre impair de points n-uples ?

Un exemple d'une telle immersion est fourni par l'immersion du plan projectif réel dans l'espace euclidien de dimension 3 dont l'image est connue sous le nom de surface de Boy [6]. C'est cette vénérable curiosité mathématique qui justifie le nom que nous donnons à la question posée ci-dessus.

Le problème des immersions de Boy orientées est résolu par P. J. Eccles dans [3] (voir aussi [9]). Dans [4] P. J. Eccles donne une solution du problème sans restriction sur l'orientation dans le cas où n n'est pas divisible par 4. Dans la présente note nous complétons cette solution.

Les travaux de P. Vogel [13] permettent de ramener le problème des immersions de Boy à une question concernant l'homomorphisme d'Hurewicz :

$$\pi_n \Omega^\infty S^\infty MO(1) \to \tilde{H}_n \Omega^\infty S^\infty MO(1),$$

aussi avant d'exposer notre solution, commençerons-nous à rassembler quelques résultats relatifs à la topologie algébrique des espaces $\Omega^\infty S^\infty X$ ou plutôt des espaces de configurations.

1. RAPPELS SUR LES ESPACES DE CONFIGURATIONS

Soit X un espace pointé, on note CX l'espace des configurations associé à X et s : $CX \longrightarrow \Omega^\infty S^\infty X$ l'application naturelle (qui est une équivalence d'homotopie si X est connexe le point base étant supposé "bon"). On note $r : CX \longrightarrow\!\!\!\!\!\!\!\!+\;\; X$ l'application stable dont l'adjointe est s (une flèche barrée : $\longrightarrow\!\!\!\!\!\!\!\!+\;\;$ désigne une application stable).

Soient $m \geqslant 1$ une entier et $E\mathfrak{S}_m$ un espace contractile sur lequel le groupe symétrique $\mathfrak{S}_m$ agit librement. On note $\mathfrak{S}_m X$ le quotient de l'espace $(E\mathfrak{S}_m)_+ \wedge (X \wedge X \wedge \ldots \wedge X)$, X m fois, par l'action diagonale de $\mathfrak{S}_m$ (si Y est un espace, Y_+ désigne la réunion disjointe de Y et d'un point base). On note $\varphi_m : CX \longrightarrow C\,\mathfrak{S}_m X$ les applications de multiplicités de [13] appelées "James-Hopf invariants" dans la terminologie anglo-saxonne [1] [8].

On note enfin respectivement $r_m : CX \longrightarrow \mathfrak{S}_m X$, $r_{m,n} : CX \longrightarrow \mathfrak{S}_m \mathfrak{S}_n X$ les compositions suivantes

$$CX \xrightarrow{\quad \varphi^m \quad} C\,\mathfrak{S}_m X \xrightarrow{\quad r \quad} \mathfrak{S}_m X$$

$$CX \xrightarrow{\quad \varphi^n \quad} C\,\mathfrak{S}_n X \xrightarrow{\ \varphi^m\ } C\,\mathfrak{S}_m \mathfrak{S}_n X \xrightarrow{\quad r \quad} \mathfrak{S}_m \mathfrak{S}_n X$$

2. QUELQUES FORMULES INSTABLES DANS LA COHOMOLOGIE DE L'ESPACE CX.

L'application $r : CX \longrightarrow X$ n'étant qu'une application stable ne commute pas avec les $\smile$-produits, la formule ci-dessous précise le défaut de commutativité.

Proposition 2.1 : Soient Λ_1, Λ_2 deux groupes abéliens et u_1, u_2 deux classes de cohomologie dans $\tilde{H}^*(X, \Lambda_1)$, $\tilde{H}^*(X; \Lambda_2)$ respectivement. On a dans $\tilde{H}^*(CX; \Lambda_1 \otimes \Lambda_2)$ la formule :

$$r^*(u_1 \smile u_2) = (r^* u_1) \smile (r^* u_2) - r_2^* \mathrm{tr}(u_1 \times u_2)$$

où tr désigne la transfert : $\tilde{H}^*(X \wedge X; \Lambda_1 \otimes \Lambda_2) \longrightarrow \tilde{H}^*(\mathfrak{S}_2 X; \Lambda_1 \otimes \Lambda_2)$.

2.2 Avant d'énoncer les deux formules suivantes il nous faut d'abord introduire certains polynômes.

2.2.1. Soients A un anneau commutatif et $\{a_i\}_{i \in I}$ une famille finie d'éléments de A. Pour toute partie non vide J de A on pose $a_J = \prod_{i \in J} a_i$. On note s_m, la $m^{\text{ème}}$ fonction symétrique des a_i :

$$s_m = \sum_{\#J = m} a_J$$

Plus généralement on pose :

$$s_{m,n} = \sum_{\#J = m} (a_J)^n$$

On note enfin $t_{m,n}$ la m^{eme} fonction symétrique des a_J, J décrivant l'ensemble des parties à n éléments de I.

On considère l'anneau de polynômes $\mathbb{Z}[T_1,T_2,\ldots,T_m,\ldots]$; on attribue à la variable T_m le poids m.

Il existe un unique polynôme $\Phi_{m,n}$ (resp. $\Theta_{m,n}$) tel que l'on ait, pour tout anneau commutatif A et toute famille $\{a_i\}_{i \in I}$ d'éléments de A :

$$s_{m,n} = \Phi_{m,n}(s_1,s_2,\ldots)$$

$$\text{resp.} \qquad t_{m,n} = \Theta_{m,n}(s_1,s_2,\ldots)$$

Les polynômes $\Phi_{m,n}$, $\Theta_{m,n}$ sont homogènement pondérés de poids mn; ce sont en fait des polynômes en $T_1, T_2,\ldots,T_{mn}$.

<u>Lemme 2.2.2</u> : Les coefficients de T_{mn} dans les polynômes $\Phi_{m,n}$ et $\Theta_{m,n}$ sont respectivement $(-1)^{m(n-1)}n$ et $(-1)^{(m-1)(n-1)}$.

Soit u une classe de $\tilde{H}^q(X;\Lambda)$ avec $\Lambda = \mathbb{Z}/2$ si q est impair et $\Lambda = \mathbb{Z}/2$ ou $\mathbb{Z}$ si q est pair, on note $P_m u$ la $m^{ème}$ puissance externe de Steenrod appartenant à $\tilde{H}^{mq}(\mathfrak{S}_m X;\Lambda)$ [12 ; p.99].

<u>Proposition 2.2.3</u> (formule de Newton) : On a dans $\tilde{H}^*(CX;\Lambda)$ la formule : $r_m^* P_m u^n = \Phi_{m,n}(r^*u, r_2^*P_2u, r_3^*P_3u,\ldots)$.

<u>Corollaire 2.2.4</u> : On a dans $\tilde{H}^*(CX;\Lambda)$ la congruence : $r_m^*P_m u^n \equiv (-1)^{m(n-1)} n\, r_{mn}^* P_{mn} u$ modulo $\tilde{H}^*(CX;\Lambda) \smile \tilde{H}^*(CX;\Lambda)$.

<u>Proposition 2.2.5</u> : On a dans $\tilde{H}^*(CX;\Lambda)$ la formule : $r_{m,n}^* P_m P_n u = \Theta_{m,n}(r^* u, r_2^* P_2 u, r_3^* P_3 u,\ldots)$.

<u>Corollaire 2.2.6</u> : On a dans $\tilde{H}^*(CX;\Lambda)$ la congruence : $r_{m,n}^* P_m P_n u \equiv (-1)^{(m-1)(n-1)} r_{mn}^* P_{mn} u$ modulo $\tilde{H}^*(CX;\Lambda) \smile \tilde{H}^*(CX;\Lambda)$.

2.3 La dernière formule de ce paragraphe fait intervenir le carré de Pontryagin $\mathfrak{P}$: $\tilde{H}^{2k}(\ ;\mathbb{Z}/2) \longrightarrow \tilde{H}^{4k}(\ ;\mathbb{Z}/4)$, elle mesure le "défaut de stabilité" de cette opération.

<u>Proposition 2.3</u> : Soit u une classe dans $\tilde{H}^{2k}(X;\mathbb{Z}/2)$, on a dans $\tilde{H}^{4k}(CX;\mathbb{Z}/4)$ la formule :

$$r^*\mathfrak{P}u = \mathfrak{P}r^*u + 2_* r_2^* P_2 u$$

2_* désignant l'application : $\tilde{H}^*(\ \ ;\mathbb{Z}/2) \longrightarrow \tilde{H}^*(\ \ ;\mathbb{Z}/4)$ induite par l'inclusion de $\mathbb{Z}/2$ dans $\mathbb{Z}/4$.

Cette formule est une conséquence de la proposition 2.2.3 avec $q = 2k$, $\Lambda = \mathbb{Z}$, $m = 1$ et $n = 2$, elle est généralisée dans [16].

3. SOLUTION DU PROBLEME DES IMMERSIONS DE BOY

Le groupe de cobordisme d'immersions de variétés fermées de dimension n-1 dans $\mathbb{R}^n$ est isomorphe au groupe d'homotopie stable $\pi_n^S MO(1) = \pi_n CMO(1)$ [14][13] et en faisant correspondre à une immersion générique le nombre modulo 2 de ses points n-uples on définit un homomorphisme $\theta_n : \pi_n CMO(1) \longrightarrow \mathbb{Z}/2$ qui d'après [13] est la composition :

$$\pi_n CMO(1) \xrightarrow{\text{Hurewicz}} H_n CMO(1) \xrightarrow{r_n^* P_n U} \mathbb{Z}/2$$

où U désigne la classe de Thom modulo 2 de MO(1).

Le problème est de déterminer les entiers positifs n pour lesquels cet homomorphisme θ_n est non trivial.

3.1 Le cas où n n'est pas divisible par 4.

Dans ce cas le problème est résolu dans [4] (à la solution près du problème de l'invariant de Kervaire!). La solution que nous en donnons ci-dessous est fondée sur la congruence 2.2.4.

3.1.1 Cas $n \equiv 1$ (mod. 2). D'après 2.2.4 (avec $m = 1$ et $\Lambda = \mathbb{Z}/2$) θ_n est aussi la composition suivante :

$$\pi_n CMO(1) \xrightarrow{\text{Hurewicz}} H_n CMO(1) \xrightarrow{r^* U^n} \mathbb{Z}/2$$

On en déduit que θ_n est la composition :

$$\pi_n CMO(1) = \pi_n^S \mathbb{R}P^\infty \xrightarrow{\lambda_*} \pi_n^S \xrightarrow{h} \mathbb{Z}/2$$

où λ_* désigne l'application induite par le plongement habituel de $\mathbb{R}P^\infty$ dans SO et où h désigne l'invariant de Hopf.

Puisque λ_* est surjectif sur la 2-composante de π_n^S [7] on obtient comme conséquence du fameux résultat sur l'invariant de Hopf :

<u>Théorème 3.1.1</u> (P. J. Eccles) : Pour n impair l'homomorphisme θ_n est non trivial si et seulement si $n = 1, 3, 7$.

3.1.2 Cas $n \equiv 2$ (mod. 4). Posons $n = 2\ell$, d'après 2.2.4 (avec $m = 2$ et $\Lambda = \mathbb{Z}/2$) θ_n est aussi la composition suivante :

$$\pi_n CMO(1) \xrightarrow{\text{Hurewicz}} H_n CMO(1) \xrightarrow{r_2^* P_2 U^\ell = r_2^*(P_2 U)^\ell} \mathbb{Z}/2$$

A ce point là le cas $n \equiv 2$ (mod. 4) se subdivise en deux sous-cas.

Si $\ell + 1$ n'est pas une puissance de 2 on utilise le lemme suivant dû à N. Boudriga et S. Zarati pour montrer que θ_n est trivial.

<u>Lemme 3.1.2.1</u> : Soient ι la classe fondamentale de $K(\mathbb{Z}/2,2)$ et I l'idéal d'augmentation de l'algèbre de Steenrod modulo 2. Si $\ell+1$ n'est pas une puissance de 2 alors ι^ℓ appartient à $IH^*(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$.

Si $\ell + 1$ est une puissance de 2 on montre à partir de la définition même des formes de Kervaire que θ_n est la composition :

$$\pi_n CMO(1) = \pi_n^S \mathbb{R}P^\infty \xrightarrow{\lambda_*} \pi_n^S \xrightarrow{\kappa} \mathbb{Z}/2$$

où κ désigne l'invariant de Kervaire.

On obtient donc :

<u>Théorème 3.1.2.2</u> (P. J. Eccles) : Pour $n \equiv 2$ (mod.4) l'homomorphisme θ_n est trivial si $n + 2$ n'est pas une puissance de 2; si $n + 2$ est une puissance de 2, θ_n est non trivial si et seulement si l'invariant de Kervaire : $\pi_n^S \longrightarrow \mathbb{Z}/2$ est non trivial (θ_n est donc non trivial en particulier si $n = 2, 6, 14, 30, 62$).

3.2 <u>Le cas où n est divisible par</u> 4.

Il reste donc à traiter le cas où n est divisible par 4. On va voir que dans le cas θ_n est non trivial si et seulement si $n = 4$. Ce résultat est suggéré dans [4] , signalons également qu'en employant une méthode analogue à celle décrite en 3.1.2 N. Boudriga et S. Zarati montrent dans [2] que pour $n \equiv 4$ (mod. 8) θ_n est trivial si $n + 4$ n'est pas une puissance de 2.

On va utiliser cette fois-ci les formules 2.2.6, 2.3 et 2.1.

<u>Théorème 3.2.1</u> : Soit k un entier positif, il existe une immersion générique $\alpha : V \longrightarrow \mathbb{R}^{4k}$, d'une variété fermée de dimension 4k-1 dans l'espace euclidien de dimension 4k, ayant un nombre impair de points 4k-uples si et seulement si $k = 1$.

Pour préparer la démonstration de ce théorème il nous faut introduire les deux propositions suivantes.

Proposition 3.2.2 : Soient M une variété fermée de dimension 2k et $\beta : M \longrightarrow \mathbb{R}^{4k}$ une immersion générique, alors le nombre M_2 de points doubles de β est donnée modulo 2 par la congruence :

$$M_2 - \frac{1}{2}\chi(\nu_\beta) \equiv \langle w_1 w_{2k-1}(\nu_M), [M]\rangle \qquad (\text{mod. } 2)$$

où ν_β désigne le fibré normal de β et $\chi(\nu_\beta)$ le nombre d'Euler de ce fibré (qui est un entier pair), ν_M le fibré normal stable de M (qui est sous-jacent à ν_β), et [M] la classe d'orientation modulo 2 de M.

Cette proposition, qui généralise des résultats de H. Whitney et M. Mahowald dans le cas de plongements [15] [10] [11], est une conséquence de la formule 2.3 et de l'expression du carré de Pontryagin de la classe de Thom modulo 2 d'une fibré de dimension 2k en fonction, de la classe d'Euler, et des classes de Stieffel-Whitney w_1 et w_{2k-1}, de ce fibré [11] .

Proposition 3.2.3 : Soient N une variété fermée de dimension 3k, avec k pair, et $\gamma : N \longrightarrow \mathbb{R}^{4k}$ une immersion générique, alors le nombre caractéristique normal $\langle w_1 w_{2k-1}(\nu_{N_2}), [N_2]\rangle$ de la variété double N_2 de γ est nul.

Démonstration : Soit ξ un fibré vectoriel et m un entier positif, on note $\mathfrak{S}_m \xi$ le fibré vectoriel $E\mathfrak{S}_m \underset{\mathfrak{S}_m}{\times} \xi^m$. Pour démontrer la proposition ci-dessus on écrit à l'aide de 2.1 la classe $r_2^* \emptyset\, w_1 w_{2k-1}(\mathfrak{S}_2\, \nu_\gamma)$ ($\emptyset$ désigne l'isomorphisme de Thom modulo 2) comme une somme de $\cup$-produits dans $\tilde{H}^*(CT\nu_\gamma ; \mathbb{Z}/2)$.

Démonstration du théorème 3.2.1 : Soient N^{3k} la variété k-uple de α, $\gamma : N \longrightarrow \mathbb{R}^{4k}$ une immersion générique régulièrement homotope à l'immersion induite par α, M^{2k} la variété double de γ et $\beta : M \longrightarrow \mathbb{R}^{4k}$ une immersion générique régulièrement homotope à l'immersion induite par γ. La formule 2.2.6 montre que le nombre de points 4k-uples de α est congru modulo 2 au nombre de points quadruples de γ ou encore au nombre de points doubles de β.

La proposition 3.2.2 donne alors :

(F) $$\theta_{4k}(\alpha) = \langle w_1 w_{2k-1}(\nu_M), [M]\rangle;$$

en effet l'entier $\chi(\nu_\beta)$ est nul puisque ν_β est image réciproque de

$\mathfrak{S}_2\ \mathfrak{S}_k\ \mu$, μ désignant le fibré canonique sur BO(1), et que donc la classe d'Euler $e(\nu_\beta)$ est de torsion.

D'autre part le nombre caractéristique $\langle w_1 w_{2k-1}(\nu_M),[M]\rangle$ est nul si k n'est pas une puissance de 2 sans aucune hypothèse supplémentaire sur M. En outre, d'après 3.2.3, comme M est la variété double de γ , ce nombre caractéristique est nul si $k \neq 1$.

Enfin pour k = 1 la formule (F) implique que la composition

$$\pi_3^S = \pi_4^S MSO(1) \longrightarrow \pi_4^S MO(1) \xrightarrow{\theta_4} \mathbb{Z}/2$$

est l'invariant de Hopf [5] et donc que θ_4 est non trivial.

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On Boy immersions.

We specify the positive integers n for which there exists a self-tranverse immersion of a closed (n-1)-manifold in the Euclidean n-space with an odd number of n-fold points. In particular it is shown that the only n divisible by 4 that fits is 4 itself; this completes the results of P. J. Eccles [4].

SMOOTHING THEORY AND FREEDMAN'S WORK ON FOUR MANIFOLDS

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Introduction: The Freedman-Casson handle theorem [F] used an unusual combination of smooth and topological techniques that resulted in the topological classification of almost smooth 1-connected closed four manifolds. (A compact connected manifold M is almost smooth, if M_0 = M - interior point, is smooth.) In this paper we combine smoothing theory with Freedman's results to further study the structure of topological and almost smooth manifolds.

In section 1 we give a preliminary discussion of almost smooth manifolds and show, using Freedman's completion of Scharlemann's transversality theorem, that if M is a compact four manifold, then $M \# k(S^2 \times S^2)$, some k, is s-cobordant to an almost smooth manifold. (Theorem A)

In sections 2-5 we give consequences of our main result that $\pi_i(Top_4/O_4) = \pi_i(Top/O)$, i = 2, 3. Some of these consequences are:

1. A smoothing of $M_0 \times R$, M a four manifold, is isotopic to a product smoothing provided M_0 admits some smoothing. (Theorem B)
2. If V is a cobordism between almost smooth four manifolds, then V has a topological handle decomposition on $\partial_- V$. (Theorem C)
3. An s-cobordism between almost smooth four manifolds becomes a topological product by adding $S^2 \times S^2$'s along the cobordism. (Theorem D)
4. Let M be a closed 5-manifold. Then the tangent microbundle of M splits off a line bundle. (Corollary of Theorem E)

Finally, in section 5 we prove our main result.

Remark: Quinn [Q] has proved that $\pi_i(Top_4/O_4) = 0$, $i = 0, 1, 2$. This implies that every four manifold is almost smoothable. Our proof that $\pi_2(Top_4/O_4) = 0$ is independent of Quinn's.

1. Remarks on Almost Smooth 4-Manifolds.

If M is a topological manifold, a smoothing of M is a pair (U,α) where U is a smooth manifold and $\alpha : M \to U$ is a homeomorphism. Two such (U_1,α_1) and (U_2,α_2) are isotopic rel ∂M if there is an isotopy $G : M \times I \to M$ such that $G_0 = 1_M$, $G_t|\partial M = 1_{\partial M}$ and $\alpha_2 G_1 \alpha_1^{-1} : U_1 \to U_2$ is a diffeomorphism (where $G_t(x) = G(x,t)$).

An almost smoothing of M is a smoothing (U,α) of M minus one interior point from each compact component. If M is compact and connected, denote an almost smoothing by (U,α,p), where p is the interior point. A homotopy class $w(p,q)$ of paths from p to q in M determines a bijective correspondence between isotopy classes of smoothings rel ∂M of $m - p$ and $M - q$. In fact there is an ambient isotopy $G : M \times I \to I$ such that $G_0 = 1_M$, $G_t|\partial M = 1_{\partial M}$, $G_1(p) = q$ and $G_t(p)$, $0 \le t \le 1$, is a path in $w(p,q)$. If (U,α) is a smoothing of $M - p$, $(U,\alpha G_1^{-1})$ is a smoothing of $M - q$. If G' is another such isotopy, then G' is isotopic rel endpoints to G" with $G''_t(p) = G_t(p)$, and it is easy to see that this implies the two smoothings of $M - q$ are isotopic. This gives an action of the fundamental group on the smoothings of $M - p$. Just as for homotopy groups we will often suppress the base point and simply write M_0 for M minus any interior point p.

If (U,α) is a smoothing of M, we will sometimes identify M with U via α, and write M_α for M with this smoothing.

Note that a four manifold is smoothable if and only if it is a handlebody. Freedman has shown there are four manifolds which are not smoothable and hence not handlebodies. This suggests we investigate the following notion:

Call a compact 4-manifold M an <u>almost handlebody</u> if one can find a compact contractible 4-manifold W in the interior of M so that $\overline{M - W}$ is a smooth manifold with boundary. Thus M is a handlebody except for one exotic 4 handle W. Clearly, every almost handlebody is almost smooth. The converse is unknown in general, but we will say more about it later. For the present we note the following:

1.1. Scharlemann's transversality theory [S] as completed by Freedman [F] allows one to deform a map $f : V^{k+4} \to T(\xi^k)$ of a topological manifold V into the Thom space of a k-plane bundle ξ to a map g topologically transverse to the zero section. This process yields an almost handlebody M^4 as preimage of the zero section.

1.2. The arguments of Freedman and Quinn [FQ] in the smooth case show that if the Wall obstruction vanishes, one may do surgery mod # $S^2 \times S^2$'s on a normal degree one map $f : M \to X$, M an almost handlebody, so as to end up with a simple homotopy equivalence of an almost handlebody M' with $X \# k(S^2 \times S^2)$, some k. In fact their method requires surgery only on 0 and 1 spheres.

<u>Theorem A</u>: If M is a compact 4 manifold, there is a k such that $M \# k(S^2 \times S^2)$ is s-cobordant rel ∂ to an almost handlebody.

<u>Proof</u>: By 1.1 with ξ the normal bundle of M and $V^{k+4} = S^{k+4}$, we obtain an almost handlebody N and a degree one normal map $f : N \to M$, normally cobordant to 1_M. By 1.2 we may assume f is a simple homotopy equivalence, when we replace M by $M \# k(S^2 \times S^2)$. Now we wish to do surgery on the normal cobordism to make it an s-cobordism, but in general there is a surgery obstruction. On the other hand, every surgery obstruction can be realized mod # $S^2 \times S^2$ by a normal cobordism of N to N', at least if N is smooth [CS]. By first removing the interior of the exotic 4-handle in N, and realizing the surgery obstruction on the resultant smooth manifold, we end up with a normal cobordism mod $S^2 \times S^2$'s of N to another almost handlebody N', such

that the surgery obstruction for the normal cobordism from N' to $M \# k(S^2 \times S^2)$, some k, vanishes, enabling us to construct an s-cobordism.

Remark: Alternately, starting with $M \times [0,\infty)$ and making the projection onto $[0,\infty)$ transverse to say $M \times 1$, we could construct an almost handlebody N in $M \times (0,1)$, and modify it so that mod $S^2 \times S^2$'s the cobordism from M to N is an s-cobordism. Compare [CSL], where the argument is done in the smooth case.

2. Bundle Reductions and the Product Structure Theorem.

Let $j : BTop_4 \to BTop$ and $j : BO_4 \to BO$ be the maps induced by the inclusion of Top_4 in Top. Note that j may be considered a map of fibrations:

$$\begin{array}{ccc} BO_4 & \xrightarrow{\hat{j}} & BO \\ \downarrow & & \downarrow \\ BTop_4 & \xrightarrow[j]{} & BTop \end{array} ,$$

with fibres Top_4/O_4 and Top/O, respectively.

Notation: If (X,A) is a relative CW complex, we let $(X,A)^i = A \cup$ cells of dimension $\leq i$.

Proposition 2.1: Let (X,A) be a relative CW complex of dimension at most four. Let $\xi : X \to BTop_4$ and suppose $\xi_0 = \xi|(X,A)^3$ lifts to $\hat{\xi}_0 : (X,A)^3 \to BO_4$. Then the correspondence $\hat{\xi}$ to $\hat{j}\hat{\xi}$ induces a surjection of the homotopy classes of lifts of ξ extending $\xi_0|A$ onto the homotopy classes of lifts of $j\xi$ extending $\hat{j}\hat{\xi}_0|A$.

Addenda: By replacing A by $(X,A)^2$ and using the fact that $\pi_i(Top/O) = 0$ for $i < 3$ we have by 2.1:

2.1a: ξ lifts to BO_4 extending $\xi_0|(X,A)^2$ if and only if $j\xi$ lifts to BO extending $j\xi_0|A$.

By replacing A by $(X,A)^3$ and using the fact that $\pi_4(\mathrm{Top}/O) = 0$ we have:

2.1b: ξ lifts to BO_4 extending $\hat{\xi}_0$ if and only if $j\xi$ lifts to BO extending $\hat{j}\hat{\xi}_0$. Any two such lifts of $j\xi$ are homotopic $\mathrm{rel}(X,A)^3$; and in particular if $\hat{\xi}$ is such a lift of ξ and $\hat{\eta}$ such a lift of $j\xi$, $\hat{j}\hat{\xi}$ is homotopic to $\hat{\eta}$ $\mathrm{rel}(X,A)^3$.

<u>Proof of 2.1</u> (using the main theorem): Since $\pi_i(\mathrm{Top}/O) = 0$ for $i < 3$ we may assume up to homotopy that any lift $\hat{\eta}$ of $j\xi$ to BO extending $j\xi_0|A$ actually extends $\hat{j}\hat{\xi}_0|(X,A)^2$. Since $\pi_3(\mathrm{Top}_4/O_4) \to \pi_3(\mathrm{Top}/O)$ is surjective, we may change $\hat{\xi}_0$ over the 3 cells of (X,A) so that $\hat{j}\hat{\xi}_0$ is homotopic $\mathrm{rel}(X,A)^2$ to $\hat{\eta}|(X,A)^3$, and hence we can assume $\hat{\eta}$ agrees with $\hat{j}\hat{\xi}_0$ over $(X,A)^3$. Since $\pi_3(\mathrm{Top}_4/O_4) \to \pi_3(\mathrm{Top}/O)$ is injective $\hat{\xi}_0$ extends to a lift $\hat{\xi}$ of ξ. Since $\pi_4(\mathrm{Top}/O) = 0$, $\hat{j}\hat{\xi}$ is homotopic to $\hat{\eta}$ $\mathrm{rel}(X,A)^3$.

If M is a 4-manifold, the Kirby-Siebenmann obstruction $\kappa \in H^4(B\mathrm{Top};Z_2)$ yields a class $\kappa(M) \in H^4(M,\partial M;Z_2)$ which can be viewed as the obstruction to smoothing $M \times R$ rel $\partial M \times R$.

The following is an immediate consequence of 2.1.

<u>Proposition 2.2</u>: Let M be a 1-connected almost smoothed closed 4-manifold with $\kappa(M) = 0$. Then the corresponding lift of τ_{M_0} to BO_4 extends to a lift of τM.

<u>Remark</u>: The proposition says there is no bundle theoretic obstruction to extending the smoothing to M. Nevertheless, a recent result of Donaldson on Spin manifolds, shows that the smoothing does not always extend.

Smoothing theory and 2.1 will allow us to prove the following weak product structure theorem:

Theorem B: Let M be an almost smoothed 4-manifold, and suppose we are given a smoothing of $M_0 \times R$ which is the product smoothing on $\partial M \times R$. Then there exists (a possibly different) smoothing of M_0, unchanged on the boundary, so that the product smoothing of $M_0 \times R$ is isotopic rel $\partial M \times R$ to the given smoothing.

Remark: The reason this theorem is called weak is that the new smoothing of M_0 is unique only up to concordance – not isotopy or even sliced concordance.

Addendum B1: Let $C \subset M$ be a proper closed subset. Under the hypothesis of Theorem B and supposing $C \subset M_0$ (which can always be arranged – see section 1) and that the smoothing of $M_0 \times R$ restricts to the product smoothing on $U \times R$, U a neighborhood of C in M_0, then we can conclude that the new smoothing of M_0 agrees with the original smoothing on a neighborhood of C.

If M is smoothed and we are given a smoothing of $M \times R$ one cannot guarantee that this smoothing is isotopic to a product smoothing on all of $M \times R$, even though all bundle obstructions vanish. However we can show:

Addendum B2: There is an integer $k \geq 0$ with the following property. With the hypothesis of Theorem B, and assuming $M = X \# k(S^2 \times S^2)$ for some smooth compact connected 4-manifold X; if we are given a smoothing of all of $M \times R$, which is the product smoothing on $\partial M \times R$, then there is a smoothing of all of M such that the product smoothing on $M \times R$ is isotopic rel $\partial M \times R$ to the given smoothing.

Remark B3: The relative version of Addendum B2 holds provided $C \subset X_0 \subset M$.

Addendum B4: Let N^4 be the twisted S^3-bundle over S'. There exists a smooth 4-manifold, M^4, and a homotopy equivalence $f : M^4 \to N^4$ which is not homotopic to a diffeomorphism iff $k = 0$.

Proof of Theorem B: The classifying map $\tau : M \to BTop_4$ of the tangent microbundle of M satisfies: a) $\tau_0 = \tau|M_0$ lifts to BO_4 – using the almost smoothing of M, b) $j\tau_0$ lifts to BO — using the smoothing of $M_0 \times R$, so that if $\hat{\tau}_0$ is the lift of τ_0 and $\hat{\eta}_0$ is the lift of $j\tau_0$, then $\hat{j}\hat{\tau}_0|\partial M = \hat{\eta}_0|\partial M$. By 2.1 there is a lift $\hat{\tau}_0'$ of τ_0 so that $\hat{\tau}_0'|\partial M = \tau_0|\partial M$ and $\hat{j}\hat{\tau}_0'$ is homotopic to $\hat{\eta}_0$ rel ∂M. It follows from smoothing theory [L], that there is a smoothing of M_0 satisfying the conclusion.

Proof of B1: Take a smooth compact submanifold $A^4 \subset U$, with $C \subset \text{Int } A$. Then the same argument as above with $\partial M \cup A$ replacing ∂M, proves B1.

Proof of B2: The classifying map for the tangent bundle of M factors up to homotopy as follows:
$M = X \# k(S^2 \times S^2) \xrightarrow{q} X \vee k(S^2 \times S^2) \xrightarrow{\tau \vee \tau'} BTop_4$, where q is the quotient map and τ (resp. τ') classifies the tangent bundle of X (resp. $k(S^2 \times S^2)$). Since $k(S^2 \times S^2)$ has a trivial stable tangent bundle, $j\tau'$ is homotopically trivial by a standard based homotopy. Since $\pi_2(Top/O) = 0$, any lift of $j\tau_M$ defines a lift of $j\tau$. The lift is unique up to homotopy since $\pi_4(Top/O) = 0$. Thus the smoothing of $M \times R$ defines a lift $\hat{\eta}$ of $j\tau$ so that if $\hat{\tau}$ is the lift of τ given by the smoothing of X, $\hat{j}\hat{\tau}|\partial X = \eta|\partial X$.

Since Top/O is a $K(Z_2,3)$, the difference between $\hat{\eta}$ and $\hat{j}\hat{\tau}$ defines a class $\alpha \subset H^3(X,\partial X;Z_2)$. We assume $\alpha \neq 0$, since otherwise the result is trivial. The dual of α is represented by a smoothly embedded S^1 in X. The normal tube of S^1 is either $E_+ = D^3 \times S^1$ or

E_- = the unoriented D^3 bundle over S^1. Let E denote whichever one we have. Then α is the image of the generator γ of $H^3(E,\partial E;Z_2) = H^3(D^3,\partial D;Z_2)$. In particular, we can assume $\hat{\eta} = \hat{j}\hat{\tau}$ on X - Int E. Then $\hat{\eta}|E$ is in the unique non-standard homotopy class δ of lifts of $j\tau_E$ rel ∂E. To realize the lift $\hat{\eta}$ of $j\tau$, it would be sufficient to change the smoothing on E rel ∂E from the standard smoothing represented by $\hat{\tau}|E$ to one defining a lift $\hat{\sigma}$, where $\hat{j}\hat{\sigma}$ is in δ. By 2.1, a lift $\hat{\sigma}$ always exists such that $\hat{j}\hat{\sigma}$ is in δ. However, in general, $\hat{\sigma}$ only defines a smoothing of $E \# k(S^2 \times S^2)$ for some k [LS], but nevertheless with the induced product smoothing of $(E \# k(S^2 \times S^2)) \times R$ representing δ. Thus if for E_+ we chose σ_+ with $\hat{j}\hat{\sigma}_+ \in \delta_+$ and similarly for E_-, then letting $k = \max(k_+,k_-)$, we can always get a smoothing of $X \# k(S^2 \times S^2)$ satisfying B2.

Proof of B4: If k = 0 we can use the homeomorphism h promised by B2 for f.

Given f it is an easy surgery theoretic calculation to show that M^4 is topologically h-cobordant to N^4 but M^4 is not smoothly h-cobordant to N^4.

By a theorem of Quinn [Q], any sufficiently large cover of this h-cobordism is a product. Here we can find a smooth manifold $\tilde{M}^4$ and a homeomorphism $h : \tilde{M}^4 \to N^4$ by taking a large odd cover. This shows $k_- = 0$. The double cover of this picture shows $k_+ = 0$.

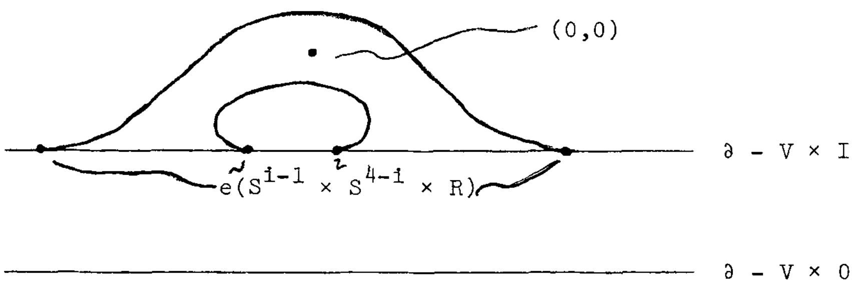
3. Handlebody Theory.

Remark: These results are largely superseded by Quinn's results [Q].

Proposition 3.1: Let V be a five dimensional compact cobordism between the four manifolds ∂_-V and ∂_+V which is a product between their boundaries. If V is a topological handlebody on ∂_-V, there is a 4-plane bundle η over V_0 such that $\eta \oplus 1 = \tau(V_0)$ and $\eta | \partial_{\pm} V = \tau(\partial_{\pm} V)$.

Proof: It suffices to prove 3.1 when $V = \partial_- V \times I \cup_f D^i \times D^{5-i}$. In fact, by induction up the handles this will construct a bundle η over $V - F$, F a finite collection of interior points, which we can assume contains at least one point from each component of V. By a standard argument V_0 may be engulfed rel ∂V into $V - F$.

Now we can always define a smooth structure on a neighborhood of $f(S^{i-1} \times D^{5-i}) \subset \partial_- V$ so the attachment is smooth with rounded corners; and in particular we have an embedding $e : S^{i-1} \times S^{4-i} \times R \to \partial_- V \times 1$.



Define η over $W = \partial_- V \times I \cup_e R^i \times S^{4-i}$ by gluing $T(R^i \times S^{4-i})$ to $T(\partial_- V) \times I$ by $T(e)$. Since $V_0 = V - (0,0)$, $(0,0) \in D^i \times D^{5-i}$, has W as a deformation retract, the result follows.

Corollary 3.2: Under the hypothesis of 3.1 and assuming $\partial_+ V \neq \emptyset$, there is a 4-plane bundle ξ over V such that $\xi \oplus 1 = \tau(V)$, $\xi | \partial_- V = \tau(\partial_- V)$ and $\xi | (\partial_+ V)_0 = \tau(\partial_+ V)_0$.

Proof: Engulf V in V_0 by pushing in on an interval from a base point in $\partial_+ V$ to the base point in V. Let ξ be the pull back of η by the engulfing.

Proposition 3.3: Let X be a 4-complex and ξ_1, ξ_2 topological 4-plane bundles over X. If a) ξ_1 and ξ_2 are stably equivalent, and

and b) ξ_1 and ξ_2 have lifts to BO_4 over the 3-skeleton X^3; then ξ_1 and ξ_2 are equivalent over X^3 and ξ_1 lifts to BO_4 over X if and only if ξ_2 does.

Proof: $\pi_i BO_4 \to \pi_i BO$ is an isomorphism for $i \leq 3$ and $\pi_i BO \to \pi_i BTop$ is an isomorphism for $i \leq 3$. Thus the lifts $\hat{\xi}_1$ and $\hat{\xi}_2$ of $\xi_1|X^3$ and $\xi_2|X^3$ to BO_4 are homotopic over X^2. Since $\pi_3 BO_4 = 0$, $\hat{\xi}_1$ and $\hat{\xi}_2$ are homotopic. Hence ξ_1 and ξ_2 are equivalent over X^3. The last statement of the proposition follows from 2.1 and hypothesis a).

Remark: In order for ξ_1 and ξ_2 to be equivalent it is necessary and sufficient that they have the same Euler class.

Proposition 3.4: Let V be a compact h-cobordism between 4-manifolds which is a product along the boundary, and suppose V is a handlebody on $\partial_- V$. Then

a) $\partial_- V$ is almost smoothable if and only if $\partial_+ V$ is almost smoothable.

b) $\tau(\partial_- V)$ reduces to a vector bundle if and only if $\tau(\partial_+ V)$ reduces to a vector bundle.

Proof: By 3.2, there is a 4-plane bundle ξ over V such that $\xi|\partial_\pm V = \tau(\partial_\pm V)$. Since V is an h-cobordism $\xi = r^*\tau(\partial_- V)$, where $r : V \to \partial_- V$ is the retraction. In particular, $\tau(\partial_+ V) = r_+^*\tau(\partial_- V)$, where $r_+ = r|\partial_+ V$. Since $\partial_+ V$ has the homotopy type of a 4-complex X with $(\partial_+ V)_0$ homotopy equivalent to X^3 [W], the result follows from 3.3 and the fact that if V is a handlebody on $\partial_- V$ then it is a handlebody on $\partial_+ V$.

Proposition 3.5: Suppose there is a compact 4-manifold which is not almost smoothable. Then

a) there is a compact s-cobordism V^5 which does not have a handle decomposition on $\partial_- V$, and

b) there is a compact manifold V^5 with boundary such that V is not a handlebody on ∂V.

<u>Proof</u>:

a) By Theorem A, if M is the compact 4-manifold of the hypothesis, then $M \# k(S^2 \times S^2)$ is s-cobordant to an almost smoothable compact manifold. But if M is not almost smoothable, neither is $M \# k(S^2 \times S^2)$. Hence by 3.4, the s-cobordism cannot have a handle decomposition.

b) Let V be the s-cobordism in a). Suppose V is a handlebody on ∂V. By 3.2, there is a 4-plane bundle ξ on V which restricts to $\tau(\partial V)_0$ on $(\partial V)_0$. Since $\xi = r^*\tau(\partial_- V)$, $\xi|(\partial V)_0 = \tau(\partial V)_0$ has a vector bundle reduction. But this implies $\partial_+ V$ is almost smoothable, giving a contradiction.

<u>Theorem</u> <u>C</u>: Let V be a compact cobordism between almost smoothable 4-manifolds which is a product along the boundary. Then V has a topological handle decomposition on $\partial_- V$.

<u>Proof</u>:

1. We may assume $\partial_- V$ and $\partial_+ V$ are non-empty:

Just remove one or two open discs from V as necessary to make $\partial_- V$ and $\partial_+ V$ non-empty. Obviously if the new V has a handle decomposition on $\partial_- V$ so does the original cobordism.

2. We may assume $\tau(V)$ reduces to a vector bundle rel L, $L = \partial(\partial_- V) \times I$ the (possibly empty) "lateral" surface of V:

Suppose the obstruction $\kappa(V) \in H^4(V,L;Z_2)$ to extending the reduction of $\tau(V)|L$ (induced by the smoothing of $\partial(\partial_- V)$) is non-zero. Let $\alpha \in H_1(V, \partial_+ V \cup \partial_- V;\, Z_2)$ be the dual class. Then it is easy to see that α is represented by a finite collection of locally flat embedded arcs going from $\partial_- V$ to $\partial_+ V$. Now each arc is the core of a 1-handle $I \times D^4$ going from $\partial_- V$ to $\partial_+ V$. Let $P^4 \subset \operatorname{Int} D^4$ be a compact contractible 4-manifold with ∂P the Poincaré homology sphere [F]. Remove $I \times \operatorname{Int} P$ from each of the above 1-handles. This gives a new compact cobordism which is a product along the boundary, and it is again obvious that if the new V has a handle decomposition on $\partial_- V$ so did the original cobordism. Since the obstruction to reducing $\tau(P)$ to a vector bundle rel ∂P is non-zero, it follows that the tangent bundle of the new V reduces to a vector bundle rel the new L.

3. If $\tau(V)$ reduces to a vector bundle rel L, V has a handle decomposition on $\partial_- V$:

The reduction of $\tau(V)$ defines stable reductions of $\tau(\partial_\pm V)$ and by 2.1, reductions of the $\tau(\partial_\pm V)$ themselves. By Lashof and Shaneson [LS], there is a compatible smoothing of $\partial_\pm V \# k(S^2 \times S^2)$, for some k. We may think of $\partial_\pm V \# k(S^2 \times S^2)$ as embedded in outside collar neighborhoods of the $\partial_\pm V$ by first adding trivial 2-handles to $\partial_\pm V \times I$ and then cancelling 3-handles. Thus we have a smoothable manifold W, $\partial_- W = \partial_- V \# k(S^2 \times S^2)$ and $\partial_+ W = \partial_+ V \# k(S^2 \times S^2)$. V is constructed by first adding trivial 2-handles to $\partial_- V \times I$ to reach $\partial_- W$, and then attaching the (smooth) handles of W to reach $\partial_+ W$, and finally attaching the dual three handles to $\partial_+ W$ to get to $\partial_+ V$.

Addendum: We may assume the handles of a given dimension are attached disjointly in order of increasing dimension.

Proof: In the handle decomposition given in the proof above, one can certainly assume the 0, 1, 2 handles are attached before any of the 3, 4, 5 handles, by taking such a handle decomposition for the smooth manifold W. Hence using only general position arguments one can arrange the 0, 1, 2 handles in order on $\partial_- V$ and the dual 0, 1, 2, handles in order on $\partial_+ V$.

Notation: Let V be a compact connected cobordism between four manifolds, and let $H \subset V$ be a 1-handle $I \times D^4$ going from $\partial_- V$ to $\partial_+ V$.

We let $V \#_H I \times k(S^2 \times S^2)$ be the cobordism from $\partial_- V \# k(S^2 \times S^2)$ to $\partial_+ V \# k(S^2 \times S^2)$ obtained by replacing H by $I \times (D^4 \# k(S^2 \times S^2))$.

Theorem D: Given a compact s-cobordism V between almost smoothable four manifolds which is a product along the boundary, then there is a 1-handle $H \subset V$ from $\partial_- V$ to $\partial_+ V$ and a k such that $V \#_H I \times k(S^2 \times S^2)$ is a topological product extending the product structure along the boundary.

Proof: As in part 2 of the proof of Theorem C, we may assume that $\tau(V)$ reduces to a vector bundle, after deleting copies of $I \times \operatorname{Int} P$. By 2.1 this defines compatible reductions of $\tau(\partial_\pm V)$. By [LS] there is a k such that $\partial_\pm V \# k(S^2 \times S^2)$ have compatible smoothings. Further there is an immersion and hence an embedding of $(1 \times D^4, -1 \times D^4, 1 \times D^4)$ in $(V, \partial_- V, \partial_+ V)$ whose differential is deformable to a linear map with respect to the above reductions. But this implies the smoothings of $\partial_\pm V \# k(S^2 \times S^2)$ extend to a smoothing of $V \#_H I \times k(S^2 \times S^2)$, where H is the above embedded handle. This gives a smooth s-cobordism; but then Quinn's stable Whitney trick [Q] shows that after possibly adding more $S^2 \times S^2$'s we get a smooth product. Thus for the original V, $V \#_H I \times k(S^2 \times S^2)$ is a topological product.

We can now discuss the extent to which almost smooth manifolds are almost handlebodies: Freedman constructs almost handlebodies [F] and since he proves a uniqueness theorem for 1-connected almost smooth closed manifolds we have:

Proposition 3.6: Any 1-connected almost smooth closed 4-manifold is an almost handlebody.

In general, all we can say is,

Proposition 3.7: If M is a compact almost smooth 4-manifold, then there is a k such that $M \# k(S^2 \times S^2)$ is an almost handlebody.

Proof: Immediate from Theorems A and D.

We can also add a little to our knowledge of homotopy RP^4's. By Theorem D, all the Cappell-Shaneson RP^4's [CS] and the Finteshel-Stern exotic RP^4 [FS] are homeomorphic to RP^4 mod connected sums with $S^2 \times S^2$'s. Secondly, we note that since $H^3(RP^4;Z_2) \neq 0$ there is an exotic almost smoothing of RP^4. The bundle obstruction to extending this smoothing over the last point is zero, as remarked above (see B2 and the proof thereof). Thus we get a non-trivial smoothing of $RP^4 \# k(S^2 \times S^2)$ by [LS]. Also note that we can assume the smoothing is standard on a neighborhood of RP^2.

4. Disc Bundles.

In [St], R. Stern did a detailed study of the problem of finding a disc bundle inside a given microbundle. He was able to deal with this question except for five dimensional bundles. We offer,

Theorem E: Let X be a 5-dimensional complex. Any 5-dimensional microbundle over X contains a topological disc bundle.

Remark: We do not claim the disc bundle is unique. That involves unknown homotopy groups of Top_4/O_4.

The following is due to Stern for $k > 2$.

Corollary 4.1: Let M^{2k+1} be a closed manifold. Then the tangent microbundle of M splits off a line bundle.

Proof of Corollary: Let $Top(I)_n$, resp. $Top(S)_n$, denote the group of homeomorphisms of I^n, resp. S^n. An n-dimensional microbundle over X contains a disc bundle if and only if $\xi : X \to BTop_n$ lifts to $BTop(I)_n$. Since the restriction map $Top(I)_n \to Top(S)_{n-1}$ is a homotopy equivalence, one has the fibration:

$$* \qquad Top_{n-1} \to Top(I)_n \to S^{n-1},$$

and hence the fibration: $S^{n-1} \to BTop_{n-1} \to BTop(I)_n$. Thus ξ will split off a line bundle if its Euler class is zero. But for $n = 2k + 1$, the Euler class of $\tau(M)$ is zero.

Proof of Theorem E: From (*) we see that $Top_4/O_4 \to Top(I)_5/O_5$ is a homotopy equivalence and that we have the fibration:

$$Top_4/O_4 \to Top_5/O_5 \to Top_5/Top(I)_5.$$

In particular, since $\pi_4(Top_5/O_5) = 0$ and $\pi_3(Top_4/O_4) \to \pi_3(Top_5/O_5)$ is an isomorphism by our main theorem, we see that $\pi_4(Top_5/Top(I)_5) = 0$ and that $\pi_3(Top_5/O_5) \to \pi_3(Top_5/Top(I)_5)$ is trivial. From the map of fibrations:

$$\begin{array}{ccccc} & & Top_5/O_5 & \longrightarrow & Top_5/Top(I)_5 \\ & & \downarrow & & \downarrow \\ & & BO_5 & \longrightarrow & BTop(I)_5 \\ & & \downarrow & & \downarrow \\ X & \xrightarrow{\xi} & BTop_5 & = & BTop_5 \end{array},$$

we see that $\xi|X^3$ lifts to BO_5 since $\pi_i(Top_5/O_5) = 0$ for $i < 3$, and that the obstruction to getting a lift to BO_5 over X^4 in $H^4(X;\pi_3(Top_5/O_5))$ maps to zero in $H^4(X;\pi_3(Top_5/Top(I)_5)$. Thus $\xi|X^4$ lifts to $BTop(I)_5$. The lift extends to X since $\pi_4(Top_5/Top(I)_5) = 0$.

5. $\pi_i(\mathrm{Top}_4/0_4)$.

Case $i = 2$:

An element $\alpha \in \pi_2(\mathrm{Top}_4/0_4)$ defines an exotic smoothing of $R \times S^2 \times S^1$ as follows: Since the tangent bundle of the standard smoothing is trivial, the classifying map $\tau : R \times S^2 \times S^1 \to B\mathrm{Top}_4$ can be taken to be the constant map to the base point. Define a lift τ_α of τ to $B0_4$ by $\tau_\alpha = ifp$, where $p : R \times S^2 \times S^1 \to S^2$ is projection, $f : S^2 \to \mathrm{Top}_4/0_4$ represents α and $i : \mathrm{Top}_4/0_4 \to B0_4$ is the inclusion of the fibre when we consider $B0_4$ as a fibre space over $B\mathrm{Top}_4$. This defines a homotopy class of lifts τ_α and hence a sliced concordance class of smoothings $(R \times S^2 \times S^1)_\alpha$ [LS].

Proposition 5.1: There is a compact 4-manifold V with $\partial V = S^2 \times S^1$ which is h-cobordant rel boundary to $D^3 \times S^1 \# k(S^2 \times S^2)$ and such that the smoothing α of $R \times S^2 \times S^1$ extends to a smoothing of $W = V \cup$ open collar (i.e., the open bicollar of ∂V in W identifies with $R \times S^2 \times S^1$).

Proof: Since $\pi_2(\mathrm{Top}_5/0_5) = 0$, the smoothing α is stably equivalent to the standard smoothing. Thus there is a smoothing β of $R \times S^2 \times S^1 \times I$, $I = [-1,1]$, which is the standard smoothing near $R \times S^2 \times S^1 \times \pm 1$, the smoothing $\alpha \times 1$ on a product neighborhood of $R \times S^2 \times S^1 \times 0$ and with β isotopic to the standard smoothing rel a product neighborhood of the boundary. Identifying $R \times S^2$ with $R^3 - 0$, we can get a smoothing γ of $R^3 \times S^1 \times I$ which is standard near $R^3 \times S^1 \times \pm 1$ and equal to β outside $D^3_\xi \times S^1 \times I$, D^3_ξ a small disc about 0 in R^3.

We can deform the projection $p : (R^3 \times S^1 \times I)_\gamma \to I$ to a smooth map p' transverse to 0 in I, rel the complement of $D^3_\xi \times S^1 \times I$ and a collar neighborhood of the boundary. Let $W = p'^{-1}(0)$. Then W is smooth and the end of W is topologically the same as the end of $R^3 \times S^1$ and has the smoothing α. The composition $qj : W \to R^3 \times S^1$, j the inclusion of W in $R^3 \times S^1 \times I$ and q the projection of $R^3 \times S^1 \times I$ onto $R^3 \times S^1$, is a proper degree one normal map. Let V be the compact topological manifold with $\partial V = S^2 \times S^1$ such that $W = V \cup$ open collar. Then qj restricts to a degree one normal map $h : (V,\partial V) \to (D^3 \times S^1, S^2 \times S^1)$, the identity on the boundary. Following [FQ] or [CS] we can do smooth framed surgery on Int V mod $S^2 \times S^2$'s so that we get a homotopy equivalence of the new V with $D^3 \times S^1 \# k(S^2 \times S^2)$ rel boundary. In fact, as in the proof of Theorem A, we can assume the homotopy equivalence is actually an h-cobordism.

<u>Lemma</u> <u>5</u>.<u>2</u>: Let α and β be smoothings of W and τ_α and τ_β the corresponding lifts of $\tau : W \to BTop_4$ to BO_4. If $\tau_\alpha \sim \tau_\beta$ on the base point $(p,q) \in S^2 \times S^1$, then $\tau_\alpha \sim \tau_\beta$ on $S^2 \times q$. ($\sim$ means homotopic through lifts.)

<u>Proof</u>: Since V is h-cobordant to $D^3 \times S^1 \# k(S^2 \times S^2)$, the Stiefel-Whitney classes $w_1(V)$ and $w_2(V)$ are zero. It follows that TW_α and hence τW is trivial. Thus we may assume τ sends W to the base point. Again since V is homotopy equivalent rel boundary to $D^3 \times S^1 \# k(S^2 \times S^2)$, the inclusion $i : S^2 \times q \to \partial V \subset W$ is homotopic to the constant map to (p,q). Since $\tau_\alpha \sim \tau_\beta$ on (p,q), $\tau_\alpha i \sim \tau_\beta i$ over τi; i.e., $\tau_\alpha \sim \tau_\beta$ on $S^2 \times q$.

<u>Proposition</u> <u>5</u>.<u>3</u>: Let V be homotopy equivalent rel boundary to $D^3 \times S^1 \# k(S^2 \times S^2)$ and let $W = V \cup$ open collar. Let $M = V \cup D^3 \times S^1$, identified along their boundaries, and let α be a smoothing of W which is standard on a neighborhood of $(p,q) \in S^2 \times S^1$ in the bicollar. If M is almost smoothable, τ_α is homotopic to the standard lift on a neighborhood of $S^2 \times q$ in the bicollar.

Proof: We identify W with an open neighborhood of V in M. Let β be an almost smoothing of M. Since τW is trivial, if $\tau_\alpha, \tau_\beta : W \to Top_4/O_4 \subset BO_4$ do not land in the same component we can always change τ_α by composition with an element g of Top_4 to achieve this; and then $\tau'_\alpha \sim \tau_\beta$ on the base point, $\tau'_\alpha = g\tau_\alpha$. By 6.2, $\tau'_\alpha \sim \tau_\beta$ on S^2. Since we can take $M_0 = M - (0,q')$, $(0,q') \in D^3 \times S^1$, $q' \neq q$, τ_β extends over $D^3 \times q$. Take the trivialization of τW to be that given by $T(M_0)_\beta$ so that $\tau : M_0 \to$ base point and $\tau_\beta : M_0 \to (1) \subset Top_4/O_4$. Since $\tau'_\alpha \sim \tau_\beta$ on $S^2 \times q$, $\tau'_\alpha | S^2 \times q$ is homotopic to the constant map onto (1). By composing with g^{-1} we see that $\tau_\alpha | S^2 \times q$ is homotopic to a constant map, and since α was standard on a neighborhood of the base point, τ_α must be homotopic to the standard lift on a neighborhood of $S^2 \times q$.

Corollary 5.4: Let $\alpha \in \pi_2(Top_4/O_4)$ and let $M = V \cup D^3 \times S^1$, where V is given by 5.1. If M is almost smoothable $\alpha = 0$.

Proposition 5.5: Let α and M be as in 5.4. If the universal cover of M is smoothable, $\alpha = 0$.

Proof: The obstruction to smoothing M_0 with a given smoothing β in a neighborhood of the base point is a class $\mathcal{O}_\beta \in H^3(M;\pi_2(Top_4/O_4))$. Indeed if β is isotopic to α on a neighborhood of the base point it extends to W, and the obstruction to extending τ_β, and hence β, to M_0 is a class $\mathcal{O}_\beta$ as above. If τ_β corresponds to a different component of Top_4/O_4 than τ_α on the base point, $\tau'_\alpha = g\tau_\alpha$ will be in the same component for some $g \in Top_4$, and hence τ_β will extend over W in any case so that we get an obstruction to smoothing M_0 as above.

If $f : \tilde{M} \to M$ is the universal cover, $f^* : H^3(M;\pi_2(Top_4/O_4)) \to H^3(\tilde{M};\pi_2(Top_4/O_4))$ is an isomorphism since M has the homotopy type of $S^3 \times S^1 \# k(S^2 \times S^2)$. If $\mathcal{O}_\beta$ is non zero, $f^*\mathcal{O}_\beta \neq 0$; but $f^*\mathcal{O}_\beta$ is the obstruction to smoothing M with the pull back smoothing $\tilde{\beta}$ on a neighborhood of the base point in M. Thus if M is smoothable $f^*\mathcal{O}_\beta = 0$ for some β, so $\mathcal{O}_\beta = 0$ and M_0 is smoothable. Hence $\alpha = 0$ by 5.4.

<u>Proposition 5.6</u>: Let α and M be as in 5.5. Then M is homeomorphic to $S^3 \times R \# \infty(S^2 \times S^2)$ and hence smoothable.

<u>Proof</u>: $M = V \cup D^3 \times S^1$ and $M - 0 \times S^1$ is homeomorphic to W. So $\tilde{M} = \tilde{V} \cup D^3 \times R$ and $\tilde{M} - 0 \times R$ is homeomorphic to $\tilde{W}$. Since W is properly h-cobordant to $R^3 \times S^1 \# k(S^2 \times S^2)$, $\tilde{W}$ is properly h-cobordant to $R^4 \# \infty(S^2 \times S^2)$. Since W is smoothable by the pull back of α, Freedman's theorem says $\tilde{W} = R^4 \# \infty(S^2 \times S^2)$. In particular, we can perform topological surgery on W to obtain R^4 and this changes M to a manifold M', the proper homotopy type of $S^3 \times R$. But Siebenmann [F] has shown that such a manifold is homeomorphic to $S^3 \times R$. But then $\tilde{M} = S^3 \times R \# \infty(S^2 \times S^2)$, connected along an embedding f of R^4 in $S^3 \times R$. The Lemma below shows that there is a homeomorphism h of $S^3 \times R$ such that hf is isotopic to the standard embedding of R^4 and hence M is homeomorphic to the standard (smooth) connected sum.

<u>Lemma 5.7</u>: If $f : R^4 \to S^3 \times R$ is any embedding, then there is a smooth embedding $g : R^4 \to S^3 \times R$ and a homeomorphism h of $S^3 \times R$ such that $hf = g$ on D^4.

<u>Proof</u>: We can smoothly identify $S^3 \times R$ with $R^4 - q$, $q \neq 0$, so that f(0) is identified with 0 in R^4. By Kister's theorem [K] there is an ambient homeomorphism k of R^4 with $k(0) = 0$ and $k|D^4 = f|D^4$. Choose $\xi > 0$ such that $q \notin D^4_\xi \cup f(D^4_\xi)$. Then we may assume $k|D^4_\xi = f|D^4_\xi$ and $k(q) = q$. Thus k restricts to a homeomorphism of $S^3 \times R$ so that

$ki = f|D^4_\xi$, where $i : D^4_\xi \to S^3 \times R$ is the smooth embedding so that composed with the inclusion of $S^3 \times R$ in R^4 it is the standard (smooth) embedding of D^4_ξ in R^4. Then $k^{-1}f|D^4$ is isotopic to a smooth embedding g and so using the isotopy extension theorem, we can find a homeomorphism h such that $hf = g$ on D^4.

Main Theorem I: $\pi_2(Top_4/O_4) = 0$.

Proof: This follows immediately from 5.5 and 5.6.

Case $i = 3$:

Let $\alpha \in \pi_3(Top_4/O_4)$, then α defines a smoothing, unique up to sliced concordance, of $S^3 \times R$ which is standard near the base point. We denote this by $(S^3 \times R)_\alpha$. In [LS], it is shown that if α is stably trivial this is the end of a smooth manifold W the proper homotopy type of $R^4 \# k(S^2 \times S^2) = (k(S^2 \times S^2))_0$. By Freedman's classification theorem the underlying topological manifold W is homeomorphic to $R^4 \# k(S^2 \times S^2)$. Since the tangent bundle of the latter is trivial, we can assume $\tau : W \to BTop_4$ is the constant map to the base point, and the standard smoothing β gives a constant lift τ_β to the base point of BO_4. Then W_α defines a lift $\tau_\alpha : W \to Top_4/O_4 \subset BO_4$ of τ. Since the inclusion of $S^3 \times 0 \subset S^3 \times R \subset W$ is homotopically trivial, $\tau_\alpha|S^3$ is homotopic to $\tau_\beta|S^3$.

We wish to show that $\alpha = 0$; but we cannot conclude this directly from the above. That is, if $h : W \to R^4 \# k(S^2 \times S^2)$ is the homeomorphism and $i : S^3 \times R \to W$ is the inclusion, then we do not know that hi is the standard inclusion of the end in $R^4 \# k(S^2 \times S^2)$ and hence we do not know that the pull back by hi of β is the standard smoothing of $S^3 \times R$. On the other hand, τ is homotopic to $\tau'|W$, where τ' is a classifying map for $\overline{W}$, the one point compactification of W. $\overline{W}$ is homeomorphic to $S^4 \# k(S^2 \times S^2)$ and we take τ' to be the constant map to the base point on a neighborhood of the point at ∞. By the covering homotopy property τ_α and τ_β are homotopic to lifts τ_α' and τ_β' of $\tau'|W$, where τ_β' extends to a lift of τ' with τ_β' constant on a neighborhood of ∞. But since the compactification of one end of $S^3 \times R$ is R^4 (with $S^3 \times R = R^4 - 0$) and since $\tau_\alpha' \sim \tau_\beta'$ on S^3, $\tau_\alpha'|S^3 \times R$ extends to a lift of $\tau'|R^4$. But $\tau'|R^4$ has only one homotopy class of lifts which is standard over a base point. Thus $\tau_\alpha'|S^3$ is standard and $\alpha = 0$.

Main Theorem II: $j_* : \pi_3(Top_4/O_4) \to \pi_3(Top/O)$ is an isomorphism.

Proof: The above argument shows the stabilization homomorphism j_* is a monomorphism. On the other hand, Freedman [F] has exhibited an almost smoothed almost parallelizable closed 1-connected manifold of index 8. It follows that the smoothing of the end of this manifold represents a stably non-trivial element of $\pi_3(Top_4/O_4)$. Hence j_* is an isomorphism.

Remark: The base point is irrelevant in the main theorems since Top_4/O_4 is homogeneous.

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A TWO STAGE PROCEDURE FOR THE CLASSIFICATION OF VECTOR BUNDLE MONOMORPHISMS WITH APPLICATIONS TO THE CLASSIFICATION OF IMMERSIONS HOMOTOPIC TO A MAP

by

Li Banghe and Nathan Habegger

§0. Introduction

0.1. Let A be a subspace of a path connected space B. Let $* \in B$ be a base point and denote by $\pi_1(B,A,*)$ the set of homotopy classes of paths $c : [0,1] \to B$ with $c(1) = *$, $c(0) \in A$. Then $\pi_1(B,*)$ acts on $\pi_1(B,A,*)$ on the right with orbit space $\pi_0(A)$. Thus the problem of calculating $\pi_0(A)$ may be divided into two stages:

I Calculate $\pi_1(B,A,*)$

II Calculate the action of $\pi_1(B,*)$ on $\pi_1(B,A,*)$

0.2. If M^m, N^n are differentiable manifolds, the space $\mathrm{Imm}(M,N)$ of immersions of M in N is a subspace of the space N^M of all maps M to N (with the compact open topology). For fixed $f \in N^M$, let $N^M_{[f]}$ denote its path component. Applying 0.1 we have that $\pi_0(N^M_{[f]} \cap \mathrm{Imm}(M,N))$ (the set of regular homotopy classes of immersions homotopic to f, which we will denote by $[M \looparrowright N]_{[f]}$) is the orbit space of $\pi_1(N^M,\mathrm{Imm},f)$ (the set of regular homotopy classes of immersions with a homotopy to f given, denoted by $[M \looparrowright N]_f$) under an action of $\pi_1(N^M,f)$.

This work is an investigation into the second stage of the classification procedure. We were motivated to look closer at step two as we had observed in the literature several misstated results due to a failure to consider this step.

In §1 we recall the notion of affine structure and affine action.

In many situations, the sets encountered come equipped with an affine structure and the group actions are affine. This additional algebraic structure facilitates the expression of the final results.

In §2 we discuss general properties of π_1 actions in lifting problems. Here we give the general homotopy theoretic framework which is then applied in §3 to the lifting problem associated to the classification of monomorphisms of vector bundles. In §4 we give examples of trivial and non-trivial affine actions of immersion theory. In the appendix we give some calculations of $\pi_1(Y^X,f)$.

§1. Affine Structures

Definition 1.1. A set X is said to be affine (over a group G) if there is a map $\mu: X \times X \to G$ satisfying

a) $\mu(x,y) \cdot \mu(y,z) = \mu(x,z)$

b) for all $x \in X$, $\mu(x,\cdot) : X \to G$ is a bijection

Remark 1.1.1. μ determines (and is determined by) a simply transitive action $r : X \times G \to X$ by the equation $r(x,\mu(x,y)) = y$.

Definition 1.2. An affine map is a pair $(f,\bar{f})$ making the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{\mu} & G \\ \Big\downarrow{\scriptstyle f \times f} & & \Big\downarrow{\scriptstyle \bar{f}} \\ X' \times X' & \longrightarrow & G' \end{array}$$

commute.

Definition 1.3. The group of affine transformations of X (over G, w.r.t. μ) will be denoted by $\mathrm{Aut}(X,G,\mu)$ (or just $\mathrm{Aut}(X)$).

Remark 1.3.1. There is a split exact sequence $1 \to G \to \mathrm{Aut}(X,G,\mu) \xrightarrow{res} \mathrm{Aut}(G) \to 1$, where $res : \mathrm{Aut}(X) \to \mathrm{Aut}(G)$ is given by $(f,\bar{f}) \to \bar{f}$. The action of G as automorphisms (on the left) of X is called translation.

§2. π_1 actions in lifting problems

2.1. π_1 actions on fibers

Given a (Serre) fibration $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ with B path connected and fiber F, one has a right action of $\pi_1(B)$ on $\pi_0(F)$ given by taking the end point of lifts of paths. The orbit space of this operation is $\pi_0(E)$.

The above situation is equivalent to that of 0.1, since if the inclusion $A \subset B$ is replaced by a fibration $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ then $\pi_1(B,A) \simeq \pi_0(F)$ and this bijection is compatible with the action of $\pi_1(B)$.

2.1.1. Naturality

If $\begin{matrix} E' & \rightarrow & E \\ \downarrow & & \downarrow \\ B' & \rightarrow & B \end{matrix}$ is a pullback (in the homotopy category)

then the action of $\pi_1(B')$ factors through $\pi_1(B)$.

2.2. Maps into fibrations

Let $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ be a (Serre) fibration and X a complex. (These assumptions will be made throughout, although this is more restrictive than necessary).

The map $\begin{matrix} E^X \\ \rho\downarrow \\ B^X \end{matrix}$ is a Serre fibration. For $f \in B^X$, the fiber Γ_f over f (possibly empty) is the space of <u>lifts over f</u>. By 2.1 $\pi_1(B^X,f)$ acts on $\pi_0(\Gamma_f)$ with orbit space $\pi_0(\rho^{-1}B^X_{[f]})$ (denoted respectively by $[X,E]_f$ and $[X,E]_{[f]}$.) Thus the homotopy classification of liftings and of liftings "up to homotopy" differ by an action of π_1.

2.2.1. Naturality

If $\begin{matrix} E' & \rightarrow & E \\ \downarrow & & \downarrow \\ B' & \xrightarrow{g} & B \end{matrix}$ is a pullback, so is $\begin{matrix} E'^X & \rightarrow & E^X \\ \downarrow & & \downarrow \\ B'^X & \rightarrow & B^X \end{matrix}$ so, by 2.1.1, the action of $\pi_1(B'^X,f)$ factors through $\pi_1(B^X, g\circ f)$.

2.3. Two stage lifting problems

Suppose $\begin{matrix}E\\\downarrow\\B\end{matrix}$ and $\begin{matrix}B\\\downarrow\\X\end{matrix}$ are fibrations. Let $\Gamma\begin{pmatrix}B\\\downarrow\\X\end{pmatrix}\subset B^X$ denote the space of sections.

The diagram

$$\begin{matrix}\Gamma\begin{pmatrix}E\\\downarrow\\X\end{pmatrix} & \subset & E^X\\ \downarrow & & \downarrow\rho\\ \Gamma\begin{pmatrix}B\\\downarrow\\X\end{pmatrix} & \subset & B^X\end{matrix}$$

is a pullback.

For $f\in\Gamma\begin{pmatrix}B\\\downarrow\\X\end{pmatrix}$ denote by $\Gamma\begin{pmatrix}B\\\downarrow\\X\end{pmatrix}_{<f>}$ the path component of f. By 2.1 $\pi_1(\Gamma\begin{pmatrix}B\\\downarrow\\X\end{pmatrix}, f)$ acts on $\pi_0(\Gamma_f)$ ($=[X,E]_f$, see 2.2) with orbit space $\pi_0(\rho^{-1}\Gamma\begin{pmatrix}B\\\downarrow\\X\end{pmatrix}_{<f>})$ (denoted by $[X,E]_{<f>}$). Thus classification of liftings of a two stage fibration involves calculating an action of π_1.

Remark. By 2.1.1, the action of $\pi_1(\Gamma\begin{pmatrix}B\\\downarrow\\X\end{pmatrix})$ factors through $\pi_1(B^X)$ so little generality is lost by considering only the situation of 2.2.

2.4. Affine Structures

Proposition 2.4.1.

Let $\begin{matrix}\widetilde{G}\\\downarrow\\B\end{matrix}$ be a local coefficient system with fiber G and let $\begin{matrix}\widetilde{B}\\\downarrow\\B\end{matrix}$ be a covering space with fiber F. Suppose we are given a fiberwise action $\widetilde{B}\times\widetilde{G}\xrightarrow{r}\widetilde{B}$ such that for $x\in F$, $r(x,\cdot):G\to F$ is a bijection, i.e., so that F is affine over G. Then the action of $\pi_1(B)$ on F is affine.

Proof: Let $\alpha\in\pi_1(B)$ and let $f_\alpha : F\to F$, $g_\alpha : G\to G$ be the maps given by path lifting. Then $g_\alpha = \overline{f}_\alpha \in \mathrm{Aut}(G)$.

Example 2.4.2.

Suppose $\begin{matrix}E\\\downarrow\\B\end{matrix}$, $\begin{matrix}T\\\downarrow\\B\end{matrix}$ are fibrations with fibers F_b,Ω_b and suppose that the Ω_b are groups (H-spaces) and that we are given a fiberwise (H-space) action $E\times T\to E$ such that for $x\in F_b$ $r(x,\cdot):\Omega_b\to F_b$ is a (homotopy) equivalence.

Then $\pi_0(\Omega_b)$ is a local coefficient system over B acting on the covering $\pi_0(F_b)$, so 2.4.1. applies.

Example 2.4.3.

Let $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ be a fibration with fiber F. Let $(SX,*)$ be a reduced suspension. Then the product $(B,*)^{(SX,*)} \times (F,*)^{(SX,*)}$ acts fiberwise on the fibration $\begin{matrix} (E,*)^{(SX,*)} \\ \downarrow \\ (B,*)^{(SX,*)} \end{matrix}$ as in 2.4.2. We get that $[(SX,*), (E,*)]_f$ is affine over the group $[(SX,*), (F,*)]$. Moreover the affine action of $\pi_1((B,*)^{(SX,*)})$ is via translations (since path lifting in products is trivial, see proof of 2.4.1).

Example 2.4.4.

Let $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$, $\begin{matrix} T \\ \downarrow \\ B \end{matrix}$ be as in 2.4.2.

Then $\begin{matrix} E^X \\ \rho \downarrow \\ B^X \end{matrix}$ $\begin{matrix} T^X \\ \downarrow \\ B^X \end{matrix}$ are also as in 2.4.2 (except that $\rho^{-1}(f)$ may possibly be empty). Applying 2.4.1 and 2.4.2 we get that $[X,E]_f$ is affine over $[X,T]_f$ and the action of $\pi_1(B^X,f)$ is affine.

2.5. Affine structures for lifting problems in the stable range

2.5.1. Notation. For spaces we have the functor Σ, the unreduced suspension with distinguished points S and N, the south and north poles. For spaces with a base point, $*$, we have the functor Ω_*, loops at $*$, and P_*, paths ending at $*$. For spaces with two base points $*_0$ and $*_1$, we have the functor $P_{*_0,*_1}$ of paths beginning at $*_0$ and ending at $*_1$. Ω_* acts on P_* from the right. Ω_{*_1} acts on $P_{*_0,*_1}$ from the right and for $c \in P_{*_0,*_1}(X)$, $r(c,\cdot) : \Omega_{*_1}(X) \to P_{*_0,*_1}(X)$ is a homotopy equivalence.

If $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is a fibration (with section s, or sections s_0,s_1) then

$\Sigma E_B, \Omega_s E_B$, $P_s E_B$, $P_{s_0,s_1} E_B$ denote the fiberwise application of the functors Σ, Ω_*, P_*, $P_{*_0,*_1}$.

Theorem 2.5.2. (Becker [Be]).

Let $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ be a fibration with $n-1$ connected fiber F. Let X be a $2n-1$ coconnected complex and $f : X \to B$ a map. Then $[X,E]_f$ is affine. Moreover, the action of $\pi_1(B^X,f)$ on $[X,E]_f$ is affine.

Lemma 2.5.3. For X $n-1$ connected, the inclusion $X \to P_{S,N}\,\Sigma X$ is $2n-1$ connected.

Lemma 2.5.4. Let $\begin{matrix} E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ B & \underset{\text{id}}{=\!=} & B \end{matrix}$ be a map of fibrations such that the map of fibers $F \to F'$ is m connected. Then if X is m coconnected, $[X,E]_f \to [X,E']_f$ is 1-1 and onto.

Proof of 2.5.2. By 2.5.3 and 2.5.4 we may replace $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ by $\begin{matrix} P_{S,N}\,\Sigma\, E_B \\ \downarrow \\ B \end{matrix}$. Now apply 2.4.4 to the fibrations $\begin{matrix} P_{S,N}\,\Sigma\, E_B \\ \downarrow \\ B \end{matrix}$, $\begin{matrix} \Omega_N\,\Sigma\, E_B \\ \downarrow \\ B \end{matrix}$.

2.6. Affine structures for lifting problems with fiber an Eilenberg MacLane space

Theorem 2.6.1.

Let $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ be a fibration with fiber F a $K(G,n)$, $n > 1$. Then the set $[X,E]_f$ has an affine structure μ with group $H^n(X,\tilde{G}_f)$ (where $\tilde{G}_f$ is the local coefficient system on X induced by f from the local coefficient system $\pi_n(F_b)$). The action of $\pi_1(B^X,f)$ on $[X,E]_f$ is affine.

Moreover let $\psi: B \to K(\pi_1(B),1) = K$ be a map inducing the isomorphism $\pi_1(B) = \pi_1(K)$. Then the composite $\pi_1(B^X,f) \to \mathrm{Aut}([X,E]_f, H^n(X,\tilde{G}_f),\mu) \to \mathrm{Aut}(H^n(X,\tilde{G}_f))$ coincides with the composite

$$\pi_1(B^X,f) \to \pi_1(K^X,\psi f) \xrightarrow[\sim]{\text{cf } 5.1.3} \mathrm{Aut}\begin{pmatrix} \tilde{G}_f \\ \downarrow \\ X \end{pmatrix} \to \mathrm{Aut}(H^n(X,\tilde{G}_f)).$$

Proof: The affine structure is classical obstruction theory. Let $\widetilde{G}$ denote an operation of π on G (or a local coefficient system over $K = K(\pi,1)$. Let $\widetilde{K}$ denote the universal cover). Let Z be a pointed $K(G,n+1)$ with based π action inducing $\widetilde{G}$ on $\pi_n(Z) = G$. Set $L(\widetilde{G},n+1) = \widetilde{K} \times_{\pi} Z$, a $K(G,n+1)$ fibration over $K = K(\pi,1)$ with section u and projection p. Consider $E = P_u L(\widetilde{G},n+1)_K$ (see 2.5.1). The map $\begin{array}{c} E \\ \downarrow \\ L(\widetilde{G},n+1) \end{array}$, given by evaluating a path at its origin, is the universal (see [Ba], page 298) $K(G,n)$ fibration. (The fiber over $x \in Z \subset L(\widetilde{G},n+1)$ is $P_{x,*}Z$, a $K(G,n)$.)

By naturality, it will be enough to prove 2.6.1 in the universal case. Note that the fibers over $u(K)$ are H-spaces (the fiber over $* \in Z \subset L(\widetilde{G},n+1)$ is ΩZ) and $u^*E = \Omega_u L(\widetilde{G},n+1)_K$. The fibrations $\begin{array}{c} E \\ \downarrow \\ L(\widetilde{G},n+1) \end{array}$ and $\begin{array}{c} p^*u^*E \\ \downarrow \\ L(\widetilde{G},n+1) \end{array}$ satisfy 2.4.4. So $[X,E]_f$ is affine over $[X,p^*u^*E]_f = [X,\Omega_u L(\widetilde{G},n+1)_K]_{pof}$. This latter is isomorphic to $H^n(X,\widetilde{G}_f)$ as groups. By 2.4.2 and 2.1.1, the action of $\pi_1(B^X,f)$ is affine. Moreover, by the proof of 2.4.1, the map $\pi_1(L(\widetilde{G},n+1)^X,f) \to \mathrm{Aut}(H^n(X,\widetilde{G}_f))$ is given by path lifting in the fibration $\begin{array}{c} (\Omega_u L(\widetilde{G},n+1)_K)^X \\ \downarrow \\ K^X \end{array}$ (more precisely, by 2.4.2, by path lifting in the associated local coefficient system over K^X with fiber $[X,\Omega_u L(\widetilde{G},n+1)_K]_{pof} = H^n(X,\widetilde{G}_f)$.) This is easily seen to be given by the coefficient automorphism.

Corollary 2.6.2.

Let X have dimension n and let $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ be a fibration with $n-1$ connected fiber F. Then $[X,E]_f$ is affine over $H = H^n(X, \pi_n^f(F))$. The map $\pi_1(B^X,f) \to \mathrm{Aut}\, H$ factors through $\mathrm{Aut}(\pi_n^f(F))$.

Proof: Let $\begin{array}{c} E_n \\ \downarrow^n \\ B \end{array}$ be the first stage of a Postnikov tower for $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ with fiber $K(\pi_n(F),n)$. Then by 2.5.4 $[X,E]_f \to [X,E_n]_f$ is a bijection. Apply 2.6.1 to $\begin{array}{c} E_n \\ \downarrow^n \\ B \end{array}$.

Proposition 2.6.3.

Let $X = S^n$, the n-sphere, and let $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ be a fibration with simply connected fiber F. Then $[X,E]_f$ has an affine structure with group $\pi_n(F)$. The composite $\pi_1(B^{S^n}) \to \mathrm{Aut}\big((\pi_n(F)\big)$ factors through $\pi_1(B)$. Furthermore, the map $\pi_{n+1}(B) \xrightarrow{\text{cf } 5.3} \ker(\pi_1(B^{S^n},f) \to \pi_1(B)) \to$ translation group of $[X,E]_f = \pi_n(F)$, is the boundary homomorphism.

Proof: Let $\to E_k \longrightarrow E_{k-1} \longrightarrow$ be a Postnikov decomposition for $\begin{smallmatrix} E \\ \downarrow \\ B \end{smallmatrix}$ where

$$\begin{tikzcd} E_k \arrow[r] \arrow[dr, "p_k"'] & E_{k-1} \arrow[d] \\ & B \end{tikzcd}$$

$\begin{smallmatrix} E_k \\ \downarrow \\ E_{k-1} \end{smallmatrix}$ has fiber an Eilenberg MacLane Space, $K(\pi_k(F),k)$. The fibers of

$\begin{smallmatrix} E^{S^n} \\ \downarrow \\ E^{S^n}_{k-1} \end{smallmatrix}$ are connected and simply connected for $k \le n-1$, hence by induction

$E^{S^n}_{n-1,f} = \{\tilde{f} : S^n \to E_{n-1}, p_{n-1} \circ \tilde{f} = f\}$ is connected and simply connected and thus

$E^{S^n}_{n-1,[f]} = \{g : S^n \to E_{n-1}, p_{n-1} \circ g \in [f]\}$ is connected and

$\pi_1(E^{S^n}_{n-1,[f]},\tilde{f}) \to \pi_1(B^{S^n},f)$ is an isomorphism, where $\tilde{f} : S^n \to E_{n-1}$ lifts f.

It follows that $[S^n,E]_{\tilde{f}} \to [S^n,E]_f$ is a bijection and one can apply 2.6.2.

The last assertion is an elementary verification.

§3. π_1 actions and monomorphisms of vector bundles

3.1. The fibration $\mathrm{Mono}(\xi,\eta) \to Y^X$

Let $\begin{smallmatrix} \xi^m \\ \downarrow \\ X \end{smallmatrix}$, $\begin{smallmatrix} \eta^n \\ \downarrow \\ Y \end{smallmatrix}$ be vector bundles and let $\mathrm{Mono}(\xi,\eta)$ be the space of all vector bundle maps ξ to η which are monomorphisms on each fiber. Each such map induced a map $X \to Y$. The map $\mathrm{Mono}(\xi,\eta) \to Y^X$ is a (Serre) fibration.

Let $E = \beta(\xi,\eta)$ be the fiber space over $X \times Y$ with fiber $\mathrm{Mono}(\xi_x, \eta_y)$ $\mathrm{Mono}(\xi_x, \xi_y) = V_{n,m}$ and structure group $O(m) \times O(n)$. Projecting onto X one has the two stage fibration

$$\begin{array}{c} E \\ \downarrow \\ X \times Y \\ \downarrow \\ X \end{array}$$

. The space of sections $\Gamma\begin{pmatrix} E \\ \downarrow \\ X \end{pmatrix}$ is homeomorphic to the space $\mathrm{Mono}(\xi,\eta)$. The space of sections $\Gamma\begin{pmatrix} X \times Y \\ \downarrow \\ X \end{pmatrix}$ is homeomorphic to the space Y^X. Thus, for $f : X \to Y$ we have (see 2.3) $\pi_1(Y^X, f)$ acts on $\pi_0(\mathrm{Mono}_f(\xi,\eta))$ (denoted by $[\xi,\eta]_f$) with quotient $\pi_0(\mathrm{Mono}_{[f]}(\xi,\eta))$ (denoted by $[\xi,\eta]_f$). (Here $\mathrm{Mono}_f(\xi,\eta)$ is the space of monomorphisms covering f and $\mathrm{Mono}_{[f]}(\xi,\eta)$ is the space of monomorphisms covering maps homotopic to f). We remark that $\mathrm{Mono}_f(\xi,\eta)$ is homeomorphic to $\mathrm{Mono}_{id}(\xi, f^*(\eta))$.)

3.2. Naturality

Suppose
$$\begin{array}{ccc} \eta & \longrightarrow & \eta' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g} & Y' \end{array}$$
is a pullback of vector bundles.

Then the diagram

$$\begin{array}{ccc} \beta(\xi,\eta) & \to & \beta(\xi,\eta') \\ \downarrow & & \downarrow \\ X \times Y & \to & X \times Y' \end{array}$$
is also a pullback

so the action of $\pi_1(Y^X, f)$ factors through $\pi_1(Y'^X, g\circ f)$.

Example 3.2.1.

Let $\psi : Y \to BO(m)$ classify η.

The action of $\pi_1(Y^X, f)$ factors through $\pi_1(BO(m)^X, \psi\circ f)$ (= $\pi_0(\mathrm{Aut}\ f^*(\eta))$, see 5.1.3.)

Example 3.2.2.

Let dimension $X = r$ and suppose η is trivial on the $r+1$ skeleton Y_{r+1} of Y. Then the action of $\pi_1(Y^X, f)$ is trivial.

Proof: We may suppose $f : X \to Y_{r+1}$. The action of $\pi_1(Y^X_{r+1}, f)$ is trivial since it factors through the trivial group $\pi_1(pt^X)$. But since

$\pi_1(Y^X_{r+1},f) \to \pi_1(Y^X,f)$ is surjective, $\pi_1(Y^X,f)$ also acts trivially.

3.3. Codimension zero monomorphisms

3.3.1. Theorem. Let $\begin{matrix}\xi & \eta\\ \downarrow & \downarrow\\ X & Y\end{matrix}$ be vector bundles of dimension n.

Let $g : Y \to BO(n)$ classify η. Then $[\xi,\eta]_f$ is affine over $\pi_1(BO(n)^X, gof)$.

3.3.2. Corollary. $[\xi,\eta]_{[f]}$ corresponds bijectively with the coset space

$$\frac{\pi_1(BO(n)^X, gof)}{\text{im } \pi_1(Y^X,f)}.$$

Proof of theorem. $\mathrm{Mono}_f(\xi,\eta) = \mathrm{Mono}_{id}(\xi, f^*(\eta))$ has $\mathrm{Aut}(f^*(\eta))$, as action group. So $[\xi,\eta]_f$ has $\pi_0(\mathrm{Aut}(f^*(\eta)))$ ($= \pi_1(BO(n)^X, gof)$, see 5.1.3) as action group.

3.4. Codimension one monomorphisms

Let $\begin{matrix}\xi & \eta\\ \downarrow & \downarrow\\ X & Y\end{matrix}$ be vector bundles with $\dim \xi+1 = \dim \eta = n$.

Let $\begin{matrix}\omega\\ \downarrow\\ X\end{matrix}$ be the bundle of dimension 1 with first Stiefel Whitney class

class $W^1(\omega) = W^1(f^*(\eta))-W^1(\xi)$. The map $\begin{matrix}\mathrm{Mono}_{[f]}(\xi\oplus\omega,\eta)\\ \downarrow\\ \mathrm{Mono}_{[f]}(\xi,\eta)\end{matrix}$ is a 2 fold covering

which is split if η is orientable (by fixing orientations of $\xi \oplus \omega$ and η and requiring an extension to preserve orientations). The 2-fold covering $\begin{matrix}\mathrm{Mono}_f(\xi \oplus \omega,\eta)\\ \downarrow\\ \mathrm{Mono}_f(\xi,\eta)\end{matrix}$ is split (by requiring orientations to be preserved at the base point). $\mathrm{Mono}_f(\xi,\eta)$ has commuting action groups $\mathrm{Aut}_+(\xi \oplus \omega)$, $\mathrm{Aut}_+(F^*(\eta))$ where $\mathrm{Aut}_+ \subset \mathrm{Aut}$ is the normal subgroup of orientation perserving automorphisms. Hence with respect to the affine structure given by the action of $\pi_0(\mathrm{Aut}_+(\xi\oplus\omega))$, $\pi_0(\mathrm{Aut}_+(f^*(\eta))$ is the translation group and the action of $\pi_0(\mathrm{Aut}(f^*(\eta))$ is affine. We have proven:

Theorem 3.4.1.

Let $\begin{matrix}\xi & \eta\\ \downarrow & \downarrow\\ X & Y\end{matrix}$ be vector bundles with $\dim \xi+1 = \dim \eta = n$.

Let $g : Y \to BO(n)$ classify η. Then $[\xi,\eta]_f$ has an affine structure with $\ker(\pi_1(BO^X(n), g\circ f) \to \pi_1(BO(n))$ acting as the translation group, and the action of $\pi_1(BO(n)^X, g\circ f)$ is affine. If η is orientable and $g : Y \to BSO(n)$ classifies η, then $[\xi,\eta]_f$ has an affine structure with $\pi_1(BSO(n)^X, g\circ f)$ acting as translation group.

Corollary 3.4.2.

If $g : Y \to BSO(n)$ classifies η, then $[\xi,\eta]_{[f]}$ corresponds bijectively with the coset space

$$\frac{\pi_1(BSO(n)^X, g\circ f)}{\operatorname{im} \pi_1(Y^X, f)} .$$

3.5. The case of a sphere

Let $\begin{matrix}\xi^m & \eta^n\\ \downarrow & \downarrow\\ S^k & Y\end{matrix}$ be a vector bundles, $m+2 \le n$, and let $g : Y \to BO(n)$ classify η. The fiber $V_{n,m}$ of $B(\xi,\eta)$ is simply-connected so 2.6.3 applies. Combining 2.6.3 with 3.2.1 we obtain:

Theorem 3.5.1.

$[\xi,\eta]_f$ is affine over $\pi_k(V_{n,m})$. The action of $\pi_1(Y^{S^k}, f)$ is affine. Moreover the diagram

$$\begin{array}{ccccc}
\pi_{k+1}(Y) & \longrightarrow & \pi_{k+1}(BO(n)) & \xrightarrow{\partial} & \pi_k(V_{n,m}) = T([\xi,\eta],\pi_k(V_{n,m}),\gamma)\\
\downarrow & & \downarrow & & \downarrow\\
\pi_1(Y^{S^k}, f) & \longrightarrow & \pi_1(BO(n)^{S^k}, g\circ f) & \longrightarrow & \mathrm{Aut}([\xi,\eta],\pi_k(V_{n,m}),\gamma)\\
\downarrow & & \downarrow & & \downarrow\\
\pi_1(Y) & \longrightarrow & \pi_1(BO(n)) & \xrightarrow{\mu_*} & \mathrm{Aut}\, \pi_k(V_{n,m})
\end{array}$$

is commutative, where ∂ is the boundary homomorphism of the fibration $V_{n,m} \to BO(n-m) \to BO(n)$. μ_* is given by post multiplication by a non-rotation.

Corollary 3.5.2.

If η is orientable or μ_* is trivial, then $[\xi,\eta]_{[f]}$ is in bijection with a coset space of $\pi_k(V_{n,m})$.

Corollary 3.5.3.

If $\pi_1(Y^{S^k},f) \to \pi_1(BO(n)) = \mathbf{Z}/2\mathbf{Z}$ and μ_* are non-trivial then the action of $\pi_1(Y^{S^k},f)$ is non-trivial.

Corollary 3.5.4.

Suppose $\partial: \pi_{k+1}(BO(n)) \to \pi_k(V_{n,m})$ is the zero homomorphism. The action of $\pi_1(Y^{S^k},f)$ is trivial if η is orientable and factors through $\pi_1(BO(n)) = \mathbf{Z}/2\mathbf{Z}$ otherwise. Suppose in addition there is $\alpha \in \pi_1(Y^{S^k},f)$ with non-trivial image in $\pi_1(BO(n))$ (i.e. $res(\alpha) \in \pi_1(Y)$ reverses orientation) which fixes some element of $[\xi,\eta]_f$. Then the action of $\pi_1(Y^{S^k},f)$ is trivial if and only if μ_* is trivial and $[\xi,\eta]_{[f]}$ is in bijection with the set of orbits of $\pi_k(V_{n,m})$ under the operation of μ_*.

Proof: Follows from 3.5.1. and 1.3.1.

Example 3.5.4. a).

If $k+1 + 2m < 2n$, $13 \le n-m$ and $k+2 \le n$ then ∂ is zero (cf. [BM]). If $k+2 \le n$ and $k = 2,4,5$ or $6 \bmod 8$ then $\pi_{k+1}(BO(n)) = \pi_{k+1}(BO) = 0$.

Example 3.5.4. b).

Let $f : S^k \to Y$ be homotopic to a constant map. Let $\xi \oplus \xi' = f^*(\eta)$ and suppose ξ' admits an orientation reversing automorphism (e.g. if dim ξ' is odd or if ξ' admits a section). Then the map $S^n \times S^1 \to S^1 \overset{c}{\to} Y$, where c is an orientation reversing loop, fixes the monomorphism $\xi \subset \xi \oplus \xi' \simeq f^*(\eta) \to \eta$.

Example 3.5.4. c).

The map μ_* on $\pi_r(V_{n,k})$ has been calculated by James [J] to be $id - u_* s_* \Delta_* - \Delta_* s_* p_*$ where $\pi_r(V_{n,k}) \xrightarrow{\Delta^*} \pi_{r-1}(S^{n-k-1}) \xrightarrow{S_*} \pi_r(S^{n-k}) \xrightarrow{u_*} \pi_r(V_{n,k})$

and $\pi_r(V_{n,k}) \xrightarrow{P_*} \pi_r(S^{n-1}) \xrightarrow{S_*} \pi_{r+1}(S^n) \xrightarrow{\Delta_*} \pi_r(V_{n,k})$ P_* is projection onto base, u_* the inclusion of fiber and Δ_* are boundary homomorphisms of the obvious fibrations. S_* is suspension.

For example if $r=k=m$ with $n \geq m+2$, then $\Delta_* S_* P_* = 0$ since $\pi_m(S^n) = 0$. Hence $\mu_* = id - u_* S_* \Delta_* = \lambda_*$ (λ_* is premultiplication by a non-rotation). Since $\lambda_*^2 = 1 = \mu_*^2$ and $\lambda_*^m = \mu_*^n$ we get $\mu_* = 0$, if $n-m$ is odd.

From [P] $\pi_n(V_{2n-2,n}) = \mathbf{Z}_2 + \mathbf{Z}_2$ if $n \equiv 2 \bmod 4$. Moreover one can check $\Delta_* : \mathbf{Z}_2 + \mathbf{Z}_2 \to \pi_{m-1}(S^{m-3}) = \mathbf{Z}_2$ is surjective, S^* is an isomorphism and $u_* : \mathbf{Z}_2 \to \mathbf{Z}_2 + \mathbf{Z}_2$ is injective. Hence $\mu_* = id - u_* S_* \Delta_*$ fixes two elements and exchanges the other 2.

From [P], $\pi_8(V_{12,8}) = \mathbf{Z}_2 + \mathbf{Z}_2 + \mathbf{Z}_2$ and one may check Δ is onto $\pi_7(S^3) = \mathbf{Z}_2$, $S^* : \pi_7(S^3) \to \pi_8(S^4) = \mathbf{Z}_2 + \mathbf{Z}_2$ is injective and $u_* : \pi_8(S^4) \to \pi_8(V_{12,8})$ is injective. Hence $\mu_* = id - u_* S_* \Delta_*$ fixes 4 elements and exchanges the other 4 in pairs.

One can also show the following:

If $n = 1 \bmod 4$ and $n \geq 5$ then $\pi_n(V_{2n-1,n}) = \mathbf{Z}_2 + \mathbf{Z}_2$ and there are 3 orbits.

If $n = 3 \bmod 4$, then $\pi_n(V_{2n-1,n}) = \mathbf{Z}/4\mathbf{Z}$ and $\mu_* = id$.

If $n = 1 \bmod 4$ and $n \geq 9$, then $\pi_n(V_{2n-3,n}) = \mathbf{Z}/12\mathbf{Z}$ and $\mu_*(X) = -x$ so there are 7 orbits.

If $k = n+2$ then $\pi_n(V_{n+2,n}) \simeq \pi_n(SO)$ and $\mu_* = id$.

3.6. The case of the first obstruction

Let $\begin{matrix} \xi^m & \eta^n \\ \downarrow & \downarrow \\ X & Y \end{matrix}$ and let $g : Y \to BO(n)$ classify η.

Suppose $n-m$ = dimension of $X = k$. The fiber $V_{n,m}$ of $B(\xi,\eta)$ is k-1 connected so 2.6.5. applies. Let $\widetilde{\pi_k(V_{n,m})}$ be the local coefficient system over X twisted by $\pi_1(X) \to \mathbf{Z}_2 \times \mathbf{Z}_2 \to \operatorname{Aut} \pi_k(V_{n,m})$ where $\pi_1(X) \to \mathbf{Z}_2 \times \mathbf{Z}_2$ is the orientation homomorphism of ξ and η, and $\mathbf{Z}_2 \times \mathbf{Z}_2 \to \mathrm{Aut}\, \pi_k(V_{n,m})$ is induced by pre and post composition with non-rotations.

Proposition 3.6.1.

In the above $[\xi,\eta]_f$ is affine over $H^k(X, \widetilde{\pi_k(V_{n,m})}) = H$. The map $\pi_1(Y^X, f) \to \mathrm{Aut}(H)$ factors through $\pi_1(BO(n)) = \mathbf{Z}_2 \xrightarrow{\mu_*} \mathrm{Aut}(\pi_k(V_{n,m}))$.

Corollary 3.6.2.

If $\pi_1(Y^X, f) \to \pi_1(BO(n)) = \mathbf{Z}_2$ is non-trivial and the coefficient automorphism μ_* induces a non-trivial automorphism of $H^k(X, \widetilde{\pi_k(V_{n,m})})$, then the action of $\pi_1(Y^X, f)$ is non-trivial.

§4. Applications to Immersion Theory

4.1. Smale Hirsch Theorem

Let M^m, N^n be differentiable manifolds τ_M, τ_N their respective tangent bundles. One has a map $\mathrm{Imm}(M,N) \to \mathrm{Mono}(\tau_M, \tau_N)$ given by taking the differential. Smale-Hirsch theory says that this map is a weak homotopy equivalence provided either $m < n$ or (if $m=n$) M has no closed components.

One may think of this theorem as saying that the inclusion $\mathrm{Imm}(M,N) \subset N^M$ (see 0.1) may be replaced by the fibration $\mathrm{Mono}(\tau_M, \tau_N) \to N^M$ (see 2.1). Thus the sets $[M \looparrowright N]_f$ and $[M \looparrowright N]_{[f]}$ are equal to the sets $[\tau_M, \tau_N]_f$, respectively $[\tau_M, \tau_N]_{[f]}$.

4.2. Immersions of surfaces in orientable 3 manifolds (cf [Li1])

Let Σ^2 be any surface and N^3 any orientable 3-manifold. If f is any map, then $[\Sigma \looparrowright N]_{[f]}$ is in bijection with $H^1(\Sigma, \mathbf{Z}_2)$.

Proof: Any map $M^m \to N^{2m-1}$ is homotopic to an immersion (cf [LP]). N^3 orientable implies N^3 parallelizable (since $\pi_2(SO(3)) = 0$) so by 3.2.2. the action of $\pi_1(N^\Sigma, f)$ is trivial. Hence $[\Sigma^2 \looparrowright N^3]_{[f]} = [\tau_\Sigma, \tau_N]_f$. The fiber of $\beta(\tau_\Sigma, \tau_N)$ over $\Sigma \times N$ is $V_{3,2}$. Since $\pi_1(V_{3,2}) = \mathbf{Z}/2\mathbf{Z}$,

$\pi_2(V_{3,2}) = 0$ the set $[\tau_\Sigma,\tau_N]_f$ has an affine structure with group $H^1(\Sigma,\mathbb{Z}_2)$

4.3. Periodic Isotopy

A periodic isotopy is a map $S^1 \times N \to N$ which is the identify for $t = * \in S^1$ and an embedding for all $t \in S^1$. It is easy to see the following.

Proposition 4.3.1.

Let $\alpha \in \pi_1(N^M,f)$ be induced by a periodic isotopy on N. Then α acts trivially on $[M \looparrowright N]_f$.

Example 4.3.2.

Let L_m^{2n-1} denote a lens space. Then each element of $\pi_1(L) = \mathbb{Z}/m\mathbb{Z}$ is induced by a periodic isotopy. If $\dim M \leq 2n-3$, then $\pi_1(L^M,f) \to \pi_1(L)$ is an isomorphism (see 5.2.3). Hence by 4.3.1, $\pi_1(L^M,f)$ acts trivially.

4.4. Immersions of disks

Let $M = D^m$. If $m < n$ then $[D^m \looparrowright N^n]$ has one element. If $m = n$ then $[D^m \looparrowright N^m]_f$ is affine over $\mathbb{Z}/2\mathbb{Z}$ and the action of $\pi_1(N^D,f)$ is trivial if and only if N is orientable.

4.5. Immersions of M^m in S^{m+1} (cf [Li 2])

Proposition 4.5.1. $[M^m \looparrowright S^{m+1}]$ is in bijection with $[M,SO]$, provided $[M^m \looparrowright S^{m+1}]$ in non empty.

Proof: Let $SO(m+1) \to SO(m+2) \to S^{m+1}$ be the natural fibration. Then $\begin{matrix} SO(m+2) \\ \downarrow \\ S^{m+1}\end{matrix}$ is the principal bundle associated to the tangent bundle (cf. [H]). In particular, in the fibration $S^{m+1} \to BSO(m+1) \to BSO(m+2)$, the inclusion of the fiber classifies the tangent bundle of S^{m+1}. By 3.4.1, $[\tau_M,\tau_S]_f$ is affine over $\pi_1(BSO(m+1)^M)$ and by corollary 3.4.2, $[\tau_M,\tau_S]_{[f]}$ is in bijection with $\dfrac{\pi_1(BSO(m+1)^M)}{\operatorname{im}\pi_1({S^{m+1}}^M)} = \pi_1(BSO(m+2)^M) = \pi_1(BSO^M) = [M,SO]$ (cf.5.1.5).

Example 4.5.2.

$$[S^m \looparrowright R^{m+1}] = \pi_m(SO(m+1))$$

while $$[S^m \looparrowright S^{m+1}] = \pi_m(SO).$$

4.6. Immersions of spheres in manifolds

Applying 3.5.1, 3.5.2, 3.5.3, for $n \geq m+2$ we have

Proposition 4.6.1.

$[S^m \looparrowright N^n]_f$ is affine over $\pi_m(V_{n,m})$ and the action of $\pi_1(M^{S^m}, f)$ is affine. If $\varepsilon : \pi_1(M) \to \mathbf{Z}_2$ is the orientation homomorphism and $\mu_* : \mathbf{Z}_2 \to \operatorname{Aut} \pi_n(V_{m,n})$ is given by postmultiplication by a non-rotation, then $\pi_1(N^{S^m}, f) \to \operatorname{Aut}(\pi_m(V_{n,m}))$ is the composite $\mu_* \circ \varepsilon \circ \mathrm{res}$ where $\mathrm{res} : \pi_1(N^{S^m}, f) \to \pi_1(N)$ is the restriction.

Corollary 4.6.2.

If μ_* or $\varepsilon \circ \mathrm{res}$ is trivial then $[S^m \looparrowright N^n]_{[f]}$ is a coset space of $\pi_n(V_{n,m})$. If $\varepsilon \circ \mathrm{res}$ and μ_* are non-trivial, the action of $\pi_1(N^{S^m}, f)$ is non-trivial.

Theorem 4.6.3. $(m+2 \leq n)$

a) Suppose the compositie $\pi_{m+1}(N) \to \pi_{m+1}(BO(n)) \xrightarrow{\partial} \pi_m(V_{n,m})$ is zero. The set of regular homotopy classes of immersion of S^m in N^n which are homotopic to a constant is in bijection with $\pi_m(V_{n,m})$ if N is orientable and with the set of orbits of the operation of μ_* on $\pi_m(V_{n,m})$ if N is non-orientable.

b) Suppose the map $\pi_{m+1}(BO(n)) \xrightarrow{\partial} \pi_m(V_{n,m})$ is zero. The action of $\pi_1(N^{S^m}, f)$ on $[S^m \looparrowright N^n]_f$ is trivial if N is orientable and factors through $\mathbf{Z}/2\mathbf{Z}$ if N is non-orientable. If there is $\alpha \in [S^m \looparrowright N^n]_f$ which is fixed by $h \in \pi_1(N^{S^m}, f)$ having non-zero image in $\pi_1(BO(m)) = \mathbf{Z}/2\mathbf{Z}$, then $[S^m \looparrowright N^n]_{[f]}$ corresponds bijectively with the set of orbits of the operation of μ_* on $\pi_m(V_{n,m})$.

Proof: b) follows from 4.6.1 and 1.3.1.

a) follows from 3.5.4 b) and 5.2.1.

4.7. Immersions of M^m in N^{2m}

Applying 3.6.1 and 3.6.2 we obtain

Proposition 4.7.1.

$[M^m \looparrowright N^{2m}]_f$ is affine over $H^m(M,\mathbf{Z}/2\mathbf{Z})$ if m is odd and $H^m(M,\mathbf{Z})$ if m is even ($\mathbf{Z}$ is the integers twisted by $f^*W^1(N) - W^1(M)$). For m odd and $f^*W^1(N) = 0$, μ_* induces multiplication by -1 on $H^n(M,\tilde{\mathbf{Z}}) = \mathbf{Z}$, hence if $\pi_1(N^M,f) \xrightarrow{res} \pi_1(N) \xrightarrow{W^1(N)} \mathbf{Z}/2\mathbf{Z}$ is non-trivial, the action of $\pi_1(N^M,f)$ is non-trivial.

Remark.

One can show [Li1] that the action is trivial if $[M^m \looparrowright N^{2m}]_f \simeq \mathbf{Z}/2\mathbf{Z}$ and factors through $\mathbf{Z}/2\mathbf{Z}$ if $[M^m \looparrowright N^{2m}]_f \simeq \mathbf{Z}$. Note that $\mathbf{Z}/2\mathbf{Z}$ can act on $\mathbf{Z}$, up to isomorphism, either by $x \to -x$ (one fixed point) or by $x \to 1-x$ (no fixed points).

§5. Appendix. Some calculations of $\pi_1(Y^X,f)$

5.1. The universal case

Let $\begin{array}{c}\eta\\ \downarrow\\ Y\end{array}$ be a vector bundle of dimension n (or a local coefficient system or other object) satisfying the following universal property: let $\begin{array}{c}\xi\\ \downarrow\\ X\end{array}$ be a n-dimensional vector bundle, $A \subset X$ and $\xi|_A \to \eta$ a bundle map which is an isomorphism on each fiber. Then there is an extension to a bundle map $\xi \to \eta$, which is an isomorphism on each fiber. (Technically, A is assumed to be a subcomplex of the complex X).

Let $\mathrm{Iso}(\xi,\eta)$ be the space of all bundle maps which are isomorphisms on each fiber.

Lemma 5.5.1.

If η is universal, then $\mathrm{Iso}(\xi,\eta)$ is contractible.

Proof: A map of the cone $C\ \mathrm{Iso}(\xi,\eta) \to \mathrm{Iso}(\xi,\eta)$ extending the identity of $\mathrm{Iso}(\xi,\eta)$ is produced as follows: Let $p : C\ \mathrm{Iso}(\xi,\eta) \times X \to X$ denote the projection and define $f : p^*\xi|_{\mathrm{Iso}(\xi,\eta) \times X} = \mathrm{Iso}(\xi,\eta) \times \xi \to \eta$ by $(\alpha,v) \to \alpha(v)$. By universality, f may be extended to all of $p^*\xi = C\ \mathrm{Iso}(\xi,\eta) \times \xi$.

If $\begin{smallmatrix}\eta\\ \downarrow\\ Y\end{smallmatrix}$ is universal and $F \in \mathrm{Iso}(\xi,\eta)$, the map $f : X \to Y$ induced by F is said to classify ξ.

Proposition 5.1.2.

Let $\begin{smallmatrix}\eta\\ \downarrow\\ Y\end{smallmatrix}$ be universal and let $f : X \to Y$ classify $\begin{smallmatrix}\xi\\ \downarrow\\ X\end{smallmatrix}$.

Then $Y^X_{[f]} = B\ \mathrm{Aut}(\xi)$.

Proof: Follows from 5.1.1 since $\mathrm{Aut}(\xi)$ acts effectively on the left of $\mathrm{Iso}(\xi,\eta)$ with orbit space $Y^X_{[f]}$.

Corollary 5.1.3. $\pi_1(Y^X,f) = \pi_0(\mathrm{Aut}(\xi))$.

Example 5.1.4.

If $f : X \to BO(n)$ is homotopically trivial then $\pi_1(BO(n)^X,f) = [X,O(n)]$.

Example 5.1.5.

Let X be a finite complex. The space BO^X is an H-space, so $\pi_1(BO^X,f) = \pi_1(BO^X,f)$ where $c : X \to BO$ is the constant map. So $\pi_1(BO^X,f) \simeq [X,O]$.

5.2. The map $\pi_1(Y^X,f) \to \pi_1(Y)$

Let X be a space with base point $*$. Restriction to $*$ yields a homomorphism $\pi_1(Y^X,f) \xrightarrow{\mathrm{Res}} \pi_1(Y,f(*))$.

Proposition 5.2.1.

Suppose f is null homotopic. Then Res is split.

Proof: We may assume f is the constant map. The projection $X \to *$ induces the splitting $\pi_1(Y) \to \pi_1(Y^X, f)$

Proposition 5.2.2.

$$\text{Image(Res)} \subset \text{centralizer of } f_*(\pi_1(X)).$$

Proof: $\pi_1(S^1)$ commutes with $\pi_1(X)$ in $\pi_1(S^1 \times X)$.

As a partial converse, we have

Proposition 5.2.3.

Suppose $\pi_i(Y) = 0$ for $2 \le i \le \dim X = m$. There is an exact sequence

$$1 \to H^m(X, \pi^f_{m+1}(Y)) \to \pi_1(Y^X, f) \to \text{centralizer of } f_*(\pi_1(X)) \to 1$$

where $\pi^f_{m+1}(Y)$ is the local coefficient system induced by f.

Proof: $\pi_1(Y^X, f)$ consists of homotopy classes of maps $S^1 \times X$ rel$* \times X$ to Y. By the assumption on $\pi_i(Y)$ any extension on the 2 skeleton of $S^1 \times X$ (rel$* \times X$) can be extended to all of $S^1 \times X$. An extension to the 2-skeleton exists if and only if there is θ making the diagram

$$\begin{tikzcd}
\pi_1(Y) \arrow[r] & \pi_1(Y \times X \times S') \arrow[dr] & \\
\pi_1(X) \arrow[u, "f_*"] \arrow[r] & \pi_1(X \times S') \arrow[u, dashed, "\theta"] \arrow[r, "="] & \pi_1(X \times S')
\end{tikzcd}$$

commute (cf. [Ba] page 265)

i.e. if and only if there is σ making

$$\begin{tikzcd}
\pi_1(X) \arrow[r, "f_*"] \arrow[d] & \pi_1(Y) \\
\pi_1(X \times S^1) \arrow[ur, dashed, "\sigma"'] &
\end{tikzcd}$$

commute

i.e. if and only if $\sigma(t)$ commutes with $f_*\pi_1(X)$ where t is the generator of

$\pi_1(S^1) = \mathbf{Z}$. This proves exactness at centralizer of $f_*(\pi_1(X))$.

Now let u be the composite $S^1 \times X \to X \xrightarrow{f} Y$. $\ker \pi_1(Y^X, f) \to \pi_1(Y)$ consists of homotopy classes of maps $S^1 \times X \to Y$ which are $(\mathrm{rel}* \times X)$ homotopic to u on $S^1 \vee X$. By our assumption on $\pi_i(Y)$, these correspond to homotopy classes of maps which are homotopic to u on the m skeleton of $S^1 \times X$. By the spectral sequence (cf [Ba] page 277) these are just $H^{m+1}(S^1 \times X, * \times X; \pi^u_{n+1}(Y)) = H^m(X, \pi^f_{n+1}(Y))$.

Example 5.2.4.

Let M^n be a connected manifold and $Y = S^{n+1}$.

Then $$\pi_1(S^{n+1^M}) = \begin{cases} \mathbf{Z} & \text{if } M \text{ is orientable and closed} \\ \mathbf{Z}/2\mathbf{Z} & \text{if } M \text{ is non-orientable and closed} \\ 0 & \text{if } M \text{ is open.} \end{cases}$$

Example 5.2.5.

Let $Y = \mathbb{R}P^{n+1}$ and $\dim X \leq n$.

Then $1 \to H^n(X;Z_f) \to \pi_1(\mathbb{R}P^{n+1^X}, f) \to \mathbf{Z}/2\mathbf{Z} \to 1$ is exact where Z_f is the integers twisted by

$$\pi_1(X) \xrightarrow{f_*} \pi_1(\mathbb{R}P^{n+1}) = \mathbf{Z}_2 = \langle t \rangle \text{ and } t \text{ acts on } \mathbf{Z} \text{ by } (-1)^n.$$

5.3. The case $X = S^n$

Proposition 5.3.1.

There is an exact sequence

$$\pi_2(Y) \xrightarrow{d} \pi_{n+1}(Y) \to \pi_1(Y^{S^n}, f) \xrightarrow{\mathrm{Res}} \pi_1(Y)$$

where d is the Whitehead product with $[f] \in \pi_n(Y)$ and image(Res) = stabilizer of $[f]$.

Corollary 5.3.2.

If f is homotopic to a constant, then

$1 \to \pi_{n+1}(Y) \to \pi_1(Y^{S^n},f) \to \pi_1(Y) \to 1$ is a split exact sequence.

<u>Proof of 5.3.1</u>. Since $\pi_1(S^1 \times S^n)$ stabilizes $\pi_n(S^1 \times S^n)$ image(Res) $\subset$ stabilizer of $[f]$. (In the following we use the decomposition $S^n \subset S^1 \vee S^n \subset S^1 \times S^n$). If an extension of f to $S^1 \vee S^n$ is given, the obstruction to extending to $S^1 \times S^n$ is just $\alpha[f]-[f] \in H^{n+1}(S^1 \times S^n, * \times S^n, \pi_n(Y)) = \pi_n(Y)$ where $\alpha \in \pi_1(Y)$ is given by $S^1 \to S^1 \vee S^n \to Y$. So image(Res) = stabilizer of $[f]$. Let u be the map $S^1 \times S^n \to S^n \xrightarrow{f} Y$. Then ker(res) is the set of maps $S^1 \times S^n \to Y$ homotopic, rel$* \times S^n$ to u on $S^1 \vee S^n$ (which is the n skeleton of $S^1 \times S^n$ rel$* \times S^n$ since there are no cells of dimension less than $n+1$). Thus by the spectral sequence ([Ba], page 277) we have $\ker(\text{res}) = \pi_{n+1}(Y)/d_n(\pi_2(Y))$. d_n is the Whitehead product ([Ba] page 285).

<u>Example 5.3.3</u>.

$1 \to \pi_{n+1}(BO) \to \pi_1(BO^{S^n},f) \to \pi_1(BO) \to 1$ is exact since Whitehead products in $\pi_i(BO)$ are trivial, and the operation of π_1 is trivial (BO is an H-space).

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LA DEMONSTRATION DE J. LEVINE DES THEOREMES DE A. PLANS

par Claude WEBER

1. Introduction

En 1927, J.W. Alexander et G.B. Briggs calculent l'homologie des revêtements cycliques ramifiés à 2 et 3 feuilles des noeuds ayant au plus 9 croisements. Cf. [1]. Ces calculs sont repris dans le livre de K. Reidemeister [9]. Un phénomène étonnant se manifeste : les coefficients de torsion pour le revêtement cyclique à 3 feuilles vont toujours par paires.

Ce n'est qu'en 1953 que l'explication est fournie par A. Plans, dans un article (en espagnol) extrêmement compliqué [8].

En 1971, C. Mc Gordon en donne une démonstration beaucoup plus simple [3].

Mais c'est J. Levine, [6], qui, par un argument très ingénieux fait toute la lumière sur les phénomènes en question. Contrairement aux démonstrations précédentes, la preuve de J. Levine ne consiste pas à trouver une présentation astucieuse de l'homologie cherchée. Il montre que la décomposition en "double direct" est intimément liée à la présence d'une forme bilinéaire alternée sur le sous-groupe de torsion.

La méthode de J. Levine a également l'avantage de s'appliquer aux revêtements cycliques d'ordre pair. (Ce cas fut traité par A. Plans, mais pas par C. Mc Gordon.)

Définition : On dira qu'un groupe abélien G de type fini est un double direct s'il existe un groupe H tel que $G \approx H \oplus H$.

Théorème (A. Plans) : Soit $K \subset \Sigma$ un noeud (apprivoisé) dans une sphère d'homologie entière, de dimension 3. Soit $\hat{X}_K^m$ le revêtement cyclique à m feuilles de Σ, ramifié sur K. Alors :

1° Si m est impair, $H_1(\hat{X}_K^m;\mathbb{Z})$ est un double direct.

2° Si m est pair, la projection de revêtement $\pi : \hat{X}_K^m \to \hat{X}_K^2$ induit un homomorphisme surjectif : $\pi_* : H_1(\hat{X}_K^m;\mathbb{Z}) \to H_1(\hat{X}_K^2;\mathbb{Z})$ dont le noyau est un double direct.

Commentaires :

1° Pour nous, un noeud est toujours connexe. Le théorème est faux pour les enlacements à plusieurs composantes.

2° La démonstration de A. Plans est écrite pour le cas $\Sigma = S^3$. Il semble bien qu'elle s'étend sans difficulté au cas d'une sphère d'homologie entière.

3° La partie du théorème qui concerne le cas où m est pair est, à peu de choses près, utilisée par R. Hartley [5]. Pour une démonstration, il réfère au commentaire de R.H. Fox dans Math. Reviews 1954, p. 147 et donc indirectement à l'article de A. Plans. La démonstration "à la Levine" que nous donnerons ici est certainement beaucoup plus simple! Nous reviendrons sur ce sujet au § 4.

2. Formes unimodulaires alternées sur les groupes finis

Soit G un groupe abélien fini. Une forme alternée sur G est une application

$\mu : G \times G \to \mathbb{Q}/\mathbb{Z}$ qui est :

a) $\mathbb{Z}$-bilinéaire;

b) alternée, c'est-à-dire telle que $\mu(x,x) = 0$ pour tout $x \in G$.

Bien sûr, ceci implique que μ est antisymétrique, c'est-à-dire $\mu(x,y) = -\mu(y,x)$ et la réciproque est vraie lorsque G n'a pas de 2-torsion.

On dira qu'une telle forme est unimodulaire si :
$\mathrm{Ad}\,\mu : G \to \mathrm{Hom}(G;\mathbb{Q}/\mathbb{Z})$ est un isomorphisme.

La proposition suivante est un classique. Elle me semble due à G. de Rham. Voir sa thèse [10], p. 165.

Proposition 1 : Supposons que le groupe abélien fini G puisse être muni d'une forme unimodulaire alternée. Alors G est un double direct.

Par souci de complétude, nous allons donner une preuve de cette proposition. C'est essentiellement celle de G. de Rham et d'ailleurs aussi celle de C.T.C. Wall [12].

Preuve de la proposition 1 :

Soit G_p la p-composante du groupe G. (p est un nombre premier.) Comme G est abélien, on a bien sûr :

$G = \underset{p}{\oplus} G_p$, la somme étant, en principe, une somme directe de groupes abéliens.

En fait, $\oplus$ est une somme orthogonale. En effet, soient p et q deux premiers distincts. Soit N un entier suffisamment grand pour que la multiplication par q^N soit nulle sur G_q. Alors, si $x \in G_p$, $y \in G_q$, on a :

$$\mu(x,y) = q^N \mu(\frac{1}{q^N} x, y) = \mu(\frac{1}{q^N} x, q^N y) = \mu(\frac{1}{q^N} x, o) = 0 .$$

$\frac{1}{q^N} x$ a un sens puisque la multiplication par q est un isomorphisme de G_p.

Il suffit donc de démontrer la proposition 1 pour $G = G_p$. Pour cela, considérons l'entier $r \geq 1$ tel que p^r soit l'ordre maximum des éléments de G. Soit $x \in G_p$ un élément dont l'ordre est exactement p^r.

<u>Affirmation</u> : Il existe un élément $y \in G$ tel que $\mu(x,y) = p^{-r}$.

<u>Preuve de l'affirmation</u> : Comme x est d'ordre maximum, le sous-groupe de G engendré par x, noté (x), est un facteur direct de G. Soit alors φ un homomorphisme de G dans $\mathbb{Q}/\mathbb{Z}$ obtenu en projetant d'abord G sur (x), puis en envoyant x sur p^{-r}.

Comme μ est unimodulaire, il existe un élément $y \in G$ tel que :

$$\varphi(z) = \mu(z,y) \text{ pour tout } z \in G.$$

En particulier $\mu(x,y) = \varphi(x) = p^{-r}$.

fin de l'affirmation.

<u>Remarque</u> : y est d'ordre exactement p^r. En effet, comme $\mu(x,y) = p^{-r}$, l'ordre de y est un multiple de p^r. Comme $G = G_p$, l'ordre de y est une puissance de p. Par maximalité, p^r est alors exactement l'ordre de y.

<u>Affirmation</u> : $(x) \cap (y) = \{0\}$.

<u>Preuve de l'affirmation</u> : Raisonnons par l'absurde en supposant qu'il existe deux entiers u et v, tels que $0 < u < p^r$ et $0 < v < p^r$ avec $ux = vy$.

Alors : $0 = \mu(x,x) = u\mu(x,x) = \mu(x,ux) = \mu(x,vy) = v\mu(x,y) = vp^{-r} \neq 0$ dans $\mathbb{Q}/\mathbb{Z}$.

Contradiction.

Par conséquent, le sous-groupe (x,y) engendré par x et y dans G est isomorphe à $\mathbb{Z}/p^r \oplus \mathbb{Z}/p^r$. Manifestement, c'est un double direct!

<u>Affirmation</u> : La restriction de μ au sous-groupe (x,y) est unimodulaire.

<u>Preuve de l'affirmation</u> : Soit $\varphi : (x,y) \to \mathbb{Q}/\mathbb{Z}$ un homomorphisme. Pour fixer les idées, supposons que $\varphi(x) = ap^{-u}$, $\varphi(y) = bp^{-v}$ pour certains entiers u et v tels que $0 \leq u \leq r$, $0 \leq v \leq r$; a et b étant premiers à p.

Considérons l'élément $z \in (x,y)$ défini par :
$z = ap^{r-u}y - bp^{r-v}x$. Calculons :

$$\mu(x,z) = \mu(x, ap^{r-u}y - bp^{r-v}x) = ap^{r-u}\mu(x,y) = ap^{r-u}p^{-r} = ap^{-u} .$$

De façon analogue $\mu(y,z) = bp^{-v}$.

$\mu(-,z)$ est donc un homomorphisme de (x,y) dans $\mathbb{Q}/\mathbb{Z}$ qui coïncide avec φ sur x et sur y. Ces deux homomorphismes sont donc égaux.

Ceci implique que :

$$\mathrm{Ad}\mu \,|\, (x,y) \to \mathrm{Hom}((x,y);\mathbb{Q}/\mathbb{Z})$$

est surjective. Elle est donc bijective, puisque (x,y) est un groupe fini.

fin de l'affirmation.

La démonstration de la proposition 1 s'achève alors par une récurrence facile, en utilisant le lemme suivant, qui est un classique des formes ± symétriques.

<u>Lemme classique</u> : Soit Γ un groupe abélien fini et soit $\mu : \Gamma \times \Gamma \to \mathbb{Q}/\mathbb{Z}$ une forme bilinéaire, ± symétrique, unimodulaire. Soit $H \subset \Gamma$ un sous-groupe tel que $\mu | H \times H \to \mathbb{Q}/\mathbb{Z}$ soit encore unimodulaire. Alors : $G = H \oplus H^{\perp}$ (la somme étant orthogonale par construction) et $\mu | H^{\perp} \times H^{\perp}$ est encore unimodulaire.

<u>Preuve du Lemme classique</u> : Bien sûr, $H^{\perp}$ est l'ensemble des $x \in \Gamma$ tels que $\mu(x,y) = 0$, pour tout $y \in H$. Considérons alors $\phi = \mu | \Gamma \times H$. L'adjointe de ϕ donne un homomorphisme :

$\mathrm{Ad}\phi : \Gamma \to \mathrm{Hom}(H,\mathbb{Q}/\mathbb{Z})$ dont le noyau est, par définition, exactement $H^{\perp}$.

De plus, comme $\mu | H \times H$ est unimodulaire, la restriction $\mathrm{Ad}\phi | H$ est un isomorphisme de H sur $\mathrm{Hom}(H,\mathbb{Q}/\mathbb{Z})$. Nous avons donc une suite exacte :

$$0 \longrightarrow H^{\perp} \longrightarrow \Gamma \xrightarrow{\quad \mathrm{Ad}\phi \quad} \mathrm{Hom}(H;\mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

et cette suite exacte est scindée. D'où :

$$\Gamma = H \oplus H^{\perp} .$$

Reste à voir que $\mu|H^{\perp} \times H^{\perp}$ est unimodulaire. Mais ceci est automatique, car on a un isomorphisme

$$\Gamma = H \oplus H^{\perp} \xrightarrow{\mathrm{Ad}\mu} \mathrm{Hom}(H \oplus H^{\perp};\mathbb{Q}/\mathbb{Z}) \ .$$

Ce dernier groupe est isomorphe à :

$\mathrm{Hom}(H;\mathbb{Q}/\mathbb{Z}) \oplus \mathrm{Hom}(H^{\perp};\mathbb{Q}/\mathbb{Z})$ est comme la décomposition $\Gamma = H \oplus H^{\perp}$ est orthogonale, $\mathrm{Ad}\mu$ respecte les facteurs. La restriction de Ad induit donc nécessairement un isomorphisme :

$$H^{\perp} \longrightarrow \mathrm{Hom}(H^{\perp};\mathbb{Q}/\mathbb{Z}) \ .$$

fin du lemme classique.

Note : La démonstration de la proposition 1 montre aussi que, si G est un double direct, il peut être muni d'une unique (à isométrie près) forme alternée unimodulaire.

3. Démonstration du théorème de A. Plans .

La proposition suivante nous ramène à démontrer le théorème de Plans pour le sous-groupe de torsion de $H_1(\hat{X}^m_K;\mathbb{Z})$, ce qui est la partie vraiment intéressante de la démonstration.

Proposition : Le rang sur $\mathbb{Z}$ de $H_1(\hat{X}^m_K;\mathbb{Z})$ est toujours pair.

Il y a plusieurs preuves de cette proposition classique. Voir, par exemple, [4].

Considérons maintenant la variété $\hat{X}^m_K$. Elle est de dimension trois, close et orientable (et même orientée par une orientation de

Σ, via la projection de revêtement).

Soit T_m le sous-groupe de torsion de $H_1(\hat{X}_K^m;\mathbb{Z})$. La dualité de Poincaré (via les enlacements) munit T_m d'une forme bilinéaire, symétrique, unimodulaire :

$$<,> \; : T_m \times T_m \to \mathbb{Q}/\mathbb{Z} \; .$$

D'autre part, $\hat{X}_K^m$ est munie d'un automorphisme de revêtement t, générateur du groupe de Galois. Comme t conserve l'orientation, t agit par isométries sur $<,>$.

<u>Définition (J. Levine)</u> : Soit $[,] \; : T_m \times T_m \to \mathbb{Q}/\mathbb{Z}$ définie par $[\alpha,\beta] = <(t-t^{-1})\alpha,\beta>$.

<u>Propriétés de [,]</u> :

1. [,] est $\mathbb{Z}$-bilinéaire. C'est évident, puisque $<,>$ l'est.
2. [,] est alternée. En effet :
   $[\gamma,\gamma] = <(t-t^{-1})\gamma,\gamma> = <t\gamma,\gamma> - <t^{-1}\gamma,\gamma> = <t\gamma,\gamma> - <\gamma,t\gamma> = 0$ puisque $<,>$ est symétrique.
3. [,] n'est pas nécessairement unimodulaire. Mais on a :

<u>Lemme 3</u> : Le radical de [,] est égal au noyau de (la multiplication par) $1-t^2$.

<u>Preuve du lemme 3</u> : Par construction, l'adjointe de [,] est la composée de la multiplication par $1-t^2$, suivie de la multiplication par t^{-1} et enfin de l'adjointe de $<,>$. Comme les deux dernières applications sont des isomorphismes, le noyau de l'adjointe de [,] est égal au noyau de $(1-t^2)$.

fin du lemme 3.

<u>Proposition 4</u> :

1^o Si m est impair, $\mathrm{Ker}(1-t^2) = 0$.

2^o Si m est pair, $T_m/\mathrm{Ker}(1-t^2)$ est isomorphe au noyau de $\pi_*|T_m \to H_1(\hat{X}^2_k,\mathbb{Z})$.

<u>Preuve de la proposition 4</u> : La suite exacte de J. Milnor pour le revêtement infini cyclique $X^\infty_K \to X_K$ fournit une suite exacte :

$$H_1(X^\infty_K;\mathbb{Z}) \xrightarrow{1-t^m} H_1(X^\infty_K;\mathbb{Z}) \to H_1(\hat{X}^m_K;\mathbb{Z}) \to 0$$

Cf. [4] et [7] pour plus de détails.

Ici, X^m_K désigne le revêtement cyclique à m feuilles non ramifié, et $\hat{X}^m_K$ le revêtement ramifié.

La dernière application de la suite exacte est essentiellement induite par la projection de revêtement. D'où une identification :

$$(*) \qquad H_1(X^\infty_K;\mathbb{Z})/\mathrm{Im}(1-t^m) \underset{\approx}{\longrightarrow} H_1(\hat{X}^m_K;\mathbb{Z})$$

Par conséquent, si $k|m$, on a une suite exacte analogue :

$$H_1(\hat{X}^m_K;\mathbb{Z}) \xrightarrow{1-t^k} H_1(\hat{X}^m_K;\mathbb{Z}) \to H_1(\hat{X}^k_K;\mathbb{Z}) \longrightarrow 0$$

et donc une suite exacte courte :

$$0 \longrightarrow H_1(\hat{X}^m_K;\mathbb{Z})/\mathrm{Ker}(1-t^k) \xrightarrow{1-t^k} H_1(\hat{X}^m_K;\mathbb{Z}) \longrightarrow H_1(\hat{X}^k_K;\mathbb{Z}) \longrightarrow 0 .$$

Posons $k = 2$ et prenons les intersections avec T_m. Nous obtenons exactement l'affirmation 2^o du théorème. Remarquer que nous n'affirmons pas que $T_m \to H_1(\hat{X}^2_K;\mathbb{Z})$ est surjective. Nous verrons au § 4 un exemple montrant que ce n'est pas nécessairement le cas.

Maintenant, si m est impair, l'identification (*) montre que la multiplication par $1-t^m$ est nulle sur $H_1(\hat{X}_K^m;\mathbb{Z})$. Supposons par l'absurde qu'il existe un $x \in H_1(\hat{X}_K^m;\mathbb{Z})$, $x \neq 0$, tel que $(1-t^2)x = 0$. On déduirait que, comme m est impair : $tx = x$, ce qui contredirait le fait que la multiplication pour $(1-t)$ est un isomorphisme de $H_1(X_K;\mathbb{Z})$ et donc aussi de $H_1(\hat{X}_K^m;\mathbb{Z})$.

fin de la proposition 4.

Nous pouvons maintenant passer à la <u>démonstration proprement dite du théorème de A. Plans</u> :

Si m est impair, nous savons par la proposition 2 que le rang de $H_1(\hat{X}_K^m;\mathbb{Z})$ est pair et que $[\,,\,]$ munit le sous-groupe de torsion T_m d'une forme alternée et unimodulaire. La proposition 1 achève alors la démonstration dans ce cas.

Si m est pair, on peut raisonner ainsi : Le rang du noyau de π_* est égal au rang de $H_1(\hat{X}_K^m;\mathbb{Z})$ puisque le groupe $H_1(\hat{X}_K^2;\mathbb{Z})$ est fini. Il est donc pair par la proposition 2. Par la proposition 4, la torsion de $\mathrm{Ker}\pi_*$ est isomorphe à $T_m/\mathrm{Ker}(1-t^2)$. Par le lemme 3, $[\,,\,]$ induit sur ce dernier groupe une forme alternée et unimodulaire. La proposition 1 achève à nouveau la démonstration.

<u>Remarques</u> :

1° Une façon un peu différente d'énoncer le théorème pour m pair consisterait à dire que $H_1(\hat{X}_K^m;\mathbb{Z})$ est un groupe de rang pair tel que $T_m \cap \mathrm{Ker}\pi_*$ est un double direct.

2^o Pour m pair, le théorème de A. Plans implique que la p-torsion de $H_1(\hat{X}^m_K;\mathbb{Z})$ est un double direct pour tout premier p qui ne divise pas $A(-1)$, puisque $A(-1)$ est, en valeur absolue, l'ordre de $H_1(\hat{X}^2_k;\mathbb{Z})$. Ici, $A(t)$ désigne le polynôme d'Alexander du noeud K.

Note : Dans ce paragraphe, nous avons utilisé à plusieurs reprises la "suite exacte de Milnor" [7] et les conséquences qu'on peut en tirer. Elles s'appliquent ici, puisque le complémentaire du noeud K dans Σ a l'homologie entière d'un cercle, à cause de la dualité d'Alexander. C'est ici qu'intervient le fait que Σ est une sphère d'homologie sur $\mathbb{Z}$ et l'on n'a pas besoin d'hypothèses plus fortes que celle-là.

4. Le cas des revêtements d'ordre pair

Ce paragraphe est consacré à quelques remarques et exemples concernant le cas m pair.

1. Pour m pair, le théorème énoncé par A. Plans n'est pas tout à fait équivalent à l'énoncé que nous avons donné ici. De fait, son énoncé est un peu plus faible. Voici quelques détails :

Un argument dû à H. Seifert [11], fournit pour un noeud K de genre h, une matrice de présentation F_m pour $H_1(\hat{X}^m_K;\mathbb{Z})$. A. Plans montre après des calculs très compliqués que, pour m pair, F_m est équivalente, comme matrice de présentation, à un produit matriciel $F_2 \cdot W$, où W est une matrice de présentation d'un double direct.

On déduit de là, comme le fait R. Hartley dans [5] qu'il existe un homomorphisme surjectif :

$$H_1(\hat{X}_K^m;\mathbb{Z}) \to H_1(\hat{X}_K^2;\mathbb{Z})$$

dont le noyau est un double direct.

Notre énoncé est un peu plus précis puisqu'il montre qu'on peut prendre pour cet homomorphisme surjectif celui induit par la projection $\hat{X}_K^m \to \hat{X}_K^2$. Réciproquement, il est très facile de voir que notre énoncé implique celui donné par A. Plans.

2. Dans l'énoncé de A. Plans que nous venons de rappeler, on ne peut pas manipuler la matrice F_2 à volonté. En principe, il s'agit de la matrice de présentation donnée par H. Seifert et on ne peut pas la remplacer arbitrairement par n'importe quelle matrice diagonale de présentation. Nous allons illustrer cette remarque par plusieurs exemples.

2A. Soit m un entier pair. Nous savons que la projection $\hat{X}_K^m \to \hat{X}_K^2$ induit un homomorphisme surjectif $H_1(\hat{X}_K^m;\mathbb{Z}) \to H_1(\hat{X}_K^2;\mathbb{Z})$. Mais, en général, cet homomorphisme π_* n'est plus surjectif si on se restreint au sous-groupe de torsion T_m. (En fait, les deux cas peuvent se produire, comme les exemples ci-dessous le montrent.)

Ceci a pour conséquence que $H_1(\hat{X}_K^2;\mathbb{Z})$ peut très bien être un double direct sans que $H_1(\hat{X}_K^m;\mathbb{Z})$ le soit.

<u>Exemples</u> : Soit K le noeud de trèfle.

On a : $H_1(\hat{X}_K^2;\mathbb{Z}) \approx \mathbb{Z}/3$

$$H_1(\hat{X}_K^{12};\mathbb{Z}) \approx \mathbb{Z} \oplus \mathbb{Z} \ .$$

(De fait, déjà $H_1(\hat{X}^6_K;\mathbb{Z}) \approx \mathbb{Z}\oplus\mathbb{Z}$. Comme le sous-groupe de torsion est nul, la restriction de π_* au sous-groupe de torsion ne peut pas être surjective. Nous allons construire des exemples où le sous-groupe de torsion est non trivial.)

Soit L un noeud ayant pour module d'Alexander :

$$\mathbb{Z}T/1-t^2+t^4 \quad .$$

On a :

$$H_1(\hat{X}^2_L;\mathbb{Z}) = 0 \quad \text{et} \quad H_1(\hat{X}^{12}_L;\mathbb{Z}) \approx \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \ .$$

Soit N un noeud ayant pour module d'Alexander :

$$\mathbb{Z}T/2t^4+t^3-5t^2+t+2 \quad .$$

On a :

$H_1(\hat{X}^2_N;\mathbb{Z}) \approx \mathbb{Z}/3$ et $H_1(\hat{X}^{12}_N;\mathbb{Z})$ est un groupe de torsion. Il ne peut donc être un double direct à cause du théorème de A. Plans.

Considérons maintenant la somme connexe $P = K \# N$. La restriction de $\pi_* : H_1(\hat{X}^{12}_P;\mathbb{Z}) \to H_1(\hat{X}^2_P;\mathbb{Z})$ au sous-groupe de torsion n'est pas surjective.

$H_1(\hat{X}^2_P;\mathbb{Z})$ est un double direct, tandis que

$H_1(\hat{X}^{12}_P;\mathbb{Z})$ ne l'est pas.

La somme connexe $Q = L \# N$ est un exemple où $H_1(\hat{X}^{12}_Q;\mathbb{Z})$ n'est pas de torsion et où la restriction de π_* au sous-groupe de torsion est encore surjective.

2B. Il se peut que $H_1(\hat{X}_K^2;\mathbb{Z})$ soit un double direct et que $H_1(\hat{X}_K^m;\mathbb{Z})$ soit de torsion sans être un double direct; de sorte que la présence de facteurs libres n'est pas seule responsable des phénomènes rencontrés dans 2A. Voici un exemple :

Soit U un noeud dont le module d'Alexander est :

$\mathbb{Z}T/(2t^2-5t+2)^2$.

On a

$H_1(\hat{X}_U^2;\mathbb{Z}) \approx \mathbb{Z}/81$ et

$H_1(\hat{X}_U^6;\mathbb{Z}) \approx \mathbb{Z}/81 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9$

Soit V un noeud dont le module d'Alexander est :

$\mathbb{Z}T/20t^2-41t+20$.

On a :

$H_1(\hat{X}_V^2;\mathbb{Z}) \approx \mathbb{Z}/81$ et

$H_1(\hat{X}_V^6;\mathbb{Z}) \approx \mathbb{Z}/243 \oplus \mathbb{Z}/3$.

Considérons alors $U \# V = Y$. On voit que :

$H_1(\hat{X}_Y^2;\mathbb{Z}) \approx \mathbb{Z}/81 \oplus \mathbb{Z}/81$ est un double direct tandis que $H_1(\hat{X}_Y^6;\mathbb{Z})$ ne l'est pas.

<u>Conclusion</u> : Ces calculs montrent que, pour avoir une certaine intuition de la situation, il faut aller au-delà des noeuds de genre 1 et des calculs donnés par R.H. Fox dans [2]. Ils montrent aussi qu'il ne faut pas faire dire au théorème de A. Plans davantage que ce qu'il dit.

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CONNECTIONS BETWEEN YONEDA AND PONTRJAGIN ALGEBRAS

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Summary

The relationship between Yoneda Ext algebras of local rings and the homology rings of loop spaces on simply connected CW complexes has been observed by several authors [7] [8] [11] [16]. Most of the work has been done over characteristic zero. In this paper we will use the Adams-Hilton construction [2] to understand this connection over arbitrary characteristics.

Fourconsequences of the resulting theory are especially noteworthy. The concept of formal spaces is generalized to non-zero characteristics. The Eilenberg-Moore spectral sequence for the homology of the loop space has $E_2 \approx E_\infty$ as algebras for many spaces, and as algebras "up to sign" for others. We compute the Poincaré series and Pontrjagin structures for the loop space on a Γ-wedge (to be defined) of suspensions. Finally, we observe that all Ext algebras of commutative monomial k-algebras occur as the Pontrjagin rings of loop spaces.

Introduction

Throughout this paper, a CW complex will be assumed to be locally finite, 1-connected, and to have 1-skeleton equal to the base point. In the well-known work [2] Adams and Hilton described a (multi-valued) functor $A(\cdot)$ from CW complexes to chain algebras, such that $H_*(A(X),d_X) \approx H_*(\Omega X)$ as graded rings. This may also be done with coefficients in any commutative ring with unity S, so as to obtain a chain algebra with homology isomorphic to $H_*(\Omega X;S)$.

Fix a CW complex X. As X is taken to be locally finite, we may label its cells as e_o(= base point), $e_1, e_2, \ldots$, where $2 \le |e_i| \le |e_{i+1}|$ for $i \ge 1$.

The algebra $A(X)$ is the free associative algebra over S with generators $\{a_1,a_2,\dots\}$, where $|a_i| = |e_i| - 1$. The differential $d_X\colon A(X) \to A(X)$ is defined on generators in accordance with the way cells are attached (see [2]) and has degree -1; there is some flexibility in the choice of d_X. Precisely, the indeterminacy for $d_X(a_i)$ is generated by the d_X-images of products of two or more generators of lower dimensions. For $Y \subseteq X$ a subcomplex, d_X may be chosen so that it is an extension of any suitable d_Y, under the natural embedding $A(Y) \hookrightarrow A(X)$. If $\{Y_j\} \subseteq X$ are subcomplexes satisfying

$d_{Y_{j_1}}|A(Y_{j_1} \cap Y_{j_2}) = d_{Y_{j_2}}|A(Y_{j_1} \cap Y_{j_2})$ for all j_1 and j_2, then d_X can be chosen

so as to simultaneously extend each d_{Y_j}. These properties motivate our unusually heavy reliance on specific CW structures for spaces, when we are ultimately seeking results which depend only on homotopy types.

We shall omit the subscript X on d_X when no confusion can result.

p-Minimal Complexes

In general, a differential d on a locally finite free S-algebra $A = S\langle a_1,a_2,\dots\rangle$ is determined by constants $c_i, c_{ij}, c_{ij\ell},\dots$, almost all being zero for each i, where

$$d(a_i) = c_i + \sum_j c_{ij}a_j + \sum_{j,\ell} c_{ij\ell}a_ja_\ell + \dots .$$

In our case, dimensional constraints imply that each $c_i = 0$. We shall find that our analysis of $H_*(A,d)$ is considerably simplified when the linear part is also zero, i.e., all the c_{ij}'s are zero. The next two lemmas prove that when S is a field, we may always choose a cell structure for X so as to make the linear part of d vanish.

Let p denote any prime (or zero), let $\mathbb{Z}_p$ be the integers mod p (or Q), and let k be any field containing $\mathbb{Z}_p$.

Def A CW complex X is p-minimal iff, in each dimension $r \geq 0$, its cells are in one-to-one correspondence with a basis for $H_r(X;\mathbb{Z}_p)$.

The universal coefficient theorem shows that $\mathbb{Z}_p$ could be replaced by k in this definition. Henceforth, whenever we are dealing with a p-minimal space X, it will be assumed unless otherwise stated that $A(X)$ is constructed with coefficients in k.

Lemma 1. Let X be any CW complex. The linear part of the differential d on $A(X)$ is zero if and only if X is p-minimal.

Proof: Let $A^1(X) = \mathrm{Span}(a_1, a_2, \dots)$ and let $d_1 : A^1(X) \to A^1(X)$ be $d_1(a_i) = \sum_j c_{ij} a_j$; then $d_1^2 = 0$. Borrowing notation from [2], it is easy to see that $(A^1(X), d_1)$ is chain isomorphic in positive dimensions by an isomorphism of degree +1 with the complex $(B(X), \pi d)$, which is a free k-module with basis corresponding to all the cells of X. The latter has homology isomorphic with $H_*(X;k)$. The cells of X correspond to a basis for $H_*(X;k)$ if and only if $\pi d = 0$, which in turn occurs if and only if $d_1 = 0$.

Lemma 2. Let X be any CW complex. There is a p-minimal CW complex Y and a p-homotopy equivalence $f: Y \to X$.

Proof: Only the outline of a proof is given, since it is a straightforward exercise in working with homotopy and homology groups. Let $R = \mathbb{Z}_{(p)}$, the integers localized at (p), and let $\pi_n(\cdot)$ denote the n^{th} homotopy group of a space or pair, tensored with R. $h: \pi_n(\cdot) \to H_n(\cdot\,; R)$ denotes the Hurewicz homomorphism.

Take $Y^1 = (*)$ and inductively construct complexes $Y^1, Y^2, Y^3, \dots$, together with maps $f_r : Y^r \to X$, satisfying all of the following properties: (1) $Y^r \subseteq Y^{r+1}$ as a subcomplex; (2) $f_r = f_{r+1}|_{Y_r}$; (3) $\dim(Y^r) \leq r+1$; (4) $f_{r\#} : \pi_m(Y^r) \to \pi_m(X)$ is an isomorphism for $m \leq r$; (5) Y^r is p-minimal; and (6) $h(\pi_{r+1}(Y^r)) \subseteq p \cdot H_{r+1}(Y^r; R)$. To obtain Y^{r+1} from Y^r, first one wedges on the minimum number of S^{r+1}'s needed so that f_r can be extended to a map f_r' inducing a surjection of π_{r+1}. Then one attaches a minimum number of r+2-cells, obtaining Y^{r+1}, so that f_r' extended over Y^{r+1} induces an isomorphism of π_{r+1}.

Letting $Y = \underrightarrow{\lim}(Y^r)$, $f = \underrightarrow{\lim}(f_r)$, one obtains a p-minimal space Y and $f\colon Y \to X$ having $f_{\#}\colon \pi_m(Y) \xrightarrow{\approx} \pi_m(X)$ for all $m \geq 0$.

Lemma 3. Let X be a p-minimal CW complex, and let Y be any subcomplex. Then Y is p-minimal, and the inclusion $i\colon Y \to X$ induces a monomorphism of homology and an epimorphism of cohomology with coefficients in k .

Proof: Because d_X may be taken to extend d_Y , Y is p-minimal if X is. $i\colon Y \hookrightarrow X$ is a cellular map, which induces an injection on the cellular chain complexes with coefficients in k for Y and X . The cellular chain complexes have trivial differentials, so this induces an injection of $H_*(\cdot;k)$. It follows that i^* is a surjection of $H^*(\cdot;k)$.

p-Quadratic Complexes

When $(A(X),d)$ is the Adams-Hilton algebra for X and the linear part of d vanishes, the quadratic part reflects the cup coproduct on X . Precisely, fix p and k with $\mathrm{char}(k) = p$ as before and note that the diagonal $\Delta\colon X \to X \times X$ induces a map $\Delta_*\colon H_*(X;k) \to H_*(X \times X;k) \xrightarrow{\approx} H_*(X;k) \otimes H_*(X;k)$. For a p-minimal X , there is a natural basis $\{b_0, b_1, b_2, \ldots\}$ for $H_*(X;k)$ in correspondence with the cells of X and consistent with $s(a_i) = b_i$ in the notation of [2]. This basis determines constants $\gamma_{ij\ell} \in k$ by the formula $\Delta_*(b_i) = b_i \otimes b_0 + b_0 \otimes b_i + \sum_{j,\ell>0} \gamma_{ij\ell}\, b_j \otimes b_\ell$. On the other hand, $d(a_i) = \sum_{j,\ell>0} c_{ij\ell} a_j a_\ell$ + (cubic and higher terms). We will show that $c_{ij\ell} = (-1)^{|b_j|}\gamma_{ij\ell}$.

Given CW complexes X with cells $\{e_i\}_{i\geq 0}$ and Y with cells $\{e'_j\}_{j\geq 0}$, there is a natural decomposition of $X \times Y$ into $\{e''_{ij}\}_{i,j\geq 0}$, where $e''_{ij} = e_i \times e'_j$. We denote the generator of $A(X \times Y)$ corresponding to e''_{ij} by a''_{ij} for $(i,j) \neq (0,0)$, and the corresponding basis element of $B(X \times Y)$ by b''_{ij} . The embedding $X = X \times e'_0 \hookrightarrow X \times Y$ sends a_i to a''_{i0} in the respective Adams-Hilton algebras; likewise $Y = e_0 \times Y \hookrightarrow X \times Y$ has a'_j going to a''_{0j}. When X and Y are p-minimal, the Kunneth formula shows that $X \times Y$ is p-minimal.

In this case the isomorphism $\theta\colon H_*(B(\cdot),\pi d) \to H_*(\cdot;k)$ has $\theta(b''_{ij}) = \theta(b_i)\otimes\theta(b'_j)$.

Lemma 4. Suppose X and Y are p-minimal CW complexes. Let I be the two-sided ideal of $A(X\times Y)$ generated by all cubic and higher monomials, as well as by all quadratic monomials not of the form $a''_{i0}a''_{0j}$. Then $d(a''_{st}) \in (-1)^{|e_s|}a''_{s0}a''_{0t} + I$.

Proof: That the term $a''_{s0}a''_{0t}$ must be present, with the stated sign, follows from the universal example [2, p.320]. To see that there are no others, let $m = |e_s|$ and $n = |e'_t|$ and consider the subcomplexes $X' = X^{m-1} \cup e_s$, $Y' = Y^{n-1} \cup e'_t$, superscripts denoting skeleta here. $X'\times Y'$ is a subcomplex of $X\times Y$, so $d_{X\times Y}(a''_{st})$ must be a valid choice for $d_{X'\times Y'}(a''_{st})$, up to an indeterminacy which lies in I by p-minimality. In $A(X'\times Y')$, however, $a''_{s0}a''_{0t}$ is the only term of the form $a''_{i0}a''_{0j}$ of dimension m+n-2. So no other $a''_{i0}a''_{0j}$ may contribute to $d_{X\times Y}(a''_{st})$.

Lemma 5 (cf [5]). Let X be p-minimal, and let $c_{ij\ell}$, $\gamma_{ij\ell}$ be defined as above. Then $c_{ij\ell} = (-1)^{|e_j|}\gamma_{ij\ell}$.

Proof: First observe that if $\Delta\colon X \to X\times X$ is the diagonal, then the induced map $\Delta_*\colon A(X) \to A(X\times X)$ has $\Delta_*(a_i) \in a''_{i0} + a''_{0i} + J$, where J is the two-sided ideal of $A(X\times X)$ generated by quadratic terms and all $a''_{j\ell}$ for $j > 0$ and $\ell > 0$. This follows from projecting $\Delta_*(a_i)$ back onto each coordinate. Let K be the cube of the augmentation ideal of $A(X)$ and let I be as in lemma 4. We use the fact that Δ_* is a chain map preserving multiplication.

Say $d(a_i) \in \sum_{j,\ell>0} c_{ij\ell}\, a_j a_\ell + K$ and $\Delta_*(b_i) = b''_{i0} + b''_{0i} + \sum_{j,\ell>0}\gamma_{ij\ell}b''_{j\ell}$.

$\Delta_* d(a_i) \in \Sigma\, c_{ij\ell}\,\Delta_*(a_j)\Delta_*(a_\ell) + \Delta_*(K) \subseteq \Sigma\, c_{ij\ell}(a''_{j0}+a''_{0j})(a''_{\ell 0}+a''_{0\ell}) + I$, or $\Delta_* d(a_i) \equiv \Sigma\, c_{ij\ell}\, a''_{j0}a''_{0\ell} \pmod{I}$. To evaluate $\Delta_*(a_i)$, we use the map s defined in [2]. Consider $s\Delta_*(a_i) = \Delta_* s(a_i) = \Delta_*(b_i) = b''_{i0} + b''_{0i} + \sum_{j,\ell>0}\gamma_{ij\ell}\, b''_{j\ell} =$

$= s(a''_{i0}) + s(a''_{0i}) + \sum_{j,\ell>0}\gamma_{ij\ell}\, s(a''_{j\ell})$. Since $\ker(s)$ is non-zero only in dimension zero, this gives $\Delta_*(a_i) = a''_{i0} + a''_{0i} + \sum_{j,\ell>0}\gamma_{ij\ell}\, a''_{j\ell}$ and

$$d\Delta_*(a_i) = d(a''_{i0}) + d(a''_{0i}) + \sum_{j,\ell>0} \gamma_{ij\ell}\, d(a''_{j\ell}) \equiv \sum_{j,\ell>0} \gamma_{ij\ell}(-1)^{|e_j|}\, a''_{j0}a''_{0\ell} \pmod{I},$$

as desired.

In view of lemma 5, any spaces for which d is concentrated entirely in its quadratic part should have properties which depend more heavily than usual on their cohomology ring structures. This is indeed true and motivates the following definition.

Def. Let X be a p-minimal CW complex. X is p-quadratic iff d may be chosen for $A(X)$ so that, for all $i > 0$, $d(a_i) = \Sigma\, c_{ij\ell}\, a_j a_\ell$ for some constants $c_{ij\ell} \in k$.

This definition does not depend upon the choice of the field k so long as $\mathrm{char}(k) = p$, since the $\{c_{ij\ell}\}$ will always lie in the canonical prime subfield of k, and thus there exists a purely quadratic d for $A(X)$ constructed over k if and only if it exists for $A(X)$ constructed over $\mathbb{Z}_p$. When X is p-quadratic, it will always be assumed that d is chosen for $A(X)$ so that $d(a_i) = \Sigma\, c_{ij\ell}\, a_j a_\ell$. By lemma 5, d_X is uniquely determined for X p-quadratic.

The connection with rational homotopy theory may be made at once for $p = 0$. By [5], the Adams-Hilton functor for 0-minimal spaces coincides with Quillen's minimal Lie algebra model [15] over the rationals. Thus our definition of 0-quadratic coincides with the much-studied concept of formal spaces [8] [13] , about which many beautiful results are known. In this way p-quadratic spaces are a natural generalization of formal spaces to non-zero characteristics.

Lemma 6 (cf [8]). Let X be p-quadratic. Then $(A(X),d)$ has a natural structure as a bigraded algebra such that d has bidegree $(+1,-1)$.

Thus $H_*(A(X),d) \simeq H_*(\Omega X;k)$ has the structure of a bigraded algebra.

Proof: Let $A = A(X)$, let $A^1 = \mathrm{Span}(a_1,a_2,\dots) \subseteq A$, and let $A^r = (A^1)^r$. Then $A = \bigoplus_{r\geq 0} A^r$ and $A^r \cdot A^s \subseteq A^{r+s}$. Because X is p-quadratic, $d(A^1) \subseteq A^2$ and then $d(A^r) \subseteq A^{r+1}$ by induction. This gives the first gradation, the second one being the gradation by dimension we have always had. Since d respects both grades, the homology $H_*(A,d)$ is also bigraded. This completes the proof.

The additional gradation is called the "lower gradation" to be consistent with the definition in [8], which coincides with ours over characteristic zero. If $A = A(X)$ with X p-quadratic, the component of A with bidegree (s,t) will be denoted $A_{s,t}$. If W is the (singly) graded k-module $W = A^1 = \bigoplus_{t\geq 0} A_{1,t}$, then $A^s = \bigoplus_{t\geq 0} A_{s,t} \approx \underbrace{W \otimes \cdots \otimes W}_{s}$, and $A(X) \approx TW$, the tensor algebra on W. Recall also that $W_t = A_{1,t} \approx \bar{H}_{t+1}(X;k)$, so W may be viewed as the desuspension of the reduced k-homology of X, and that $d(A_{s,t}) \subseteq A_{s+1,t-1}$.

Fix a p-quadratic space X, let $A = A(X)$, and set $R = H^*(X;k)$. We want to compare the bigraded algebras $\mathrm{Ext}_R^{*,*}(k,k) = \bigoplus_{s,t\geq 0} \mathrm{Ext}_R^{s,t}(k,k)$ and $H_{*,*}(\Omega X;k) = \bigoplus_{s,t\geq 0} H_{s,t}(\Omega X;k)$, where $H_{s,t}(\Omega X;k)$ denotes the component of $H_t(\Omega X;k)$ with lower grade s. Eilenberg and Moore [6] have given a spectral sequence with $E^2_{s,t} = \mathrm{Tor}^R_{s,t}(k,k)$ and $\bigoplus_{t-s=n} E^\infty_{s,t} \approx H^n(\Omega X;k)$. Dualizing this gives a spectral sequence with $E_2^{s,t} = \mathrm{Ext}_R^{s,t}(k,k)$ and $\bigoplus_{t-s} E_\infty^{s,t} \approx H_n(\Omega X;k)$. In this paper, it is the latter spectral sequence we refer to when we speak of the Eilenberg-Moore spectral sequence.

Let $\bar{R}$ be the augmentation ideal of R, $R = \bar{H}^*(X;k)$. There is a natural free R-module resolution of k, called the bar resolution [12]:
$0 \leftarrow k \leftarrow R \xleftarrow{\mu} \bar{R}\otimes R \xleftarrow{\hat{\mu}} \bar{R}\otimes\bar{R}\otimes R \xleftarrow{\hat{\mu}} \cdots$, in which $\mu\colon \bar{R}\times R \to R$ is (cup) multiplication and $\hat{\mu}(x_1 \otimes \cdots \otimes x_s) = \sum_{j=1}^{s-1} (-1)^{j-1} x_1 \otimes \cdots \otimes \mu(x_j \otimes x_{j+1}) \otimes \cdots \otimes x_s$

Taking $\mathrm{Hom}(\cdot,k)$ of this sequence after omitting the term k gives a chain complex whose cohomology is $\mathrm{Ext}_R^{*,*}(k,k)$. It is
$0 \to k \xrightarrow{0} U \xrightarrow{\mu^*} U\otimes U \xrightarrow{\hat{d}} U\otimes U\otimes U \xrightarrow{\hat{d}} \cdots$, where $U = \mathrm{Hom}(\bar{R},k) \approx \bar{H}_*(X;k)$, μ^* is dual to $\mu_{|\bar{R}\otimes\bar{R}}$, and

$$(1)\quad \hat{d}(u_1 \otimes \cdots \otimes u_s) = \sum_{j=1}^{s} (-1)^{j-1} u_1 \otimes \cdots \otimes \mu^*(u_j) \otimes \cdots \otimes u_s .$$

Furthermore, the pairing $(\underbrace{U \otimes \ldots \otimes U}_{s_1}) \otimes (\underbrace{U \otimes \ldots \otimes U}_{s_2}) \to \underbrace{U \otimes \ldots \otimes U}_{s_1+s_2}$ makes this complex into a bigraded cochain algebra and induces the Yoneda multiplication on $\mathrm{Ext}_R^{*,*}(k,k)$ [1]. Let $F_s = \underbrace{U \otimes \ldots \otimes U}_{s}$ and let $F_{s,t}$ be the component of F_s in total degree t, so $\hat{d}(F_{s,t}) \subseteq F_{s+1,t}$.

At this point, observe that $A_{s,t-s}$ and $F_{s,t}$ are isomorphic k-modules for each (s,t), since $A \approx TW$, $F = \bigoplus_s F_s \approx TU$, and W is U desuspended. Is there a chain isomorphism? Indeed there is: letting the basis element b_i of U correspond to the basis element a_i of W, we define $\phi\colon F_{s,t} \to A_{s,t-s}$ by setting

$$\phi(b_{n_1} \otimes \ldots \otimes b_{n_s}) = (-1)^{\sigma(b_{n_1} \otimes \ldots \otimes b_{n_s})} a_{n_1} \ldots a_{n_s}$$

and extending linearly. Here $\sigma(b_{n_1} \otimes \ldots \otimes b_{n_s}) = \sum_{s-j \text{ odd}} |b_{n_j}|$, the sum equaling $|b_{n_1}| + |b_{n_3}| + \ldots + |b_{n_{s-1}}|$ for s even and equaling $|b_{n_2}| + |b_{n_4}| + \ldots + |b_{n_{s-1}}|$ for s odd. To prove that this is a chain isomorphism, recall first from lemma 5 that

$$d(a_{n_i}) = \sum_{j,\ell} (-1)^{|b_j|} \gamma_{n_i j \ell}\, a_j a_\ell \text{, and}$$

$$d(a_{n_1} \ldots a_{n_s}) = \sum_{i=1}^{s} (-1)^{|a_{n_1} \ldots a_{n_{i-1}}|} a_{n_1} \ldots d(a_{n_i}) \ldots a_{n_s} ,$$

while

$$\hat{d}(b_{n_i}) = \sum_{j,\ell} \gamma_{n_i j \ell}\, b_j \otimes b_\ell \quad \text{and}$$

$\hat{d}(b_{n_1} \otimes \ldots \otimes b_{n_s})$ is given by formula (1).

Then if $b = b_{n_1} \otimes \ldots \otimes b_{n_s}$,

$$d\phi(b) = (-1)^{\sigma(b)} \sum_{i=1}^{s} (-1)^{|a_{n_1} \ldots a_{n_{i-1}}|} \sum_{j,\ell} (-1)^{|b_j|} \gamma_{n_i j \ell}\, a_{n_1} \ldots a_j a_\ell \ldots a_{n_s} .$$

The sign on the term $\gamma_{n_i j \ell}\, a_{n_1} \ldots a_j a_\ell \ldots a_{n_s}$ has the parity of

$$\sigma(b) + |a_{n_1}| + \ldots + |a_{n_{i-1}}| + |b_j| = \sum_{\substack{s-m \\ \text{odd}}} |b_{n_m}| + (i-1) + \sum_{m=1}^{i-1} |b_{n_m}| + |b_j| = (i-1) + \sigma(b_{n_1} \otimes \ldots \otimes b_j \otimes b_\ell \otimes \ldots \otimes b_{n_s}) .$$

On the other hand,

$$\phi\hat{d}(b) = \phi\Big(\sum_{i=1}^{s} (-1)^{i-1} \sum_{j,\ell} \gamma_{n_i j\ell}\, b_{n_1} \otimes \dots \otimes b_j \otimes b_\ell \otimes \dots \otimes b_{n_s}\Big) =$$

$$= \sum_{i=1}^{s} \sum_{j,\ell} (-1)^{i-1} (-1)^{\sigma(b_{n_1}\otimes\dots\otimes b_j \otimes b_\ell \otimes \dots \otimes b_{n_s})} \gamma_{n_i j\ell}\, a_{n_1} \dots a_j a_\ell \dots a_{n_s} ,$$

as desired.

Because a chain isomorphism induces an isomorphism on homology, we have at once

Lemma 7. For a p-quadratic space X with cohomology ring $R = H^*(X;k)$, $\mathrm{Ext}_R^{s,t}(k,k)$ is isomorphic with $H_{s,t-s}(\Omega X;k)$ for each bidegree (s,t).

An immediate consequence is

Theorem 1. Let X have the p-homotopy type of a p-quadratic CW complex. Then the Eilenberg-Moore spectral sequence for X degenerates, i.e., $E_2 \approx E_\infty$ as k-modules.

We can go further, however, with the isomorphism $\phi\colon F_{s,t} \xrightarrow{\approx} A_{s,t-s}$. Does this induce an isomorphism of algebras? Perhaps surprisingly, it does not; a simple counterexample follows.

Example. Let $X = S^2 \times S^3$. X is p-quadratic for all p and $R = H^*(X;k)$ is the commuting polynomial ring on two generators whose squares vanish. It follows that $\mathrm{Ext}_R^{*,*}(k,k)$ is a polynomial ring (non-vanishing squares) on two anti-commuting generators β_1 and β_2 of bidegrees $(1,2)$ and $(1,3)$. However, $H_*(\Omega X;k)$ is a commutative polynomial ring on two generators α_1 and α_2 of bidegrees $(1,1)$ and $(1,2)$. The problem is that, although both algebras are commutative with the convention $xy = (-1)^{(\deg x)(\deg y)}yx$ when $\{x,y\}$ is the generating set, **"deg" is understood to mean homological degree (corresponding to lower degree) in the Ext algebra, but total degree for the loop space homology!**

Def. Let $A = \bigoplus_{s,t} A_{s,t}$ and $A' = \bigoplus_{s,t} A'_{s,t}$ be bigraded k-algebras. An isomorphism $f\colon A_{s,t} \to A'_{s,t-s}$ of bigraded vector spaces is called an algebra isomorphism up to sign iff $f(x\,y) = (-1)^{s't} f(x)f(y)$ for $x \in A_{s,t}$ and $y \in A_{s',t'}$.

In our case, if $b = b_{n_1} \otimes \dots \otimes b_{n_s}$ and $b' = b_{n_{s+1}} \dots b_{n_{s+s'}}$,

$$\phi(b\otimes b') = (-1)^{\sigma(b\otimes b')}\, a_{n_1}\cdots a_{n_{s+s'}} = (-1)^{\sigma(b)}(-1)^{\sigma(b')}(-1)^{s't}\, a_{n_1}\cdots a_{n_{s+s'}} =$$

$$= (-1)^{s't}\,\phi(b)\,\phi(b')\ ,$$

where $t = |b_{n_1}| + \dots + |b_{n_s}|$ is the total degree of b .

It is clear that a chain algebra isomorphism up to sign induces an isomorphism of algebras up to sign on homology. We have proved

Theorem 2. Let X have the p-homotopy type of a p-quadratic CW complex. Then for $R = H^*(X;k)$, $\bigoplus_{s,t} \mathrm{Ext}_R^{s,t}(k,k)$ and $\bigoplus_{s,t} H_{s,t-s}(\Omega X;k)$ are isomorphic as algebras up to sign. If $p = 2$ or if $H^*(X;k)$ is concentrated in even degrees only, it is an isomorphism of bigraded algebras.

In applying theorem 2, we must of course be careful to allow for the shift in bidegrees from the Ext algebra to the Pontrjagin ring.

The next two lemmas provide examples of p-quadratic spaces. Still further examples will be offered in the last section.

Lemma 8. Suppose X is a suspension. Then X has the p-homotopy type of a p-quadratic CW complex for each p .

Proof: By lemma 2 we may choose Y , with the p-homotopy type of X , to be p-minimal. $H_*(\Omega X;k) \approx H_*(\Omega Y;k) \approx H_*(A(Y),d_Y)$. $A(Y)$ is isomorphic to a tensor algebra on the desuspended reduced k-homology of X . By [9] $H_*(\Omega X;k)$ is also isomorphic to this tensor algebra. Thus we must have $d_Y = 0$.

Lemma 9 (cf [8]). Suppose there is some $r \geq 1$ such that $H^m(X;k) = 0$ for $m \leq r$ and for $m > 3r+1$. Then X has the p-homotopy type of a p-quadratic CW complex.

Proof: By lemma 2, let Y be p-minimal with the p-homotopy type of X . The non-trivial cells of Y have $r+1 \leq |e_i| \leq 3r+1$, hence $r \leq |a_i| \leq 3r$ and $r-1 \leq d(a_i) \leq 3r-1$. It follows that only the linear and quadratic parts of d may be non-zero. As Y is p-minimal, it must be p-quadratic.

Application to Poincaré Series

Let R be a locally finite commutative graded k-algebra. Define the double Poincaré series of R by

$$P'_R(y,z) = \sum_{s\geq 0}\ \sum_{t\geq 0} \operatorname{rank}(\operatorname{Tor}^R_{s,t}(k,k))y^s z^t .$$

The usual Poincaré series is then $P_R(y) = P'_R(y,1)$. Similarly, if ΩX is the loop space on a p-quadratic complex, its homology is doubly graded by lemma 6 and we may define

$$P'_{\Omega X}(y,z) = \sum_{s\geq 0}\ \sum_{t\geq 0} \operatorname{rank}(H_{s,t}(\Omega X;k))y^s z^t .$$

The usual Poincaré series is then $P_{\Omega X}(z) = P'_{\Omega X}(1,z)$. These two double series are closely related.

<u>Lemma 10</u>. Suppose X has the p-homotopy type of a p-quadratic space. Then if $R = H^*(X;k)$,

$$P'_{\Omega X}(y,z) = P'_R(yz^{-1},z) \ , \quad \text{and}$$

$$P'_{\Omega X}(yz,z) = P'_R(y,z) \ .$$

<u>Proof</u>: These are an immediate consequence of the vector space isomorphism of bigraded modules in lemma 7. Let $h_{s,t} = \operatorname{rank}(\operatorname{Tor}^R_{s,t}(k,k)) = \operatorname{rank}(\operatorname{Ext}_R^{s,t}(k,k)) = \operatorname{rank}(H_{s,t-s}(\Omega X;k))$ and set $h_{s,t} = 0$ for $s < 0$ or $t < 0$. Then

$$P'_R(y,z) = \sum_{s,t} h_{s,t}\, y^s z^t \quad \text{and}$$

$$P'_{\Omega X}(y,z) = \sum_{s,t} h_{s,t}\, y^s z^{t-s} = \sum_{s,t} h_{s,t}(yz^{-1})^s z^t = P'_R(yz^{-1},z) \ .$$

Substituting yz for y gives the other formula.

A certain special case of lemma 10 has been much celebrated. Let X be a CW complex with cells in dimensions two and four only, and suppose that $R = H^*(X;k)$ is generated as a ring in degree two. Then X^2 is a wedge of n S^2's with $H_*(\Omega X^2;k) \approx k\langle\alpha_1,\ldots,\alpha_n\rangle$, and the attaching maps of the four-cells determine certain primitive elements $\beta_1,\ldots,\beta_m \in H_2(\Omega X^2;k)$. Calling I the two-sided ideal in $k\langle\alpha_1,\ldots,\alpha_n\rangle$ generated by $\{\beta_1,\ldots,\beta_m\}$, let G^X be the graded Hopf algebra $k\langle\alpha_1,\ldots,\alpha_n\rangle/I$. By filtering the bar resolution for R , Lemaire obtained in [11] a formula which in our notation gives rise to

(2) $P'_{\Omega X}(y,z) = (1+y^{-1}z)G^X(yz)^{-1} - y^{-1}z(1-nyz+my^2z^2)$,

where $G^X(u) = \sum_{j\geq 0} \mathrm{rank}(G^X_j)u^j$ is the Hilbert series for G^X .

Using purely algebraic methods, Roos obtained in [16] a double Poincaré series formula when R has the property that its maximal ideal cubed vanishes. After replacing z by z^2 to account for R occurring in grades two and four instead of one and two, it is

(3) $P'_R(y,z) = (1+y^{-1})G^R(yz^2)^{-1} - y^{-1}(1-nyz^2+my^2z^4)$,

where G^R is the subalgebra of $\mathrm{Ext}^*_R(k,k)$ generated by $\mathrm{Ext}^1_R(k,k)$. In the situation we are considering here, G^R is easily shown to be isomorphic with G^X.

When $\dim(X) \leq 4$, X is p-quadratic by an application of lemma 9 with $r = 1$. The equivalence between formulas (2) and (3) is thus seen to be a special case of lemma 10.

Products

Theorem 3. Suppose X and Y are p-quadratic CW complexes. Then $X \times Y$ is p-quadratic.

Proof: $X \times Y$ is p-minimal because X and Y are. As before, we let the cells of $X \times Y$ be $\{e''_{ij} = e_i \times e'_j\}$, where $X = \bigcup_{i\geq 0} e_i$ and $Y = \bigcup_{j\geq 0} e'_j$. Let a''_{st} be a generator of $A(X \times Y)$. If $t = 0$ we define $d_{X\times Y}(a''_{s0})$ to be consistent with the identification $X \times e'_0 \simeq X$, likewise if $s = 0$. For $s > 0$ and $t > 0$ let $m = |e_s|$, $n = |e'_t|$, and let $X' = X^{m-1} \cup e_s$, $Y' = Y^{n-1} \cup e'_t$. We will show that if d may be chosen to be quadratic on $A(X' \times Y') - e''_{st})$, then d extends quadratically over $X' \times Y'$. This provides the inductive step in the proof, for if we know d to be quadratic on $A((X\times Y)^r)$ and apply this extension separately to each $r+1$-cell, then the extensions may be pieced together for a d which is valid on all of $A((X \times Y)^{r+1})$. Consequently it suffices to prove that $X \times Y$ is p-quadratic when $X = X^{m-1} \cup e_s$, $Y = Y^{n-1} \cup e'_t$, and $Z = X \times Y - e''_{st}$ is p-quadratic. We now restrict our attention to this situation. Let $R = H^*(X;k)$ and

$S = H^*(Y;k)$, so that $H^*(X\times Y;k) \approx R\otimes S$.

For a CW complex V , let $L(V)$ denote the Pontrjagin ring $H_*(\Omega V;k)$. When V is known to be p-quadratic we take $L(V)$ to be bigraded. For a graded k-algebra P , let $F_{\bar{s},\bar{t}}(P) = \mathrm{Ext}_P^{\bar{s},\bar{t}}(k,k)$ and $F(P) = \bigoplus_{\bar{s},\bar{t}\geq 0} F_{\bar{s},\bar{t}}(P)$, so $F(P)$ is a bigraded algebra. From purely algebraic considerations one sees that $F(R\otimes S) \approx F(R)\otimes F(S)$ as algebras, where multiplication in the bigraded $F(R)\otimes F(S)$ is defined by $(x_1\otimes y_1)\cdot(x_2\otimes y_2) = (-1)^{s_1s_2+t_1t_2}(x_1x_2\otimes y_1y_2)$ if $y_1\in F_{s_1,t_1}(S)$, $x_2\in F_{s_2,t_2}(R)$. As X and Y are p-quadratic, there are isomorphisms of algebras up to sign, $\phi_R:F(R)\to L(X)$, $\phi_S:F(S)\to L(Y)$. Together these define a map $\phi_{R\otimes S}:F(R\otimes S)\to L(X)\otimes L(Y)$ which is also an isomorphism up to sign, the bigrading on $L(X)\otimes L(Y)$ coming from the bigradings on $L(X)$ and $L(Y)$, and the algebra structure on $L(X)\otimes L(Y)$ being chosen so as to be consistent with its natural identification with $L(X\times Y)$.

By dualizing the bar resolution for $R\otimes S$ (the reverse of the process used in proving theorem 2) we obtain a bigraded chain algebra (A',d'), with $A' = A(X\times Y)$, which is the natural candidate for $(A(X\times Y),d)$. (A',d') has the following properties:

(1) $d'(a''_{ij})$ is purely quadratic for each generator $a''_{ij}\in A(X\times Y)$;

(2) $d'|_{A(Z)}$ is the unique quadratic differential which is a suitable choice for d_Z on $A(Z)$;

(3) $H_{*,*}(A',d')$ is isomorphic as a bigraded algebra with $H_{*,*}(\Omega X;\ k)\otimes H_{*,*}(\Omega Y;\ k)$.

We want to be certain that $d'(a''_{st})$ is a suitable choice for $d(a''_{st})$.

Let $\delta = d'(a''_{st})\in A(Z)$. δ is a quadratic cycle in $A(Z)$. If δ were not a boundary in $(A(X\times Y),d)$, then it would represent a sum of products of non-zero homology in $H_*(\Omega X;\ k)$ and $H_*(\Omega Y;\ k)$, since all the homology in $H_*(A(X\times Y),d) \approx H_*(\Omega(X\times Y);k)$ has this form. But then δ would also be non-zero in $H_*(A',d')$, which it is not. So $\delta = d(\tilde{a})$ for some $\tilde{a}\in A(X\times Y)$.

From dimensional considerations we know that $\tilde{a} = c\cdot a''_{st} + \tilde{a}'$, where $c \in k$ and $\tilde{a}' \in A(Z)$. $c = 0$ would contradict δ being quadratic, so $c \neq 0$, and $d(a''_{st}) = c^{-1}d(\tilde{a}) - c^{-1}d(\tilde{a}') = c^{-1}\delta - c^{-1}d(\tilde{a}')$. The flexibility in the choice of d allows us to add elements which are boundaries in $A(Z)$. Thus we may choose d so that $d(a''_{st}) = c^{-1}\delta$.

This proves d to be quadratic. As a final remark, $c = 1$ by lemma 5 and by observation that $d'(a''_{st}) = \delta$ and $d(a''_{st}) = c^{-1}\delta$ must both correspond to the cup coproduct in $X \times Y$.

Theorem 3 greatly increases the class of spaces known to be p-quadratic. We obtain even more examples by taking subcomplexes of products in the most general way. To facilitate this we have the following definition.

Def. Let Γ be a simplicial complex with n vertices labeled $1,\ldots,n$. Let $X_1,\ldots,X_n$ be pointed spaces. The Γ-wedge of $(X_1,\ldots,X_n)$ is the space $X_\Gamma = \{(x_1,\ldots,x_n) \in X_1 \times\ldots\times X_n | \{i| \ x_i \neq *\} \in \Gamma\}$.

When Γ is an n-1 simplex σ^{n-1}, then X_Γ is the product $X_1 \times\ldots\times X_n$. When Γ is n disjoint points, X_Γ is the usual wedge $X_1 \vee\ldots\vee X_n$. When Γ is the j-skeleton of σ^{n-1}, X_Γ is the "generalized fat wedge" considered by Porter [14] , Lemaire [10], and others. The Γ-wedge is therefore a natural generalization of this already generalized concept.

If $X_i \underset{g_i}{\overset{f_i}{\rightleftarrows}} Y_i$ are homotopy equivalences of pointed spaces for $i = 1,\ldots,n$, then X_Γ and Y_Γ are homotopy equivalent. One simply restricts the homotopy between $g \circ f$ and id: $X_1 \times \ldots \times X_n \to X_1 \times \ldots \times X_n$ to $X_\Gamma \times I$, and likewise for Y_Γ .

Since most of our results involve p-homotopy equivalence, we wish to show that Γ-wedges preserve p-homotopy equivalence. Let X_i and Y_i, $i = 1,\ldots,n$, be CW complexes, and suppose $g_i : X_i \to Y_i$ are p-homotopy equivalences. By Whitehead's theorem this is equivalent to $g_{i*}: C_*(X_i) \to C_*(Y_i)$ inducing an isomorphism of homology for each i, where $C_*(\cdot)$ denotes the cellular chain complex with coefficients in $\mathbb{Z}_{(p)}$. As each g_{i_*} induces an isomorphism of homology and cellular chain complexes consist of free $\mathbb{Z}_{(p)}$-modules, there are chain homotopy inverses $\varphi_i\colon C_*(Y_i) \to C_*(X_i)$.

Omitting details, the proof proceeds as follows. Let τ denote the n-1-simplex σ^{n-1} and let Γ be any simplicial subcomplex containing all n vertices. The $\{g_i\}$ define $g_\tau\colon X_\tau \to Y_\tau$ and $g_\Gamma = \bar{g}_{\tau|X_\Gamma}\colon X_\Gamma \to Y_\Gamma$ while $\{\varphi_i\}$ induce $\varphi_\tau\colon C_*(Y_\tau) \to C_*(X_\tau)$ and $\varphi_\Gamma = \varphi_{\tau|C_*(Y_\Gamma)}\colon C_*(Y_\Gamma) \to C_*(X_\Gamma)$. Chain homotopies G_i between $\varphi_i g_{i_*}$ and $(id_{X_i})_*$ induce G_Γ between $\varphi_\Gamma g_{\Gamma_*}$ and $(id_{X_\Gamma})_*$, and likewise for $g_{\Gamma_*}\varphi_\Gamma$. We deduce that $g_\Gamma\colon X_\Gamma \to Y_\Gamma$ induces an isomorphism of cellular homology (coefficients still $\mathbb{Z}_{(p)}$) and hence is a p-homotopy equivalence. We have shown

<u>Lemma 11</u>. Suppose that for $i = 1,\dots,n$, X_i and Y_i are CW complexes with the same p-homotopy type and let Γ be any simplicial complex on n vertices. Then X_Γ and Y_Γ have the same p-homotopy type.

<u>Lemma 12</u>. Let Γ be a simplicial complex on n vertices and let $X_1,\dots,X_n$ be p-quadratic CW complexes. Then X_Γ has a CW decomposition for which it is p-quadratic.

<u>Proof</u>: Let X_i have cells $\{e_0^{(i)} = *, e_1^{(i)}, e_2^{(i)}, \dots\}$. A natural cellular decomposition for X_Γ is $\{e_{m_1}^{(1)} \times e_{m_2}^{(2)} \times \dots \times e_{m_n}^{(n)} \mid \{i \mid m_i \neq 0\} \in \Gamma\}$. With this decomposition X_Γ is a subcomplex of $X_1 \times \dots \times X_n$.

We may write $X = \bigcup_{\sigma\in\Gamma} X_\sigma$, the union being over all faces of Γ. The proof is then by induction on the number of faces of Γ. Let $\bar{\Gamma}$ be obtained by removing from Γ a face σ of maximal dimension, and suppose inductively that $X_{\bar{\Gamma}}$ and $X_{\bar{\Gamma}\cap\sigma}$ are p-quadratic. X_σ, being a product of p-quadratic spaces, is p-quadratic, so the differentials $d_{X_{\bar{\Gamma}}}$ and d_{X_σ} both extend the unique quadratic differential for $X_{\bar{\Gamma}\cap\sigma}$. It follows that $d_{X_{\bar{\Gamma}}}$ and d_{X_σ} together define a quadratic differential which is a valid choice for d_{X_Γ}.

A corollary of lemmas 11 and 12 is that Γ-wedges of locally finite formal spaces are formal.

Theorem 4. Let Γ be a simplicial complex on n vertices, let $SX_1,\dots,SX_n$ be CW complexes which are suspensions, and let X_Γ be the Γ-wedge of $(SX_1,\dots,SX_n)$. Then (a) X_Γ has the p-homotopy type of a p-quadratic complex for any p ; (b) the double Poincaré series of ΩX_Γ using any coefficient field is a rational function of two variables; and (c) the Pontrjagin ring $H_*(\Omega X_\Gamma;k)$ is finitely presented as an algebra for any field k .

Proof: Part (a) is immediate from lemmas 8, 11, and 12. Parts (b) and (c) rely on some theorems of Backelin [3] and of Backelin and Roos [4] on the structure of Ext algebras of monomial rings.

Fix k , let $R_i = H^*(SX_i;k)$ and $R_\Gamma = H^*(X_\Gamma;k)$, and set $R = H^*(SX_1 \times \dots \times SX_n;k) \approx R_1 \otimes \dots \otimes R_n$. Each R_i is finitely generated and has a presentation as $R_i = k[\alpha_{i1},\dots,\alpha_{it_i}]/[\alpha_{ij_1}\alpha_{ij_2}]$, the ideal of relations being generated by all products of two generators. Consequently,
$R = k[\alpha_{ij} | 1 \le i \le n \ , \ 1 \le j \le t_i]/[\alpha_{ij_1}\alpha_{ij_2} | 1 \le i \le n \ , \ 1 \le j_1 \le j_2 \le t_i]$.
By lemma 3, R_Γ is a quotient of R . One sees easily that
$R_\Gamma = R/[\alpha_{i_1 j_1}\cdots\alpha_{i_r j_r} \mid \{i_1,\dots i_r\} \notin \Gamma, \ 1 \le j_\ell \le t_\ell]$. So $R_\Gamma = k[\alpha_{ij}]/[M_1,\dots,M_s]$,
where $M_1,\dots,M_s$ are monomials in the anti-commuting generators $\{\alpha_{ij}\}$. In [3] it is shown that local rings of this form always have rational double Poincaré series, i.e., the series are the expansions about the origin of quotients of polynomials. [3] actually deals only with the commutative case, but the methods of proof are easily generalized to anti-commutative rings as well. Part (b) of the theorem follows from this and lemma 10 and part (a).

In [4] Backelin and Roos observe that $\mathrm{Ext}_S^{*,*}(k,k)$ is finitely presented as an algebra when S is an anti-commutative Noetherian graded k-algebra whose ideal of relations is generated by a set of monomials in the generators. Since $H_{*,*}(\Omega X_\Gamma;k)$ is isomorphic with $\mathrm{Ext}_{R_\Gamma}^{*,*}(k,k)$ as algebras up to sign, it too is finitely presented.

Naturality and Examples

Let $f:X \to Y$ be a map between two p-quadratic CW complexes. Let $R = H^*(X;k)$ and $S = H^*(Y;k)$. $f^*:S \to R$ induces $f^{**}:\mathrm{Ext}_R^{*,*}(k,k) \to \mathrm{Ext}_S^{*,*}(k,k)$. If ϕ_* is the isomorphism of theorem 2, the diagram

$$\begin{array}{ccc} \mathrm{Ext}_R^{*,*}(k,k) & \xrightarrow{f^{**}} & \mathrm{Ext}_S^{*,*}(k,k) \\ \phi_*\downarrow & & \phi_*\downarrow \\ H_*(\Omega X;k) & \xrightarrow{(\Omega f)_*} & H_*(\Omega Y;k) \end{array} \tag{4}$$

in general does not commute. A simple example is the Hopf map $\eta:S^3 \to S^2$, for which $(\Omega\eta)_*$ is injective but η^* and hence η^{**} are zero. However, in a special case we have commutativity.

Lemma 13. Let $f:X \to Y$ be a map between p-quadratic CW complexes. Suppose that the chain map $f_*:(A(X),d_X) \to (A(Y),d_Y)$ induced by f respects the lower gradation, i.e., suppose f_* sends generators of $A(X)$ to linear combinations of generators in $A(Y)$. Then (4) commutes as a diagram of bigraded algebras.

Proof: We always have a diagram

$$\begin{array}{ccc} TR_+ & \xrightarrow{(f^*)^*} & TS_+ \\ \phi\downarrow & & \phi\downarrow \\ T\Sigma^{-1}R_+ & \xrightarrow{f_*} & T\Sigma^{-1}S_+ \end{array} , \tag{5}$$

"Σ^{-1}" denoting desuspension and the vertical maps being chain isomorphisms of algebras. Under the stated hypotheses (5) commutes for generators, hence it commutes. The lemma follows by taking homology of each chain complex.

We conclude with the observation that all monomial rings with degree two generators and all "monomial maps" between them are covered by p-quadratic theory. We wish to consider quotients of polynomial rings whose ideals of relations are generated by monomials, and maps between them whose kernels are also generated by monomials. One way to formalize this is to consider a category whose objects are graded rings, together with specified minimal sets of generators with respect to which the rings are monomial. A simpler formalization, which disallows permutations of generators but still allows any monomial maps up to isomorphism, is given next.

Let C be any (presumably large) set, and think of C as a set of degree-two generators for $k[C]$. Let I,J denote ideals of $k[C]$ generated by monomials in C. Let $\mathcal{M}$ (resp. $\mathcal{M}_0$) be the category whose objects are rings (resp. finitely generated rings) of the form $k[C]/I$. A morphism in $\mathcal{M}$ (resp. $\mathcal{M}_0$) is a composition of any projections $k[C]/I \to k[C]/(I+J)$ and of injections $k[C]/(I+J) \to k[C]/I$ when J is the ideal of $k[C]$ generated by a subset $C_J \subseteq C$.

For the counterpart in topology, let $\{X_\alpha\}$ for $\alpha \in C$ be copies of $\mathbb{C}P^\infty$ with the usual cell structure, and for a finite subset $L \subseteq C$ let $X_L = \prod_{\alpha \in C} Y_\alpha$, where $Y_\alpha = X_\alpha$ if $\alpha \in L$ and $Y_\alpha = *$ if $\alpha \notin L$. For any $P \subseteq C$, $X_P = \varinjlim_{\substack{L \subseteq P \\ L \text{ finite}}} X_L$ is a CW complex which comes with a natural projection $\pi_P : X_C \to X_P$. Let the category $\mathcal{C}$ (resp. $\mathcal{C}_0$) have as objects all subcomplexes (resp. all locally finite subcomplexes) of X_C. A morphism in $\mathcal{C}$ (or $\mathcal{C}_0$) is a composition of subcomplex inclusions among the objects and projections $\pi_P|_Y : Y \to Y \cap X_P$ for $P \subseteq C$.

To define a functor $G: \mathcal{M}, \mathcal{M}_0 \to \mathcal{C}, \mathcal{C}_0$, consider the free abelian monoid $M = M(C)$ on the set C. Elements of M may be denoted as sums $\Sigma n_i \alpha_i$, where $n_i \geq 0$, $\alpha_i \in C$, and only finitely many n_i's are non-zero. $k[C]$ is the monoid algebra over M, i.e., there is a natural monoid homomorphism $g: M \to k[C]$ whose image is a k-basis. For the topology, when the 2j-cell of X_α is identified with $j \cdot \alpha$ and products of cells are identified with sums in M, we obtain a bijection between M and the cells of X_C. For $B \subseteq M$ a monoid ideal (this means that $\beta + \gamma \in B$ whenever $\beta \in B$ and $\gamma \in M$), let $R_B = k[C]/[g(B)]$ and let $X_{(B)} = \cup\{\text{cells corresponding to } M - B\}$. As B runs through all proper monoid ideals of M (resp. proper monoid ideals containing almost all $1 \cdot \alpha$ for $\alpha \in C$), $\{R_B\}$ runs through $\operatorname{Ob} \mathcal{M}$ (resp. $\operatorname{Ob} \mathcal{M}_0$)

and $\{X_{(B)}\}$ runs through Ob C(resp. Ob C_0). [Note that we must include the empty ideal as one possibility for B.] We may define $G:M,M_0 \to C,C_0$ by $G(R_B) = X_{(B)}$; it is clear that $f:R_{B_1} \to R_{B_2}$ corresponds to a unique $G(f):X_{(B_2)} \to X_{(B_1)}$ for $f \in \operatorname{Hom} M$ (resp. $\operatorname{Hom} M_0$) such that G is a contravariant functor.

<u>Theorem 5</u>. The functors $G:M,M_0 \to C,C_0$ and $H = H^*(\cdot\,;k):C,C_0 \to M,M_0$ are inverses. Restricting to M_0 and C_0 , let $E = \operatorname{Ext}^{*,*}_{(\cdot)}(k,k)$ and $L = H_{*,*}(\Omega\cdot;k)$. Then there is a natural equivalence $\phi_*:E \to L\circ G$. That is, each map between Ext algebras of monomial rings which is induced by a monomial map is realized by a map between bigraded Pontrjagin algebras. Furthermore, the algebras which occur in the image of L are finitely presented.

<u>Proof</u>: That GH and HG are the identity functors comes easily out of our definitions of R_B and $X_{(B)}$. That each $X_{(B)} \in \operatorname{Ob} C$ is p-quadratic follows from the fact that $\mathbb{C}P^\infty$ is p-quadratic for any p (see [10]). The subcomplex inclusions and projections which generate Hom C induce homomorphisms preserving both gradations as needed for lemma 13, so the diagram (4) always commutes for $f \in \operatorname{Hom} C_0$. Since all $X_{(B)}$'s have cohomology in even degrees only, ϕ_* is in fact an equivalence of algebras. Finally, the remark about these algebras being finitely presented is a consequence of the corresponding fact about monomial rings [4], as observed before.

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A construction of p-local H-spaces

by

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0. Introduction

The existence of an H-space structure on an odd dimensional sphere which has been localized at an odd prime has been useful in homotopy theory. One way of exhibiting this H-space structure is to use Serre's decomposition [9] of ΩS^{2n} at odd primes into $S^{2n-1} \times \Omega S^{4n-1}$. The point of view of this paper is that Serre's decomposition is a prototype which extends to other examples. For certain X, we split $\Omega\Sigma X$ at an odd prime and obtain the sorts of low rank torsion free H-spaces previously given by Cooke-Harper-Zabrodsky [4] and by Mimura-Toda [7].

1. The results

Our standing assumptions are that spaces have a basepoint, are simply connected, have the homotopy type of a CW complex, and are localized at an odd prime p.

Let X be the localization at p of a cell complex $S^{n_1} \cup e^{n_2} \cup \ldots \cup e^{n_\ell}$ with $n_i \leq n_{i+1}$ and n_i odd. Our main theorem is:

Theorem 1.1. If ℓ is less than p-1, then there is a p-local H-space M(X) and a map $\iota\colon X \to M(X)$ such that $H_*(M(X))$ is the exterior Hopf algebra generated by the injection of $\overline{H}_*(X)$.

Remarks: i) M is a functor on homotopy categories and ι is a natural transformation. ii) We construct a natural map $\rho\colon \Omega\Sigma X \to M(X)$ together with a natural section $s\colon M(X) \to \Omega\Sigma X$ such that ι is the composite $\rho\Sigma\colon X \to \Omega\Sigma X \to M(X)$ and the H-space multiplication on M(X) is the composite $\rho(\mathrm{mult})(s\times s)$: $M(X) \times M(X) \to \Omega\Sigma X \times \Omega\Sigma X \to \Omega\Sigma X \to M(X)$. iii) Since $H_*(X)$ is a trivial coalgebra with free $Z_{(p)}$ module basis 1, $u_1, \ldots, u_\ell$ where u_i has degree n_i, $H_*(M(X))$ is the primitively generated exterior Hopf algebra $\Lambda(u_1, \ldots, u_\ell) = \Lambda(\overline{H}_*(X))$.

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There do exist H-spaces which cannot be constructed by the method of 1.1. For example, if we localize at 2 and let $X = S^3 \cup_\eta e^5$, then $SU(3) = X \cup e^8$ is not a retract of $\Omega\Sigma X$. [3]

However, 1.1 is more general than we have yet indicated. For example, localize at 3 and let λ be an element of $\pi_{2k}(S^{2n+1})$ such that λ is in the image of the double suspension Σ^2_* and such that $\Sigma^2_*(\lambda) = 0$. If $X = S^{2n+1} \cup_\lambda e^{2k+1}$, then a 3-local H-space $S^{2n+1} \cup_\lambda e^{2k+1} \cup e^{2n+2k+2}$ exists.

The functor M converts certain cofibrations into fibrations. In particular, let X and X' be the respective localizations of $S^{n_1} \cup e^{n_2} \cup \dots \cup e^{n_j}$ and $X \cup e^{n_{j+1}} \cup \dots \cup e^{n_\ell}$ where all n_i are odd and let X" be the cofibre of the inclusion $X \to X'$.

Proposition 1.2. If ℓ is less than p-1, then M(X) is the homotopy theoretic fibre of $M(X') \to M(X'')$.

Since $M(S^{2n+1})$ has the homotopy type of S^{2n+1} at p, 1.2 shows that M(X) can be constructed by successive fibrations over localized spheres.

If X' is the bouquet $X \vee X''$, then 1.2 shows that the natural map $M(X \vee X'') \to M(X) \times M(X'')$ is a homotopy equivalence. Hence, if X is again the localization of $S^{n_1} \cup e^{n_2} \cup \dots \cup e^{n_\ell}$, then the following is a formal consequence of naturality and of the H-maps $M(X \vee X) \to M(X)$ and $M(X \vee X \vee X) \to M(X)$ induced by folding.

Corollary 1.3. If 2ℓ is less than p-1, then M(X) is homotopy commutative. If 3ℓ is less than p-1, then it is homotopy associative.

We also have analogues of the double suspension:

Proposition 1.4. If ℓ is less than p-1, then there exist natural maps $E^2: M(X) \to \Omega^2 M(\Sigma^2 X)$ such that, if X is the localization of S^{2n-1}, then E^2 is the localized double suspension $\Sigma^2: S^{2n-1} \to \Omega^2 S^{2n+1}$.

The homotopy type of $\Sigma M(X)$ can be expressed simply in terms of W_k, a contractible space on which the symmetric group Σ_k on k letters acts freely, and $X^{[k]}$, the k-fold smash of X.

Proposition 1.5. If ℓ is less than p-1, then $\Sigma M(X)$ has the homotopy type of the bouquet $\bigvee_{k=1}^{\ell} \Sigma(W_k \times_{\Sigma_k} X^{[k]})$.

2. The method

Throughout this paper, we adopt the notation of the next three paragraphs.

X is the localization at an odd prime p of $S^{n_1} \cup e^{n_2} \cup \dots \cup e^{n_\ell}$ with n_i odd. $V = \bar{H}_*(X)$ is the free $Z_{(p)}$ module with basis $u_1, \dots, u_\ell$ where u_i has degree n_i.

Since $H_*(X)$ is a trivial coalgebra, $H_*(\Omega\Sigma X)$ is the primitively generated tensor Hopf algebra $T(V) = T(u_1,\dots,u_\ell)$ generated by V. Also, $T(V) = UL$, the universal enveloping Hopf algebra of the free Lie algebra $L = L(V)$ generated by V.

We shall construct $M(X)$ so that $H_*(M(X))$ is the primitively generated exterior Hopf algebra $\Lambda(V) = \Lambda(u_1,\dots,u_\ell)$. In turn, $\Lambda(V) = UV$, the universal enveloping Hopf algebra of the abelian Lie algebra V. Note that the abelianization of L is $ab(L) = L/[L,L] = V$.

Our method involves two steps. The first step is to construct a space $\lambda(X)$ and a map $\theta\colon \lambda(X) \to \Sigma X$ such that, if $M(X)$ is the homotopy theoretic fibre of θ and $\Omega\lambda(X) \to \Omega\Sigma X \to M(X) \to \lambda(X) \to \Sigma X$ is the associated fibration sequence, then $\Omega\theta_*$ maps $H_*(\Omega\lambda(X))$ isomorphically onto $U[L,L]$. This is done in section 3, but we shall show now that this determines the coalgebra $H_*(M(X))$.

Consider the principal fibration sequence $\Omega\lambda(X) \to \Omega\Sigma X \to M(X)$. Since $H_*(\Omega\Sigma X) = UL$ is a free $H_*(\Omega\lambda(X)) = U[L,L]$ module [2], the Eilenberg-Moore spectral sequence with abutment $H_*(M(X))$ collapses. That is, $E^2 = \mathrm{Tor}^{U[L,L]}(Z_{(p)},UL) = Z_{(p)} \otimes_{U[L,L]} UL = Uab(L) = UV = \Lambda(V)$. Hence, $H_*(M(X)) = \Lambda(V)$ as a coalgebra.

The second step is to show that, if ι is the composite $\rho\Sigma\colon X \to \Omega\Sigma X \to M(X)$, then the suspension $\Sigma\iota\colon \Sigma X \to \Sigma M(X)$ admits a retraction $r\colon \Sigma M(X) \to \Sigma X$. This is done in section 4, but we shall show now that this implies that $\rho\colon \Omega\Sigma X \to M(X)$ has a section s and hence that $M(X)$ is an H-space.

Define s' to be the composite $(\Omega r)\Sigma\colon M(X) \to \Omega\Sigma M(X) \to \Omega\Sigma X$ and note that $\rho s'\iota = \iota$. Thus, the induced map $(\rho s')_*$ on $H_*(M(X)) = \Lambda(V)$ restricts to an isomorphism on the primitives V. Hence, $(\rho s')_*$ is an isomorphism and $\rho s'$ is a homotopy equivalence. If $s = s'(\rho s')^{-1}$, then s is a section for ρ.

If we use the section s to define the H-space multiplication on $M(X)$ by the composite $M(X) \times M(X) \to \Omega\Sigma X \times \Omega\Sigma X \to \Omega\Sigma X \to M(X)$, then $H_*(M(X)) = \Lambda(V)$ as a Hopf algebra. In other words, $\Omega\lambda(X) \to \Omega\Sigma X \to M(X)$ induces in homology the short exact sequence of homology Hopf algebras $U[L,L] \to UL \to Uab(L)$. [8]

All of our constructions will be natural. Thus, this completes the proof of 1.1, subject to constructing θ and r. It is worthwhile to stress that we do not construct the finite H-space $M(X)$ directly but rather we construct an infinite complement $\Omega\lambda(X)$ for $M(X)$ in $\Omega\Sigma X$.

3. Constructing $\lambda(X)$

Let R_k be $Z_{(p)}[\Sigma_k]$, the group ring of the symmetric group on k letters. Since Σ_k acts on the k-fold smash $X^{[k]}$ and we localize spaces at p, it is easy to see that R_k "acts" on the suspension $\Sigma X^{[k]}$. More precisely, there is a map $R_k \to [\Sigma X^{[k]}, \Sigma X^{[k]}]$ such that the induced map $R_k \to \mathrm{Hom}(\overline{H}_*(\Sigma X^{[k]}), \overline{H}_*(\Sigma X^{[k]}))$ gives the natural action of R_k on $\overline{H}_*(\Sigma X^{[k]}) = \sigma(\bigotimes_{j=1}^{k} \overline{H}_*(X)) = \sigma(\bigotimes_{j=1}^{k} V)$, where σ indicates the suspension of a module.

Suppose R_k splits into a direct sum of right ideals, $R_k = I_1 \oplus \ldots \oplus I_s$. Write $1 = e_1 + \ldots + e_s$ with e_i in I_i. Then the e_i are orthogonal idempotents and $I_i = e_i R_k$. Furthermore, if M is a left R_k module, then $M = I_1 M \oplus \ldots \oplus I_s M = e_1 M \oplus \ldots \oplus e_s M$ as a $Z_{(p)}$ module. The method in [1] extends to the following geometric realization of this splitting.

Lemma 3.1. $\Sigma X^{[k]}$ has the homotopy type of a bouquet $I_1(\Sigma X^{[k]}) \vee \ldots \vee I_s(\Sigma X^{[k]})$ with $\overline{H}_*(I_i(\Sigma X^{[k]})) = I_i(\overline{H}_*(\Sigma X^{[k]}))$.

Proof: Let $I_i(\Sigma X^{[k]})$ be the mapping telescope of $\Sigma X^{[k]} \xrightarrow{e_i} \Sigma X^{[k]} \xrightarrow{e_i} \Sigma X^{[k]} \to \ldots$. Add the maps $\Sigma X^{[k]} \to I_i(\Sigma X^{[k]})$ to get a map $f\colon \Sigma X^{[k]} \to I_1(\Sigma X^{[k]}) \vee \ldots \vee I_s(\Sigma X^{[k]})$. Since homology commutes with direct limits, $\overline{H}_*(I_i(\Sigma X^{[k]})) = e_i \overline{H}_*(\Sigma X^{[k]})$ and f is a homology isomorphism. //

To give an example, define the Dynkin-Specht-Wever element β_k in R_k recursively as follows: $\beta_2 = 1-(1,2), \beta_k = (1-(k,k-1,\ldots,2,1))(1 \otimes \beta_{k-1})$ where $\otimes\colon R_i \otimes R_{k-i} \to R_k$ denotes the natural pairing. When R_k acts on $\bigotimes_{j=1}^{k} V$, $\beta_k(x_1 \otimes \ldots \otimes x_k) = [x_1,[x_2,\ldots[x_{k-1},x_k]\ldots]] = \mathrm{ad}(x_1)\mathrm{ad}(x_2)\ldots\mathrm{ad}(x_{k-1})(x_k)$ where $[x,y] = \mathrm{ad}(x)(y) = x \otimes y - (-1)^{\deg x \deg y}\, y \otimes x$. We have $\beta_k^2 = k\beta_k$. [5] Hence, if $k < p$, then $e_1 = (1/k)\beta_k$ and $e_2 = 1-e_1$ are orthogonal idempotents in R_k. Thus, $R_k = I_1 \oplus I_2$ with $I_i = e_i R_k$, $I_1 = \beta_k R_k$, and

$$\bigotimes_{j=1}^{k} V = e_1(\bigotimes_{j=1}^{k} V) \oplus e_2(\bigotimes_{j=1}^{k} V), \text{ where } e_1(\bigotimes_{j=1}^{k} V) = \beta_k(\bigotimes_{j=1}^{k} V) = L(V) \cap (\bigotimes_{j=1}^{k} V) =$$

the module generated by length k Lie brackets in L(V) which is embedded in T(V) = UL(V). [5] Hence, $\Sigma X^{[k]}$ splits into a bouquet where the homology of the first piece is the suspension of the length k Lie tensors in T(V).

To continue we must compute [L,L] explicitly where $L = L(V) = L(u_1,\ldots,u_\ell)$ and degree u_i is odd. We need the following lemma, whose proof is implicit in [2].

Now, $D_k u$ is the intersection of $[[L,L],[L,L]]$ with those length k Lie tensors S with exactly one occurence of each u_i. Likewise, $B_k u$ is the intersection of $[L,L]$ with S. Since $[[L,L],[L,L]]$ is a $Z_{(p)}$ summand of $[L,L]$, it follows that D_k is a $Z_{(p)}$ summand of B_k.

Since $[L,L]$ is a $Z_{(p)}$ summand in L and L is one in T(V), it follows that B_k is a $Z_{(p)}$ summand in R_k. //

For $k < p$, write $R_k = C_k \oplus \bar{G}_k \oplus D_k$ and use 3.1 to obtain a bouquet summand $\lambda(X)_k = G_k(\Sigma X^{[k]})$ of $\Sigma X^{[k]}$ with $\bar{H}_*(\lambda(X)_k)$ = the suspension σW_k. Define θ_k: $\lambda(X)_k \to \Sigma X$ as follows. Let Σ: $X \to \Omega\Sigma X$ be the suspension map and let $ad^{k-1}(\Sigma)(\Sigma)$: $X^{[k]} \to \Omega\Sigma X$ be the k-fold iterated Samelson product. Let ϕ_k: $\Sigma X^{[k]} \to \Sigma X$ be the adjoint and let θ_k be the restriction of ϕ_k.

If $\ell < p-1$, let $\lambda(X)$ be the bouquet of $\lambda(X)_k$, $2 \leq k \leq \ell+1$. Let θ: $\lambda(X) \to \Sigma X$ restrict to θ_k on each summand. In the fibration sequence

$$\Omega\lambda(X) \xrightarrow{\Omega\theta} \Omega\Sigma X \to M(X) \to \lambda(X) \to \Sigma X,$$

we have:

Lemma 3.5. $H_*(\Omega\lambda(X))$ is mapped isomorphically onto $U[L,L]$ in $UL = H_*(\Omega\Sigma X)$.

Proof: Since $\lambda(X)_k$ is a retract of $\Sigma X^{[k]}$, $H_*(\lambda(X)_k)$ is a trivial coalgebra and, in the Eilenberg-Moore spectral sequence which abuts to $H_*(\Omega\lambda(X)_k)$, $E^2 = \mathrm{Cotor}^{H_*(\lambda(X)_k)}(Z_{(p)},Z_{(p)}) = T(\sigma^{-1}\bar{H}_*(\lambda(X)_k)) = T(W_k)$. Since this spectral sequence retracts from the one which abuts to $H_*(\Omega\Sigma X^{[k]})$, $E^2 = E^\infty$. Hence, $H_*(\Omega\lambda(X)_k) = T(W_k)$ and, since it retracts from $H_*(\Omega\Sigma X^{[k]})$, it is primitively generated.

Similarly, $H_*(\Omega\lambda(X)) = T(W)$ as a Hopf algebra.

Examine the homology suspension to see that $\Omega\lambda(X)_k \to \Omega\Sigma X^{[k]}$ maps W_k isomorphically to W_k modulo decomposable primitives in $H_*(\Omega\Sigma X^{[k]})$. Since $\Omega\phi_k$ restricts to $ad^{k-1}(\Sigma)(\Sigma)$ and $ad^{k-1}(\Sigma)(\Sigma)$ induces β_k in homology, it follows that $\Omega\theta_k$ maps W_k isomorphically to W_k in $H_*(\Omega\Sigma X)$ modulo primitive tensors of length greater than k. Note also that $H_*(\Omega\Sigma X^{[k]})$ is mapped into $U[L,L] = UL(W)$.

Now, a simple filtration argument shows that $H_*(\Omega\lambda(X))$ is mapped isomorphically onto $U[L,L]$. //

4. Constructing the retraction

We have a map ρ: $\Omega\Sigma X \to M(X)$ which induces in homology the natural map $T(u_1,\ldots,u_\ell) \to \Lambda(u_1,\ldots,u_\ell)$. We need only assume that $\ell < p$ to get the existence

Suppose that the degree of u is odd and consider the map $L(u,u_\alpha) \to \langle u\rangle$ to the abelian Lie algebra generated by u which sends u to u and u_α to 0.

Lemma 3.2. The kernel of this map is the free Lie algebra generated by u_α, $[u,u]$, and $[u,u_\alpha]$.

Apply 3.2 to the short exact sequences $L_1 \to L \to \langle u_\ell\rangle$, $L_2 \to L_1 \to \langle u_{\ell-1}\rangle$, ... , $L_\ell \to L_{\ell-1} \to \langle u_1\rangle$ and note that $L_\ell = [L,L]$. This gives:

Lemma 3.3. $[L,L]$ is the free Lie algebra generated by: $[u_i,u_j]$, $[u_{k_1},[u_i,u_j]]$, $[u_{k_2},[u_{k_1},[u_i,u_j]]]$, ... with $1 \le j \le i \le \ell$ and $1 \le k_t < k_{t-1} < \dots < k_2 < k_1 < i$.

For example, if $\ell = 2$, then $[L,L]$ is generated by a basis for the length 2 and length 3 Lie tensors. In general, it is at least true that $[L,L]$ is generated by some Lie tensors of lengths between 2 and $\ell+1$. We express this as follows. Suppose that $[L,L] = W \oplus [[L,L],[L,L]]$ as $Z_{(p)}$ modules, where W is homogeneous with respect to length in L. Then W is a direct sum of homogeneous modules W_k of length k with $2 \le k \le \ell+1$. And $[L,L] = L(W)$ = the free Lie algebra generated by W.

Set $[\beta_i,\beta_j] = (1-\sigma_{ij})(\beta_i \otimes \beta_j)$ in R_k where $i+j = k$, $\sigma_{ij}(s) = j+s$ if $1 \le s \le i$, and $\sigma_{ij}(s) = s-i$ if $i < s \le k$. Let D_k be the right ideal in R_k generated by all $[\beta_i,\beta_j]$ with $i+j = k$ and $2 \le i,j \le k-2$, and let B_k be the right ideal generated by β_k. Then $B_k(\bigotimes_{j=1}^{k} V)$ is the intersection of $[L,L]$ with length k in L, $k \ge 2$. Hence, $D_k(\bigotimes_{j=1}^{k} V)$ is the intersection of $[[L,L],[L,L]]$ with lenth k in L.

If G_k is a right ideal such that $B_k = G_k \oplus D_k$, $W_k = G_k(\bigotimes_{j=1}^{k} V)$, and W is the direct sum of W_k with $2 \le k \le \ell+1$, then $[L,L] = L(W)$.

Lemma 3.4. If $k< p$, then there exist right ideals G_k and C_k in R_k such that $B_k = G_k \oplus D_k$ and $R_k = C_k \oplus B_k$.

Proof: Recall that, if $k < p$, then R_k is semi-simple in the sense that a submodule has an R_k module complement if and only if it has a $Z_{(p)}$ module complement.

If V is free on a basis $u_1, \dots ,u_k$, then R_k acts faithfully on $u = u_1 \otimes \dots \otimes u_k$, that is, $\alpha u = \beta u$ implies that $\alpha = \beta$.

of a retraction $r: \Sigma M(X) \to \Sigma X$. This follows at once from 4.1 below.

If $k < p$, let $\sigma_k = (1/k!) \sum_{\sigma \in \Sigma_k} \sigma$ in R_k. Then σ_k and $1-\sigma_k$ are orthogonal idempotents which generate right ideals S_k and T_k with $R_k = S_k \oplus T_k$.

Lemma 4.1. If $\ell < p$, then $\Sigma M(X)$ has the homotopy type of the bouquet of $S_k(\Sigma X^{[k]})$, $1 \leq k \leq \ell$, and $S_1(\Sigma X) = \Sigma X$.

Proof: Recall that $\Sigma\Omega\Sigma X$ has the homotopy type of the bouquet of $\Sigma X^{[k]}$, $1 \leq k \leq \ell$, and the homology of $\Sigma X^{[k]}$ corresponds to the suspension of the length k tensors. Since the homology of $\Sigma M(X)$ is the suspension of the symmetric tensors, 3.1 shows that, if we split $\Sigma X^{[k]}$ into $S_k(\Sigma X^{[k]}) \vee T_k(\Sigma X^{[k]})$ for $k < p$, then $\Sigma\rho: \Sigma\Omega\Sigma X \to \Sigma M(X)$ gives a homotopy equivalence between the bouquet $\bigvee_{k=1}^{\ell} S_k(\Sigma X^{[k]})$ and $\Sigma M(X)$. It is clear that $S_1(\Sigma X) = \Sigma X$. //

Proposition 4.2. If $k < p$ and W is a contractible space on which Σ_k acts freely, then $S_k(\Sigma X^{[k]})$ has the homotopy type of $\Sigma(W \times_{\Sigma_k} X^{[k]})$.

Proof: The map $W \times X^{[k]} \to W \times_{\Sigma_k} X^{[k]}$ is a covering map. Since p does not divide k!, it induces in mod p homology the natural map $H_*(X^{[k]};Z/pZ) \to Z/pZ \otimes_{\Sigma_k} H_*(X^{[k]};Z/pZ)$. Thus, the suspension of this map restricts to a homotopy equivalence $S_k(\Sigma X^{[k]}) \to \Sigma(W \times_{\Sigma_k} X^{[k]})$. //

5. Cofibrations and fibrations

If $X \to X' \to X''$ is a cofibration sequence which satisfies the hypotheses of 1.2, then we shall show that the sequence $M(X) \to M(X') \to M(X'')$ is a fibration sequence up to homotopy.

Let F be the homotopy theoretic fibre of $M(X') \to M(X'')$. Since $H^*(M(X');Z/pZ)$ is a free $H^*(M(X'');Z/pZ)$ module, the mod p cohomology Serre spectral sequence of $F \to M(X') \to M(X'')$ collapses at E_2. Hence, the mod p Euler-Poincare series of F is the same as that of $M(X)$.

Since $M(X) \to M(X'')$ is null, we can factor $M(X) \to F \to M(X')$. But, $M(X) \to M(X')$ is a mod p homology monomorphism and $M(X) \to F$ must be a mod p homology isomorphism.

6. Analogues of the double suspension

If $\ell < p-1$, then we generalize the double suspension to a map $E^2: M(X) \to \Omega^2 M(\Sigma^2 X)$.

Since $M(X)$ is the fibre of $\theta: \lambda(X) \to \Sigma X$ where θ factors through a generalized Whitehead product $\bigvee_{k=2}^{\ell+1} \Sigma X^{[k]} \to \Sigma X$, the composite $\Sigma\theta: \lambda(X) \to \Sigma X \to \Omega\Sigma^2 X$ is null homotopic. Hence, there is a homotopy commutative diagram

$$\begin{tikzcd} \lambda(X) \arrow[r] \arrow[d, "\theta"'] & * \arrow[d] \\ \Sigma X \arrow[r, "\Sigma^2"] & \Omega^2\Sigma^3 X \end{tikzcd}$$

and a map of homotopy theoretic fibres $\alpha: M(X) \to \Omega^3\Sigma^3 X$. Let E^2 be the composite $(\Omega^2\rho)\alpha: M(X) \to \Omega^3\Sigma^3 X \to \Omega^2 M(\Sigma^2 X)$.

If X is the localization of S^{2n-1}, then the transgression shows that E^2 is degree one on the bottom cell and hence is the double suspension.

7. Extensions of the method

An examination of sections 3 and 4 shows that, if ℓ is arbitrary, then an H-space $M(X)$ can be constructed with $H_*(M(X)) = \Lambda(\overline{H}_*(X)) = \Lambda(u_1,\ldots,u_\ell)$ provided that two things can be done.

First, find a space $\overline{\lambda}(X)$ and a map $\overline{\theta}: \overline{\lambda}(X) \to \Omega\Sigma X$ which maps $\overline{H}_*(\overline{\lambda}(X))$ isomorphically onto a generating module for $U[L,L]$ in $UL = H_*(\Omega\Sigma X)$. This corresponds to section 3 by setting $\lambda(X) = \overline{\lambda}(X)$ and θ = the adjoint of $\overline{\theta}$.

Second, find a space $\mu(X)$ and a map $\mu(X) \to \Omega\Sigma X$ which induces an isomorphism of $\overline{H}_*(\mu(X))$ onto the exterior products $\sigma\ \overline{\Lambda}(u_1,\ldots,u_\ell)$. In section 4, we did this for $\ell < p$.

It may happen in other cases that we can still do these two things. For example, in the trivial case that $X = \Sigma Y$ and that $\Sigma^2 X$ is a bouquet of localized spheres, $\Sigma X^{[k]}$ is also a bouquet of localized spheres and the two steps are easy to perform.

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DESUSPENSION AT AN ODD PRIME

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§ 0.

It is classical that for $r < 2np-3$, $\pi_r(S^{2n-1})$ is isomorphic to the stable group and $\pi_r(S^{2n})$ contains the stable group as a direct summand at the prime p. It is natural to ask to what extend this result can be extended to other spaces. We will investigate spaces X which satisfy the following <u>map desuspension condition</u>: X is $2n-2$ connected and $\pi_r(X) \to \pi_r^S(X)$ is epic for $r < 2np-3$ and split for $r < 2np-4$. Not all spaces satisfy this condition (e.g., -a product of spheres) but we will show how to construct spaces that do.

Given a desuspension theorem for maps, there ought to be a corresponding desuspension theorem for spectra. Such a theorem would input a spectrum and output a space whose suspension spectrum is the given spectrum. We will begin with a theorem of this type. The desuspension thus constructed will be shown to satisfy the map desuspension theorem.

Let us assume for the remainder of this paper that all spaces and spectra are of finite type and localized at p.

<u>Theorem 1</u>. Suppose that $\underline{X}$ is a $2n-2$ connected spectrum and $\dim \underline{X} < 2np-1$. Then there is a space $d_n(\underline{X})$ such that $\Sigma^\infty d_n(\underline{X}) = \underline{X}$.

We need to refine the map desuspension condition.

<u>Definition 2</u>. A space X is n-semistable if

1) X is $2n-2$ connected; $\pi_1(X)$ is abelian if $n = 1$.

2) There exists $F \xrightarrow{\varphi} \Omega^{3n-1}X$ such that the composite

$$F \to \Omega^{3n-1}X \to \Omega^{3n-1}Q(X)$$

is a $(2p-3)n-2$ equivalence.

Here $Q(X) = \Omega^{\infty}\Sigma^{\infty}X$, and a map is an r-equivalence if it induces isomorphisms in homology in dimensions less than r and an epimorphism in dimension r.

Being n-semistable clearly implies the map desuspension condition if $r = k+3n-1$, $k \geq 0$. If $r < 3n-1$, $\pi_r(X) \simeq \pi_r^S(X)$.

<u>Theorem 3</u>. $\Sigma^k d_n(\underline{X})$ is $n + [\frac{k}{2}]$ semistable.

We apply these results to the X_k construction of Barratt and Mahowald and obtain some technical information about $\omega_n \in \pi_{2np-3}(S^{2n-1})$, the first element in the kernel of E^2.

<u>§ 1</u>.

We will describe a spectrum or a space as a *bit* if it is a finite one point union of localized spheres and Z_{p^r} Moore spectra (spaces). The following lemma is easily obtained.

<u>Lemma 4</u>. Suppose $\underline{B}$ is $2n-2$ connected and $\dim \underline{B} < 2n+q-1$. Then $\underline{B}$ is a bit spectrum and $\underline{B} = \Sigma^{\infty}(b)$ where b is the corresponding bit space.

<u>Proof of Theorem 1</u>. We will use induction on n. The case $n = 1$ is implied by Lemma 4 with $n = 1$. Suppose $n > 1$ and $B(\underline{X}) = X^{2n+q-2}$. Then $B(\underline{X}) = \Sigma^{\infty}(b)$. Let $\underline{X}' = \Sigma^{-2p}(\underline{X}/B(\underline{X}))$. $\underline{X}'$ is $2n-4$ connected and has dimension $<2(n-1)p-1$. Thus by induction $\underline{X}' = \Sigma^{\infty}d_{n-1}(\underline{X}') = \Sigma^{\infty}d'$. Thus

$$\underline{X} = B(\underline{X}) \cup_f C\, \Sigma^{2p-1}\underline{X}'.$$

In the adjointness $[\Sigma^\infty A, \Sigma^\infty B] \approx [A, Q(B)]$ f corresponds to a map $\tilde{f}\colon \Sigma^{2p-1}d' \to Q(b)$. We introduce a lemma upon which the remainder of the proof rests.

Lemma 5. Suppose b is a $2n-2$ connected bit space. Then $\exists F$, $\varphi\colon F \to \Omega^2 b$ so that the composite

$$F \xrightarrow{\varphi} \Omega^2 b \longrightarrow \Omega^2 Q(b)$$

is a $2np-5$ equivalence.

Returning to our proof, let $g\colon \Sigma^{2p-3}d' \to \Omega^2 Q(b)$ be the double loop adjoint of $\tilde{f}$. Since $\dim(\Sigma^{2p-3}d') \leq 2p-3+2(n-1)p-2 = 2np-5$, g factors trough F and hence $\Omega^2 b$. Thus $\tilde{f}$ is the suspension of a map $h\colon \Sigma^{2p-1}d' \to b$ and hence $\underline{X}$ is homotopy equivalent to the suspension spectrum of the mapping cone of h. Let this be $d_n(\underline{X})$

$$\begin{tikzcd}
\Sigma^{2p-3}d' \arrow[r] \arrow[dr, "h^{**}"] \arrow[ddr, "\theta"'] & \Omega^2 Q(b) \\
& \Omega^2(b) \arrow[u] \\
& F \arrow[u]
\end{tikzcd}$$

Observe in the proof that, by induction, the homotopy type of $d_n(\underline{X})$ is unambiguously determined if $\dim \underline{X} < 2np-2$.

Proof of Lemma 5:

Case 1. $b = S^{2n-1}$. Let $F = \Omega^2 b$ and $\varphi = 1$.

Case 2. $b = S^{2n}$. Let $F = \Omega S^{2n-1}$ and $\varphi\colon \Omega S^{2n-1} \to \Omega^2 S^{2n}$ be the loops on the inclusion.

Case 3. Let $b = S^{2n-1} \cup_d e^{2n}$ with $d = p^r$. We then have a commutative diagram in which the upper and lower horizontal sequences are fiberings:

$$\begin{array}{ccccc}
S^{2n-1}(d) & \longrightarrow & S^{2n-1} & \xrightarrow{d} & S^{2n-1} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega(S^{2n-1} \cup_d e^{2n}) & \longrightarrow & \Omega S^{2n} & \xrightarrow{d} & \Omega S^{2n} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega Q(S^{2n-1} \cup_d e^{2n}) & \longrightarrow & Q(S^{2n-1}) & \longrightarrow & Q(S^{2n-1})
\end{array}$$

The right and center vertical composites are $2np-3$ equivalences; hence the left verticle composite is a $2np-4$ equivalence. Now let $F = \Omega S^{2n-1}(d)$.

<u>Case 4</u>. Let $b = S^{2n} \cup_d e^{2n+1}$ with $d = p^r$. Here we set $F = S^{2n-1}(d)$ and take the composite

$$F \to \Omega(S^{2n-1} \cup_d e^{2n}) \to \Omega^2(S^{2n} \cup_d e^{2n+1}) = \Omega^2 b \to \Omega^2 Q(b).$$

<u>Case 5</u>. Suppose $b = \vee b_i$ is a finite wedge of spaces for which Lemma 5 holds. We thus have, for each i, $F_i \to \Omega^2 b_i$. Let $F = \Pi F_i \to \Omega^2(\Pi b_i) \xrightarrow{\alpha} \Omega^2(\vee b_i)$ where α is the loop sum of the compositions $\Omega^2(\Pi b_i) \to \Omega^2 b_i \to \Omega^2(\vee b_i)$, one for each i.

This completes the proof of Lemma 5. Note that in Case 3 and 4 the spaces F are universal for coextensions. Thus we might say that the mapping h in Theorem 1 is chosen so that it has no component in the "cross terms" of the wedge (e.g., no Whitehead products) and is a coextension onto each of the Moore spaces. Since the mapping cone of a coextension is also the mapping cone of an extension, these spaces are obtained by attaching spaces to spheres, i.e., building upside down. Whether or not this is sufficient for Theorem 3 eludes me for the time being.

Let us designate $F = F(b)$.

<u>Lemma 6</u>. $\exists$ a map $\beta\colon F(b) \to \Omega F(\Sigma b)$ so that the diagrams

$$\begin{tikzcd} \Omega^2 b \arrow[r] & \Omega^3(\Sigma b) \\ F(b) \arrow[u, "\varphi"] \arrow[r, "\beta"] & \Omega F(\Sigma b) \arrow[u, "\Omega\rho'"'] \end{tikzcd}$$

homotopy commutes.

Proof. We will use the same cases as in Lemma 5. In Case 1 we let $\beta = 1$ as $F(b) = \Omega F(\Sigma b) = \Omega^2 S^{2n-1}$. In Case 2, β is the inclusion $\Omega S^{2n-1} \subset \Omega^3 S^{2n+1}$. In Case 3, $\beta = 1$ as again $F(b) = \Omega F(\Sigma b)$. In Case 4, we must establish a homotopy commutative diagram:

$$\begin{tikzcd} \Omega^2(S^{2n} \cup_d e^{2n+1}) \arrow[r] & \Omega^3(S^{2n+1} \cup_d e^{2n+2}) \\ S^{2n-1}(d) \arrow[u] \arrow[r, "\beta"] & \Omega^2 S^{2n+1}(d) \arrow[u] \end{tikzcd}$$

In fact, for any $f\colon X \to Y$ let F_f be the fiber of f and $c_f\colon F_f \to \Omega(Y \cup_f CX)$ the universal coextension. Then we have a commutative diagram:

$$\begin{tikzcd} F_f \arrow[r, "\sqrt{\beta}"] \arrow[d, "c_f"] & \Omega F_{\Sigma f} \arrow[d, "\Omega c_{\Sigma f}"] \\ \Omega(Y \cup_f CX) \arrow[r] & \Omega^2(\Sigma Y \cup_{\Sigma f} C\Sigma X) \end{tikzcd}$$

where $\sqrt{\beta}(y,w)(s) = ((y,s),\lambda)$ with $\lambda(t) = (w(t),s)$. Applying this diagram twice, the map β in the first square is constructed.

In Case 5 we simply take $\beta = \pi b_i$.

Corollary 7. There is a commutative diagram:

$$\begin{tikzcd} \Omega^2 b \arrow[r] & \Omega^{k+2}(\Sigma^k b) \\ F(b) \arrow[u, "\varphi"] \arrow[r, "\beta_k"] & \Omega^k F(\Sigma^k b) \arrow[u, "\Omega^k\varphi'"'] \end{tikzcd}$$

Lemma 8. There exists $E(k,n) \to \Omega^{3n-1}(\Sigma^k d_n)$ such that the composite

$$E(k,n) \to \Omega^{3n-1}(\Sigma^k d_n) \to \Omega^{3n-1}Q(\Sigma^k d_n)$$

is a $(2p-3)n + 2p[\frac{k}{2}] - 2$ equivalence.

Proof of Theorem 3. Apply Lemma 8 with $F = \Omega^{3\,[k/2]}E(k,n)$.

Proof of Lemma 8. In case $p = 2$ this follows readily since $n+2k-2$ skeletons of $\Omega^{3n-1}(\Sigma^k d_n)$ and $\Omega^{3n-1}Q(\Sigma^k d_n)$ are equivalent. In case $p > 2$, we first observe that Corollary 7 implies the commutativity of the triangle:

$$\begin{tikzcd} & \Omega^2(\Sigma^k b) \\ \Sigma^{k+2p-3}d' \arrow[ur, "(\Sigma^k h)^{**}"] \arrow[dr, "\delta"'] & \\ & F(\Sigma^k b) \arrow[uu] \end{tikzcd}$$

where δ is the k-fold adjoint of $\beta_k \circ \theta$. Thus we have a commutative square:

$$\begin{tikzcd} \Omega^2(\Sigma^{k+2p-1}d') \arrow[r, "\Omega^2\Sigma^k h"] & \Omega^2(\Sigma^k b) \\ \Sigma^{k+2p-3}d' \arrow[u] \arrow[ur, "(\Sigma^k h)^{**}"] \arrow[r, "\delta"] & F(\Sigma^k b) \arrow[u] \end{tikzcd}$$

We now construct the diagram:

$$\begin{tikzcd}
 & \Omega^{3n-4}Q(\Sigma^{k+2p-3}d') \arrow[r] \arrow[d, "\wr"] & \Omega^{3n-2}Q(\Sigma^{k}b) \arrow[d, "\wr"] \\
\Omega^{3n-1}Q(\Sigma^{k}d_n) \arrow[r] & \Omega^{3n-1}Q(\Sigma^{k+2p}d') \arrow[r] & \Omega^{3n-1}Q(\Sigma^{k+1}b) \\
\Omega^{3n-1}(\Sigma^{k}d_n) \arrow[r] \arrow[u] & \Omega^{3n-1}(\Sigma^{k+2p}d') \arrow[r] \arrow[u] & \Omega^{3n-1}(\Sigma^{k+1}b) \arrow[u] \\
\Omega^{3n-2}E \arrow[r] \arrow[u] & \Omega^{3n-2}(\Sigma^{k+2p-1}d') \arrow[r, "\Omega^{3n-2}(\Sigma^{k}h)"] \arrow[u] & \Omega^{3n-2}(\Sigma^{k}b) \arrow[u] \\
\Omega^{3n-4}E' \arrow[r] \arrow[u] & \Omega^{3n-4}(\Sigma^{k+2p-3}d') \arrow[r, "\Omega^{3n-4}\delta"] \arrow[u] & \Omega^{3n-4}F(\Sigma^{k}b) \arrow[u] \arrow[d, equal] \\
E(k,n) \arrow[r] \arrow[u] & E(k+2p-3,\ n-1) \arrow[r, "\gamma"] \arrow[u] & \Omega^{3n-4}F(\Sigma^{k}b)
\end{tikzcd}$$

where E is the fiber of $\Sigma^k h$, E' is the fiber of δ and $E(k,n)$ is the fiber of γ. Thus all horizontal sequences except the second are fiber sequences.

We now use induction on n. In case $n=1$, $\Sigma^k d$ is a k connected bit space, so $E(k,1) = F(\Sigma^k d_1)$. The composite is thus a $2([\frac{k}{2}]+1)p-5$ equivalence by Lemma 5, as desired. In the inductive step we apply Lemma 5 to see that the right hand vertical sequence is a $2(n+[\frac{k}{2}])p-5-(3n-4)$ equivalence. By induction the middle vertical sequence is a $(2p-3)(n-1) + 2p[\frac{k+2p-3}{2}]-2$ equivalence. Since the top and bottom horizontal sequences are fiberings, the left hand vertical sequence is an m-equivalence where m = minimum $-$ 1

$$\begin{aligned} m-1 &= \min(2(n+[\tfrac{k}{2}])p-5-(3n-4),\ (2p-3)(n-1) + 2p[\tfrac{k+2p-3}{2}]-2) \\ &= \min((2p-3)n + 2p[\tfrac{k}{2}]-1,\ (2p-3)n + 2p([\tfrac{k+2p-3}{2}]-1)+1). \end{aligned}$$

Thus it suffices to show that $2p[\frac{k}{2}]-1 \leq 2p([\frac{k+2p-3}{2}]-1)+1)$. This follows immediately of $k = 2s+1$, while if $p \geq 3$ it also follows in case $k = 2s$.

§ 2.

In this section we will apply the result above to the desuspension of the Baratt-Mahowald construction X_k. Let us recall that for $\phi \in \pi_{k+n-1}(S^n)$ they construct complexes

$$X_k = S^{kn} \cup e^{kn+r} \cup \ldots \cup e^{k(n+r)}$$

$\Sigma^n X_k \subset X_{k+1}$ where the relative attaching maps of the cells are ϕ, 2ϕ, $3\phi, \ldots, k\phi$. This is constructed similarly to the James construction: The spectrum $X_\infty = \varinjlim \Sigma^{-kn} X_k$ is in fact the free monoid in the category of spectra over S^o.

We concern ourselves with the case $\phi = \alpha_1 \in \pi_{2p}(S^3)$. Thus

$$X_k(\alpha_1) = S^{3k} \cup \ldots \cup e^{k(2p+1)}.$$

Let $\underline{X}_k(\alpha_1) = \Sigma^{-(k-1)} X_k(\phi)$

$$= S^{2k+1} \cup \ldots \cup e^{2kp+1}.$$

$\underline{X}_k(\phi)$ satisfies the conditions of Theorem 1, with $n = k+1$, thus $d_n(\underline{X_k}(\phi))$ is defined.

Theorem 9. There exist complexes

$$Y_k = S^{2k+1} \cup_{\alpha_1} e^{2k+q+1} \cup_{2\alpha_1} \cdots \cup_{k\alpha_1} e^{2kp+1}$$

whose suspension spectrum is $\underline{X_k}(\alpha_1)$. Y_k is not a suspension if $p-1 \nmid k$ and Y_k is never a double suspension.

Proof. It remains to prove the second statement. Since $P^k \neq 0$ in the Z_p cohomology of $\underline{X}_k(\alpha_1)$ by construction, if $Y_k = \Sigma Y$, the p^{th} power map must be nonzero in Y. Thus the two remarks immediately follow.

The first two desuspensions are

$$Y_1 = S^3 \cup_{\alpha_1} e^{2p+1}$$

$$Y_2 = S^5 \cup_{\alpha_1} e^{2p+3} \cup_{2\alpha_1} e^{4p+1}.$$

We will now use this desuspension to make some remarks about $\omega_n \in \pi_{2np-3}(S^{2n-1})$, the first element in the kernel of the double suspension.

By construction, $Y_n = S^{2n+1} \cup_{g_n} C\Sigma^{2p-1}Y'_n$ where $g_n\colon \Sigma^{2p-1}Y'_n \to S^{2n+1}$ is not a double suspension. Consider the double adjoint

$$\Sigma^{2p-3}Y'_n \xrightarrow{g_n^{**}} \Omega^2 S^{2n+1} = S^{2n-1} \cup_{\omega_n} e^{2np-2} \cup \ldots$$

Since $\dim(\Sigma^{2p-3}Y'_n) = 2np-2$, g_n^{**} factors through $S^{2n-1} \cup_{\omega_n} e^{2np-2}$ and since it is not a double suspension, it does not factor through S^{2n-1}. Consequently it induces a nonzero homomorphism in $H_{2np-2}(\ ;Z_p)$. In order to proceed we make two direct observations:

(1) $d_n(\Sigma^2\underline{X}) = \Sigma^2 d_{n-1}(\underline{X})$ when both are defined.

(2) If $\underline{X}$ is torsion free, $d_n(X^k) = d_n(X)^k$ where the superscript denotes the skeleton.

Consequently $\qquad \Sigma^2 Y_n \subset Y_{n+1}$

and $\qquad \Sigma^2 Y'_n \subset Y'_{n+1}$.

Now $Y'_{n+1} = \Sigma^2 Y'_n \cup_{f_n} e^{2np+1}$. Suppose $g_n\colon \Sigma^{2p-1}Y'_n \to S^{2n+1}$ extends over $\Sigma^{2p-3}Y'_{n+1}$

$$\begin{tikzcd} \Sigma^{2p-3}Y'_{n+1} \arrow[dr, dashed, "g"] & \\ \Sigma^{2p-1}Y'_n \arrow[u, hook] \arrow[r, "g_n"] & S^{2n+1} \end{tikzcd}$$

We claim that g may be chosen so that $\Sigma^2 g = g_{n+1}$. Both of these are extensions of $\Sigma^2 g_n$, thus the difference lies in $\pi_{2(n+1)p}(S^{2n+3})$, but $E^2\colon \pi_{2(n+1)p-2}(S^{2n+1}) \to \pi_{2(n+1)p}(S^{2n+3})$ is onto, so g may be modified if necessary so that $\Sigma^2 g = g_{n+1}$. However since $Y_{n+1} \neq \Sigma^2 Y$, this is impossible. Thus such an extension does not exist. It follows that the composite

$$S^{2(n+1)p-3} \xrightarrow{\Sigma^{2p-3}f_n} \Sigma^{2p-1}Y'_n \xrightarrow{g_n} S^{2n+1}$$

is nonzero. Since the double suspension of this composite is zero, we set this equal to ω_{n+1}.

We now examine the fiber sequences of Toda [3].

$$\Omega S^{2n}_{(p-1)} \xrightarrow{i} \Omega^2 S^{2n+1} \xrightarrow{H} \Omega^2 S^{2np+1}$$

$$S^{2n-1} \xrightarrow{j} \Omega S^{2n}_{(p-1)} \xrightarrow{k} \Omega S^{2np-1}$$

$H \circ g_n^{**} : \Sigma^{2p-3}Y'_n \to \Omega^2 S^{2n+1} \to \Omega^2 S^{2np+1}$ is null homotopic for dimensional reasons. Thus $g_n^{**} = i \circ h_n$. If $k \circ h_n = 0$, g_n^{**} would factor through S^{2n-1} which is impossible; $k \circ h_n$ could only be a multiple of the composition of the projection on S^{2np-2} followed by the inclusion into S^{2np-1}. This multiple cannot be divisible by p, for otherwise we could alter h_n to make it 0. Thus we have

<u>Proposition 10</u>. $H'(\omega_{n+1}) = u(n+1)\alpha_1$, $u \not\equiv 0 \pmod p$

$$\begin{tikzcd}
S^{2(n+1)p-5} \arrow[r] \arrow[dr, "(n+1)\alpha_1"'] & \Sigma^{2p-3}Y'_n \arrow[r, "g_n^{**}"] \arrow[d] \arrow[dr, "h_n"] & \Omega^2 S^{2n+1} \\
& S^{2np-2} \arrow[dr, "u"] & \Omega S^{2n}_{(p-1)} \arrow[u, "i"'] \\
& & \Omega S^{2np-1} \arrow[u, "k"']
\end{tikzcd}$$

This was conjectured in [1]. In case $p \mid n+1$, consider the diagram

$$\begin{array}{ccccc}
S^{2np+1} & \xrightarrow{\Sigma f_n} & \Sigma^3 Y'_n & \xrightarrow{g^*_n} & \Omega^{2p-4}S^{2n+1} \\
 & & \cup & & \cup \\
 & & \Sigma^5 Y'_{n-1} & \xrightarrow{g^*_{n-1}} & \Omega^{2p-6}S^{2n-1} \\
 & & \cup & & \cup \\
 & & \vdots & & \vdots \\
 & & \Sigma^{2p-1} Y'_{n-p+2} & \xrightarrow{g_{n-p+2}} & S^{2n-2p+5}
\end{array}$$

Using techniques of Mahowald [2] it can be shown that Σf_n factors through $\Sigma^{2k+3}Y'_{n-k}$ iff $p^k | n$ at least for $k \leq p-2$ and the factorization projects onto a unit multiple of α_k, the generator of Im J. Consequently we have

Theorem 11. If $k \leq p-2$, $w_{sp^k} = \Sigma^{2k}x$ with $H'(x) = s\alpha_k$.

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H-spaces and self-maps

J. Harper and A. Zabrodsky

INTRODUCTION. An outstanding feature of the classical Lie groups is their appearance as iterated bundles over spheres. The question arises whether (torsion free) finite H-spaces display similar structure. We give examples showing that such structure is not the case in general. Both the construction and analysis of the examples depend on some recent results concerning power like self-maps of H-spaces. Here is our main example.

Theorem 1. For each prime $p \geq 5$, there is a mod p H-space W of rank $p+1$ such that

$$H^*(W, Z/p) = \Lambda(x_{2p+1}, \ldots, P^i x_{2p+1}, \ldots, P^p x_{2p+1})$$

$0 \leq i \leq p$. Furthermore, there is no map $f: W \to X^{2p^2+1}$ with $\deg f \equiv 1 \bmod p$.

Section 1. The construction of W

In order to construct W, we make use of a result from [HZ]. Suppose X is an H-space with $H^*(X)$ primitively generated in dimensions $< n$ (suppressed coefficients are Z/pZ and p is an odd prime). With a given choice of generators, we define $G_t \subset \tilde{H}^*(X), * < n$, $t = 1, 2, \ldots$, to be the subspace spanned by t-fold products of primitive generators. Let ϕ be the λ-th power map of X where λ is a primitive $(p-1)$-st root of unity. Let E_λ be the eigenspace of λ-eigenvectors of $(\phi \wedge \phi)^*$ on $H^n(X \wedge X)$. We can write

$$E_\lambda \subset \bigoplus_{s \equiv 1 (p-1)} \bigoplus_{a+b=s} G_a \otimes G_b$$

Next, we write $s \equiv 1(p-1)$ in the form $s = \ell(p-1)+1$ and introduce $u = [\frac{\ell}{2} + 1](p-1)$ where $[\ ]$ is the greatest integer function. From [HZ] we have

Theorem 2. If $E_\lambda \subset \bigoplus_{\ell \geq 1} \bigoplus_{s-u<r<u} G_r \otimes G_{s-r}$, then X has a multiplication such that $H^*(X)$ is primitively generated in dimension $<n+1$. We now

apply Theorem 2 to construct the H-space W. Let X_0 be an H-space of rank $p+2$ such that

$$H^*(X_0) = \Lambda(x_3, P^1x_3, y_{4p-1}, \dots, P^{p-2}y_{4p-1}, P^pP^1x_3).$$

Such a space arises from Nishida's decomposition of $SU(p^2+1)$. Let X_1 be the 3-connective cover of X_0. By standard methods one obtains

$$H^*(X_1)=\Lambda(x_{2p+1}, P^1x_{2p+1}, \dots, P^px_{2p+1})\otimes Z/p[z_{2p^3}]\otimes\Lambda(\beta z_{2p^3}).$$

Note that for $p \geq 3$, $2p^3 > \dim H^*(W)=(p+1)(p^2+p+1)$. We can take W to be the homology approximation through this dimension (even this skeleton would suffice). Then we have a rational equivalence $W \to X_1$ and the pair is $2p^3-1$ connected. Now we kill un-wanted cohomology, beginning with X_1, to produce a pair $W \to X$ such that the inclusion of W is a rational equivalence and the connectivity of the pair is greater than twice the dimension of W. The inductive step is displayed

$$\begin{tikzcd} & X_{k+1} \arrow[d] & \\ W \arrow[ur] \arrow[r] & X_k \arrow[r, "f_k"] & K(Z/p, n_k) \end{tikzcd}$$

with $n_k \geq 2p^3$ and $H^*(X_k) \cong H^*(W)$ for $* < n_k$. To apply Theorem 2, note that $G_r = 0$ for $r>p+1$ and $G_{p-1}=0$ in dimensions $> \dim (P^2x_{2p+1} \cup \dots \cup P^px_{2p+1}) = p^3+2p^2-4p+1$. Hence $G_1 \otimes G_{p-1} = 0$ in dimensions $> p^3+4p^2-4p+2$ which is smaller than $2p^3$ for $p \geq 3$. For $p \geq 5$, $2p-2 < p+1$, so for $p \geq 5$ the hypotheses of Theorem 2 are satisfied. Hence, X_k has a multiplication for which f_k is an H-map.

Remark. The case $p=3$ involves making explicit calculation through a range of X_k in order to analyze the obstructions corresponding to $s=2p-1$.

Section 2. Here we show that there is no map $f: W \to S^{2p^2+1}$ with $\deg f \equiv 1 \bmod p$, or in other words, every map from W to S^{2p^2+1} must induce 0 on mod p cohomology. The results of this section are based on the study of a certain p-th order unstable cohomology operation. The details are deferred to a longer paper. Here we state a useful relation which emerges from that study.

Definition. A Q_λ-space X is a space with a self-map ϕ such that ϕ^* induces multiplication by the mod p integer λ on $QH^*(X)$. In practice, we choose λ to be a primitive (p-1)-st root of unity.

Theorem 3. Let X be a Q_λ-space with $QH^*(X)$ concentrated in dimensions congruent to $d \bmod (2p-2)$ where d is odd and $d \not\equiv -1 \bmod p$. If $x \in H^{2n+1}(X)$ is λ-characteristic and $P^n x = 0$. Then

$$x \cup P^1 x \cup \ldots \cup P^{p-1} x = P^1 y$$

for some y (we allow the possibility $y = 0$).

We apply Theorem 3 to obtain the rest of Theorem 1. If a map $f : W \to S^{2p^2+1}$ exists with $\deg f \equiv 1 \bmod p$, then by the lifting theorem of [Z], there is a map $\tilde{f}$ inducing the same map in mod p cohomology as f and for sufficiently large k, $\tilde{f}$ commutes up to homotopy with the λ^{p^k}-th power maps of the source and target. Let $F = \text{fiber } \tilde{f}$. Then F is a Q_λ-space with

$$H^*(F) = \Lambda(x_{2p+1}, P^1 x_{2p+1}, \ldots, P^{p-1} x_{2p+1}).$$

But Theorem 3 rules out such a cohomology algebra for Q_λ-spaces.

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Unstable algebras over the Steenrod algebra

S.P. Lam

§1. Introduction

Let X be a 1-connected space whose loop space is equivalent to a finite complex; a notable example is $X = BG$ where G is a compact and connected Lie group. Then what is the algebraic structure of $H^*(X, F_p)$? Throughout p is a prime and F_p is the field with p elements.

In [Ad-W], Adams and Wilkerson have given results that are "best possible" when the prime p is sufficiently large. Rector [R] has suggested a plausible extension of this program; when the prime p is small, one should seek to generalize the following result due to Quillen.

<u>Theorem 1.1</u> [Q, 7.1] Let $\mathcal{A}_G$ be the category of elementary abelian p-subgroups of a compact Lie group G, with morphisms generated by conjugations and inclusions. Then the following induced homomorphism is a purely inseparable isogeny.

$$H^*(BG, F_p) \longrightarrow \varprojlim_{\mathcal{A}_G} H^*(BV, F_p) \qquad ||$$

Here a homomorphism $f: A \longrightarrow B$ of algebras over F_p is a purely inseparable isogeny if (i) $\operatorname{Ker} f$ consists of nilpotent elements, and (ii) for each $z \in B$, there is some $m \geq 0$ s.t. $z^{p^m} \in \operatorname{Im} f$.

Rector's idea is that $H^*(X, F_p)$ might also be determined up to inseparable isogeny by a finite category $\mathcal{C}$ with the following properties:

(i) ob $\mathcal{C}$ consists of elementary abelian p-groups;

(ii) $\mathrm{Mor}_{\mathcal{C}}(V', V)$ consists of group homomorphisms $V' \longrightarrow V$ for $V, V' \in \mathrm{ob}\,\mathcal{C}$.

For simplicity, we shall write EA for elementary abelian; thus an EA p-group means an elementary abelian p-group.

There is a construction, due to Rector, which leads from an unstable algebra over the Steenrod algebra H^* to an associated canonical category $\mathcal{C}(H^*)$ with properties (i) & (ii) such that $\mathcal{C}(H^*(BG, F_p))$ is equivalent to $\mathcal{Q}_G$ for a compact Lie group G; moreover there is an induced homomorphism

$$H^* \longrightarrow \varprojlim \mathcal{C}(H^*) .$$

Here $\varprojlim \mathcal{C}(H^*)$ means $\varprojlim_{\mathcal{C}(H^*)} H^*(BV, F_p)/\mathrm{Nil}^*$; this is an algebra consisting of compatible families $\{z(V) \in H^*(BV, F_p)/\mathrm{Nil}^*: V \in \mathrm{ob}\,\mathcal{C}(H^*)\}$ in the sense that if $\varphi: V' \longrightarrow V$ is a morphism in $\mathcal{C}(H^*)$, then $\varphi^*: H^*(BV, F_p)/\mathrm{Nil}^* \longrightarrow H^*(BV', F_p)/\mathrm{Nil}^*$ takes $z(V)$ to $z(V')$; Nil^* means nilradical.

Rector proves the following.

<u>Theorem 1.2</u> [R] Suppose H^* is a finitely generated unstable algebra over the Steenrod algebra. Then

(i) the category $\mathcal{C}(H^*)$ is finite and every morphism in $\mathcal{C}(H^*)$ is a group monomorphism;

(ii) the induced homomorphism $H^* \longrightarrow \varprojlim \mathcal{C}(H^*)$ is a purely inseparable isogeny.

In this paper, we will show that Theorem 1.2 follows from a seemingly unrelated result, Theorem 1.3, which is a new result and is interesting as a result in pure algebra. We shall indicate how one can obtain 1.3 by calculations with Steenrod operations.

All proofs are to be found in later sections.

Our approach to 1.2 is different from that of Rector, and it has the advantage that we can generalize 1.2(ii). We shall comment on this in §6.

For brevity, we shall write $S(V^{\#})$ for $H^*(BV, F_p)/Nil^*$ for an EA p-group V. $S(V^{\#})$ is the symmetric algebra of $V^{\#}$, the dual of V; we also identify $V^{\#}$ with $H^2(BV, F_p)/Nil^2$ (or $H^1(BV, F_2)$ if $p = 2$).

Theorem 1.3 Let V be an EA p-group. Let H^* be a subalgebra of $S(V^{\#})$ which is closed under the Steenrod algebra action and closed under purely inseparable extension. Suppose $H^* \hookrightarrow S(V^{\#})$ is an algebraic extension with Galois group W and H^* is finitely generated. Then

$$c_o(V)\{S(V^{\#})\}^W \subset H^* .$$

Here $c_o(V)$ is the product of all non-zero elements in $V^{\#}$.

We now comment on 1.3. Adams and Wilkerson [Ad-W] have proved that in the situation of 1.3, the extension $H^* \hookrightarrow S(V^{\#})$ is normal and separable; hence every $x \in S(V^{\#})^W$ can be written as u/v with $u, v \in H^*$. However, we have $H^* \subsetneq S(V^{\#})^W$ in general. A question is: what denominators are needed to express all $x \in S(V^{\#})^W$ as such quotients? Theorem 1.3 says only one is needed, namely, $c_o(V)$.

There are generalizations of Theorems 1.2 and 1.3 in which the finitely generated assumption is replaced by some weaker conditions. In §6, we shall comment on the finitely generated assumption, the generalizations and the methods we used here. The content of this paper is based on the talk given by the author at the Fourth Aarhus Topology Conference in August, 1982.

We shall follow the convention of [Ad-W] so that in all graded objects, only homogeneous elements are considered unless the contrary is mentioned. This means in particular that an ideal of a graded algebra is a graded ideal. We say that an ideal of an algebra over the Steenrod algebra is invariant if this ideal is closed under the action of the Steenrod algebra on the given algebra.

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Notations and abbreviations:

(1) EA is the abbreviated form for elementary abelian.

(2) $S(V^{\#})$ means $H^*(BV, F_p)/Nil^*$ for an EA p-group V.

(3) A NF algebra means an algebra that is free of non-zero nilpotent element.

(4) When V is an EA p-group, $c_o(V)$ stands for the product of all the non-zero elements of $V^{\#}$.

§2. Preliminaries

The Steenrod algebra A_p^* is generated by operations β, P^1, P^2, ... (or Sq^1, ... if $p = 2$). We consider a subalgebra B_p^* (or simply B^*) of A_p^* ; B_2^* is A_2^* and B_p^* $(p > 2)$ is the subalgebra generated by $P^1, \ldots$. Throughout, the term Steenrod algebra stands for B_p^*.

We shall consider only graded commutative algebras over F_p. An algebra is called an algebra over B^* (or a B^*-algebra) if it is a graded module over B^*, in which the Steenrod operations on products satisfy the Cartan formula.

Let H^* be a B^*-algebra. An element $x \in H^i$ is said to be unstable if

$$P^k x = \begin{cases} x^p & \text{if } 2k = i \\ 0 & \text{if } 2k > i \end{cases} \qquad (p > 2)$$

or

$$Sq^k x = \begin{cases} x^2 & \text{if } k = i \\ 0 & \text{if } k > i \end{cases} \qquad (p = 2) .$$

We say that a B^*-algebra is unstable if every element of it is unstable.

In [Ad-W], the authors consider commutative algebras over F_p, so that in the case when $p > 2$, the operation β acts on the algebras trivially. We see that all the results in [Ad-W] remain valid if we replace A_p^* by B_p^* and $H^*(BT, F_p)$ by $S(V^{\#})$ where V is the EA p-subgroup of the torus T consisting of all elements of order p . (Note that Nil^* is an invariant ideal.) In future, whenever we mention a result in [Ad-W], we always mean the modified version of that result.

We omit the proof of the following lemma; details can be found in [L] .

Lemma 2.1 Suppose H^* is a B^*-algebra. Let Nil^* be the subset of H^* consisting of all homogeneous nilpotent elements of H^*. Then Nil^* is a 2-sided ideal of H^* which is invariant under the B^*-action.

Moreover, the ring $\oplus H^n/Nil^n$ has no non-zero nilpotent element. ||

Quillen's result (1.1) suggests that we are only interested in the quotient $H^*(X, F_p)/Nil^*$ which is an algebra over B^* by 2.1. For convenience, an algebra which is free of non-zero nilpotent element is called a NF algebra. In particular, a NF algebra H^* is evenly graded if $p > 2$; thus H^* is commutative.

Let H^* be an algebra. Suppose P is a minimal ideal of $\oplus H^n$. Let us write $P_h = \oplus P \cap H^n$. It is easy to show that P_h is a prime ideal of $\oplus H^n$. Details can be found in [L]. Since P is a minimal prime, $P_h = P$. It follows that P is homogeneous. Therefore a minimal prime of $\oplus H^n$ corresponds uniquely to a (graded) prime of H^* which is necessarily minimal.

One can easily see that if $\{P_\alpha^*\}$ is the collection of all the minimal (graded) primes of H^*, then $\cap P_\alpha^* = 0$.

Next, suppose H^* is an unstable NF algebra over B^*. Let P^* be a minimal prime of H^*. We claim P^* is invariant.

Let $I(P^*)^*$ be the set consisting of all $x \in H^*$ s.t. $P^k x \in P^*$, for all $k \geqq 0$. It is easily verified that $I(P^*)^*$ is an ideal of H^* contained in P^*. Moreover, using the Cartan formula for P^k and the unstable condition on H^*, we can readily show that $I(P^*)^*$ is indeed prime. Then the minimality of P^* forces $I(P^*)^* = P^*$; consequently P^* is invariant. Details can be found in [L]. The following lemma follows easily from the above discussion.

<u>Lemma 2.2</u> Let Q^* be an invariant ideal of H^*, an unstable NF algebra over B^*. Suppose $\{P_\alpha^*\}$ is the collection of all invariant primes of H^* that contain Q^*. Then the kernel of the homomorphism

$$H^*/Q^* \longrightarrow \times H^*/P^*_\alpha$$

is the nilradical of H^*/Q^*. Here the homomorphism is the product of the various projections.

In particular, if $Q^* = 0$, then the kernel is 0. (H^* has no non-zero nilpotent element.) ||

§ 3. The Category $\mathcal{C}(H^*)$

We shall construct $\mathcal{C}(H^*)$ when H^* is finitely generated. This construction is due to D. Rector [R].

By working with the quotient H^*/Nil^* if necessary, we may assume H^* is a NF algebra.

Let P^* be an invariant prime of H^*. Then H^*/P^* is an integral domain with an unstable B^*-action. Moreover, the transcendence degree of H^*/P^* over F_p is finite. Then by [Ad-W, 1.1 and 1.7], we can find an algebraic extension $H^*/P^* \hookrightarrow S(V^{\#}_{P^*})$ which is compatible with the B^*-action, for some suitable EA p-group V_{P^*}. Then ob $\mathcal{C}(H^*)$ is taken to be the collection of these V_{P^*}, one for each invariant prime P^* of H^*.

By construction, there is an one-one correspondence between ob $\mathcal{C}(H^*)$ and the collection of invariant primes of H^*. If V corresponds to P^*, we shall write V_{P^*} for V and P^*_V for P^* whenever necessary.

Suppose $V, V' \in$ ob $\mathcal{C}(H^*)$. Suppose $P^*_V \subset P^*_{V'}$. Then $\mathrm{Mor}_{\mathcal{C}(H^*)}(V', V)$ is taken to be the finite set consisting of all group homomorphisms $\varphi: V' \longrightarrow V$ s.t. the following diagram commutes.

$$
(3.1) \qquad
\begin{tikzcd}
 & H^*/P^*_V \arrow[r, hook] \arrow[d] & S(V^\#) \arrow[d, "\varphi^*"] \\
H^* \arrow[ur] \arrow[dr] & & \\
 & H^*/P^*_{V'} \arrow[r, hook] & S(V'^\#)
\end{tikzcd}
$$

If $P^*_V \not\subset P^*_{V'}$, then $\mathrm{Mor}_{\mathcal{C}(H^*)}(V', V) = \emptyset$. By [Ad-W, 1.10], $\mathrm{Mor}_{\mathcal{C}(H^*)}(V', V)$ is non-empty if $P^*_V \subset P^*_{V'}$ because H^*/P^* is finitely generated.

Since $S(V^\#)$ is a polynomial algebra, the algebra homomorphism φ^* in 3.1 is uniquely determined by its restriction $\varphi^*|V^\#: V^\# \longrightarrow V'^\#$. Suppose $\mathrm{Im}(\varphi^*|V^\#) \subsetneq V'^\#$. Since φ^* is an algebra homomorphism, $\mathrm{Im}\,\varphi^*$ is a polynomial algebra generated by $\mathrm{Im}(\varphi^*|V^\#)$. Hence $S(V'^\#)$ is not integral over $\mathrm{Im}\,\varphi^*$; i.e. it is not integral over $S(V^\#)$. This gives a contradiction because $S(V'^\#)$ is integral over H^* as H^* is a finitely generated algebra [Ad-W, 1.8]. It follows that φ^* is epi; thus φ is mono. This proves half of 1.2(i). Next we prove the other half of 1.2(i).

It remains to show that ob $\mathcal{C}(H^*)$ is finite, it is equivalent to show that H^* has finitely many invariant primes.

H^* has finitely many minimal invariant primes. It suffices to show that for each minimal (invariant) prime I^* of H^*, there are only finitely many invariant primes of H^* containing I^*. To do so, it is enough to assume $I^* = 0$, i.e. H^* is an integral domain. Suppose P^* is an invariant prime of H^*. Then we can find the following diagram in which the horizontal arrows are algebraic extensions.

$$
\begin{tikzcd}
H^* \arrow[r, hook] \arrow[d] & S(V^\#) \arrow[d, dashed, "\varphi^*"] \\
H^*/P^* \arrow[r, hook] & S(V^\#_{P^*})
\end{tikzcd}
$$

As in the construction of $\mathcal{C}(H^*)$, there is at least one

$\varphi : V_{P^*} \longrightarrow V$ s.t. the above diagram commutes. It follows that $P^* = (\mathrm{Ker}\,\varphi^*) \cap H^*$. It is clear that $\mathrm{Ker}\,\varphi^*$ is an invariant prime of $S(V^\#)$. Since $S(V^\#)$ has only finitely many invariant primes [Se, Prop. 1], so has H^*. This completes the proof of 1.2(i).

We have the following commutative diagram.

$$\begin{array}{ccc} H^* & \longrightarrow & \varprojlim \mathcal{C}(H^*) \\ \downarrow & & \downarrow \\ \times H^*/P^*_\alpha & \longrightarrow & \times S(V^\#_{P^*_\alpha}) \end{array}$$

where P^*_α run over all invariant primes of H^*. The left hand vertical arrow is mono by 2.2. The lower horizontal arrow is also mono. Hence the upper horizontal arrow is mono. Then half of 1.2(ii) follows easily.

Let I^* be an invariant ideal of H^*. Then it is easily seen that we can regard $\mathcal{C}(H^*/I^*)$ as a full subcategory of $\mathcal{C}(H^*)$. We have also an induced homomorphism

$$\varprojlim \mathcal{C}(H^*) \longrightarrow \varprojlim \mathcal{C}(H^*/I^*)$$

which takes an element $\{z(V) \in S(V^\#) : V \in \mathrm{ob}\,\mathcal{C}(H^*)\} \in \varprojlim \mathcal{C}(H^*)$ to $\{z(V) \in S(V^\#) : V \in \mathrm{ob}\,\mathcal{C}(H^*/I^*)\} \in \varprojlim \mathcal{C}(H^*/I^*)$.

Suppose G is a compact Lie group. Then Rector [R] has shown that $\mathcal{C}(H^*(BG, F_p)/\mathrm{Nil}^*)$ is equivalent to $\mathcal{A}_G$. A different proof of this equivalence for finite G has been obtained by the author in [L]. We shall not give the details here.

§4. A Chinese Remainder Theorem

Proposition 4.1 Let H^* be an algebra, $P^*_1, \ldots, P^*_t$ be ideals of

H^*. Consider the homomorphisms

$$\varphi : H^* \longrightarrow H^*/P_1^* \times \cdots \times H^*/P_t^*$$

and

$$\psi : \times H^*/P_i^* \longrightarrow \underset{i<j}{\times} H^*/P_i^* + P_j^*$$

where $\varphi(a) = (a + P_1^*, \ldots , a + P_t^*)$ and

$$\psi(a_1 + P_1^*, \ldots , a_t + P_t^*) = ((a_i - a_j) + P_i^* + P_j^*)_{i<j} .$$

Then for any $\underline{z} \in \operatorname{Ker} \psi$, there is $n \geq 0$ s.t. $\underline{z}^{p^n} \in \operatorname{Im} \varphi$.

This result is proved by induction over t. We leave the details to our readers.

<u>Corollary 4.2</u> Let H^* be an unstable NF algebra over B^*, $P_1^*, \ldots , P_t^*$ be invariant primes of H^*. Suppose $x_1, \ldots ,x_t \in H^*$ are s.t. $x_i - x_j \equiv 0 \mod Q^*$ for each invariant prime Q^* containing $P_i^* + P_j^*$, $1 \leq i < j \leq t$. Then we can find $x \in H^*$ and $n \geq 0$ s.t.

$$x \equiv x_i^{p^n} \mod P_i^*, \quad i = 1, 2, \ldots,t.$$

<u>Proof</u> Consider the following commutative diagram.

$$\begin{tikzcd}
 & \operatorname{Ker}\alpha \arrow[d, hook] & \\
H^* \arrow[r, "\varphi"] & H^*/P_1^* \times \cdots \times H^*/P_t^* \arrow[r, "\psi"] \arrow[d, "\alpha"] & \underset{i<j}{\times} H^*/P_i^*+P_j^* \arrow[dl, bend left, "\beta = \times \beta_{ij}"] \\
 & \underset{i<j}{\times} \Big(\underset{\substack{Q^* \supset P_i^*+P_j^* \\ Q^* \text{ invariant prime}}}{\times} H^*/Q^* \Big) &
\end{tikzcd}$$

where φ, ψ are as in 4.1,

$$\beta_{ij} : H^*/P_i^*+P_j^* \longrightarrow \underset{\substack{Q^* \supset P_i^*+P_j^* \\ Q^* \text{ invariant prime}}}{\times} H^*/Q^*$$

is the product of the various projections, and α sends

$(x_1 + P_1^*, \ldots, x_t + P_t^*)$ to the element $\{(x_i - x_j) + Q^*: Q^*$ run over all the invariant primes of H^* containing $P_i^* + P_j^*$, $1 \leq i < j \leq t\}$.

Note that β is an algebra homomorphism and ψ respects p^{th} powers. Since the B^*-action on H^* is unstable, $\mathrm{Ker}\,\beta_{ij}$ consists of nilpotent elements (this is an easy consequence of 2.2); hence $\mathrm{Ker}\,\beta$ consists of nilpotent elements.

The assumption on the given x_i's is such that $\underline{z} = (x_1 + P_1^*, \ldots, x_t + P_t^*) \in \mathrm{Ker}\,\alpha$. Commutativity of the above diagram implies $\psi(\underline{z}) \in \mathrm{Ker}\,\beta$; thus we can find $M \geq 0$ s.t. $\psi(\underline{z}^{p^M}) = \psi(\underline{z})^{p^M} = 0$. By 4.1, we can find $N \geq 0$ s.t. $(\underline{z}^{p^M})^{p^N} \in \mathrm{Im}\,\varphi$. The corollary then follows. ||

§5. Proofs of Main Results

Lemma 5.1 Let V be an EA p-group. Let $x \in S(V^{\#})$ be a non-zero element s.t. $x \equiv 0 \mod P^*$ for each invariant prime P^* of $S(V^{\#})$. Then there is some $u \in S(V^{\#})$ s.t.

$$c_o(V)u = x^p .$$

Proof It is clear that each non-zero element of $V^{\#}$ is prime, hence divides x by hypothesis. It follows easily that the product of all the non-zero elements of $V^{\#}$ divides x^{p-1}. The result then follows. ||

Proof of 1.2 assuming 1.3 Let H^* be a finitely generated unstable algebra over B^*. By working with H^*/Nil^* if necessary, we may assume H^* is a NF algebra. In §3, we have proved 1.2(i) and and half of 1.2(ii); thus the induced homomorphism $H^* \longrightarrow \varprojlim \mathcal{C}(H^*)$ is mono.

Let $\underline{z} = \{z(V) \in S(V^{\#}): V \in \mathrm{ob}\,\mathcal{C}(H^*)\} \in \varprojlim \mathcal{C}(H^*)$. Then the subfamily $\underline{z}_{P*} = \{z(V) \in S(V^{\#}): V \in \mathrm{ob}\,\mathcal{C}(H^*/P^*)\} \in \varprojlim \mathcal{C}(H^*/P^*)$, where P^* is any invariant prime of H^*. (See the remark at the end of §3.) We claim $\underline{z}_{P*}^{p^M} \in H^*/P^*$ for some $M \geq 0$.

Recall from §3 that we have algebraic extensions $H^*/P^* \hookrightarrow S(V_{P*}^{\#})$, where $V_{P*} \in \mathrm{ob}\,\mathcal{C}(H^*)$. We shall prove the claim by induction over rank V_{P*}.

If rank $V_{P*} = 0$, the claim is trivially true. Suppose rank $V_{P*} > 0$, and assume the claim is true for each invariant prime P'^* of H^* with rank $V_{P'*} <$ rank V_{P*}. Then we have

$$\underline{z}_{P'*}^{p^m} \in H^*/P'^*$$

for each such P'^*. Since there are only finitely many invariant primes of H^*, we can find a single m which works for all $\underline{z}_{P'*}$.

Choose $x_{P'*} \in H^*/P^*$ s.t. $x_{P'*} \equiv \underline{z}_{P'*}^{p^m} \mod P'^*/P^*$. These (finitely many) $x_{P'*}$ satisfy the hypotheses of 4.2, thus we can find $x \in H^*/P^*$ and $n \geq 0$ s.t.

$$x \equiv x_{P'*}^{p^n} \equiv (\underline{z}_{P'*}^{p^m})^{p^n} \mod P'^*/P^* .$$

Let I^* be a non-zero invariant prime of $S(V_{P*}^{\#})$. Then $I^* \cap (H^*/P^*)$ is a non-zero invariant prime of H^*/P^*; it is of the form P'^*/P^* for some invariant prime P'^* of H^* containing P^*. It is clear that

$$\underline{z}_{P*} \equiv \underline{z}_{P'*} \mod I^* .$$

It follows easily that

$$x \equiv \underline{z}_{P*}^{p^{m+n}} \mod I^* .$$

Then by 5.1, there is some $u \in S(V_{P*}^{\#})$ s.t.

$$c_o u = x^p - \underline{z}_{P*}^{p^{m+n+1}} ,$$

where $c_o = c_o(V_{P*})$. Since x, $\underline{z}_{P*}$ and c_o are invariant under the action of the Galois group of the extension $H^*/P^* \hookrightarrow S(V^{\#}_{P*})$, so is u. Thus it follows from 1.3 that there is some $k \geqq 0$ s.t.

$$(c_o u)^{p^k} \in H^*/P^* .$$

Now it is clear that $\underline{z}_{P*}^{p^{m+n+k+1}} \in H^*/P^*$. This completes the induction.

Now, we have $\underline{z}_{P*}^{p^M} \in H^*/P^*$ for each invariant prime P^* of H^*. Again, since H^* has only finitely many invariant primes, we can find one single M which works for all $\underline{z}_{P*}$. Choose $y_{P*} \in H^*$ s.t. $y_{P*} \equiv \underline{z}_{P*}^{p^M} \bmod P^*$. These (finitely many) y_{P*} satisfy the hypotheses of 4.2, thus we can find $y \in H^*$ and $N \geqq 0$ s.t.

$$y \equiv y_{P*}^{p^N} \equiv \underline{z}_{P*}^{p^{M+N}} \mod P^* .$$

It follows that y and $\underline{z}^{p^{M+N}}$ have the same image in $S(V^{\#}_{P*})$ for each invariant prime P^* of H^*. Hence $y = \underline{z}^{p^{M+N}}$, and 1.2(ii) follows.

<u>Sketch proof of 1.3</u> First we recall that when V is an EA p-group of rank n, $S(V^{\#})^{GL(V)}$ is a polynomial algebra $F_p[c_o, c_1, \ldots, c_{n-1}]$ with $\deg c_i = 2(p^n - p^i)$ (or $2^n - 2^i$ if $p = 2$) [D].

Let $H^* \hookrightarrow S(V^{\#})$ be the given algebraic extension with Galois group W. Consider the set

$$P^* = \{x \in I(S(V^{\#})^W) : xS(V^{\#})^W \subset H^*\}$$

where $I(\ )$ means augmentation ideal.

<u>Lemma 5.2</u>

(i) P^* is a non-zero ideal of $S(V^{\#})^W$; it is contained in H^*.

(ii) If $x \in S(V^{\#})$ and $x^p \in P^*$, then $x \in P^*$.

(iii) P^* is closed under the action of B^* on $S(V^{\#})$.

(iv) P^* contains a non-zero homogeneous polynomial

$$f(c_o, \ldots, c_{n-1}) .$$

<u>Proof</u> For convenience, we write K^* for $S(V^{\#})^W$.

(i) That P^* is an ideal contained in H^* follows easily from the definition. We omit the proof here.

Since K^* is a finitely generated algebra, it follows from [At-M, 5.2] that K^* is a finitely generated module over H^*.

Suppose $z_1, \ldots, z_N$ generate K^* as a module over H^*. Then as pointed out in [Ad-W, p.140], any element in K^* can be expressed as a quotient u/v of two elements in H^*. So we can find $h_i, h_i' \in H^*$ with $h_i' \neq 0$, and $\deg h_i' > 0$ s.t. $z_i = h_i/h_i'$, $i = 1, 2, \ldots, N$. The non-zero element $x = h_1' \ldots h_N'$ has positive degree. Any $z \in K^*$ can be expressed as $k_1z_1 + \ldots + k_Nz_N$ for some $k_i \in H^*$. It is clear that $xz \in H^*$. That is $x \in P^*$.

(ii) We leave this as an exercise for our readers.

(iii) Let $x \in P^*$. We shall show by induction over k that $P^kx \in P^*$. We have $P^ox = x$; this gives the basis for induction.

Assume $P^ix \in P^*$ for $i < k$. Take $y \in K^*$. The Cartan formula gives

$$P^k(xy) = (P^kx)y + \sum_{\substack{i+j=k \\ i<k}} (P^ix)(P^jy) .$$

Since $xy \in H^*$, $P^k(xy) \in H^*$; by induction assumption, $(P^ix)(P^jy) \in H^*$ for $i < k$. Hence $(P^kx)y \in H^*$, proving $P^kx \in P^*$.

(iv) Let x be a non-zero element in P^*. Clearly $\deg x > 0$. Consider the element

$$\pi = \prod_{\substack{g \in GL(V) \\ g \neq 1}} gx .$$

We claim $\pi \in K^*$. The element $x\pi$ is invariant under $GL(V)$, so it is invariant under W.

For $w \in W$, $x\pi = w(x\pi) = (wx)(w\pi) = x(w\pi)$. This implies $\pi = w\pi$ as $S(V^{\#})$ is an integral domain. Since $x \in P^*$, $\pi \in K^*$ and P^* is an ideal of K^*, $x\pi \in P^*$. Evidently $x\pi$ is a non-zero polynomial in $c_o, c_1, \dots, c_{n-1}$. ||

We continue to sketch the proof of 1.3. Let Q^r be the Steenrod operation defined in [Ad-W, p.102], $r \geq 1$. (Replace P^i by Sq^i if $p = 2$.) These are derivations. Each of the p^n elements of $V^{\#}$ is a root of the equation [Q, 11.6]

$$x^{p^n} + c_{n-1}x^{p^{n-1}} + \dots + c_1 x^p + c_o x = 0.$$

By applying Q^r to this equation, we easily obtain the following formula which is well-known to experts in this area:

$$(5.3) \qquad Q^r c_i = \begin{cases} 0 & \text{for } i \neq r,\ r < n \\ -c_o & \text{for } i = r,\ r < n \\ c_o c_i & \text{for } r = n . \end{cases}$$

(The following calculation is suggested by Frank Adams.) Let $f(c_o, \dots, c_{n-1})$ be a non-zero homogeneous polynomial in P^*. By operating on f with suitably chosen operations Q^r, we obtain a non-zero homogeneous polynomial

$$g(c_o, \dots, c_{n-1})$$

which also lies in P^* and whose degree in each of $c_1, \dots, c_{n-1}$ is no greater than that of f, and which is a p^{th} power. Taking the p^{th} root of g, we get a non-zero homogeneous polynomial

$$h(c_o, \ldots, c_{n-1})$$

which also lies in P^* and whose degree in each of $c_1, \ldots, c_{n-1}$ is actually less than that of f (unless the corresponding degree of f is zero). By repeating this process, we obtain a non-zero homogeneous polynomial in P^* which is in fact λc_o^m with $0 \neq \lambda \in F_p$. We now repeat the same process by taking p^{th} root if p divides m and applying the operation Q^n on λc_o^m otherwise so that we decrease m and eventually proving $c_o \in P^*$. The above description is only a sketch, the reader may find working out for himself the case $n = p = 2$ instructive. We actually have to use some of the properties of P^* listed in 5.2 and formula 5.3 in the calculations.

Since $c_o \in P^*$, $c_o S(V^{\#})^W \subset H^*$ by definition of P^*. ||

§6. Comments

Let X be a space with properties as given in §1. We are still unable to verify the finitely generated assumption for $H^*(X, F_p)$. This is only known to be true for $X = BG$ [Ve]. It is therefore natural to seek for generalizations of 1.2 and 1.3. It turns out that in 1.2(ii), it is necessary and sufficient to assume the following conditions:

(A) H^*/Nil^* has finite transcendence degree over F_p;

(B) H^* has finitely many minimal primes.

These follow when H^* is a finitely generated algebra; C. W. Wilkerson has an argument which shows that $H^*(X, F_p)$ satisfies (A). In the generalized version of 1.2, the category $\mathcal{C}(H^*)$ is more complicated, and the morphisms need not be injective (group homomorphisms).

The generalized version of 1.2(ii) follows from generalized versions of 1.3 and [Ad-W, 1.10] in which we drop the finitely generated assumption. They have been obtained by J. F. Adams and the author. The spirit of the proof of the generalized version of 1.3 is the same as that given in §5 for the special case; however we need to know more about the action of Steenrod operations on c_i, and the calculations are much longer and harder. The full account is too long to be included here.

We give sketch proofs for the following two consequences of the generalized version of 1.3; 6.1 is suggested by J.F. Adams.

Proposition 6.1 Let V be an EA p-group. Let $H^* \hookrightarrow S(V^{\#})$ be an extension s. t.

(i) H^* is closed under the B^*-action;

(ii) $H^* \subset S(V^{\#})^{GL(V)}$;

(iii) H^* is closed under taking p^{th} roots in $S(V^{\#})$.

Then there exists an EA p-subgroup V' of V s.t. the following diagram is a pull-back.

$$\begin{array}{ccccc} H^* & & \longrightarrow & & F_p \\ \downarrow & & & & \downarrow \\ S(V^{\#})^{GL(V)} & \hookrightarrow & S(V^{\#}) & \longrightarrow & S(V'^{\#}) \end{array}$$

Sketch proof Using 1.3(generalized version), we deduce easily that in this situation, either $H^* = F_p$ or $H^* \hookrightarrow S(V^{\#})$ is an algebraic extension.

Then 6.1 follows from 1.3(generalized version) and an inductive argument over rank V. $\|$

Proposition 6.2 (Going-up Theorem) Let V be an EA p-group. Let $H^* \hookrightarrow S(V^{\#})$ be an extension which satisfies 6.1(i). Then for any invariant prime P^* of H^*, there is an invariant prime

Q^* of $S(V^{\#})$ s.t. $Q^* \cap H^* = P^*$.

This result generalizes [Ad-W, 1.10]. This result also follows from 1.3(generalized version) and an inductive argument over rank V. Slightly more precisely, we may assume the given extension is algebraic and we use 1.3(generalized version) to show that **any** non-zero invariant prime P^* of H^* contains some $H^* \cap K^*$ where

$$K^* = \mathrm{Ker}\{S(V^{\#}) \longrightarrow S(V'^{\#})\}$$

for some EA p-subgroup $V' \subset V$ with rank $V' = n-1$. Then we can apply induction assumption to

$$H^*/H^* \cap K^* \hookrightarrow S(V'^{\#}) .$$

The result then follows easily. Details will appear elsewhere.

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A remark on the Kervaire invariant problem

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Let A denote the mod 2 Steenrod algebra. Let $\psi_{i,i}$ be the secondary mod 2 cohomology operation based on the Adem relation

$$Sq^{2^i} Sq^{2^i} + \sum_{j=0}^{i-1} Sq^{2^{i+1}-2^i} Sq^{2^i} = 0$$

in A ([2]). Let S^n denote the sphere spectrum in stable dimension n ([3]). Call a homotopy class $S^{2^{i+1}-2} \longrightarrow S^0$ θ_i if $\psi_{i,i}$ is non-zero in $H^*(S^0 \cup_{\theta_i} e^{2^{i+1}-1})$ where H*( ) is the mod 2 cohomology functor. θ_i exists if and only if in the mod 2 Adams spectral sequence $\{E_r^{s,t}\}$ for the stable homotopy groups of spheres ([1]) the class $h_i^2 \in Ext_A^{2,2^{i+1}}(H^*(S^0),\mathbb{Z}_2)=E_2^{2,2^{i+1}}$ survives in the spectral sequence. It is classical ([4]) that θ_i exists for $0 \leq i \leq 3$: h_0^2 detects 4ι where ι generates $\pi_0(S^0) = \mathbb{Z}$ while h_1^2, h_2^2 and h_3^2 detect the **squares** η^2, ν^2 and σ^2 of the Hopf classes η,ν and σ respectively. Mahowald and Tangora ([8]) have shown that θ_4 also exists. The Kervaire invariant problem can be stated as follows: Does θ_i exist for $i \geq 5$? For the geometric roots of the problem and for its relation to manifold theory we refer to Browder [5].

From the homotopy point of view it is natural to study a stronger version of the problem : Does there exist a θ_i with $2\theta_i = 0$? This stronger version of the problem was suggested

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by Barratt and Mahowald in [7] where they proposed an inductive approach toward the problem. Some of their results announced there which were obtained in an effort to settle the problem are now proved in [6].

In this note we introduce another approach*. In this approach we construct for each $i \geq 5$ a class in $\pi_{2^{i+1}-2}(S^0)$ which should be a θ_i of order 2 if such a class exists. It is plausible that the constructed classes are the correct classes, but this is still under investigation.

We fix an $i \geq 5$. Consider the following minimal Adams resolution of S^0

(1)
$$\begin{tikzcd}
\Sigma^{-1}K_2 \arrow[r, "j_3"] & X_3 \arrow[d, "p_3"] & \\
\Sigma^{-1}K_1 \arrow[r, "j_2"] & X_2 \arrow[r, "f_2"] \arrow[d, "p_2"] & K_2 \\
\Sigma^{-1}K_0 \arrow[r, "j_1"] & X_1 \arrow[r, "f_1"] \arrow[d, "p_1"] & K_1 \\
S^0 = & X_0 \arrow[r, "f_0"] & K_0 = K
\end{tikzcd}$$

up to the 3^{rd} stage in a range of dimensions which is described as follows.

Let K denote the Eilenberg-MacLane spectrum $K(\mathbb{Z}_2)$ and let $f_0 : S^0 \longrightarrow K$ be the non-trivial map. Write $K_1 = K \times \Sigma K \times \Sigma^3 K \times \cdots \times \Sigma^{2^{i+1}-1}K$ and define $f_1 = (Sq^1, Sq^2, \cdots, Sq^{2^i}, Sq^{2^{i+1}})$. The set of the classes in f_1 corresponds to the $\mathbb{Z}_2$-base $\{h_j | j \leq i+1\}$ of $\mathrm{Ext}_A^{1,*+1}(H^*(S^0), \mathbb{Z}_2)$ for $* \leq 2^{i+1}-1$. Let B_2 be a minimal set of

*The contents of this article is different from what was talked at the Aarhus conference.

generators of $H^*(X_2)$ over the mod 2 Steenrod algebra A for $* \leq 2^{i+2} - 2$. Then B_2 is in one-to-one correspondence with a $\mathbb{Z}_2$-base of $\mathrm{Ext}_A^{2,2+*}(H^*(S^0),\mathbb{Z}_2 0)$ for $* \leq 2^{i+1} - 2$. J.F. Adams ([2]) has shown that in this range of dimensions a $\mathbb{Z}_2$-base for the Ext groups is $\{h_\ell h_m \mid 0 \leq \ell \leq m \leq i + 1,\ \ell \neq m - 1\}$. Let $\psi_{\ell,m} \in B_2$ be the class corresponding to $h_\ell h_m$. Consider the subset $\bar{B}_2 = \{\psi_{\ell,m} \mid \ell \leq i,\ m \leq i\}$ of B_2. f_2 in (1) is defined to be the map which kills all the classes in $\bar{B}_2$ and $K_2 = K \times \Sigma^2 K \times \cdots \times \Sigma^{2^i+2^{i-2}-2}K \times \Sigma^{2^{i+1}-2}K$ is the corresponding product of the spectra $\Sigma^j K$ where the last factor $\Sigma^{2^{i+1}-2}K$ corresponds to $\psi_{i,i}$. Let B_3 be a minimal set of generator of $H^*(X_3)$ over A for $* \leq 2^{i+1} - 2$. Then B_3 is one-to-one correspondent with a $\mathbb{Z}_2$-base of $\mathrm{Ext}_A^{3,3+*}(H^*(S^0),\mathbb{Z}_2)$ for $* \leq 2^{i+1} - 2$. The calculations by Adams in [2] and by Wang in [9] show that $\mathrm{Ext}_A^{3,2^{i+1}+1}(H^*(S^0),\mathbb{Z}_2) \cong \mathbb{Z}_2$ and is generated by $h_i^2 h_0$. Let $\psi_{i,i,0} \in B_3$ be the class corresponding to $h_i^2 h_0$; $\psi_{i,i,0}$ is the "last" element in B_3.

Let $\Sigma^{2^{i+1}-2}K \xrightarrow{i_2} \Sigma^{-1}K_1 = \Sigma^{-1}K \times \Sigma K \times \cdots \times \Sigma^{2^{i+1}-2}K$ be the inclusion and let $\bar{j}_2 = j_2 \circ i_2 : \Sigma^{2^{i+1}-2}K \longrightarrow X_2$. It is clear that the composite $\Sigma^{2^{i+1}-2}K \xrightarrow{\bar{j}_2} X_2 \xrightarrow{f_2} K_2$ is trivial and that there is a unique lifting $\sigma_2 : \Sigma^{2^{i+1}-2}K \longrightarrow X_3$ of $\bar{j}_2$ as indicated in the following portion of diagram (1)

$$\begin{tikzcd} & X_3 \arrow[d, "p_3"] \\ \Sigma^{2^{i+1}-2}K \arrow[ur, dashed, "\sigma_2"] \arrow[r, "\bar{j}_2"'] & X_2 \end{tikzcd}$$

Let $\bar{h}_{i+1}$ be the generator of $H^{2^{i+1}-2}(\Sigma^{2^{i+1}-2}K) = \mathbb{Z}_2$. Adams theorem on the Hopf invariant ([2]) implies $\sigma_2^*(\psi_{i,i,0}) = \bar{h}_{i+1}$ since $i \geq 5$. Let $\tau_3 : X_3 \longrightarrow \Sigma^{2^{i+1}-2}K$ be the map which classifies $\psi_{i,i,0}$. Then $\tau_3 \circ \sigma_2 : \Sigma^{2^{i+1}-2}K \longrightarrow \Sigma^{2^{i+1}-2}K$ is a self-homotopy equivalence. We may assume $\tau_3 \circ \sigma_2$ is the identity map. Thus $\Sigma^{2^{i+1}-2}K$ is a retract of X_3. So there is a decomposition $X_3 = X_3' \vee \Sigma^{2^{i+1}-2}K$, and we can choose it so that the projection of $\Sigma^{2^{i+1}-2}K \xrightarrow{\sigma_2} X_3$ to the factor X_3' is trivial and so that the other projection is the identity map. The projection of the class $\psi_{i,i,0} \in H^{2^{i+1}-2}(X_3)$ to $H^{2^{i+1}-2}(\Sigma^{2^{i+1}-2}K) = \mathbb{Z}_2$ is the generator which we denote by $\bar{\psi}_{i,i,0}$.

Let

$$\Sigma^{2^{i+1}-3}K \xrightarrow{i_3} \Sigma^{-1}K_2 = \Sigma^{-1}K \times \Sigma K \times \cdots \times \Sigma^{2^i+2^{i-2}-3}K \times \Sigma^{2^{i+1}-3}K$$

be the inclusion and let $\bar{j}_3 = j_3 \circ i_3 : \Sigma^{2^{i+1}-3}K \longrightarrow X_3$. Let $\overline{h_i^2}$ be the generator of $H^{2^{i+1}-3}(\Sigma^{2^{i+1}-3}K) = \mathbb{Z}_2$. Then $\bar{j}_3^*(\bar{\psi}_{i,i,0}) = Sq^1\overline{h_i^2}$; this is easily verified. We may assume that $\Sigma^{2^{i+1}-3}K$ has a cell structure of the form $\Sigma^{2^{i+1}-3}K = S^{2^{i+1}-3} \cup_{2\iota} e^{2^{i+1}-2} \cup \cdots$ where the cells not indicated begin with dimension $2^{i+1}-1$. Let $g = \bar{j}_3 | S^{2^{i+1}-3} \cup_{2\iota} e^{2^{i+1}-2}$ and let

$$g_1 : S^{2^{i+1}-3} \cup_{2\iota} e^{2^{i+1}-2} \longrightarrow X_3'$$

$$(\text{resp. } g_2 : S^{2^{i+1}-3} \cup_{2\iota} e^{2^{i+1}-2} \longrightarrow \Sigma^{2^{i+1}-2}K)$$

be its projection to the factor X_3'(resp. $\Sigma^{2^{i+1}-2}K$). Then we still have $g_2^*(\bar{\psi}_{i,i,0}) = Sq^1\overline{h_i^2}$

Let $\iota : S^{2^{i+1}-2} \longrightarrow \Sigma^{2^{i+1}-2}K$ be the non-trivial map and let $\{h_{i+1}\}$ denote the composite $S^{2^{i+1}-2} \xrightarrow{\iota} \Sigma^{2^{i+1}-2}K \xrightarrow{j_2} X_2$. The class $\{h_{i+1}\}$ has order 2. Our problem is to find a class θ_i of order 2 in $\pi_{2^{i+1}-2}(S^0)$ which is detected by h_i^2 (if it exists). This is equivalent to looking for a class $\bar{\theta}_i \in \pi_{2^{i+1}-2}(X_2)$ detected by h_i^2 such that $2\bar{\theta}_i = \{h_{i+1}\}$.

<u>Lemma A</u>: $\bar{\theta}_i$ exists if and only if

$$g_1 : S^{2^{i+1}-3} \cup_{2\iota} e^{2^{i+1}-2} \longrightarrow X_3'$$

is trivial.

<u>Proof</u>: Consider the diagram

$$\begin{tikzcd}
\Sigma^{-1}K_2 = \cdots \times \Sigma^{2^{i+1}-3}K \arrow[r, "j_3"] & X_3 = X_3' \vee \Sigma^{2^{i+1}-2}K \arrow[r, "p_3"] & X_2 & \\
& S^{2^{i+1}-3} \cup_{2\iota} e^{3^{i+1}-2} \arrow[ul, dashed, bend left, "\bar{g}_2"] \arrow[ul, "\bar{g}"'] \arrow[u, "g"] \arrow[u, bend right, "g_2"'] \arrow[r, "p"] & S^{2^{i+2}-2} \arrow[u, "\{h_{i+1}\}"] \arrow[r, "\times 2"] & S^{2^{i+1}-2} \arrow[ul, dashed, "\bar{\theta}_i"']
\end{tikzcd}$$

where p is the pinching map so that the lower sequence is a cofibration and $\bar{g}$ is the inclusion map to the factor $\Sigma^{2^{i+1}-3}K$. Then $j_3\bar{g} = g = g_1 + g_2$. Since $p_3j_3 = 0$ it follows that $p_3g_2 = -p_3g_1$.

If $g_1 = 0$ then $p_3g_2 = 0$. But p_3g_2 is just $\{h_{i+1}\}p$ (this follows from our construction of the decomposition for X_3); so $\{h_{i+1}\}p = 0$. Thus there is a $\bar{\theta}_i : S^{2^{i+1}-2} \longrightarrow X_2$ to make the right hand triangle commutative; i.e., $2\bar{\theta}_i = \{h_{i+1}\}$.

Conversely, suppose such a $\bar{\theta}_i$ exists. Then $p_3 g_2 = \{h_{i+1}\}p = 0$. Thus g_2 can be lifted to a map $\bar{g}_2 : S^{2^{i+1}-3} \cup_{2\iota} e^{2^{i+1}-2} \longrightarrow \Sigma^{2^{i+1}-3}K$ as indicated in the diagram. Since

$S^{2^{i+1}-2} \xrightarrow{\bar{\theta}_i} X_2 \xrightarrow{f_2} K_2 = \cdots \times \Sigma^{2^{i+1}-2}K$ is non-trivial on the last factor it follows that $\bar{g}_2 = \bar{g}$. **Now** $g_1 = 0$ since $g_2 = j_3\bar{g}_2 = j_3\bar{g} = g = g_1 + g_2$. Q.E.D.

We now briefly describe the basic idea in our approach. We "skip" the map g_1(i.e. let $g_1 = 0$) to construct a spectrum $\bar{X}_2$ so as to get a $\bar{\theta}_i \in \pi_{2^{i+1}-2}(\bar{X}_2)$ as above and then try to compare $\bar{X}_2$ with X_2.

To construct $\bar{X}_2$ let Z denote the product of the factors in K_2 except $\Sigma^{2^{i+1}-2}K$; so $\Sigma^{-1}K = \Sigma^{-1}Z \times \Sigma^{2^{i+1}-3}K$. Let

$j_3'' = j_3|\Sigma^{-1}Z : \Sigma^{-1}Z \longrightarrow X_3$. Consider the subspectrum

$\Sigma^{-1}Y = \Sigma^{-1}Z \times (S^{2^{i+1}-3} \cup_{2\iota} e^{2^{i+1}-2})$ of $\Sigma^{-1}K$. Define

$j_3' : \Sigma^{-1}Y \longrightarrow X_3$ by $j_3'|\Sigma^{-1}Z = j_3''$ and $j_3'|S^{2^{i+1}-3} \cup_{2\iota} e^{2^{i+1}-2} = g_2$.

$\bar{X}_2$ is defined to be the cofiber of j_3' so that there is a cofiber sequence

$$\Sigma^{-1}Y \xrightarrow{j_3'} X_3 = X_3' \vee \Sigma^{2^{i+1}-2}K \xrightarrow{\bar{p}_3} \bar{X}_2 \xrightarrow{\bar{f}_2} Y. \tag{2}$$

The image of the generator $\iota \in \pi_{2^{i+1}-2}(\Sigma^{2^{i+1}-2}K) = \mathbb{Z}_2$ in $\bar{X}_2$ is still denoted by $\{h_{i+1}\}$. The proof of Lemma A shows that there is a class $\bar{\theta}_i : S^{2^{i+1}-2} \longrightarrow \bar{X}_2$ such that the composite

$$S^{2^{i+1}-2} \xrightarrow{\bar{\theta}_i} \bar{X}_2 \xrightarrow{\bar{f}_2} Y = Z \times (S^{2^{i+1}-2} \cup_{2\iota} e^{2^{i+1}-1})$$

$$\xrightarrow{p} S^{2^{i+1}-2} \cup_{2\iota} e^{2^{i+1}-1}$$

is the inclusion $S^{2^{i+1}-2} \longrightarrow S^{2^{i+1}-2} \cup_{2\iota} e^{2^{i+1}-1}$ and such that

$2\bar{\theta}_i = \{h_{i+1}\}$ where the last map p is the projection.

We can map the cofiber sequence (2) to the cofiber sequence

$$\Sigma^{-1}K = \cdots \times \Sigma^{2^{i+1}-2}K \xrightarrow{j_2} X_2 \xrightarrow{p_2} X_1 \xrightarrow{f_1} K_1$$

to get a commutative diagram

$$\begin{tikzcd}
\Sigma^{-1}Y \arrow[r, "j_3'"] \arrow[d, "p"] & X_3 \arrow[r, "\bar{p}_3"] \arrow[d, "p_3"] & \bar{X}_2 \arrow[r, "\bar{f}_2"] \arrow[d, "\bar{p}_2"] \arrow[dl, dashed, "f"'] & Y \arrow[d, "\Sigma p"] \\
\cdots \times \Sigma^{2^{i+1}-2}K=\Sigma^{-1}K_1 \arrow[r, "j_2"] & X_2 \arrow[r, "p_2"] & X_1 \arrow[r, "f_1"] & K_1
\end{tikzcd} \tag{3}$$

where $p|\Sigma^{-1}Z = 0$ and $p|S^{2^{i+1}-3}\cup_{2\iota} e^{2^{i+1}-2}$ is the composite

$$S^{2^{i+1}-3}\cup_{2\iota} e^{2^{i+1}-2} \xrightarrow{p} S^{2^{i+1}-2} \xrightarrow{\iota} \Sigma^{2^{i+1}-2}K \xrightarrow{i_2} \Sigma^{-1}K_1.$$

Let $\theta_i \in \pi_{2^{i+1}-2}(S^0)$ be defined to be the class

$S^{2^{i+1}-2} \xrightarrow{\bar{\theta}_i} \bar{X}_2 \xrightarrow{\bar{p}_2} X_1 \xrightarrow{p_1} S^0$. Then $2\theta_i = 0$

<u>Conjecture B</u>. $\theta_i \neq 0$ and is detected by h_i^2.

To see the plausibility of the truth of this conjecture we note that $f_1\bar{p}_2 = (\Sigma p)\bar{f}_2 = 0$: this is clear. So $\bar{p}_2$ can be lifted to a map $f : \bar{X}_2 \longrightarrow X_2$ as indicated in diagram (3). We have $H^*(\bar{X}_2) \cong H^*(X_2)$ for $* \leq 2^{i+1}-2$. This isomorphism follows by dimensional reason. If f^* induces the isomorphism (in this range of dimensions) then $\underset{\sim}{B}$ is true. Extensive calculations reveal that f^* could be an isomorphism. Conclusive work (if possible), however, remains to be done.

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Massey-Peterson Towers and Maps from Classifying Spaces

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The "Sullivan conjecture" [10] asserts that, given a finite-dimensional connected CW complex X and a finite group G, the space X^{BG} of pointed maps from the classifying space BG to X has the weak homotopy type of a point. This conjecture was resolved in the affirmative in [9]. It is then natural and important to ask about the mapping space X^{BG} for infinite dimensional spaces X. The situation then appears to be far more complex, even, for instance, when we take X to be the classifying space of a connected topological group. In this paper I shall stage a raid into this area. As proof of the riches to be found there, I offer the following:

Theorem A. For any elementary Abelian 2-group E, the classifying space functor B induces a weak homotopy equivalence

$$\mathrm{Hom}(E,SU_2) \to BSU_2^{BE}$$

from the discrete space of group homomorphisms from E to SU_2 to the indicated pointed mapping space. In particular, $\mathrm{Hom}(E,SU_2) \to [BE,BSU_2]$ is bijective.

The techniques used actually depend only on $H^*(X;\mathbb{F}_2)$, but operate only under the assumption that X is simply connected. Notice that if X is a simply connected CW complex whose mod 2 cohomology is polynomial on a single 4-dimensional generator, then ΩX is 2-locally equivalent to SU_2. A natural question arises: is X 2-locally equivalent to BSU_2? The following result shows that as far as maps from BE are concerned X and BSU_2 are indistinguishable.

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Theorem B. Let X be a simply connected CW complex whose mod 2 cohomology is a polynomial algebra on a single 4-dimensional class. Then $[B\mathbb{Z}/2,X]$ contains exactly two elements, one of which, call it f, induces a nontrivial map in mod 2 cohomology. Moreover for any elementary Abelian 2-group E, the map

$$\bar{H}^1(E;\mathbb{Z}/2) = [BE,B\mathbb{Z}/2] \to [BE,X]$$

induced by f is a bijection.

I will use the obstruction theory of Massey and Peterson ([8], [1], [7]). This theory applies to simply connected spaces whose mod p cohomology is "very nice" [4]. An unstable algebra B over the mod p Steenrod algebra A is <u>very nice</u> provided that it is of finite type and admits a simple system of generators whose vector space span is closed under the action of A. This is admittedly an awkward condition, but it does include the classical Lie groups and the complex and quaternonic Stiefel varieties, and at 2, the real Stiefel varieties as well. The Massey-Peterson theory should be regarded as a piece of light artillery, with which one can move quickly and execute small ambushes before wheeling in the heavy simplicial guns of Bousfield and Kan [5]. I note that a very elementary application of this theory shows that the algebraic theorem from [9], quoted below as (3.1), yields the Sullivan conjecture for elementary Abelian p-groups and simply connected spaces whose mod p cohomology is finite and very nice; see (3.2) below. This result is in part contained in:

Theorem C. Let E be an elementary Abelian p-group. Evaluation of mod p cohomology induces a map

$$[BE,X] \to \mathrm{Hom}(H^*X,H^*BE)$$

to the indicated set of A-algebra maps. If X is simply connected and H^*X is very nice, then this map is bijective.

Conjecture: This is still true if X is any simply connected space whose mod p homology is of finite type.

The Massey-Peterson theory will be reviewed in Section 1, with some improvements, due largely to J. R. Harper and A. Zabrodsky. A couple of technical results are proved in Section 2, and a convergence theorem, due to A. K. Bousfield, appears in Section 4. The theorems stated above are proved in Section 3, by application of an algebraic result from [9].

I am very grateful to John Harper, who tutored me patiently on Massey-Peterson theory, and to Alex Zabrodsky, who suggested a proof of the key Lemma 1.11 in conversation at Aarhus and later proposed the marvellous property (1.7) of Massey-Peterson towers used here to prove (1.11). I am also indebted to Pete Bousfield, Gunnar Carlsson, Mark Mahowald, and Jeff Smith, for their help. Finally, I thank the Mathematics Departments of Northwestern University and the University of Cambridge for their hospitality.

§1. Obstruction theory.

I shall begin by recalling briefly the theory of Massey and Peterson [8], [1], with improvements due to Harper [7] and Zabrodsky. Unless otherwise specified, $H^*(X)$ denotes the mod p cohomology of X, p an arbitrary prime.

Mod p cohomology in its richest form is a functor from pointed spaces to the category $\mathcal{A}^*$ of augmented unstable algebras over the Steenrod algebra A. Let $\mathcal{A}^*_{ft}$ denote the full subcategory of those of finite type. Formation of the augmentation ideal gives a functor I to the category $\mathcal{U}^*_{ft}$ of unstable left A-modules of finite type, and this functor has a left adjoint U [8], [1]. It is easy to verify that an object of $\mathcal{A}^*_{ft}$ is very nice in the sense of the introduction iff it is of the form $U(M)$ for some $M \in \mathcal{U}^*_{ft}$.

The functor U helps to relate algebra to geometry. The category $\mathcal{U}^*_{ft}$ has enough projective objects, and there is a contravariant association $P \mapsto K(P)$ of a mod p generalized Eilenberg-MacLane space to a projective in $\mathcal{U}^*_{ft}$, equipped with compatible natural isomorphisms

$$\pi_t(K(P)) \cong \mathrm{Hom}_A(P,S(t)) \tag{1.1}$$

$$H^*(K(P)) \cong U(P), \tag{1.2}$$

where $S(t) = \bar{H}^*(S^t)$.

There is a functor $\Omega : \mathcal{U}^*_{ft} \to \mathcal{U}^*_{ft}$ left adjoint to suspension Σ :

$$\mathrm{Hom}_A(\Omega M,N) \cong \mathrm{Hom}_A(M,\Sigma N). \tag{1.3}$$

Since Σ is exact, Ω carries projectives to projectives, and the isomorphism (1.2) is naturally compatible with Ω:

$$K(\Omega P) \simeq \Omega K(P). \tag{1.4}$$

Now let X be a simply connected space such that $H^*(X) \cong U(M)$, and let $M \leftarrow P_\bullet$ be a projective resolution of M in $\mathcal{U}^*_{ft}$. There is a tower of principal fibrations under X:

$$\begin{tikzcd}
 & \vdots \arrow[d, phantom] & \\
 & X_2 \arrow[r, "k_2"] \arrow[d] & K(\Omega^2 P_3) \\
 & X_1 \arrow[r, "k_1"] \arrow[d] & K(\Omega P_2) \\
X \arrow[r, "j_0"] \arrow[ur, "j_1"] \arrow[uur, "j_2"] & K(P_0) \arrow[r, "k_0"] & K(P_1)
\end{tikzcd} \tag{1.5}$$

such that

(1.6) $$\ker(H^*(X_s) \to H^*(X)) = \ker(H^*(X_s) \to H^*(X_{s+1}));$$ and

(1.7) k_s is induced by a null-homotopy of $d_s k_{s-1}$. That is, there exists a commutative square

$$\begin{tikzcd}
X_{s-1} \arrow[r, "h_s"] \arrow[d, "k_{s-1}"] & PK(\Omega^{s-1}P_{s+1}) \arrow[d, "\pi"] \\
K(\Omega^{s-1}P_s) \arrow[r, "d_s"] & K(\Omega^{s-1}P_{s+1}),
\end{tikzcd}$$

in which π is the path-space fibration, such that the induced map $X_s \to K(\Omega^s P_{s+1})$ of homotopy fibers is homotopic to k_s. Here and below I write d_s for any map induced by d_s.

Property (1.7) was suggested by Zabrodsky. It appears to be a fundamental feature of Massey-Peterson towers, and it may be possible to give a treatment of the subject in which it occupies a central position. For the present, however, I give a derivation of it from other known properties in the next section, and treat it as an axiom in this section.

By applying π_* to (1.5) one obtains a spectral sequence with

$$E_2^{s,t} = \mathrm{Ext}^s(M,S(t)) \Rightarrow \pi_{t-s}(X).$$

The Ext group here, and below, is computed in the category $\mathcal{U}^*_{ft}$, or, equivalently, in $\mathcal{U}^*$. The goal of the present paper is to show that under certain circumstances, the Massey-Peterson machinery allows one to draw conclusions about $[Y,X]$ for Y not even a suspension, given the assumptions one expects to demand by analogy with this spectral sequence.

Theorem 1.8. Let Y be a connected CW complex such that $\bar{H}_*(Y;\mathbb{Z})$ is of finite type and p-torsion, and let X be a simply connected space such that $H^*(X)$ is of finite type and isomorphic to $U(M)$. Consider the map

$$H^* : [Y,X] \to \mathrm{Hom}_A(M,\bar{H}^*(Y)).$$

Then H^* is (a) monic if $\mathrm{Ext}^s(M,\bar{H}^*(\Sigma^s Y)) = 0$ for all $s > 0$ and (b) epic if $\mathrm{Ext}^{s+1}(M,\bar{H}^*(\Sigma^s Y)) = 0$ for all $s > 0$.

This theorem and the method of proof presented below are for the most part due to Harper ([7] 2.2.1, for example). I have chosen a different set of convergence conditions. Moreover, an improvement will be noticed in part (a), for Harper proves only that, under the stated assumptions, $f \simeq *$ if $f^* = 0$. That proof is easier, requiring, aside from (1.6), only the elementary fact that $k_s i_s \simeq d_s$, where $i_s : K(\Omega^s P_s) \to X_s$ is the inclusion of the fiber over $*$. This restricted form of Theorem 1.8(a) is in fact all that is needed to prove the cases of the Sullivan conjecture considered here, Theorems 3.2 and 3.3. The full strength of (1.8) is required, however, to prove the theorems stated in the introduction.

Before starting the proof of Theorem 1.8, it is convenient to record a couple of consequences of Zabrodsky's observation (1.7). They both involve principal actions, for which I need some notation. Given a map $k : X \to B$, I shall write $\alpha_k : \Omega B \times E_k \to E_k$, or just α, for the action of ΩB on the homotopy fiber E_k of k. Also, given $f : Y \to E_k$ and $h : Y \to \Omega B$, I shall write $h * f$ for the composite

$$Y \xrightarrow{\Delta} Y \times Y \xrightarrow{h\times f} \Omega B \times E_k \xrightarrow{\alpha} E_k$$

The following lemma is a restatement of "primitivity of the principal action" [7] 1.2.6.

Lemma 1.9. The k-invariants are linear over the algebraic differential. That is, the following diagram is homotopy commutative.

$$\begin{array}{ccc} K(\Omega^s P_s) \times X_s & \xrightarrow{\alpha} & X_s \\ \downarrow d_s \times k_s & & \downarrow k_s \\ K(\Omega^s P_{s+1}) \times K(\Omega^s P_{s+1}) & \xrightarrow{\mu} & K(\Omega^s P_{s+1}). \end{array}$$

Proof. Use naturality of the principal actions resulting from (1.7), and the fact that $\mu \simeq \alpha_\pi$. □

Corollary 1.10. Let $f : Y \to X_s$ and $h : Y \to K(\Omega^s P_s)$. Then

$$k_s(h*f) \simeq d_s h * k_s f. \quad \square$$

Lemma 1.11. The following diagram is homotopy-commutative.

$$\begin{array}{ccc} K(\Omega^s P_{s-1}) \times X_s & \xrightarrow{d_s \times 1} & K(\Omega^s P_s) \times X_s \\ & \searrow^{pr_2} & \downarrow \alpha \\ & & X_s \end{array}$$

It is in order to prove Lemma 1.11 that property (1.7) was introduced here. However, the proof of this lemma involves a technical result about compatibility of various principal actions which, in order not to further delay presentation of the proof of Theorem 1.8, I have placed in the next section.

Proof of Theorem 1.8. I shall prove part (a), and leave the proof of part (b), which is similar and somewhat easier, to you. So let Y be a connected CW complex such that $H_*(Y)$ is of finite type, let X and M be as in the statement of the theorem, and suppose that $f,g : Y \to X$ induce the same map in cohomology. Then the composites $f_0, g_0 : Y \to X \to K(P_0)$ are homotopic. I will now show that $f_s, g_s : Y \to X \to X_s$ are homotopic provided f_{s-1} and g_{s-1} are. By principality of $X_s \to X_{s-1}$, there is a map $h : Y \to K(\Omega^s P_s)$ such that $g_s \simeq h * f_s$. Thus by (1.10), $k_s g_s \simeq d_s h * k_s f_s$. Now f_s and g_s both lift to X_{s+1}, so $k_s f_s$ and $k_s g_s$ are both null-homotopic, and since $[Y, K(\Omega^s P_{s+1})]$ is a group under $*$, it follows that $d_s h \simeq *$. Thus $h^* | \Omega^s P_s \in \mathrm{Hom}_A(\Omega^s P_s, \bar{H}^*(Y)) = \mathrm{Hom}_A(P_s, \bar{H}^*(\Sigma^s Y))$ is a cocycle. By assumption, it is therefore also a coboundary; that is, h factors through $d_{s-1} : K(\Omega^s P_{s-1}) \to K(\Omega^s P_s)$. Lemma 1.11 then implies that $h * f_s$ is homotopic to f_s, as claimed.

Now the issue of whether the homotopies $f_s \simeq g_s$ together yield a homotopy $f \simeq g$ is a question of convergence, and will be dealt with in Section 4. This finishes my treatment of Theorem 1.8. $\square$

§2. Two proofs.

It is now time to prove (1.7). The proof is based on:

Lemma 2.1. The composite

$$\delta : K(\Omega^s P_s) \times X \xrightarrow{1 \times j_s} K(\Omega^s P_s) \times X_s \to X_s$$

induces a monomorphism in cohomology.

Proof. This follows from a comparison of the "fundamental sequences" [7] associated to the vertical fibration sequences in the homotopy commutative diagram

$$\begin{array}{ccccc}
K(\Omega^s P_s)\times X & \longrightarrow & K(\Omega^s P_s)\times X & \xrightarrow{\alpha} & X_s \\
\downarrow & & \downarrow & & \downarrow \\
PK(\Omega^{s-1}P_s)\times X & \longrightarrow & PK(\Omega^{s-1}P_s)\times X_{s-1} & \xrightarrow{pr_2} & X_{s-1} \\
\downarrow & & \downarrow & & \downarrow \\
K(\Omega^{s-1}P_s) & \xrightarrow{in_1} & K(\Omega^{s-1}P_s)\times K(\Omega^{s-1}P_s) & \xrightarrow{\mu} & K(\Omega^{s-1}P_s),
\end{array}$$

by analogy with the proof of [7] 1.2.6. □

Lemma 2.2. The k-invariant k_s may be characterized as the unique map k such that (a) the diagram

$$\begin{array}{ccc}
K(\Omega^s P_s)\times X_s & \xrightarrow{\alpha} & X_s \\
\downarrow d_s\times k & & \downarrow k \\
K(\Omega^s P_{s+1})\times K(\Omega^s P_{s+1}) & \xrightarrow{\mu} & K(\Omega^s P_{s+1})
\end{array}$$

is homotopy commutative and (b) $kj_s : X \to K(\Omega^s P_{s+1})$ is null-homotopic.

Proof. The k-invariant k_s satisfies (a) by virtue of primitivity of the principal action, [7], and (b) since j_s lifts to j_{s+1}. On the other hand, (a) and (b) together allow one to compute that
$\delta^* k = d_{s+1}pr_1 : K(\Omega^s P_s)\times X \to K(\Omega^s P_{s+1})$; but δ^* is monic by Lemma 2.1. □

Proof of (1.7). It follows easily from the compatability of the splitting of the fundamental sequence with the k-invariant k_{s-1} that $d_s k_{s-1} \simeq *$. Pick a null-homotopy h, and look at the commutative diagram

$$\begin{tikzcd}
 & X_s \arrow[r, "k"] \arrow[d] & K(\Omega^s P_{s+1}) \arrow[d] \\
X \arrow[ur, "j_s"] \arrow[r, "j_{s-1}"] & X_{s-1} \arrow[r, "h"] \arrow[d, "k_{s-1}"] & PK(\Omega^{s-1} P_{s+1}) \arrow[d, "\pi"] \\
 & K(\Omega^{s-1} P_s) \arrow[r, "d_s"] & K(\Omega^{s-1} P_{s+1}).
\end{tikzcd}$$

Any such k satisfies (a) of Lemma 2.2, as noted in the proof of Lemma 1.9. To complete the proof, it therefore suffices to alter h to another null-homotopy h_s such that the map k' induced on homotopy fibers satisfies $k'j_s \simeq *$. Since j_{s-1} is epic in cohomology, kj_s factors as ℓj_{s-1} for some $\ell : X_{s-1} \to K(\Omega^s P_{s+1})$. If χ reverses paths and $*$ juxtaposes them, then $h_s = \chi\ell * h$ has the desired property. $\square$

<u>Proof of Lemma 1.11</u>. This is based on the following technical result about principal actions.

<u>Proposition 2.3</u>. Let h be a null-homotopy of a composite gf, and construct homotopy fibers to produce a commutative diagram

$$\begin{tikzcd}
 & F \arrow[r, "k"] \arrow[d] & \Omega Z \arrow[d] \\
H \arrow[r] \arrow[d, "\ell"] & X \arrow[r, "h"] \arrow[d, "f"] & PZ \arrow[d, "\pi"] \\
G \arrow[r] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Then the homotopy fibers of k and of ℓ are identical, and if we call this common space E, then the following diagram is homotopy commutative.

$$\begin{tikzcd}
\Omega^2 Z\times\Omega G\times E \arrow[r, "\Omega\alpha_g\times 1"] \arrow[d, "1\times\alpha_\ell"] & \Omega G\times E \arrow[d, "\alpha_\ell"] \\
\Omega^2 Z\times E \arrow[r, "\alpha_k"] & E
\end{tikzcd}$$

To prove this proposition, draw pictures of the elements of the spaces involved; you will see that the homotopy required is similar to the one showing that a double loop space is homotopy commutative. It is convenient to remember that when ΩZ is regarded as the homotopy fiber of π, it maps to PZ by sending a loop to the reverse of its second half.

By including $*$ into ΩG, we find that α_k factors as $\alpha_\ell(\Omega i\times 1)$ where $i : \Omega Z \to G$ is the natural map. Since $i \circ \Omega g \simeq *$, this implies:

<u>Corollary</u> 2.4. Let h be a null-homotopy of a composite gf, and construct homotopy fibers to produce a commutative diagram

$$\begin{tikzcd}
F \arrow[r, "k"] \arrow[d] & \Omega Z \arrow[d] \\
X \arrow[r, "h"] \arrow[d, "f"] & PZ \arrow[d, "\pi"] \\
Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Then

$$\begin{tikzcd}
\Omega^2 Y\times E \arrow[r, "\Omega^2 g\times 1"] \arrow[dr, "pr_2"'] & \Omega^2 Z\times E \arrow[d, "\alpha_k"] \\
& E
\end{tikzcd}$$

is homotopy-commutative. □

Lemma 1.11 follows from an application of this Corollary to (1.7). $\square$

§3. Applications.

To apply Theorem 1.8 when Y is a suspension of the classifying space BE of an elementary Abelian p-group, I recall from [9] a basic vanishing theorem.

Theorem 3.1. Let M be an unstable left A-module of finite type. Then

$$\mathrm{Ext}^s(M,\bar{H}^*(\Sigma^n BE)) = 0$$

(a) for any $s > n \geq 0$ and for $s = n > 0$; and

(b) for any $s,n \geq 0$ if M is finite.

Theorem C follows immediately from this and Theorem 1.8. Notice, by the way, that since

$$\pi_n(X^{BE},*) = [\Sigma^n BE,X],$$

these theorems also imply:

Theorem 3.2. If E is an elementary Abelian p-group and X a simply connected space whose mod p cohomology is finite and very nice, then X^{BE} is weakly contractible.

Moreover:

Theorem 3.3. The Sullivan conjecture is valid for elementary Abelian p-groups and spheres.

Proof. Since $[\Sigma^n BG,S^1] = \bar{H}^1(\Sigma^n BG;\mathbb{Z}) = 0$ for any $n \geq 0$ and any finite group G, the Sullivan conjecture for G arbitrary is trivial for $X = S^1$. The case of $X = S^m$ for $m > 1$ with m odd or $p = 2$ is covered by Theorem

3.1. The remaining case is dealt with using the following trick, which I owe to J. R. Harper.

Let $J_{p-1}S^{2k}$ denote the skeleton of the James construction on S^{2k} for which $H^*(J_{p-1}S^{2k}) = U(S(2k))$. Let F_{2k} be the homotopy fiber of the natural map $S^{2k} \to J_{p-1}S^{2k}$. Then an easy computation shows that $H^*(F_{2k}) = U(M)$ where

$$M = \langle x_{4k-1}, y_{(2pk-2)p^i} : i \geq 0\rangle$$

with trivial A-action. Since Ext is additive, we find that

$$\mathrm{Ext}^s(M, \bar{H}^*(\Sigma^n\ BE)) = 0, \qquad n,s \geq 0,$$

and so, from Theorem 1.8, F_k^{BE} is weakly contractible. Since $(J_{p-1}S^{2k})^{BE}$ is too, from Theorem 3.2, the result follows from the homotopy long exact sequence of a fibration. □

Many other spaces which are not $U(M)$'s may be handled by analogous tricks.

To prove Theorem B, let M_k be the A-module generated over $\mathbb{F}_2$ by $\{x_i : i \geq k\}$, with $|x_i| = 2^i$ and $Sq^{2^i} x_i = x_{i+1}$. Then $U(M_2)$ is the unique A-algebra which as an $\mathbb{F}_2$-algebra is polynomial on a single 4-dimensional generator. Thus Theorem C shows that

$$[BE, X] \xrightarrow{\cong} \mathrm{Hom}_A(M_2, \bar{H}^* BE).$$

With $E = \mathbb{Z}/2$, the latter set clearly has order two, proving the first assertion. Note that $H^*(B\mathbb{Z}/2) \cong U(M_0)$, and that the nontrivial map $B\mathbb{Z}/2 \to X$ induces $U(i)$ in cohomology, where $i : M_2 \to M_0$ is the inclusion. Now the rest of Theorem B follows from the commutative diagram

$$\begin{array}{ccc} [BE,B\mathbb{Z}/2] & \xrightarrow{\cong} & \mathrm{Hom}_A(M_0,\bar{H}^*(BE)) \\ f\downarrow & & i\downarrow\cong \\ [BE,X] & \xrightarrow{\cong} & \mathrm{Hom}_A(M_2,\bar{H}^*(BE)) \end{array}$$

in which the bottom arrow is iso by Theorem C. □

Theorem 3.4. In the situation of Theorem B, the component of X^{BE} which contains the trivial map is weakly contractible.

Proof. There are short exact sequences

$$0 \to M_k \to M_0 \to M_0^{k-1} \to 0$$

of A-modules, with M_0^{k-1} finite. The long exact sequence induced in $\mathrm{Ext}^*(-,\bar{H}^*(\Sigma^n BE))$, together with Theorem 3.1, shows that

$$\mathrm{Ext}^s(M_k,\bar{H}^*(\Sigma^n BE)) \to \mathrm{Ext}^s(M_0,\bar{H}^*(\Sigma^n BE))$$

is an isomorphism. But M_0 is projective in the category $\mathcal{U}^*$, so we conclude that for all $s > 0$, $n \geq 0$, and $k \geq 0$,

$$\mathrm{Ext}^s(M_k,\bar{H}^*(\Sigma^n BE)) = 0.$$

It is easy to see that for $n > 0$, this group is also zero when $s = 0$; so, putting $k = 2$, Theorem 1.8 gives

$$\pi_n(X^{BE},*) = [\Sigma^n BE,X] = 0. \qquad \square$$

It would be interesting to get information on the homotopy type of the other components of X^{BE}. When $X = BSU_2$, one may argue as follows. Since the center of SU_2 is Z_2, there is a group homomorphism $Z_2 \times SU_2 \to SU_2$, inducing a pointed map $BZ_2 \times BSU_2 \to BSU_2$. Pass to spaces of pointed maps from BE; the Abelian group BZ_2^{BE} acts on BSU_2^{BE}. If $h : E \to Z_2$ is a homomorphism, then the action by Bh provides a homotopy equivalence from the component of BSU_2^{BE} containing the trivial map to the component containing hf, where $f : BZ_2 \to BSU_2$ is induced by the inclusion. This completes the proof of Theorem A.

§4. Convergence.

The final task is to prove a convergence theorem. While Massey and Peterson [8] did important work on this issue, it seems better to appeal to the now standard work of Bousfield and Kan [5]; so move to the simplicial framework by passing to singular simplicial sets. To relate a Massey-Peterson tower for X to the p-adic completion $(\mathbb{Z}/p)_\infty X$ of [5], we have:

Lemma 4.1. Let X be a simply-connected space such that $H^*(X)$ is of finite type and very nice. Let (1.5) be a Massey-Peterson tower for X. Then $\{X_i\}$ and $\{(\mathbb{Z}/p)_i X\}$ are weakly equivalent prosystems.

Proof. By [5] III §5.5, p. 84 and induction, each X_i is $\mathbb{Z}/p$-nilpotent. By (1.6), the first image prosystem $\{\mathrm{Im}(H_*(X_i) \to H_*(X_{i-1}))\}$ is the constant system $\{H_*(X)\}$. Thus $\{X_i\}$ is a $\mathbb{Z}/p$-tower for X, so the result follows from [5] III §6.4, p. 88. □

According to [5] VIII §3, homotopy classes of maps agree in the categories of CW complexes and of simplicial sets; so the following theorem is sufficient for our purpose.

Theorem 4.2. Suppose that X is connected and nilpotent and that Y is connected with $\bar{H}_*(Y;\mathbb{Z}[\frac{1}{p}]) = 0$. Then the map $X \to (\mathbb{Z}/p)_\infty X$ induces an equivalence of pointed mapping spaces

$$X^Y \to ((\mathbb{Z}/p)_\infty X)^Y.$$

The statement of the theorem in this generality and the proof given here are both due to A. K. Bousfield, and I am grateful to him for allowing me to reproduce them.

<u>Proof</u>. Recall from [6] that there is, up to homotopy, a fiber square

$$\begin{array}{ccc} X & \to & \Pi_\ell X_{\mathbb{Z}/\ell} \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \to & (\Pi_\ell X_{\mathbb{Z}/\ell})_{\mathbb{Q}} \end{array}$$

where X_A denotes the Bousfield $H_*(-;A)$-localization of X [2]. Thus there is, up to homotopy, an analogous fiber square of pointed function spaces with source space Y. Now Proposition 12.2 of [2] easily implies that C^B is contractible whenever B is h_*-acyclic and C is h_*-local. Taking $h_*(-) = H_*(-;\mathbb{Z}[\frac{1}{p}])$, it follows that $(Z_A)^Y \simeq *$ for any space Z, where $A = \mathbb{Q}$ or $A = \mathbb{Z}/\ell$ with ℓ prime to p. Thus the fiber square implies that the map

$$X^Y \to (X_{\mathbb{Z}/p})^Y$$

is an equivalence, and the proposition follows since $X_{\mathbb{Z}/p} \cong (\mathbb{Z}/p)_\infty X$ by §4 of [2]. □

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MULTIPLICATIVE STRUCTURE OF FINITE RING SPECTRA

AND STABLE HOMOTOPY OF SPHERES

Shichiro Oka

Throughout this paper, p will denote a prime with $p \geqq 5$.

In this paper, we develope ideas, in somewhat general situation, with which we have proved [15] that the stable homotopy group of spheres, π_*^S, has arbitrarily many p-primary generators in dimension t for sufficiently large t. There is a 4-cell ring spectrum K_s with BP-homology $BP_*(K_s) = BP_*/(p, v_1^s)$ for $s \geqq 1$ [14], and if $s \equiv 0$ mod p it has a property that $K_s \wedge K_s$ splits into four copies of K_s [15]. We called K_s with such splitting a split ring spectrum. The generators of π_*^S in our results in [15] are constructed from stable self-maps of split ring spectra K_s, i.e., K_s with $s \equiv 0$ mod p, and they are detected in Ext^2 of Adams-Novikov's E_2 term.

In this paper, we analyse the structure of $K_s{}^*K_s$ including the case $s \not\equiv 0$ mod p, i.e., non-split case, with which the self-maps required are constructed. It makes our construction easier in the following points. The construction here depends only on the structure of the subring of K_s-module maps and on checking the coboundary $\delta' \in K_s{}^*K_s$ to be a derivation as a cohomology operation, though the whole of $K_s{}^*K_s$ is discussed in [15]. This is due to a suggestion given by Professor J. F. Adams on the occasion of the conference (see Remark 2.4, 6.7). Secondly, including the case $K_1 = V(1)$ of smallest dimension allows us to reduce extremely our input of induction constructing elements. In fact, Input I at the end of §2 is obtainable with information of π_*^S through dimension $2(p^2-1)$, even though the dimension reaches at least $4p(p^2-1)$ to get Input III, the input required in [15].

The expanded method given here is even stronger. It is applicable to finite ring spectra and self-maps of them having similar property. We will give in §4 an application giving the "fringe family" (in the sense of [24]) for the so-called gamma family. One more application, which is concerned with higher order elements in π_*^S of BP-filtration 3, will appear elsewhere.

Our method for K_s is explained in §2, but the proof of Theorem 2.5, which asserts that K_s is commutative and associative and δ' is a derivation for all $s \geqq 1$, is given later. Section 1 is the recollection on the Adams-Novikov spectral sequence which is used to detect our homotopy elements. A shorter proof of the aforementioned result on π_*^S is given in §3. A $V(2)$ analogue of a weak form of §§1-3 is given in §4. In sections 5-6, we give a strong form of commutativity and associativity of K_s (Theorems 6.5, 6.6) including the proof of Theorem 2.5 with an analogous discussion of [15]§4.

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§1. The Adams-Novikov spectral sequence

Let BP denote the Brown-Peterson spectrum at $p > 3$, [16]. It defines a homology theory with coefficient ring

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots], \quad |v_i| = 2(p^i-1).$$

The BP homology of a spectrum X, $BP_*(X)$, is equipped with natural comodule structure over the coalgebra (Hopf algebroid) BP_*BP over BP_* [2]. For a BP_*BP-comodule M, let us denote

$$H^sM = \mathrm{Ext}^s_{BP_*BP}(BP_*, M),$$

where the Ext is the extension in the category of BP_*BP-comodules. There is a spectral sequence due to Adams [1] and Novikov [9], with E_2 term $H^*BP_*(X)$, converging, if X is connective, to $\pi_*(X)_{(p)}$. The E_2 term for $X = S^0$, H^*BP_*, has been determined in (cohomological) dimensions 1 [9] and 2 [7].

For $x \in BP_*$ and ideals I, J in BP_* with $I \subset J$, the multiplication by x induces a BP_*-homomorphism $BP_*/I \longrightarrow BP_*/J$, $y \bmod I \mapsto xy \bmod J$, which is also denoted by x. By [5], $BP_*/(p, v_1)$ is a BP_*BP-comodule and

$$H^0BP_*/(p, v_1) = \mathrm{Hom}^*_{BP_*BP}(BP_*, BP_*/(p, v_1)) = \mathbb{Z}/p[v_2].$$

This implies $v_2^{p^n} \in H^0BP_*/(p, v_1^{p^n})$ [24], and hence,

(1.1) $$v_2^t \in H^0BP_*/(p, v_1^s) \quad \text{if} \quad s \leq p^{\nu(t)},$$

where $\nu(t)$ is the highest exponent ν such that $p^\nu \mid t$. We define

(1.2) $$\beta_{t/s} = \delta\delta_s(v_2^t) \in H^2BP_*, \qquad 1 \leq s \leq p^{\nu(t)}, \quad t \geq 1,$$

where δ and δ_s are, respectively, the coboundaries of the long exact sequences of Ext associated to the following short exact sequences:

(1.3) $$E: \; 0 \longrightarrow BP_* \xrightarrow{p} BP_* \longrightarrow BP_*/(p) \longrightarrow 0,$$

$$E_s: \; 0 \longrightarrow BP_*/(p) \xrightarrow{v_1^s} BP_*/(p) \longrightarrow BP_*/(p, v_1^s) \longrightarrow 0.$$

The internal dimension of $\beta_{t/s}$ is $(t(p+1) - s)q$, $q = 2(p-1)$, so different betas may occure in the same dimension.

Lemma 1.1. *The elements* $\beta_{t/s}$ *are non-zero and of order* p. *They are linearly independent over* $\mathbb{Z}/p$.

This is proved by Zahler [24], Theorem 1,(a).Miller, Ravenel and

Wilson [7], [8] proved that more elements are needed to describe the subgroup of H^2BP_* consisting of elements of order p. Moreover they proved

<u>Lemma</u> 1.2. [7] p <u>divides</u> $\beta_{t/s}$ <u>if and only if</u> $\nu(t) \geq 2$, $s = s'p$ <u>with</u> $1 \leq s' \leq p^{\nu(t)-2} + p^{\nu(t)-3} - 1$ ($s' = 1$ <u>if</u> $\nu(t) = 2$).

Observing the dimension of $\beta_{t/s}$, we see that the p-rank of $H^{2,t}BP_*$ becomes arbitrarily large; for example, if $t = p^n q$ $H^{2,t}BP_*$ has at least (and exactly, by [7]) $\left[\frac{n+1}{2}\right]$ generators $\beta_{a_k p^{n+1-2k}/p^{n+1-2k}}$, $1 \leq k \leq \left[\frac{n+1}{2}\right]$, where $a_k = (p^{2k-1}+1)/(p+1)$.

§2. Constructing homotopy elements

We shall try to realise geometrically each step in constructing $\beta_{t/s}$ with further restriction of s, say $1 \leq s \leq N(t)$, where $N(t) \longrightarrow \infty$ as $t \longrightarrow \infty$, to keep elements in π_*^S in the same dimension as many as we want.

Let M be the mod p Moore spectrum $S^0 \cup_p e^1$. The cofibration

$$C: \quad S^0 \xrightarrow{p} S^0 \xrightarrow{i} M \xrightarrow{j} S^1,$$

with last map j omitted, clearly realises the exact sequence E in (1.3) as its BP homology. By [17], $v_1 : BP_*/(p) \longrightarrow BP_*/(p)$ is realised by the Adams-Toda map

$$\alpha : \Sigma^q M \longrightarrow M, \quad q = 2(p-1),$$

Let $\phi = \alpha^s : \Sigma^{sq} M \longrightarrow M$ and K_s be the cofibre of ϕ. Then the cofibration

$$C_s : \Sigma^{sq} M \xrightarrow{\phi} M \xrightarrow{i'} K_s \xrightarrow{j'} \Sigma^{sq+1} M,$$

with j' omitted, realises the second exact sequence E_s.

Now, realising the element v_2^t of (1.1) is equivalent to constructing a stable map

$$(2.1) \qquad f_{t,s} : S^{tq'} \longrightarrow K_s \quad \text{such that} \quad (f_{t,s})_* = v_2^t,$$

where $q' = 2(p^2-1) = (p+1)q$. If we were done it, the construction of homotopy element which corresponds with $\beta_{t/s}$ is immediate, that is,

<u>Lemma</u> 2.1. <u>If</u> $f_{t,s}$ <u>exists for some</u> t, s, <u>then</u> $\beta_{t/s}$ <u>is a permanent cycle which is represented at</u> E_∞ <u>by the element</u> $b_{t/s} = jj'f_{t,s} \in \pi_*^S$. <u>Moreover</u> $b_{t/s}$ <u>is non-trivial, of order</u> p, <u>and a scalor multiple of an indecomposable; if it is divisible by</u> p <u>then</u> $\nu(t) \geq 2$ <u>and</u> $s = s'p$ <u>with</u> $s' \leq p^{\nu(t)-2}+p^{\nu(t)-3}-1$ $(s' = 1$ <u>if</u> $\nu(t) = 2)$.

<u>Proof</u>. By Lemma 2.10, Theorem 8.18 in [7] and Lemma 1.2. □

As we proved in [14]Example 5.7, K_s has a multiplication $\mu : K_s \wedge K_s \longrightarrow K_s$ with unit $i_0 = i'i : S^0 \longrightarrow K_s$. For a map $f : S^m \longrightarrow K_s$, set

$$\bar{f} = \mu(f \wedge 1_K) : \Sigma^m K_s \longrightarrow K_s \wedge K_s \longrightarrow K_s.$$

Then $\bar{f}i_0 = \mu(1 \wedge i_0)f = f$, i.e., $\bar{f}$ is an extension of f. Since i_0 induces the canonical projection $BP_* \longrightarrow BP_*/(p, v_1^s)$, $\bar{f}_*$ is the multiplication by the same element in BP_* as of f_*. As is well known (e.g. [10]Lemma 1.5), the composite $\Sigma^{-1}K_s \xrightarrow{j'} \Sigma^{sq}M \xrightarrow{i'} \Sigma^{sq}K_{s'}$ induces the cofibration

$$C' : \qquad \Sigma^{sq}K_{s'} \longrightarrow K_{s+s'} \xrightarrow{\rho} K_s,$$

which realises the short exact sequence

$$E' : \quad 0 \longrightarrow BP_*/(p,v_1^{s'}) \xrightarrow{v_1^s} BP_*/(p,v_1^{s+s'}) \longrightarrow BP_*/(p,v_1^s) \longrightarrow 0,$$

where the last map ρ_* is the canonical projection. Our construction of $f_{t,s}$ is then made with the following three procedures:

<u>Construction</u> I (<u>Decreasing</u> s). <u>If</u> $f_{t,s}$ <u>exists, then</u> $f_{t,s'}$ <u>exists for all</u> $s' \leq s$.

<u>Construction</u> II(<u>Increasing</u> t). <u>If</u> $f_{t,s}$ <u>exists, then</u> $f_{nt,s}$ <u>exists for all</u> $n \geq 1$.

<u>Construction</u> III (<u>Increasing</u> s <u>with</u> t). <u>If</u> $f_{t,s}$ <u>exists and its extension</u> $\bar{f}_{t,s}$ <u>satisfies</u> $(\bar{f}_{t,s})^p\delta' = \delta'(\bar{f}_{t,s})^p$, <u>where</u> $\delta' = i'j' : \Sigma^{-sq-1}K_s \longrightarrow K_s$, <u>then there exists</u> $g_{tp,2s} \in \pi_*(K_{2s})$ <u>such that</u> $(g_{tp,2s})_* = v_2^{tp} + v_1^s x$ <u>for some</u> $x \in BP_*$. <u>With the same assumption</u>, $f_{tp^2,2s}$ <u>exists</u>.

<u>Proof of</u> I, II, III. We may put $f_{t,s'} = \rho f_{t,s}$ for I and $f_{nt,s} = (\bar{f}_{t,s})^n i_0$ for II, where $\rho: K_s \longrightarrow K_{s'}$ is the map in C'. For III, the commutativity with δ' defines a self-map $\bar{g}$ of the cofibre K_{2s} of δ', which induces, by E', $v_2^{tp} + y$ with $v_1^s y \equiv 0 \mod (p, v_1^{2s})$, i.e. $y \equiv v_1^s x \mod (p, v_1^{2s})$ for some x. Put $g_{tp,2s} = \bar{g} i_0$, $f_{tp^2,2s} = \bar{g}^p i_0$. □

The extra term $v_1^s x$ in III may possible be non-trivial if t is large. In order to include such a $g_{tp,2s}$, we modify the property (2.1). Consider a map $g_{t,s} : S^{tq'} \longrightarrow K_s$ with the property

(2.1)' $\quad (g_{t,s})_* = v_2^t + v_1^r x$ for some r with $rp \geqq s$ and some $x \in BP_*$.

Then we can immediately get the following analogue for g with (2.1)'.

<u>Constructions</u> I',II',III'. <u>If</u> $g_{t,s}$ <u>exists, then</u>

I' : $\quad g_{t,s'}$ <u>for all</u> $s' \leqq s$,

II': $\quad g_{nt,s'}$, $f_{ntp,s}$ <u>for all</u> $n \geqq 1$,

<u>exist. If moreover</u> $(\bar{g}_{t,s})^p \delta' = \delta' (\bar{g}_{t,s})^p$, <u>then</u>

III': $\quad g_{tp,2s}$,

<u>and hence</u> $g_{ntp,2s}$, $f_{ntp^2,2s}$ <u>by</u> II', <u>exist</u>.

To complete the induction, we shall try to remove the extra condition $(\bar{g}_{t,s})^p \delta' = \delta' (\bar{g}_{t,s})^p$ in III'. As usual, we put

$$K_s * K_s = [K_s, \Sigma^* K_s], \quad K_s^* = \pi_{-*}(K_s).$$

<u>Definition</u> 2.2. Mod $= \{ f \in K_s * K_s \mid \mu(f \wedge 1_K) = f\mu \}$,

Der $= \{ f \in K_s * K_s \mid \mu(f \wedge 1_K) + \mu(1_K \wedge f) = f\mu \}$,

<u>that is</u>, Mod <u>consists of right</u> K_s<u>-module maps (not necessarily associative) and</u> Der <u>consists of elements which behave as a derivation on the cohomology defined by</u> K_s.

<u>Lemma</u> 2.3.(a) <u>Assume the multiplication</u> μ <u>of</u> K_s <u>is commutative. Then</u> Mod <u>is a commutative subring of</u> $K_s * K_s$ <u>and</u> $[f, g] \in$ Mod, $f^p g = g f^p$ <u>for</u> $f \in$ Mod, $g \in$ Der; <u>in particular</u> $f^p \delta' = \delta' f^p$ <u>for</u> $f \in$ Mod <u>if</u> $\delta' \in$ Der. <u>Here the multiplication of</u> $K_s * K_s$ <u>is, as usual, the one given by the composition of maps, and</u> $[f, g]$ <u>denotes the (graded) commutator</u> $fg - (-1)^{mn} gf$, $m = |f|$, $n = |g|$.

(b) <u>Assume the multiplication of</u> K_s <u>is associative. Then</u> $(i_0)^* : K_s * K_s \longrightarrow K_s^*$ <u>induces an isomorphism</u> Mod $\cong K_s^*$, <u>and</u>

$$K_s * K_s = \mathrm{Mod} \oplus \mathrm{Ker}(i_0)^*.$$

Proof. (a) A right K_s-module map is a left K_s-module map. The commutativity of Mod then follows from $(f \wedge 1)(1 \wedge g) = \pm(1 \wedge g)(f \wedge 1)$. $[\mathrm{Mod}, \mathrm{Der}] \subset \mathrm{Mod}$ is immediate from the definition. Then $[f, [f, g]] = 0$ for $f \in \mathrm{Mod}$, $g \in \mathrm{Der}$. This implies $f^p g = g f^p$ since $K_s * K_s$ is a $\mathbb{Z}/p$-module, cf. Proof of Theorem 1.5 in [15].

(b) As mentioned above, $(i_0)^*$ is split epi with splitting s given by $s(f) = \bar{f} = \mu(f \wedge 1)$. Then $K_s * K_s = \mathrm{Im}\, s \oplus \mathrm{Ker}(i_0)^*$ and $\mathrm{Mod} \subset \mathrm{Im}\, s$. The associativity implies the converse $\mathrm{Im}\, s \subset \mathrm{Mod}$. □

Remark 2.4. Put $\delta_0 = i'ijj'$. Then we see $\mathrm{Der} \subset \mathrm{Ker}(i_0)^*$, $(\mathrm{Mod})\delta_0 \subset \mathrm{Ker}(i_0)^*$ and $\mathrm{Der} \cap (\mathrm{Mod})\delta_0 = 0$. A stronger version of associativity implies that Der and $(\mathrm{Mod})\delta_0$ generate $\mathrm{Ker}(i_0)^*$, and hence, $K_s * K_s = \mathrm{Mod} \oplus \mathrm{Der} \oplus (\mathrm{Mod})\delta_0$ (see Remark 6.7 in §6). I would like to thank Prof. J. F. Adams for pointing out this stronger decomposition in case $s = 1$.

We will claim all the assumptions in Lemma 2.3 are fullfilled. The proof of the following theorem will be given in §6.

Theorem 2.5. *K_s has a commutative and associative multiplication for which $\delta' \in \mathrm{Der}$.*

Corollary 2.6. *For any $f \in K_s * K_s$, there is $f' \in K_s * K_s$ such that $f' i_0 = f i_0$, $(f')_* = f_*$ and $(f')^p \delta' = \delta'(f')^p$.*

Proof. Define f' to be the component of f in Mod. □

As a consequence, we have been successiful in removing the extra assumption in III'.

Construction III". *If $g_{t,s}$ exists, then $g_{ntp,2s}$, $f_{ntp^2,2s}$ exist for $n \geq 1$.*

At the last place, we shall assemble results on existence of $f_{t,s}$ as our inputs of the construction.

Input I. [17] *$f_{1,1}$ exists.*

Input II. [10], [24], [18] *For $1 \leq s \leq p-1$, $f_{p,s}$ exists.*

Input III. [11] *For $t \geq 2$, $f_{tp,p}$ exists.*

§3. p-Rank of the stable homotopy of spheres

We begin with Input I. By Construction III", we are able to construct $g_{np^m, 2^m}$ with $g_{1,1} = f_{1,1}$ and then $f_{np^{m+1}, 2^m}$ by II' for all $n \geqq 1$, $m \geqq 0$. Then Construction I leads to the existence of $f_{np^{m+1}, s}$ subject to $1 \leq s \leqq 2^m$. By Lemmas 2.1 and 1.1, we conclude the following

Theorem 3.1. *The element* $\beta_{t/s}$ *in* (1.2) *subject to*

$$(3.1) \qquad 1 \leqq s \leqq 2^{\nu(t)-1}, \quad t \geqq 1 \text{ with } \nu(t) \geqq 1,$$

is a permanent cycle in the Adams-Novikov spectral sequence for S^0 *which converges to an indecomposable (up to scalar multiple) element* $b_{t/s} \in \pi^S_{(tp+t-s)q-2}$ *of order* p. *Moreover* $b_{t/s}$*'s are linearly independent over* $\mathbb{Z}/p$. *If* $s \not\equiv 0 \bmod p$, $b_{t/s}$ *is a generator of a direct summand of* $\pi^S_{(tp+t-s)q-2}$.

Let $B(k)$ be the subgroup of π^S_k generated by the elements $b_{t/s}$ with $\dim b_{t/s} = (tp+t-s)q-2 = k$, and let $B'(k)$ be the subgroup of $B(k)$ generated by $b_{t/s}$ such that $s \not\equiv 0 \bmod p$. $B(k)$ and $B'(k)$ are frequently trivial, in non-trivial case, though, the same argument as in [15], §7, Theorem II leads to an important property of them.

Corollary 3.2. π^S_k *has a direct summand* $B'(k)$ *which is generated by indecomposable elements and whose* p-*rank becomes arbitrarily large for sufficiently large* k.

If we begin by Input II, the resultant elements are subject to

$$1 \leqq s \leqq 2^{\nu(t)-1}(p-1), \quad t \geqq 1 \text{ with } \nu(t) \geqq 1,$$

which is shaper than (3.1). With an attention similar to [15] §7, Remark after Proof of Theorem I, this recovers [10] Theorem A'. The Input III produces [15] Theorem I except for p-divisibility of the resultant elements. However it is easy to see from [15] Lemma 4.5, Theorem 4.2 that the element δ in [15] (2.5) is a derivation, thus the p-divisibility given in [15] Theorem I may also be obtained from our construction here. There is one more interesting derivation : Toda's element $\alpha'' \in K_1 * K_1$ ($K_1 = V(1)$ in traditional notation), [20] Lemma 3.1, is easily checked to be a derivation. This implies the p-divisibility of the elements $\alpha_1\beta_{tp}$

as mentioned in [10] Corollary 7.6 and [13] Theorem 5.5. In case $s \equiv 0 \bmod p$, we have obtained in [15] Theorem 5.5 more about the decomposition in Lemma 2.3 (a) and Remark 2.4; Mod coincides with $\mathscr{L}_*$ in [15] and Der coincides with the middle two factors in the decomposition in [15]. Although our method here is essentially the same as in [15] in many parts, it has interesting applications more than [15] including the case $s \not\equiv 0 \bmod p$, the other element in Der mentioned above, and analogous elements for $V(2)$ as we will do in the next section.

§4. A third order periodicity family

Following traditional notation [7], [17], [19], [20], $M = V(0)$, $K_1 = V(1)$, and $V(2)$ is the cofibre of the periodicity element

$$\beta = \bar{f}_{1,1} = s(f_{1,1}) : \Sigma^{2(p^2-1)}V(1) \longrightarrow V(1).$$

If $p \geqq 7$, there is the third periodicity element

$$\gamma : \Sigma^{2(p^3-1)}V(2) \longrightarrow V(2),$$

which induces v_3, and its cofibre $V(3)$ realises $BP_*/(p, v_1, v_2, v_3)$ [19]. The iterated composite γ^t defines the element $\gamma_t \in \pi_*^S$, which is known to be non-trivial for all $t \geqq 1$ [7]. The γ_t is so called third order periodic element because of its detection at H^3BP_*.

In [7] Corollary 7.8, more gammas in H^3BP_* are constructed in a similar manner as of (1.1), (1.2); for example, the element

$$v_3^t \in H^0BP_*/(p, v_1, v_2^s), \quad 1 \leqq s \leqq p^{\nu(t)}, \quad t \geqq 1,$$

defines

$$\gamma_{t/s} \ (= \gamma_{t/s,1} \text{ in [7]}) \in H^3BP_*,$$

which is non-trivial, of order p and linearly independent to each other unless $1 < s = t = p^{\nu(t)}$. The internal dimension of $\gamma_{t/s}$ is $2t(p^3-1) - 2s(p^2-1) - 2(p-1)$. It is hoped that this again leads Corollary 3.2. Unfortunately we do not have a substitute for Theorem 2.5 with s arbitrary. Here we merely check it for small s by low dimensional computation of homotopy.

By Toda [19], [20],

$$\pi_*(V(1)) = \mathbb{Z}/p[\beta, \beta'] \otimes A, \quad \text{up to dimension } p^2q-3,$$

where $q = 2(p-1)$, $\dim \beta = (p+1)q$, $\dim \beta' = pq-2$ and A is a $\mathbb{Z}/p$-module generated by 6 elements of dimensions 0, $q-1$, $pq-1$, $(p+2)q-2$,

$(2p+1)q-2$, $(2p+2)q-3$. Let L_s be the cofibre of $\beta^s : \Sigma^{s(p+1)q}V(1) \longrightarrow V(1)$ and δ be the coboundary $L_s \longrightarrow \Sigma^{s(p+1)q+1}V(1) \longrightarrow \Sigma^{s(p+1)q+1}L_s$. Then, up to dimension p^2q-3,

(4.1) $$\pi_*(L_s) \cong \mathbb{Z}/p[\beta]/(\beta^s) \otimes \mathbb{Z}/p[\beta'] \otimes A.$$

The following lemma in case $s = 1$ ($L_1 = V(2)$) is obtained by Yosimura [22] (3.4), [23] (1.3), $(\Lambda_2)_2$.

<u>Lemma</u> 4.1. <u>Let</u> $p \geq 7$ <u>and</u> $s \leq \left[\frac{p-2}{3}\right]$. <u>Then</u> L_s <u>has a commutative and associative multiplication such that</u> δ <u>is a derivation in</u> $L_s{*}L_s$.

<u>Proof</u>. Let X and Y be the cofibres of the inclusions $V(1)\wedge V(1) \longrightarrow L_s\wedge L_s$ and $V(1)\wedge V(1)\wedge V(1) \longrightarrow L_s\wedge L_s\wedge L_s$. Checking non-trivial dimensions in (4.1) leads us that $\pi_i(L_s) = 0$, $i = \dim e$, for every cell e in $\Sigma^{-1}X$. Therefore $[\Sigma^{-1}X, L_s] = 0$. Similarly we have $[X, L_s] = 0$, $[Y, L_s] = 0$. The composite $V(1)\wedge V(1) \xrightarrow{\mu} V(1) \longrightarrow L_s$ has then an extension $\bar{\mu} : L_s\wedge L_s \longrightarrow L_s$ by $[\Sigma^{-1}X, L_s] = 0$, which is unique by $[X, L_s] = 0$. The $\bar{\mu}$ is a multiplication of L_s for which the inclusion $V(1) \longrightarrow L_s$ is a map of ring spectrum. It is commutative by the uniqueness. The associativity of μ may be extended to $L_s\wedge L_s\wedge L_s$, by $[Y, L_s] = 0$. It is not hard to compute $[L_s, \Sigma^{s(p+1)q+1}L_s]$ and the last statement is now clear. □

We mention that if $v_3^p + xv_2 \in H^0BP_*/(p, v_1, v_2^2)$, x must be trivial by dimensional reason. In the same fashon as before, the following theorem is now immediate.

<u>Theorem</u> 4.2. <u>Let</u> $p \geq 7$.

(i) <u>There is a map</u> $f : \Sigma^{2p(p^3-1)}L_2 \longrightarrow L_2$ <u>which induces</u> v_3^p.

(ii) <u>There is a</u> 16-<u>cell complex</u> X <u>with</u> $BP_*(X) = BP_*/(p, v_1, v_2^2, v_3^{tp})$.

(iii) <u>The element</u> $\gamma_{tp/2} \in H^3BP_*$ <u>is a permanent cycle in the Adams-Novikov spectral sequence for</u> S^0.

(iv) <u>The corresponding element in</u> $\pi^S_{2tp(p^3-1)-4(p^2-1)-2(p-1)-3}$, <u>which we call</u> $\gamma_{tp/2}$ <u>again, is non-trivial, of order</u> p <u>and the Toda bracket</u> $\{\gamma_{tp/2}, p, \alpha_1, p, \beta_1\}$ <u>contains</u> γ_{tp}.

For p bigger, Lemma 4.1 produces little more elements $\gamma_{t/s}$, but we shall make no attempt to describe them.

§5. Some consequences of the theorem of Haynes Miller

In this section, we shall discuss about the structure of the ring of stable self-maps of M, $[M, M]_* = M^{-*}M$, in connection with a result of Miller [6] which assures that elements in $\pi_*(M)$, not in the image nor counter image of $\mathrm{Im}\, J\ (\subset \pi_*^S)$, are annihilated by some power of $\alpha : \Sigma^q M \longrightarrow M$. We shall first extend it to $[M, M]_*$. Then exact sequences used to compute $[K_s, K_s]_* = K_s^{-*}K_s$ from $[M, M]_*$ become short exact in lower dimensions. We shall secondly make computation of $[K_s, K_s]_*$ within our necessity in proving Theorem 2.5.

As in [15] we shall use the notation $[X, Y]_*$ for $Y^{-*}(X)$ so that the grading fits into the grading of homotopy groups. The multiplication of the ring $[X, X]_*$ is understood to be given by composition of maps unless otherwise stated.

Let $\delta = ij \in [M, M]_{-1}$ and A_* be the subring of $[M, M]_*$ generated by δ and the Adams-Toda element α. There is a relation

(5.1) $$\delta\alpha - \alpha\delta = \alpha_1 \wedge 1_M \quad (\alpha_1 = j\alpha i),$$

which is, therefore, in the centre of $[M, M]_*$, cf. [15] Lemma 2.5. The ring structure of A_* is determined by Yamamoto [21]:

(5.2) $$A_* = \mathbb{Z}/p[\alpha] \otimes E(\delta\alpha - \alpha\delta,\ \delta),$$

where E denotes an exterior algebra over $\mathbb{Z}/p$ and the relation $\alpha(\delta\alpha - \alpha\delta) = (\delta\alpha - \alpha\delta)\alpha$ as mentioned in (5.1) is understood to complete the ring structure. We put

$$N(s) = s(p^2 - 3p + 1) + pq - 2.$$

<u>Proposition</u> 5.1. <u>If</u> $k < N(s)$,

$$\alpha^s[M, M]_k = [M, M]_k\alpha^s = \alpha^s A_k = A_k\alpha^s = A_{k+sq}.$$

<u>Proof</u>. The multiplication μ of M makes $\pi_*(M)$ a commutative ring in usual way : $ab = \mu(a \wedge b)$, $a, b \in \pi_*(M)$. Since α is an M-module map : $\mu(\alpha \wedge 1) = \alpha\mu = \mu(1 \wedge \alpha)$, A_*i is a subring of $\pi_*(M)$ and

$$i^* : A_* \longrightarrow A_*i$$

is a ring homomorphism. Since $\delta i = 0$ and $(\delta\alpha - \alpha\delta)i = i\alpha_1$,

$$A_*i = \mathbb{Z}/p[\alpha i] \otimes E(i\alpha_1)$$

as a ring, by (5.2). Let $E_r^{s,t}$ be the Adams spectral sequence based on $H\mathbb{Z}/p$ [1] which converges to $\pi_*(M)$. Haynes Miller [6] proved that

(5.3) $\quad E_\infty = \mathbb{Z}/p[q_1] \otimes E(h_{1,0}) \quad$ for $\quad t-s \leqq (p^2-p-1)(s+1)$,

where $\deg q_1 = (1, q+1)$, $\deg h_{1,0} = (1, q)$, and q_1 and $h_{1,0}$ are represented by αi and $i\alpha_1$. Hence $A_* i \longrightarrow \mathbb{Z}/p[q_1] \otimes E(h_{1,0})$ is isomorphic.

For any element $\xi \in [M, M]_k$, $k > 0$, let $x \in E_\infty^{a,b}$ be the class of $\xi i \in \pi_*(M)$, where $b-a = k$ and we may assume $a \geqq 1$. If $k \leqq N(s)$, $k+sq \leqq (p^2-p-1)(s+a+1)$ and hence $q_1^s x \in \mathbb{Z}/p[q_1] \otimes E(h_{1,0})$ by (5.3). This means $\alpha^s \xi i \in A_* i$, so $\alpha^s \xi \delta \in A_* \delta \subset A_*$.

Let $d : [M, M]_j \longrightarrow [M, M]_{j+1}$ be the Hoffman-Toda derivation [4], [20], [12]. Then ξ is uniquely expressed as $\xi = \xi_0 + \delta \xi_1$ with $\xi_i \in [M, M]_{k+i}$, $d(\xi_i) = 0$. Since $d(\delta) = -1$, $d(\alpha) = 0$, A_* is invariant under d : $d(A_*) \subset A_*$. Cf. [4], [12]. Then $d(\xi) = \pm \xi_1$ and

(5.4) $\quad \alpha^s \xi \pm \alpha^s \xi_1 \delta \in A_*, \quad k \leqq N(s)$.

We first consider the case $d(\xi) = 0$, i.e., $\xi_1 = 0$. By (5.4) and the commutativity of Ker d [4], $\xi \alpha^s = \alpha^s \xi \in A_*$ for $k \leqq N(s)$. In general case, this implies $\xi_1 \alpha^s = \alpha^s \xi_1 \in A_*$ for $k+1 \leqq N(s)$, hence $\alpha^s \xi \in A_*$ for $k < N(s)$, by (5.4). $\xi \alpha^s \in A_*$ follows from the decomposition of ξ. $\square$

<u>Remark</u> 5.2. The proposition is also true for $p = 3$. Consult the modified Hoffman decomposition for $[M, M]_*$ [12] §8, [14] §3.

<u>Remark</u> 5.3. It is likely that generalising it to the mod p^n Moore spectrum is not hard. However the substitute for A_* is more complicate.

Let ξ be an element in $[M, M]_*$ which is non-zero in $[M, M]_*/A_*$. By (5.2) and Proposition 5.1, there exist unique non-zero elements ξ_L, ξ_R such that

(5.5) $\quad \alpha^s \xi_L = 0, \quad \xi_R \alpha^s = 0, \quad \xi_L \equiv \xi_R \equiv \xi \mod A_*$.

Since α^s commutes with elements in A_* of dim $\not\equiv -1 \mod q$, $\xi_L = \xi_R$ if $\dim \xi \not\equiv -1 \mod q$. Let L_* (resp. R_*) be the two-sided ideal of $[M, M]_*$ generated by indecomposables ξ other than α, δ such that $\alpha^s \xi = 0$ (resp. $\xi \alpha^s = 0$) for some s. Then

(5.6) $\quad [M, M]_* = R_* \oplus A_* = L_* \oplus A_*$ as additive group,

$R_* = L_*$ in $\dim \not\equiv -1 \mod q$,

$[M, M]_* = R_* = L_*$ in $\dim \not\equiv 0, -1$ or $-2 \mod q$.

Since $\mathrm{Im}\, J = \pi_*^S$ for $\dim < pq-2$ when localised at p, we have also

(5.7) $\quad$ For $\dim < pq-3$, $[M, M]_* = A_*$, $R_* = L_* = 0$.

Remark 5.4. It is not hard to interpret (5.5) as the relations in π_{*}^{s}. Let $\alpha_s = j\alpha^s i$ and $\zeta \in \pi_k^S$ be an arbitrary element of BP-filtration at least 2. Then (5.5) implies that if $k \leqq N(s)$

$$\alpha_s \zeta = 0, \quad \{p, \alpha_s, \zeta\} \equiv 0 \quad \text{mod indeterminacy,}$$

and if moreover $p\zeta = 0$ and $k < N(s)$,

$$\{\alpha_s, p, \zeta\} \equiv 0, \quad \{p, \alpha_s, p, \zeta\} \equiv 0 \quad \text{mod indeterminacy.}$$

Here $\{\ \ \}$ denotes the Toda bracket. These are almost equivalent to (5.5). The indeterminacies of above may be too large to deduce (5.5), which claims that the elements in the above Toda brackets which are defined in terms of the specific extension of α_s are trivial.

Now, let K_s be the cofibre of $\phi = \alpha^s$ as before. ϕ satisfies

(5.8) $$\alpha_s \wedge 1_M = \delta\phi - \phi\delta = s\alpha^{s-1}(\delta\alpha - \alpha\delta).$$

Let $i' : M \longrightarrow K_s$ be the inclusion and put $A_*^! = A_*/\alpha^s A_*$. Every element in $(i')_* A_*^! = (i')_* A_*$ except, in case $s \not\equiv 0$ mod p, for $i'\delta$ is in the image of $(i')^* : [K_s, K_s]_* \longrightarrow [M, K_s]_*$. Its counter image in $[K_s, K_s]_*$ is given as follows. K_s is an associative M-module spectrum with module multiplication $m : M \wedge K_s \longrightarrow K_s$. Then an element $\bar{m} : \Sigma K_s \longrightarrow M \wedge K_s$ is uniquely associated to m in such a way that $(j \wedge 1_K)\bar{m} = 1_K$, $m\bar{m} = 0$. They define a derivation d of $[K_s, K_s]_*$ and an operator $\lambda : [M, M]_* \longrightarrow [K_s, K_s]_{*+1}$: $d(f) = m(1_M \wedge f)\bar{m}$, $\lambda(\xi) = m(\xi \wedge 1_K)\bar{m}$, cf. [20], [12]. We put

$$\bar{\alpha} = \lambda(\alpha\delta) = m(\alpha i \wedge 1_K) \in [K_s, K_s]_q \quad (s \geqq 2),$$

$$\alpha' = \lambda(\delta\alpha\delta) = \alpha_1 \wedge 1_K \in [K_s, K_s]_{q-1}.$$

By [20] Theorem 2.4, $\bar{\alpha} = \lambda(\delta\alpha) = (j\alpha \wedge 1_K)\bar{m}$ and $\bar{\alpha}^j = \lambda(\alpha^j\delta)$. Therefore $\bar{\alpha}^s = m(\phi i \wedge 1_K) = 0$ by [14], §4. Also they satisfy

(5.9) $$\bar{\alpha}^j i' = i'\alpha^j, \quad j'\bar{\alpha}^j = \alpha^j j',$$
$$\alpha' i' = i'(\delta\alpha - \alpha\delta), \quad j'\alpha' = (\alpha\delta - \delta\alpha)j',$$

and $(\alpha')^2 = 0$, $\bar{\alpha}\alpha' = \alpha'\bar{\alpha}$. There exist elements $\alpha'' \in [K_s, K_s]_{q-2}$, $\bar{\alpha}' \in [K_s, K_s]_{q-1}$ ($\bar{\alpha}' = 0$ if $s = 1$) with properties

(5.10) $$\alpha'' i' = i'\delta\alpha\delta, \quad j'\alpha'' = \delta\alpha\delta j', \quad d(\alpha'') = -\alpha',$$
$$\bar{\alpha}' i' = i'\alpha\delta, \quad j'(\bar{\alpha}' - s\alpha') = -\alpha\delta j', \quad d(\bar{\alpha}') = -\bar{\alpha}.$$

If $s \equiv 0$ mod p, there exists $\bar{\delta} \in [K_s, K_s]_{-1}$ with

(5.11) $$\bar{\delta} i' = i'\delta, \quad j'\bar{\delta} = -\delta j', \quad d(\bar{\delta}) = -1_K$$

[15], (2.5), and we see easily that

$$(5.12) \qquad \alpha'' = \bar{\delta}\bar{\alpha}\bar{\delta}, \quad \alpha' = \bar{\delta}\bar{\alpha} - \bar{\alpha}\bar{\delta}, \quad \bar{\alpha}' = \bar{\alpha}\bar{\delta}.$$

Define a subgroup $\bar{A}_*$ of $[K_s, K_s]_*$ to be

$$\mathbb{Z}/p[\bar{\alpha}]/(\bar{\alpha}^s) \otimes \{1, \alpha'', \alpha', \bar{\alpha}'\} \qquad \text{for} \quad s \not\equiv 0 \mod p,$$

$$\mathbb{Z}/p[\bar{\alpha}]/(\bar{\alpha}^s) \otimes \{\bar{\delta}, 1, \bar{\delta}\bar{\alpha}\bar{\delta}, \bar{\delta}\bar{\alpha}\} \qquad \text{for} \quad s \equiv 0 \mod p.$$

By (5.2), (5.9)-(5.12), we conclude that

$$\bar{A}_* \xrightarrow{(i')^*} (i')_* A''_* \xleftarrow{(i')_*} A''_* \ (\subset [M, M]_*)$$

are isomorphisms, where $A''_* = A'_*$ if $s \equiv 0 \mod p$, and $A''_* = A'_*/\{\delta\}$ if $s \not\equiv 0 \mod p$.

We put

$$\Gamma_* = \text{image of } (i')_*(j')^* : [M, M]_{*+sq+1} \longrightarrow [K_s, K_s]_*.$$

Clearly, $\bar{A}_* \cap \Gamma_* = 0$ and $\bar{A}_* \oplus \Gamma_* \subset [K_s, K_s]_*$. Set

$$(5.13) \qquad \phi_1 = j\phi i \ (= \alpha_s), \quad \bar{\phi} = \phi_1 \wedge 1_K.$$

By (5.8) and (5.9), we then have

$$(5.14) \qquad \bar{\phi} = s\bar{\alpha}^{s-1}\alpha', \quad \bar{\phi}i' = i'\delta\phi, \quad j'\bar{\phi} = -\phi\delta j'.$$

The following lemmas are now immediately obtained by assembling above results, in particular Proposition 5.1, (5.5), (5.6), (5.7), (5.14).

<u>Lemma</u> 5.5.

$\phi_* = 0 : [K_s, M]_k \longrightarrow [K_s, M]_{k+sq}$ <u>for</u> $k < pq-3$ <u>with</u> $k \neq -sq-2$,

$\phi^* = 0 : [M, K_s]_k \longrightarrow [M, K_s]_{k+sq}$ <u>for</u> $k < sq+pq-2$ <u>with</u> $k \neq -1$,

$\bar{\phi}_*\bar{A}_* = 0, \ \bar{\phi}^*\bar{A}_* = 0,$

$\bar{\phi}_*\Gamma_k = \bar{\phi}^*\Gamma_k = 0$ <u>for</u> $k < N(s)-sq-1$ <u>with</u> $k \neq -sq-2$.

We mention that if $s \equiv 0 \mod p$, $\phi \wedge 1_K = 0$, $\bar{\phi} = 0$ [15], and hence the lemma holds with no restriction of k.

<u>Lemma</u> 5.6. (i) <u>For</u> $k < sq + pq - 3$,

$$(i')_*(j')^* : R_{k+sq+1} \oplus A'_{k+sq+1} = L_{k+sq+1} \oplus A'_{k+sq+1} \longrightarrow \Gamma_k$$

<u>is isomorphic</u>.

(ii) <u>If</u> $k < pq - 3$, $[K_s, K_s]_k = \bar{A}_k \oplus \Gamma_k$, <u>and if moreover</u> $k \neq -sq-2$

$$\bar{\phi}_*[K_s, K_s]_k = \bar{\phi}^*[K_s, K_s]_k = 0.$$

Remark 5.7. In case $s \not\equiv 0 \mod p$, we need the restriction $k \neq -sq-2$ in (ii). In fact, $[K_s, K_s]_{-sq-2} = \mathbb{Z}/p$ is generated by $\delta_0 = i'\delta j'$, and $\bar{\phi}\delta_0 = \delta_0\bar{\phi} = s\bar{\alpha}^{s-1}\alpha''\delta' \neq 0$ ($\delta' = i'j'$).

§6. Commutativity and associativity

In this section, the exponent s in $\phi = \alpha^s \in [M, M]_{sq}$ will be fixed and we will omit the index s in $K = K_s$, the cofibre of ϕ.

Recall the cofibrations

$$S^0 \xrightarrow{i} M \xrightarrow{j} S^1, \quad M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{sq+1}M,$$

and the elements $\phi_1 = j\phi i \in \pi^S_{sq-1}$, $\bar{\phi} = \phi_1 \wedge 1_K \in [K, K]_{sq-1}$ in (5.13). Traditional name of ϕ_1 is α_s. If $s \equiv 0 \mod p$, $\bar{\phi}$ is trivial [15], which leads to $\phi \wedge 1_K = 0$ and the existence of the element δ_K in [15] (2.5). The first fact then makes $K \wedge K$ a wedge $\vee \Sigma^{n_i}K$ giving one more "derivation" d' in $[K, K]_*$, and the second fact induces a decomposition of $[K, K]_*$ given in [15], Corollary 2.7. Although such strong evidences may not be expected in case $s \not\equiv 0 \mod p$, Lemmas 5.5, 5.6 allow us to develop analogue of [15]§4 without $s \equiv 0 \mod p$.

Let $m : M \wedge K \longrightarrow K$ be an <u>associative</u> M-module multiplication (see [14] Lemma 3.11) and $\bar{m} : \Sigma K \longrightarrow M \wedge K$ the associated element with $(j \wedge 1_K)\bar{m} = 1_K$, $m\bar{m} = 0$ [12], [15]. They give a factorisation of $1_{M \wedge K}$:

(6.1) $$(i \wedge 1_K)m + \bar{m}(j \wedge 1_K) = 1_{M \wedge K},$$

or equivalently a homotopy equivalence $M \wedge K = K \vee \Sigma K$. By [14] Examples 4.5, 5.7, the obstruction coset $o(\phi)$ for K to be a ring spectrum is zero modulo $J_{2sq+1}(\phi)$, the $2sq+1$ dimensional part of two-sided ideal generated by ϕ. By (5.6), $J_{2sq+1}(\phi) = 0$ and hence $\lambda(\phi\delta) = m(\phi \wedge 1_K)(i \wedge 1_K) = 0$ (for any m) by [14] Definition 4.1. Since $\lambda d = 0$ ([20] Theorem 2.4(v)) and $\alpha = -d(\delta\alpha)$, $\delta\alpha - \alpha\delta = d(\delta\alpha\delta)$, we have also $m(\phi \wedge 1_K)\bar{m} = 0$ and $(j \wedge 1_K)(\phi \wedge 1_K)\bar{m} = \lambda(\delta\phi) = 0$. Therefore, by (6.1),

(6.2) $$\phi \wedge 1_K = \bar{m}\bar{\phi}m.$$

By Lemma 5.6 (ii),

(6.3) $$(\phi \wedge 1_K)^* : [M \wedge K, K]_k \longrightarrow [M \wedge K, K]_{k+sq}$$ is trivial for $k < pq-4$ with $k \neq -sq-3$.

We denote by L the cofibre of ϕ_1. Let

$$S^0 \xrightarrow{i''} L \xrightarrow{j''} S^{sq}$$

be the cofibration induced by ϕ_1. Since the cofibres of $\phi \wedge 1_K$ and $\bar{\phi}$ are $K \wedge K$ and $L \wedge K$, (6.2) gives rise to a homotopy equivalence

$$K \wedge K = K \vee (\Sigma L \wedge K) \vee \Sigma^{sq+2} K,$$

more precisely, there exist

$$\mu : K \wedge K \longrightarrow K, \qquad \mu_2 : K \wedge K \longrightarrow \Sigma L \wedge K,$$
$$\nu : \Sigma^{sq+2} K \longrightarrow K \wedge K, \quad \nu_2 : \Sigma L \wedge K \longrightarrow K \wedge K$$

such that

(A) $\mu(i' \wedge 1_K) = m, \qquad (j' \wedge 1_K)\nu = \bar{m},$

(B) $\mu_2(i' \wedge 1_K) = (i'' \wedge 1_K)(j \wedge 1_K), \qquad (j' \wedge 1_K)\nu_2 = (i \wedge 1_K)(j'' \wedge 1_K),$

(C) $(j'' \wedge 1_K)\mu_2 = m(j' \wedge 1_K), \qquad \nu_2(i'' \wedge 1_K) = (i' \wedge 1_K)\bar{m},$

(D) $\mu\nu_2 = 0, \quad \mu\nu = 0, \quad \mu_2\nu = 0, \quad \mu_2\nu_2 = 1_{L \wedge K}.$

If we put

$$i_0 = i'i : S^0 \longrightarrow K, \qquad j_0 = jj' : K \longrightarrow S^{sq+2},$$

(A) and (B) imply

(A)' $\mu(i_0 \wedge 1_K) = 1_K, \qquad (j_0 \wedge 1_K)\nu = 1_K,$

(B)' $\mu_2(i_0 \wedge 1_K) = 0, \qquad (j_0 \wedge 1_K)\nu_2 = 0,$

which, together with (D) and the obvious relation $(j_0 \wedge 1_K)(i_0 \wedge 1_K) = 0$, provide a factorisation of $1_{K \wedge K}$:

(6.4) $\qquad (i_0 \wedge 1_K)\mu + \nu_2\mu_2 + \nu(j_0 \wedge 1_K) = 1_{K \wedge K},$

(see [13] Proposition 2.5 in case $s = 2$).

There can be choices of (μ, μ_2, ν, ν_2) with properties (A)-(D). Among them, we will find a choice giving μ for which Theorem 2.5 is satisfied. Throughout the discussion we will fix m (and hence $\bar{m}$) to be associative.

Lemma 6.1. Let $(\tilde{\mu}, \tilde{\mu}_2, \tilde{\nu}, \tilde{\nu}_2)$ be an another choice of (μ, μ_2, ν, ν_2). Then there are unique elements $\alpha_i \in [K, K]_{sq+i}$ $(i = 1, 2)$ and $\beta_i \in [K, K]_{sq+i}$ $(i = 0, 1)$ such that

$$\tilde{\mu} - \mu = (\alpha_1 m + \alpha_2(j \wedge 1_K))(j' \wedge 1_K),$$
$$\tilde{\mu}_2 - \mu_2 = (i'' \wedge 1_K)(\beta_0 m + \beta_1(j \wedge 1_K))(j' \wedge 1_K),$$
$$\tilde{\nu} - \nu = -(i' \wedge 1_K)(\bar{m}\beta_1 + (i \wedge 1_K)\alpha_2),$$
$$\tilde{\nu}_2 - \nu_2 = -(i' \wedge 1_K)(\bar{m}\beta_0 + (i \wedge 1_K)\alpha_1)(j'' \wedge 1_K).$$

Proof. By (A) and (6.1), $\tilde{\mu} - \mu$ is expressed as required. The uniqueness of α_i follows from (6.3). By Lemma 5.6 (ii) and (6.1), $\bar{\phi}_* = 0$: $[M\wedge K, K]_{-sq} \longrightarrow [M\wedge K, K]_{-1}$, hence $(i''\wedge 1)^* : [M\wedge K, K]_{-1} \longrightarrow [M\wedge K, L\wedge K]_{-1}$ is monic. Therefore $\tilde{\mu}_2 - \mu_2$ is uniquely expressed as above. Similarly, by (A), (B), (C), we have the expression of $\tilde{\nu} - \nu$, $\tilde{\nu}_2 - \nu_2$ with certain unique "coefficients of $\bar{m}$ and $i\wedge 1_K$", which are given as above, by (D). □

Recall the coboundaries

$$\delta = ij \in [M, M]_{-1}, \quad \delta' = i'j' \in [K, K]_{-sq-1}, \ \delta_0 = i'ijj' \in [K,K]_{-sq-2}.$$

They satisfy ([12], [20])

$$d(\delta) = -1_M, \quad d(\delta') = 0, \quad d(\delta_0) = \delta'.$$

Lemma 6.2. *There exist elements*

$$\tilde{\Delta} \in [K, L\wedge K]_{-1}, \quad \bar{\Delta} \in [L\wedge K, K]_{-sq-1}$$

such that

(i) $(j''\wedge 1_K)\tilde{\Delta} = \delta'$, $\quad \bar{\Delta}(i''\wedge 1_K) = \delta'$;

(ii) $\tilde{\Delta}i' = (i''\wedge 1_K)i'\delta$, $\quad j'\bar{\Delta} = \delta j'(j''\wedge 1_K)$;

(iii) $(1_L\wedge j')\tilde{\Delta} = -(i''\wedge 1_M)\delta j'$, $\quad \bar{\Delta}(1_L\wedge i') = -i'\delta(j''\wedge 1_M)$;

(iv) $\bar{\Delta}\tilde{\Delta} = 2\delta_0$;

(v) $d(\tilde{\Delta}) = -i''\wedge 1_K$, $\quad d(\bar{\Delta}) = j''\wedge 1_K$,

where the M-module structure of $L\wedge K$ *giving* d *of above is the composite* $M\wedge L\wedge K \longrightarrow L\wedge M\wedge K \xrightarrow{1\wedge m} L\wedge K$.

Proof. Put $\tilde{\Delta} = \mu_2(1_K\wedge i_0)$, $\bar{\Delta} = (1_K\wedge j_0)\nu_2$. They are independent of the choice of μ_2, ν_2 and satisfy (i), (ii) by (B) and (C). Estimate the choice of $\tilde{\Delta}$, $\bar{\Delta}$ with (i), (ii) and compute the d-images of (i) and (ii), then one can get $\tilde{\Delta}$, $\bar{\Delta}$ with (i), (ii) and (v). The computation of d then leads to (iii) and (iv). □

The smash product $K\wedge K$ has the M-module structures given from each factor:

$$M\wedge K\wedge K \xrightarrow{m\wedge 1} K\wedge K,$$

$$M\wedge K\wedge K \xrightarrow{T\wedge 1} K\wedge M\wedge K \xrightarrow{1\wedge m} K\wedge K.$$

As in [15]Lemma 1.6,(ii), for an M-module spectrum X, each module structure defines the derivation of $[K\wedge K, X]_*$ (and of $[X, K\wedge K]_*$), which we denote by d_i according to the i-th factor (i = 1, 2). We also write

$$\mathscr{M}_i[K \wedge K, X]_* = \mathrm{Ker}\, d_i \ (i = 1, 2), \qquad \mathscr{M}_{1,2}[K \wedge K, X]_* = \mathscr{M}_1 \cap \mathscr{M}_2.$$

The d_1, d_2, $\mathscr{M}_1$, $\mathscr{M}_2$, $\mathscr{M}_{1,2}$ can be defined for $[M \wedge K, X]_*$, $[M \wedge M, X]_*$ $[X, K \wedge K]_*$ etc, in the same way. By (6.3), we have the short exact sequence:

$$(6.5)_k \quad 0 \longrightarrow [M \wedge K, K]_{k+sq+1} \xrightarrow{(j' \wedge 1)^*} [K \wedge K, K]_k \xrightarrow{(i' \wedge 1)^*} [M \wedge K, K]_k \longrightarrow 0$$

for $k < pq-5$ with $k \neq -sq-4, -sq-3$ and for $s \equiv 0 \mod p$ all k [15]. We shall deduce exact sequences of $\mathscr{M}_i$ from $(6.5)_k$, which, in case $s \equiv 0 \mod p$, hold for all k and played an important role in computation in [15].

Lemma 6.3. (i) For $k < pq-6$ with exception $k = -sq-5, -sq-2$ and for $i = 1, 2$, the following sequences are exact:

$$D_i : \quad [K \wedge K, K]_{k-1} \xrightarrow{d_i} [K \wedge K, K]_k \xrightarrow{d_i} [K \wedge K, K]_{k+1}.$$

(ii) For $k < pq-7$ with exception $-sq-6 \leq k \leq -sq-2$ and for $i =$ 1, 2, (1,2), the following sequences are short exact:

$$M_i : \mathscr{M}_i[M \wedge K, K]_{k+sq+1} \xrightarrow{(j' \wedge 1)^*} \mathscr{M}_i[K \wedge K, K]_k \xrightarrow{(i' \wedge 1)^*} \mathscr{M}_i[M \wedge K, K]_k.$$

Proof. We notice that

$$d_1 d_1 = 0, \ d_2 d_2 = 0, \ d_1 d_2 + d_2 d_1 = 0$$

[15]Lemma 1.6,(ii) and

$$(6.6) \qquad \mathrm{Ker}\, d_1 = \mathrm{Im}\, d_1 \quad \text{in } [M \wedge X, Y]_*$$

[12],[14]. Consider in general a commutative diagram

$$\begin{array}{llll}
A: & A_1 \xrightarrow{a_1} & A_2 \xrightarrow{a_2} & A_3 \\
 & \downarrow f_1 & \downarrow f_2 & \downarrow f_3 \\
B: & B_1 \xrightarrow{b_1} & B_2 \xrightarrow{b_2} & B_3 \\
 & \downarrow & \downarrow g_2 & \downarrow \\
C: & C_1 \xrightarrow{c_1} & C_2 \xrightarrow{c_2} & C_3 \\
 & & & \\
 & \mathrm{I} & \mathrm{II} & \mathrm{III}.
\end{array}$$

It is easy to see that if a_2 is epic, B is exact, c_1 is monic, I and III are exact, and II is a chain complex (i.e., $g_2 f_2 = 0$), then II is exact. One can apply this with $A = (6.5)_{k-1}$, $B = (6.5)_k$, $C =$

$(6.5)_{k+1}$ and vertical arrows d_1 to get exact sequence D_1. The exactness of D_2 follows from the commutative diagram

$$\begin{array}{ccc} [K\wedge K, K]_* & \xrightarrow{d_1} & [K\wedge K, K]_* \\ \downarrow T^* & & \downarrow T^* \\ [K\wedge K, K]_* & \xrightarrow{d_2} & [K\wedge K, K]_*, \end{array}$$

where $T : K\wedge K \longrightarrow K\wedge K$ is the map switching factors.

Next consider the case $A = (6.5)_k$, $B = (6.5)_{k+1}$, $C = (6.5)_{k+2}$ with d_1 vertical arrows in the above. The exact sequence $0 \longrightarrow \operatorname{Ker} f_1 \longrightarrow \operatorname{Ker} f_2 \longrightarrow \operatorname{Ker} f_3 \longrightarrow 0$ we want comes from checking that A is short exact, b_1 and c_1 are monic, I is exact and that II is a chain complex. To discuss M_2 in this way, we only need a substitute for (6.6):

(6.6)' $\quad \operatorname{Ker} d_2 = \operatorname{Im} d_2$ in $[M\wedge K, K]_*$ in appropriate dimensions.

If we replace $M\wedge K$ by $M\wedge M$, (6.6)' holds with no restriction of dimension ([12] §7, [14] §3). There is an exact sequence

$$[M\wedge M, K]_* \xrightarrow{(1\wedge j')^*} [M\wedge K, K]_* \xrightarrow{(1\wedge i')^*} [M\wedge M, K]_*,$$

which becomes short exact in appropriate dimensions by Lemma 5.5. In a similar argument as above, one may verify (6.6)' in dimension $< sq+pq-5$ with exceptions $0, -1, -3, -4$. Proving M_2.

Now the decomposition $[M\wedge K, K]_* = \operatorname{Im} d_1 \oplus (\operatorname{Im} d_1)(\delta\wedge 1_K)$ given in [12] §7, [14] §3 is compatible with d_2, that is,

$$\operatorname{Ker} d_2 = \operatorname{Im} d_1 \cap \operatorname{Ker} d_2 \;\oplus\; (\operatorname{Im} d_1 \cap \operatorname{Ker} d_2)(\delta\wedge 1_K)$$

in $[M\wedge K, K]_*$. Then

(6.7) $\qquad \operatorname{Im} d_1 \cap \operatorname{Im} d_2 = \operatorname{Im} d_1 d_2 \quad$ in $[M\wedge K, K]_*$.

The exactness of $M_{1,2}$ is now clear. $\square$

Remark. The elements

$$x_1 = \bar{\alpha}^{s-1}\alpha''\delta'(j_0\wedge 1_K), \quad x_2 = \bar{\alpha}^{s-1}\alpha''\delta'(1_K\wedge j_0) \in [K\wedge K, K]_{-sq-5},$$

$$y_1 = 1_K\wedge j_0, \quad y_2 = j_0\wedge 1_K \in [K\wedge K, K]_{-sq-2}$$

satisfy $d_i(x_i) = 0$, $d_i(y_i) = 0$ but, if $s \not\equiv 0 \bmod p$, $x_i \notin \operatorname{Im} d_i$, $y_i \notin \operatorname{Im} d_i$. So the above few exception of k is necessary for D_i to be exact. Similar examples may be found for M_i. Another immediate consequences of (6.7) are the exact sequences

$$\mathscr{M}_1 \xrightarrow{d_2} \mathscr{M}_1 \xrightarrow{d_2} \mathscr{M}_1, \quad \mathscr{M}_2 \xrightarrow{d_1} \mathscr{M}_2 \xrightarrow{d_1} \mathscr{M}_2$$

in appropriate dimensions for $[K\wedge K, K]_*$.

We shall turn back Lemma 6.1. Let $T : K\wedge K \longrightarrow K\wedge K$ be the map switching factors.

Lemma 6.4. There is a choice of (μ, μ_2, ν, ν_2) which satisfy (A), (B), (C), (D) and

(E) $\mu T(i'\wedge 1_K) = m, \qquad (j'\wedge 1_K)T\nu = \bar{m};$

(F) $\mu_2 T(i'\wedge 1_K) = -(i''\wedge 1_K)(j\wedge 1_K) + \tilde{\Delta} m,$

$(j'\wedge 1_K)T\nu_2 = -(i\wedge 1_K)(j''\wedge 1_K) + \bar{m}\bar{\Delta};$

(G) $d_i(\mu) = 0, \quad d_i(\nu) = 0, \quad i = 1, 2;$

(H) $d_1(\mu_2) = -(i''\wedge 1_K)\mu, \quad d_2(\mu_2) = 0,$

$d_1(\nu_2) = -\nu(j''\wedge 1_K), \quad d_2(\nu_2) = 0.$

Proof. By Lemma 6.3, there is a choice of μ with $\mu \in \mathscr{M}_{1,2}$, since $m \in \mathscr{M}_{1,2}[M\wedge K, K]_0$ ([15], Lemma 1.6). From Lemma 6.3, we have a short exact sequence

$$0 \to \mathscr{M}_i[M\wedge K, L\wedge K]_{sq} \xrightarrow{(j'\wedge 1)^*} \mathscr{M}_i[K\wedge K, L\wedge K]_{-1} \xrightarrow{(i'\wedge 1)^*} \mathscr{M}_i[M\wedge K, L\wedge K]_{-1} \to 0,$$

$i = 1, 2, (1,2)$. Then there is a μ_2 with $d_2(\mu_2) = 0$. By Lemma 6.2 and (B), $d_1(\mu_2 - \tilde{\Delta}\mu)(i'\wedge 1) = 0$. Then we may choose a μ_2 with $d_1(\mu_2 - \tilde{\Delta}\mu) = 0$ (i.e., $d_1(\mu_2) = -(i''\wedge 1_K)$) keeping $d_2(\mu_2) = 0$ in a similar way as in getting exact sequence of $\mathscr{M}_{1,2}$ in Lemma 6.3. The relations (A)- (D) provide that if μ, μ_2 satisfy (G), (H) then so do the ν, ν_2 associated to μ, μ_2. Therefore there is a (μ, μ_2, ν, ν_2) which satisfies (G) and (H). The choice of such (μ, μ_2, ν, ν_2) is given by

(6.8) $\alpha_2 = 0, \beta_1 = -\alpha_1, \quad d(\alpha_1) = 0, \quad d(\beta_0) = 0$ with variables α_1, β_0 in Lemma 6.1.

Next we have

$\mu T(i'\wedge 1) \in [M\wedge K, K]_0 = \{m\} \oplus D,$

$\mu_2 T(i'\wedge 1) \in [M\wedge K, L\wedge K]_{-1} = \{(i''\wedge 1)(j\wedge 1), \tilde{\Delta} m\} \oplus D',$

where

$$D = i'[M, M]_{sq+1}j'm \oplus i'[M, M]_{sq+2}j'(j\wedge 1),$$

$$D' = (i''\wedge 1)i'[M, M]_{sq}j'm \oplus (i''\wedge 1)i'[M, M]_{sq+1}j'(j\wedge 1).$$

We mention the relation

$$(j'\wedge 1_K)T(i'\wedge 1_K) = (i\wedge 1_K)\delta' m + \bar{m}\delta'(j\wedge 1_K) + \bar{m}\delta_0 m,$$

cf.[13], Lemma 1.11,(i). With this in computation, it is not hard to find a μ(resp. μ_2) such that the component of $\mu T(i'\wedge 1_K)$ (resp. $\mu_2 T(i'\wedge 1_K)$) in $D\cap\mathscr{M}_{1,2}$ (resp. $D'\cap\mathscr{M}_{1,2}$) is trivial. They satisfy (G) and (H). A similar discussion for ν, ν_2 completes the proof. □

The choice of (μ, μ_2, ν, ν_2) satisfying (A)-(H) then becomes (6.8) with restriction $\alpha_1 = i'\alpha j'$, $\beta_0 = i'\beta j'$ with unique $\alpha\in\mathscr{M}[M,M]_{sq+2}$ and $\beta\in\mathscr{M}[M, M]_{sq+1}$ ($\mathscr{M}$ stands for the kernel of d).

<u>Theorem</u> 6.5. <u>There is a choice of</u> (μ, μ_2, ν, ν_2) <u>such that</u>

$$\mu T = \mu, \qquad T\nu = \nu,$$

$$\mu_2 T = -\mu_2 + \tilde{\Delta}\mu, \qquad T\nu_2 = -\nu_2 + \nu\bar{\Delta}.$$

<u>Proof</u>. We define an involution τ on the set S of all possible (μ, μ_2, ν, ν_2) with the properties (A)-(H) as follows:

$$\tau(\mu, \mu_2, \nu, \nu_2) = (\tau(\mu), \tau(\mu_2), \tau(\nu), \tau(\nu_2)),$$

$$\tau(\mu) = \mu T, \ \tau(\mu_2) = -\mu_2 T + \tilde{\Delta}\mu T,$$

$$\tau(\nu) = T\nu, \ \tau(\nu_2) = -T\nu_2 + T\nu\bar{\Delta}.$$

As mentioned above, there is a bijection between S and $\mathscr{M}[M, M]_{sq+1} \oplus \mathscr{M}[M, M]_{sq+2}$, and hence, S is finite and consists of <u>odd</u> number of elements. Since τ is an involution, there is a fixed point, for which the theorem is satisfied. □

<u>Theorem</u> 6.6. <u>Any</u> μ <u>in Theorem</u> 6.5 <u>is associative</u>.

<u>Proof</u>. Notice that (μ, μ_2, ν, ν_2) in Theorem 6.5 also satisfies (G) and (H) in Lemma 6.4. One may develop an analogous discussion as in [15], Lemmas 4.6,4.8 to estimate the element

$$(1_L\wedge\mu_2)(T_{K,L}\wedge 1_K)(1_K\wedge\mu_2) + (T_{L,L}\wedge 1_K)(1_L\wedge\mu_2)(\mu_2\wedge 1_K)$$

in $[K\wedge K\wedge K, L\wedge L\wedge K]_{-2}$, which is just the $(i''\wedge i''\wedge 1_K)_*$-image of $\bar{\mu}_{2,2} + \mu_{2,2}$ in [15], Lemma 4.6 in case $s \equiv 0 \mod p$, with expression in terms of elements in $[K, L\wedge L\wedge K]_*$ ($T_{X,Y} : X\wedge Y \longrightarrow Y\wedge X$ is the twisting map); then compute $d_1 d_2$ of the estimate to get

$$(*)_1 \qquad (i''\wedge i''\wedge 1_K)(\mu(\mu\wedge 1_K) - \mu(1_K\wedge\mu)) = \gamma\mu_M(\mu_M\wedge 1_M)(j'\wedge j'\wedge j')$$

for some $\gamma\in[M\wedge M\wedge M, L\wedge L\wedge K]_{3sq+3}$, where μ_M is the multiplication of M. In case $s\equiv 0 \mod p$, $(*)_1$ is the same as the image of the first equation in [15], Lemma 4.8.

Let $P : X\wedge X\wedge X \longrightarrow X\wedge X\wedge X$ be the cyclic permutation : $P(x_1,x_2,x_3) = (x_3,x_1,x_2)$. Since μ_M is commutative and associative, $\mu_M(\mu_M\wedge 1_M)$ is invariant under P : $\mu_M(\mu_M\wedge 1_M)P = \mu_M(\mu_M\wedge 1_M)$, hence,

$$(*)_2 \qquad \mu_M(\mu_M\wedge 1_M)(1 + P + P^2) = 3\mu_M(\mu_M\wedge 1_M).$$

By the commutativity of μ, we have $\mu(\mu\wedge 1_K) = \mu T(1_K\wedge\mu)P = \mu(1_K\wedge\mu)P$ and $(\mu(\mu\wedge 1_K) - \mu(1_K\wedge\mu))(1 + P + P^2) = (\mu(\mu\wedge 1_K) - \mu(1_K\wedge\mu))(P^3 - 1) = 0$. By $(*)_1$ and $(*)_2$, the right side, and hence the left side, of $(*)_1$ is trivial. (Notice that we always assume $p\geqq 5$). One may use Lemma 5.6 repeatedly to show that

$$(i''\wedge i''\wedge 1_K)_* : [K\wedge K\wedge K, K]_0 \longrightarrow [K\wedge K\wedge K, L\wedge L\wedge K]_0$$

is monic. □

Remark 6.7. More detailed discussion leads to the following "associativity":

$$\mu_2(1_K\wedge\mu) = (1_L\wedge\mu)(\mu_2\wedge 1_K),$$
$$(1_L\wedge\mu)(T_{K,L}\wedge 1_K)(1_K\wedge\mu_2) = -(1_L\wedge\mu)(\mu_2\wedge 1_K) + \mu_2(\mu\wedge 1_K),$$
$$(1_L\wedge\mu_2)(T_{K,L}\wedge 1_K)(1_K\wedge\mu_2) = -(T_{L,L}\wedge 1_K)(1_L\wedge\mu_2)(\mu_2\wedge 1_K),$$

cf.[13], Lemma 2.8, in case $s = 2$. They give stronger about the decomposition of K^*K. (6.4) and (A)', (B)', (D) define a decomposition

$$\pi_*(K\wedge K) = \pi_*(K)\oplus\pi_{*-1}(L\wedge K)\oplus\pi_{*-sq-2}(K),$$

which is shown to be a $\pi_*(K)$-**right** module isomorphism. The elements $\nu i_0 : S^{sq+2} \longrightarrow K\wedge K$, $j_0\mu : K\wedge K \longrightarrow S^{sq+2}$ give the duality $K^*K \cong \pi_{sq+2-*}(K\wedge K)$, which is an isomorphism of $\pi_*(K)$-bimodule, and via which the above decomposition of $\pi_*(K\wedge K)$ becomes

$$K^*K = (\mathrm{Mod})\delta_0 \oplus \mathrm{Der} \oplus \mathrm{Mod},$$

see Remark 2.4.

Proof of Theorem 2.5. Immediate from Theorems 6.5, 6.6, in particular, $\delta'\in\mathrm{Der}$ follows from $\mu_2 + \mu_2 T = \tilde{\Delta}\mu$ and Lemma 6.2. □

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Homotopy Invariance of A_∞ and E_∞ Ring Spaces

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0. Introduction

In [2] May introduced the notion of an E_∞ ring space. This is a space on which a multiplicative E_∞ operad $\mathcal{G}$ and an additive E_∞ operad $\mathcal{C}$ act. These two actions are not independent, they are connected by distributivity relations. Although it is intuitively fairly clear how these relations should look like, a detailed description is quite elaborate. The additive structure of such a ring space gives rise to a spectrum which inherits the multiplicative structure. This in turn can be exploited for calculations.

Waldhausen's algebraic K-theorey of topological space ([8],[9]), which has applications to the investigation of pseudo isotopy spaces, needed the weaker version of an A_∞ ring space which was subsequently defined by May in [4]. Here the multiplication is encoded in an A_∞ operad rather than an E_∞ operad. An E_∞ ring space corresponds to an A_∞ ring space like a commutative ring to a non-commutative one.

The purpose of this paper is to solve the problem of homotopy invariance of A_∞ and E_∞ ring spaces posed by May in [3]. It seems to us that May's definitions, in particular his distributivity relations, are too rigid to allow spaces of the homotopy type of an A_∞ or E_∞ ring space to carry an A_∞ or E_∞ ring space structure. So we modify his definition in the spirit of [1] and bring it closer to the notion of a theory as known from universal algebra. Our definition has the advantage to be transparent and easy to memorize. In particular, we need not list any particular relation such as distributivity. Our results will show that we stay close enough to May's theory: Any A_∞ or E_∞ ring space in his sense is one in ours, and any A_∞ or E_∞ ring space in our sense is homotopy equivalent to one in his, and the homotopy equivalence preserves this structure up to coherent homotopies. Sc we do not have to worry about the machine which associates to each such space an A_∞ or E_∞ ring spectrum: We just can take May's machine.

The central technical device from which all follows is a ring theoretic lifting theorem similar to the lifting theorem 3.17 of [1]. Its proof is far more complicated than the one of [1] because ring theories are considerably more complicated than theories associated with monoids.

In section 1 we introduce A_∞ and E_∞ ring theories and indicate the reasons why it took such a long time to find a reasonable definition. Section 2 is devoted to the proof of the lifting theorem just mentioned. In section 3 we deal with homotopy homomorphisms between ring spaces while the main results on homotopy invariance are listed and proved in section 4. As pointed out they are an almost immediate consequence of the lifting theorem. Their proofs are essentially the same as those of [1]. A final section deals with A_∞ and E_∞ ring spaces. We apply the results of section 4 and investigate the connection of our notion with the one of May.

Needless to say that we have to work in a suitable category *Top* of topological spaces, e.g. compactly generated spaces in the sense of [7], to avoid point set topological inaccuracies.

We also should mention that we found the key ingredients for the lifting theorem while we were studying Steiner's paper on loop structures on the algebraic K-theory of spaces [6].

1. Ring theories

Let us recall the definition of a theory. It is the category of all operations that can be written down in the particular theory in question. Each theory contains a distinguished collection of operations, the set operations. Let $\mathcal{S}$ be the category of finite sets $\underline{n} = \{1,2,\ldots,n\}$ and all maps. For each $\alpha \in \mathcal{S}(\underline{m},\underline{n})$ we have a set operation

(1.1) $\qquad \alpha^* : X^n \longrightarrow X^m$

given by $\alpha^*(x_1,\ldots,x_n) = (x_{\alpha 1},\ldots,x_{\alpha m})$

1.2 Definition: A theory is a category Θ with objects $0,1,2,\ldots.,$ topologized morphism sets $\Theta(m,n)$, and products, together with a faithful functor $\mathcal{S}^{op} \to \Theta$ preserving objects and products. Composition is continuous and the canonical map $\Theta(m,n)\to\Theta(m,1)^n$ is a homeomorphism.

A Θ-space is a continuous product preserving functor $X:\Theta\to\mathcal{T}op$ such that the composite $\mathcal{S}^{op} \to \Theta \to \mathcal{T}op$ gives the set operations (1.1) on $X(1)$. The space $X(1)$ is called the underlying space of X. A homomorphism of Θ-spaces is a natural transformation of such functors.

A theory functor $\Theta_1 \to \Theta_2$ is a continuous functor of theories such that

$$\begin{tikzcd} & \mathcal{S}^{op} \arrow[dl] \arrow[dr] & \\ \Theta_1 \arrow[rr] & & \Theta_2 \end{tikzcd}$$

commutes.

In abuse of notation we often denote the underlying space of a Θ-space X by the same symbol X (instead of $X(1)$), and say "X admits a Θ-structure".

Examples of theories are the theories of monoids, groups, rings etc. One should think of $\Theta(n,1)$ as the space of all n-ary operations in the particular theory. S^{op} is the trivial theory. Each topological space evidently admits an S^{op}-structure.

In any theory the set operations have the useful property that they can be pushed from left to right:

(1.3) For $\sigma \in S(\underline{m},\underline{n})$, $(a_1,\ldots,a_n) \in \Theta(k,1)^n$ and $b_i \in \Theta(k_i,1)$, $i=1,\ldots,n$, we have

$$\sigma^* \circ (a_1,\ldots,a_n) = (a_{\sigma 1},\ldots,a_{\sigma m})$$
$$\sigma^* \circ (b_1 \times \ldots \times b_n) = (b_{\sigma 1} \times \ldots \times b_{\sigma n}) \circ \sigma(k_1,\ldots,k_n)^*$$

where

$$\sigma(k_1,\ldots,k_n) : \coprod_{i=1}^{n} k_{\sigma i} \longrightarrow \coprod_{i=1}^{n} k_i$$

maps the i-th summand $k_{\sigma i}$ identically onto the $\sigma(i)$-th summand $k_{\sigma i}$ of these ordered disjoint unions.

We are interested in a reasonable definition of a ring theory up to homotopy. In [1] Boardman and Vogt associated a canonical homotopical theory $W\Theta$ with each given theory Θ. Loosely speaking, $W\Theta$ is the free theory over Θ with the relations in Θ put back up to coherent homotopies. Making these relations up to homotopy into strict ones again one obtains a theory functor

$$\varepsilon : W\Theta \to \Theta,$$

which is a homotopy equivalence (on morphism spaces). More generally a Θ-space up to homotopy is a Ψ-space, where Ψ is a theory admitting a theory functor $\Psi \to \Theta$ which is a homotopy equivalence.

A result of Boardman and Vogt [1; Thm. 4.58] shows that under mild restrictions any such Ψ-space is a deformation retract of a Θ-space. Hence this definition is too restrictive for our purposes: We think of an infinite loop space as a homotopy commutative and homotopy associative monoid such that the homotopies satisfy "all" coherence conditions. Hence an infinite loop space theory should be the theory of a commutative monoid up to homotopy. The results of Boardman and Vogt show that the homotopical theory $W\Theta_{cm}$ associated with the theory Θ_{cm} of commutative monoids characterizes all spaces which are homotopy equivalent to a weak product of Eilenberg-MacLane spaces. So not every infinite loop space is a $W\Theta_{cm}$-space.

The correct version of theory characterizing infinite loop spaces is obtained as follows: Let $S \subset \Theta_{cm}$ be the subcategory generated under composition and $\times$ by permutation set operations and the operations

$$\text{(1.4)} \qquad \begin{aligned} \lambda_n &: (x_1,\ldots,x_n) \longmapsto x_1 + x_2+\ldots+x_n, \ n>0 \\ \lambda_o &: * \longmapsto 0 \end{aligned}$$

(This category is isomorphic to S defined above). An E_∞ theory is a theory Ψ equipped with a theory functor

$$F : \Psi \longrightarrow \Theta_{cm}$$

such that

$$F|F^{-1}(S) : F^{-1}(S) \longrightarrow S$$

is a homotopy equivalence. Such E_∞ theories describe infinite loop spaces. The difference between an E_∞ theory and a Θ_{cm}-theory up to coherent homotopies is explained in [1; section 6.4]. Here we only mention that set operations from epimorphisms - they operate as all sorts of diagonal inclusions - are responsible for this fundamental difference.

The problem with ring theories is to find the correct notion of a corresponding E_∞ theory. As essential operations we here have the additive operations λ_n, $n \geq 0$, of (1.4) and the multiplicative operations

$$(1.5)\qquad \begin{aligned} \mu_n &: (x_1,\dots,x_n) \longmapsto x_1\cdot x_2\cdot\ldots\cdot x_n, \quad n>0 \\ \mu_0 &: \qquad * \qquad \longmapsto 1 \end{aligned}$$

To write down the relations we need permutation set operations for the commutatively relations, diagonal maps for the distributive law and - since we work with semirings rather than rings - projections to write down the relation $0\cdot x=0$. Since diagonal maps are set operations from epimorphisms and projections from monomorphisms we cannot discard any set operation as one does for E_∞ theories, where one restricts ones attention to permutations.

This led May to consider the multiplicative and additive structures separately and connect them by a number of distributive formulas. We adopt a different approach: Let Θ_r and Θ_{cr} be the theories of semirings and commutative semirings respectively.

Let $\mathcal{R}$ respectively $\mathcal{CR}$ be the union of the two subcategories of Θ_r and Θ_{cr} the first of which is generated under composition and $\times$ by the operations μ_n, $n \geq 0$, and the second of which is generated under composition and $\times$ by the operations μ_n and λ_k with $n > 0$ and $k \geq 0$. We stress that $\mathcal{R}$ and $\mathcal{CR}$ are unions of subcategories, but they are not subcategories themselves. Both contain all set operations from monomorphisms. Set operations from epimorphisms occur in connection with the distributive law. For $n \geq 2$ the operation $x \mapsto n\cdot x$ is not contained in $\mathcal{R}$ or $\mathcal{CR}$. In particular, each operation in $\mathcal{R}$ or $\mathcal{CR}$ is simple in the sense of Steiner [6;Def.1.2].

The subsets $\mathcal{R}$ and $\mathcal{CR}$ of operations in Θ_r and Θ_{cr} take the part $\mathcal{S}$ had for infinite loop spaces.

1.6 <u>Definition</u>: A <u>theory</u> <u>over</u> Θ_r is a theory Θ equipped with a surjective theory functor $F_\Theta\colon \Theta \to \Theta_r$. Moreover, $\operatorname{ob}\Theta \subset \operatorname{mor}\Theta$ is supposed to be a closed cofibration.

A *functor* $\Theta \to \Psi$ *over* Θ_r is a functor H such that $f_\Psi \circ H = F_\Theta$.

In the commutative case substitute the suffix r by cr.

1.7 *Definition*: An A_∞ or E_∞ *ring theory* is a theory Θ over Θ_r respectively Θ_{cr} such that

$$F_\Theta : F_\Theta^{-1}(\mathcal{R}) \to \mathcal{R}$$

(resp. $\mathcal{CR}$) is a homotopy equivalence.

This special type of homotopy equivalence will play a central role in our theory, so we introduce an extra name for it.

1.8 *Definition*: A theory functor $H: \Theta \longrightarrow \Psi$ over Θ_r (or Θ_{cr}) is called a *ring equivalence*, if

$$H : F_\Theta^{-1}(\mathcal{R}) \longrightarrow F_\Psi^{-1}(\mathcal{R})$$

is a homotopy equivalence on the morphism spaces. It is called an *equivariant ring equivalence* if

$$H : F_\Theta^{-1}(\mathcal{R})(p,1) \longrightarrow F_\Psi^{-1}(\mathcal{R})(p,1)$$

is a Σ_p-equivariant homotopy equivalence for all $p \geq 2$, where Σ_p is the symmetric group of p symbols acting by composition with permutation set operations.

From the point of view of homotopy invariance a homomorphism $f: X \to Y$ of Θ-spaces is not the appropriate notion of a morphism because any other map in the homotopy class of f is most likely not a homomorphism. To deal with this, we also have to consider the categories

$$\Theta \times \mathcal{I}s \longrightarrow \Theta \times \mathcal{L}_n$$

where $\mathcal{I}s$ is the category with two objects and an isomorphism

$$\begin{tikzcd} 0 \arrow[r, bend left] & 1 \arrow[l, bend left] \end{tikzcd}$$

between them and L_n is the linear category

$$0 \to 1 \to 2 \to \ldots \to (n-1) \to n.$$

The notions of a $(\Theta\times I\mathfrak{s})$-space and a $(\Theta\times L_n)$-space are the obvious extensions of the definition of a Θ-space. In particular, a $(\Theta\times I\mathfrak{s})$-space determines and is determined by an isomorphism of Θ-spaces $X_o \cong X_1$, while a $(\Theta\times L_n)$-space determines and is determined by a sequence of Θ-spaces and homomorphisms

$$X_o \to X_1 \to X_2 \to \ldots \to X_{n-1} \to X_n$$

Hence the homotopical categories $W(\Theta\times I\mathfrak{s})$ and $W(\Theta\times L_n)$ clearly codify isomorphisms and homomorphisms up to coherent homotopies. In particular, an isomorphism up to coherent homotopies is a homotopy equivalence where the maps and homotopies involved preserve the algebraic structures $W\Theta$ up to coherent homotopies.

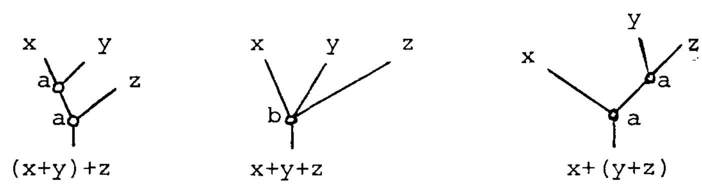
Also note, that there are two canonical inclusions of L_1 into $I\mathfrak{s}$, which we shall use without explicit mentioning.

2. The lifting theorem

In [1] the crucial tool in the analysis of homotopy invariant algebraic structures associated with the theory Θ_{cm} of commutative monoids is the lifting theorem [1;Thm. 3.17]. We try to prove a similar result in our situation.

Let us first recall the definition of the theory $W\Theta$ mentioned above.

(2.1) The theory $T\Theta$: Let Θ be any theory. We consider each operation $a\in\Theta(n,1)$ as an electrical box with n inputs and one output. Composite operations are obtained by wiring boxes together. E.g. let $a\in\Theta_{cm}(2,1)$ be the addition and $b\in\Theta_{cm}(3,1)$ be the addition of three elements; then

(2.2) 

In Θ_{cm}, these three circuits all represent the same operation, because of associativity. The shape of these circuits suggests to call them trees with the inputs on the top and the output at the bottom. In the associated homotopical theory the three trees in (2.2) do not coincide but are homotopic. To account for this we give each connection between boxes a length between 0 and 1 and allow edges of length 0 be shrunk away by combining the boxes at the ends (see below).

Given a particular shape λ of trees we now have the following data: Each box with n inputs is labelled by an element $a\in\Theta(n,1)$; and each connection has a length $t\in I$, I being the unit intervall. Hence the set of all trees of shape λ can be topologized as a product of spaces $\Theta(n,1)$ and a cube I^r.

There is one additional datum to a tree: There is a map from the inputs of the tree to some set $\underline{n}$. We give each input a digit, namely its image under this map.

Let $T\Theta(n,1)$ be the space of all pairs (θ,τ), where θ is any tree and $\tau:\underline{k} \to \underline{n}$ is the function from the k inputs of θ (identified with $\{1,2,\ldots,k\}$ from left to right) to $\underline{n}$. Define $T\Theta(n,m) = (T\Theta(n,1))^m$. We allow the trivial tree

in $T\Theta(1,1)$ and the stumps

$, a\in\Theta(0,1)$

in $T\Theta(0,1)$.

For $(\theta,\tau)\in T\Theta(n,1)$ and $(\psi_i,\sigma_i)\in T\Theta(m,1)$, $1\le i\le n$, we define the composition

$$(\theta,\tau)\circ((\psi_1,\sigma_1),\ldots,(\psi_n,\sigma_n))$$

in $T\Theta$ by wiring (ψ_i,σ_i) onto each input with digit $i\in\underline{n}$. The new connections obtain the lengths 1, the digits of the ψ_i define the map of the inputs of the new tree into $\underline{m}$.
The product of two trees $(\theta_i,\sigma_i) \in T\Theta(n_i,1)$ is

$$(\theta_1,\sigma_1) \times (\theta_2,\sigma_2) = ((\theta_1,j_1\circ\sigma_1),(\theta_2,j_2\circ\sigma_2))$$

where $j_i:\underline{n}_i \longrightarrow \underline{n}_1 \sqcup \underline{n}_2 = \underline{n_1+n_2}$ is the inclusion and $\underline{n}_1 \sqcup \underline{n}_2$ is identified with $\underline{n_1+n_2}$ in order preserving blocks.
From this it is clear that

$$(\theta,\tau) = (\theta,\mathrm{id})\circ\tau^*$$

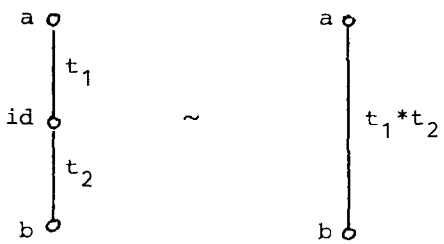
We have obtained a new theory $T\Theta$.

(2.3) <u>The theory</u> $W\Theta$: To relate $T\Theta$ to the given theory Θ we impose relations on $T\Theta$.

(2.4) (a) Any connection of length 0 can be shrunk away by composing its ends:

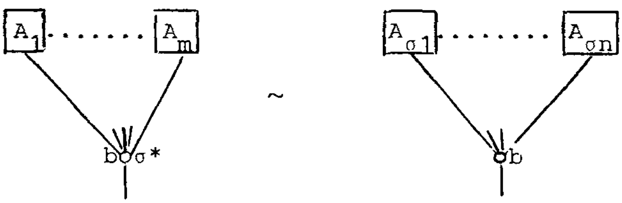
$c = b \circ (id \times a \times id)$ in Θ

(b) Any box with label $id \in \Theta(1,1)$ can be removed



with $t_1 * t_2 = \max(t_1, t_2)$. (For inputs and output we assume the lengths 1, by convention).

(c) Any box $b \circ \sigma^*$ can be replaced by b by changing the tree above it



for $\sigma \in S(\underline{n}, \underline{m})$.

We define $W\Theta$ to be the quotient theory of $T\Theta$ modulo these relations.

Define the augmentation

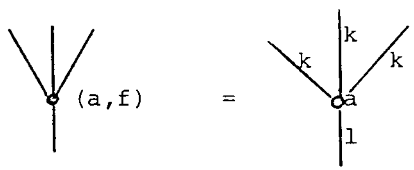
$$\varepsilon = \varepsilon_\Theta : W\Theta \longrightarrow \Theta$$

by mapping each tree to its composition operation in Θ, i.e. $\varepsilon(\theta,\tau) = \varepsilon(\theta,\mathrm{id}) \circ \tau^*$, where $\varepsilon(\theta,\mathrm{id})$ is the composite operation represented by θ if we neglect the lengths of the connections.

2.5 <u>Proposition</u>: The functor $\varepsilon: W\Theta \to \Theta$ is a homotopy equivalence of theories. (The homotopy inverse is not a functor!)

For a proof see [1;3.6].

Since we need the lifting theorem not only for theories Θ but also for the "generalized" case $\Theta\times\mathcal{I}\!s$ and $\Theta\times L_n$, we have to consider a minor generalization. Let $C=\mathcal{I}\!s$ or L_n. Then $W(\Theta\times C)$ is obtained from $\Theta\times C$ in the same manner as $W\Theta$ from Θ. Since there is at most one morphism between any two objects of C, we can alternate the description of a tree in $T(\Theta\times C)$ as follows: Let $a\in\Theta(n,1)$ and $f\in C(k,l)$. Instead of labeling a box in a tree θ by $(a,f)\in(\Theta\times C)((n,k),(1,l))$ we label it by $a\in\Theta(n,1)$ and give its inputs besides lengths an additional label k and its output an additional label l.



$f: k \to l$

We call a subtree of θ to be of level k if its edges all have the label k.

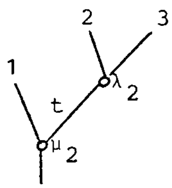
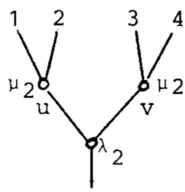
The lifting theorem is going to be applied in the following situation. We are given a diagram of theory functors

$$\begin{array}{ccc} W\Theta & \overset{\overline{F}}{\dashrightarrow} & \Psi \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle G} \\ \Theta & \xrightarrow{\;F\;} & \Lambda \end{array} \tag{2.6}$$

where G is a homotopy equivalence of some kind, and a Ψ-space $X: \Psi \to \mathcal{T}op$. We would like to lift X to a $W\Theta$-space $X\circ\overline{F}$. This ought to be possible because the subcategory $W_o\Theta$ of $W\Theta$ represented by all trees whose connections all have the lengths 1 is the free theory over Θ. Hence $\overline{F}: W_o\Theta \to \Psi$ exists making the square (2.6) commute up to homotopy. Now filter $W\Theta$ by subcategories $W_n\Theta$ and extend $\overline{F}|W_o\Theta$ inductively over the filtration.

This procedure only has a chance to work if the following crucial condition is satisfied: A composite operation $a\circ b:n\to 1$ has (at least) the filtration of the maximum of the filtrations of its factors. Otherwise we simply do not have any chance to extend $\overline{F}$ as a functor.
In the case of theories related to Θ_{cm} one can restrict ones attention to the parts over $S \subset \Theta_{cm}$ where one only has permutations as set operations. Here the obvious skeleton filtration of $W\Theta$ makes everything work. The distributive law causes trouble in the case of ring theories.

2.7 Example: Consider the trees

$\theta_1 =$  and $\theta_2 =$ 

They do not have anything to do with each other. But if we compose θ_2 with the operation from the epimorphism

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

then θ_1 and $\theta_2 \circ \sigma^*$ are connected for $t=u=v=0$ via the distributive law.

There are two possibilities to get around this difficulty. Instead of lifting the functor F to a theory functor $\overline{F}$ we could try to pull back $X : \Psi \to Top$ to a $W\Theta$-space $\overline{X}$ which is X on objects but which not necessarily factors through Ψ. For this we have to assume that the underlying space X and the morphism spaces $\Theta(n,1)$ are locally equiconnected in a strong sense. The precise assumptions are those of [1; Thm. 4.58].

We, for our purposes, favour a different approach. We prove the lifting theorem for a special kind of theory over Θ_r or Θ_{cr}, namely for those theories whose morphisms can be decomposed almost uniquely into an additive and a multiplicative morphism. The following definition is motivated by results of Steiner [6].

2.8 Notation: Let R_a and R_m denote the subcategories of Θ_r generated under composition and $\times$ by all λ_n and all permutations respectively by all μ_n. We use the same notation in the commutative case with the difference that permutations also appear in R_m.

2.9 Definition: A theory $F:\Theta\to\Theta_r$ (or Θ_{cr}) is called factorial if it contains subcategories Θ^a of "additive" operations and Θ^m of "multiplicative" operations such that

1. Θ^a is a PROP in the sense of [1;2.44]

2. Θ^m is a PROP in the commutative and a PRO in the sense of [1;2.42] in the non-commutative case

3. $\Theta^a(k,1)$ and $\Theta^m(k,1)$ are closed subspaces of $\Theta(k,1)$. They are disjoint for $k\neq 1$ and $\Theta^a(1,1)\cap\Theta^m(1,1)=\{id\}$. The inclusion of id into $\Theta^a(1,1)$ and $\Theta^m(1,1)$ is a cofibration.

4. The functor F maps Θ^a and Θ^m surjectively onto R_a and R_m

5. Each operation $c\in\Theta(n,1)$ admits a factorization

$$c= a\circ(m_1\times\ldots\times m_k)\circ\alpha^*$$

with $a\in\Theta^a(k,1)$, $m_r\in\Theta^m(j_r,1)$ for $r=1,\ldots,k$, and α^* a set operation, uniquely up to the relation

$(a\circ\tau^*)\circ[(m_1\circ\pi_1^*)\times\ldots\times(m_k\circ\pi_k^*)]\circ\alpha^*$

$$= a\circ[m_{\tau 1}\times\ldots\times m_{\tau k}]\circ[\alpha\circ\tau(j_1,\ldots,j_k)\circ(\pi_{\tau 1}\times\ldots\times\pi_{\tau k})]^*$$

with $\tau\in\Sigma_k$, $\pi_r\in\Sigma_{j_r}$. In the non-commutative case, each $\pi_r=\mathrm{id}$.

Examples of factorial theories are the theory Θ_r or Θ_{cr} itself and, by a result of Steiner [6; Lemma 1.10], each theory arising from an operad pair in the sense of May [2; chapt. VI].

Given a factorial theory, we call a decomposition 2.9.5 of a morphism c a standard factorization of c.
For later use we state an elementary result.

2.10 <u>Lemma</u>: Let $a\circ(m_1\times\ldots\times m_l)\circ\alpha^*$ be a standard factorization of $c=m\circ(a_1\times\ldots\times a_k)$ with $m\in\Theta^m(k,1)$ and $a_r\in\Theta^a(jr,1)$, $r=1,\ldots,k$. Then $l=j_1\cdot\ldots\cdot j_k$ and $m_r\in\Theta^m(k,1)$, $r=1,\ldots,k$. Hence $\alpha\in S(\underline{k\cdot j_i\cdot\ldots\cdot j_k}, \underline{j_1+\ldots+j_k})$

<u>Proof</u>: $F(c) = \mu_k\circ(\lambda_{j_1}\times\ldots\times\lambda_{j_k}) = \lambda_l\circ(\mu_{i_1}\times\ldots\times\mu_{i_l})\circ\alpha^*$
with $\mu_{i_r} = F(m_r)$. By the distributive law,

$$\mu_k\circ(\lambda_{j_1}\times\ldots\times\lambda_{j_k}) = \lambda_p\circ(\mu_k\times\ldots\times\mu_k)\circ\beta^*$$

with $p=j_1j_2\ldots j_k$ and some suitable set operation β^*. The uniqueness of the standard decomposition of F(c) up to permutations implies, that $p=l$, $i_r=k$, and α and β differ by a permutation.

For factorial theories Θ we can construct a filtration of $W(\Theta\times C)$, $C=L_n$ or $I\mathcal{S}$, which serves our purposes. It is bound to be fairly complicated. A guide line should be that the tree θ_2 of (2.7) has to have lower filtration than the tree θ_1, because the composite $\theta_2\circ\sigma^*$ has something to do with θ_1.

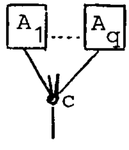
Observe that $W\Theta$ never is factorial, which makes Definition 2.9 look artificial. Indeed, (2.9) is a technical condition, but we shall see that in the case of A_∞ and E_∞ ring theories it suffices to consider factorial theories.

For the rest of this section we restrict our attention to the commutative case. The development of the non-commutative case is completely analoguous, one simply omits the permutations in Θ^m.
Let $T'(\Theta\times C)$ be the subcategory of $T(\Theta\times C)$ of those trees having box labels in Θ^a or Θ^m only. So, given a box label $c\circ\tau^*$, the set operation τ^* comes from a permutation. Any other set operation only occurs as digiting function of the inputs. The morphism spaces of $T'(\Theta\times C)$ are closed subspaces of those of $T(\Theta\times C)$.

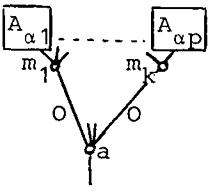
Any representing tree in $T(\Theta\times C)$ can in a canonical way (up to permutations) be brought into a tree in $T'(\Theta\times C)$ by introducing redundant connections of lengths 0. If a box is labelled c with $c \notin \Theta^a \cup \Theta^m$ we choose a standard factorization

$$c=a\circ(m_1\times\ldots\times m_k)\circ\alpha^*, \qquad \alpha\in S(\underline{p},\underline{q})$$

and substitute



by



Starting at the output and working upwards we can change each representing tree inside its equivalence class to one of the required form. Hence $W(\Theta\times C)$ is the quotient of $T'(\Theta\times C)$ by the following modified relations:

(2.11) Call a box with label in Θ^a or Θ^m an a-box respectively an m-box.

(a) Any connection of length 0 between two a-boxes or two m-boxes may be shrunk by using relation (2.4a)

(b) = (2.4b), but note that the incoming and outgoing edges must have the same labels

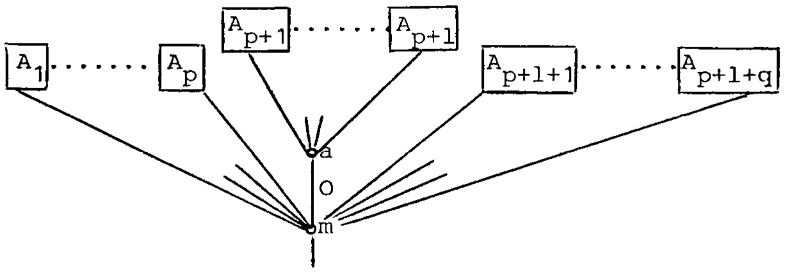
(c)=(2.4c) but for permutations only

(d) Given $m\in\Theta^{m}(k,1)$ and $a\in\Theta^{a}(1,1)$, and a standard factorization

$$m\circ(id_p\times a\times id_q) = a'\circ(m_1\times\ldots\times m_r)\circ\alpha^*$$

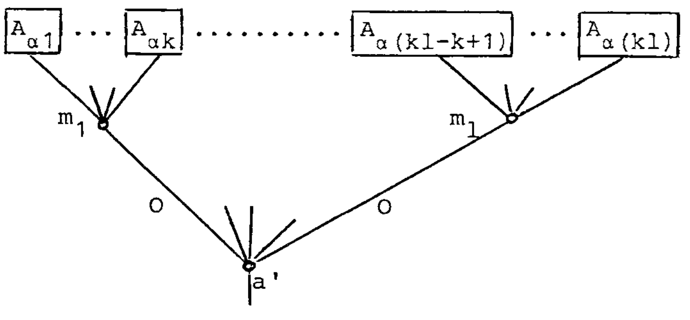
then the following subtree

(I)



can be substituted by

(II)



and vice versa.

We call the process (2.11d) "pushing up an m-box". It is the relation (2.4a) for connections between m- and a-boxes in the spaces $T'(\Theta\times C)$ of representatives.

Observe, that the different choices of standard factorizations made in relating a tree in $T(\Theta\times C)$ "canonically" to a tree in $T'(\Theta\times C)$ are taken care of by (2..11c).

(2.11c) A filtration of $W(\Theta\times C)$: As mentioned above we have to take care of the distributivity relation, which complicates the filtration. The shape λ of a tree Θ in $T'(\Theta\times C)$ is defined to be the underlying circuit of Θ together with a specification of edge labels (in ob C) and a partition of the boxes into a-boxes and m-boxes. With θ we associate a sequence of integers: Define the height of an m-box in θ to be the number of a-boxes between it and the output of θ. Let p_k be the number of m-boxes of height k. Starting with the m-boxes of maximal height we push up all m-boxes through all a-boxes different from a nullary operation using relation (2.11d). We end up with a shape $h(\theta)$ where all m-boxes have maximal height. Let l denote the number of connections of this shape. Thus, we can associate with each tree θ of shape λ the same sequence of integers

$$s(\theta) = (l;p_o,p_1,p_2,\ldots).$$

We order these sequences lexicographically. This ordering is not as infinite as it looks, because $p_i=0$ for $i>l$ and each $p_i\leq l$. We obtain an induced filtration of $W(\Theta\times C)$: For $\nu=(l;p_1,p_2\ldots)$ define F_ν to be the subcategory of $W(\Theta\times C)$ generated under composition and the product $\times$ by all set operations and all elements represented by trees θ with $s(\theta)\leq\nu$.

It should be observed that id, which is an additive as well as a multiplicative operation, behaves like an additive one as far as this filtration is concerned.
The next result shows that our filtration satisfies the fundamental requirement mentioned above.

2.13 Proposition: If $c=c_1\circ c_2$ is in F_ν, so are c_1 and c_2.

Proof: Let (θ,α) be a representing tree of c_1 with lowest possible $s(\theta)$. Since $(\theta,\alpha) = (\theta,id)\circ\alpha^*$, it suffices to consider the case that c_1 is represented by (θ,id). If c_2 is not a set operation, c is represented by a composite tree ψ of θ and other trees ρ_i, so that $s(\theta) < s(\psi)$ and $s(\rho_i)<s(\psi)$. Here observe that pushing up m-boxes does not decrease the number of connections, so that $h(\psi)$ has at least as many connections as $h(\theta) \circ (h(\rho_1)\times\ldots\times h(\rho_r))$.

If $c_2=\tau^*$ and τ is a permutation then (θ,τ) is not related to a tree ψ with $s(\psi)< s(\theta)$because permutations do not change $h(\theta)$ or any p_k. If τ is a monomorphism the relation, that any element of $\theta^a(0,1)$ behaves like a strict multiplicative 0, can influence (θ,τ). But the related tree has more m-boxes and hence is of higher filtration than θ. If τ is an epimorphism, relation (2.11d) can possibly be applied: A subtree of type (II) can be replaced by a tree of type (I). But (I) has an m-box of height 0 which (II) does not have. Hence this representative is of higher filtration than θ.

We now can preceed almost in complete analogy to [1; section III.3]. Given a tree shape λ, the space M'_λ of all trees of this shape is a product

$$M'_\lambda = (\prod_{i,j} \theta^m(k_i,1)\times \theta^a(l_j,1)\times I^r \times S(\underline{p},\underline{q})$$

(2.14) An element $(\theta,\tau)\in M'_\lambda$ represents an element of filtration less than $s(\theta)$ iff one of the following conditions holds

(a) some connection between two a-boxes or two m-boxes has length 0

(b) some box with the same input and output labels is labelled by $id \in \theta(1,1)$

(c) some connection from an a-box down to an m-box has length 0

(d) sufficiently many connections have length 1 to decompose θ in $T'(\theta\times C)$.

Explanation: (a) reduces θ by relation (2.11a). The resulting tree ψ has less a-boxes or m-boxes. Hence, in particular, $h(\psi)<h(\theta)$. (b) reduces by relation (2.11b). Again, $h(\psi)< h(\theta)$ for the resulting tree ψ. In case (c) relation (2.11d) applies. Here we distinguish two cases: The a-box is labelled by a nullary operation e. Since any nullary operation in a factorial theory is multiplicatively a strict zero the adjacent m-box disappears, and again $h(\psi) < h(\theta)$ for the resulting tree ψ. In the second case the a-box is not a nullary operation. Then the m-box can be pushed up to lower the filtration. (2.13d) is clear.

Trees which are related by (2.11c) have the same tuple and represent elements of the same filtration. We have to account for this: Let Λ be the set of all tree shapes obtained from λ by applying relation (2.11c). Let $\Sigma_p \subset S(\underline{p},\underline{p})$ be the subgroup of permutations, and

$$M_\lambda = (\prod_{i,j} \Theta^m(k_i,1)\times \Theta^a(l_j,1))\times I^r\times \Sigma_p \subset M'_\lambda .$$

There is a group G' acting canonically on the space

$$M_\Lambda = \bigcup_{\lambda\in\Lambda} M_\lambda$$

A subgroup Σ_r permutes the coordinates of I^r, subgroups Σ_{k_i} and Σ_{l_j} operate via composition with set operations on $\Theta^m(k_i,1)$ and $\Theta^a(l_j,1)$, a subgroup Σ_p operates from the right on the factor Σ_p, and an additional subgroup permutes the factors of the product

$$\Pi\ \Theta^m(k_i,1) \times \Theta^a(l_j,1)$$

Let G be the subgroup of G', generated by all $g \in G'$, which map M_λ into itself and for which the trees θ and $g(\theta)$ are related by a single application of (2.11c). We call G the symmetry group of the shape λ.

For a given filtration F_ν of $W(\Theta\times C)$ let DF_ν be the subcategory of F_ν generated by elements represented by trees (θ,α) with $s(\theta)<\nu$. Let $N_\lambda \subset M_\lambda$ be the subspace of all trees to which (2.14) applies. Clearly, N_λ is a G-NDR of M_λ (G-equivariant neighbourhood deformation retract), and the map

$$N_\lambda \longrightarrow DF_\nu$$

factors through N_λ/G.

To obtain F_ν from DF_ν we have to attach a quotient of M'_λ, one for each shape orbit Λ. Note that Σ_p operates from the left on M_λ by

$$\tau\cdot(\theta,\sigma) = (\theta,\sigma)\circ\tau^* = (\theta,\tau\circ\sigma)$$

and on the right of $\mathcal{S}(\underline{p},\underline{q})$ by

$$\alpha\cdot\tau = \alpha\circ\tau$$

The attaching map of N_λ/G is equivariant with respect to the Σ_p-action, and we have to attach $M_\lambda/G\times_{\Sigma_p}\mathcal{S}(\underline{p},\underline{q})$ relative to $N_\lambda/G\times_{\Sigma_p}\mathcal{S}(\underline{p},\underline{q})$. We, in fact, attach M_λ/G and extend "equivariantly" using the set operations. To avoid considering the G-action and the Σ_p-action on M_λ independently we combine the two. Let $P_{\lambda,\xi} \subset M_\lambda$ be the subspace with Σ_p-coordinate ξ. An element $g\in G$ maps $P_{\lambda,\xi}$ to some space $P_{\lambda,\eta}$. The correspondence $g\mapsto \xi^{-1}\circ\eta$ defines a homomorphism

$$\phi : G \longrightarrow \Sigma_p$$

We define a G-action on $P_\lambda := P_{\lambda,\mathrm{id}}$ by

$$G\times P_\lambda \longrightarrow M_\lambda \longrightarrow P_\lambda$$

The first map is $g\cdot(\theta,\mathrm{id}) = (\theta\cdot g^{-1},\ \phi(g^{-1}))$, the second maps (θ,σ) to (θ,id). The characteristic map

$$\mu_\lambda : P_\lambda \longrightarrow F_\nu$$

sending each tree to its corresponding operation then is G-equivariant, where the G-action on $W(\Theta\times C)((p,c),(1,d))$ is given by

$$g\cdot c = c \circ \phi(g)^*$$

If we try to construct a theory functor $W(\Theta\times C) \longrightarrow \Psi$ inductively by extending a data-preserving functor $DF_\nu \to \Psi$ over F_ν it hence suffices to find G-equivariant maps $P_\lambda \to \Psi$ extending the given maps on

$$Q_\lambda := P_\lambda \cap N_\lambda$$

one for each shape orbit, where $s(\lambda) = \nu$.

Now almost all is set for the proof of the lifting theorem. Since we need a relative version let us introduce the notion of an admissible subcategory of $W(\Theta\times C)$. Let $V \subset W(\Theta\times C)$ be a subcategory and let $V_\lambda \subset P_\lambda$ be the subspace of those trees (θ,id) representing an element of V.

2.15 Definition: $V \subset W(\Theta\times C)$ is called admissible if the following holds:

(i) If $c_1 \circ c_2$ is in V then c_1 and c_2 are in V

(ii) If $c_1 \times c_2$ is in V then c_1 and c_2 are in V

(iii) $(P_\lambda, V_\lambda \cup Q_\lambda)$ is a G-NDR and, if $F_{\Theta\times C} \circ \varepsilon \circ \mu_\lambda(P_\lambda) \notin CR$, a G-SDR (strong deformation retract)

The empty category is admissible because (P_λ, Q_λ) is a G-NDR. If P_λ does not sit over CR it contains an a-box with an input on which a tree is sitting having no inputs but only stumps with labels in Θ^m. Then P_λ has "free" lower faces and there exists an equivariant deformation retraction $P_\lambda \to Q_\lambda$.

2.16 Lifting Theorem: Given a diagram

$$\begin{tikzcd}
V \arrow[d, phantom, "\cap"] \arrow[rrd, "H'"] \arrow[rrdd, "K'(t)"'] & & \\
W(\Theta\times C) \arrow[d, "\varepsilon"] & & \Psi \arrow[d, "G"] \\
\Theta\times C \arrow[rr, "F"] & & \Lambda
\end{tikzcd}$$

of theories $\Theta\times C, \Lambda, \Psi$ over $\Theta_{cr}\times C$ and data preserving functors satisfying the following conditions

(a) Θ is factorial

(b) V is admissible

(c) $K'(t)$, $t\in I$, is a homotopy through data preserving functors from $F\circ(\varepsilon|V)$ to $G\circ H'$

(d) One of the following conditions hold

 (I) G is an equivariant ring equivalence

 (II) G is a ring equivalence, and $\Theta^{a}(p,1)$ and $\Theta^{m}(p,1)$ are numerable principal Σ_p-spaces for all $p\geq 2$.

Then: (A) There exist extensions $H:W(\Theta\times C) \longrightarrow \Psi$ and $K(t):W(\Theta\times C) \longrightarrow \Lambda$ of H' and $K'(t)$ to theory functors over $\Theta_{cr}\times C$, such that $K(t):F\circ\varepsilon\simeq G\circ H$.

(B) Given a data preserving homotopy $H'(u)$: $V\to\Psi$ of H', extensions H_o, $H_1:W(\Theta\times C)\to\Psi$ of $H'(0)$ and $H'(1)$ to theory functors over $\Theta_{cr}\times C$, homotopies $K(i,t):W(\Theta\times C)\to \Lambda, i=0,1$, of theory functors from $F\circ\varepsilon$ to $G\circ H_i$ and a data preserving homotopy of homotopies $L'(u,t)$: $V \longrightarrow \Lambda$ from $K(0,t)|V$ to $K(1,t)|V$ such that $L'(u,0)=F\circ(\varepsilon|V)$ and $L'(u,1) = G\circ H'(u)$. Then there exist extensions $H(u):W(\Theta\times C)\longrightarrow \Psi$ and $L(u,t):W(\Theta\times C)\longrightarrow \Lambda$ of H' and L' to homotopies of theory functors over $\Theta_{cr}\times C$, such that $L(u,0) = F\circ\varepsilon$, $L(u,1)= H(u)$, $H(0) = H_o$, and $H(1) = H_1$.

Part (A) proves the existence of a lift up to homotopy and Part (B) shows that it is unique up to homotopy.

<u>Proof</u>: We construct H by induction on the filtration, starting with $\nu=(0,1,0,\ldots)$. Then the elements of F_ν are represented by a tree with a single box. Let (θ,id) be a representing tree of $c\in F_\nu$. If $c\in V$ define $H(c) = H'(c)$. Otherwise, put $H(c) = \overline{G}\circ F\circ\varepsilon(c)$ where $\overline{G}$ is any homotopy inverse of G. The homotopy $Id \simeq G\circ\overline{G}$ induces $K(t)(c)$. We extend to all of F_ν by composing with set operations

from the right.

In the inductive step we need G-equivariant maps

$$h_\lambda : P_\lambda \longrightarrow \Psi((p,c),\ (1,d))$$

and G-equivariant homotopies

$$k_\lambda(t) :\ P_\lambda \longrightarrow \Lambda((p,c),\ (1,d))$$

which are already given on $Q_\lambda \cup V_\lambda$ and which satisfy $k_\lambda(0) = F \circ \varepsilon \circ u_\lambda$ and $k_\lambda(1) = G \circ h_\lambda$, where u_λ is the characteristic map of P_λ.
If P_λ maps to $CR \times C$ under $F_{\Theta\times C} \circ \varepsilon \circ u_\lambda$ we apply [1;Thm. 3.5, p.241] to prove (dI). Otherwise we use the strong deformation retraction $P_\lambda \to V_\lambda \times Q_\lambda$ given by assumption. In the case (dII) we show that P_λ is a numerable principal G-space in exactly the same manner as in [1;p.85] and proceeds in the same way.

The proof of part (B) is exactly the same if one substitutes the G-NDR pair $(P_\lambda, Q_\lambda \cup V_\lambda)$ by the G-NDR pair $(P_\lambda, Q_\lambda \cup V_\lambda) \times (I, \partial I)$. This completes the proof of the lifting theorem.

Using the same induction and the G-equivariant homotopy extension property of the pair $(P_\lambda, Q_\lambda \cup V_\lambda)$ we obtain the following extension result.

2.17 <u>Extension Theorem</u>: Let Θ be factorial, $V \subset W(\Theta\times C)$ admissible, $H: W(\Theta\times C) \longrightarrow \Psi$ a theory functor over $\Theta_{cr} \times C$ and $K(t): V \longrightarrow \Psi$ a data preserving homotopy of functors such that $K(0) = H|V$. Then there is an extension $H(t): W(\Theta\times C) \longrightarrow \Psi$ of $K(t)$ to a homotopy of theory functors over $\Theta_{cr} \times C$ such that $H(0) = H$.

<u>Remark</u>: We can prove the lifting and the extension theorem for a weaker type of factorial theory. We do not need the uniqueness part of 2.9.5. It suffices to have a "decomposition function" which maps each operation c to a "standard factorization". A slight refinement of the filtration makes the same proofs work in this case.

3. Homotopy homomorphisms

Let $[n]$ be the ordered set $\{0,1,\ldots,n\}$. The correspondence

$$[n] \mapsto \{W(\Theta\times L_n)\text{-spaces}\} := K_n\Theta$$

defines a semisimplicial class $K\Theta$. Note that an order preserving map $\alpha:[m] \to [n]$ defines a functor $L_m \to L_n$ and hence a functor $[\alpha] : W(\Theta\times L_m) \to W(\Theta\times L_n)$. Composition with $[\alpha]$ gives the structure maps of $K\Theta$. Let

$$d^i : K_n\Theta \longrightarrow K_{n-1}\Theta \qquad 0\le i\le n$$

$$s^i : K_n\Theta \longrightarrow K_{n+1}\Theta \qquad 0\le i\le n$$

be the standard face and degeneracy maps.

3.1 Definition: Let $X,Y:W\Theta \to \mathit{Top}$ be $W\Theta$-spaces. A homotopy homomorphism (h-morphism) is a $W(\Theta\times L_1)$-space ρ such that $d^0\rho = Y$ and $d^1\rho = X$.

The map $f:X(1)\longrightarrow Y(1)$ given by $\rho(\theta,\mathrm{id})(x) = f(x)$ with

$$\theta = \begin{array}{c|} 0 \\ \bullet \\ 1 \end{array}\ \mathrm{id}$$

is called the underlying map of ρ (here 0 and 1 are edge-labels).

It is clear how to define the composite of an h-morphism and a homomorphism of $W\Theta$-spaces, but it is far from clear, how to define the composite of two h-morphisms. For this we need a special property of $K\Theta$.

3.2 Lemma: Let Θ be factorial. Given a horn of $(n-1)$-simplices $\rho_0,\rho_1,\ldots,\rho_{r-1},\ \rho_{r+1},\ldots,\rho_n$ in $K\Theta$ with the 1st or $(n-1)^{st}$ simplex ρ_r missing. Then there exists an n-simplex $\sigma\in K_n\Theta$ such that $d^i\sigma = \rho_i$ for $0\le i\le n, i\neq r$.

In [1;Thm. 49] this is proved for $r=1$. By treating n and $n-1$ the same way as 0 and 1 in this proof, one obtains the result for $r=n-1$. The proof is exactly the same in our case, if we use the lifting theorem 2.16. We should remark that one can obtain the more general result that $K\Theta$ satisfies the restricted Kan extension condition with some more effort. For our purposes Lemma 3.2 suffices.

3.3 Definition: Given h-morphisms $\alpha: X \longrightarrow Y$ and $\beta: Y \longrightarrow Z$ of $W\Theta$-spaces. Call an h-morphism $\gamma: X \longrightarrow Z$ a composite of α and β if there is a $\sigma \in K_2\Theta$ such that $d^0\sigma = \beta$, $d^1\sigma = \gamma$, $d^2\sigma = \alpha$.

By (3.2) a composite of α and β always exists if Θ is factorial. It need not be unique, but it is unique up to homotopy.

3.4 Definition: Two h-morphisms $\alpha, \beta: X \longrightarrow Y$ are called homotopic if there is a $\sigma \in K_2\Theta$ such that $d^0\sigma = \alpha$, $d^1\sigma = \beta$, $d^2\sigma = s^0X$.

In view of (3.2) the proof of the following results is standard (e.g. see [1; p.104 ff]).

3.5 Lemma: If Θ is factorial the following hold:

(a) The notion of homotopy is an equivalence relation

(b) Composition is unique up to homotopy

(c) The homotopy class of a composite of α and β depends on the homotopy classes of α and β only.

3.6 Proposition: If Θ is factorial the $W\Theta$-spaces and homotopy classes of h-morphisms form a category C_Θ.

When dealing with h-morphisms from $W\Theta$-spaces to Θ-spaces we prefer a variant of the above notion. For this we have to modify the W-construction.

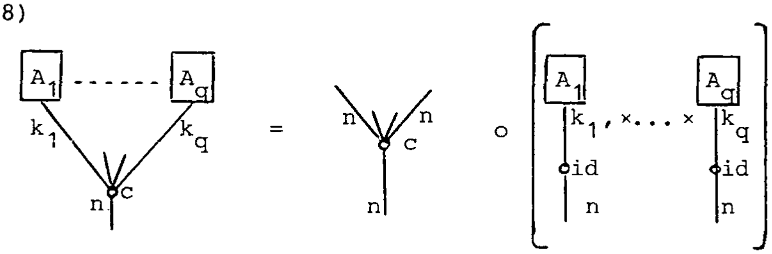
Let $W_r(\Theta \times L_n)$ be the quotient of $W(\Theta \times L_n)$ obtained by adding the following relation to (2.4).

(3.7) Any tree θ is related to the tree obtained from θ by changing the lengths of all connections with label n to 0.

Each tree in $T(\Theta\times L_n)$ is related to a tree having no connection of label n. We call such a tree a reduced tree. Hence, the full subcategory of objects (k,n), k=0,1,..., in $W_r(\Theta\times L_n)$ is canonically isomorphic to Θ, the isomorphism being introduced by $\varepsilon : W_r(\Theta\times L_n) \to \Theta\times L_n$. The spaces $W_r(\Theta\times L_1)$ will be used in section 4 to prove that a $W\Theta$-space is isomorphic in C_Θ to a Θ-space.

Note that any reduced tree with output label n decomposes canonically:

(3.8)



For the reduced version we also have a lifting result. We only state it in the generality we are going to use.

3.9 <u>Proposition</u>: Given a diagram of theory functors over $\Theta_{cr}\times L_n$

$$\begin{tikzcd}
 & V \arrow[dl, hook', "i"'] \arrow[dr, "H'"] \arrow[dd, "K'(t)"] & \\
W_r(\Theta\times L_n) \arrow[dr, "\varepsilon"'] & & \Psi \arrow[dl, "G"] \\
 & \Theta\times L_n &
\end{tikzcd}$$

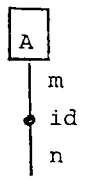
where Θ is factorial, V is generated by some of the faces $d^iW_r(\Theta\times L_n)$ and $K'(t)$ is a homotopy through theory functors from $\varepsilon|\mathcal{V}$ to $G \circ H'$. Let $\mathcal{V}'$ and Ψ' be the full subcategories of $\mathcal{V}$ and Ψ of all objects (k,n), $k\in\mathbb{N}_o$. Assume

(a) G is an equivariant ring equivalence and on Ψ' an isormorphism

(b) $H'|\mathcal{V}'=G^{-1}\circ(\varepsilon|\mathcal{V}')$.

Then there exists a theory functor extension $H:W_r(\Theta\times L_n)\longrightarrow\Psi$ of H' and a homotopy $K(t):W_r(\Theta\times L_n)\longrightarrow\Psi$ of theory functors from ε to $G\circ H$.

The proof is exactly the same as the proof of the lifting theorem with one modification. H is already defined on the subcategory of all objects (k,n), $k\in\mathbb{N}_o$. When considering trees with output label n we only consider reduced trees of the form



The extension to all trees with output label n is given by (3.8).

We apply this result to a special type of h-morphism from a $W\Theta$-space to a Θ-space.

3.10 Definition: Let X and Y be $W\Theta$-spaces and Z a Θ-space.
A reduced h-morphism $Y\longrightarrow Z$ is a $W_r(\Theta\times L_1)$-space α with $d^o\alpha=Z$ and $d^1\alpha=Y$.
Two reduced h-morphisms $\alpha_o,\alpha_1:Y\to Z$ are called homotopic, if there is a $W_r(\Theta\times L_2)$-space σ with $d^o\sigma=\alpha_o$, $d^1\sigma=\alpha_1$, and $d^2\sigma=s^oY$.
A reduced h-morphism $\gamma:X\longrightarrow Z$ is called a composite of the h-morphism $\beta:X\longrightarrow Y$ with the reduced h-morphism $\alpha:Y\longrightarrow Z$ if there is a $W_r(\Theta\times L_2)$-space ρ with $d^o\rho=\alpha$, $d^1\rho=\gamma$, and $d^2\rho=\beta$.

The proof of (3.5) carries over to reduced h-morphisms:

3.11 Proposition: Let Θ be factorial. Then the following hold:

(a) Homotopy between reduced h-morphisms is an equivalence relation

(b) If $\beta:X\to Y$ is an h-morphism of $W\Theta$-spaces and $\alpha:Y\to Z$ a reduced h-morphism from Y to a Θ-space Z, then there is a composite $\alpha\circ\beta$. It is unique up to homotopy of reduced h-morphisms. Its homotopy class depends on the homotopy classes of α and β only.

(c) Composition of two h-morphisms and a reduced h-morphism is associative up to homotopy.

A Θ-space Z is canonically a $W\Theta$-space by pulling back via ε. Hence we have the notion of an h-morphism $Y\to Z$, where Y is a $W\Theta$-space. From the reduced version of (3.2) we obtain

3.12 Proposition: Let Θ be factorial, Y a $W\Theta$-space and Z a Θ-space. Then there is a reduced h-morphism in each homotopy class of h-morphisms $Y\to Z$. It is unique up to homotopy of reduced h-morphisms.

Proof: Let $\alpha:Y\to Z$ be an h-morphism. Obviously id_Z is a reduced h-morphism. By (3.11) there is a reduced h-morphism $\beta:Y\to Z$ which is the composite of id_Z and α. By construction, β is homotopic to α. Since any reduced h-morphism $Y\to Z$ homotopic to α can be interpreted as composite of α with id_Z, β is unique up to homotopy of reduced h-morphisms.

We finally mention a result for later use.

3.13 Proposition: Let Θ be factorial. Then two h-morphisms $\alpha,\beta:X\to Y$ of $W\Theta$-spaces are homotopic iff there is a homotopy $F(t):W(\Theta\times L_1)\to \mathcal{Top}$ of $W(\Theta\times L_1)$-spaces such that $F(0)=\alpha$ and $F(1)=\beta$.

The proof is the same as the one of [1;4.13].

4. Homotopy invariance

Let X and Y be $W\Theta$-spaces. In this section we try to impose h-morphism structures on given maps $f:X\to Y$. Therefore it is useful to add the underlying map to the notation of an h-morphism, i.e. in this section an h-morphism is a pair

$$(f,\alpha):X\longrightarrow Y$$

where α is the $W(\Theta\times L_1)$-space and f its underlying map.

4.1 <u>Proposition</u>: Suppose Θ is factorial and $(f,\alpha):X\longrightarrow Y$ an h-morphism. If $g:X\longrightarrow Y$ is homotopic to f, it can be given the structure of an h-morphism $(g,\beta):X\longrightarrow Y$ such that $(f,\alpha)\simeq(g,\beta)$.

The proof is exactly the same as in [1; Prop. 4.14] using our extension theorem 2.17.
As in [1;p.110], we have the following consequence

4.2 <u>Corollary</u>: Let Θ be factorial and $(f,\alpha):X\longrightarrow Y$, $(g,\beta):Y\longrightarrow Z$ h-morphisms of $W\Theta$-spaces. Then there is a composite $(H,\gamma):X\longrightarrow Z$ of (f,α) and (g,β) such that $h=g\circ f$.

We now follow closely the development of [1;p.110 ff]. Lemma 4.16 of [1] reads in our terminology.

4.3 <u>Lemma</u>: Let S^{op} be the trivial theory and let $p:X\longrightarrow Y$ be a homotopy equivalence of spaces. Then p extends to a $W(S^{op}\times I_\delta)$-space, i.e. there is a $W(S^{op}\times I_\delta)$-space α such that

$$\alpha\left(\Big|\begin{matrix}0\\ \mathrm{id}\\ 1\end{matrix}\right)(x) = p(x)$$

Note that a $W(\Theta\times I_\delta)$-space α gives rise to two h-morphisms in the obvious way, and these two are homotopy inverse to each other in the sense of homotopy of h-morphisms. Hence the following two results provide the main theorems of homotopy invariance.

4.4 Theorem: Let Θ be factorial and $\Psi \subset \Theta$ a subtheory such that $W\Psi$ is an admissible subcategory of $W\Theta$. Let X be a $W\Psi$-space, Y a $W\Theta$-space and $p:X\to Y$ a homotopy equivalence. Suppose p admits a $W(\Psi\times I\mathcal{s})$-space structure $\bar{\rho}$ extending the $W\Psi$-structures X and $Y|W\Psi$. Then there is an extension $\rho:W(\Theta\times I\mathcal{s}) \to \mathcal{T}op$ of $\bar{\rho}$ and Y. In particular, the $W\Psi$-structure on X can be extended to a $W\Theta$-structure, and ρ defines an isomorphism between X and Y in $\mathcal{C}_\Theta$.

The subcategory of $W(\Theta\times I\mathcal{s})$ generated by $W(\Psi\times I\mathcal{s})$ and $d^{o}(W\Theta)$ is admissible. Hence the proof of this result is exactly the same as the one of [1;4.18] if we use our lifting theorem.

The next result is far more difficult to prove. It most certainly holds under considerably weaker assumptions on the subtheory Ψ than those we are going to introduce now. But our result suffices for the applications we have in mind.

4.5 Definition: A subtheory Ψ of a factorial theory Θ is called factorially admissible if

(i) for each n, $(\Theta^{a}(n,1),\ \Psi(n,1)\cap\Theta^{a}(n,1))$ is a Σ_n-NDR unless $\Psi(n,1)\cap\Theta^{a}(n,1) = \emptyset$, and the same with Θ^{a} substituted by Θ^{m}

(ii) If $a \circ (m_1\times\ldots\times m_k)\circ\sigma^*$ is a standard factorization in Θ of an operation in Ψ, then $a, m_1, \ldots, m_k$ are in Ψ.

4.6 Theorem: Let Θ be factorial and Ψ be a factorially admissible subtheory of Θ. Let $(p,\alpha):X\to Y$ be an h-morphism of $W\Theta$-spaces with p a homotopy equivalence, such that the restriction of α to $W(\Psi\times L_1)$ has an extension $\beta':W(\Psi\times I\mathcal{s}) \longrightarrow \mathcal{T}op$. Then there exists an extension $\beta:W(\Theta\times I\mathcal{s})\to\mathcal{T}op$ of β' and α.

Before we say anything about the proof, let us draw a number of conclusions.

4.7 Proposition: Let Θ be factorial and $\Phi \subset \Theta$ be factorially admissible. Let X be a $W\Phi$-space and Y a $W\Theta$-space such that X and $Y|W\Phi$ are homotopy equivalent as $W\Phi$-spaces (i.e. they are isomorphic in C_Φ). Then the $W\Phi$-structure on X can be extended to a $W\Theta$-structure and the homotopy equivalence $X \to Y|W\Phi$ to a homotopy equivalence of $W\Theta$-spaces.

Proof: By assumption, there is an h-morphism $(p,\alpha): X \to Y|W\Phi$ of $W\Phi$-spaces with p a homotopy equivalence. By (4.3), p extends to a $W(S^{op} \times I\!\!\mathit{\Delta})$-space, compatibly with α. Now apply (4.6) with $\Theta = \Phi$ and $\Psi = S^{op}$ to extend α to a $W(\Phi \times I\!\!\mathit{\Delta})$-space $\bar{\alpha}$. Now use (4.4) to extend $\bar{\alpha}$ and Y to a $W(\Theta \times I\!\!\mathit{\Delta})$-space which provides the required homotopy equivalence of $W\Theta$-spaces.

4.8 Proposition: Let Θ be factorial and $\Phi \subset \Theta$ factorially admissible. Let $(p,\alpha): X \to Y$ be an h-morphism of $W\Theta$-spaces and p a homotopy equivalence. Let $(q,\beta): Y|W\Phi \to X|W\Phi$ be an h-morphism of $W\Phi$-spaces which is homotopy inverse to $(p,\alpha)|W(\Phi \times L_1)$ in the category of $W\Phi$-spaces. Then (q,β) can be extended to an h-morphism of $W\Theta$-spaces $Y \to X$ homotopy inverse to (p,α).

Proof: By (4.3), p extends to a $W(S^{op} \times I\!\!\mathit{\Delta})$-space compatible with α. This action extends by 4.6 to a $W(\Theta \times I\!\!\mathit{\Delta})$-space ρ compatible with α. In particular, ρ provides a homotopy inverse $(q',\beta'): Y \to X$ of (p,α). Its restriction to $W(\Phi \times I\!\!\mathit{\Delta})$ has to be homotopic to (q,β). By (3.13), there is a homotopy (q_t,β_t) through h-morphisms of $W\Phi$-spaces from the restriction of (q',β') to (q,β). We extend this homotopy and the constant homotopy on X and Y using (2.17) to obtain the required homotopy inverse.

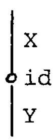
Proof of (4.6): We cannot take the proof of [1;Thm.4.19] because the filtration of the morphism spaces given there is messed up by the distributive law.

Let $\Lambda \subset W(\Theta \times I\delta)$ be the subtheory generated by $W(\Theta \times L_1)$ and $W(\Psi \times I\delta)$, where L_1 is included into $I\delta$ as $0 \to 1$. The functors α and β' define a Λ-space. Since Λ is an admissible subcategory, it suffices to show that $\varepsilon|\Lambda : \Lambda \to \Theta \times I\delta$ is an equivariant ring equivalence, so that the lifting theorem can be applied.

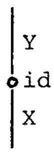
To distinguish lengths and labels we call an edge labelled 0 an X-edge and an edge labelled 1 a Y-edge. As in [1] it suffices to show that ε is an equivariant ring equivalence for spaces

$$\Lambda((n,1),\ (1,1)) \longrightarrow (\Theta \times I\delta)((n,1),(1,1))$$

To prove this we use a variant of the theory $T'(\Theta \times I\delta)$ and its filtration of section 2. Call the boxes

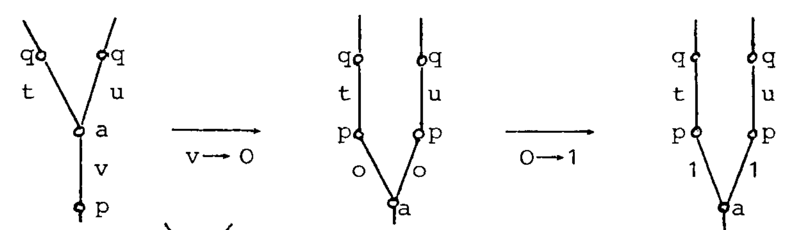


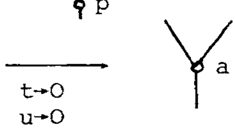
and



a p-box respectively a q-box. The idea of the proof is the following: The subspace of $\Lambda((n,1),(1,1))$ of all operations represented by trees with no p-box is homeomorphic to $W\Theta(n,1)$, so that the restriction of ε to this subspace is an equivariant homotopy equivalence by (2.5). If a representing tree has a p-box it also has a q-box. We push up this p-box in $\Lambda((n,1),(1,1))$ to be next to the q-box so that they cancel out. Thus we deform $\Lambda((n,1),(1,1))$ by induction on the number of p- and q-boxes into the subspace $W\Theta(n,1)$.

4.9 Example:

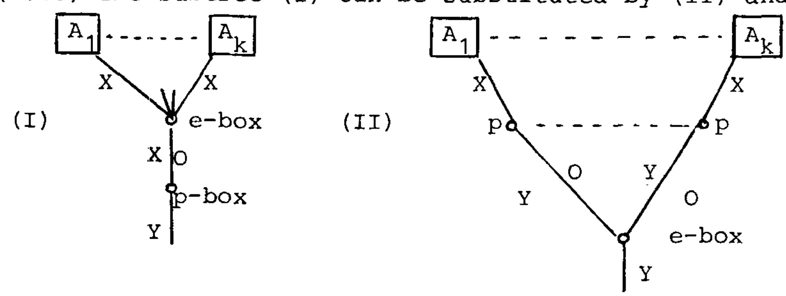




(since poq=id)

It is a little complicated to make this simple idea compatible with all relations (2.11). For this we need a tricky filtration. We introduce a subtheory $T''=T''(\Theta\times I\Delta)$ of $T'(\Theta\times I\Delta)$. Its trees have only boxes of the following types: p-boxes, q-boxes, further a-boxes and m-boxes whose inputs have the same label as their output. Call these a- and m-boxes also e-boxes. On T'' we impose the relations (2.11) with the exception that a connection of length 0 between an e-box and a p- or q-box cannot be shrunk. Instead, we have the notion of pushing up a p- or q-box "through" an e-box similar to relation (2.11d), (compare also Example 4.9):

(4.10) The subtree (I) can be substituted by (II) and vice versa



Similarly for q-boxes connected with e-boxes.

Clearly, in view of relation (2.4a), $W(\Theta\times I\Delta)$ is the quotient of T'' modulo the relations (2.11) with the just mentioned modification (4.10) of relation (2.11a) for connections between e-boxes and p- or q-boxes.
The set of e-boxes divides into two classes: Those with labels in $\Theta-\Psi$, and those with labels in Ψ. The trees in T'' representing elements of Λ are those that satisfy the separation condition: In any directed edge-path between a q-box and a box in $\Theta-\Psi$ there is a connection of length 1.

We now filter T'', thus inducing a filtration of $W(\Theta\times I\Delta)$: Given a tree θ of shape λ in T'', we have the sequence $(p_o,p_1,\dots.)$, where p_k is the number of m-boxes of height k. Then there is a sequence $(q_o,q_1,\dots)$, where q_i is the number of p- and q-boxes having exactly i e-boxes between them and the tree output. Finally we associate with θ (or better λ) a new shape $k(\theta)$ as follows: First push up all p- and q-boxes through all e-boxes as described in (4.10), then push up all m-boxes through all a-boxes different to a nullary operation as described in (2.12). The resulting shape is denoted by $k(\theta)$. Let l be the number of connections of $k(\theta)$. We order

$$t(\theta) = (l;p_0,p_1,\ldots;q_0,q_1,\ldots)$$

lexigraphically. Since $(p_0+p_1+\ldots) + (q_0+q_1+\ldots) \le l$, this ordering is countable. For $\nu = (l;p_0,p_1,\ldots,q_0,q_1,\ldots)$ we have an induced filtration F_ν of the spaces $\Lambda((n,1),(1,1))$, consisting of all elements which can be represented by trees θ with $t(\theta)\le\nu$. Let $DF_\nu \subset F_\nu$ be the subspace of all elements of lower filtration than ν.

(4.11) A tree θ with $t(\theta)=\nu$ represents an element of DF_ν iff one of the following conditions holds:

(a) one of the conditions (2.11a,b,c) holds

(b) some connection from an e-box down to a q- or p-box has length 0 (because then (4.10) applies)

(c) some connection between a p-box and a q-box has length 0 (because then they cancel out)

The proof now proceeds as in [1;p.118 ff]

Finally we mention that each $W\Theta$-space is homotopy equivalent to a Θ-space. The proof is exactly the same as the one of [1;Thm.4.49]. Here we use the models $W_r(\Theta\times L_1)$ introduced in section 3. More precisely, the proof of [1; Thm 4.49] applied to our situation leads to the following result:

4.12 <u>Theorem</u>: Let Θ be any theory over Θ_{cr} (or Θ_r). Then for any $W\Theta$-space X there exists a Θ-space MX and a reduced h-morphism $(i,\alpha):X \longrightarrow MX$ such that

(a) $i: X \to MX$ embeds X as a strong deformation retract

(b) any reduced h-morphism $(f,\beta): X \to Y$ is the canonical composite of (i,α) and a unique homomorphism $h_\beta: MX \to Y$ of Θ-spaces

(c) If Θ is factorial and (g,γ) is homotopic to (f,β) in the sense of (3.10) then there is a homotopy through homomorphism from h_β to h_γ.

From this it is easy to deduce

4.13 Proposition: If Θ is factorial, the construction M defines a functor from the category $\mathcal{C}_\Theta$ to the category $\mathcal{H}_\Theta$ of Θ-spaces and homotopy classes of homomorphisms. It is left adjoint to the canonical inclusion $\mathcal{H}_\Theta \subset \mathcal{C}_\Theta$.

5. A_∞ and E_∞ ring spaces

In this section we show that the results of section 3 and section 4 apply to A_∞ and E_∞ ring spaces, and we investigate the relationship of our notion to the one of May ([2],[4]).

Let (C,G) be an A_∞ or E_∞ operad pair in the sense of [2;chapt.VI]. Then we have a theory $\Theta(C,G)$ or (C,G)-spaces. Its morphisms from m to n are all natural transformations from Y^m to Y^n, where Y is a (C,G)-space (for a definition see [2; p.145]). As mentioned before, each such theory is factorial: One takes $\Theta(C,G)^a(n,1)=C(n)$ and $\Theta(C,G)^m(n,1) = G(n)$.

Now let (H_∞,L) be Steiner's canonical operad pair [5]. In the A_∞ theory we take L without permutations, in the E_∞ theory with permutations. By Steiner's analysis of $H_\infty(p)$, it is a contractible numerable principal Σ_p-space. It is well-known that the same holds for $L(p)$. Hence we obtain from the lifting theorem part II

5.1 <u>Theorem</u>: Any A_∞ or E_∞ ring space is a $W\Theta(H_\infty,L)$-space, and the homotopy-invariance results can be applied.

Here, of course, we have to distinguish between L with or without permutations.

5.2 <u>Corollary</u>: Any A_∞ or E_∞ ring space X is homotopy equivalent to an (H_∞,L)-space Y in the sense of [2;p.145]

<u>Proof</u>: By (5.1), X can be given a $W\Theta(H_\infty,L)$-structure. Hence Y=MX has a $\Theta(H_\infty,L)$-structure. By definition of $\Theta(H_\infty,L)$ the space Y is nothing but a (H_∞,L)-space.
From this and from [6; Lemma 1.7] we obtain

5.3 <u>Proposition</u>: Any A_∞ or E_∞ space in the sense of [2] is an A_∞ or E_∞ space in the sense of (1.7). Conversely, any A_∞ or E_∞ space in the sense of (1.7) is homotopy equivalent to an A_∞ or E_∞ space in the sense of [2] .

Our homotopy invariance results have a number of consequences. We prove just one, which is of some importance for the algebraic K-theory of A_∞ ring spaces.

5.4 <u>Proposition</u>: For any $W\Theta(\mathcal{H}_\infty, \mathcal{L})$-space X there exists a $W\Theta(\mathcal{H}_\infty, \mathcal{L})$-space Y and a homotopy equivalence $(f,\alpha)\colon X \to Y$ of $W\Theta(\mathcal{H}_\infty, \mathcal{L})$-spaces such that the multiplication on Y is a strict monoid structure. ($\mathcal{L}$ without permutations)

<u>Proof</u>: Let Θ_m be the theory of monoids (written multiplicatively), and $\mathcal{A} \subset \Theta_m$ be the subcategory generated under composition and $\times$ by all μ_n. The unique maps $\mathcal{L}(n) \to \mathcal{A}(n,1) = \{\mu_n\}$ induce a theory functor $H\colon \Theta(\mathcal{L}) \to \Theta_m$, which is a homotopy equivalence. Applying the lifting theorem [1;3.17] we obtain a homotopy commutative diagram

$$\begin{tikzcd} W\Theta_m \arrow[r, "F"] \arrow[d, "\varepsilon_1"'] & W\Theta(\mathcal{L}) \arrow[d, "\varepsilon_2"] \\ \Theta_m & \Theta(\mathcal{L}) \arrow[l, "H"'] \end{tikzcd}$$

By the uniqueness part of the same theorem, we have

$$F \circ WH \simeq \mathrm{Id}$$

through functors, because both functors make the diagram

$$\begin{tikzcd} W\Theta(\mathcal{L}) \arrow[r] \arrow[d, "\varepsilon_2"'] & W\Theta(\mathcal{L}) \arrow[d, "H \circ \varepsilon_2"] \\ \Theta(\mathcal{L}) \arrow[r, "H"] & \Theta_m \end{tikzcd}$$

commute up to homotopy.

By restriction, X is a $W\Theta(L)$-space, and hence $X\circ F$ a $W\Theta_m$-space. By [1; Thm. 4.49] there is a homotopy equivalence $(f,\beta):X\circ F\to Y$ of $W\Theta_m$-spaces from $X\circ F$ to a strict monoid Y(i.e. Y is a Θ_m-space and β an h-morphism from $X\circ F$ to $Y\circ\varepsilon_1$). Pulling back via WH, we obtain a homotopy equivalence of $W\Theta(L)$-spaces

$$(f,\beta^*):\ X\circ F\circ WH \longrightarrow Y\circ\varepsilon_1\circ\ WH$$

By [1;6.23], there is an h-morphism of $W\Theta(L)$-spaces

$$(id,\gamma):X \longrightarrow X\circ F\circ WH$$

since $Id \simeq F\circ WH$. Composing these two, we have a homotopy equivalence of $W\Theta(L)$-spaces

$$(f,\delta):X \longrightarrow Y\circ\varepsilon_1\circ WH$$

Since $\Theta(L) \subset \Theta(H_\infty,L)$ is factorially admissible, the results (4.4) and (4.6) can be applied to give an extension of (f,δ) to a homotopy equivalence of $W\Theta(H_\infty,L)$-spaces. In particular, Y admits the structure of a $W\Theta(H_\infty,L)$-space with a monoid multiplication.

5.5 <u>Corollary</u>: Any A_∞ ring space is homotopy equivalent to an A_∞ ring space with a monoid multiplication.

5.6 <u>Remark</u>: Using the methods of sections 3 and 4 we could improve (5.4) to a functorial version. Let C be the category of $W\Theta(H_\infty,L)$- spaces and homotopy classes of h-morphisms. Let D be the category of $W\Theta(H_\infty,L)$-spaces Y with monoid multiplication, i.e. the restriction of Y to $W\Theta(L)$ factors through Θ_m. As morphisms of D we take homotopy classes of h-morphisms of $W\Theta(H_\infty,L)$ -spaces, which are homomorphisms when restricted to $W\Theta(L)$, and hence homomorphisms of the multiplication monoids. There is an obvious functor $J:D\to C$. Our construction in the proof of (5.4) induces a functor $M:C\to D$ and a natural isomorphism $Id \longrightarrow J\circ M$.

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PREREQUISITES (ON EQUIVARIANT STABLE HOMOTOPY) FOR CARLSSONS'S LECTURE.

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§1. Introduction. Three things might be done to help those who wish to understand Carlsson's work on Seqal's Burnside Ring Conjecture [8]. First, one might attempt a general exposition about Seqal's Burnside Ring Conjecture, both in its non-equivariant and in its equivariant forms. Secondly, one might explain the results which Gunawardena, Miller and myself have obtained by calculation for the case $G = (Z_p)^n$. (This is relevant because Carlsson uses these results.) Thirdly, one might attempt a general introduction to equivariant stable homotopy.

In the lecture I gave in Aarhus, I tried to say something on all three topics, but for lack of time I was forced to omit an important part of what I had prepared. In this published text I shall omit the first and second topics, and try to do better justice to the third.

In fact, when I first saw [8] - apart from rejoicing - I thought, "Oh dear; now I shall have to work to understand the fundamentals of this subject". Since I more or less do understand them now (at least, so far as they seem to be needed for Carlsson's work) it may save other topologists trouble if I try to pass on my understanding.

I should stress that I do not claim any originality for what follows; everything I shall explain is or should be known to those who reckon to know about such things. (There is a possible exception in Theorems 5.4 and 8.5, which are recent, and where my statements differ slightly from those in my incoming mail - which I shall acknowledge in due course.)

My survey is arranged as follows. In §2 I shall review unstable equivariant homotopy theory. In §3 I discuss the G-suspension theorem. In §4 I discuss the G-Spanier-Whitehead category. In §5 I discuss certain theorems for reducing problems involving a group G to problems involving a smaller group. In §6 I discuss theories graded over the representation ring $RO(G)$. In §7 I say very little about G-spectra. In §8 I discuss G-Spanier-Whitehead duality.

I shall use the words "ordinary" and "classical" to refer to the non-equivariant case, $G = 1$.

I am very grateful to many correspondents, including G. Carlsson, T. tom Dieck, C. Kosniowski, L.G. Lewis, J.P. May, A. Ranicki and G. Segal. I am particularly grateful to L.G. Lewis for tutorials on G-spectra and to J.P. May for many letters.

§2. Unstable equivariant homotopy. This section must recall how the most elementary part of unstable homotopy theory carries over to the equivariant case.

Let G be a finite group. (Ideally it is desirable to arrange the foundations of equivariant topology so as to cater for compact Lie groups, but for present purposes I will not bother.) A "G-space" is a space on which G acts; for definiteness we agree that groups normally act on the left of spaces. Let X and Y be G-spaces; then a map

$f: X \longrightarrow Y$ is a G-map if

$$f(gx) = g(fx)$$

for all $g \in G$, $x \in X$.

Two G-maps are "G-homotopic" if they are homotopic through G-maps. Alternatively, we can define G-homotopy in terms of G-maps of cylinders; for this purpose, if X is a G-space, we make G act on $I \times X$ by

$$g(t,x) = (t,gx)$$

for all $g \in G$, $t \in I$, $x \in X$.

With these definitions one can carry over a good deal of ordinary homotopy theory. Ordinary homotopy-theory often needs a base-point; at the corresponding places in G-homotopy theory, we suppose given a base-point fixed under G. We then define $[X,Y]^G$ in terms of G-maps and G-homotopies which preserve the base-point.

There are a few simple cases in which problems over G can be reduced to problems over a subgroup H of G. Naturally, we leech onto them to use them in inductive proofs. The technical statement is that if H is a subgroup of G, then the "forgetful functor" from G-spaces to H-spaces has a left adjoint. More precisely, if $i: H \longrightarrow G$ is the inclusion and Y is a G-space, we write i^*Y for the H-space in which H acts on the same space Y via i. Then we have the following natural (1-1) correspondence.

(2.1) $$\text{H-Map}\,(X, i^*Y) \longleftrightarrow \text{G-Map}\,(G \times_H X, Y).$$

Here X is supposed to be an H-space, and $G \times_H X$ is the quotient of $G \times X$ in which (g,hx) is identified with (gh,x). In (2.1) we have no base-points; if we wish to have base-points, then the natural (1-1) correspondence is as follows.

(2.2) $\quad$ $\text{Ptd-H-map}(X, i^*Y) \longleftrightarrow \text{Ptd-G-Map}((G \sqcup P)\wedge_H X, Y)$.

Here $G \sqcup P$ is the disjoint union of G and a base-point P fixed under G, and $\wedge_H$ is defined as $\times_H$ was before. If we can take X in the form i^*Z where Z is a G-space, then we have the following natural G-homeomorphism.

(2.3) $$(G \sqcup P)\wedge_H i^*Z \longleftrightarrow (G/H \sqcup P)\wedge Z .$$

It is given by

$$(g,z) \longmapsto (g,gz)$$
$$(g,g^{-1}z) \longleftarrow\!\shortmid (g,z).$$

The distinctive features of the equivariant theory begin with the study of fixed-point sets. Let X be a G-space, and let H be a subgroup of G; then the fixed-point set X^H is defined by

$$X^H = \{x \in X \mid hx = x, \forall h \in H\} .$$

The action of $g \in G$ gives a homeomorphism from X^H to $X^{gHg^{-1}}$; in particular, X^H admits operations from $N(H)/H$, where $N(H)$ is the normaliser of H in G. Any G-map $f: X \longrightarrow Y$ must carry X^H into Y^H and preserve the action of $N(H)/H$; we usually write $f^H: X^H \longrightarrow Y^H$ for the map induced by f on the fixed-point set. If $H \subset K$, then $X^H \supset X^K$.

Many proofs in equivariant homotopy theory are done by induction up the fixed-point sets, beginning with the smallest, X^G, and finishing with the largest, $X^1 = X$. A convenient class of G-spaces in which to do such proofs is the class of G-CW-complexes. We will come to these soon, but first we must mention cells and spheres.

By a "representation of G", we shall mean a finite-dimensional real inner-product space V on which G acts (linearly, and preserving the inner product). Such a representation V has a unit sphere

$S(V)$ and a unit cell $E(V)$ defined by $\|v\|=1$ and $\|v\|\le 1$ respectively. The usual homeomorphism between $E(V) \times E(W)$ and $E(V\oplus W)$ is equivariant and gives no more nuisance than usual. If we want to use base-points, we normally define S^V to be the one-point compactification of V and put the base-point at infinity. In representation-theory we often write "n" to indicate the representation in which G acts trivially on R^n ; so the new meaning of S^n is S^{R^n} , that is, the old S^n with G acting trivially on it.

The first theorem in ordinary homotopy theory is the theorem that $\pi_r(S^n) = 0$ for $r < n$.

<u>Proposition 2.4</u>. If $\dim V^H < \dim W^H$ for all H , then
$$[S^V, S^W]^G = 0 .$$

The proof will become clear as soon as I have introduced the relevant ideas.

In general, suppose that in the classical case we have some invariant like "dim", which assigns to each suitable space X a value dim (X) which may be an integer or ∞ . Then the analogue in the equivariant case is to consider "dim X" as a function which assigns to each subgroup $H \subset G$ the value $\dim (X^H)$ (taking equal values on conjugate subgroups H). So the assumption of (2.4) should be thought of as

$$\text{"}\dim (S^V) \le \dim (S^W) - 1\text{"}$$

(with the obvious interpretation of inequality between functions of H). Similarly for the "Hurewicz dimension of X , Hur X" (defined to be the greatest n such that $\pi_r(X) = 0$ for $r < n$). For spheres we have

$$\mathrm{Hur}\ (S^{V^H}) = \dim\ (S^{V^H}) .$$

The following result generalises (2.4).

<u>Proposition 2.5</u>. If $\dim (X^H) \le \mathrm{Hur}\,(Y^H) - 1$ for all H, then

$$[X,Y]^G = 0 .$$

Here it will be prudent to assume that X is a generalised CW-complex of some sort.

On the usual definitions, G-CW-complexes are constructed just like CW-complexes; but instead of using cells of the form E^n, S^{n-1} one uses G-cells of the form

$$(G/H) \times E^n, \ (G/H) \times S^{n-1} .$$

The usual reference for G-CW-complexes is the work of Matumoto [20]. The G-complexes of Bredon [3,5] served the same purpose earlier (for G finite). There is also a thesis by Illman [16], although theses are not usually easily available.

I thank J.P. May for pointing out two possible objections to the usual approach to G-CW-complexes. The first is that the definition of "G-cell" is not wide enough to accommodate the "cells" introduced above. Certainly it would seem worthwhile to make our machinery accept "G-cells" of the form

$$G\times_H E(V), \ G\times_H S(V)$$

where V is a representation of H. This doesn't affect the class of G-spaces considered, because any G-cell of the more general form can be subdivided into G-cells of the special form.

The second objection may be seen from the following example. If H is a subgroup of G, we would like to say that a G-CW-complex "is" an H-CW-complex. Unfortunately, we can't display the G-cell G as a union of H-cells H without choosing coset representatives. This nuisance recurs with products; if X and Y are G-CW-complexes, then $X \times Y$ (with the CW-topology) is likely to come as a complex over the group $G \times G$, and we want it as a complex over the diagonal subgroup

$G \xrightarrow{\Delta} G \times G$.

For present purposes we seem to have a workable way out (though it only serves when G is discrete). We stipulate that the given structure of CW-complex X includes characteristic maps

$$\chi_\alpha : E^{n(\alpha)} \longrightarrow X \ .$$

If X comes as a G-space, we ask for a commutative diagram of the following form for each $g \in G$ and α .

$$\begin{tikzcd} E^{n(\alpha)} \arrow[r, "\chi_\alpha"] \arrow[d, "\ell(\alpha,g)"'] & X \arrow[d, "g"] \\ E^{n(\beta)} \arrow[r, "\chi_\beta"] & X \end{tikzcd}$$

Clearly β will be unique (so that G will implicitly act on the set of indices α); $\ell(\alpha,g)$ will also be unique, and we ask that it be linear and preserve the inner product. (The last clause is actually redundant.) Then we can choose to organise our characteristic maps into G-orbits

$$G \times_H E(V) \longrightarrow X \ ,$$

but no such choice is part of the given structure. If we want to insist on G-cells of the form $G/H \times E^n$, we can impose an axiom that $\beta = \alpha$ implies $\ell(\alpha,g) = 1$. If so, we get back to Bredon's G-complexes.

In order to carry over the standard arguments about CW-complexes, one needs to be able to manipulate G-maps of G-cells. Let Y be a G-space; from (2.1) we get the following natural (1-1) correspondence.

(2.6) $$\text{G-Map } ((G/H)\times E^n, Y) \longleftrightarrow \text{Map } (E^n, Y^H) .$$

If on the left we wish to prescribe the values of the G-map on $(G/H)\times S^{n-1}$, that corresponds on the right to prescribing the values of the map on S^{n-1} . So the standard arguments for CW-complexes carry over to G-CW-complexes, using induction over the G-cells plus ordinary homotopy theory in fixed-point subspaces Y^H.

Since our object is to reduce to ordinary homotopy theory, we generally carry out these arguments with G-cells $(G/H)\times E^n$ rather than $G\times_H E(V)$, reducing to that case by subdivision if necessary.

We will forgo a long discussion of those results on CW-complexes which carry over with little change. (For example, the inclusion of a G-subcomplex in a G-CW-complex has the G-homotopy-extension property.)

The first result we do need to mention is the "theorem of J.H.C. Whitehead". Recall that in the ordinary case, a map $f: X \longrightarrow Y$ between path-connected spaces is called an n-equivalence if

$$f_*: \pi_r(X) \longrightarrow \pi_r(Y)$$

is iso for $r < n$ and epi for $r = n$. (If the spaces are not path-connected we modify this definition in an obvious way; see [24 p404].) In the equivariant case, suppose given a function n which assigns to each subgroup $H \subset G$ a value $n(H)$ which may be an integer or ∞, subject to the condition $n(gHg^{-1}) = n(H)$. Then a G-map $f: X \longrightarrow Y$ is an n-equivalence if $f^H: X^H \longrightarrow Y^H$ is an ordinary $n(H)$-equivalence for each H.

Proposition 2.7. Let W be a G-CW-complex and let $f: X \longrightarrow Y$ be a G-map which is an n-equivalence. Then the induced map

$$f_*: [W,X]^G \longrightarrow [W,Y]^G$$

is onto if

$$\dim W^H \le n(H) \quad \text{for all } H ;$$

it is a (1-1) correspondence if

$$\dim W^H \le n(H) - 1 \quad \text{for all } H .$$

As with (2.4), one should think of the assumptions as "dim $W \le n$" and "dim $W \le n-1$", where dim W and n are functions of H.

Results of this sort go back to Bredon [5 Chap. II §5]; see also Matumoto [20 §5], Illman [16 Chapter I §3] and Namboodiri [29 Corollary 2.2].

§3. The G-suspension theorem. This section must recall how the most elementary result of suspension theory carries over to the equivariant case.

When we suspend in equivariant homotopy theory, we have available a variety of actions of G on the suspension coordinates we introduce. For an unreduced suspension, of G-spaces without base-point, it is reasonable to take the join $S(V)*X$. For a reduced suspension, of G-spaces with base-point, it is natural to take the smash product $S^V \wedge X$. Here the action of G on a smash product $X \wedge Y$ is defined by

$$g(x,y) = (gx,gy) ,$$

and similarly for the join.

The relationship between smash and join is much as in the classical case. In fact, classical comparison maps, such as the ordinary quotient map

$$X*Y \longrightarrow X\wedge S^1\wedge Y ,$$

are commonly natural, and therefore equivariant. They can be proved to be G-equivalences by (2.7), provided the following conditions are satisfied.

(a) The restriction of the comparison map to a fixed-point-set is another instance of the same comparison map, for example,

$$X^H * Y^H \longrightarrow X^H \wedge S^1 \wedge Y^H .$$

(b) The comparison map is classically a weak equivalence.

(c) The G-spaces involved are G-CW-complexes.

For this section we will use $S^V \wedge X$. By taking the smash product with the identity map of S^V , we get a function

$$S^V\colon [X,Y]^G \longrightarrow [S^V \wedge X,\ S^V \wedge Y]^G .$$

We wish to show that S^V is a (1-1) correspondence under suitable conditions. We can follow the classical approach by introducing a function-space. Let $\Omega^V(Z)$ be the space of pointed maps $\omega\colon S^V \longrightarrow Z$; we make G act on this function space by

$$(g\omega)(s) = g(\omega(g^{-1}s)) .$$

As in the classical case, it is sufficient to study the canonical map

$$Y \longrightarrow \Omega^V(S^V \wedge Y)$$

and prove that it is an n-equivalence for some suitable n . We choose our function $n = n(H)$ so that it has the following properties.

(3.1) For each subgroup $H \subset G$ such that $V^H > 0$ we have

$$n(H) \le 2\ \mathrm{Hur}(Y^H) - 1 .$$

(3.2) For each pair of subgroups $K \subset H \subset G$ such that $V^K > V^H$ we have

$$n(H) \le \mathrm{Hur}(Y^K) - 1 .$$

<u>Theorem 3.3</u>. Under these conditions, the map

$$Y \longrightarrow \Omega^V(S^V \wedge Y)$$

is an n-equivalence. It follows that the function

$$S^V\colon [X,Y]^G \longrightarrow [S^V \wedge X,\ S^V \wedge Y]^G$$

is onto if X is a G-CW-complex and $\dim (X^H) \leq n(H)$ for each H ; it is a (1-1) correspondence if X is a G-CW-complex and $\dim (X^H) \leq n(H) - 1$ for each H .

The second sentence follows from the first by using (2.7), as in the ordinary case.

For $G = Z_2$ the result is due to Bredon [4]. If I may count [4,6] as one paper, then this is the first paper in equivariant stable homotopy theory, and I think it may deserve more credit than it has received. To promote understanding of this subject, I recommend study of the special case $G = Z_2$.

It would have been good if the G-suspension theorem could have come next after [4,6]. The proof does contain ingredients which are additional to those well-known in the ordinary case, including, of course, the use of condition (3.2). The standard reference is Hauschild [14]; see also Namboodiri [29, Theorem 2.3].

One use of the suspension theorem is to show that certain limits are attained. For this purpose we must decide what class of representations to use when we suspend. For the moment we keep our options open; we suppose given some class of "allowable representations" of G , so that our class is closed under passage to sums and summands, and also under passage from any representation to an isomorphic one. We order the allowable representations, writing $W \geq V$ if $W \cong U \oplus V$ for some U . Let X be a finite-dimensional G-CW-complex, and Y a G-space.

<u>Theorem 3.4</u>. There exists an allowable $W_0 = W_0(X)$ such that for any allowable $W \geq W_0$ and any allowable V the map

$$S^V\colon [S^W \wedge X,\ S^W \wedge Y]^G \longrightarrow [S^V \wedge S^W \wedge X,\ S^V \wedge S^W \wedge Y]^G$$

is a (1-1) correspondence. Indeed, the map

$$S^V\colon [X',\ S^W \wedge Y]^G \longrightarrow [S^V \wedge X',\ S^V \wedge S^W \wedge Y]^G$$

is a (1-1) correspondence for any subcomplex X' of $S^W \wedge X$ or of any subdivision of $S^W \wedge X$.

The final sentence about X' is intended to help with [8 p45].

The result will follow from Theorem 3.3, provided we can satisfy the following inequalities on the dimensions.

(i) If for some H there is an allowable V with $V^H > 0$, then

$$\dim W^H + \dim X^H \leq 2 \dim W^H - 2 .$$

It is clear that if there is any allowable V with $V^H > 0$, then by putting sufficiently many copies of it into W we can increase $\dim W^H$ till this inequality is satisfied; it then holds for all larger W .

(ii) If for some $K \subset H$ there is any allowable V with $V^K > V^H$, then

$$\dim W^H + \dim X^H \leq \dim W^K - 2 .$$

It is clear that if there is any allowable V with $V^K > V^H$, then by putting sufficiently many copies of it into W we can increase $\dim W^K - \dim W^H$ till this inequality is satisfied; it then holds for all larger W .

Of course, we have to satisfy inequalities of type (i) for a finite number of subgroups H , and inequalities of type (ii) for a finite number of pairs $K \subset H$, but we can satisfy all these conditions if W is sufficiently large. This proves Theorem 3.4.

§4. The G-analogue of the Spanier-Whitehead category. This section must review how the original approach to stable homotopy theory carries over to the equivariant case.

We wish to pass to a limit and consider stable classes of maps. To take a "limit" of groups we must supply a system of groups and homomorphisms, or equivalently, a functor defined on some suitable category.

We take the objects of our category C to be all "allowable" representations V of G (see §3); we take the morphisms $f: V \longrightarrow W$ in C to be the R-linear G-maps which preserve inner products. (Such maps f are necessarily mono.)

Suppose given two G-spaces X, Y with base-points. To each object V of C we associate the set

$$[S^V \wedge X,\ S^V \wedge Y]^G.$$

For any morphism $i: V \longrightarrow W$ in C, we first use i to identify W with $U \oplus V$, where U is the orthogonal complement of the image $i(V)$ under the inner product. We now associate to i the following composite function.

$$\begin{array}{ccc} [S^V \wedge X,\ S^V \wedge Y]^G & \xrightarrow{S^U} & [S^U \wedge S^V \wedge X,\ S^U \wedge S^V \wedge Y]^G \\ & & \updownarrow \\ & & [S^W \wedge X,\ S^W \wedge Y]^G \end{array}$$

(Notice that we can identify $S^U \wedge S^V$ with $S^{U \oplus V}$ and so with S^W.) We get a functor from C to sets.

Next we must check that this functor is such that we can take its limit. First we need to see that if U and V are objects in C, then there is an object in C which receives morphisms from both. This is immediate: it is enough to take $U \oplus V$. Secondly we need to see that if $f,g: U \longrightarrow V$ are two morphisms in C , then there is a further morphism $h:V \longrightarrow W$ such that our functor assigns equal values to hf,hg . It is easy to reduce to the case in which f is an automorphism of V and $g = 1$. One can see by counter-examples that it is not sufficient to take $h = 1$; that is, the composite

$$S^V \wedge X \xrightarrow{f^{-1}\wedge 1} S^V \wedge X \xrightarrow{\phi} S^V \wedge Y \xrightarrow{f\wedge 1} S^V \wedge Y$$

need not be G-homotopic to ϕ . However, we take h to be the injection of V as the second factor in $V\oplus V$. Clearly we have $hf = (1\oplus f)h$. But $1\oplus f$ is homotopic through G-isomorphisms to $f\oplus 1$, for example by

$$\begin{bmatrix} \mathrm{Cos}t & -\mathrm{Sin}t \\ \mathrm{Sin}t & \mathrm{Cos}t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & f \end{bmatrix} \begin{bmatrix} \mathrm{Cos}t & \mathrm{Sin}t \\ -\mathrm{Sin}t & \mathrm{Cos}t \end{bmatrix} \qquad (0 \le t \le \tfrac{1}{2}\pi).$$

For $f\oplus 1$ we see that $(f\oplus 1)h$ and h induce the same function

$$[S^V \wedge X,\ S^V \wedge Y]^G \longrightarrow [S^{V\oplus V} \wedge X,\ S^{V\oplus V} \wedge Y]^G .$$

This completes the checks.

We may therefore pass to the limit, and define

$$\{X,Y\}^G = \varinjlim_{V\in C} [S^V \wedge X,\ S^V \wedge Y]^G .$$

This definition is due to Segal [23]. Some of my correspondents would prefer to see the category C replaced by an equivalent small category before the limit is taken. In the applications X is finite-dimensional, and the limit is equivalent to taking the common value of $[S^V \wedge X,\ S^V \wedge Y]^G$ for all sufficiently large V - which is perfectly safe, however many V there are.

To continue, composition of G-maps $X \longrightarrow Y \longrightarrow Z$ is compatible with suspension, so it is clear how to define composition of stable maps and make the sets $\{X,Y\}^G$ into the hom-sets of a category.

The analogues of (2.4), (2.5) are as follows.

Proposition 4.1. If $\dim V^H < \dim W^H$ for all H, then

$$\{S^V, S^W\}^G = 0 .$$

Proposition 4.2. If $\dim (X^H) \le \mathrm{Hur}\,(Y^H) - 1$ for all H,

$$\text{then } \{X,Y\}^G = 0 .$$

In fact, in each case one has to take a limit of sets which are all trivial, by (2.4) or (2.5) as the case may be.

The definition of $\{X,Y\}^G$ given above clearly follows that of Spanier and Whitehead for the classical case. Therefore one can only expect it to be useful when X is finite-dimensional. In this case we expect the following result.

Proposition 4.3. If X is a finite-dimensional G-CW-complex, then the limit $\{X,Y\}^G$ is attained by $[S^W \wedge X, S^W \wedge Y]^G$ for all sufficiently large W.

This follows immediately from Theorem 3.4.

For later use, we also need to assure ourselves that our category is really "stable", by verifying that a suitable "external" suspension is a (1-1) correspondence. For this purpose, suppose given two G-spaces X, Y and an allowable representation U. For each object V of C we have a function

$$[S^V \wedge X, S^V \wedge Y]^G \xrightarrow{\mathrm{Susp}_V} [S^V \wedge X \wedge S^U, S^V \wedge Y \wedge S^U]^G$$

carrying f to $f \wedge 1_U$. This function commutes with the maps of our direct system, and defines a function

$$\begin{array}{ccc} \{X,Y\}^G & & \{X\wedge S^U,\ Y\wedge S^U\}^G \\ \| & & \| \\ \underset{\overrightarrow{V\in C}}{\mathrm{Lim}}\ [S^V\wedge X,\ S^V\wedge Y]^G & \longrightarrow & \underset{\overrightarrow{V\in C}}{\mathrm{Lim}}\ [S^V\wedge X\wedge S^U,\ S^V\wedge Y\wedge S^U]^G . \end{array}$$

<u>Lemma 4.4</u>. If X is a finite-dimensional G-CW-complex then this function

$$\{X,Y\}^G \xrightarrow{\text{Susp}} \{X\wedge S^U,\ Y\wedge S^U\}^G$$

is a (1-1) correspondence.

In fact, the functions Susp_V whose limit is taken are (1-1) correspondences for all sufficiently large V, by Theorem 3.4.

So far we have not said that our sets $\{X,Y\}^G$ are groups. To introduce addition one needs a suspension coordinate on which G acts trivially. From now on we assume that trivial representations are allowable; in this case the sets $\{X,Y\}^G$ become additive groups, and in fact the hom-sets of a preadditive category.

In the applications it is important to have suitable finiteness theorems.

<u>Theorem 4.5</u>. Suppose X is a finite G-CW-complex and Y is a G-space for which each fixed-point-set Y^H is an (ordinary) CW-complex with finitely many cells of each dimension. Then $\{X,Y\}^G$ is a finitely-generated abelian group.

The crucial point is that this limit is attained by

$$[S^W\wedge X,\ S^W\wedge Y]^G$$

for some W, according to (4.3). It is fairly clear how to prove that this group is finitely-generated, by combining the methods indicated in §2 with standard finiteness theorems in the classical case.

§5. Theorems on changing groups. In the unstable case, it is useful to have results such as (2.2) which allow one to reduce suitable problems involving G to problems involving a smaller group. In the stable case, there are more such theorems which are useful; in this section we will consider them.

First suppose given a homomorphism $\theta: G_1 \longrightarrow G_2$. For any G_2-space X, $\theta^* X$ will mean the same space considered as a G_1-space, with G_1 acting via θ. In particular, if V is a representation of G_2, then $\theta^* V$ is a representation of G_1. We assume that if V is an "allowable" representation of G_2, then $\theta^* V$ is an "allowable" representation of G_1. The operation $\theta^* V$ commutes with suspension, in the sense that

$$\theta^*(S^V \wedge X) = S^{\theta^* V} \wedge \theta^* X .$$

Therefore θ^* gives a functor from the G_2-Spanier-Whitehead category to the G_1-Spanier-Whitehead category. Here the objects of the "G-Spanier-Whitehead category" are the finite G-CW-complexes; the hom-sets are the sets $\{X,Y\}^G$ introduced in §4.

Until further notice, all representations are allowable.

I will state four results about θ^* before pausing to discuss them. To analyse θ^*, it is reasonable to factor θ through an epimorphism and a monomorphism, and tackle the factors separately. So first, let $i: H \longrightarrow G$ be the inclusion of a subgroup H in the group G; let X run over the H-Spanier-Whitehead category and let Y run over the G-Spanier-Whitehead category.

Theorem 5.1. There is a natural (1-1) correspondence

$$\{X, i^* Y\}^H \longleftrightarrow \{(G \lrcorner P) \wedge_H X, Y\}^G .$$

Theorem 5.2. There is a natural (1-1) correspondence

$$\{i^*Y,\ X\}^H \longleftrightarrow \{Y,\ (G \sqcup P) \wedge_H X\}^G .$$

Secondly, let $j: G \longrightarrow \bar{G}$ be the projection of G on a quotient group $\bar{G}$, and let $N = \operatorname{Ker} j$. (The case most useful for the applications is that in which $N = G$ and $\bar{G} = 1$; but there seems to be no harm in looking for a natural level of generality.) We let X run over finite G-CW-complexes in which the subgroup $N \subseteq G$ acts freely away from the base-point; more precisely, we let X run over the full subcategory of the G-Spanier-Whitehead category determined by these N-free objects. We let Y run over the $\bar{G}$-Spanier-Whitehead category.

Theorem 5.3. There is a natural (1-1) correspondence

$$\{X,\ j^*Y\}^G \longleftrightarrow \{X/N,\ Y\}^{\bar{G}} .$$

Theorem 5.4. There is a natural (1-1) correspondence

$$\{j^*Y,\ X\}^G \longleftrightarrow \{Y,\ X/N\}^{\bar{G}} .$$

Here X/N is of course the usual orbit space.

I will discuss these four results before I proceed to necessary technical details. Theorem 5.1 is a simple analogue of (2.2), and is widely known. Given (5.1), (5.2) says that the forgetful functor from the G-stable world to the H-stable world has a right adjoint which coincides with its known left adjoint. I first heard this principle from L.G. Lewis; of course his "stable worlds" were worlds of spectra, which makes for a better theorem. In that form, the result is to appear in [18]. That work uses the name "Wirthmüller isomorphism", thus giving credit to Wirthmüller for the result from which the authors started; it was only slightly less general than theirs, but I accept that it is more illuminating to state the result in the form I have quoted from Lewis.

I turn to motivation for (5.3). In the Atiyah-Segal theorem [2] one wishes to know that the ordinary K-theory of the classifying space BG coincides with the equivariant K-theory of the corresponding total space EG . In studying Segal's Burnside Ring Conjecture, one wishes similarly to know that the ordinary cohomotopy of BG coincides with the equivariant cohomotopy of EG . In suitable notation this reads

$$\{EG \sqcup P, S^0\}^G \longleftrightarrow \{\frac{EG \sqcup P}{G}, S^0\}^1 .$$

This is an instance of (5.3) (with $N = G$, $\bar{G} = 1$) except that (5.3) only gives the result for finite approximations to EG , BG ; the result for EG requires the analogue of (5.3) for spectra, unless you pass to limits from the result for finite approximations. Such results are probably widely known to those who have started work on Segal's conjecture; after (5.1), I regard (5.3) as the second easiest of the four.

Theorem 5.4 is necessary to maintain symmetry, as well as being needed for applications later. I am grateful to L.G. Lewis and J.P. May for letters, to which I owe the case $N = G$, $\bar{G} = 1$ (for the world of spectra).

Theorems 5.3 and 5.4 are not as satisfactory as (5.1) and (5.2); (5.1) and (5.2) each give an honest pair of adjoint functors, but (5.3) and (5.4) do not. (In (5.3) and (5.4) X is restricted to be N-free, but j^*Y cannot be N-free except in trivial cases.) One might like to see the statements of (5.3) and (5.4) improved in some way; I am open to suggestions.

In the rest of this section I will begin by giving some necessary technical details to complete the statements of (5.1) - (5.4), and continue with the proofs. The reader should consider skipping to §6.

The main technical detail concerns the sense in which $(G \sqcup P) \wedge_H X$ is functorial for stable H-maps of X rather than for unstable maps, and similarly for X/N. The careful reader should not take this for granted.

First suppose given a stable H-map $\phi: X_1 \longrightarrow X_2$. Since representations of the form i^*V are cofinal among representations of H, we may suppose given a representative H-map

$$f: S^{i^*V} \wedge X_1 \longrightarrow S^{i^*V} \wedge X_2 .$$

We now define the stable G-map

$$1 \wedge_H \phi: (G \sqcup P) \wedge_H X_1 \longrightarrow (G \sqcup P) \wedge_H X_2$$

to be the class of the following composite.

$$\begin{tikzcd} S^V \wedge ((G \sqcup P) \wedge_H X_1) \arrow[r, "\varepsilon^{-1}"] & (G \sqcup P) \wedge_H (S^{i^*V} \wedge X_1) \arrow[d, "1 \wedge_H f"] \\ S^V \wedge ((G \sqcup P) \wedge_H X_2) & (G \sqcup P) \wedge_H (S^{i^*V} \wedge X_2) \arrow[l, "\varepsilon"'] \end{tikzcd}$$

Here the G-homeomorphism

$$(G \sqcup P) \wedge_H (S^{i^*V} \wedge X) \xrightarrow{\varepsilon} S^V \wedge ((G \sqcup P) \wedge_H X)$$

is given by

$$\varepsilon(g, (s, x)) = (gs, (g, x)) .$$

Of course we have to check that the result depends only on ϕ, and that $(G \sqcup P) \wedge_H X$ becomes a functor as stated.

We now wish to copy this procedure for X/N; the difficulty is that representations of the form j^*W are not cofinal among representations of G. We need the following crucial result.

Proposition 5.5. If X is N-free away from the base-point, then $\{X, Y\}^G$, as defined allowing suspensions of the form S^{j^*W} only, agrees with $\{X, Y\}^G$ as defined using all suspensions S^V.

Proof. Let me begin by fine-tuning some results I explained earlier. In the "theorem of J.H.C. Whitehead", Proposition 2.7, we do not really need the assumption that all the cells of W^H are of dimension $\leq n(H)-1$; it is sufficient if all G-cells of W of the form $(G/H) \times E^m$ have $m \leq n(H)-1$. The same applies to any deduction from (2.7), including the G-suspension theorem, Theorem 3.3. These remarks are due to U. Namboodiri [28,29].

Our object is now to show that the map

$$[S^{j^*W} \wedge X,\ S^{j^*W} \wedge Y]^G \xrightarrow{S^V} [S^V \wedge S^{j^*W} \wedge X,\ S^V \wedge S^{j^*W} \wedge Y]^G$$

is iso for all V if the representation W of $\bar{G}$ is sufficiently large (depending on X). Here $S^{j^*W} \wedge X$ is also N-free away from the base-point. Therefore it can have cells $(G/H) \times E^m$ only if $N \cap H = 1$. So it will be sufficient to impose a suitable bound on the dimension of $(S^{j^*W} \wedge X)^H$ just for those subgroups H which satisfy $N \cap H = 1$. We now wish to satisfy the following inequalities on the dimensions.

(i) If H is a subgroup with $N \cap H = 1$ then

$$\dim\, (j^*W)^H + \dim\, X^H \leq 2 \dim\, (j^*W)^H - 2 \ .$$

We can satisfy this condition by putting sufficiently many copies of the trivial representation into W .

(ii) If $K \subset H$ is a pair of subgroups such that $N \cap H = 1$ and $V^K > V^H$ for some representation V , then

$$\dim\, (j^*W)^H + \dim\, X^H \leq \dim\, (j^*W)^K - 2 \ .$$

If $V^K > V^H$ for some V then we must have $K < H$. If $N \cap H = 1$ then the images of K, H in $\bar{G}$ satisfy $\bar{K} < \bar{H}$. Therefore there is a representation U of $\bar{G}$ for which $U^{\bar{K}} > U^{\bar{H}}$ (for example, the permutation representation on the cosets of $\bar{K}$). We can satisfy the condition by putting sufficiently many copies of U into W .

Of course, we have to satisfy inequalities of type (i) for a finite number of subgroups H, and inequalities of type (ii) for a finite number of pairs $K \subset H$, but we can satisfy all these conditions if W is sufficiently large. This proves (5.5).

We can now return to the task of making X/N functorial for stable G-maps of X. According to (5.5), any stable G-map $\phi\colon X_1 \longrightarrow X_2$ has a representative of the form

$$f\colon S^{j^*W} \wedge X_1 \longrightarrow S^{j^*W} \wedge X_2 .$$

We now define the stable $\overline{G}$-map $\overline{\phi}\colon X_1/N \longrightarrow X_2/N$ to be the class of the following composite.

$$\begin{tikzcd}
S^W \wedge X_1/N \arrow[r, leftrightarrow] & \dfrac{S^{j^*W} \wedge X_1}{N} \arrow[d, "\overline{f}"] \\
S^W \wedge X_2/N \arrow[r, leftrightarrow] & \dfrac{S^{j^*W} \wedge X_2}{N}
\end{tikzcd}$$

(Here the $\overline{G}$-homeomorphisms of spaces are the obvious ones.) We have to check that the result depends only on ϕ, and that X/N becomes a functor as stated, but these points are trivial.

This completes the technical details needed to explain (5.1)-(5.4).

In what follows we will omit the symbols i^*, j^* which show which groups are supposed to be acting on a given space; it is always easy to work out which groups are supposed to be acting, and these symbols only complicate the notation.

Proof of Theorem 5.1. The (1-1) correspondence is induced by the same two maps that serve in the unstable category. These are the H-map

$$X \xrightarrow{\alpha} (G \sqcup P) \wedge_H X$$

given by $\alpha(x) = (1,x)$, and the G-map

$$(G \sqcup P) \wedge_H Y \xrightarrow{\gamma} Y$$

given by $\gamma(g,y) = gy$. These maps make the following diagrams commute.

$$\begin{tikzcd}
S^V \wedge X \arrow[r, "S^V\alpha"] \arrow[dr, "\alpha"'] & S^V \wedge ((G \sqcup P) \wedge_H X) \\
& (G \sqcup P) \wedge_H (S^V \wedge X) \arrow[u, "\varepsilon"']
\end{tikzcd}$$

$$\begin{tikzcd}
S^V \wedge ((G \sqcup P) \wedge_H Y) \arrow[r, "S^V\gamma"] & S^V \wedge Y \\
(G \sqcup P) \wedge_H (S^V \wedge Y) \arrow[u, "\varepsilon"] \arrow[ur, "\gamma"']
\end{tikzcd}$$

It follows that α is natural, not only for unstable H-maps of X , but also for stable H-maps of X ; similarly, γ is natural, not only for unstable G-maps of Y, but also for stable G-maps of Y . Therefore these maps induce natural transformations in the usual way.

These two natural transformations are inverse because the composites

$$Y \xrightarrow{\alpha} (G \sqcup P) \wedge_H Y \xrightarrow{\gamma} Y$$

$$(G \sqcup P) \wedge_H X \xrightarrow{1 \wedge_H \alpha} (G \sqcup P) \wedge_H (G \sqcup P) \wedge_H X \xrightarrow{\gamma} (G \sqcup P) \wedge_H X$$

are already identity maps unstably.

<u>Proof of Theorem 5.2</u>. I have given the proof of (5.1) in the form above so that I can transcribe it by using arrow-reversing duality. First we shall need a stable H-map

$$(G \sqcup P) \wedge_H X \xrightarrow{\alpha} X .$$

This comes as an unstable map; we set

$$\alpha\ (h,\ x) = hx$$

$$\alpha\ (g,\ x) = x_o \quad \text{if} \quad g \notin H ,$$

where x_o is the base-point. We shall also need a stable G-map

$$Y \xrightarrow{\gamma} (G \sqcup P) \wedge_H Y ,$$

and this we now construct.

First choose an embedding $G/H \longrightarrow W$ of G/H in a representation W of G. For definiteness, we may take W to be a permutation representation, with the elements of G/H as an orthonormal base. Next choose an open equivariant tubular neighbourhood N of G/H in W; for definiteness, we may take the discs of radius 1/2 around the points of G/H. Consider the quotient map of $S^W = W \cup \infty$ in which we identify to a point the complement of N; we obtain a G-map

$$\beta : S^W \longrightarrow S^W \wedge (G/H \sqcup P) .$$

This map β is fixed once for all and does not depend on Y.

Given β, we define γ to be the following composite.

$$\begin{array}{ccc} S^W \wedge Y & \xrightarrow{\beta \wedge 1} & S^W \wedge (G/H \sqcup P) \wedge Y \\ & & \Big\downarrow 1 \wedge \eta^{-1} \\ & & S^W \wedge ((G \sqcup P) \wedge_H Y) . \end{array}$$

Here the G-homeomorphism

$$(G \sqcup P) \wedge_H Y \xrightarrow{\eta} (G/H \sqcup P) \wedge Y$$

is defined by

$$\eta\ (g,\ y) = (g,\ gy)$$

as in (2.3).

The properties of the maps α and γ are as follows.

<u>Lemma 5.6</u> (i) The following diagram is commutative.

$$\begin{tikzcd}
S^V \wedge ((G \sqcup P) \wedge_H X) \arrow[r, "S^V\alpha"] & S^V \wedge X \\
(G \sqcup P) \wedge_H (S^V \wedge X) \arrow[u, "\varepsilon"] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

(ii) The following diagram is commutative.

$$\begin{tikzcd}
S^W \wedge S^V \wedge Y \arrow[r, "S^V\gamma"] \arrow[dr, "\gamma"'] & S^W \wedge S^V \wedge (G \sqcup P) \wedge_H Y \\
& S^W \wedge ((G \sqcup P) \wedge_H (S^V \wedge Y)) \arrow[u, "1 \wedge \varepsilon"']
\end{tikzcd}$$

(iii) The composite

$$Y \xrightarrow{\gamma} (G \sqcup P) \wedge_H Y \xrightarrow{\alpha} Y$$

is the identity as a stable H-map.

(iv) The composite

$$(G \sqcup P) \wedge_H X \xrightarrow{\gamma} (G \sqcup P) \wedge_H (G \sqcup P) \wedge_H X \xrightarrow{1 \wedge_H \alpha} (G \sqcup P) \wedge_H X$$

is the identity as a stable G-map.

In part (ii), the G-map $S^V\gamma$ is obtained by suspending

$$S^W \wedge Y \xrightarrow{\gamma} S^W \wedge ((G \sqcup P) \wedge_H Y)$$

according to the inclusion $W \longrightarrow W \oplus V$.

Assuming Lemma 5.6, we can complete the proof of Theorem 5.2 as follows. The map α is clearly natural for unstable H-maps of X ; using (5.6) (i), we see that α is natural for stable H-maps of X , just as in the proof of (5.1). Similarly, the map γ is natural for unstable G-maps of Y ; using (5.6) (ii), and taking a little more care that we are really following the definitions laid down in §4, we see that γ is natural for stable G-maps of Y . Now we follow the standard routine for adjoint functors. The transformation

$$\{Y, X\}^H \longrightarrow \{Y, (G \sqcup P) \wedge_H X\}^G$$

carries a stable H-map

$$Y \xrightarrow{f} X$$

to the composite

$$Y \xrightarrow{\gamma} (G \sqcup P) \wedge_H Y \xrightarrow{1 \wedge_H f} (G \sqcup P) \wedge_H X .$$

The transformation

$$\{Y, (G \sqcup P) \wedge_H X\}^G \longrightarrow \{Y, X\}^H$$

carries a stable G-map

$$Y \xrightarrow{f} (G \sqcup P) \wedge_H X$$

to the composite

$$Y \xrightarrow{f} (G \sqcup P) \wedge_H X \xrightarrow{\alpha} X .$$

These transformations are inverse by (5.6) (iii), (iv).

<u>Proof of Lemma 5.6</u>. It is straightforward to verify parts (i) and (ii) from the definitions.

To prove part (iii), we introduce the map

$$\zeta : (G/H) \sqcup P \longrightarrow S^0$$

which carries g to the base-point if $g \notin H$, to the non-base-point if $g \in H$. We check that the composite

$$S^V \xrightarrow{\beta} S^V \wedge ((G/H) \sqcup P) \xrightarrow{1 \wedge \zeta} S^V$$

is H-homotopic to the identity, and that the diagram

$$\begin{tikzcd}
((G/H) \sqcup P) \wedge Y \arrow[r, "\zeta \wedge 1"] & Y \\
(G \sqcup P) \wedge_H Y \arrow[u, "\eta"] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

is strictly commutative. The result follows by combining these facts.

To prove part (iv), we first note an associative law. Suppose H acts on the right of $A \wedge B$ by acting on the right of B. Then the identity map of $A \wedge B \wedge C$ passes to the quotient to give an identification

$$(A \wedge B) \wedge_H C \longleftrightarrow A \wedge (B \wedge_H C) .$$

Up to this identification, the map

$$\begin{tikzcd}
S^W \wedge (G/H \sqcup P) \wedge (G \sqcup P) \wedge_H X \arrow[d, "1 \wedge \eta^{-1}"] \\
S^W \wedge (G \sqcup P) \wedge_H (G \sqcup P) \wedge_H X \arrow[d, "1 \wedge 1 \wedge_H \alpha"] \\
S^W \wedge (G \sqcup P) \wedge_H X
\end{tikzcd}$$

which occurs in (iv) may be written as $1 \wedge \delta \wedge 1_X$, where

$$(G/H \sqcup P) \wedge (G \sqcup P) \xrightarrow{\delta} (G \sqcup P)$$

carries (g_1, g_2) to g_2 if $g_1H = g_2H$, to the base-point otherwise.

It is now sufficient to check that the composite

$$\begin{array}{c} S^W \wedge (G \sqcup P) \\ \Big\downarrow \beta \wedge 1 \\ S^W \wedge (G/H \sqcup P) \wedge (G \sqcup P) \\ \Big\downarrow 1 \wedge \delta \\ S^W \wedge (G \sqcup P) \end{array}$$

is equivariantly homotopic to the identity, where the word "equivariant" means that we preserve both the actions of G on the left of these spaces and the action of H on the right.

In this composite, the map $\beta \wedge 1$ may be regarded as the map of $(S^W \times G)/(\infty \times G)$ which collapses to a point the complement of a tubular neighbourhood of $G/H \times G$. To apply $1 \wedge \delta$ we replace the relevant parts of this map by parts which map to the base-point; we thus obtain the map which collapses to a point the complement of a tubular neighbourhood of G, embedded via $g \longmapsto (gH, g)$. Clearly this embedding is homotopic to the zero cross-section by a linear homotopy $g \longmapsto (tgH, g)$ $(0 \leq t \leq 1)$. In this way we obtain a homotopy with the required equivariance property. This completes the proof of Lemma 5.6, and so finishes the proof of Theorem 5.2.

Proof of Theorem 5.3. In (5.3) and (5.4) we do not have an honest adjunction, and we can only expect to construct a natural transformation in one direction. In (5.3), the transformation is induced by an unstable G-map, namely the quotient map

$$X \xrightarrow{q} X/N .$$

For spaces it is trivial that the induced map

$$[S^W \wedge X/N,\ S^W \wedge Y]^{\overline{G}} \xrightarrow{q^*} [S^W \wedge X,\ S^W \wedge Y]^G$$

is a (1-1) correspondence, because every G-map of $S^W \wedge X$ into a space fixed by N must factor through $S^W \wedge X/N$. (Here W is of course a representation of $\overline{G}$.) Passing to limits, we see that

$$\varinjlim_W [S^W \wedge X/N,\ S^W \wedge Y]^{\overline{G}} \xrightarrow{q^*} \varinjlim_W [S^W \wedge X,\ S^W \wedge Y]^G$$

is a (1-1) correspondence. The left-hand side is $\{X/N,\ Y\}^{\overline{G}}$, and the right-hand side is $\{X,\ Y\}^G$ by Proposition 5.5. This proves Theorem 5.3.

<u>Proof of Theorem 5.4</u>. We shall construct for each finite G-CW-complex X on which N acts freely away from the base-point a stable G-map

$$tr_X \in \{X/N,\ X\}^G$$

with suitable properties. We shall then use this map to induce a natural transformation

$$\{Y,\ X/N\}^{\overline{G}} \longrightarrow \{Y,\ X\}^G .$$

The map tr_X is a "transfer" corresponding to the "covering" $X \longrightarrow X/N$. (I write "covering" because it fails to be an honest covering at the base-point.)

To construct tr_X , we first replace X by an equivalent G-CW-complex if that is thought to ease the next step. We choose a G-embedding

$$\begin{tikzcd} X \arrow[rr, "{(e,q)}"] \arrow[dr, "q"'] & & V \times X/N \arrow[dl, "\pi"] \\ & X/N & \end{tikzcd}$$

of the quotient map $q: X \longrightarrow X/N$ in the projection π of a trivial vector-bundle, whose fibre V is of course a representation of G. We also choose a G-invariant function $\varepsilon: X \longrightarrow R$ which is continuous, zero at the base-point and positive elsewhere, so that as x runs over X the points $(v, qx) \in V \times X/N$ with $\| v - e(x) \| \leq \varepsilon(x)$ make up a "tubular neighbourhood" $N(X)$ of X, which is just like an ordinary tubular neighbourhood except that its radius tends to zero as x approaches the base-point. For example, one may choose

$$\varepsilon(x) = \frac{1}{3} \operatorname*{Min}_{n \in N} \| e(nx) - e(x) \| .$$

We now perform the usual "Pontryagin-Thom" construction, and collapse the complement of the open tubular neighbourhood $N(X)$ to the base-point. We obtain a G-map

$$tr_X : S^V \wedge (X/N) \longrightarrow S^V \wedge X .$$

Here we get $S^V \wedge X$ rather than $(S^V \times X)/(\infty \times X)$ precisely because the radius of the tubular neighbourhood goes to zero at the base-point.

<u>Lemma 5.7</u>. The class $tr_X \in \{X/N, X\}^G$ is independent of the choices made in its construction, and natural for unstable G-maps of X.

<u>Proof</u>. It is more or less clear that the choice of ε affects the result only up to a G-homotopy, so it remains to discuss the dependence on V and on the embedding. We handle this together with the proof that tr_X is natural for (unstable) G-maps.

Suppose then that we are given a G-map $f: X_1 \longrightarrow X_2$ and embeddings

$$\begin{tikzcd}
X_1 \arrow[rr, "(e_1{,}q_1)"] \arrow[dr, "q_1"'] & & V_1 \times X_1/N \arrow[dl, "\pi"] \\
& X_1/N &
\end{tikzcd}$$

$$\begin{tikzcd}
X_2 \arrow[rr, "(e_2{,}q_2)"] \arrow[dr, "q_2"'] & & V_2 \times X_2/N \arrow[dl, "\pi"] \\
& X_2/N &
\end{tikzcd}$$

yielding G-maps

$$tr_1 : S^{V_1} \wedge X_1/N \longrightarrow S^{V_1} \wedge X_1$$

$$tr_2 : S^{V_2} \wedge X_2/N \longrightarrow S^{V_2} \wedge X_2 .$$

Then we can embed both embeddings in an embedding

$$\begin{tikzcd}
X_3 \arrow[rr, "(e_3{,}q_3)"] \arrow[dr, "q_3"'] & & V_3 \times X_3/N \arrow[dl, "\pi"] \\
& X_3/N &
\end{tikzcd}$$

in which X_3 is the mapping-cylinder of f and the injections $X_1 \longrightarrow X_3$, $X_2 \longrightarrow X_3$ are the usual ones. For example, we can take $V_3 = V_1 \times V_2$ and

$$e_3(t, x_1) = ((1-t)\, e_1(x_1) , te_2(fx_1))$$

$$e_3(x_2) = (0 , e_2(x_2)) .$$

Performing the same construction on this embedding, we get the following diagram.

$$\begin{tikzcd}
S^{V_3} \wedge X_1 \arrow[rr, "S^{V_3}f"] \arrow[dr] & & S^{V_3} \wedge X_2 \arrow[dl, "\simeq"'] \\
& S^{V_3} \wedge X_3 & \\
& S^{V_3} \wedge X_3/N \arrow[u, "tr_3"'] & \\
S^{V_3} \wedge X_1/N \arrow[uuu, "S^{V_3/V_1} tr_1"] \arrow[ur] \arrow[rr, "S^{V_3}\overline{f}"'] & & S^{V_3} \wedge X_2/N \arrow[uuu, "S^{V_3/V_2} tr_2"'] \arrow[ul]
\end{tikzcd}$$

This proves Lemma 5.7.

We now wish to show that tr_X has suitable properties for suspension, and of course our discussion is modelled on (5.6) (ii). Suppose given an embedding

$$\begin{tikzcd}
X_1 \arrow[rr, "{(e_1, q_1)}"] \arrow[dr, "q_1"'] & & V \times X_1/N \arrow[dl, "\pi"] \\
& X_1/N &
\end{tikzcd}$$

leading to the G-map

$$tr_1 : S^V \wedge (X_1/N) \longrightarrow S^V \wedge X_1 .$$

Suppose given also a representation W of $\overline{G} = G/N$. We wish to obtain the map tr_2 for the space $X_2 = S^W \wedge X_1$.

Lemma 5.8. There is a choice of tr_2 which makes the following diagram G-homotopy-commutative.

$$\begin{tikzcd}
S^V \wedge S^W \wedge X_1/N \arrow[r, "S^W tr_1"] \arrow[d, leftrightarrow] & S^V \wedge S^W \wedge X_1 \\
S^V \wedge \dfrac{S^W \wedge X_1}{N} \arrow[ur, "tr_2"'] &
\end{tikzcd}$$

As in (5.6) (ii), the G-map $S^W tr_1$ is obtained by suspending tr_1 according to the inclusion $V \longrightarrow V \oplus W$.

Proof. Let us decompose S^W into the hemisphere $E(W)_o$ given by $\| w \| \leq 1$ and the hemisphere $E(W)_\infty$ given by $\| w \| \geq 1$. Let us choose a real-valued function $\eta(w)$ on S^W which is continuous, G-invariant, 0 at ∞ and positive elsewhere, and 1 on $E(W)_o$; for example, we may take

$$\eta(w) = \frac{1}{\| w \|} \quad \text{on} \quad E(W)_\infty .$$

Then we can construct an embedding

$$\begin{tikzcd}
X_2 \arrow[rr, "(e_2, q_2)"] \arrow[dr, "q_2"'] & & V \times (X_2/N) \arrow[dl, "\pi"] \\
& X_2/N &
\end{tikzcd}$$

for $X_2 = S^W \wedge X$, by taking

$$e_2\ (w, x) = \eta(w)\ e_1(x) .$$

The map tr_2 is given by the corresponding collapsing map, and may be described as follows. For each point $w \in E(W)_o$ we get a copy of

$$tr_1 : S^V \wedge (X_1/N) \longrightarrow S^V \wedge X_1 .$$

Over $E(W)_\infty$ we get some map

$$S^V \wedge E(W)_\infty \wedge (X_1/N) \longrightarrow S^V \wedge E(W)_\infty \wedge X_1 ,$$

but these spaces are G-contractible and so it does not matter what the map is; our map tr_2 is G-homotopic to $S^W tr_1$. This proves Lemma 5.8.

Corollary 5.9. tr_X is natural for G-stable maps of X.

Given (5.5), this follows formally from (5.7) and (5.8). The argument is the same as that for γ in the proof of (5.2).

We now wish to know how tr_X behaves when X is an N-free G-sphere $(G/H \sqcup P) \wedge S^n$. The condition for this G-sphere to be N-free is $N \cap H = 1$; that is, H maps isomorphically to a subgroup $\overline{H}$ of $\overline{G}$. With $X = (G/H \sqcup P) \wedge S^n$ we have $X/N = (\overline{G}/\overline{H} \sqcup P) \wedge S^n$.

We first consider the case $n = 0$. We choose an embedding

$$\begin{tikzcd} X = G/H \sqcup P \arrow[r] \arrow[dr] & V \times (\overline{G}/\overline{H} \sqcup P) \arrow[d] \\ & \overline{G}/\overline{H} \sqcup P \end{tikzcd}$$

from which to construct tr_X. Since X is discrete, the tubular neighbourhood will consist of a set of discs centred at the points of G/H.

Lemma 5.10. If α_1 and α_2 are as in the proof of (5.2) then the diagram

$$\begin{tikzcd} S^V \wedge (\overline{G}/\overline{H} \sqcup P) \arrow[r, "tr_X"] \arrow[dr, "S^V\alpha_1"'] & S^V \wedge (G/H \sqcup P) \arrow[d, "S^V\alpha_2"] \\ & S^V \end{tikzcd}$$

is H-homotopy commutative.

Recall that α_1 is only an $\overline{H}$-map and α_2 is only an H-map.

Proof. tr_X is given by the usual collapsing map. To apply $S^V\alpha_2$, we must change our map to the base-point on all discs except that centred on the coset H/H. The result maps $S^V \times \bar{g}\bar{H}$ to the base-point unless $\bar{g}\bar{H}$ is the coset $\bar{H}$; then we get a map of S^V which is H-homotopic to the identity. After this homotopy we reach $S^V\alpha_1$.

Corollary 5.11. If $X = (G/H \sqcup P) \wedge S^n$ then the diagram

$$\begin{tikzcd} S^V \wedge (\bar{G}/\bar{H} \sqcup P) \wedge S^n \arrow[rr, "tr_X"] \arrow[dr, "S^V\alpha_1"'] & & S^V \wedge (G/H \sqcup P) \wedge S^n \arrow[dl, "S^V\alpha_2"] \\ & S^V \wedge S^n & \end{tikzcd}$$

is H-homotopy-commutative.

Proof. Apply the trivial suspension S^n to (5.10).

Corollary 5.12. The natural transformation

$$\{Y, X/N\}^{\bar{G}} \longrightarrow \{Y, X\}^G$$

induced by tr_X is iso when $X = (G/H \sqcup P) \wedge S^n$.

Proof. Consider the following diagram.

$$\begin{array}{ccc}
\{Y, (\bar{G}/\bar{H} \sqcup P) \wedge S^n\}^{\bar{G}} & \longrightarrow & \{Y, (G/H \sqcup P) \wedge S^n\}^G \\
{\scriptstyle (5.2)}\updownarrow & & \updownarrow{\scriptstyle (5.2)} \\
\{Y, S^n\}^{\bar{H}} & \longleftrightarrow & \{Y, S^n\}^H
\end{array}$$

Here the vertical arrows come from Theorem 5.2, and the lower horizontal isomorphism comes because $\bar{H}$ is isomorphic to H. Corollary 5.11 shows that the diagram is commutative, and the result follows.

Corollary 5.13. The natural transformation

$$\{Y, X/N\}^{\bar{G}} \longrightarrow \{Y, X\}^G$$

induced by tr_X is iso whenever X can be built up by the successive attachment of cones on G-spheres $(G/H \sqcup P) \wedge S^n$ with $N \cap H = 1$.

This follows from (5.12) by an obvious induction over the number of cones, using the five lemma.

Unfortunately, not every finite G-CW-complex can be built up by the successive attachment of cones on G-spheres (think of a finite approximation to $EG \sqcup P$); this is one of the well-known snags of the subject. However, a single trivial suspension S^1 is sufficient to turn any finite G-CW-complex into one which can be constructed in this way. Thus the natural tranformation

$$\{Y \wedge S^1, \overline{\frac{X \wedge S_1}{N}}\}^{\overline{G}} \longrightarrow \{Y \wedge S^1, X \wedge S^1\}^G$$

is iso. However, the whole result commutes with S^1, so this proves (5.4).

§6. Groups graded over the representation ring RO(G). In this section we will consider the question of theories graded over $RO(G)$.

Lately I've noticed authors writing sentences of the following general form. "Write $\alpha \in RO(G)$ in the form $\alpha = V - W$; then we define

$$\{X, Y\}^G_\alpha = \{S^V \wedge X, S^W \wedge Y\}^G \text{ ".}$$

If you catch anyone writing a sentence like that, make a note that you do not trust his critical faculties. The sentence in quotes is not sufficient. It implies that it is possible to verify that the result obtained depends only on α and does not depend on the choice of V and W; but it is not possible to verify this. In fact, suppose that α is the class of a representation, and that at one point the author wishes to use a representation V, and suppose (as is likely) that at another point he wishes to use a different but isomorphic representation V'. Then he must choose an isomorphism $V \cong V'$ to use in identifying $\{S^V \wedge X, Y\}^G$ with $\{S^{V'} \wedge X, Y\}^G$; and he must say which, for if he chooses a different one it will change his identification by an invertible element of the coefficient ring $\{S^0, S^0\}^G$. If he doesn't say which, then he doesn't know what he is doing and nor do we.

I will list three options suggested for overcoming this difficulty.

(i) Retreat to a notation which displays V and W explicitly.

(ii) Follow the classical precedent. A graded group such as $\pi_n(X)$ is not defined by allowing the use of any old vector space of dimension n; it is defined by using the specific space R^n which is under our control. This suggestion, then, involves an initial choice of preferred representatives. Presumably one begins by choosing one specific irreducible representation in each isomorphism class of irreducible representations.

(iii) "It may appear that $\{X,Y\}_*^G$ is intended to be a function which assigns to each $\alpha \in RO(G)$ a group $\{X,Y\}_\alpha^G$. Indeed, for purposes of planning strategy I like to think of it that way, and I hope you will do the same. But for purposes of rigourous proof, I suggest that $\{X,Y\}_*^G$ is a functor, which assigns a group to each object of some godawful category, and assigns to each morphism in that category a different way of identifying the groups in question".

The merit of (i) is that it is manifestly honest. The drawback is that it does not succeed in justifying notation such as $\{X,Y\}_\alpha^G$, which might be convenient.

The drawback of (ii) is that it may involve unattractive technicalities. Nevertheless, this is probably the best way if anyone seriously needs notation graded over $RO(G)$.

Mathematically, (iii) is indistinguishable from (i). Linguistically, notation with very strong associations, which are totally different from its declared logical meaning, is misleading notation. I suggest we should use misleading notation only when we wish to mislead, for example, on April 1st. Since mathematicians do not normally intend to deceive, misleading notation is especially dangerous to authors capable of self-deception.

Now I will turn to the published record. Bredon's work [4,6] involves homotopy groups which clearly need to be indexed over RO(G) for $G = Z_2$, and it is rigourous by option (ii) because it starts from the two actions of Z_2 on the reals. Notation graded over RO(G) , and the difficulty above, goes back to [23 p60]. Those who read German already possessed the means of implementing option (ii), because the work of tom Dieck [9,10] explicitly says that you choose actual representations, not isomorphism classes. When those who read German started to want to use notation graded over RO(G) , they remembered this [25 p373]; but they forgot it as soon as they could.

Next I point out that questions may arise which need checking from the definitions. I thank J.P. May for drawing my attention to the point which follows.

Authors who write about generalised cohomology theories commonly assume that for each $\alpha \in RO(G)$ and each X there is given a group $\tilde{H}^{\alpha}(X)$. (So, whatever else they are doing, they are not following option (iii).) Such a cohomology theory should come provided with suspension isomorphisms

$$\sigma^V : \tilde{H}^{\alpha}(X) \longrightarrow \tilde{H}^{\alpha+[V]}(S^V \wedge X)$$

where $[V]$ is the class of V in RO(G) . Clearly, $\alpha+[V]+[W]$ is logically the same element of RO(G) as $\alpha+[W]+[V]$, and apart from an axiom saying $\sigma^V\sigma^W = \sigma^{V\oplus W}$, we need an axiom about a diagram of the following form.

$$\begin{array}{ccc}
\tilde{H}^{\alpha}(X) & \xrightarrow{\sigma^V} & \tilde{H}^{\alpha+[V]}(S^V \wedge X) \\
 & & \downarrow \sigma^W \\
\sigma^W \downarrow & & \tilde{H}^{\alpha+[V]+[W]}(S^W \wedge S^V \wedge X) \\
 & & \downarrow (\tau \wedge 1)_* \\
\tilde{H}^{\alpha+[W]}(S^W \wedge X) & \xrightarrow{\sigma^V} & \tilde{H}^{\alpha+[W]+[V]}(S^V \wedge S^W \wedge X)
\end{array}$$

Here $S^W \wedge S^V \xrightarrow{\tau} S^V \wedge S^W$ is of course the switch map. In the ordinary case this diagram only commutes up to a sign $(-1)^{pq}$; in the equivariant case, it has to commute up to an invertible element of the coefficient ring $\{S^0, S^0\}^G$, and we must be told which. For example, if $V = W$, then the composite $\sigma^W \sigma^V$ is logically the same as $\sigma^V \sigma^W$, and the required element is the class of τ . Now τ can be replaced by the map

$$(+1) \oplus (-1): V \oplus V \longrightarrow V \oplus V .$$

Let us write $\varepsilon(V) \in \{S^0, S^0\}^G$ for the element represented by

$$(-1): V \cup (\infty) \longrightarrow V \cup (\infty) \ ;$$

then the answer in this case must be $\varepsilon(V)$. It can be shown by example that this element may be different both from $+1$ and from -1 (take $G = Z_2$ and take V to be the non-trivial action on the reals).

If any author on this subject had wished to inspire confidence, he should have faced this problem and not tried to skirt it. The answer I would like to see is

$$\prod_{\rho} \varepsilon(\rho)^{\langle\rho,V\rangle \langle\rho,W\rangle};$$

here the product runs over irreducibles ρ , and $\langle\rho,V\rangle$, $\langle\rho,W\rangle$ are the multiplicities of ρ in V,W respectively.

Of course, the correctness of such an answer, for some well-defined function $\tilde{H}^\alpha$ of $\alpha \in RO(G)$, can only be proved by checking from the definition. However, the inconsistency of certain other answers can be proved without.

I now invite the reader to try to audit works such as [ ] and [ ], and try to determine whether their statements are checked from definitions. In my opinion, uncritical use of $RO(G)$-gradings is likely to lead to treatments which cannot be accepted as satisfactory.

The relevance of all this is as follows. Carlsson's preprint [8] uses groups graded over $RO(G)$. It is suggested by Caruso and May that it might be profitable to rewrite more of Carlsson's proof as an exercise in RO(G)-graded generalised homology and cohomology. Of course, Caruso and May provide a rigourous foundation for the small use of RO(G)-grading they propose. However, we must also consider RO(G)-gradings elsewhere in the subject.

At this point I should perhaps point out one other thing well known to the experts, as follows. This is going to be a splendid subject, but we need to cure it of a certain tendency to minor sloppiness.

Question 6.1. Hey! Wouldn't it be better to deal with that in private?

Answer. I did try, but things seem to have gone too far. Only the other day one of my graduate students brought me his work, and when I checked the main reference, I found it was open to the objections I have explained; and this was from a source I had not previously regarded as suspect. (I wouldn't mind if the only results affected were either (a) so easy that anyone can prove them correctly, or (b) so dull that nobody would ever quote them. But as a defence of mathematical work, "de minimis non curat lex" is less popular than it might be.)

Now, I earnestly desire that if there are going to be theorems in this subject, then this subject should fall in with the rest of topology, and get itself written so that innocent graduate students can tell, without extravagantly much work, what is rigourously proved and what is not. It was so in 1972, why not now?

Question 6.2. But surely anyone can make a mistake?

Answer. Yes, of course, anyone can make a mistake. And anyone can put it right, by publishing some correction or addition to his work. But you want to do it before twenty other people have followed you into the same pitfall. After seven years, the way things move now, your paper is up for the judgement of history.

Question 6.3. But isn't it dangerous to make such sweeping generalizations? You will have all manner of upright citizens pressing you to publish the statement that you intended no slur on their care, rigour or professional standards.

Answer. (a) Everyone knows I don't mean [13] or [18]. The proportion of papers in this subject which are wholly satisfactory is well above the proportion of righteous men for which the Lord would have spared Sodom [30]. (b) I've tried lectures which don't name names and I've tried drafts which do name names, and nothing will please everyone.

I have consulted older and wiser men, and I am moved to preach a sermon to this subject. So, if such of my friends as have favourite pieces of minor sloppiness will please put them down and walk quietly away from them, I will begin.

I earnestly desire that people should not copy out of previous papers without pausing to think whether the passages to be copied make sense. And when we write a sentence which implies that one checks A and B, then we shall take scrap paper and check A and B - from the definitions. And for those of us who have the care of graduate students, I recommend that we give them critical faculties first and their Ph.D.'s afterwards. Here ends my sermon.

§7. G-spectra. In the classical case, the advantage of doing stable homotopy theory in a category of spectra are by now well understood. In this section we will consider very briefly the corresponding equivariant theory.

G-spectra were introduced by tom Dieck in [9,10]; the published account, [12], is less explicit. However, tom Dieck introduced G-spectra merely in order to obtain the associated generalised cohomology theories; he did not treat them as a category in which to do equivariant stable homotopy theory.

A good category of G-spectra exists [18,19]. "Just as the nonequivariant stable category is "Boardman's category", and is still Boardman's category no matter whose construction one actually quotes, so the equivariant stable category is that of Lewis and May". One can have confidence that the work of these authors will be careful, accurate and reliable, and we may hope that it will appear soon.

I need to draw attention to only one snag, and to explain it I must make some preliminary remarks.

For G-spaces X we have fixed-point subspaces X^H and we know what they do under suspension; we have

$$(S^V \wedge X)^H = (S^{V^H}) \wedge X^H .$$

Therefore, passage to fixed-point subspaces defines a functor, say T, from the G-Spanier-Whitehead category of §4 to the $N(H)/H$-Spanier-Whitehead category, where $N(H)$ is the normaliser of H in G.

Any good category of G-spectra must contain the G-Spanier-Whitehead category embedded in it as a full subcategory. In particular, the category of Lewis and May does so. Similarly, the category of N(H)/H-spectra must contain the $N(H)/H$-Spanier-Whitehead category.

We can now consider the following conditions on a hypothetical functor U from G-spectra (whatever they are) to $N(H)/H$-spectra (whatever they are).

(7.1). U extends the functor T defined above.

(7.2). U permits one to carry over to spectra the result (2.6) for spaces, say in the form of a (1-1) correspondence

$$\{(G/H \sqcup P) \wedge S^n,\ Y\}^G \longrightarrow \pi_n(U(Y))$$

(whatever π_n is).

The snag is that these two conditions are inconsistent; you cannot have both and so you must choose.

Lewis and May attach great importance to (7.2). "Since the reduction of equivariant problems to non-equivariant ones by passage to fixed-point spaces is probably the most basic tool in equivariant homotopy theory, it is clearly desirable" to carry over that tool from spaces to spectra. I freely concede the great mathematical interest of the objects $U(Y)$ which Lewis and May construct and call fixed-point spectra. Lewis and May argue further that, to avoid confusion, it would be highly undesirable for anyone to try to attach the name "fixed-points" to a functor U satisfying (7.1).

The relevance of this is as follows. Carlsson, in his preprint [8] p9, says that he will work in a category of G-spectra, and specifically in the category of [18]. If so, then by [8] p44 he wants a functor U with the property (7.1) and he has little or no interest in (7.2). Now, this seems to me a most reasonable request; I see no reason on earth why Carlsson should not have a functor with the property (7.1), and in the first draft of this section I constructed him one.

The reason I have cut this section since the first draft is that it now appears that most of Carlsson's proof can be done without G-spectra.

§8. Equivariant S-duality. In this section we will study the equivariant analogue of ordinary Spanier-Whitehead duality.

In the classical case, there are two standard approaches. In the first, which was historically prior, you suppose given a finite complex X. You choose a good embedding of X in the sphere S^{n+1}, and the complement gives the S-dual of X, up to a shift of n dimensions. In the second, one works not with embeddings, but with structure maps

$X^* \wedge X \longrightarrow S^n$. The standard reference is to Spanier's exercises [24 pp 462-463]. Both approaches carry over to the equivariant case. Of course, in the first, you embed in a sphere with G-action, and in the second, you map to a sphere with G-action. Both approaches have all the good properties a reasonable man would expect.

The standard reference is to Wirthmüller [26], who implements the second approach. [26] was a paper worth writing properly. It is written in RO(G)-graded notation; and at the time it was written, there was no adequate rigourization of RO(G)-graded notation in print so far as I know. Wirthmüller might have written one; alternatively, he might have used different notation. If he had done either, [26] could have been a splendid paper. As it is, I report that it can clearly be rewritten so as to become completely satisfactory.

For the embedding method, I thank J.P. May for recommending a reference to Section 3 of [27].

I will begin by summarising some basic material on G-S-duality.

Question 8.1. We need to begin with the duals of cells and spheres. What is the G-S-dual of $(G/H) \sqcup P$?

Answer. It is $(G/H) \sqcup P$ again. For example, you can embed $(G/H) \sqcup P$ in the sphere corresponding to the permutation representation of G on the elements of G/H ; then the complement is (up to G-equivalence) a wedge, indexed by G/H , of copies of the reduced permutation representation.

If you wish to avoid the embedding method, I suggest that you rely on (5.1) and (5.2) for the following natural (1-1) correspondences.

$$\begin{aligned} &\{(G/H \sqcup P) \wedge X, Y\}^G \\ &\qquad \longleftrightarrow \{X, Y\}^H \\ &\qquad \longleftrightarrow \{X, (G/H \sqcup P) \wedge Y\}^G . \end{aligned}$$

One has to remember that this answer needs modification when G is a compact Lie group. See [26 p 428].

Question 8.2. Should we expect G-S-duality to have good behaviour on cofiberings?

Answer. Yes. Of course, both query and answer beg the question of what we mean by "good behaviour", but we mean, "the same behaviour as for $G = 1$ ".

With the method of embedding in spheres, it is almost clear that the dual of a Mayer-Vietoris diagram

$$\begin{tikzcd} & X \arrow[dr] & \\ X \cap Y \arrow[ur] \arrow[dr] & & X \cup Y \\ & Y \arrow[ur] & \end{tikzcd}$$

is another Mayer-Vietoris diagram

$$\begin{tikzcd} & X^* \arrow[dl] & \\ X^* \cup Y^* & & X^* \cap Y^* \arrow[ul] \arrow[dl] \\ & Y^* \arrow[ul] & \end{tikzcd} .$$

Now take Y (and therefore Y^*) stably G-contractible; we see that the dual of a cofibre diagram

$$A \xrightarrow{f} X \xrightarrow{i} X \cup_f CA = B$$

is another cofibre diagram

$$X^* \cup_{i^*} CB^* \xleftarrow{f^*} X^* \xleftarrow{i^*} B^* .$$

With the Spanier approach, it's one of the lemmas which have to be proved before the method works. See [26 p 429].

I thank J.P. May for pointing out that if you want a cofibering to include all three relevant maps, then on a suitably precise definition, the statement "the S-dual of a cofibering is a cofibering" is actually false if you worry about signs - there is a sign which won't go away. But this is just the same as in the classical case.

Question 8.3. Should we expect G-S-duality to have good behaviour under the forgetful functor i^*, under j^* and under passage to fixed-point sets?

Answer. Yes. Suppose given a homomorphism $\theta: G_1 \longrightarrow G$, and suppose you can embed G-spaces X, X^* in a G-sphere S. Then you can apply θ^* and regard them as G_1-spaces θ^*X, θ^*X^* embedded in the G_1-sphere θ^*S. Similarly, you can pass to fixed-point sets and obtain X^H, $(X^*)^H$ embedded in S^H.

With the Spanier approach, you start from a structure map $X^* \wedge X \longrightarrow S$ and you can again apply θ^* or pass to fixed-point sets. See [26 p 427, p 431].

We turn to more interesting results. First suppose that X is a finite G-CW-complex which is free (away from the base-point) over a normal subgroup $N \subset G$.

Theorem 8.4. Then X admits a G-S-dual $D_G X$ with the following properties.

(i) $D_G X$ is also N-free (away from the base-point).

(ii) The duality is with respect to a "dimension" which is a representation of G/N.

Proof. We first avoid the standard snag mentioned at the end of §5 by the same device used there; by passing to $X \wedge S^1$ if necessary, we can assume that X is constructed by the successive attachment of cones on G-spheres $(G/H \sqcup P) \wedge S^n$ with $N \cap H = 1$.

We now apply (8.2). By (8.1) the dual of $(G/H \sqcup P) \wedge S^n$ with respect to dimension n is $(G/H \sqcup P)$. By (5.5), all the stable attaching maps required to build up $D_G X$ can be realised by maps of spaces at the price of suspension S^{j^*W} .

Let X and $D_G X$ be as in (8.4), that is, N-free and G-S-dual with respect to a dimension j^*W .

Theorem 8.5. Then X/N and $(D_G X)/N$ are G/N-dual with respect to dimension W .

It seems that this was in doubt until recently. I owe the case $N = G$, $G/N = 1$ to letters from L.G. Lewis and J.P. May.

Proof. Let $\bar{G} = G/N$ and let Y run over the $\bar{G}$-Spanier-Whitehead category. Then we have the following (1-1) correspondences natural in Y .

$$
\begin{aligned}
&\{Y \wedge X/N,\ S^W\}^{\bar{G}} \\
&\qquad \longleftrightarrow \{j^*Y \wedge X,\ S^{j^*W}\}^G && (5.3) \\
&\qquad \longleftrightarrow \{j^*Y,\ D_G X\}^G && \text{(G-S-duality)} \\
&\qquad \longleftrightarrow \{Y,\ \tfrac{D_G X}{N}\}^{\bar{G}} && (5.4)
\end{aligned}
$$

This characterises $\frac{D_G X}{N}$ as the $\bar{G}$-S-dual of X/N with respect to dimension W .

I next recall that A. Ranicki [21] has given an "unconventional" treatment of G-S-duality, in which the group G need not be finite, but the G-complexes must be G-free (away from the base-point).

<u>Theorem 8.6</u>. If G is finite then the Ranicki dual of a G-free space X agrees with the conventional one.

Proof. Let Y run over the G-Spanier-Whitehead category; Y need not be G-free. Let X^* be a Ranicki n-dual of X. In view of (5.5), the Ranicki n-dual X^* is characterised by the first of the following two (1-1) correspondences which are natural in Y.

$$\{X^*, Y\}^G$$

$$\longleftrightarrow \{S^n, \frac{X \wedge Y\ G/G}{G}\} \qquad \text{(Ranicki duality)}$$

$$\longleftrightarrow \{S^n, X \wedge Y\}^G \qquad (5.4)\ .$$

But this characterises X^* and X as conventional n-duals.

One of my correspondents suggests that the results presented above make it unnecessary for me (or Carlsson) to mention Ranicki duality. I take this point.

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ON THE RATIONAL HOMOTOPY OF CIRCLE ACTIONS

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At the conference the second-named author gave a talk in which he advertised methods from deformation theory of algebraic structures (associative algebras, chain complexes, Lie algebras) to study problems in cohomology and rational homotopy of transformation groups. In this context some results concerning cohomology can be found in [10], [11]. They rely on Borel's approach to P.A. Smith theory based on the localization theorem for equivariant cohomology.

Using Sullivan's theory of minimal models the first-named author gave analogous localization results for rational homotopy of torus actions (s. [1], [2]) which can also be interpreted in deformation theoretical terms to give further insight into certain problems in this area (s. [11]). In this note we pursue some questions mentioned in the talk at the conference by combining methods and results from [1], [2] and [10], [11].

Let X be a simply connected, finite S^1-CW complex such that $\dim_{\mathbb{Q}}(\pi_*(X) \otimes \mathbb{Q})$ is finite, and let $F \subset X$ be a simply connected π-full (i.e. $\dim_{\mathbb{Q}}(\pi_*(X) \otimes \mathbb{Q}) = \dim_{\mathbb{Q}}(\pi_*(F) \otimes \mathbb{Q})$ s. [2]) component of the fixed point set of X. For a (simply connected) space Y let $L_*(Y) := \Sigma^{-1}(\pi_*(Y) \otimes \mathbb{Q})$ denote the (connected) graded Lie algebra over $\mathbb{Q}$ given by the "desuspension" of the rational homotopy of Y where the Lie product corresponds to Whitehead product. Then one has

PROPOSITION: *$L_*(F)$ as a Lie algebra is a deformation (s. [5], [8]) of the graded Lie algebra $L_*(X)$. More precisely: There exists a graded Lie algebra $L_*^{S^1}(X)$ (isomorphic to $L_*(X) \otimes \mathbb{Q}[t]$ as a graded $\mathbb{Q}[t]$-module, $|t| = -2$, but with "twisted" Lie bracket)[1] over the graded polynomial ring*

[1] I.e. the Lie bracket of $L_*^{S^1}(X)$ does not coincide in general with the canonical $\mathbb{Q}[t]$-bilinear extension of the Lie bracket of $L_*(X)$.

$\mathbb{Q}[t]$ ($|t| = -2$) such that

$$L_*^{S^1}(X) \underset{\mathbb{Q}[t]}{\otimes} \mathbb{Q}^\varepsilon \cong \begin{cases} L_*(X) & \text{for } \varepsilon = 0 \\ L_*(F) & \text{for } \varepsilon \neq 0 \end{cases}$$

where the $\mathbb{Q}[t]$-module structure on $\mathbb{Q}^\varepsilon \cong \mathbb{Q}$ (as $\mathbb{Q}$-vector space) is given by $p^\varepsilon: \mathbb{Q}[t] \to \mathbb{Q};\ t \mapsto \varepsilon,\ \varepsilon \in \mathbb{Q}$.

This result follows from [2] and is stated in [11] for the dual situation, i.e. for the "pseudo-dual rational homotopy" of X and F as co-Lie algebras over $\mathbb{Q}$.

The situation in rational homotopy described by the above proposition is analogous to the "totally non-homologous to zero" case in cohomology (s. [2], [10]). Our aim is to establish a result on the minimal number of generators and relations for $L_*(F)$ in comparison with those for $L_*(X)$ which is analogous to the results in cohomology (s. [4],[6],[9],[10]). In the following lemma we collect some facts on presentations of (graded) Lie algebras which are more or less well known (s. [7]). Let L(V) denote the free Lie algebra over the $\mathbb{Q}$-vector space V.

LEMMA 1: a) If $L(W) \xrightarrow{\beta} L(V) \xrightarrow{\alpha} L \longrightarrow 0$ is a presentation of a connected ($L_0=0$) graded Lie algebra over $\mathbb{Q}$ such that V is a finite dimensional $\mathbb{Q}$-vector space then:

(i) $\dim_{\mathbb{Q}} V \geq \dim_{\mathbb{Q}} {}^{L}/_{[L,L]}$ = minimal number of Lie algebra generators of L

(ii) $\dim_{\mathbb{Q}} W - \dim_{\mathbb{Q}} V \geq \operatorname{cid}(L)$, where $\operatorname{cid}(L) :=$ min. number of relations of L - min. number of generators of L

b) For every finitely generated, connected graded Lie algebra L there exists a minimal presentation, i.e. a presentation $L(W) \to L(V) \to L \to 0$ such that

(i) $\dim_{\mathbb{Q}} V = \dim_{\mathbb{Q}} {}^{L}/_{[L,L]} < \infty$, (ii) $\dim_{\mathbb{Q}} W - \dim_{\mathbb{Q}} V = \operatorname{cid} L$.

Remark 1: Lemma 1 is a reformulation of some of the results given in [7], chap. 1 for $\mathbb{N}$-graded Lie algebras and grading preserving presentations. For grading preserving morphisms α,β the sequence $L(W) \xrightarrow{\beta} L(V) \xrightarrow{\alpha} L \longrightarrow 0$ is a presentation of the connected graded Lie algebra L if and only if

$\alpha\circ\beta = 0$ and the induced linear maps $[\alpha]: V \to L/_{[L,L]}$ and $[\beta]: W \longrightarrow \ker\alpha/_{[\ker\alpha, L(V)]}$ are surjective (s. [7]). For morphisms α,β which only preserve the $\mathbb{Z}/_{2\mathbb{Z}}$-grading given by the odd and even part of the $\mathbb{Z}$-grading of L these last conditions may no longer guarantee that α,β give a presentation of L, but if the connected graded Lie algebra L has finite total dimension as a vector space over $\mathbb{Q}$ (and hence is nilpotent), one can see (by similar arguments as in the grading preserving case) that they suffice to obtain the inequalities in part a) of Lemma 1 (the nilpotency of L together with the surjectivety of $[\alpha]$ implies that α is surjective).

Let $L(W) \xrightarrow{\beta} L(V) \xrightarrow{\alpha} L \longrightarrow 0$ be a grading preserving presentation (with $\dim_{\mathbb{Q}} V < \infty$) of a connected graded Lie algebra L over $\mathbb{Q}$ which has finite total dimension as a $\mathbb{Q}$-vector space. Let $\tilde{L}$ be a "one-parameter family of deformations of L" i.e. $\tilde{L} \cong L \otimes \mathbb{Q}[t]$ ($|t| = -2$) as a graded $\mathbb{Q}[t]$-module but with a twisted Lie bracket such that $\tilde{L}$ is a graded Lie algebra over $\mathbb{Q}[t]$ and $\tilde{L} \underset{\mathbb{Q}[t]}{\otimes} \mathbb{Q}^0 = L$ (s. [5],[8]).

<u>LEMMA 2</u>: <u>The morphisms</u> α,β <u>can be lifted to morphisms of graded Lie algebras</u> (<u>over</u> $\mathbb{Q}[t]$) $\tilde{\alpha}: \tilde{L}(V) := L(V) \otimes \mathbb{Q}[t] \longrightarrow \tilde{L}$, $\tilde{\beta}: \tilde{L}(W) := L(W) \otimes \mathbb{Q}[t] \to \tilde{L}(V)$ <u>such that</u>:

(i) $\tilde{\alpha} \underset{\mathbb{Q}[t]}{\otimes} \mathrm{id}_{\mathbb{Q}^0} = \alpha$, $\tilde{\beta} \underset{\mathbb{Q}[t]}{\otimes} \mathrm{id}_{\mathbb{Q}^0} = \beta$

(ii) $\tilde{\alpha}$ <u>is surjective, in particular the induced</u> $\mathbb{Q}[t]$-<u>linear map</u> $[\tilde{\alpha}]: V\otimes\mathbb{Q}[t] \longrightarrow \tilde{L}/_{[\tilde{L},\tilde{L}]}$ <u>is surjective</u>.

(iii) $\tilde{\alpha}\circ\tilde{\beta} = 0$ <u>and</u> $\tilde{\beta}$ <u>induces a surjective</u> $\mathbb{Q}[t]$-<u>linear map</u>

$$[\tilde{\beta}]: W \otimes \mathbb{Q}[t] \longrightarrow \ker\tilde{\alpha}/_{[\ker\tilde{\alpha},\, \tilde{L}(V)]}.$$

<u>Proof</u>: Since $p^0: \tilde{L} \longrightarrow L = \tilde{L} \underset{\mathbb{Q}[t]}{\otimes} \mathbb{Q}^0$ is surjective one can choose a (degree preserving) $\mathbb{Q}$-linear map $\tilde{\alpha}: V \to \tilde{L}$ such that $p^0\circ\tilde{\alpha} = \alpha$. This $\mathbb{Q}$-linear map has a unique extension (again denoted by $\tilde{\alpha}$) to a morphism of graded Lie algebras (over $\mathbb{Q}[t]$) $\tilde{\alpha}: \tilde{L}(V) \to \tilde{L}$. By construction $\tilde{\alpha} \underset{\mathbb{Q}[t]}{\otimes} \mathrm{id}_{\mathbb{Q}^0} = \alpha$ and the composition

$$\tilde{L}(V) \xrightarrow{\tilde{\alpha}} \tilde{L} \longrightarrow L = L \underset{\mathbb{Q}[t]}{\otimes} \mathbb{Q}^0 = \tilde{L}/_{t\cdot\tilde{L}}$$ is surjective.

By a simple induction argument one gets that the composition $\tilde{L}(V) \xrightarrow{\tilde{\alpha}} \tilde{L} \longrightarrow \tilde{L}/_{t^q\tilde{L}}$ is surjective for each $q \in \mathbb{N}$.

Since L is a positively graded Lie algebra of finite total dimension over $\mathbb{Q}$ and $\tilde{L} = L \otimes \mathbb{Q}[t]$ as a graded $\mathbb{Q}[t]$- module ($|t| = -2$) for every fixed dimension n there exists a $q(n)$ such that $(t^{q(n)}\tilde{L})_n = 0$. Hence $\tilde{\alpha}$ is surjective in every dimension which proves (ii). Since $\tilde{L}$ is a free $\mathbb{Q}[t]$-module there is a splitting of $\tilde{\alpha}$ as a morphism of $\mathbb{Q}[t]$-modules. It follows that $\ker\tilde{\alpha}$ is mapped surjectively onto $\ker\alpha$ under the map $p^o\colon \tilde{L}(V) \longrightarrow L(V) = \tilde{L}(V) \underset{\mathbb{Q}[t]}{\otimes} \mathbb{Q}^o$.

Therefore one can choose a (degree preserving) $\mathbb{Q}$-linear map $\tilde{\beta}\colon W \to \ker\tilde{\alpha}$ which extends uniquely to a morphism of graded Lie algebras (over $\mathbb{Q}[t]$) $\tilde{\beta}\colon \tilde{L}(W) \to \tilde{L}(V)$. By construction $\tilde{\alpha}\circ\tilde{\beta} = 0$ and $\tilde{\beta} \underset{\mathbb{Q}[t]}{\otimes} \mathbb{Q}^o = \beta$. It remains to show that

$[\tilde{\beta}]\colon W \otimes \mathbb{Q}[t] \longrightarrow \ker\tilde{\alpha}/_{[\ker\tilde{\alpha},\tilde{L}(V)]}$ is surjective.

Again by construction (and right exactness of the tensor product) the composition

$$W\otimes\mathbb{Q}[t] \xrightarrow{[\tilde{\beta}]} \ker\tilde{\alpha}/_{[\ker\tilde{\alpha},\tilde{L}(V)]} \to \left(\ker\tilde{\alpha}/_{[\ker\tilde{\alpha},\tilde{L}(V)]}\right)\underset{\mathbb{Q}[t]}{\otimes}\mathbb{Q}^o =$$

$$\ker\tilde{\alpha}\underset{\mathbb{Q}[t]}{\otimes}\mathbb{Q}^o \Big/ [\ker\tilde{\alpha},\tilde{L}(V)]\underset{\mathbb{Q}[t]}{\otimes}\mathbb{Q}^o = \ker\alpha/_{[\ker\alpha,L(V)]}$$

is surjective (since $[\beta]\colon W \longrightarrow \ker\alpha/_{[\ker\alpha,L(V)]}$ is surjective).

Since L has finite total dimension as a $\mathbb{Q}$-vector space the morphism $\tilde{\alpha}\colon \tilde{L}(V) \to \tilde{L} = L\otimes\mathbb{Q}[t]$ is trivial in sufficiently high dimensions ($|t| = -2$(!)), i.e. $(\ker\tilde{\alpha})_m = \tilde{L}(V)_m$ for large enough m. On the other hand $[\tilde{L}(V),\tilde{L}(V)]_m = \tilde{L}(V)_m$ for large enough m because $\dim_{\mathbb{Q}}V < \infty$. It follows that $\left(\ker\tilde{\alpha}/_{[\ker\tilde{\alpha},\tilde{L}(V)]}\right)_m = 0$ for large enough m.

By an argument similar to that above one gets $W\otimes\mathbb{Q}[t] \xrightarrow{[\tilde{\beta}]} R \longrightarrow R/_{tR}$ is surjective where $R := \ker\tilde{\alpha}/_{[\ker\tilde{\alpha},\tilde{L}(V)]}$ and therefore by induction $W\otimes\mathbb{Q}[t] \xrightarrow{[\tilde{\beta}]} R \longrightarrow R/_{t^qR}$ is surjective for all $q \in \mathbb{N}$ which implies the surjectivity of $[\tilde{\beta}]$ since $(t^q R)_n = 0$ if $q = q(n)$ is large enough.

<u>THEOREM</u>: <u>Under the hypothesis of the proposition above one has</u>:

(i) $\dim_{\mathbb{Q}} L_*(F)/[L_*(F),L_*(F)] \leq \dim_{\mathbb{Q}} L_*(X)/[L_*(X),L_*(X)]$
<u>i.e. the minimal number of Lie algebra generators of</u> $L_*(F)$ <u>is smaller or equal to the minimal number of Lie algebra generators of</u> $L_*(X)$.

(ii) $\mathrm{cid}(L_*(F)) \leq \mathrm{cid}(L_*(X))$. <u>In particular</u>:
<u>the minimal number of relations of</u> $L_*(F)$ <u>is smaller or equal to the minimal number of relations of</u> $L_*(X)$.

<u>Proof</u>: Choose a minimal presentation of $L := L_*(X)$
$L(W) \xrightarrow{\beta} L(V) \xrightarrow{\alpha} L \longrightarrow 0$ (in particular $\dim_{\mathbb{Q}} V = \dim_{\mathbb{Q}} L/[L,L]$ = min. number of generators of $L_*(X)$, $\dim_{\mathbb{Q}} W - \dim_{\mathbb{Q}} V = \mathrm{cid}(L_*(X)$, $\dim_{\mathbb{Q}} W$ = min. number of relations of $L_*(X)$).

By the above proposition and Lemma 2 there exists a sequence of Lie algebra morphisms

$$(*)\ \tilde{L}(W) \xrightarrow{\tilde{\beta}} \tilde{L}(V) \xrightarrow{\tilde{\alpha}} \tilde{L} = L_*^{S^1}(X) \longrightarrow 0 \text{ with } \tilde{\alpha}\circ\tilde{\beta} = 0$$
$[\tilde{\alpha}]: V\otimes\mathbb{Q}[t] \longrightarrow \tilde{L}/[\tilde{L},\tilde{L}]$ and
$[\tilde{\beta}]: W\otimes\mathbb{Q}[t] \longrightarrow \ker \tilde{\alpha}/[\ker \tilde{\alpha},\tilde{L}(V)]$ surjective.

Tensoring the sequence (*) with $\mathbb{Q}^{\varepsilon}$ over $\mathbb{Q}[t]$ one gets a sequence of $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebras and morphisms

$$\tilde{L}(W) \underset{\mathbb{Q}[t]}{\otimes} \mathbb{Q}^{\varepsilon} \xrightarrow{\tilde{\beta}^{\varepsilon}} \tilde{L}(V) \underset{\mathbb{Q}[t]}{\otimes} \mathbb{Q}^{\varepsilon} \xrightarrow{\tilde{\alpha}^{\varepsilon}} \tilde{L} \underset{\mathbb{Q}[t]}{\otimes} \mathbb{Q}^{\varepsilon} = L_*(F) \longrightarrow 0. (\varepsilon \neq 0)$$

By the right exactness of the tensor product one has: $\tilde{\alpha}^{\varepsilon}$ surjective, in particular $[\tilde{\alpha}^{\varepsilon}]: V \to L_*(F)/[L_*(F),L_*(F)]$ surjective and $[\tilde{\beta}^{\varepsilon}] = [\tilde{\beta}]^{\varepsilon}: W \to \ker \tilde{\alpha}^{\varepsilon}/[\ker \tilde{\alpha}^{\varepsilon}, \tilde{L}(V) \underset{\mathbb{Q}[t]}{\otimes} \mathbb{Q}^{\varepsilon}]$ surjective ($\ker(\tilde{\alpha}^{\varepsilon}) = (\ker \tilde{\alpha}) \underset{\mathbb{Q}[t]}{\otimes} \mathbb{Q}^{\varepsilon}$, because $\tilde{\alpha}$ splits as a $\mathbb{Q}[t]$-linear map).
Hence the theorem follows by applying Remark 1.

Remark 2: By induction on the dimension of the torus one immediately gets similar results for torus actions.

Remark 3: The analogous result for the "totally non-homologous to zero" case in cohomology mentioned above including the result on the "complete intersection defect" (cid) (s. [11], Cor(1.2)) can be obtained by analogous arguments for commutative associative graded algebras.

Remark 4: In the absence of the π-fullness assumption on F one can use deformation theoretical methods with respect to the differential of the minimal model to derive results on the rational homotopy and cohomology of S^1 (resp. torus)-actions (s. [11] for a sketch of some simple proofs of certain known results, which have been proved before by different methods [1],[12]). Recent work of G. Carlsson (s. [3]) allows one to use similar methods for the cohomology of $\mathbb{Z}/p\mathbb{Z}$ (resp. finite p-group)-actions (p prime).

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SEMIFREE TOPOLOGICAL ACTIONS OF FINITE GROUPS ON SPHERES

by

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Let S^{n+k} be the unit sphere in R^{n+k+1} and $S^k = S^{n+k} \cap R^{k+1}$ where R^{k+1} is the linear subspace of R^{n+k+1} spanned by the last (k+1)-coordinates. Let G be a finite group. It is the object of this paper to study the existence and desuspension problems for semifree topological actions of G on S^{n+k} with fixed point set S^k. (Henceforth, we shall denote the fixed point set by Fix(G).) Before we state in a precise form the problems we consider, we fix an identification of $S^{n+k} - S^k$ with $S^{n-1} \times R^{k+1}$ and record an easy lemma.

Lemma: _If_ G _acts semifreely on_ S^{n+k} _with_ $\mathrm{Fix}(G) = S^k$, _then_

i) G _has periodic cohomology of period_ d _and_ $n = rd$ _for some_ $r \geq 1$ _and_;

ii) _If_ $n \geq 4$, _then_ $X = S^{n+k} - S^k/G$ _is a polarized complex_.

We refer the reader to Madsen, Thomas, Wall [9] or [16] for the definition of a polarized complex.

Proof: Part i) follows immediately from Cartan and Eilenberg [3] since $S^{n+k} - S^k = S^{n-1} \times R^{n+1}$. For ii) it is clear that there is an identification of $\pi_1(X)$ with G and a preferred homotopy equivalence of the universal cover $\tilde{X}$ of X with S^{n-1}. Furthermore, X is finitely dominated by a result of Edmonds [5].

The significance of part i) is that it limits the groups G that can occur in this problem. In the case when $k = -1$ (i.e. $\mathrm{Fix}(G) = \emptyset$), Milnor's theorem [10] imposes the further limitation that every subgroup of G of order 2p (p a prime) must be cyclic and in this case the following theorem holds:

Theorem (Madsen, Thomas, Wall). Let G be a group with periodic cohomology of period d and such that every subgroup of G of order $2p$ is cyclic. Then G acts freely on S^{2d-1}.

In most, but not all, cases it is also known that G acts freely on S^{d-1}, i.e. in the "period dimension." It is hoped that the results presented here will help to resolve this "period dimension" problem in some of these outstanding cases. For example, we expect that some of these groups cannot act even semifreely in the period dimension--that is, they cannot act semifreely on S^{d+k} with $\mathrm{Fix}(G) = S^k$.

If $k \geq 0$, then Milnor's theorem can no longer be applied. (This was pointed out to us by Julius Shaneson.) In particular, it is not known at this writing whether G must satisfy the 2p-condition described above and the question of whether this is so becomes an interesting problem.

Henceforth in this paper we shall assume that $n \geq 4$. We note that since $n = rd$, where d is the cohomological period of G, and d must be even [3], this is a restriction only when $d = 2$ and is a mild one at that. In fact, if $d = 2$, then G is cyclic and linear actions provide a complete solution for the existence problem that we consider. In this case, then, only the desuspension problem is of interest.

The significance of part ii) of the above lemma is that it introduces a new parameter into the existence problem. In particular, in [16] it is shown that there is a bijection between polarized complexes with universal cover homotopy equivalent to S^{n-1} and generators of $H^n(G;Z) = Z_{|G|}$ given by taking the first k-invariant. Thus, if $\chi \in H^n(G;Z)$, we shall let $X(\chi)$ be the corresponding polarized complex. We remark that $X(\chi)$ is a finitely dominated

Poincaré complex of formal dimension n-1 whose Spivak normal fibre space admits at TOP reduction and we fix one such reduction $\nu(\mathcal{K}): X(\mathcal{K}) \to BTOP$.

The problems considered in this paper may now be stated precisely as follows:

Existence Problem: For which quadruples $(G,n,k,\mathcal{K})$ does there exist a semifree topological action of G on S^{n+k} with $Fix(G) = S^k$ and $S^{n+k} - S^k/G$ homotopy equivalent to $X(\mathcal{K})$?

Desuspension Problem: Suppose G acts semifreely on S^{n+k} with $Fix(G) = S^k$. Is the given action the suspension of a semifree action of G on S^{n+k-1} with $Fix(G) = S^{k-1}$?

Our solutions to these problems are the following theorems:

Theorem A: Consider the following statements:

i) There exists an element $x \in [X(\mathcal{K});G/TOP]$ whose surgery obstruction $\theta(x) = 0$ in $L^{-k}_{n-1}(G)$.

ii) There is a semifree topological action of G on S^{n+k} with $Fix(G) = S^k$ and $S^{n+k} - S^{+k}/G$ homotopy equivalent to $X(\mathcal{K})$.

If $n \geq 4$, $k \geq 0$, and $n+k \geq 5$, then i) implies ii). If $n \geq 5$, then ii) implies i). Thus i) and ii) are equivalent for $n \geq 5$.

Remarks: a) In this theorem, $\mathcal{K} \in H^n(G;Z)$ for some $n = rd$ and $r \geq 1$.

b) The functors $L^{-k}_{n-1}(\ )$ used in this theorem are defined inductively by setting $L^0_{n-1}(\ ) = L^p_{n-1}(G)$, the Wall group based on projective modules, and then setting

$$L^{-(k+1)}_{n-1}(G) = \operatorname{coker}\{\sigma_* : L^{-k}_{n-1}(G) \to L^{-k}_{n-1}(G \times Z)\}$$

where $\sigma : G \to G \times Z$ is the obvious inclusion. These groups have been investigated by Ranicki [14]. In particular, he shows that they fit into a Rothenberg type exact sequence

$$\cdots \to H^n(Z_2;K_{-(k+1)}(G)) \to L^{-k}_{n-1}(G) \xrightarrow{\rho_k} L^{-(k+1)}_{n-1}(G) \to H^{n-1}(Z_2;K_{-(n+1)}(G)) \to \cdots$$

Thus, although the surgery obstruction $\theta(x)$ of this theorem originally lies in $L^p_{n-1}(G)$, it is its image in $L^{-k}_{n-1}(G)$ that is of interest here.

c) It seems quite likely that the condition that $n \geq 5$ in the second half of this theorem can be replaced by $n \geq 4$. Thus, it is likely that i) and ii) are equivalent for $n \geq 4$.

Theorem B. Let G act semifreely on S^{n+k} with $\mathrm{Fix}(G) = S^k$. Let $n+k \geq 6$. Suppose that either $k \geq 2$ or that $0 \leq k \leq 1$ and that an obstruction $\mathcal{O} \in K'_{-k}(G)$ vanishes. Then there is a G-invariant subspace M of S^{n+k} meeting S^k in S^{k-1} and an ambient isotopy $h_t : S^{n+k} \to S^{n+k}$ $(0 \leq t \leq 1)$ such that $h_0 = 1$, $h_1(M) = S^{n+k-1}$, and $h_t | S^k = 1$ for all t. In particular, the given action is isotopic to the suspension of an action of G on S^{n+k-1} with $\mathrm{Fix}(G) = S^{k-1}$.

Remarks: a) In this theorem $K'_{-k}(G)$ is $\tilde{K}_0(G)$ if $k = 0$ and $K_{-1}(G)$ if $k = 1$.

b) If $k = 0$, the obstruction $\mathcal{O}$ is a Siebenmann end invariant [15] and has been investigated by Edmonds [5]. If $k = 1$, $\mathcal{O}$ is the Quinn invariant ([12] and [13]) of a certain parametrized end which is described later in this paper.

c) We could state this theorem more generally in terms of an obstruction $\mathcal{O} \in K_{-k}(G)$ for any $k \geq 1$. If $k \geq 2$, however, Carter [4] has shown that $K_{-k}(G) = 0$ for any finite group; thus this obstruction vanishes.

d) In this theorem $S^{n+k-1} = S^{n+k} \cap R^{n+k}$, where $R^{n+k} \subset R^{n+k-1}$ is spanned by the first $(n+k)$-coordinates, and $S^{k-1} = S^{n+k-1} \cap S^k$.

As an immediate corollary of Theorem B, we have

Corollary C. Suppose that the finite group G acts on S^{n+k} with $\mathrm{Fix}(G) = S^k$. If $k \geq 2$, then G acts on S^{n+1} with $\mathrm{Fix}(G) = S^1$.

This paper contains sketches of the proofs of these results. The authors expect to publish full details of these proofs elsewhere.

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1. The Proof of Theorem A: i) implies ii).

In this section we begin the sketches of the proofs of Theorems A and B by sketching the proof that i) implies ii) in Theorem A when $n \geq 4$, $k \geq 0$, and $n+k \geq 5$. We begin this sketch by observing that, by using the cobordism interpretation of the L^p group given in [11], taking the cartesian product with S^1 yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L_n^p(G) \arrow[r] \arrow[d, "\times S^1"] & L_n^p(G \times Z) \arrow[r] \arrow[d, "\times S^1"] & L_{n-1}^{-1}(G) \arrow[r] \arrow[d, dashed, "\chi_1"] & 0 \\
0 \arrow[r] & L_n^h(G \times Z) \arrow[r] & L_n^h(G \times Z \times Z) \arrow[r] & L_n^p(G \times Z) \arrow[r] & 0
\end{tikzcd}$$

and induces a homomorphism χ_1 which is a natural transformation of functors. We may then proceed by induction to construct homomorphisms

$$\chi_k : L_{n-1}^{-k}(G) \to L_n^{-k+1}(G \times Z)$$

which define a natural transformation from the functor $L_{n-1}^{-k}(\ )$ to $L_n^{-k+1}(\ \times Z)$. Furthermore, it can easily be shown that $\chi_{k+1}\rho_k = \rho_{k+1}\chi_k$, where $\rho_k : L_{n-1}^{-k}(G) \to L_{n-1}^{-(k+1)}(G)$ is the homomorphism in the Rothenberg sequence, and that χ_k is a monomorphism. We shall use this latter fact in Section 4.

With this information in hand, it can be shown that the following diagram commutes

$$\begin{array}{ccccc}
 & & [X(\mathcal{K});G/TOP] & \xrightarrow{\Theta} & L^{-k}_{n-1}(G) \\
 & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \chi} \\
hS^h(X(\mathcal{K}) \times T^{k+1}) & \longrightarrow & [X(\mathcal{K}) \times T^{k+1};G/TOP] & \xrightarrow{\Theta'} & L^h_{n+k}(G \times Z^{k+1})
\end{array}$$

where π^* is induced by the projection $\pi : X(\mathcal{K}) \times T^{k+1} \to X(\mathcal{K})$, $\chi = (\times S^1)\chi_1 \cdots \chi_k$, and Θ' is the surgery obstruction map. It follows that if $x \in [X(\mathcal{K}),G/TOP]$ is such that $\Theta(x) = 0$, then there is a homotopy structure $f : M^{n+k} \to X(\mathcal{K}) \times T^{k+1}$.

Let $\tilde{M}$ and $\hat{M}$ be the covering spaces of M with fundamental group trivial and Z^{k+1} respectively. Let $q : \tilde{M} \to \hat{M}$ be a covering map and notice that $\tilde{M}$ and $\hat{M}$ both support free actions of G relative to which q is equivariant.

Lemma: <u>There exists a homeomorphism</u> $\hat{d} : \hat{M} \to S^{n-1} \times T^{k+1}$. <u>The action of</u> G <u>on</u> $S^{n-1} \times T^{k+1}$ <u>induced by this homeomorphism induces the identity on fundamental groups</u>.

Proof: Lift f to $\hat{f} : \hat{M} \to \widetilde{X(\mathcal{K})} \times T^{k+1}$, where $\widetilde{X(\mathcal{K})}$ is the universal cover of $X(\mathcal{K})$, and compose with $h \times 1 : \widetilde{X(\mathcal{K})} \times T^{k+1} \to S^{n-1} \times T^{k+1}$, where h is the preferred homotopy equivalence coming from the polarization of $X(\mathcal{K})$. Since n is even, the results of [7; Section 10] can be applied to the homotopy structure $(h \times 1)\hat{f}$ to prove the lemma.

If we now lift $\hat{d}$ to universal covers, we obtain a homeomorphism $\tilde{d}$ such that the following diagram commutes

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{d}} & S^{n-1} \times R^{k+1} \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle 1 \times e} \\
\hat{M} & \xrightarrow{\hat{d}} & S^{n-1} \times T^{k+1}
\end{array}$$

where e is the exponential map. In particular, there is an induced action of G on $S^{n-1} \times R^{k+1}$ relative to which $1 \times e$ is equivariant. It follows that since each $g \in G$ acts trivially on $\pi_1(S^{n-1} \times T^{k+1})$, the action of g on $S^{n-1} \times R^{k+1}$ is bounded in the R^{k+1} direction (cf. [8] or [1]). Thus we have a homomorphism $\rho' : G \to \mathrm{Homeo}_b(S^{n-1} \times R^{k+1})$, the group of bounded homeomorphisms of $S^{n-1} \times R^{k+1}$. Since the compactification arguments of [1] or [2], yield a homomorphism $\gamma : \mathrm{Homeo}_b(S^{n-1} \times R^{k+1}) \to \mathrm{Homeo}(S^{n+k}, S^k)$, the group of homeomorphisms of S^{n+k} that are the identity of S^k, setting $\rho = \gamma\rho'$ completes the proof that i) implies ii) in Theorem A.

2. Some Ends of an Action.

In this section we describe several ends associated with a semifree action of G on S^{n+k} with $\mathrm{Fix}(G) = S^k$. It is the analysis of these ends that leads to the proofs of Theorem B and of the fact that ii) implies i) in Theorem A when $n \geq 5$.

We first define a map $c_1 : S^{n+k} \to D^{k+1}$ by fixing identifications of S^{n+k} with $S^{n-1} * S^k$ and D^{k+1} with $v * S^k$ and letting c_1 correspond to $c * 1$ where $c : S^{n-1} \to v$ is a collapse map. We let rD^{k+1} be the disk of radius r about the origin and set $M_1 = S^{n+k} - c_1^{-1}(\mathrm{Int}\, 1/3\, D^{k+1} \cup S^k)$. Finally we let $e_1 : M_1 \to S^k$ be the composite $c_1| : M_1 \to D^{k+1} - \mathrm{Int}(1/3\ D^{k+1}) \xrightarrow{\rho} S^k$ where ρ is a radial retraction. It is easy to see that there is a commutative diagram

$$\begin{tikzcd} M_1 \arrow[r, "e_1"] \arrow[d, "h"'] & S^k \arrow[d, equal] \\ S^{n-1} \times S^k \times (0,1] \arrow[r, "p_2"] & S^k \end{tikzcd}$$

where h is a homeomorphism and p_2 is projection on the second factor. Thus, if e_1 is regarded as an end in the sense of [12], it is 0-LC and tame (cf. [12] for definitions). Furthermore, e_1 admits the nicest sort of completion, namely one corresponding to the projection $p_2 : S^{n-1} \times S^k \times [0,1] \to S^k$.

Now define $c_2 : S^{n+k} \to D^{k+1}$ by $c_2(x) = |G|^{-1} \Sigma\, c_1(gx)$ where the summation runs over $g \in G$. Again we set $M_2 = S^{n+k} - c_2^{-1}(\mathrm{Int}\, 1/3\, D^{k+1} \cup S^k)$

and let $e_2 : M_2 \to S^k$ be the composite $c_2| : M_2 \to D^{k+1} - \operatorname{Int} 1/3\, D^{k+1} \xrightarrow{\rho} S^k$. An elementary argument based on comparing e_1 and e_2, using the product structure on e_1 (i.e. the fact that e_1 is essentially a projection), and little point set topology shows that the end e_2 is 1-LC and tame. We call e_2 the <u>equivariant end</u> of S^{n+k} near S^k.

Finally, we note that M_2 is a G-invariant subspace of S^{n+k} and that $e_2 : M_2 \to S^k$ is G-equivariant if S^k is given the trivial action. Thus, e_2 induces a map $e_3 : M_3 \to S^k$ where $M_3 = M_2/G$. Since e_2 is a regular cover of e_3, it is easy to see that the end e_3 is 0-LC, has constant fundamental group G, and is tame. (The last condition follows from [12; Proposition 1.7].) We call e_3 the <u>end of the orbit space</u> S^{n+k}/G near the singular set S^k.

The ends that are actually used in this paper are obtained from the ends that we have just described by restriction. Specifically, let $\rho : R^k \to D^k$ and $a : D^k \to S^k$ be given by $\rho(x) = \frac{1}{1+|x|}x$ and $a(x) = ((1-|x|^2)^{1/2}, x)$ respectively. Then ρ is a homeomorphism onto $\operatorname{Int} D^k$ and a is an embedding whose image we denote by D^k_+. Let $N_i = e_i^{-1}(\operatorname{Int} D^k_+)$ and $f_i = \rho^{-1}a^{-1}e_i : N_i \to R^k$ $(i = 1,2,3)$. We call f_i the part of e_i over R^k $(i = 1,2,3)$.

Since f_i is obtained from e_i by restriction to an open subset, f_i is ℓ-LC (ℓ=0 or 1) whenever e_i is, is tame, and has constant fundamental group. We observe that there is a commutative diagram

$$\begin{tikzcd} N_1 \arrow[r, "f_1"] \arrow[d, "h"'] & R^k \arrow[d, equal] \\ S^{n-1} \times R^k \times (0,1] \arrow[r, "p_2"] & R^k \end{tikzcd}$$

where h is a homeomorphism and p_2 is projection on the second factor. We also observe that f_2 and f_3 are related by a commutative diagram

$$\begin{tikzcd} N_2 \arrow[d, "q"'] \arrow[dr, "f_2"] & \\ N_3 \arrow[r, "f_3"'] & R^k \end{tikzcd}$$

in which q is a principal G-bundle and hence a covering map.

Proposition 2.1. i) <u>The end</u> $f_3 : N_3 \to R^k$ <u>has a completion</u> $\overline{f_3} : \overline{N_3} \to R^k$ <u>if and only if an obstruction</u> $\mathcal{O} \in K'_{-k}(G)$ <u>vanishes</u>.

ii) <u>If</u> $n+k \geq 6$, <u>the end</u> $N_3 \times S^1 \xrightarrow{p} N_3 \to R^k$ <u>has a completion where</u> p <u>is projection on the first factor</u>.

Remarks. a) In the lemma, $K'_{-k}(G) = K_{-k}(G)$ if $k \geq 1$ and $K'_0(G) = \tilde{K}_0(G)$.

b) We recall that a completion of an end $e : M \to X$ consists of a manifold $\overline{M}$ with $M \subset \overline{M}$ and $\overline{M} - M \subset \partial\overline{M}$ and a proper map $\overline{e} : \overline{M} \to X$ extending e.

Corollary 2.2. <u>If</u> $k \geq 2$, <u>then</u> $f_3 : N_3 \to R^k$ <u>has a completion</u> $\overline{f}_3 : \overline{N}_3 \to R^k$.

Proof: If $k \geq 2$, then $K_{-k}(G) = 0$ by a result of Carter [4].

Proof of 2.1: It follows from [13] that f_3 has a completion if and only if a sequence of obstructions $\mathcal{O}_j \in \check{H}^{\ell f}_j(R^k; K'_{-j}(G))$ vanish. These homology groups, however, vanish except when $j = k$ in which case the group is just $K'_{-k}(G)$. Part i) follows.

Part ii) follows from the product formula of [13; Proposition 1.8].

Lemma 2.3. <u>If the end</u> f_3 <u>has a completion</u> $\overline{f}_3 : \overline{N}_3 \to R^k$, <u>then there is a principal</u> G-<u>bundle</u> $\overline{q} : \overline{N}_2 \to \overline{N}_3$ <u>such that</u> $\overline{f}_2 = \overline{f}_3\overline{q}$ <u>is a completion of</u> f_2 <u>and such that</u> $\overline{q}|N_2 = q$.

This lemma is obvious.

3. The Proof of Theorem B

In this section we sketch the proof of Theorem B. We begin by defining the notion of a "tapered embedding" that underlies the geometric construction on which this proof is based.

A <u>tapering</u> $t : R^k \to (0,1]$ is a continuous function such that $\lim_{x \to \infty} t(x) = 0$.

Let $\overline{f} : \overline{N} \to R^k$ be a completion of the end $f : N \to R^k$, set $\partial_0\overline{N} = \overline{N} - N$, and fix a collar $c : \partial_0\overline{N} \times [0,1] \to \overline{N}$ such that $c(x,0) = x$. Let $t : R^k \to (0,1]$ be a tapering. The embedding $T : \partial_0\overline{N} \times (0,1] \to N$ given by $T(x,s) = c(x,(t\overline{f}(x))s)$ is called the <u>tapered embedding associated with</u> t. Notice that if T_i is a tapered embedding associated with t_i $(i = 1,2)$ and $t_2(x) < t_1(x)$ for all $x \in R^k$, then the function $h : \partial_0\overline{N} \times [0,1] \to \operatorname{Im} T_1 - \operatorname{Int} \operatorname{Im} T_2 = V$ given by $h(x,s) = c(x,(1-s)t_2\overline{f}(x) + st_1\overline{f}(x))$ is a homeomorphism. We call h the <u>canonical product structure</u> on V.

The geometric construction underlying the proof of Theorem B begins with the assumption that the end $f_3 : N_3 \to R^k$ has a completion. (In particular, this holds if either $k \geq 2$ or $k \leq 1$ and the obstruction $\mathcal{O} \in K'_{-k}(G)$ of 2.1 vanishes.) We now fix completions $\overline{f}_i : \overline{N}_i \to R^k$ $(i = 1,2,3)$ such that there is a commutative diagram

$$\begin{array}{ccc} \overline{N}_1 & \xrightarrow{\overline{f}_1} & R^k \\ \downarrow{\scriptstyle \overline{h}} & & \| \\ S^{n-1} \times R^k \times [0,1] & \xrightarrow{p_2} & R^k \end{array}$$

where $\overline{h}$ is a homeomorphism and p_2 is projection on the second factor and such that there is a principal G-bundle $\overline{q} : \overline{N}_2 \to \overline{N}_3$ with $\overline{f}_2 = \overline{f}_3\overline{q}$ as in 2.3. We also fix collars $c_i : \partial_0\overline{N}_i \times I \to \overline{N}_i$ with $c_i(x,0) = x$ $(i = 1,2,3)$ and such that $\overline{h}c_1 = (\overline{h}|\partial_0\overline{N}_1) \times 1$, where $\overline{h}$ is the homeomorphism above, and $c_3(\overline{q} \times 1) = \overline{q}c_2$. Then corresponding to any tapered embedding $T_3 : \partial_0\overline{N}_3 \times (0,1] \to N_3$, there is a unique tapered embedding $T_2 : \partial_0\overline{N}_2 \times (0,1] \to N_2$ satisfying $T_3(\overline{q} \times 1) = \overline{q}T_2$. We say T_2

is <u>associated with</u> T_3.

We let k_i $(i = 1,2,3)$ be the inclusion of N_i in S^{n+k} (if $i = 1,2$) or in S^{n+k}/G (if $i = 3$).

The main result needed to prove Theorem B is the following proposition whose proof is temporarily deferred:

Proposition 3.1. <u>There exist tapered embeddings</u> $T_i : \partial_0\overline{N}_i \times (0,1] \to N_i$ $(i = 1,3)$ <u>such that</u>

i) $\operatorname{Im} k_2T_2 \subset \operatorname{Int} \operatorname{Im} k_1T_1$ <u>where</u> $T_2 : \partial_0\overline{N}_2 \times (0,1] \to N_2$ <u>is the tapered embedding associated with</u> T_3.

ii) <u>There exists a homeomorphism of pairs</u>

$$h : ((S^{n-1} \times I) * S^{k-1}, S^{k-1}) \to (W \cup S^{k-1}, S^{k-1})$$

<u>with</u> $h|S^{k-1} = 1$ <u>where</u> $W = \operatorname{im} k_1T_1 - \operatorname{Int} \operatorname{Im} k_2T_2$.

Remark: a) In ii) we have identified S^{k-1} with its image in the join $(S^{n-1} \times I) * S^{k-1}$. We are also considering $W \cup S^{k-1}$ as a subspace of S^{n+k} and topologizing it as such.

b) The following figure may be of help to the reader. The region inside of the indicated S^k should be thought of as $S^{n+k} - S^k$.

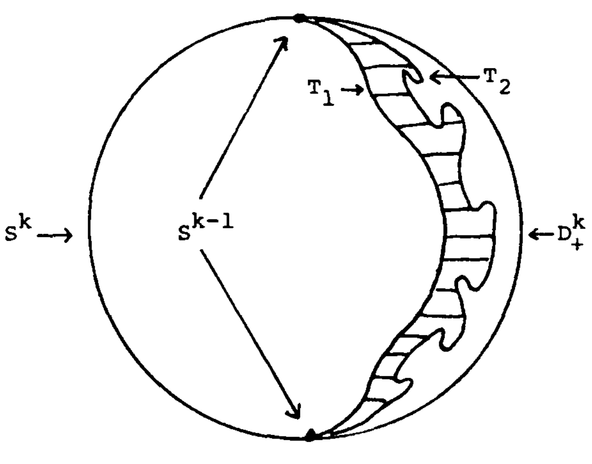

FIGURE 1

The image of k_iT_i $(i=1,2)$ lies to the right of the curve labelled T_i $(i=1,2)$ and W is the shaded region.

Proof of Theorem B assuming 3.1. Set $M = k_2T_2(\partial_0\bar{N}_2 \times 1) \cup S^{k-1}$ (i.e. the right hand boundary of W above union S^{k-1}). Then M is G-invariant since $k_2T_2(\partial_0\bar{N}_2 \times 1)$ covers $k_3T_3(\partial_0\bar{N}_3 \times 1)$ and M meets S^k in S^{k-1}. The ambient isotopy is obtained by using the product structure on W coming from 3.1 together with canonical product structures extending slightly to the left (respectively, right) of the left (respectively, right) hand boundary of W to push M across W onto $k_1T_1(\partial_0\bar{N}_1 \times 1) \cup S^{k-1}$. It is then easy to ambient isotope $k_1T_1(\partial_0\bar{N}_1 \times 1) \cup S^{k-1}$ onto S^{n+k-1}. This completes the proof of Theorem B assuming 3.1.

The key ingredient in the proof of 3.1 is the following lemma:

Lemma 3.2: <u>Let</u> $\delta > 0$ <u>be given. Then there exist tapered embeddings</u> T_i $(i=1,2,3)$ <u>satisfying part</u> i) <u>of</u> 3.1 <u>such that</u> $f_1 : W' \to R^k$ <u>is a</u> (δ,h)-<u>cobordism where</u> $W' = k_1^{-1}(W) \subset N_1$ <u>and is</u> $(\delta,1)$-<u>connected</u>.

The reader should see [12; Section 2] for the definitions of a (δ,h)-cobordism and $(\delta,1)$-connectedness.

Sketch of proof: The idea of the proof is to construct tapered embeddings $T_{i,j} : \partial_0\bar{N}_i \times (0,1] \to N_i$ $(i,j=1,2,3)$ such that $T_{2,j}$ is associated with $T_{3,j}$; $\operatorname{Im} k_2T_{2,j} \subset k_1(N_1)$, and the images of $T_{1,j}$ and $T_{2,j}$ are nested as indicated in the following figure:

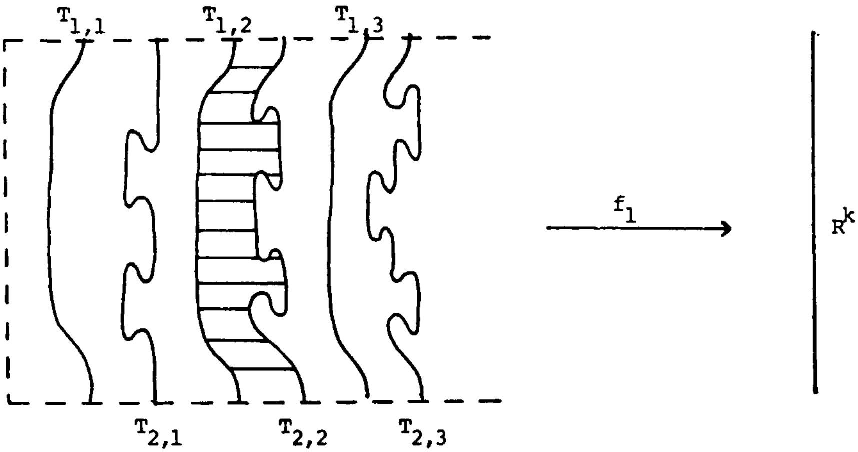

FIGURE 2

In this figure we have actually sketched the images of $T_{1,j}$ and $k_1^{-1}k_2T_{2,j}$ in N_1 (i.e. we used the fact that $\operatorname{Im} k_2T_{2,j} \subset k_1N_1$) and indicated N_1 by the dotted lines. In addition, the curves that we have labelled as $T_{i,j}$ are actually only the images of $\partial_0\bar{N}_i \times 1$ under $k_1^{-1}T_{i,j}$ $(i = 1,2;\ j = 1,2,3)$. We will suppress the k_1^{-1} in the sequel to simplify notation. The full image of $T_{i,j}$ should be thought of as lying to the right of these curves. Finally, the reader should notice that when Figure 2 is embedded in S^{n+k} via k_1, R^k becomes the interior of D_+^k and the curves $T_{i,j}$ "converge" to $S^{k-1} = \partial D_+^k$ as in Figure 1.

Let W' be the closed region between $T_{1,2}$ and $T_{2,2}$ (i.e. the shaded region in Figure 2). By using the canonical product structures on the regions between $T_{i,j}$ and $T_{i,j+1}$ $(i,j = 1,2)$ it is easy to construct deformation retractions $R_i : W' \to T_{i,2}(\partial_0\bar{N}_i \times 1)$ $(i = 1,2)$. Thus, W' is an h-cobordism. By making sure that the tapering occurs fast enough (i.e. that the images of these embeddings are close enough to $\operatorname{Int} D_+^k$ in S^{n+k}), we can bound the diameters of these deformation retractions in R^k (i.e. under f_1) by δ. Thus $f_1 : W' \to R^k$ is a (δ,h)-cobordism. If we now set $T_i = T_{i,2}$ $(i = 1,2,3)$, then $W = k_1(W')$ and the first part of the lemma is established.

To see that $f_1 : W' \to R^k$ is $(\delta,1)$-connected, let (R,S) be a relative 2-complex and suppose given maps r and s such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "s"] \arrow[d, phantom, "\cap"] & W' \arrow[d, "f_1"] \\ R \arrow[r, "r"] & R^k \end{tikzcd}$$

We must find $g : R \to W'$ with $g|S = s$ and such that $|f_1 g(x) - r(x)| < \delta$ for all $x \in R$. Using the canonical product structure h' on the region V' between $T_{1,2}$ and $T_{1,3}$ and the fact that $f_1 : N_1 \to R^k$ is topologically equivalent to the projection on the second factor $p_2 : S^{n-1} \times R^k \times (0,1] \to R^k$ (cf. section 2), we obtain the commutative diagram

$$\begin{tikzcd} S \arrow[r, "s"] \arrow[d, phantom, "\cap"] & W' \arrow[d, "f_1"] & \subset & V' \arrow[d, "f_1"] \arrow[r, "h'"] & S^{n-1} \times R^k \times I \arrow[d, "p_2"] \\ R \arrow[r, "r"] & R^k & = & R^k \arrow[r, "1"] & R^k \end{tikzcd}$$

Using this diagram and the fact that $n \geq 4$, it is now easy to find a map $g' : R \to V'$ such that $g'|S = s$ and $f_1 g' = r$.

On the other hand the canonical product structure on the region R' between $T_{2,2}$ and $T_{2,3}$ allows us to construct a deformation retraction $\rho' : R' \to \partial_L R'$ where $\partial_L R'$ is the left hand boundary of R'. Thus, we can also construct a deformation retraction $\rho : U' \to W'$ where U' is the region between $T_{1,2}$ and $T_{2,3}$ by setting $\rho = 1$ on W' and ρ' on R'. Let g be the composite $R \xrightarrow{g'} V' \subset U' \xrightarrow{\rho} W'$. Clearly $g|S = s$ and notice that the difference between $f_1 g$ and $f_1 g'$ arises entirely from the behavior of $f_1 \rho$ on R'; i.e. from $f_1 \rho'$. This map, however, is one of the key maps that is used in showing that W' is a (δ,h)-cobordism. In particular, the tapering of $T_{i,j}$ $(i,j = 1,2,3)$ was chosen to gain control over this map (among others). The fact that $|f_1g(x)-r(x)| < \delta$ for all x now follows from this control. Hence $f_1 : W' \to R^k$ is $(\delta,1)$-connected and 3.2 follows.

Sketch proof of 3.1: Let $\epsilon > 0$ be given. A slight strengthening of the Thin h-cobordism Theorem [12; Theorem 2.7] allows us to conclude that there is a $\delta > 0$ such that the (δ,h)-cobordism of 3.2 has an ϵ-product structure. Thus there is a homeomorphism $h_1 : \partial_L W' \times I \to W'$ with $h_1|\partial_L W' \times 0 = 1$ (where $\partial_L W'$ is the left hand boundary of W') such that the paths $f_1h_1(x \times I)$ $(x \in \partial_L W')$ all have diameter $< \epsilon$. Since the tapering provides a homeomorphism $\partial_0\bar{N}_1 \to \partial_L W'$ and $\partial_0\bar{N}_1$ can be identified with $S^{n-1} \times R^k$, we obtain a homeomorphism of pairs $h_2 : S^{n-1} \times R^k \times (I,0) \to (W', \partial_L W')$ such that the paths $f_2h_2(z \times I)$ have diameter $< \epsilon$ for all $z \in S^{n-1} \times R^k$.

We now notice that there is a quotient map $q : S^{n-1} \times D^k \times I \to (S^{n-1} \times I) * S^{k-1}$ and that the composite $S^{n-1} \times R^k \times I \xrightarrow{1 \times c \times 1} S^{n-1} \times D^k \times I \xrightarrow{q} (S^{n-1} \times I) * S^{k-1}$ defines an embedding onto $(S^{n-1} \times I) * S^{k-1} - S^{k-1}$. We call this embedding e and define $h : (S^{n-1} \times I) * S^{k-1} \to k_1(W') \cup S^{k-1} = W \cup S^{k-1}$ by setting $h|\mathrm{Im}e = k_1h_2e^{-1}$ and $h|S^{k-1} = 1$. Then h is the desired homeomorphism and the proof of 3.1 is completed.

The reader will notice that this proof is a variation on the compactification arguments of [1] and [2].

4. The Proof of Theorem A: ii) implies i).

This section sketches the proof that ii) implies i) in Theorem A when $n \geq 5$. It is based on the following observations.

Lemma 4.1. Let $n \geq 5$. Suppose there exists a semifree action of G on S^{n+1} with $\mathrm{Fix}(G) = S^1$. Set $S^{n+1} - S^1/G = X(\mathcal{K})$. Then $hS^h(X(\mathcal{K}) \times T^2) \neq \emptyset$.

Proof: Let $f_3 : N_3 \to R^k$ be the end of the orbit space near R^k. By 2.1, the composite $N_3 \times S^1 \xrightarrow{p} N_3 \xrightarrow{f_1} R^k$ has a completion $F : V \to R^k$. Let $\partial_0 V = \partial V - \partial(N_3 \times S^1)$ and include $\partial_0 V$ in $N_3 \times S^1$ via the "inner end" of a collar on $\partial_0 V$ in V. The composite k given by $\partial_0 V \subset N_3 \times S^1 \subset X(\mathcal{K}) \times S^1$ is then a homotopy equivalence.

Now $\partial_0 V$ is a manifold with two tame Siebenmann ends. Hence $\partial_0 V \times S^1$ has a Siebenmann completion W with two compact boundary components $\partial_0 W$ and $\partial_1 W$. Include $\partial_0 W$ in $\partial_0 \bar{V} \times S^1$ as the "inner end" of a collar. Then the composite $\partial_0 W \subset \partial_0 V \times S^1 \xrightarrow{k \times 1} X(\mathcal{K}) \times S^1 \times S^1$ defines a homotopy structure on $X(\mathcal{K}) \times T^2$.

Lemma 4.2. _Let_ $n \geq 5$. _If_ $hS^h(X(\mathcal{K}) \times T^2) \neq \emptyset$, _then there exists an element_ $x \in hS^h(X(\mathcal{K}) \times T^2)$ _whose normal invariant lies in_ $\mathrm{Im}\{\pi^* : [X(\mathcal{K});G/TOP] \to [X(\mathcal{K}) \times T^2;G/TOP]\}$ _where_ $\pi : X(\mathcal{K}) \times T^2 \to X(\mathcal{K})$ _is projection on the first factor_.

Proof: Let $h : V \to X(\mathcal{K}) \times T^2$ be a homotopy structure. Then the map $\hat{h} : \hat{V} \to X(\mathcal{K}) \times S^1 \times R^1$ of infinite cyclic covers is a homotopy equivalence. Since $\hat{V}$ has two tame ends, crossing with S^1, completing, and embedding a boundary component as in 4.1 yields a new homotopy structure $W \to X(\mathcal{K}) \times T^2$. The normal invariant of this structure lies in $\mathrm{Im}\{\pi_1^* : [X(\mathcal{K}) \times S^1;G/TOP] \to [X(\mathcal{K}) \times T^2;G/TOP]\}$ where $\pi_1 : X(\mathcal{K}) \times S^1 \times S^1 \to X(\mathcal{K}) \times S^1$ is projection on the first two factors. Repeat this procedure with W and the cover corresponding to $X(\mathcal{K}) \times R^1 \times S^1$ to obtain x.

Proof of Theorem A ii) implies i): Let $n \geq 5$. If $k = -1$ (i.e. $\mathrm{Fix}\, G = \emptyset$) the result is standard; while if $k = 0$, it follows from the arguments of [6] and [11]. Thus suppose $k \geq 1$. By Corollary C, there exists an action of G on S^{n+1} with $\mathrm{Fix}\, G = S^1$ and $S^{n+1} - S^1/G = X(\mathcal{K})$. Hence by 4.1 and 4.2 there is a homotopy structure $f : M^{n+1} \to X(\mathcal{K}) \times T^2$ with normal invariant in

$\mathrm{Im}\{\pi^* : [X(\mathcal{K});G/TOP] \to [X(\mathcal{K}) \times T^2;G/TOP]\}$. Then $f \times 1 : M^{n+1} \times T^{k-1} \to X(\mathcal{K}) \times T^{k+1}$ is a homotopy structure y whose normal invariant $\eta(y) = \pi_1^*(x)$ for some $x \in [X(\mathcal{K});G/TOP]$ where $\pi_1 : X(\mathcal{K}) \times T^{k+1} \to X(\mathcal{K})$ is projection. The result now follows by chasing the diagram

$$\begin{array}{ccccc} & & [X(\mathcal{K});G/TOP] & \xrightarrow{\theta} & L^{-k}_{n-1}(G) \\ & & \downarrow{\pi_1^*} & & \downarrow{\chi} \\ hS^h(X(\mathcal{K}) \times T^{k+1}) & \longrightarrow & [X(\mathcal{K}) \times T^{k+1};G/TOP] & \xrightarrow{\theta} & L^h_{n+k}(G \times Z^{k+1}) \end{array}$$

where $\chi = (\times S^1)\chi_1 \cdots \chi_k$ (cf. section 2) and using the fact that χ is monomorphic.

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EXTENSIONS LIBRES DES ACTIONS DE GROUPES FINIS DANS

LES VARIETES SIMPLEMENT CONNEXES

par

Amir H. ASSADI*)

Introduction.

Soit $(W^n, \partial W^n)$ une variété compacte connexe, et soit $(M^k, \partial M^k) \subset (W^n, \partial W^n)$ une sous-variété. Supposons que M^k soit munie d'une action $\varphi : G \times M \to M$ (différentiable, PL , ou topologique). On dit qu'une action $\psi: G \times W \to W$ étend φ , si $\psi|G \times M = \varphi$. Une extension ψ de φ est dite "libre", si l'action de G dans $W - M$ est libre. Dans cet article on étudiera le problème de la construction des extensions libres d'une action donnée, pour certains cas particuliers.

L. Jones a déjà étudié le cas particulier de l'action semi-libre d'un groupe cyclique $\mathbb{Z}/q\mathbb{Z}$ où $\varphi : \mathbb{Z}/q\mathbb{Z} \to M^k$ est l'action triviale et $W^n = D^n$ [J]. Au moyen de différentes méthodes, W. Browder et moi-même avons étudié les problèmes de l'existence et de la classification des extensions libres d'une action donnée pour un groupe fini G où $W^n = D^n$ (ou S^n) .

Dans les chapitres 1 et 2, on donnera les motivations et un résumé de quelques résultats de [A-B] qu'on utilisera par la suite. Le chapitre 3 contient les énoncés et les résumés des démonstrations des théorèmes qui généralisent quelques résultats [A-B] , dans le cas où W^n est une variété compacte connexe à bord avec $\pi_1(W^n) = \pi_1(\partial W^n) = 1$. P. Vogel et moi-même avons obtenu la caractérisation des extensions libres dans le cas des actions "simples", où $\pi_1 W^n$ et $\pi_1(\partial W^n)$ sont isomorphes. L'énoncé de ce résultat se trouve au chapitre 4. Les demonstration et les applications de ces résultats va paraît dan [A-B] et [A-V].

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§1. Soit D^n le disque de dimension n, et soit $U^n \subset D^n$ une sous-variété de codimension zéro, $C = D^n$-intérieur(U^n), $\partial_+ C = C \cap U = \partial_+ U$ et $\partial_- C =$ C-intérieur $(\partial_+ C)$. Supposons que $\varphi : G \times U^n \to U^n$ soit une action topologique dans U telle que la restriction $\varphi | G \times \partial_+ U$ soit libre. On veut étudier l'extension de φ à une action topologique $\overline{\varphi} : G \times D^n \to D^n$; c'est-à-dire trouver $\overline{\varphi}$ tel que $\overline{\varphi} | G \times U^n = \varphi$. Ce qui est équivalent à trouver une action $\psi : G \times C \to C$ telle que $\psi | G \times \partial_+ C = \varphi | G \times \partial_+ C$.

D'abord, on remarquera que pour qu'un tel ψ existe, il faut qu'il existe une action homotopique α qui étende $\varphi | G \times \partial_+ C$. C'est-à-dire, chaque application $\varphi(g) : \partial_+ C \to \partial_+ C$ s'étend à une application $\alpha(g) : C \to C$ qui est une équivalence d'homotopie, et telle que $\alpha(gg') \sim \alpha(g) \circ \alpha(g')$. Ainsi nous obtenons la première condition nécessaire qui est une condition homotopique et que l'on prendra comme point de départ.

La deuxième condition est homologique-algébrique et elle suit des résultats généraux de [A1] (voir aussi [OP] et [A2]). Il existe un groupe abélien fini $\Omega(F, F-\{1\})$ qui s'identifie avec un sous-groupe de $\widetilde{K}_o(\mathbb{Z}G)$ - où F est la famille des sous-groupes d'isotropie, ou également des stabilisateurs qui ont la propriété suivante : à chaque G-GW complexe U on associe un élément $\sigma(U) \in \Omega(F, F-\{1\})$ qui s'annule si et seulement s'il existe un G-CW complexe fini contractile X, avec U comme sous-complexe invariant et tel que X-U ait une G-action libre. Dans le cas particulier qui nous intéresse, i.e. $\overline{H}_*(U;\mathbb{Z}/q\mathbb{Z}) = 0$, $q = |G|$, on le calcule comme suit. Soit R la catégorie des G-modules cohomologiquement triviaux , et soit $K_o'(\mathbb{Z}G)$ le groupe de Grothendick de R . On définit une application $\psi : R \to K_o(\mathbb{Z}G)$ par $\psi(A) = [P_o]-[P_1]$ où la suite $0 \to P_1 \to P_o \to A \to 0$ est exacte, $A \in R$, et P_i sont $\mathbb{Z}G$-projectifs. Pour tout $A \in R$ une telle suite existe après un résultat de D.S. Rim [R] . Ψ donne un homomorphisme Ψ' de $K_o'(\mathbb{Z}G)$ dans $K_o(\mathbb{Z}G)$ qui est bien défini grâce au lemme de Schanuel [S] . En fait, Ψ' est un isomorphisme.

Alors, $H_i(U)$ a une structure de G-module induite par φ , et qui est cohomologiquement triviale (pour tout i 0) comme un G-module , car $\widetilde{H}_*(U;\mathbb{Z}/q\mathbb{Z}) = 0$, $q = |G|$. Ainsi, on calcule que

$$\sigma(U) = \sum_{i>0} (-1)^i \Psi(H_i(U))$$

(voir [A1], page 49) . On peut donc prouver directement que l'élément

$$\sum_{i>0} (-1)^i \Psi(H_i(U)) \in \widetilde{K}_o(\mathbb{Z}G)$$

s'annule, s'il existe une action cellulaire libre $\psi : G \times C \to C$ telle que $\psi|G \times \partial_+ C = \varphi|G \times \partial_+ C$.

Il faut seulement observer que le complexe des chaînes $C_*(C)$ est $\mathbb{Z}G$-libre , et alors

$$\begin{aligned} 0 = \sum_{i>0}(-1)^i \Psi(C_i(C)) &= \Sigma(-1)^i \Psi(H_i(C)) \\ &= \Sigma(-1)^{i+1} \Psi(H_{i+1}(D^n, C)) = \sum_{i>0} (-1)^i \Psi(H_i(U)) . \end{aligned}$$

La deuxième condition nécessaire est que

$$\sigma(U) = \Sigma (-1)^i \psi(H_i(U) \in \widetilde{K}_o(\mathbb{Z}G)$$

s'annule.

Plus tard, on trouvera que $\sigma(U)$ sera calculé comme l'obstruction de finitude de Wall pour réaliser topologiquement l'action homotopique de G dans C (ou dans X qui est homotopiquement équivalent à C comme ci-dessus) par un complexe G-libre fini.

§2. Soit X un espace topologique, et $H(X)$ le monoïde des équivalences d'homotopie de X. Une action homotopique dans X est un homomorphisme de groupes $\xi : G \to \pi_0(H(X)) = E(X)$. On dit que deux actions homotopiques (X_1,ξ_1) et (X_2,ξ_2) sont équivalentes, s'il existe une équivalence d'homotopie $f : X_1 \to X_2$ telle que le diagramme ci-dessous commute :

$$\begin{tikzcd} & E(X_1) \arrow[d, "f\#"] \\ G \arrow[ur, dashed, "\xi"] \arrow[r, "\xi_2"'] & E(X_2) \end{tikzcd}$$

où f est l'application évidente induite par f.

Evidemment chaque action topologique induit une action homotopique, et l'action homotopique (X,ξ) est dite munie d'une réalisation topologique si (X,ξ) est équivalent à une action homotopique induite par une action topologique*).

Soient $\xi : G \to E(X)$ et $\xi' : G \to E(Y)$ deux actions homotopiques. Notons par $\alpha(g)$ et $\beta(g)$ des représentants de $\xi(g)$ et $\xi'(g)$ respectivement. On dit qu'une application $f : X \to Y$ est homotopie-équivariante (ou h-équivariante) si $\beta(g) \circ f \sim f \circ \alpha(g)$.

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*) Le problème de formuler une théorie d'obstruction formelle pour réaliser des actions homotopiques par des actions topologiques a été étudié par G. Cooke, mais nous ne l'utiliserons pas.

On généralise les notions ci-dessus aux cas relatifs des actions homotopiques dans une paire ou un diagramme des espaces comme suit.

Dans les situations qui nous intéressent, on peut réaliser topologiquement une action homotopique ξ dans une paire (X,Y) où la restriction de ξ sur Y est déjà une action topologique donnée. On définit les notions relatives comme suit.

Soient $H(X,Y)$ le monoïde des équivalences d'homotopie de la paire (X,Y) et $E(X,Y) = \pi_o(H(X,Y))$. Une action homotopique relative est donnée par un homomorphisme $G \to E(X,Y)$. Egalement, pour un diagramme Δ obtenu par l'inclusion $(X,X_o) \subset (Y,Y_o)$

$$(\Delta)\quad \begin{array}{ccc} X_o & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y_o & \longrightarrow & Y \end{array}$$

On définit la notion d'une action homotopique par un homomorphisme $G \to E(\Delta)$ où $H(\Delta)$ est le monoïde des équivalences d'homotopie de Δ, et $E(\Delta) = \pi_o(H(\Delta))$. On a les restrictions $E(\Delta) \to E(X_o,Y_o)$ et $E(X,Y) \to E(Y)$. Donc, une action homotopique dans Δ donne une action homotopique dans (X_o,Y_o) etc. telle que les applications d'inclusion soient h-équivariantes.

<u>Définition</u>. Soient $\xi_i : G \to E(X_i)$ $i = 1,2$ deux actions homotopiques. Soit $\alpha_i(g) : X_i \to X_i$ une représentation de $\xi_i(g)$ pour chaque $g \in G$. On dit que ξ_i est une "extension" de ξ_2 si $X_2 \subset X_1$ et les $\alpha_1(g)$ sont des extensions de $\alpha_2(g)$. On dit que ξ_1 est une "extension extérieure" si on peut choisir une extension $\alpha_1(g)$ de $\alpha_2(g)$ telle que

$$\alpha_1(g)(Y-X) \subset Y-X .$$

Pour deux paires $(Y_2,X_2) \subset (Y_1,X_1)$ ou deux diagrammes équivalents, on définit les notions "d'extension" et "d'extension extérieure" de la même façon que ci-dessus.

<u>Définition</u>. Soient $\varphi : G \times X \to X$ et $\psi : G \times Y \to Y$ des actions topologiques et $X \subset Y$. On dit que ψ est une "extension libre" de φ, si $\psi | G \times X = \varphi$ et si l'action ψ sur $Y-X$ est libre.

Les théorèmes suivants nous permettent de construire des réalisations topologiques et des actions homotopiques relatives.

<u>2.1. Théorème</u>. Soit $\xi : G \to E(X)$ une action homotopique, et $\varphi : G \times Y \to Y$ une action topologique libre. Supposons que X et Y soient simplement connexes, et qu'il existe une application h-équivariante $f : Y \to X$ qui induise un isomorphisme en homologie à coefficient dans $\mathbb{Z}/q\mathbb{Z}$ (où $q = |G|$). Alors, il existe une action topologique libre $\psi : G \times X' \to X'$, une équivalence d'homotopie h-équivariante $h : X' \to X$ et une application G-équivariante $f' : Y \to X'$ telle que $h \circ f' \sim f$, et que $h \circ \psi = \xi$.

<u>Résumé de la démonstration</u>. On trouvera par récurrence les invariants de Postnikov d'un espace $\overline{X}$ tel que $\overline{X}$ contienne l'espace quotient Y/G (que l'on désignera par $\overline{Y}$) et tel que l'homomorphisme induit par inclusion $\overline{Y} \to \overline{X}$ soit un isomorphisme

$$\pi_1(\overline{Y}) \simeq \pi_1(\overline{X}) \simeq G .$$

De plus, il faut que le revêtement universel de $\overline{X}$ muni de l'action naturelle de $\pi_1(\overline{X})$, soit X' qui aurait les propriétées énoncées au théorème 2.1.

Supposons que $\overline{X}_n$ soit la n-ème étape de la décomposition de Postnikov de $\overline{X}$ et tel que le diagramme ci-dessous soit commutatif à homotopie près, où les applications sont induites de façon évidente ou bien construites par récurrence. Prenons

$$BG = K(G,1) = K(\pi_1(\bar{Y}),1) = K(\pi_1(X),1)$$

comme la première étape de la décomposition de Postnikov de $\bar{Y}$ et $\bar{X}$, où $\bar{Y} = Y/G$.

$$\begin{tikzcd}
 & X_n & \\
Y_n \arrow[ur, "f_n"] \arrow[rr] \arrow[d, "p"] & & X'_n \arrow[ul, "h_n"'] \arrow[d, "p"] \\
\bar{Y}_n \arrow[rr, "\bar{f}_n"] \arrow[d] & & \bar{X}_n \arrow[d] \\
BG \arrow[rr] & & BG
\end{tikzcd}$$

Il est plus commode de supposer que $\bar{Y}_n \subset \bar{X}_n$, et d'identifier X'_n et X_n par abus de notation. Posons les notations suivantes :

$$j : \pi_*(Y) \to \pi_*(X) ,$$

$$j_* : H^*(Y_n;\pi_*(Y)) \to H^*(Y_n;\pi_*(X)) ,$$

$$f_n^* : H^*(X_n;\pi_*(X)) \to H^*(Y_n;\pi_*(X)) ,$$

$$p^* : H^*(\bar{X}_n;\pi_*(X)) \to H^*(X_n;\pi_*(X)) ,$$

$k(X)$, $k(\bar{Y})$ etc. sont les invariants de Postnikov, comme $k(Y) \in H^{n+2}(Y_n;\pi_{n+1}(Y))$ et $k(X) \in H^{n+2}(X_n;\pi_{n+1}(X))$ etc. En général, la cohomologie est à coefficient dans un système local et les invariants de Postnikov sont tordus.

Nous trouverons $k \in H^{n+2}(\bar{X}_n;\pi_{n+1}(X))$ tel que $\bar{f}_n^*(k) = j^*(k(\bar{Y}))$ et $p^*(k) = k(X)$, pour construire la $(n+1)$-ème étape de Postnikov et obtenir un diagramme comme ci-dessus qui terminera (l'étape) la récurrence. On utilisera les diagrammes ci dessous.

(Diagramme I).

$$\begin{tikzcd}
{} \arrow[r] & H^{n+2}(X_n,Y_n) \arrow[r] & H^{n+2}(X_n) \arrow[r] & H^{n+2}(Y_n) \arrow[r] & \ldots \\
{} \arrow[r] & H^{n+2}(\bar{X}_n,\bar{Y}_n) \arrow[r] \arrow[u] & H^{n+2}(\bar{X}_n) \arrow[r] \arrow[u] & H^{n+2}(\bar{Y}_n) \arrow[r] \arrow[u] & \ldots
\end{tikzcd}$$

où la cohomologie est à coefficients dans les systèmes locaux induits par $\{\pi_{n+1}(Y)\}$. Pour le diagramme parallèle avec cohomologie à coefficients dans les systèmes induits par le système $\{\pi_{n+1}(X)\}$, on obtient la factorisation par les éléments invariants et par l'action de G :

$$\begin{tikzcd}
\ldots \arrow[r] & H^{n+2}(X_n,Y_n) \arrow[r] & H^{n+2}(X_n) \arrow[r] & H^{n+2}(Y_n) \arrow[r, "\delta"] & H^{n+3}(X_n,Y_n) \arrow[r] & {} \\
\ldots \arrow[r] & H^{n+2}(X'_n,Y_n)^G \arrow[r] \arrow[u] \arrow[d, shift right, "t"'] & H^{n+2}(X'_n)^G \arrow[r] \arrow[u] & H^{n+2}(Y_n)^G \arrow[r, "\delta"] \arrow[u] & H^{n+3}(X'_n,Y_n)^G \arrow[r] \arrow[u] \arrow[d, shift right, "t"'] & {} \\
\ldots \arrow[r] & H^{n+2}(\bar{X}_n,\bar{Y}_n) \arrow[r] \arrow[u, shift right, "\simeq"'] & H^{n+2}(\bar{X}_n) \arrow[r] \arrow[u] & H^{n+2}(\bar{Y}_n) \arrow[r] \arrow[u] & H^{n+3}(\bar{X}_n,\bar{Y}_n) \arrow[r] \arrow[u, shift right, "\simeq"'] & {}
\end{tikzcd}$$

où $t : H^*(X'_n,Y_n)^G \to H^*(\bar{X}_n,\bar{Y}_n)$ est l'application transferts; c'est un isomorphisme d'après la suite spectrale de Cartan-Leray, grâce à l'hypothèse que $\pi_*(f) \otimes (\mathbb{Z}/|G|\mathbb{Z}) = 0$. On obtient aussi des homomorphismes entre les deux diagrammes induits par $j : \pi_{n+1}(Y) \to \pi_{n+1}(X)$. L'observation importante est que les invariants de Postnikov $k(X)$ sont invariants par l'action de $G \simeq \pi_1(\bar{X}_n)$. (Voir Baues [B] ou McClendon [Mc]). On exclura le diagramme obtenu à partir des diagrammes ci-dessus et les homomorphismes entre eux. On en déduira qu'un tel élément k existe. □

2.II. Théorème. Supposons que $\xi : G \to \mathcal{E}(X)$ est une action homotopique, $\varphi : G \to \mathrm{Homeo}\,(Y)$ est une action topologique libre, et $f : X \to Y$ est une application h-équivariante telle que

$$f_* : H_*(X;\mathbb{Z}/q\mathbb{Z}) \to H_*(Y;\mathbb{Z}/q\mathbb{Z})$$

soit un isomorphisme, (où $q = |G|$) et $\pi_1(X) = \pi_1(Y) = \pi_2(f) = 0$. Alors, il existe une action topologique ψ dans un espace X' et une application équivariante $f' : X' \to Y$, et une équivalence d'homotopie $h : X' \to X$ homotopiquement équivariante telle que $f \circ h \sim f'$.

La méthode de cette démonstration ressemble à celle du Théorème 2.1. On remarquera qu'il est plus commode de trouver d'abord les invariants de Postnikov localisés en q , puis hors de q , et ensuite les invariants rationnels. En utilisant le diagramme Cartésien :

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}[\frac{1}{q}] \arrow[d] \\
\mathbb{Z}(q) \arrow[r] & \mathbb{Q}
\end{tikzcd}$$

On trouvera les invariants désirés.

2. III. Proposition. Dans la situation du Théorème II, supposons que $A \subset X$ possède déjà une action topologique telle que l'action induite par G sur $H^*(A)$ soit triviale, et telle que l'inclusion $A \to X$ soit h-équivariante. Si $f|A$ est équivariante, alors, on peut construire X' qui contient A comme sous-espace invariant tels que $f'|A$ et $f|A$ soient G-homotopes.

Résumé de la démonstration: Soit F le fibre homotopique de $f' : X' \to Y$. Alors, dans le diagramme suivant, on a la section σ et on cherchera une section $\bar{\sigma}$:

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "="'] & X' \arrow[rr, "f'"] \arrow[d] & & Y \arrow[d] \\
 & & A \arrow[ul, "\sigma"'] \arrow[ur, "f|A"'] \arrow[d] & \\
F \arrow[r] & X'/G \arrow[rr] & & Y/G \\
 & & A/G \arrow[ul, dashed, "\bar{\sigma}"] \arrow[ur] &
\end{tikzcd}$$

Car $\pi_*(F) \otimes \mathbb{Z}/q\mathbb{Z} = 0$, $q = |G|$, on montrera que les obstructions successives $\omega(\bar{\sigma}) \in H^{n+1}(A/G;\pi_n(F)$ s'annulent lorsque l'on peut prendre successivement $\bar{\sigma}$ telle qu'elle soit compatible avec σ sur le n-skelette de A/G et de A .

<u>3.1. Théorème</u>. Soit $U^n \subset W^n$ une sous-variété de codimension zéro, et posons $C = W^n - \mathrm{int}(U^n)$, $\partial_+ U = C \cap U$, $\partial_+ C = C \cap \partial W$. Supposons que

(a) $\varphi : G \times U \to U$ soit une application topologique telle que $\varphi | G \times \partial_+ U$ est libre, et l'action induite par G dans $H_*(\partial(\partial_+ U))$ est triviale.

(b) $\pi_1(\partial_+ C) = \pi_1(C) = 0$ et $H_*(W,Y;\mathbb{Z}/q\mathbb{Z}) = 0$ où $q = |G|$.

Alors, les conditions suivantes sont nécessaires et suffisantes pour qu'il existe une variété compacte $(W_o^n, \partial W_o)$ équivalente à $(W,\partial W)$ à homotopie tangentielle près et munie d'une action topologique $\psi : G \times W_o \to W_o$ telle que W_o contienne U^n comme sous-variété G-invariante, et ψ soit une extension libre de φ ; (on a les même résultats pour les catégories différentiables et PL) :

(1) Il existe une action homotopique $\xi : G \to E(W,\partial W)$ qui est une extension extérieure de φ .

(2) L'invariant $\sum_i (-1)^i \Psi(H_i(W,U)) \in K_o(\mathbb{Z}G)$ s'annule.

<u>Démonstration</u>. On montrera que les conditions (1) et (2) sont suffisantes. Puisque $H_*(C,\partial U) \simeq H_*(W,U)$ par excision, et l'inclusion $\partial_+ U \to C$ est une équivalence d'homologie modulo q , on peut utiliser le Théorème 2.1 pour réaliser topologiquement l'action homotopique $\xi(C) : G \to E(G)$ (induite par restriction de ξ dans C). Donc, il existe un espace C' muni d'une action topologique libre, tel que $\partial_+ U$ est un sous-complexe G-invariant, et tel qu'il existe une équivalence d'homotopie h-équivariante $f : C \to C'$. D'autre part, on remplacera C par C' (à homotopie près) et donc "l'inclusion" $\partial_+ C \to C'$ est une équivalence d'homologie modulo q car

$$H_*(C',\partial_+C) \simeq H_*(C,\partial_+C) \simeq H_{n-*}(C,\partial_+U)$$

d'après le théorème de dualité de Poincaré-Lefschetz. Donc, l'action homotopique (induite par ξ) de G dans ∂_+C est aussi topologiquement réalisable en utilisant le Théorème 2.II. Alors, il existe un espace topologique $(\partial_+C)'$ muni d'une action topologique libre et des applications h-équivariantes $g' : (\partial_+C)' \to C'$ et $f' : \partial_+C \to (\partial_+C)'$ où f' est une équivalence d'homopie et "l'inclusion" $\partial_+C \to C'$ et $g' \circ f'$ sont homotopiques. D'après la Proposition 2.III, on peut supposer que $\partial(\partial_+C)$ est un G-sous-complexe de $(\partial_+C)'$. Donc $(C',(\partial_+C)' \cup \partial_+U)$ est une paire de complexes de Poincaré équivalente à (C,∂_+C) à homotopie près, et elle possède une action topologique libre qui réalise topologiquement ξ .

Soit (X,Y) l'espace quotient $(C'/G,((\partial_+C)' \cup \partial_+U)/G)$. Il suit d'un théorème de Bieri-Eckmann (voir aussi Vogel [V] et Browder [Br]) que Y et X sont dominés par des complexes finis. On calcule que dans les deux cas, l'obstruction de finitude de Wall s'annule puisqu'elle est égale à

$$\sum_i (-1)^i \bar{\Psi}(H_i(C,\partial_+U)) = \sum_i (-1)^i \Psi(H_i(W,U)) = 0$$

par l'hypothèse (2). Alors on peut supposer que (X,Y) est une paire de complexes finis, et qui donne une paire de Poincaré finie dans le sens de Wall [W].

On a la fibration sphérique de Spivak γ sur (X,Y) qui se restreint au fibré normal sphérique (stable) de ∂_+U/G . Donc, il reste à trouver un relèvement d'application classifiant de γ , disons f_γ dans le diagramme suivant.

$$\begin{tikzcd} & BO \arrow[d, "\pi"] \\ X \arrow[ur, dashed] \arrow[r, "f_\gamma"] & BF \end{tikzcd}$$

où BF est l'espace classifiant de Stasheff pour les fibrations sphériques stables, et $BO = \varinjlim_n BO(n)$. D'après le théorème de Boardman-Vogt, F/O (le fibré de π) est un espace Ω^∞ et donne une théorie de cohomologie généralisée h . Alors, l'obstruction de relèvement de f_γ est un élément $\omega \in h^*(X)$, où $h^*(X)$ est de type fini. Comme l'inclusion $\partial_+ U \to C$ est une équivalence de cohomologie modulo q , il vient

$$h^*(\partial_+/G) \otimes \mathbb{Z}/q\mathbb{Z} \simeq h^*(X) \otimes \mathbb{Z}/q\mathbb{Z} .$$

Alors ω est un élément de torsion d'ordre r , où $(r,q) = 1$. Le transfert $t : h^*(X) \otimes \mathbb{Z}/r\mathbb{Z} \to h^*(C') \otimes \mathbb{Z}/r\mathbb{Z}$ est un monomorphisme, et $t(\omega) = 0$ puisque C' est équivalent à C à homotopie près, et C est une variété différentiable. Donc, $\omega = 0$ et il existe un relèvement de f_γ dans BO , qui donne une application normale de degré 1 (rel. $\partial_+ C/G$)

$$\beta : (V_o^n, \partial V_o^n) \to (X,Y) \quad .$$

Mais $$\pi_1(Y - \partial_+ U/G) \longrightarrow \pi_1(X) \simeq G ,$$

donc l'obstruction de chirurgie de β s'annule. On en déduit qu'il existe une variété $(V_1^n, \partial V_1{}^n)$ équivalente à (X,Y) à homotopie près, telle que

$$\partial_+ U/G \subset \partial V_1 .$$

Maintenant le revêtement universel $(\widetilde{V}_1, \partial\widetilde{V}_1)$ possède une action libre de G qui étend l'action donnée de G dans $\partial_+ U$. On montre aisément que la variété $W_o = U \cup_{\partial_+ U} \widetilde{V}_1$ possède les propriétés requises. Les cas PL et topologiques ressemblent au cas précédent. Les conditions (1) et (2) sont aussi nécessaires ce que l'on vérifie facilement. □

D'après la théorie de la chirurgie, il est possible de choisir W_o(PL) homéomorphe ou difféomorphe à W . Au lieu de poursuivre dans cette direction, nous remarquerons que le théorème suivant est un corollaire du

Théorème 3.I. Selon ce théorème, les conditions nécessaires et suffisantes pour l'existence d'une extension libre d'une action donnée φ dans une sous-variété M de W sont : d'abord l'existence d'une extension linéaire de φ dans un voisinage tubulaire de M dans W (qui est une condition en K_G-théorie), et l'existence d'une extension homotopique de φ à W (condition vérifiable par la théorie de l'obstruction) et enfin la nullité d'une obstruction projective dans $\widetilde{K}_o(\mathbb{Z}G)$.

3.II. Théorème. Soit $(W^n,\partial W^n)$ une variété différentiable compacte telle que

$$\pi_1(\partial W) = \pi_1(W) = 0 \quad \text{et} \quad H_i(W) = 0 \quad , \ i \geq \frac{n}{2} \quad , \ n \geq 6 \ .$$

Supposons que $(M^k,\partial M) \subset (W,\partial W)$ soit une sous-variété de codimension $n-k > 2$ avec le fibré normal ν . Supposons que $\varphi : G\times M \to M$ soit une action différentiable, telle que l'action de G dans $H_*(\partial W)$ soit triviale, et $H_*(W,M;\mathbb{Z}/q\mathbb{Z}) = 0$, où $q = |G|$. Alors, il existe une action différentiable $\varphi : G\times W \to W$ qui est une extension libre de φ , si et seulement si :

(1) Il existe une action homotopique $\xi : G \to E(W,\partial W)$ telle que ξ est une extension extérieure de φ , et

(2) ν est muni d'une structure de G-fibré vectorielle (M,φ) , telle que l'action induite sur le fibré sphérique $S(\nu)$ soit libre et

(3) L'invariant $\sum_i(-1)^i\overline{\Psi}(H_i(W,M) \in \widetilde{K}_o(\mathbb{Z}G)$ s'annule.

4. Soit A un groupe abélien fini tel que $(|A|,|G|) = 1$. On peut considérer A comme G-module avec l'action triviale. Alors, $\overline{\Psi}(A) \in \widetilde{K}_o(\mathbb{Z}G)$ est bien défini comme toujours (voir §1). On vérifie que $\Psi(A)$ ne dépend pour sa valeur que de l'ordre de A comme unité dans le groupe multiplicatif $(\mathbb{Z}/|G|\mathbb{Z})^*$. Alors, on définit $\sigma_G : (\mathbb{Z}/|G|\mathbb{Z})^* \longrightarrow \widetilde{K}_o(\mathbb{Z}G)$ par $\sigma_G(r) = \Psi(\mathbb{Z}/r\mathbb{Z})$. Par abus de notation, on notera $\sigma_G(A)$ au lieu de $\Psi(A)$

pour spécifier une telle situation particulière. σ_G s'appelle l'homomorphisme de Swan (voir [S]) , σ_G est trivial pour les groupes cycliques, mais en général, il n'est pas nul ; par exemple, $\mathrm{Im}(\sigma_G) \simeq \mathbb{Z}/2\mathbb{Z}$ pour le groupe de quaternions d'ordre 8.

Par la suite, on considère que la localisation de X est la localisation au-desssus de $\pi_1(X)$, c'est-à-dire, on localise le revêtement universel de façon équivariante par rapport à l'action de $\pi_1(X)$. C'est possible, si on utilise une méthode de localisation assez fonctorielle, par exemple , la localisation de Bousfield-Kan [B-S].

<u>Définition</u>. Soit $\varphi : G \times X \to X$ une action topologique; φ est dite action "simple" si $E_G \times_G X \sim B_G \times X$, où $E_G \to BG$ est le fibré principal universel pour G , et $E_G \times_G X \to BG$ est le fibré associé à la fibre X . On dit que φ est (q)-simple si $((E_G \times_G X)_{(q)} \sim BG \times X_{(q)}$.

<u>Exemple</u>. Toute action $\varphi : G \times X \to X$ est (q)-simple si X est acyclique modulo q .

4.1. <u>Théorème</u>. Soit $(W^n, \partial W^n)$ une variété différentiable compacte connexe telle que $\pi_1(W) = \pi_1(\partial W)$ soient nilpotents, $n \geq 6$. Soit $(F^k, \partial F^k) \subset (W, \partial W)$ une sous-varieté à fibré normale ν, de codimension $n-k > 2$. Alors, il existe une action différentiable semi-libre (q)-simple $\varphi : G \times W' \to W'$ telle que $W'^G = F$ et $W' \simeq W$ si et seulement si

(1) $H_*(W,F;\mathbb{Z}/q\mathbb{Z}) = 0$, $q = |G|$.

(2) $\sum_i (-1)^i \sigma_G(H_i(W,F)) \in \widetilde{K}_o(\mathbb{Z}G)$ s'annule.

(3) ν possède une structure de G-fibré vectoriel avec une G-représentation libre dans le fibré.

La démonstration de ce théorème se trouve dans [A-V].

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Die Picard-Gruppe des Burnside-Ringes

Tammo tom Dieck

Die geometrische Bedeutung der Picard-Gruppe Pic A(G) des Burnside-Ringes A(G) einer endlichen Gruppe G wurde in den Arbeiten tom Dieck - Petrie [1978], [1982] dargelegt. Es sei kurz daran erinnert: Für zwei G-CW-Komplexe X und Y, deren sämtliche Fixpunktmengen X^H und Y^H n(H)-dimensionale, zur Sphäre $S^{n(H)}$ homotopie-äquivalente Komplexe sind, ist die äquivariante stabile Homotopiemenge $\omega(X,Y)$ ein projektiver Modul vom Rang eins über A(G), repräsentiert also ein Element in der Picard-Gruppe Pic A(G) solcher projektiven Moduln mit dem Tensorprodukt als Verknüpfung. Der Modul $\omega(X,Y)$ ist vollständig durch die Angabe einer stabilen Abbildung $f : X \longrightarrow Y$ mit zur Gruppenordnung $|G|$ teilerfremden Abbildungsgraden d(H) von $f^H : X^H \longrightarrow Y^H$ bestimmt. Jede Funktion $d : \emptyset(G) \longrightarrow \mathbb{Z}$ von der Menge $\emptyset(G)$ der Konjugationsklassen (H) von Untergruppen H in die ganzen Zahlen mit zu $|G|$ teilerfremden Werten läßt sich durch die Grade einer geeigneten Abbildung $f : X \longrightarrow Y$ realisieren (tom Dieck - Petrie [1982], Theorem 6.5). Die Picard-Gruppe Pic A(G) ist die algebraische Lösung des Problems, wann zwei solche Funktionen $d_1, d_2 : \emptyset(G) \longrightarrow \mathbb{Z}$ in demselben Modul $\omega(X,Y)$ vorkommen. Eine explizite Berechnung der Picard-Gruppe liefert demnach Abbildungsgradinvarianten für Abbildungen $f : X \longrightarrow Y$, die nur vom stabilen Homotopietyp von X und Y abhängig sind.

In dieser Arbeit beschreibe ich eine Berechnungsmethode für Pic A(G). Es stellt sich heraus, daß dieses durch gewisse multiplikative Kongruenzen für aus dem Burnside-Ring abgeleitete Einheitengruppen geschehen kann. Grundlage für die Gewinnung dieser Kongruenzen ist die Arbeit tom Dieck

[1978], wo der Zusammenhang zur Picard-Gruppe des rationalen Darstellungsringes hergestellt wurde.

Um nicht in das Problem der Einheiten des Burnside-Ringes verwickelt zu werden, betrachte ich eine orientierbare Version von Pic A(G), die Gruppe Inv A(G) der invertierbaren A(G)-Moduln. Es gibt eine exakte Sequenz (tom Dieck - Petrie [1978], (33))

(O.1) $$1 \longrightarrow A(G)^{*} \longrightarrow C(G)^{*} \longrightarrow \mathrm{Inv}\, A(G) \longrightarrow \mathrm{Pic}\, A(G) \longrightarrow 1.$$

Darin bezeichnet generell S^{*} die Einheiten gruppe eines Ringes S und $C(G) = C(\emptyset(G), \mathbb{Z})$ ist der Ring der Funktionen $\emptyset(G) \longrightarrow \mathbb{Z}$.

Qualitativ erhält man als Ergebnis:

(O.2) <u>Die Gruppe</u> Inv A(G) <u>ist in bestimmter Weise isomorph zum Produkt</u>.

(O.3) $$\prod_{(H) \in \emptyset(G)} (\mathbb{Z}/|WH|\,\mathbb{Z})^{*} .$$

Wie üblich ist WH = NH/H und NH der Normalisator von H in G.

Was man unter "in bestimmter Weise" zu verstehen hat, ist nicht so kurz mitzuteilen, aber gerade das Ziel der Arbeit. Für abelsche G wurde Inv A(G) in tom Dieck [1978] berechnet.

Im nächsten Abschnitt reduzieren wir das Problem. Danach geben wir die Kongruenzen für p-Gruppen $p \neq 2$ an. Der dritte Abschnitt behandelt die 2-Gruppen.

1. Die Ordnung von Inv A(G).

Wir zeigen zunächst, daß Inv A(G) dieselbe Ordnung wie die in (0.3) angegebene Gruppe hat. Danach können wir erläutern, worum es bei der Gewinnung von Kongruenzen für Inv A(G) geht.

Sei generell C ein endliches direktes Produkt von Ringen $\mathbb{Z}$ und $A \subset C$ ein Unterring , der als abelsche Gruppe maximalen Rang in C hat. Der Quotient C/A ist dann endlich und werde etwa durch $c > 0$ annuliert. Es sei Inv A die Gruppe der invertierbaren A-Untermoduln von C: Das Produkt MN zweier solcher Moduln werde wie üblich von allen $m\,n$, $m \in M$, $n \in N$ erzeugt und M heißt <u>invertierbar</u>, wenn es ein N mit MN = A gibt. Das kanonische Diagramm

$$
(1.1) \qquad \begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ A/c\,C & \longrightarrow & C/c\,C \end{array}
$$

ist ein Pullback und man erhält eine exakte Sequenz (tom Dieck [1979], 10.3.11)

$$
(1.2) \qquad 1 \longrightarrow (A/cC)^{*} \longrightarrow (C/cC)^{*} \longrightarrow \mathrm{Inv}\ A \longrightarrow 1.
$$

Der Burnside-Ring A = A(G) ist in C = C(G) enthalten: Jedem $x \in A(G)$ wird die Funktion $(H) \longmapsto |x^H|$ zugeordnet (tom Dieck [1979], 1.2.2). Ist $F \subset \emptyset(G)$ eine Menge, die mit jedem (H) auch alle (K), $H \subset K$, enthält, so betrachten wir den Quotienten $A(G) \longrightarrow A(G;F)$, das Bild von A(G) in $C(F, \mathbb{Z})$ bei der Restriktion $C(\emptyset(G), \mathbb{Z}) \longrightarrow C(F, \mathbb{Z})$ von Funktionen. Seien $F \subset F'$ (H)-benachbart, d. h. $F' \setminus F$ bestehe aus einer Konjugationsklasse (H). Man hat dann die Projektion $p : A(G,F') \longrightarrow A(G,F)$.

Genauer gilt: Es gibt ein Pullback von Ringen

$$
(1.3) \qquad
\begin{array}{ccc}
A(G,F') & \xrightarrow{\ p\ } & A(G,F) \\
{\scriptstyle d_H}\big\downarrow & & \big\downarrow{\scriptstyle s} \\
\mathbb{Z} & \xrightarrow{\ r\ } & \mathbb{Z}/|WH|
\end{array}
$$

<u>Beweis</u>. Es ist r die Reduktion modulo $|WH|$. Es ist d_H die Abbildung $x \longmapsto |x^H|$. Eine Funktion $d \in C(G)$ liegt genau dann in $A(G)$, wenn gewisse Kongruenzen

$$d(H) \equiv - \sum_K n(H,K)\; d(K) \quad \mod |WH|$$

erfüllt sind (tom Dieck [1979], 1.3.5). Man nehme als s die Abbildung $d \longmapsto - \sum n(H,K)d(K) \mod |WH|$.

Ein Pullback wie (1.3) führt zu einer exakten Folge von Inv-Gruppen, die im wesentlichen die Mayer - Vietoris - Folge für die Picard-Gruppe (Bass [1968], IX. 5.3) ist. Wir erhalten deshalb als Folgerung:

(1.4) Das Diagramm (1.3) führt zu einer exakten Sequenz

$$1 \longrightarrow \mathbb{Z}/|WH|^* \longrightarrow \mathrm{Inv}\ A(G,F') \longrightarrow \mathrm{Inv}\ A(G,F) \longrightarrow 1.$$

Durch Induktion über die Mengen F erhält man also

(1.5) <u>Lemma</u>. Inv A(G) <u>hat die Ordnung von</u>

$$\prod_{(H) \in \phi(G)} \mathbb{Z}/|WH|^* \quad .$$

Sei W_pH die p-Sylow-Gruppe von WH. Wir verwenden die folgende Sprechweise.

(1.6) <u>Definition</u>. Ein <u>System von (primären) Kongruenzen</u> für Inv A(G) ist ein System von surjektiven Homomorphismen

$$m(H,p) : C/|G|\, C^{*} \longrightarrow \mathbb{Z}/|W_pH|^{*},$$

die $A(G)/|G|\, C^{*}$ im Kern enthalten ((H) $\in \emptyset(G)$, p Teiler von $|WH|$.)

Wegen (1.2) und (1.5) liefert ein System m(H,p) von Kongruenzen einen Isomorphismus

$$\mathrm{Inv}(G) \cong \prod \mathbb{Z}/|W_pH|^{*},$$

das Produkt über alle (H) $\in \emptyset(G)$ und alle p gebildet. Es kommt also für die Präzisierung von (O.2) darauf an, ein geeignetes System von m(H,p) zu finden.

Ein m(H,p) hat <u>Standard-Form</u>, wenn es sich als ein Produkt der Evaluationen bei K, $d_K : C \longrightarrow \mathbb{Z}$, für $K \geq H$ schreiben läßt. Wir werden sehen, daß für ungerade p immer Standard-Form möglich ist, während für p = 2 gewisse Ausnahmegruppen auftreten.

2. Kongruenzen für ungerade Primzahlen.

Sei also G eine p-Gruppe und p ungerade. Wir verwenden eine p-adische Beschreibung von Inv A(G). Die Sequenz (1.2) gilt für jedes Vielfache c von G, falls A = A(G) und C = C(G) ist. Verwenden wir $c = |G|^n$ und gehen zum Limes $n \longrightarrow \infty$ über, so erhalten wir eine exakte Folge

$$(2.1) \qquad 1 \longrightarrow A_p^* \longrightarrow C_p^* \longrightarrow \mathrm{Inv}\, A \longrightarrow 1,$$

in welcher der Index p die p-adische Komplettierung bedeutet. Die Evaluation bei H, $d_H : C/cC^* \longrightarrow \mathbb{Z}/c^*$ liefert eine Evaluation $d_H : C_p^* \longrightarrow \mathbb{Z}_p^*$ in die p-adischen Einheiten.

Sei R(G,Q) der rationale Darstellungsring von G. Indem wir jeder endlichen G-Menge die zugehörige Permuationsdarstellung zuordnen, erhalten wir einen Homomorphismus $A(G) \longrightarrow R(G,Q)$ und induzierte Abbildungen

$$A(G)_p \longrightarrow R(G,Q)_p, \quad \pi : A(G)_p^* \longrightarrow R(G,Q)_p^* .$$

Mit der Augmentation $a : R(G,Q) \longrightarrow \mathbb{Z}$, $x \longmapsto \dim x$ erhalten wir eine induzierte Abbildung $a_p^* : R(G,Q)_p^* \longrightarrow \mathbb{Z}_p^*$, deren Kern die Gruppe $1 + I(G,Q)_p$ ist, mit der Komplettierung $I(G,Q)_p$ des Augmentationsideals $I(G,Q) = \text{Kern } a$. Der Charakter $\pi(x)$ hat an der Stelle g den Wert $d_{\langle g\rangle}(x)$, mit der von g erzeugten Untergruppe $\langle g \rangle$. Multiplikative Kongruenzen für die Charakterwerte $\pi(x)(g)$ liefern deshalb entsprechende Relationen für die $d_{\langle g\rangle}(x)$.

Da p ungerade ist, haben wir nach Atiyah - Tall [1969] einen Isomorphismus

$$(2.2) \qquad \rho = \rho_k : I(G,Q)_p \longrightarrow 1 + I(G,Q)_p.$$

Mittels Charakteren läßt sich ϱ_k so schreiben:

$$(2.3) \qquad \varrho_k(x) = k^{\dim x^g} .$$

(Bemerkung zu dieser Schreibweise: Für $x \in I(G,Q)$ ist $\dim x^g \equiv 0 \bmod (p-1)$. Die Funktion $(p-1)\mathbb{Z} \longrightarrow 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p$, $x \longmapsto k^x$ ist p-adisch stetig und liefert deshalb $\mathbb{Z}_p \longrightarrow 1 + p\mathbb{Z}_p$, $x \longmapsto x^k$. Diese Funktion wird in (2.3) verwendet.)

Wegen des Isomorphismus (2.2) suchen wir zunächst nach geeigneten additiven Konguenzen für Charaktere. Es gilt

(2.4) Lemma. Sei G eine nicht-zyklische p-Gruppe. Es gibt ganze Zahlen $u(C)$, $C \subset G$ zyklisch, mit den folgenden Eigenschaften:

i) $u(1) \neq 0 \bmod p$.

ii) Für alle $x \in R(G,Q)_p$ gilt

$$\sum_{\substack{C \\ C \subset G \text{ zyklisch}}} u(C) \dim x^C = |G|/p \dim x^G$$

Beweis. Wir beginnen mit der Orthogonalitätsrelation

$$|G| \dim x^G = \sum_{g \in G} x(g)$$

für Charaktere. Diese läßt sich umschreiben in

$$(2.5) \qquad |G| \dim x^G = \sum_C \Big(\sum_{D \subset C} \mu(|C/D|)\, |D| \dim x^D \Big),$$

mit der Möbius- Funktion μ der elementaren Zahlentheorie. Der Koeffizient von $\dim x$ ist $1 - s_p(G)$, worin $s_p(G)$ die Anzahl der Untergruppen der Ordnung p von G ist. Für die in Rede stehenden Gruppen G gilt aber

(2.6) $$s_p(G) \equiv 1 + p \mod p^2 .$$

(Huppert [1967], p. 314). Es ist deshalb $1 - s_p(G) = p\, u(1)$, $u(1) \neq 0 \mod p$. Die Koeffizienten von $\dim x^D$, $D \neq 1$, sind wegen des Faktors $|D|$ alle durch p teilbar. Also können wir in (2.5) durch p dividieren und erhalten eine Relation der gewünschten Art.

Mit den Bezeichnungen aus (2.4) gilt

(2.7) Lemma. Sei G eine nicht-zyklische p-Gruppe. Für $x \in A_p^*$ gilt

$$\prod_C d_C(x)^{u(C)} \equiv d_G(x)^{|G|/p} \mod |G| .$$

Beweis. Sei zunächst x aus dem Kern von $d_1 : A_p^* \longrightarrow \mathbb{Z}_p^*$, so daß also $\pi(x) \in 1 + I(G,Q)_p$. Nach (2.2) gilt $\pi(x) = \rho(y)$ und deshalb für $C = \langle g \rangle$

$$d_C(x) = \pi(x)(g) = \rho(y)(g) = k^{\dim y^g} = k^{\dim y^C}$$

und folglich

$$\prod_C d_C(x)^{u(C)} = k^t, \quad t = \sum u(C) \dim y^C.$$

Nach (2.4) ist $t \equiv 0 \mod |G|/p$ und nach der (2.3) folgenden Bemerkung ist $t \equiv 0 \mod (p-1)$, so daß also $k^t = 1$ in $\mathbb{Z}/|G|^*$ ist. Dasselbe gilt für $d_G(x)$.

Ein beliebiges $x \in A_p^*$ können wir in der Form zx_1 schreiben, wobei $z \in \mathbb{Z}_p \cdot 1 \subset A_p^*$ ist und auf x_1 die voranstehenden Überlegungen zutreffen. Nun ist aber $d_H(z) = z$ für alle $H \subset G$ und es gilt deshalb statt der Kongruenz in (2.7) sogar die Gleichheit für z statt x. Damit ist das Lemma bewiesen.

(2.8) Lemma. *Sei G eine p-Gruppe, p ungerade. Es gibt ganze Zahlen $u(H)$, $H \subset G$, derart, daß der Homomorphismus*

$$m : C_p^* \longrightarrow \mathbb{Z}/|G|^* : x \longmapsto \prod_H d_H(x)^{u(H)}$$

surjektiv ist und A_p^* im Kern enthält.

Beweis. Falls G nicht zyklisch ist, so verwenden wir als $u(C)$ für zyklische C die Zahlen aus (2.4), setzen $u(G) = -|G|/p$ und $u(H) = 0$ für alle anderen H. Wegen (2.7) liegt dann A_p^* im Kern von m. Es ist $\mathbb{Z}/|G|^*$ zyklisch von der Ordnung $(p-1)\,|G|/p$. Da aber $u(1) \not\equiv 0 \bmod p$ und $u(G) \not\equiv 0 \bmod p-1$ ist, so ist m surjektiv.

Falls G zyklisch ist, setzen wir $m(x) = d_1(x)\, d_C(x)^{-1}$, wobei $|C| = p$.

Bemerkung. Homomorphismen vom Typ m in (2.8) sind natürlich nicht eindeutig bestimmt. Für abelsche G habe ich andere in tom Dieck [1978] angegeben. Sie haben die Form

$$\prod_H d_H(x)^{\mu(1,H)}$$

wobei μ die Möbius-Funktion des Untergruppenverbandes ist. Dieser Typ von Produkten ist leider nicht für jede p-Gruppe geeignet. Die Homomorphismen aus (2.8) sind die besten mir bekannten, die einen strukturellen Sinn haben. Interessanterweise sind sie grundverschieden von den Homomorphismen (tom Dieck - Petrie [1982], Proposition (11.3)), die für die Endlichkeitshindernisse relevant sind. Darauf komme ich bei anderer Gelegenheit zurück.

3. Kongruenzen für 2-Gruppen.

Hier treten die üblichen Ausnahmegruppen auf: Dieder-, Quaternionen-, Semidiedergruppen. Wir behandeln diese zunächst.

(3.1) Diedergruppen. Es handelt sich um die Gruppen der Ordnung 2^n, $n \geq 3$,

$$G = \langle A, B \mid A^{2^{n-1}} = 1 = B^2,\ BAB^{-1} = A^{-1} \rangle .$$

Sei $C_i \subset \langle A \rangle$ die Untergruppe der Ordnung 2^i, $i = 0,\ldots,n-1$. Als Konjugationsklassen von Untergruppen hat man die C_i und die Untergruppen $\langle C_i, B \rangle = H_i$, $\langle C_i, AB \rangle = K_i$. Für $i < n-1$ ist der Normalisator von H_i gleich H_{i+1} und derjenige von K_i gleich K_{i+1}. Man hat unter anderem deshalb die folgenden Kongruenzen für den Bunrside-Ring (tom Dieck [1979], 1.3.5)

$$\sum_{i=o}^{n-1} \varphi(|C_i|)z(C_i) + 2^{n-2}z(H_o) + 2^{n-2}z(K_o) \equiv 0 \quad (2^n)$$

$$\sum_{i=1}^{n-1} \varphi(|C_i|/2)z(C_i) + 2^{n-3}z(H_1) + 2^{n-3}(K_1) \equiv 0 \quad (2^{n-1}).$$

Darin ist φ die Eulersche Funktion. Multipliziert man die zweite Kongruenz mit 2 und subtrahiert sie von der ersten, so erhält man

$$z(1) \equiv z(C_1) + 2^{n-2}z(H_o) + 2^{n-2}z(K_o) - 2^{n-2}z(H_1) - 2^{n-2}z(K_1) \bmod 2^n$$

Nun setzen wir voraus, daß die Werte z(H) ungerade sind, also

$z(H) = 1 + 2a(H)$. Dann wird aus der letzten Kongruenz

$z(1) \equiv 1 + 2a(C_1) + 2^{n-1}(a(H_o) + a(H_1) + a(K_o) + a(K_1)) \mod 2^n$.

Ferner stellt man fest, daß die rechte Seite kongruent zu

$$z(C_1)(1 + 2^{n-1}a(H_o))(1 + 2^{n-1}a(H_1))(1 + 2^{n-1}a(K_o))(1 + 2^{n-1}a(K_1))$$

ist. Sei $r : \mathbb{Z}/|G|^* \longrightarrow \mathbb{Z}/|G|^*$ der Homomorphismus $1 + 2k \longmapsto 1 + (|G|/2)k$. Dann erhalten wir also die Kongruenz

$$(3.2) \quad z(1) \equiv z(C_1)rz(H_o)rz(H_1)rz(K_o)rz(K_1) \mod 2^n$$

für Elemente $z \in A(G) \subset C(G)$ mit ungeraden Werten. Aus (3.2) ergibt sich offenbar ein surjektiver Homomorphismus $u : C_2^* \to \mathbb{Z}_2^*$ der A_2^* in seinem Kern hat.

(3.3) Quaternionengruppen. Es handelt sich um die Gruppen der Ordnung 2^n, $n \geq 3$,

$$G = \langle A,B \mid A^{2^{n-2}} = B^2,\ A^{2^{n-1}} = 1,\ BAB^{-1} = A^{-1} \rangle .$$

Verwenden wir analoge Bezeichnungen C_i, H_i, K_i für Untergruppen wie in (3.1), so stellt man fest, daß immer noch die Kongruenz (3.2) gilt.

(3.4) Semi-Diedergruppe. Es handelt sich um die Gruppe der Ordnung 2^n, $n \geq 4$,

$$G = \langle A,B \mid A^{2^{n-1}} = 1 = B^2,\ BAB^{-1} = A^{-1+2^{n-2}} \rangle .$$

Auch hier gilt wieder die Kongruenz (3.2), bei analogen Bezeichnungen und mit analogem Beweis.

(3.5) Zyklische Gruppen. Ist G zyklisch und $C \subset G$ die Untergruppe der Ordnung 2, so haben wir wiederum den Homomorphismus $C_2^* \longrightarrow \mathbb{Z}_2^* : z \longmapsto z(1)z(C)^{-1}$, der A_2^* im Kern enthält.

Im folgenden nehmen wir nun an, daß G nicht von bisher betrachteten Art ist. Sei $R_o(G,Q) \subset R(G;Q)$ der Unterring, der von den Darstellungen mit orientierungserhaltender Operation erzeugt wird. Sei $A_o(G) \subset A(G)$ der Unterring, der von den G-Mengen erzeugt wird, auf denen jedes $g \in G$ durch gerade Permutationen wirkt. Nach Atiyah - Tall [1969] hat man wiederum einen Isomorphismus

(3.6) $$\varrho : I_o(G;Q)_2 \longrightarrow 1 + I_o(G;Q)_2 ,$$

worin I_o das Augmentationsideal von R_o bezeichnet. Mittels Charakteren läßt sich ϱ durch

(3.7) $$\varrho(x)(g) = 3^{\frac{1}{2}\dim x^g}$$

berechnen. Für $x \in R(G;Q)_2$ gilt nun wiederum eine Relation der Form

(3.8) $$|G|/2 \dim x^G = \sum_C u(c) \dim x^C ,$$

$C \subset G$ zyklisch mit ungeradem $u(1)$. Der Beweis erfolgt wie für 2.4 , wenn man berücksichtigt, daß für die in Rede stehenden Gruppen $s_2(G) \equiv 3(4)$ ist.

Falls also $x \in A_o(G)_2^*$ ein Bild in $1 + I_o(G;Q)_2$ hat, so gilt $\pi(x) = \varrho(y)$ für ein $y \in I_o(G;Q)_2$ und deshalb

$$\prod_C d_C(x)^{u(C)} = \prod 3^{u(C)\frac{1}{2}\dim x^g}$$

$$(3.9) \qquad = 3^{|G|/4 \dim x^G} \equiv 1 \bmod |G|.$$

Ist $x \in A(G)_2^*$ beliebig, so schreiben wir zunächst $x = ax_o$, $d_1(x_o) = 1$, und $a \in \mathbb{Z}_2^*$. Für a gilt

$$(3.10) \qquad \prod_C d_C(a)^{u(C)} = a^{\Sigma\, u(C)} = a^{|G|/2} ,$$

letzteres nach (3.8). Ist x_o nicht in $A_o(G)_2^*$ enthalten, so liefert die Vorzeichendarstellung von x_o einen Homomorphismus $s(x_o)\colon G \longrightarrow \mathbb{Z}^*$ (siehe tom Dieck [1979], Seite 78) mit Kern H und $[G/H] - 1 = e$ ist eine Einheit von A(G). Es ist $x_oe \in A_o(G)_2^*$ und $d_1(x_oe) = 1$, so daß für x_oe statt x die Kongruenz (3.9) gilt. Für e gilt diese Kongruenz aber auch. Deshalb haben wir mittels (3.9) und (3.10) insgesamt

(3.11) <u>Lemma</u>. <u>Ist</u> G <u>keine</u> <u>Ausnahmegruppe</u> <u>wie</u> <u>in</u> (3.1) - (3.5), <u>so</u> <u>wird</u> <u>durch</u> $z \longmapsto \prod_C z(C)^{u(C)}$ <u>ein</u> <u>surjektiver</u> <u>Homomorphismus</u> $C_2^* \longrightarrow \mathbb{Z}/|G|^*$ <u>gegeben</u>, <u>der</u> A_2^* <u>im</u> <u>Kern</u> <u>enthält</u>.

Die Surjektivität folgt daraus, daß u(1) ungerade ist. Insgesamt haben wir somit ein System von 2-primären Kongruenzen gefunden.

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Even dimensional s-Smith equivalent representations

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1. Introduction

Let G be a finite group and V and W representations of G. We say that V and W are s-Smith equivalent if G acts semilinearly and smoothly on a homotopy sphere Σ such that $\Sigma^G = p \amalg q$, $T_p\Sigma = V$ and $T_q\Sigma = W$. (By definition G acts semilinearly on Σ if and only if Σ^K is a homotopy sphere for all subgroups K of G, see [R1].) This notion of s-Smith equivalence was introduced by Petrie in [P1, P2], and [P2] discusses the history of work on the classification of representations up to s-Smith equivalence; in particular the work of Atiyah-Bott, Milnor, Bredon, Sanchez, Cappell-Shaneson and Siegel is discussed in this paper.

From now on we restrict ourselves to cyclic groups. We use the following notation. Let $G = Z_n$ be the cyclic group with n elements, and ξ a primitive n^{th} root of 1, fixed once and forever. The representation t^j has C, the complex numbers, as underlying space and the generator g of G acts on z in C by $(g,z) \longrightarrow \xi^j \cdot z$.

Our main result on s-Smith equivalent representations is

Theorem B: Let $G = Z_{8 \cdot s}$ where s is an odd number, $s > 1$. The representations

$$V = 2a\ t^{4s} + 2^b(t^i + t^{i+4s+8}) + 2c\ t^2 \quad \text{and}$$
$$W = 2a\ t^{4s} + 2^b(t^{i+8} + t^{i+4s}) + 2c\ t^2$$

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are s-Smith equivalent if (i) $a > 2$, (ii) $c > a$ or $c = 0$, (iii) $b > 6a + 3$, (iv) $2b > 4(a+c)$, and $(i,8s) = (i+8,8s) = 1$. If $i < 4s$, V and W are not isomorphic.

E.g. choose $a = 2$, $b = 15$, $c = 0$, $i = 1$ to obtain the non-isomorphic s-Smith equivalent representations of $G = Z_{40}$

$$V = 4\cdot t^{20} + 2^{15}(t^{1} + t^{29}) \quad \text{and} \quad W = 4\cdot t^{20} + 2^{15}(t^{9} + t^{21}) .$$

Let us see that we can find many examples of appropriate combinations of s and i. Here are a few.

<u>If</u> $s \not\equiv 0(3)$	<u>choose</u> $i = 1$.
<u>If</u> $s \not\equiv 0(5)$, $s \not\equiv 0(13)$, $4s > 5$	<u>choose</u> $i = 5$.
<u>If</u> $s \not\equiv 0(11)$, $s \not\equiv 0(19)$, $4s > 11$	<u>choose</u> $i = 11$.
<u>If</u> $s \not\equiv 0(23)$, $s \not\equiv 0(31)$, $4s > 23$	<u>choose</u> $i = 23$.

By these examples all groups, cyclic of order $8\cdot s$ of order < 2760 and those where $s \not\equiv 0(3)$ are covered. To construct more examples look for pairs of primes p_1, p_2 such that $p_1 + 8 = p_2$, then we can construct examples for all s such that $4s > p_1$, $s \not\equiv 0(p_1)$, $s \not\equiv 0(p_2)$ by choosing $i = p_1$. Altogether this yields infinitely many examples of infinitely many groups of nonisomorphic s-Smith equivalent representations.

A counting argument that even dimensional examples exist is given by Petrie in [P2, 3.12]. It is the strength of Theorem A (see later in the introduction and section 2) which allows us to explicitly construct them. If $c = 0$ we will have these isotropy groups for the representations, $Iso(V) = Iso(W) = \{Z_{8\cdot s}, Z_{4\cdot s}, 1\}$, if $c \neq 0$ we have $Iso(V) = Iso(W) = \{Z_{8s}, Z_{4s}, Z_2, 1\}$. Examples with more isotropy groups can be constructed rather easily.

We need two main references for our study, [DR2] and [P2]. In [P2] Petrie approaches the study of s-Smith equivalent representations in a general way. He gives a number of strong necessary conditions for two representations to be s-Smith equivalent. Then he constructs nonisomorphic s-Smith equivalent representations. This is done in two steps. The first step is to construct a manifold X such that $X^G = p \amalg q$ and $T_pX = V$, $T_qX = W$ (see also section 3). The second step is to convert X (by surgery) into a semilinear homotopy sphere Σ such that $\Sigma^G = p \amalg q$ and $T_p\Sigma = V$, $T_q\Sigma = W$. It is this second step where we apply the results of [DR2] to obtain improved results over those of [P2] in the case where $\dim V^K = \dim W^K \equiv 0(4)$ for all $K \subseteq G$.

Remarks

1.) The work of Cappell and Shaneson [CS] also yields a classification for s-Smith equivalent in some of the odd-dimensional cases they study. At the present time our techniques are not sufficiently developed to allow a comparable classification for our even dimensional cases.

2.) The representations V and W in Theorem B are topologically equivalent but not linearly so. This fact can be derived directly from the Cappell-Shaneson work on nonlinear similarity. On the other hand, one can also give an alternate argument based upon Theorem B and standard equivariant engulfing techniques.

The main technical result, proved as a consequence of [DR2], is Theorem A (see section 2). Suppose $\dim V^K = \dim W^K \equiv 0(4)$ for all $K \subseteq G$. In Theorem A we give sufficient conditions when V and W are s-Smith equivalent. More generally this theorem gives a criterion for when a given normal map can be converted into a G homotopy equivalence. The obstruction is expressed in terms of easy a priori invariants of the domain and range of the normal map. These invariants are equivariant signatures and Whitehead torsion invariants. The vanishing of the invariant computed from Whitehead torsion is expressed by using the equivariant J homomorphism (over a point). This last fact follows from [DR2] where the torsion part of the surgery obstruction, the torsion of the Poincare duality map (between surgery kernels and cokernels) and algebraic data of the underlying spaces are related.

Theorem A is actually more general than is needed for the applications of this paper. It would allow us to study this problem. Let M be any G homotopy type. Describe $\{T_xN \mid x \in N^G\} \subseteq RO(G)$ for any smooth G manifold N in M. This problem has been raised in [P3]. The special case where M is a G homotopy type of a semilinear homotopy sphere with two fixed points is the case we shall deal with in detail; however we note that Theorem A is relevant to the general problem.

Here are some problems concerning s-Smith equivalent representations. Are there examples where a non 2 group is an isotropy group (except the group of index 2 in G and assuming an effective action)? Are there examples where the group of index 2 is not an isotropy group?

Are there examples with the following set of isotropy groups:

$$\{1, Z_{2k}, Z_d, G\}\ ,\ \text{where}\quad k > 1 \quad\text{and}\quad 2d = G\ ?$$

I want to thank T. Petrie for helpful discussions at all stages of this project.

2. Normal maps and Surgery obstructions.

The normal maps considered here are ht normal maps in the sense of [DP] and (Diff,ht) normal maps in the sense of [DR1]. We are going to make a few additional assumptions which simplify the notation. Let us give the basic definition and then explain the concepts involved.

<u>Definition 2.1</u>: A normal map $W = (X,f,b,c)$ consists of the following data:

(i) $f\colon X \longrightarrow Y$ is a map between smooth G manifolds.

(ii) For all $K \subseteq G$, Y^K and X^K are oriented and either connected or zero dimensional.

(iii) For all $K \subseteq G$, $\dim X^K = \dim Y^K$, and $\deg(f^K\colon X^K \longrightarrow Y^K)$ is one (if $\dim X^K \geq 1$) or f^K is a bijection (if $\dim X^K = 0$) .

(iv) Y satisfies the Gap hypothesis (if $Y^K \neq Y^L$ and $Y^K \subset Y^L$ then $2 \dim Y^K + 1 < \dim Y^L$) and $\dim Y^K = 0,1,$ or ≥ 5 .

(v) Y^K is 1-connected if $\dim Y^K \geq 2$.

(vi) We are given a virtual G vector bundle α over Y and a stable G vector bundle isomorphism $b\colon TX \longrightarrow f^*(\alpha)$.

(vii) We are given a $\Pi = \Pi(Y)$ vector bundle η over Y and a Π bundle isomorphism $c\colon \nu(X) \longrightarrow f^*(\eta)$ such that $s(c) = \Pi(b)$. ((b,c) is called bundle data.)

A $\Pi(Y)$ bundle η over Y consists of a collection of bundles $\{\eta_\beta \mid \beta \in \Pi(Y)\}$ and $\Pi(Y) = \amalg \pi_0(Y^H)$, $H \subseteq G$. There are relations between these bundles, see e.g. [DP, section 3]. These are the obvious ones as they appear in $\nu(Y) = \{\nu(Y^H,Y) \mid H \subset G\}$ where $\nu(Y^H,Y)$ is the NH/H normal bundle of Y^H in Y . Similiarly there are relations between the bundle isomorphisms, expressed in $s(c) = \Pi(b)$. This means stably c is $\Pi(b)$, the collection of isomorphisms obtained from b . We shall not need any technical statements about these bundles but we should point out some facts:

(i) Bundle data allow us to do equivariant surgery, and if we do surgery then the resulting normal map has again bundle data [DP, section 4].

(ii) If we construct f via equivariant transversality, then we obtain bundle data [DP, section 8 and P2, 1.13].

We state our main result (Theorem A) on the obstruction for converting a normal map W by surgery into a G homotopy equivalence (i.e. the underlying function is a G homotopy equivalence). After this we discuss some material which is needed for the proof, and finally we prove the theorem. The representations V and W are J_G equivalent ($J_G(V-W) = 0$) if $S(V)$ and $S(W)$ are stably G homotopy equivalent via a map which is of degree 1 on the H fixed set for all $H \subseteq G$. So J_G will denote this J homomorphism.

Theorem A: Suppose G is cyclic and $W = (X,f,b,c)$ is a G normal map, $f\colon X \longrightarrow Y$. Suppose $\dim X^K \equiv 0(4)$ for all $K \subseteq G$ and $T_pX = T_{f(p)}Y$ (as G_p representations), except for at most one point q. (Then q is a fixed point.) Suppose

(i) $\mathrm{Sign}(G,X^K) = \mathrm{Sign}(G,Y^K)$ for all $K \subseteq G$

(ii) $T_qX - T_{f(q)}Y = 2u$ for some $u \in \mathrm{Ker}\, J_G$.

There exists a manifold Σ G homotopic to Y, such that $\Sigma^G = X^G$, under a canonical bijection, and $T_p\Sigma = T_pX$ for all p in Σ^G. For condition (ii) see a footnote at the end of the paper.)

Let $S(G)$ be the set of subgroups of G, it is a G set via conjugation. A subset $H \subseteq S(G)$ is closed if H is G invariant and $K \in H$ and $L \supset K$ implies that L is in H. The notion of generalized Whitehead torsion has been introduced in [I,R2], we refer the reader to [DR1, section 5a] for a simple description. Briefly: Suppose $f\colon X \longrightarrow Y$ is a G homotopy equivalence of finite G CW complexes. Then f defines a G based acyclic complex $C_*(M_f,X)$, M_f denotes the mapping cylinder of f, whose class is $\tau(f)$ is the generalized Whitehead group, $\widetilde{Wh}(G)$. There is a direct sum decomposition $\widetilde{Wh}(G) = \bigoplus Wh(N_GK/K)$ where the sum ranges over subgroups of G, one in each conjugacy class of subgroups. Say that $\tau \in \widetilde{Wh}(G)$ corresponds to $\{\tau_K\}$ where $\tau_K \in Wh(N_GK/K)$. Then

$$\tau_K(f) = \tau(C_*(M_f^K, X^K \cup M_f^{K,s}))$$

where $M^{K,s} = \{y \in M_f^K | G_y \neq K\}$ denotes the singular points of the K fixed point set.

If $W = (X,f,b,c)$ is a normal map and H a closed subset of $S(G)$, then we say that W <u>is</u> H <u>good</u> if for all $K \in H$ the map f^K is an N_GK/K homotopy equivalence and $\tau(f^K) \in \widetilde{Wh}(N_GK/K)$ vanishes.

Suppose W is H good, and K is minimal in $S(G)-H$ ($K < L$ iff $K \supseteq L$). Set $\dim X^K = d$. With these assumptions and if $d > 5$ we have: (If $L^s \longrightarrow L^h$ is not injective some care is required and part b.) has to be stated in a more subtle way.)

<u>Theorem</u> <u>2.2</u> a.) [See DR2, Theorem 4.3] There exists an obstruction $\sigma_K(f)$ in $L_d^h(Z[N_GK/K],w)$, such that $\sigma_K(f)$ vanishes if and only if W can be converted by surgery of type K into a G normal map $W' = (X',f',b',c')$ which is H good and f^K is a N_GK/K homotopy equivalence. b.) Let $\alpha\colon L_d^h(Z[NK/K],w) \longrightarrow H^d(Z_2,Wh(NK/K))$ be the map in the Rothenberg sequence. If $\alpha(\sigma_K(f)) = 0$, then f defines an element $\sigma_K^s(f)$ in $L_d^s(Z[NK/K],w)$ such that $\sigma_K^s(f)$ vanishes if and only if W can be converted by surgery of type K into a G normal map W' which is H $G\cdot K$ good.

<u>Proof</u>: The first part is explicit in [DP2, 7.3]. For the second part suppose $\alpha(\sigma_K(f)) = 0$. We have to choose a class in $L_d^s(Z[NK/K],w)$ which maps to $\sigma_K(f)$. In fact we just have to choose basis to do so. If d is even we choose a basis for the surgery kernel in the middle dimension such that the intersection form expressed in this basis has a matrix whose torsion vanishes. If $d = 2m+1$ is odd the basis for the sub Lagrangians in the formation $(K_m(\partial U), K_{m+1}(U,\partial U), K_{m+1}(X_0,\partial U))$ is such that in the based sequence

$$0 \longrightarrow K_{m+1}(X_0,\partial U) \longrightarrow K_m(\partial U) \longrightarrow K_m(X_0) \longrightarrow 0$$

the duality map composed with the isomorphism from the universal coefficient theorem $K_{m+1}(X_0,\partial U) \longrightarrow K_m(X_0)$ is an isomorphism with vanishing torsion. The notation is as in [W, section 6] or [DR2, section 7]. Picking such a basis in either case (this is possible by the assumption that $\alpha(\sigma_K(f)) = 0$) provides us with $\sigma_K^s(f)$. It is obvious that $\sigma_K^s(f)$ has the properties we stated.

Our next step is to describe $\alpha(\sigma_K(f))$ in geometric terms. This was done in [DR2] and we recall the essential points. Without loss of generality we suppose $K = 1$. For this discussion we make the assumptions which we put down before Theorem 2.2, and as in Theorem A we assume that $T_qX \neq T_{f(q)}Y$ for at most one point q in X .

It follows from s-duality (see [DR1, 5c]) that there is a G homotopy equivalence ϕ: $S(T_qX \oplus U) \longrightarrow S(T_{f(q)}^T \oplus U)$ for some large complex representation U , ϕ is induced by the bundle data. (If G is abelian we can choose $U = 0$, see [R2, 4.10]). The torsion $\tau(\phi) \in \widetilde{W}h(G)$ is determined by T_qX and $T_{f(q)}Y$ alone if G is a product of a abelian 2 group with an odd order group [DR2]. This is seen as follows. Any two G homotopy equivalences ϕ,ϕ' between $S(T_q(X))$ and $S(T_{f(q)}(X))$ will differ by a self G homotopy equivalence β of $S(T_q(X))$. As $\tau(\phi) = \tau(\phi') + \tau(\beta)$ it suffices to show that $\tau(\beta) = 0$. It is a fairly easy computation that (stably) β is homotopic to a G diffeomorphism [DR1,2], and thus it has vanishing torsion. There we also showed (use that ϕ is its own Spanier-Whitehead dual) that $\tau(\phi) = \varepsilon\bar{\tau}(\phi)$, - is the conjugation on $\widetilde{W}h(G)$, see [DR2, 6.1]; and ε is a collection of signs, one for each summand $Wh(N_GK/K)$ of $Wh(G)$, see [DR1, 5.c.6]. This implies that $\tau(\phi)$ represents a class in $H^*(Z_2,\widetilde{W}h(G))$, more precisely $\tau_K(\phi)$ $H^{d_K}(Z_2,Wh(N_GK/K))$, d_K denotes the dimension of the K fixed point set. For cyclic groups the action of Z_2 on $Wh(L)$ is trivial [B, Prop. 7.3, page 623], so above symmetry for $\tau(\phi)$ would follow also algebraically. Some care is required in defining the conjugation on $Wh(L)$. One has to assume a conjugation on the group ring $Z[L]$. The standard conjugation will depend on the orientation homomorphism $W: L \longrightarrow Z_2$. In the first case use W given by the group action, in the second case W is assumed to be trivial.

<u>Theorem 2.3</u> With above assumptions and notation

$$\alpha(\sigma_1(f)) = [\tau_1(\phi)] .$$

The proof is in [DR2, section 10], τ_1 is the component of τ in $Wh(G) = Wh(N_G 1/1)$. The assumption of Th.A on T_pX and $T_{f(p)}(X)$ is essential.

<u>Theorem 2.4</u> Suppose that G is cyclic and $W - V = 2u$ for some u in $\operatorname{Ker} J_G$. Then there exists a G homotopy equivalence $\phi: S(W) \longrightarrow S(V)$ whose torsion $\tau(\phi) \in \widetilde{Wh}(G)$ depends on W and V. This element $\tau(\phi)$ defines an element $[\tau(\phi)]$ in $H^*(Z_2, \widetilde{Wh}(G))$ and this element vanishes.

<u>Proof</u>: Set $u = A - B$, then $W \oplus 2B \cong V \oplus 2A$. By assumption we have a G homotopy equivalence $\psi * \psi: S(2B) \longrightarrow S(2A)$ where $\psi: S(B) \longrightarrow S(A)$ is a G homotopy equivalence. The symbol $*$ denotes the join. By [R2, 4.10] we can destabilize $Id: S(W \oplus 2B) \longrightarrow S(W \oplus 2B)$ and find a G homotopy equivalence $\phi: S(W) \longrightarrow S(V)$ the torsion of ϕ is independent of the particular choice of ϕ by [DR1, 5.c.11] which is based on the understanding of the units in $\Omega(G)$. Thus $\tau(\phi * \psi * \psi) = 0$ as $\phi * \psi * \psi$ maps a representation sphere to itself. We say that a representation C is <u>orientable</u> if g in NH/H acts trivially on $H_*(S(C^H))$. Suppose first that V (and hence also W) is orientable. We can also choose A and B to be orientable. Let $\overline{K}_1(B(G); \tau)$ be the group which is defined in [R2] and in which the generalized Reidemeister torsion

lives. As G is cyclic the Reidemeister torsion λ is additive for orientable representations (see proof of 4.4 in [R2]). If $\alpha: S(U) \longrightarrow S(U')$ is a G homotopy equivalence of representation spheres we set $\lambda(\alpha) = \lambda(S(U)) - \lambda(S(U'))$. As $\lambda(\phi*\psi*\psi) = 0$ we find that $-\lambda(\phi) = \lambda(\psi*\psi) = 2\lambda(\psi)$, because of the additivity of λ. The direct sum decomposition of $\widetilde{W}h(G)$ and $\overline{K}_1(B(G);\tau)$, see [R2, 1.18 and 1.24] together with [M, 8.1] (conclude that γ: $Wh(G) \longrightarrow \overline{K}_1(B(G);\tau)$ is injective) implies that the generalized Whitehead torsion is also additive i.e. $\tau(\psi*\psi) = 2\tau(\psi) = -\tau(\phi)$. From the symmetry property discussed before 2.3 it follows that $\tau(\phi)$ represents a class in $H^*(\mathbf{Z}_2, \widetilde{W}h(G))$ which vanishes by definition as $2\tau(\psi) = -\tau(\phi)$.

If V is not orientable we reduce the computation to the above computation. Let $\mathbf{R}_\pm$ be the G representation whose underlying space is $\mathbf{R}$ and the generator of G acts by multiplication with ± 1. Let $\overline{\Phi} = \phi*\mathrm{Id}_{\mathbf{R}}: S(W \oplus \mathbf{R}_+) \longrightarrow S(V \oplus \mathbf{R}_+)$. We show that $\tau(\overline{\Phi}) = -\tau(\phi)$. To see this note first that $S(W \oplus \mathbf{R}_+) = D(W) \cup_{S(W)} D(W)$. Let $\phi_D: D(W) \longrightarrow D(V)$ be the radial extension of ϕ. Then $\overline{\Phi} = \phi_D \cup_\phi \phi_D$. For a map $\alpha: A \to B$ we abbreviate $C_*(M_\alpha, A)$ by $C_*(\alpha)$. There is a based short exact sequence of G based acyclic chain complexes

$$0 \longrightarrow C_*(\phi) \longrightarrow C_*(\phi_D) \oplus C_*(\phi_D) \longrightarrow C_*(\overline{\Phi}) \longrightarrow 0 \quad .$$

From this it follows that $\tau(\phi) + \tau(\overline{\Phi}) = 2\tau(\phi_D)$. As $\tau(\phi_D) = 0$ the above claim follows. We refer to such an argument as Mayer Vietoris argument. Hence it suffices to show that $[\tau(\overline{\Phi})] = 0 \in H^*(\mathbf{Z}_2, Wh(G))$. So we can assume that V and W have a summand $\mathbf{R}_+$. If V is not orientable, then $V \oplus \mathbf{R}_-$ is orientable. So let

$$\phi' = \phi*\mathrm{Id}_{\mathbf{R}_-}: S(W \oplus \mathbf{R}_-) \longrightarrow S(V \oplus \mathbf{R}_-)$$

We compute the torsion of ϕ'. As $S(W \oplus \mathbf{R}_-) = S(W) \times S(\mathbf{R}_+ \oplus \mathbf{R}_-)/S(W) \vee S(\mathbf{R}_+ \oplus \mathbf{R}_-)$ we have a based short exact sequence of G based acyclic chain complexes

$$0 \longrightarrow C_*(\phi v \mathrm{Id}_{+,-}) \longrightarrow C_*(\phi \times \mathrm{Id}_{+,-}) \longrightarrow C_*(\phi * \mathrm{Id}_-) \longrightarrow 0 .$$

Here $\mathrm{Id}_{+,-}$ is the identity map on $S(\mathbf{R}_+ \times \mathbf{R}_-)$, similarly Id_- . Thus $\tau(\phi v \mathrm{Id}_{+,-}) + \tau(\phi * \mathrm{Id}_-) = \tau(\phi \times \mathrm{Id}_-)$. As $\tau(\phi v \mathrm{Id}_{+,-}) = \tau(\phi) + \tau(\mathrm{Id}_{+,-})$, $\tau(\mathrm{Id}_{+,-}) = 0$, and $\phi * \mathrm{Id}_- = \phi * \mathrm{Id}_{R_-} = \phi'$:

$$\tau(\phi) + \tau(\phi') = \tau(\phi \times \mathrm{Id}_{+,-}) .$$

To compute the right hand side of this equation we take the decomposition $S(W) \times S(\mathbf{R}_+ \oplus \mathbf{R}_-) = S(W) \times [D(\mathbf{R}_-) \cup_{S(\mathbf{R}_-)} D(\mathbf{R}_-)]$. From a Mayer Vietoris argument and as $\tau(\phi \times \mathrm{Id}_{D(\mathbf{R}_-)}) = \tau(\phi)$ we find:

$$\tau(\phi \times \mathrm{Id}_{S(\mathbf{R}_-)}) + \tau(\phi \times \mathrm{Id}_{+,-}) = 2\ \tau(\phi)$$

Let $H \subset G$ be of index 2. Obviously $\phi \times \mathrm{Id}_{S(R-)} = G \times_H \phi$. Thus $\tau(\phi \times \mathrm{Id}_{S(\mathbf{R}_-)}) = \mathrm{Ind}^G \mathrm{Res}_H \tau(\phi)$. Ind^G means that we induce an action up to a G action, Res_H means that we restrict an action to an H action. Thus

$$\tau(\phi \times \mathrm{Id}_{+,-}) = 2\tau(\phi) - \mathrm{Ind}^G \mathrm{Res}_H \tau(\phi)$$

and

$$\tau(\phi') = \tau(\phi) - \mathrm{Ind}^G \mathrm{Res}_H \tau(\phi) .$$

As $\mathrm{Res}_H \tau(\phi) = \tau(\mathrm{Res}_H\ \phi)$, and as $\mathrm{Res}_H V$ and $\mathrm{Res}_H W$ are orientable we find by the first part of our argument that $\mathrm{Ind}^G \mathrm{Res}_H \tau(\phi)$ represents zero in $H^*(\mathbf{Z}_2, \widetilde{W}h(G))$. As we assumed that $W \oplus \mathbf{R}_-$ and $V \oplus \mathbf{R}_-$ were orientable it also follows that $\tau(\phi')$ represents zero in $H^*(\mathbf{Z}_2, \widetilde{W}h(G))$ Hence $\tau(\phi)$ also does.

<u>Proof of Theorem A</u>. The proof proceeds by induction over the linear ordering of $S(G)$, the set of subgroups of G . We will convert $\mathbf{W}$ by surgery into a simple G homotopy equivalence. this will be done in such a fashion that X^G is unchanged. This implies that the assumptions of the theorem are maintained at each step of the proof and

thus the second part of our claim follows immediately.

Suppose H is a closed subset of $S(G)$ and W is H good. Let K be minimal in $S(G)-H$. We want to make $\mathbf{W}$ $H \cup K$ good. By Theorem 2.2 $\sigma_K(f)$ is defined. Because of assumption (ii) in Theorem A it follows from 2.3 and 2.4 that $\alpha(\sigma_K(f)) = 0$. Hence $\sigma_K(f)$ is in the image of $L^s = L_0^s(Z[G/K],w)$ in $L_0^h(Z[G/K],w)$. By [Ba] elements in L^s are detected by the equivariant signature. We assumed that the equivariant signature vanished, $\mathrm{Sign}\,(G,X^K) = \mathrm{Sign}\,(G,Y^K)$. Then $\mathbf{W}$ defines an element $\sigma_K^s(f) \in L^s$ which vanishes. Apply 2.2 to make $\mathbf{W}$ $H \cup G \cdot K$ good.

3. Construction of normal maps

This section is a summary of results from [P2]. Let G be the cyclic group Z_{2d} and H the subgroup of order d. Let G_2 and H_2 be their respective 2-Sylow subgroups. We make the following assumptions on V:

3.1.α.) V satisfies the Gaphypothesis (i.e. if $V^K \subset V^L$ and $V^K \neq V^L$, then $2 \dim V^K + 1 < \dim V^L$; $K, L \subset G$).

β.) V is saturated (i.e. if K is a maximal isotropy group of $V - V^H$, then $\mathrm{Res}_K V$ contains a regular representation of K, see [P2, 1.7]).

γ.) V is stable (i.e. for each $K \in \mathrm{Iso}(V)$ and for each nontrivial representation χ of K either the multiplicity $m_\chi(V)$ of χ in V is zero or $d_\chi m_\chi(V) > m_1(V) = \dim_{\mathbf{R}} V^K$. Here d_χ is $\dim_{\mathbf{R}} D_\chi$ where D_χ is the algebra of real K endomorphisms of χ.)

δ.) $V^{G_2} = 0$ and $V^H = V^{H_2}$ and $H \in \mathrm{Iso}(V)$ (see [P2, 1.1]).

ε.) $K \in \mathrm{Iso}(V-V^H)$ implies that $K \subset G_2$ (see [P2, 5.13]).

Let V be a real G representation, then we set $\overline{V} = V \otimes \mathbf{C} + t^d V \otimes \mathbf{C}$. We say that the representation $\Gamma < V$ (Γ and

V are real or complex) if and only if for the underlying real representations Γ is a summand of $n \cdot V$ for some n. $R(G,V)$ is the summand of $R(G)$ generated by representation $\Gamma < V$. $K_G(X,V)$ is the summand of $K_G(X)$ generated by those G vector bundles $\xi < X \times V$; i.e. for each point $x \in X$, $\xi_x < \mathrm{Res}_{G_x} V$. J_V is the equivariant J homomorphism in the K-theory $K_G(X,V)$; over a point x we have a (stable) G_x homotopy equivalence in the fibre. Real equivalents $RO(G,V)$, $KO(X,V)$, and JO_V can also be considered. For each $L \subset H_2$ we define ideals $I(L)$ and $I'(L)$ in $R(G)$ by

$$I(L) = \mathrm{Ker}(R(G) \xrightarrow{\mathrm{Res}_H \times \mathrm{Fix}_L} R(H) \times R(G/L))$$

$$I'(L) = I(L) \cap \mathrm{Ker}(\mathrm{Res}_{G_2}) .$$

By [P2, 5.10] $I(L)$ is the free abelian group generated by $\{t^i - t^{i+d} \mid |L| \text{ does not divide } i\}$ and $I'(L)$ contains a subgroup generated by $\{t^i + t^{i+d+m} - t^{i+m} - t^{i+d} \mid |L| \text{ does not divide } i\}$. Here m is the order of G_2.

Set $n(V) = \langle 1/2 \dim_{\mathbf{R}} V^H - 1 \rangle$ ($\langle x \rangle$ denotes the smallest integer which is greater or equal to x). For $\nu(V)$ see [P2, 4.4], we will only use $n(V)$ explicitly. To $I = I(L)$ or $I'(L)$ and a representation V of G satisfying 3.1.δ we attach integers

$$a(I,V) = \begin{cases} \nu(V) & \text{if } I = I(L) \\ n(V) & \text{if } I = I'(L) \end{cases}$$

$$b(I) = \begin{cases} 1 & \text{if } G/L \equiv 0(8) \\ 2 & \text{if } 2 < |G/L| \text{ and } |G/L| \not\equiv 0(8) \\ \infty & \text{otherwise} \end{cases}$$

$$c(V) = \begin{cases} 0 & \text{if } \dim V \text{ is odd} \\ 1 & \text{if } \dim V \text{ is even} \end{cases}$$

$$d(V) = \begin{cases} \dim V^H + 1 & \text{if } \dim V^H \text{ is even} \\ \dim V^H & \text{if } \dim V^H \text{ is odd} \end{cases}$$

$\Delta(I,V) = a(I,V) + b(I) + c(V) + d(V)$.

Let $\rho\colon R(G,V) \longrightarrow R(G)$ be the inclusion. Ted Petrie shows [P2, 5.19]:

<u>Theorem 3.2</u> Let I be $I(L)$ or $I'(L)$ and let V be a G representation which satisfies 2.1 δ and ε if $I = I'(L)$. Set $\lambda = a(I,V) + b(I) + c(V)$. Then if $z \in I$, $Y = S(V \oplus R)$ and $i\colon Y^G \longrightarrow Y$ the inclusion, $(0,2^{\lambda}z) = i^*(\xi)$ for some $\xi \in K_G(Y)$ with $J(\xi) = 0$ and in addition $\mathrm{Res}_H(\xi) = 0$ if $\dim V$ is even. If in addition V is saturated and $z \in I \cap R(G,\overline{V})$ then $(0,2^{\Delta(I,V)}z) = i^*\xi$ with $\xi = \rho(\xi')$ for some $\xi' \in K_G(Y,\overline{V})$ with $J_V(\xi') = 0$. If $\dim V$ is even we can again suppose that $\mathrm{Res}_H\xi = 0$.

<u>Theorem 3.3</u> (Compare [P2, 2.6 and 3.4]) Suppose V satisfies the conditions in 3.1, $\dim V^H > 5$, and $u = W - V = 2^{\Delta(I,V)}z$ where $z \in I \cap R(G,\overline{V})$. Then there exists a normal map $W = (X,f,b,c)$, $f\colon X \longrightarrow Y = S(V \oplus R)$ such that $X^G = p \amalg q$ and $T_pX = V$, $T_qX = W$ and $T_xX \cong T_{f(x)}Y$ if x is not a fixed point. If G acts orientation preservingly on V

$$\mathrm{Sign}(G,X^K) = \mathrm{Sign}(G,Y^K) = 0 \quad \text{for all} \quad K \subseteq G .$$

(Note that $u = 2w$ with $J(w) = 0$, hence assumption (ii) in Theorem A holds.)

<u>Proof</u>: The assumptions of this theorem imply those of Theorem 3.2. We use the conclusion there and apply G transversality theory (this is where we need that V is stable) to find W, compare [P2, 2.6]. The signature computation is carried out in [P2, 3.4]. $\mathrm{Res}_H\,\xi = 0$, see 3.2, $T_xX = T_{f(x)}Y$ if x is not a fixed point.

4. Construction of examples

Let $G = Z_{2d}$ be as in the previous section, there we also defined $\bar{V}$ for a real representation V. If V is complex set $\bar{V} = \overline{r(V)}$, where r denotes realification. Let m be the order of G_2.

Lemma 4.1 Suppose $u = t^i + t^{i+d+m}$. Then $t^i + t^{i+d+m} - t^{i+d} - t^{i+m} < \bar{u}$. The proof is obvious.

Proof of Theorem B:

We verify the assumptions 3.1 for V. For this we study the isotropy groups of V. In general $\mathrm{Iso}(t^j) = \{G, Z_k\}$ where $k = (|G|, j)$. This observation together with the fact that $\mathrm{Iso}(A \oplus B) = \{K_A \cap K_B | K_A \in \mathrm{Iso}(A)$ and $K_B \in \mathrm{Iso}(B)\}$ allows us to compute $\mathrm{Iso}(V)$. As we assumed that $(i,8s) = (i+8,8s) = 1$, it follows that $\mathrm{Iso}(V) = \mathrm{Iso}(W) = \{1,H,G\}$ if $c = 0$ (H is of index 2 in G) and $\mathrm{Iso}(V) = \mathrm{Iso}(W) = \{1,Z_2,H,G\}$ if $c \neq 0$. Now 3.1 $\alpha,\delta,\varepsilon$ are trivial checks. To check saturation (3.1.β.) observe that $K = Z_2$ ($c\neq 0$) or $K = 1$ ($c=0$) in the notation of 3.1. The assumptions are then obviously satisfied. 3.1.γ is also straightforward to check. By construction $V-W = 2^b z$ where $z \in I'(L)$ and $L = Z_2$. Then

$$\Delta(I,V) < \langle {}^1/2 \dim_{\mathbf{R}} V^H - 1\rangle + 2 + 1 + \dim_{\mathbf{R}} (V^H) + 1$$
$$= {}^3/2 \dim_{\mathbf{R}} (V^H) + 3 = 6a + 3 .$$

Hence $V-W = 2^b z$ where $z \in I'(L)$ and by the previous lemma $z \in R(G,\bar{V})$. Apply Theorem 3.3 to produce a normal map. The assumptions of Theorem A are satisfied by this normal map. Hence the representations V and W are s-Smith equivalent. It is an easy check to see that V and W are not isomorphic if $i < 4s$.

Footnote: This condition assumes that the torsion invariant in 2.3 vanishes, as it is proved in 2.4. A related torsion invariant has been discussed before in Lemma 3 of [CS]. In fact the vanishing of

$\alpha(\sigma_1(f))$ in 2.3 is also a necessary assumption. This was used in the more specialized setting of [CS] to obtain the necessary and sufficient assumptions 1 and 2 of Theorem 1 in [CS].

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FIXED POINTS AND GROUP ACTIONS

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The theme of this paper is illustrated by the following:

If M is a G-manifold satisfying certain conditions then M is a G-boundary.

Several examples will be given to elucidate the statement. Throughout, all manifolds are compact and smooth.

1. Unoriented manifolds.

The first example is well known:

If M is a $\mathbb{Z}/2$-manifold with no fixed points then M is a $\mathbb{Z}/2$-boundary.

Indeed the mapping cylinder from M to the quotient $M/(\mathbb{Z}/2)$ is a $\mathbb{Z}/2$-manifold whose boundary is M.

Equally well known is the following result:

If M is a $\mathbb{Z}/2$-manifold in which the fixed point set has codimension 1 then M is a $\mathbb{Z}/2$-boundary.

Once again, the mapping cylinder provides the required $\mathbb{Z}/2$-manifold with boundary M.

For the next result let dim F denote the maximal dimension of all the fixed point set components.

If M is a $\mathbb{Z}/2$-manifold for which dim M > (5/2) dim F then M is a $\mathbb{Z}/2$-boundary.

This result is usually referred to as "Boardman's five-halves theorem". J.M. Boardman proved a slightly weaker result in [3]. The full equivariant version was proved in [11] by C. Kosniowski and R.E. Stong. That paper also contains a number of related results. For example:

If M is a $\mathbb{Z}/2$-manifold such that the fixed point set has constant dimension and dim M > 2dim F then M is a $\mathbb{Z}/2$-boundary.

The next result concerns larger groups. Let G be a finite abelian group or a torus group and let G_2 be the subgroup of G generated by the elements of order 2.

If M is a G-manifold for which G_2 acts with no fixed points then M is a G-boundary.

This result generalises the first result mentioned in this section. R.E. Stong proved the result when $G = G_2$ in [14] although a slightly weaker version was proved by P.E. Conner and E.E. Floyd in [4]. S.S. Khare obtained the more general result in [7]. A proof, together with further generalisations, is given in [8].

2. Two fixed points.

In this section we assume that all manifolds are oriented and that all group actions are orientation preserving. The group in question will be either $\mathbf{Z}/p$ with p an odd prime number, or S^1 the circle group. In most cases there are analogous results for the group $\mathbf{Z}/p^n$.

Back in 1968, M.F. Atiyah and R. Bott published a result in [1] that has become a basic result in transformation groups:

If $\mathbf{Z}/p$ acts on a homology sphere with two isolated fixed points then the representations of $\mathbf{Z}/p$ on the tangent space at each fixed point are the same.

Since the representations are the same (together with a change in orientation) the manifold is $\mathbf{Z}/p$-bordant to one with no fixed points, that is, to a free $\mathbf{Z}/p$-manifold. A free $\mathbf{Z}/p$-manifold is $\mathbf{Z}/p$-bordant to p copies of some manifold with $\mathbf{Z}/p$ action being the obvious permutation. In such a situation we shall say that the $\mathbf{Z}/p$-manifold is a $\mathbf{Z}/p$-boundary mod p.

A reformulation of the Atiyah-Bott result is given below:

If $\mathbf{Z}/p$ acts on a homology sphere with two isolated fixed points then M is a $\mathbf{Z}/p$-boundary mod p.

In fact we don't need a homology sphere but simply any $\mathbf{Z}/p$ manifold for which the G-signature vanishes, see [5].

It is rather surprising that the same result holds for any smooth manifold, provided that the dimension is not too small.

Let M be a 2n-dimensional $\mathbf{Z}/p$-manifold with two isolated fixed points. If $n > p - 3$ then M is a $\mathbf{Z}/p$-boundary mod p.

This result was proved by J. Ewing and C. Kosniowski in [6].

The results above have interpretations in terms of lens spaces. A <u>mod-p lens space</u>

is the orbit of a free linear action of $\mathbf{Z}/p$ on an odd dimensional sphere. An immediate corollary of the Atiyah-Bott result is that two lens spaces which are h-cobordant (compatible with the preferred generators and orientations) are in fact isomorphic; see [12]. Similarly, the result of Ewing-Kosniowski implies that two lens spaces of dimension $2n - 1 > 2p - 6$ which are cobordant in $O_*(B\mathbf{Z}/p)$ are in fact isomorphic.

For the circle group we have the following result.

Let M be an S^1-manifold with two fixed points; then there is an integer r such that 2^r copies of M bound as an S^1-manifold.

This result is not a corollary of the corresponding result for the group $\mathbf{Z}/p$.

A proof of this result is given as an appendix to this paper.

3. Isolated fixed points.

In this section M will be a 2n-dimensional oriented manifold having a $\mathbf{Z}/p$ action with isolated fixed points.

We know from the previous section that if the number of fixed points is two and $n > p - 3$ then M is a $\mathbf{Z}/p$-boundary. More generally we have the following result:

Let M be an oriented $\mathbf{Z}/p$ manifold with isolated fixed points. If the number of isolated fixed points F satisfies

$$F < 1 + \frac{2\{n/2\}}{(p-1)\{\log_p n\}-2}$$

where $\{x\}$ denotes the least integer greater than or equal to x, then M is a $\mathbf{Z}/p$-boundary mod p.

A proof of this result appears in [6]. We conjecture that the result holds if F satisfies $F < n^{f(n,p)}$ for some approximately linear function $f(n,p)$ of n and p. Note that if $p = 3$ then this is indeed so with $f(n,3) = [n/2]$.

Notice that if M has a pair of fixed points where the representations of $\mathbf{Z}/p$ on the tangent space at each fixed point are isomorphic by an orientation reversing isomorphism then M is $\mathbf{Z}/p$-bordant to a $\mathbf{Z}/p$-manifold with two less fixed points. The bordism is achieved by removing a disk about each of the fixed points and attaching a handle equivariantly. Thus, in the theorem above, we may ignore such pairs of fixed points in the number F of fixed points.

4. Higher dimensional fixed point sets.

Once again we asssume that M is an oriented $\mathbb{Z}/p$-manifold. There are not many general results of the type in which are interested in if the dimension of the fixed point set is positive. We shall mention one proved by J. Ewing in [5].

Suppose that M is a $\mathbb{Z}/p$-manifold for which the G-signature is zero and for which the fixed point set is a rational homology sphere of dimension 2k. If $k \neq 1$, or if 2 has even order in the multiplicative group $(\mathbb{Z}/p)^*$ then M is a $\mathbb{Z}/p$-boundary mod p.

This theorem applies, for instance, to homology spheres with $\mathbb{Z}/p$-actions. Notice that the condition 2 has even order in $(\mathbb{Z}/p)^*$ is always true if $p = 3$ or $5 \bmod 8$, and never true if $p = 7 \bmod 8$. R. Schultz (and others) constructed infinitely many examples of non-bounding $\mathbb{Z}/7$-manifolds with G-signature zero and with fixed point set a sphere of dimension 2.

5. Unitary manifolds.

Our final examples concern S^1-actions on unitary (stably almost complex) manifolds. The results we mention were proved by C. Kosniowski in [10].

Let M be a unitary S^1-manifold of dimension not equal to 2 or 6. If the fixed point set is a homology sphere then M is an S^1-boundary.

This result applies, for instance, to the case when M is a homology sphere, and in particular to the case when M is a standard sphere. Note that up to bordism (in fact up to h-bordism) the unitary structures on the standard sphere S^{4k+2} are in a one-to-one correspondence with the integers $\mathbb{Z}$, the boundary corresponding to zero. The result shows that apart from dimension 2 and 6, the unitary spheres S^{4k+2} which admit an S^1-action correspond to zero. This is related to the the fact that among the spheres only S^2 and S^6 have almost complex structures.

Another result proved in [10] is the following one.

Let M be a unitary S^1-manifold of dimension not equal to 6. If the fixed point set has the integral homology of a product of two odd dimensional spheres then M is an S^1-manifold.

This result applies in the case when M itself has the integral homology of a product of two odd dimensional spheres.

6. Appendix.

The purpose of this section is to give a proof of the following result which was mentioned in section 2.

Let M be an S^1-manifold with two fixed points; then there is an integer r such that 2^r copies of M bound as an S^1-manifold.

Let x, y be the two fixed points of an S^1-action on a 2n-dimensional oriented manifold. The representation of S^1 on TM_x, the tangent space of M at x, is equivalent to

$$o(x) \quad V_{x(1)}V_{x(2)}\cdots V_{x(n)}$$

for some positive integers x(i), where $V_{x(i)}$ is the standard irreducible one dimensional complex representation of S^1 with an element t acting on $V_{x(i)}$ by multiplication by $t^{x(i)}$. The number o(x) is either 1 or -1 depending on whether the standard orientation of $V_{x(1)}V_{x(2)}\cdots V_{x(n)}$ agrees or disagrees with the orientation of TM_x.

We call the numbers x(1), x(2), ..., x(n) the **rotation numbers** of the fixed point x, and we call o(x) the **orientation number** of x. At the point y we also have n rotation numbers y(1), y(2), ..., y(n) and an orientation number o(y).

To prove that M is an S^1-boundary it is sufficient to show that the rotation numbers of x and y coincide (up to order) and that the orientation numbers satisfy o(x) = -o(y). Because then M is equivariantly bordant to an S^1-manifold with no fixed points, and a result of E. Ossa in [13] provides the final step.

To show that the rotation numbers coincide and that o(x) = -o(y) we apply some well known formulae of M.F. Atiyah and I.M. Singer [2, section 8]; see also [9]. These tell us that if f is a symmetric polynomial in n variables of degree 2d then

$$o(x)\,\frac{f(x(1),x(2),\ldots,x(n))}{\Pi x(i)} + o(y)\,\frac{f(y(1),y(2),\ldots,y(n))}{\Pi y(i)} = 0$$

as long as $0 <= 2d < n$.

When f is the constant polynomial (degree = 0) we deduce that

$$o(x) \quad \Pi y(i) \quad = \quad -o(y) \quad \Pi x(i)$$

Since the x(i) and y(i) are positive integers we deduce that o(x) = -o(y). Furthermore we see that

$$f(x(1),x(2),\dots,x(n)) = f(y(1),y(2),\dots,y(n))$$

for all symmetric polynomials in n variables of degree $2d < n$.

Let m be the maximum value of the rotation numbers $x(1), x(2), \dots, x(n), y(1), y(2), \dots, y(n)$. There is an induced action of $\mathbf{Z}/m$ on M. The fixed point set $M^{\mathbf{Z}/m}$ contains x and y. We claim that x and y are in the same component of $M^{\mathbf{Z}/m}$. To see this suppose that F_x is a component of $M^{\mathbf{Z}/m}$ which contains x. The S^1-action on M induces an S^1-action on F_x since $S^1/(\mathbf{Z}/m)$ is isomorphic to S^1. If y is not in F_x then the fixed point set of F_x under the induced S^1-action is precisely x. This is impossible, thus y must also belong to F_x.

Looking at the $\mathbf{Z}/m$-equivariant normal bundle N of F in M it follows that the number of times that m occurs in $x(1), x(2), \dots, x(n)$ is the same as the number in $y(1), y(2), \dots, y(n)$. This is because the number of times that m occurs in each of $x(1), x(2), \dots, x(n)$ and $y(1), y(2), \dots, y(n)$ is the (complex) dimension of the bundle N at x and y respectively. Furthermore, it follows that the rotation numbers at x are the same, up to sign, modulo m, as the rotation numbers at y. We may therefore rewrite the rotation numbers as

$$m, m, \dots, m, a(1), a(2), \dots, a(k)$$
$$m, m, \dots, m, b(1), b(2), \dots, b(k)$$

where $0 < a(i) < m$, $0 < b(i) < m$ and $a(i)^2 = b(i)^2 \bmod m$.

We now apply the Atiyah-Singer formulae using the polynomials $z_1^{2d} + z_2^{2d} + \dots + z_n^{2d}$. The following relations are obtained:

$$(n-k)m^{2d} + \sum a(i)^{2d} = (n-k)m^{2d} + \sum b(i)^{2d} \qquad \text{if } 0 <= 2d < n$$

or more simply

$$\sum a(i)^{2d} = \sum b(i)^{2d} \qquad \text{for } 0 <= 2d < n$$

We want to show that the a(i) and b(i) coincide. Suppose therefore that $a(i) \neq b(i)$ for all i (otherwise we delete them from consideration). Since $a(i)^2 = b(i)^2 \bmod m$, $0 < a(i) < m$ and $0 < b(i) < m$ we deduce that $b(i) = m - a(i)$. Thus

$$\sum a(i)^{2d} = \sum (m-a(i))^{2d} \qquad \text{for } 0 <= 2d < n$$

Expanding the right hand side we obtain, inductively, the following

$$\sum a(i)^{2d-1} = \sum (m-a(i))^{2d-1} \qquad \text{for } 0 < 2d < n$$

We therefore have the following relations

$$\sum a(i)^s \quad = \quad \sum(m-a(i))^s \qquad \text{for } 0 <= s < n$$

But, since the number of a(i) is k and k < n, it follows immediately that the a(i) and m-a(i) coincide, up to order. This completes the proof of the result.

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BORSUK-ULAM THEOREMS AND K-THEORY DEGREES OF MAPS

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1. STATEMENT OF RESULTS. Let G be a compact Lie group, E and F unitary finite dimensional representations of G. We denote the unit sphere in E by S(E).

THEOREM 1. Let E and F be unitary representations of the nontrivial compact Lie group G such that G operates freely on S(E) and S(F). If there exists a G-map $f: S(E) \longrightarrow S(F)$ then $\dim E \leq \dim F$.

This is a generalization of the theorem of Borsuk and Ulam [1] which asserts that if $f: S^m \longrightarrow S^n$ is a map with $f(-x) = -f(x)$ for all x then $m \leq n$. The generalization here is to linear spherical space forms and illustrates the use of K-theory degrees of maps. The theorem is true for topological spherical space forms - an easy proof is presented in [6].

Given a G-map $f: S(E) \longrightarrow S(F)$ we use radial extensions to obtain a proper G-map $f: E \longrightarrow F$. If V is a unitary representation of G, we obtain a proper G-map $f \times 1 : E \times V \longrightarrow F \times V$. If G acts freely on S(E) and S(F), then $f \times 1$ has degree 1 on the fixed point set. This is part of the motivation for the following definition.

DEFINITION. The limit over all unitary representations U of G

$$^1\{E,F\}_G = \lim_U \, ^1[E \oplus U, F \oplus U]_G$$

is called the set of stable classes of proper maps of degree one on fixed point sets.

We shall give results about proper maps of degree one on fixed point sets for the groups $G = S^1$ and $Z/(n)$. We let B be the birth certificate representation of S^1 (the identity map of S^1) and of $Z/(n)$ (the inclusion $Z/(n) \longrightarrow S^1$ which maps $T=\underline{1}$ to $w = \exp(2\pi i/n)$).

THEOREM 2. If n is a natural number and $G = Z/(n)$ (with the convention that $G = S^1$ if n=0), then

$$^1\{B^a, B^b\}_G \neq \emptyset$$

if and only if $\underline{b} \in (\underline{a}) \subset Z/(n)$.

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It is fitting to ask about the image of the degree function

$$\deg : {}^1\{B^a, B^b\}_G \longrightarrow Z$$

THEOREM 3. If $G = Z/(n)$, $b = ka \bmod (n)$, $(a,n) = 1$, then the image of $\deg : {}^1\{B^a, B^{ka}\}_G \longrightarrow Z$ is the coset $k + (n)$.

The main tool in this paper is equivariant K-theory which allows us to attach to a G-map $f: E \oplus V \longrightarrow F \oplus V$ its K-degree $K(f) \in R(G)$. This enables us to prove results by examining class functions on G. For example, Theorem 2 for $n \neq 0$ is a consequence of the following algebraic result. Let $X: Z/(n) \longrightarrow C$ be the character of the birth certificate representation B of $Z/(n)$, and let $w = \exp(2\pi i/n) = X(T)$, where $T = \underline{1}$ is the standard generator of $Z/(n)$.

THEOREM 4. If $\underline{a} \neq \underline{0}$ in $Z/(n)$, then the function

$$\frac{1 - X^b}{1 - X^a} : D \longrightarrow C$$

(where $D = \{\underline{k} \in Z/(n) \mid \underline{ka} \neq \underline{0}\}$) has image in $Z[w]$ (the subring generated by w in the complex numbers C) if and only if $\underline{b} \in (\underline{a}) \subset Z/(n)$.

The paper is organized as follows: 2. introduces the K-theory techniques ([1], [10], [7], [8]) used in the proofs and does the easy case of $G = S^1$ as an example; 3. proves Theorem 1 and introduces the algebraic techniques for studying $G = Z/(n)$ for $n \neq 0$; 4. proves Theorem 4 and shows how this implies Theorems 2 and 3.

The author wishes to thank Ted Petrie for willing to share his insights freely and generously. Thanks also go to R.G.Swan and D.Ramakrishnan for conversations about units in $Z[w]$ and cyclotomic polynomials in $F_p[x]$. The reader is referred to [4], [5], [8], [9] for a deeper study of degrees of equivariant maps between spheres. For a complete discussion of which groups can act linearly and freely on spheres, see [11].

2. K-THEORY DEGREE.

Suppose $f: E \longrightarrow F$ is a proper G-map of the unitary representations E and F. The work of Atiyah [2] and Segal [10] shows that $K_G(E)$ is a free R(G)-module on a canonical generator λ_E. Thus the map $f^! : K_G(F) \longrightarrow K_G(E)$ is completely described if we know the element $K(f) \in R(G)$ defined by

$f^!\lambda_F = K(f)\lambda_E$. The K-theory degree $K(f)$ can be considered as a class function $K(f) : G \longrightarrow C$. The work of Petrie [8] shows how to calculate the value $K(f)(g)$ at an element g of G. Let $C = \overline{\langle g \rangle}$ be the closure of the cyclic group generated by g in G, and split $E = E_C \oplus E^C$, $F = F_C \oplus F^C$ as C representations, where M^C is the trivial component of a C-representation M. Let $\lambda_{-1}(M) = \sum_i (-1)^i \lambda^i M$. Then we have (see [8] §3, or [4] p.284) $\lambda_{-1}(E_C)(g) \neq 0$, and

$$K(f)(g) = \frac{\lambda_{-1}(F_C)(g)}{\lambda_{-1}(E_C)(g)} \cdot \deg f^C ,$$

where $f^C : E^C \longrightarrow F^C$ is the restriction of f to the fixed point sets of C (with the convention that $\deg f^C = 0$ if $\dim E^C \neq \dim F^C$). In the case of $G = S^1$ this gives

$$K(f) = \frac{\lambda_{-1}(F_G)}{\lambda_{-1}(E_G)} \cdot \deg f^G$$

as an element of $R(G)$.

Let us prove Theorems 2 and 3 for $G = S^1$ as an example of the use of $K(f)$. Suppose $a \neq 0$ and $f : B^a \oplus U \longrightarrow B^b \oplus U$ is a proper G-map with $\deg f^G = 1$. This means $(B^a \oplus U)_G = B^a \oplus U_G$, $(B^b \oplus U)_G = B^b \oplus U_G$, and

$$K(f) = \frac{\lambda_{-1}(B^b)\lambda_{-1}(U_G)}{\lambda_{-1}(B^a)\lambda_{-1}(U_G)} = \frac{\lambda_{-1}(B^b)}{\lambda_{-1}(B^a)} = \frac{\prod_{d \mid b} \Phi_d}{\prod_{e \mid a} \Phi_e} = \frac{1-X^b}{1-X^a}$$

where $\Phi_n \in R(G) = Z[X,X^{-1}]$ is the n-th cyclotomic polynomial in X. Now $R(G)$ in this case is a unique factorization domain and the cyclotomic polynomials Φ_e are all irreducible, so this means that each divisor e of a occurs among the divisors of b - but this is precisely the condition that $b \in (a) \subset Z$, proving Theorem 2 for $G = S^1$. Now suppose that $b = ka$, then

$$K(f) = \frac{1 - X^{ka}}{1 - X^a} = 1 + X^a + \dots + X^{(k-1)a} ,$$

and the value of this at $1 \in S^1$ is $k = \deg f$, proving Theorem 3 for $G = S^1$.

Theorems 2 and 3 for $G = S^1$ should be ascribed to Meyerhoff and Petrie [7] , who have these arguments, although they were interested in a different question.

3. PROOF OF THEOREM 1 . Suppose G acts freely on the unit spheres in the representations E and F, and suppose $f: S(E) \longrightarrow S(F)$ is a G-map. Since G is nontrivial, there exists a cyclic subgroup C of order p a prime. Now $f: S(E) \longrightarrow S(F)$ is a C-map and C operates freely on the two spheres. In particular, this means that as representations of C,

$$E = \bigoplus_{i=1}^{\dim E} B^{a_i}$$

$$F = \bigoplus_{j=1}^{\dim F} B^{b_j} ,$$

where all a_i, b_j are relatively prime to p.

We let $w = \exp(2\pi i/p) = X(T)$, T a generator of C. We then have

$$K(f)(T) = \frac{\prod_j (1 - w^{b_j})}{\prod_i (1 - w^{a_i})} .$$

Since $K(f) \in R(C)$, we must have $K(f)(T) \in Z[w]$.

LEMMA 5. If $(a,p) = 1$, then $1 - w^a = (1 - w).y$ for a unit $y = 1 + w + \dots + w^{a-1}$ in $Z[w]$.

The proof is easy: $ab = 1 \bmod (p)$, so $1 - w = 1 - w^{ab} = (1-w^a).z$ with $z = 1 + w^a + \dots + w^{a(b-1)}$. Hence $1-w = (1-w)yz$, so $1 = yz$.

We now go back to $K(f)(T)$. We have $1 - w^{a_i} = (1 - w)x_i$, $1 - w^{b_j} = (1 - w)y_j$ for suitable units x_i, y_j in $Z[w]$, so

$$K(f)(T) = (1 - w)^{\dim F - \dim E}.u ,$$

where u is a unit in $Z[w]$. We will show that $(1 - w)^{-1}$ is not in $Z[w]$, that is $1 - w$ is <u>not</u> a unit in $Z[w]$.

COROLLARY 6. $(1 - w)^{p-1} = px$ where x is a unit in $Z[w]$.

To prove this, let's examine the cyclotomic polynomial

$$\Phi_p = 1 + x + \dots + x^{p-1}$$

$$= \prod_{(a,p)=1} (x - w^a) .$$

We have just proved that $1 - w^a = (1 - w).y$, y a unit in $Z[w]$, so

$$p = \Phi_p(1) = \prod_{(a,p)=1} (1 - w^a) = (1 - w)^{p-1}.u$$

for a unit u in $Z[w]$.

This of course completes the argument: $1 - w$ is not a unit in $Z[w]$, since $(1 - w)^{p-1} = 0$ in $F_p[w] = F_p \otimes_Z Z[w]$.

4. PROOF OF THEOREM 4. We will start by proving a slight generalization of Lemma 5 and Corollary 6.

LEMMA 7. Let w be a primitive n-th root of 1, $(a,n)=1$, then $1 - w^a = (1 - w).y$ for a unit y in $Z[w]$.

The proof is immediate - just substitute n for p in the proof of Lemma 5.

COROLLARY 8. Let v be an n-th root of unity in $Z[w]$. If the order of v is p^r, p a prime, then

$$(1 - v)^{p^{r-1}(p-1)} = p.u$$

for some unit $u \in Z[w]$. If the order of v is a composite then $1 - v$ is a unit in $Z[w]$.

The proof consists in noticing that

$$1+x+\ldots +x^{n-1} = \prod_{\substack{d \mid n \\ d \neq 1}} \Phi_d ,$$

so $\Phi_d(1) = 1$ if d is a composite, $\Phi_d(1) = p$ if $d = p^r$, p a prime. Let m be the order of v, $\varphi(m)$ the order of U_m, the group of units in the ring $Z/(m)$. We then have: there exists a unit in $Z[w]$ such that

$$\Phi_m(1) = (1 - v)^{\varphi(m)}.u ,$$

and the corollary follows if we remember that $\varphi(p^r) = p^{r-1}(p-1)$.

Now we launch ourselves into the proof of Theorem 4. First, if $b = ka + mn$, then

$$\frac{1 - X^b}{1 - X^a} = \frac{1 - X^{ka}}{1 - X^a} = 1 + X^a + \ldots + X^{a(k-1)} ,$$

so the function does take values in $Z[w]$.

The converse will take longer to prove. We now suppose that

$$\frac{1 - w^{kb}}{1 - w^{ka}} \in Z[w]$$

for all k such that $w^{ka} \neq 1$, where $w = \exp(2\pi i/n)$. We wish to show that $\underline{b} \in (\underline{a}) \subset Z/(n)$. The proof will be divided into three easy steps.

STEP 1. We can assume that a is a divisor of n .

Let $\alpha =$ order $\underline{a}$ in $Z/(n)$. Then $n = \alpha c$, and $\underline{c}$ has order α, so we have that $\underline{a}$ and $\underline{c}$ generate the same cyclic subgroup of order α. Since the quotient map $Z/(\alpha c) \longrightarrow Z/(\alpha)$ induces a quotient map $U_{\alpha c} \longrightarrow U_{\alpha}$ of the groups of units (this is a consequence of the Chinese Remainder Theorem and the special case $n=p^r$), there exists a $k \in Z$ with $(k,n) = 1$ such that $\underline{a} = k\,\underline{c}$, so

$$\frac{1 - X^a}{1 - X^c} = 1 + X^c + \ldots + X^{(k-1)c}$$

is a unit in $Z[X] = Z[x]/(x^n - 1)$. We also have $(\underline{a}) = (\underline{c})$ (since $k. : Z/(n) \longrightarrow Z/(n)$ is a ring automorphism). This means that Theorem 4 for $(1 - X^b)/(1 - X^a)$ is equivalent to that for $(1 - X^b)/(1 - X^c)$, where c divides n .

We now suppose that a divides n, but is not equal to n . Let p be a prime dividing n/a. Write $a = p^r a'$, $b = p^s b'$, $n = ap^t c' = p^{r+t} a'c'$, where a', b', c' are prime to p.

STEP 2. Under our hypotheses

$$(1 - w^{bc'})(1 - w^{ac'})^{p^{t-1}(p-1)-1} \in (p) \subset Z[w] .$$

Contemplate

$$\frac{1 - X^b}{1 - X^a}(T^{c'}) = \frac{1 - w^{bc'}}{1 - w^{ac'}} \in Z[w] ,$$

and remember that $n = ac'p^t$. This means that the order of $w^{ac'}$ is precisely p^t, where $t > 0$ by choice. Corollary 8 now says that

$$(1 - w^{ac'})^{p^{t-1}(p-1)} = p.u$$

for some unit u in $Z[w]$, so we have

$$\frac{1 - w^{bc'}}{1 - w^{ac'}} = \frac{(1 - w^{bc'})(1 - w^{ac'})^{p^{t-1}(p-1)-1}}{p.u} \in Z[w]$$

as claimed.

STEP 3. $(1 - w^{bc'})(1 - w^{ac'})^{p^{t-1}(p-1)-1} = 0$ in $F_p[w] = Z[w]/(p)$ implies that $b \in (a)$.

We let f_m be the image of Φ_m under the quotient map $Z[x] \longrightarrow F_p[x]$. We have $F_p[w] = F_p[x]/(f_n)$. Thus what we want to prove is:

$$(1 - x^{bc'})(1 - x^{ac'})^{p^{t-1}(p-1)-1} \in (f_n)$$

implies $b \in (a)$, or what is equivalent to this: in $a = p^r a'$, $b = p^s b'$ we have $r \leq s$, $b' \in (a')$.

LEMMA 9. If $(m,p) = 1$, $f_m \in F_p[x]$ is a product of distinct irreducible polynomials of the same degree k, where k is the order of $\underline{p}$ in U_m, the group of units in $Z/(m)$. Each irreducible polynomial in $F_p[x]$ divides precisely only one f_m with $(m,p) = 1$.

The proof is easy: the derivative test gives us that $x^m - 1$ is separable in $F_p[x]$. Let K be a splitting field of $x^m - 1$ over F_p, that is

$$x^m - 1 = \prod_{i=1}^{m} (x - \lambda^i) ,$$

since the roots form a cyclic group of order m. Indeed we have

$$f_m = \prod_{i \in U_m} (x - \lambda^i) .$$

We look at the orbits of the roots of f_m in K under the Frobenius automorphism $\varphi : K \longrightarrow K$ given by $\varphi(y) = y^p$. Since $\underline{p} \in U_m$ has order k, this means that the orbits of φ are just the cosets of the cyclic group $\langle \underline{p} \rangle$ in U_m, that is each orbit has precisely k elements. Given an element y in K, if its orbit under φ is $y_1, \ldots, y_r$, then the minimal polynomial of y over F_p is

$$(x - y_1)\ldots(x - y_r) .$$

This proves our claim that f_m splits into $\varphi(m)/k$ distinct irreducible polynomials of degree k in $F_p[x]$. Each irreducible polynomial q of degree d in $F_p[x]$ occurs as a factor of a unique f_m with $(m,p) = 1$. Indeed, q is a factor of $x^{p^d-1} - 1$, hence a factor of an f_m with m dividing p^d-1 (and not dividing $p^{d-1}-1$). Suppose q divides f_m and $f_{m'}$ with $(mm',p)=1$, then q^2 divides $x^{mm'} - 1$ which is ridiculous.

What can we say about f_m for a general m ?

LEMMA 10. If $m=p^r m'$, $(m',p) = 1$, then $f_m = (f_{m'})^{p^{r-1}(p-1)}$.

Let K be the splitting field of $x^{m'} - 1$. Then K is also a splitting field of $x^m - 1 = (x^{m'} - 1)^{p^r}$. The group of roots of $x^m - 1$ is cyclic of order m'. The Chinese Remainder Theorem gives us $U_m \approx U_{p^r} \times U_{m'}$, so if λ is a primitive m'-th root of 1

$$f_{m'} = \prod_{i \in U_{m'}} (x - \lambda^i) \in K[x] ,$$

and each $(x - \lambda^i)$ occurs $\varphi(p^r) = p^{r-1}(p-1)$ times in f_m , so

$$f_m = \prod_{i \in U_{m'}} (x - \lambda^i)^{\varphi(p^r)} = (f_{m'})^{\varphi(p^r)} ,$$

as claimed.

This means that we should concentrate on the f_m with $(m,p) = 1$.

LEMMA 11. Suppose that $(m,p) = 1$, then $x^b - 1 \in (f_m)$ if and only if $b \in (m)$.

If $b \in (m)$, then certainly $x^b - 1 \in (f_m)$, since

$$x^b - 1 = \prod_{d|b} f_d .$$

Conversely, if $x^b-1 \in (f_m)$ and L is a splitting field of x^m-1 with $\lambda \in L$ a primitive m-th root of 1, then $\lambda^b = 1$, so $b \in (m)$, since the order of λ is m .

We are now ready to complete Step 3. We want to show that

$$(1 - x^{b'c'})^{p^s}(1 - x^{a'c'})^{p^{r+t-1}(p-1)-p^r} \in (f_n)$$

implies $p^s b' \in (p^r a')$. Since $n = ap^t c' = p^{r+t} a'c'$, Lemma 10 gives that

$$f_n = f_{a'c'}{}^{p^{r+t-1}(p-1)}$$

so (using Lemma 9) our condition becomes:

$$(1 - x^{b'c'})^{p^s} \in (f_{a'c'}{}^{p^r}) ,$$

and using Lemma 11 we get $b'c' \in (a'c')$ (so $b' \in (a')$) and $p^s \geq p^r$. That is, $b \in (a)$, as was to be shown.

The reader will understand the argument best if he does the special case $a = p^r$, $n = p^{r+t}$, since here

$$f_n = (x - 1)^{p^{r+t-1}(p-1)} .$$

It remains to show how Theorem 4 implies Theorems 2 and 3.

Suppose $f: B^a \oplus U \longrightarrow B^b \oplus U$ is of degree 1 on the fixed point set - in particular we can assume that $\underline{a} \neq 0$ (for if $\underline{a} = 0$, then $\underline{b} = 0$ - why?). We have seen in 2. that $K(f) \in R(Z/(n))$ is given by

$$K(f) \Big| D = \frac{1 - X^b}{1 - X^a}$$

on $D = \{ \underline{k} \in Z/(n) \mid \underline{ka} \neq 0 \}$, so in particular it takes values in $Z[w]$. Theorem 4 is now applicable, and so we conclude that $\underline{b} \in (\underline{a}) \subset Z/(n)$.

Conversely, if $b = ka + mn$, the map $f: B^a \longrightarrow B^{ka}$ defined by $f(z) = z^k$ is a $Z/(n)$ - map, hence the proof of Theorem 2 is completed.

If $b = ka \bmod (n)$, the map $h: B^a \longrightarrow B^{ka}$ given by $h(z)=z^{k+sn}$ is a proper $Z/(n)$-map of degree $k+sn$. This shows that the image of

$$\deg : {}^1\{B^a, B^{ka}\}_{Z/(n)} \longrightarrow Z$$

contains $k + (n)$. For the converse inclusion we remember the additional hypothesis that $(a,n)=1$, so $Z/(n)$ acts freely on $S(B^a)$. Thus given $f, f' \in {}^1\{B^a, B^{ka}\}_{Z/(n)}$ we have $K(f)(g) = K(f')(g)$ for all $g \neq e$ in $G = Z/(n)$. This means that $K(f)$ differs from $K(f')$ by some multiple s of the regular representation R of $Z/(n)$. Then $\deg f = K(f)(e) = K(f')(e) + sR(e) = \deg f' + sn$, and $\deg f = \deg f' \bmod (n)$, completing the proof of Theorem 3.

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The completion conjecture in equivariant cohomology

by J. P. May

Consider an $RO(G)$-graded cohomology theory k_G^*. We shall not insist on a detailed definition; suffice it to say that there is a suspension isomorphism for each real representation of G. The first examples were real and complex equivariant K-theory KO_G^* and K_G^*. The next example was equivariant stable cohomotopy theory π_G^*. There are $RO(G)$-graded ordinary cohomology theories with coefficients in Mackey functors.

The study of these theories is still in its infancy. They can all be defined for arbitrary compact Lie groups, but we shall restrict our attention to finite groups. When we localize away from the order of G, there are very powerful algebraic devices for the reduction of the calculation of k_G^* to nonequivariant calculations. If we localize at a prime dividing the order of G, there are techniques for reducing calculations to consideration of p-groups contained in G. There are no known general procedures for the calculation of k_G^* at p for p-groups G. Largely for this reason, the reservoir of known calculations is almost empty.

Let $A(G)$ be the Burnside ring of finite G-sets. Then k_G^* takes values in the category of $A(G)$-modules. For some purposes, this is the main reason for interest in the $RO(G)$-grading. The assertion is false for Z-graded equivariant cohomology theories which fail to extend to $RO(G)$-graded theories. Let EG be a free contractible G-CW complex and let $\varepsilon\colon EG \to * = \{pt\}$ be the trivial map. We have an induced homomorphism of $A(G)$-modules

$$\varepsilon^*\colon k_G^*(*) \longrightarrow k_G^*(EG).$$

(We use unreduced theories until otherwise specified.) The completion conjecture for k_G^* asserts that, on integer gradings, ε^* becomes an isomorphism upon completion with respect to the topology given by the powers of the augmentation ideal of $A(G)$. As we shall shortly make precise, when G is a p-group and the completion conjecture holds for k_G^*, the p-adic completion of k_G^* is computable in nonequivariant terms.

The first theorem in this direction was due to Atiyah [4]; see also Atiyah and Segal [5]. Their results are stated in terms of representation rings but, since G is finite, the Burnside ring gives the same topology.

<u>Theorem 1.</u> The completion conjecture holds for real and complex equivariant K-theory.

However, the completion conjecture certainly fails to hold in general.

Counterexample 2. The completion conjecture fails for ordinary cohomology $H_G^*(?;\underline{Z})$ with coefficients in the constant coefficient system $\underline{Z}$. On integer gradings, $H_G^*(*;\underline{Z}) = H_G^0(*;\underline{Z}) = Z$, whereas $H_G^*(EG;\underline{Z}) = H^*(BG;Z)$, the ordinary integral cohomology of the classifying space BG.

The Segal conjecture is the completion conjecture for equivariant cohomotopy theory. The central step in its proof has been supplied in a beautiful piece of work by Gunnar Carlsson [6].

Theorem 3. The completion conjecture holds for equivariant stable cohomotopy theory.

There is an equivalent nonequivariant reformulation and an interesting implied generalization that I will discuss at the end.

When I first heard about the Segal conjecture, my instinct was that it was unlikely to be true in general. I also felt that it was a much less important problem than the general one of explaining for which theories k_G^* the completion conjecture would or would not hold. However, Carlsson's work not only completed the proof of Theorem 3, it also led to very substantial progress on the general problem. This development is joint work of Jeff Caruso and myself [9] and follows up our simplification of Carlsson's work [8], which was undertaken in hopes of just such a generalization. My understanding of these matters also owes a great deal to joint work and conversations with Jeremy Gunawardena, Gaunce Lewis, Stewart Priddy, and Stefan Waner.

My purpose here is to explain Carlsson's work, our generalization of it, and related matters in conceptual terms, without getting bogged down in details of proofs. None of the steps presents any great difficulty any more, all of the work lying very close to the foundations, but there are quite a few steps. We shall stick to the main line of development, and this means that a great deal of earlier work on the Segal conjecture will go unmentioned. Adams [1] summarized what was known late in 1980.

We shall first explain the force of the completion conjecture and its reduction to a question about p-groups, following May and McClure [19].

For each subgroup H of G, there is an $RO(H)$-graded theory k_H^* associated to k_G^* (depending not just on H as an abstract group but on the inclusion $H \subset G$). In particular, with $H = e$, there is an associated nonequivariant cohomology theory k^*. Modulo interpretation in terms of restriction

$RO(G) \longrightarrow RO(H)$,

$$k_H^*(Y) = k_G^*(G \times_H Y).$$

The projection $G \times Y \to Y$ induces a map $\pi: k_G^*(Y) \to k^*(Y)$ of Z-graded cohomology theories on spaces Y. We say that k_G^* is split if there is a map $\zeta: k^*(Y) \to k_G^*(Y)$ of cohomology theories such that $\pi\zeta$ is the identity. We have a notion of an RO(G)-graded ring-valued cohomology theory. If k_G^* is a ring theory, so is each k_H^*. We say that k_G^* is a split ring theory if ζ is a map of ring-valued cohomology theories, and each k_H^* is then also a split ring theory.

We say that k_G^* is of finite type if each $k_H^q(*)$, $q \in Z$ and $H \subset G$, is a finitely generated Abelian group. This ensures that $k_G^\alpha(X)$ is finitely generated for all $\alpha \in RO(G)$ and all finite G-CW complexes X. The most interesting examples are split ring theories of finite type.

The connection between equivariant and nonequivariant cohomology is established by the following observation [19, lemma 12].

Lemma 4. If k_G^* is split and Y is a free G-CW complex, then, on integer gradings, $k_G^*(Y)$ is naturally isomorphic to $k^*(Y/G)$.

Thus the completion conjecture relates $k_G^*(*)$ to $k^*(BG)$. The relevance of the completion conjecture to the general calculation of $k_G^*(X)$ is given by the following result [19, Prop. 15].

Proposition 5. Assume that k_G^* is a split ring theory of finite type and that $\lim^1 k^*(BH^n) = 0$ for each $H \subset G$ (where BH^n denotes the n-skeleton of BH). If the completion conjecture holds for k_H^* for each H, then the projection $EG \times X \to X$ induces an isomorphism

$$\hat{k}_G^\alpha(X) \longrightarrow k_G^\alpha(EG \times X)$$

for all $\alpha \in RO(G)$ and all finite G-CW complexes X, where the left side is the completion of $k_G^\alpha(X)$ with respect to the Burnside ring topology.

Of course, by the lemma, $k_G^q(EG \times X) \cong k^q(EG \times_G X)$ for $q \in Z$.

For clarity of exposition, we shall henceforward restrict attention to integer gradings.

The completed Burnside ring Green functor satisfies induction with respect to the subgroups of G of prime power order. Don't worry if you don't understand the previous sentence. As a matter of pure algebra, it leads to a proof that the Burnside ring rapidly disappears from the picture. See [19, Thm 13 and Prop 14] and also Laitinen [14,15].

Theorem 6. The completion conjecture holds for k_G^* if it holds for k_H^* for all subgroups H of prime power order.

Proposition 7. If G is a p-group and k_G^* is split, then the completion conjecture holds for k_G^* if and only if $\varepsilon^*\colon k_G^*(*) \to k_G^*(EG)$ induces an isomorphism upon passage to p-adic completion.

Henceforward (until otherwise specified near the end), G is to be a p-group; $\hat{k}$ and cognate symbols will indicate completion at p. To avoid constant repetition of hypotheses, we assume once and for all that all theories k_G^*, given or constructed, are split and of finite type.

We are at the starting point of Carlsson's work, and some preliminary philosophical comments are in order. The stable part of algebraic topology has three main branches: homology and cohomology theory on spaces, homotopy theory on spectra, and infinite loop space theory. Carlsson's preprint [6] was written from the second point of view. Specifically, Carlsson worked in the stable category of G-spectra constructed by Lewis and myself [17]. This was done with my encouragement, and I must apologise to Carlsson for giving him very bad advice. As Caruso and I discovered, the mathematics simplifies considerably when the first point of view is taken, and the changed point of view is crucial to our generalization of Carlsson's work that is the theme of this paper.

Carlsson's theorem asserts that the Segal conjecture holds for all p-groups if it holds for all elementary Abelian p-groups, that is, for all p-groups of the form $(Z_p)^n$. The Segal conjecture was proven by Lin [18] for Z_2, by Gunawardena [13] for Z_p, and by Adams, Gunawardena, and Miller [3] for $(Z_p)^n$ with $n \geq 2$. These authors actually prove the equivalent nonequivariant reformulation of the conjecture to be discussed later. My work with Caruso led to work with Priddy that gives a geodesic proof of the Segal conjecture for $(Z_p)^n$ within Carlsson's context ([20] plus later corrections).

We shall see that a version of the reduction to elementary Abelian groups goes through for all theories k_G^* such that k_* is bounded below, in the sense that $k_q(*) = 0$ for all sufficiently small q. (This hypothesis serves only to ensure the convergence of certain Adams spectral sequences.) While we shall be able to shed some light on the elementary Abelian case, the general picture is still obscure. We shall describe a satisfactory necessary and sufficient condition for the completion conjecture to hold when k_* is cohomologically bounded above, in the sense that $H^q(k) = 0$ for all sufficiently large q, but without this unpleasant hypothesis it seems that "calculation is the way to the truth". Here k denotes the spectrum which represents the nonequivariant theory k^*.

We need some definitions to give content to the discussion. Let U be a countably infinite dimensional real G-inner product space which contains infinitely many copies of each irreducible representation of G. We take $U = R^\infty \oplus U'$, where U' contains no copies of the trivial representation and so fix $R^s \subset U$, $s \geq 0$. By an indexing G-space, we understand a finite dimensional G-inner product subspace of U. We assume given a G-prespectrum k_G, namely a collection of based G-spaces $k_G V$ for indexing G-spaces V and based G-maps $\sigma\colon \Sigma^{W-V} k_G V \to k_G W$ for $V \subset W$; here $W-V$ denotes the orthogonal complement of V in W. As usual, $\Sigma^V X = X \wedge S^V$, where S^V is the 1-point compactification of V; similarly, $\Omega^V X$ is the G-space of based maps $S^V \to X$. We require technical conditions on the spaces $k_G V$ and maps σ, but these result in no loss of generality and need not concern us here. For based G-CW complexes X and Y and for an integer q, write $q = r-s$, where $r \geq 0$, $s \geq 0$, and $r = 0$ or $s = 0$ (to avoid separate cases) and define

$$k_q^G(X;Y) = [\Sigma^r X, \operatorname*{colim}_{V \supset R^s} \Omega^{V-\mathbb{R}^s}(Y \wedge k_G V)]_G.$$

This is the Z-graded bitheory associated to k_G. It specializes to

$$k_q^G(Y) = k_q^G(S^0;Y) \quad \text{and} \quad k_G^q(X) = k_{-q}^G(X;S^0).$$

These are reduced theories, to which we switch henceforward. To define stable homotopy and cohomotopy, we take k_G to be the sphere G-prespectrum; its $V^{\underline{th}}$ space is S^V, and $\sigma\colon \Sigma^{W-V} S^V \to S^W$ is the evident identification.

Carlsson's first step was joint work with Cusick [7] and involved reduction from a problem in cohomotopy to a more tractable problem in homotopy. Independently and concurrently, Caruso and Waner arrived at an extremely elegant way of carrying out essentially the same reduction. They had observed earlier [10] that a model for EG could be obtained by taking the union over V of certain smooth compact G-manifolds with boundary $M(V)$ embedded with codimension zero in V. The essential property of $M(V)$ is that $M(V)/\partial M(V)$ is equivalent to S^V/T^V, where T^V denotes the singular set of S^V (namely the set of points with non-trivial isotropy subgroup). Using this model and an easy Spanier-Whitehead duality argument, one finds that $\varepsilon\colon \hat{k}_G^*(S^0) \to \hat{k}_G^*(EG^+)$ can be identified with a certain natural map of inverse limits

$$\lim \hat{k}_*^G(S^V;S^V) \longrightarrow \lim \hat{k}_*^G(S^V;S^V/T^V).$$

A simple cofibration argument then gives the following conclusion.

Proposition 8. The completion conjecture holds for k_G^* if and only if

$$\lim \hat{k}_*^G(S^V;T^V) = 0.$$

For Carlsson's second step, it is convenient to introduce the notation

$$\hat{k}_q^G(Y \wedge \underline{W}) = \lim_j \hat{k}_q^G(S^{jW};Y).$$

Here Y is a G-CW complex, W is a representation of G, and the inverse limit is taken over the homomorphisms

$$\hat{k}_q^G(S^{(j+1)W};Y) \xrightarrow{(1\wedge e)_*} \hat{k}_q^G(S^{jW}\wedge S^W;Y\wedge S^W) \cong \hat{k}_*^G(S^{jW};Y),$$

where $e\colon S^0 \to S^W$ is the evident inclusion. The case $Y = S^0$ is particularly important. By clever use of interchange of limits applied to bi-indexed limits involving smash products, Carlsson uses the criterion of the previous proposition to obtain the following conclusion.

Proposition 9. If the completion conjecture holds for k_H^* for all subgroups H of G, then $\hat{k}_*^G(\underline{W}) = 0$ for all $W \neq 0$. Conversely, if the completion conjecture holds for k_H^* for all proper subgroups H of G and $\hat{k}_*^G(\underline{W}) = 0$ for any one $W \neq 0$ such that $W^G = \{0\}$, then the completion conjecture holds for k_G^*.

Carlsson fixes a good choice of W with $W^G = 0$, which we shall call Z. Let $\overline{G}$ be the elementary Abelianization $G/[G,G] \otimes Z_p$ of G and take Z to be the pullback to G of the reduced regular representation of $\overline{G}$. The restriction of Z to any proper subgroup H contains a copy of the trivial representation, and a theorem of Serre [29] (or its consequence due to Quillen and Venkov [26]) implies that the Euler class $\alpha(Z)$ is nilpotent if G is not elementary Abelian. Proposition 9 has the following consequence.

Corollary 10. Assume that the completion conjecture holds for k_H^* for all proper subgroups H of G. Then the completion conjecture holds for k_G^* if and only if $\hat{k}_*^G(\underline{Z}) = 0$.

At this point, Carlsson introduces the key simplification. Let $\widetilde{EG}$ be the unreduced suspension of EG with one of its cone points as basepoint. Equivalently, $\widetilde{EG}$ is the cofibre of the evident map $EG^+ \to S^0$, and one obtains the fundamental long exact sequence

$$(*) \quad \cdots \longrightarrow \hat{k}_q^G(EG^+ \wedge \underline{Z}) \longrightarrow \hat{k}_q^G(\underline{Z}) \longrightarrow \hat{k}_q^G(\tilde{E}G \wedge \underline{Z}) \xrightarrow{\partial} \hat{k}_{q-1}^G(EG^+ \wedge \underline{Z}) \longrightarrow \cdots .$$

Carlsson assumes inductively that the Segal conjecture holds for all proper subquotients of G and proves that both $\hat{\pi}_*^G(\tilde{E}G \wedge \underline{Z}) = 0$ and $\hat{\pi}_*^G(EG^+ \wedge \underline{Z}) = 0$ if G is not elementary Abelian. This implies that $\hat{\pi}_*^G(\underline{Z}) = 0$ and thus that the Segal conjecture holds for G. Carlsson observes that his vanishing theorems fail when G is elementary Abelian and suggests the possibility of a direct proof that the connecting homomorphism ∂ is an isomorphism in this case. Priddy and I provide such a proof.

The generalization to k_G^* requires us to introduce a bitheory k_*^J associated to any subquotient J of G. Here $J = N/H$, where H is normal in N. For J-CW complexes X and Y and $q = r-s$ (as above), we define

$$k_q^J(X;Y) = [\Sigma^r X, \operatorname*{colim}_{V \supset R^s} \Omega^{V^H - \mathbb{R}^s}(Y \wedge (k_G V)^H)]_J$$

and specialize as before to obtain $k_*^J(Y)$ and $k_J^*(X)$. Everything said so far works equally well with k_*^G replaced by k_*^J. It is vital to recognize that, in general, k_*^J depends on H and N and not just on J. In particular, we write h_* for the nonequivariant bitheory obtained by taking $H = N$ (and thus $J = e$) in the above definition; we write k_* and k_*^1 for the nonequivariant bitheories so obtained from $H = e$ and $H = G$, respectively. An example may clarify the definition.

Example 11. For a G-space X, let $S_G X$ be the G-prespectrum with $V^{\underline{th}}$ space $\Sigma^V X$ and with $\sigma\colon \Sigma^{W-V}\Sigma^V X \longrightarrow \Sigma^W X$ the evident identification. The theory represented by $S_G X$ is split if there exists a map $\zeta\colon X \to X^G$ whose composite with the inclusion $X^G \longrightarrow X$ is homotopic to the identity. Since every representation of J occurs in V^H for some indexing G-space V, we conclude by cofinality that $(S_G X)_*^J$ is just a copy of the bitheory $(S_J X^H)_*$. Note the particular case $X = S^0$.

Now Carlsson's vanishing theorems generalize as follows.

Theorem 12. If G is not elementary Abelian and the completion conjecture holds for k_J^* for all proper subquotients J of G, then $\hat{k}_*^G(\tilde{E}G \wedge \underline{Z}) = 0$.

Theorem 13. If G is not elementary Abelian and k_* is bounded below, then $k_*^G(EG^+ \wedge \underline{Z}) = 0$.

We have stated these differently since the proof of the second is direct rather than inductive. The same results are valid if we start with some k_*^J as ambient theory, and we deduce the following generalization of Carlsson's theorem by induction. Remember that all theories in sight are assumed to be split and of finite type.

Theorem 14. If G is not elementary Abelian, all h_* are bounded below, and the completion conjecture holds for k_J^* for all elementary Abelian subquotients of G, then the completion conjecture holds for k_G^* and all other k_J^*.

Before discussing the proofs of Theorems 12 and 13, we describe what happens in the elementary Abelian case. Let M_n denote the free $\hat{Z}_p$-module on $p^{n(n-1)/2}$ generators, where $\hat{Z}_p$ denotes the p-adic integers; we take tensor products over $\hat{Z}_p$ below. In the case of stable homotopy, the following theorem is more or less implicit in Carlsson's work. The general case is due to Caruso and myself.

Theorem 15. If $G = (Z_p)^n$ and the completion conjecture holds for k_J^* for all proper subquotients J of G, then

$$\hat{k}_*^G(\tilde{E}G \wedge \underline{Z}) \cong M_n \otimes \Sigma^{1-n}\, \hat{k}_*^!(S^0).$$

The following theorem is due to Priddy and myself, although most of the work is in an Ext calculation due to others and discussed below.

Theorem 16. If $G = (Z_p)^n$ and k_* is bounded below and cohomologically bounded above, then

$$\hat{k}_*^G(EG^+ \wedge \underline{Z}) \cong M_n \otimes \Sigma^{-n}\, \hat{k}_* S^0.$$

In the absence of the bounded above hypothesis, there is an inverse limit of Adams spectral sequences such that

$$E_2 = \mathrm{Ext}_A(H^*(BG)[\alpha(Z)^{-1}] \otimes H^*(k), Z_p)$$

and $\{E_r\}$ converges to $\hat{k}_*^G(EG^+ \wedge \underline{Z})$.

Here the Euler class $\alpha(Z) \in H^{2(p^n-1)}(BG)$ has an obvious explicit description.

At this point, the virtues of naturality manifest themselves. Suppose that k_*^G is a split ring theory with unit $e_*\colon \pi_*^G \to k_*^G$. We have the following commutative square.

$$\begin{array}{ccc} \hat{\pi}^G_*(\tilde{E}G \wedge \underline{Z}) & \xrightarrow{\partial} & \hat{\pi}^G_*(EG^+ \wedge \underline{Z}) \\ \downarrow e_* & & \downarrow e_* \\ \hat{k}^G_*(\tilde{E}G \wedge \underline{Z}) & \xrightarrow{\partial} & \hat{k}^G_*(EG^+ \wedge \underline{Z}) \end{array}$$

To deduce the Segal conjecture for $G = (Z_p)^n$, it suffices to find a theory k^*_G for which the completion conjecture holds and $e_*: \hat{\pi}_0(S^0) \to \hat{k}'_0(S^0)$ is non-trivial mod p. Indeed, ∂ on the top is a morphism of $\hat{\pi}_0(S^0)$-modules whose domain and target are each freely generated by a copy of the free $\hat{Z}_p$-module M_n. Given k^*_G, Corollary 10, a bit of calculation along the lines of Theorem 15, and a chase of the diagram show that ∂ restricts to an isomorphism between the respective copies of M_n and is therefore an isomorphism. In [20], we thought that equivariant K-theory would do for k^*_G, but Costenoble has since proven the astonishing fact that in this case $\hat{k}'_*(S^0) = 0$. A choice which does work is specified by $k^*_G(X) = H^*_G(EG \times X; \underline{Z}_p)$. The completion conjecture holds trivially for theories so constructed by crossing with EG.

With k^G_* a split ring theory, the exact sequence (*) is one of $\hat{k}_*(S^0)$-modules and the isomorphisms of Theorems 15 and 16 are isomorphisms of $\hat{k}_*(S^0)$-modules. The general case of the diagram and the now established truth of the Segal conjecture lead to the second part of the following result, the first part being evident.

<u>Theorem 17</u>. If $G = (Z_p)^n$, k_* is bounded below and cohomologically bounded above, and the completion conjecture holds for k^*_J for all proper subquotients J of G, then the following conclusions hold.

(i) If $\hat{k}_*(S^0)$ and $\hat{k}^!_*(S^0)$ are not isomorphic as $\hat{Z}_p$-modules, then the completion conjecture fails for k^*_G.

(ii) If k^G_* is a split ring theory and $\hat{k}_*(S^0)$ and $\hat{k}^!_*(S^0)$ are isomorphic as $\hat{k}_*(S^0)$-modules, then the completion conjecture holds for k^*_G.

By Example 11 and part (i), we conclude that the completion conjecture generally fails for the theories $(S_G X)^*$ when X is a finite G-CW complex with non-trivial action by G. We shall later point out an important class of infinite G-CW complexes for which the completion conjecture holds. In these examples, $\hat{k}_*(S^0)$ and $\hat{k}^!_*(S^0)$ are not isomorphic, so we cannot expect the isomorphism of Theorem 16 to hold without the bounded above hypothesis.

We have another rather startling example of the failure of the isomorphism of Theorem 16. For any k^*_G, there is an associated connective G-cohomology theory for which all of the associated nonequivariant theories h^* are connective, $h_q(S^0) = 0$ for $q < 0$. When $G = (Z_p)^n$ and k_G^* is connective, $\hat{k}_q^G(\tilde{E}G \wedge \underline{Z}) = 0$ for $q < 1-n$.

Counterexample 18. The completion conjecture fails for real and complex connective equivariant K-theory kO_G^* and kU_G^* when $G = Z_2$. Here the domain of ∂ is zero in negative degrees, but Davis and Mahowald [11] have calculated its target groups and shown that they are periodic.

The completion conjecture for equivariant cobordism theories is under investigation.

We turn to the proofs of the four calculational theorems above. The starting point for Theorems 12 and 15 is Carlsson's observation that elementary obstruction theory implies a natural isomorphism

$$[X, \tilde{E}G \wedge Y]_G = [TX, Y]_G \ ,$$

where X is a finite G-CW complex, TX is its singular set, and Y is any G-space. Thus $\hat{k}_q^G(\tilde{E}G \wedge \underline{Z})$ is the inverse limit over j of the p-adic completions of the colimits over $V \supset \mathbb{R}^s$ of the groups

$$(**) \qquad [T(\Sigma^r S^{jZ} \wedge S^{V-\mathbb{R}^s}), k_G V]_G$$

The hard work in Carlsson's preprint is his analysis of TX. Following up Gunawardena's insistence that Quillen's work on posets ought to be relevant, Caruso and I found an almost trivial way of carrying out essentially the same analysis.

Let $\mathcal{A}$ be the poset of non-trivial elementary Abelian subgroups of G. As observed by Quillen [25], $B\mathcal{A}$ is contractible. In fact, $\mathcal{A}$ is a G-category via conjugation of subgroups and $B\mathcal{A}$ is G-contractible. We construct a topological G-category $\mathcal{A}[X]$ by parametrizing $\mathcal{A}$ by fixed points of X; the objects of $\mathcal{A}[X]$ are pairs (A,x), where A is an elementary Abelian subgroup of G and $x \in X^A$. There is an evident projection functor from $\mathcal{A}[X]$ to TX (regarded as a trivial topological G-category), and the induced map $B\mathcal{A}[X] \to TX$ is a G-homotopy equivalence by a simple application of Quillen's theorem A [24] to fixed point categories. It is convenient to factor out the contractible G-space $B\mathcal{A}[*]$ since it maps trivially to TX. Thus set $AX = B\mathcal{A}[X]/B\mathcal{A}[*]$. We have a natural G-homotopy equivalence $AX \to TX$. Now AX comes with an evident natural finite filtration. If we set $B_mX = F_mAX/F_{m-1}AX$, then we find by immediate inspection of definitions that

$$B_mX = \bigvee_{[\omega]} G^+ \wedge_{N(\omega)} \Sigma^m X^{A(\omega)}.$$

Here $[\omega]$ runs through the G-orbits of strictly ascending chains $\omega = (A_0 \subset \cdots \subset A_m)$

of non-trivial elementary Abelian subgroups of G, $A(\omega) = A_m$, and $N(\omega)$ is the isotropy group of ω under the action of G (of which $A(\omega)$ is a normal subgroup).

We plug this analysis into the description of $\hat{k}_q^G(\tilde{E}G \wedge \underline{Z})$ above. We may replace the functor T by the functor A in (**) and then pass to colimits, completion, and inverse limits. The filtration of A gives a finite sequence of long exact sequences in which the third terms come by substitution of B_m for T in (**). Inspection of definitions shows that the $[\omega]^{\underline{th}}$ wedge summand of the functor B_m contributes a copy of $\hat{k}_{q+m}^{J(\omega)}(\underline{Z}^{A(\omega)})$, where $J(\omega) = N(\omega)/A(\omega)$. If $J(\omega) \neq e$, then these groups are zero by Proposition 9 and the induction hypothesis. If $J(\omega) = e$ and $A(\omega) \neq G$, then these groups are zero since $Z^{A(\omega)}$ contains a copy of the trivial representation, so that the maps $e\colon S^0 \to S^{Z^{A(\omega)}}$ which give the inverse limit system are nonequivariantly null homotopic. Theorem 12 follows immediately. For Theorem 15, we are still left with those ω such that $A(\omega) = G = (Z_p)^n$. Since $Z^G = \{0\}$, a check of definitions shows that the $\omega^{\underline{th}}$ wedge summand contributes $\hat{k}'_{q+m}(S^0)$. Here we view our finite sequence of long exact sequences as an exact couple and obtain a spectral sequence converging to $\hat{k}_*(\tilde{E}G \wedge \underline{Z})$. Its E^1-term is the direct sum of a (reindexed) copy of $k_*^{\wedge}(S^0)$ for each ω with $A(\omega) = G$. Recall that the Tits building $\mathrm{Tits}(G)$ is the classifying space of the poset of non-trivial proper subgroups of G. We regard ω as a chain of $\mathrm{Tits}(G)$ by forgetting $A(\omega) = G$ and obtain an isomorphism

$$E^1_{m,q-m} = C^+_{m-1} \otimes k'_{q+m}(S^0),$$

where C^+_* denotes the augmented simplicial chains of $\mathrm{Tits}(G)$. The isomorphism carries d^1 to $d \otimes 1$ and Theorem 15 now follows from standard facts about Tits buildings [25,32].

The proofs of Theorems 13 and 16 are based on the existence of certain (nonequivariant) spectra with good properties.

<u>Theorem 19</u>. There exist spectra BG^{-V} and maps $f\colon BG^{-W} \to BG^{-V}$ for $V \subset W$ which satisfy the following properties.

(1) $k_*(BG^{-V})$ is isomorphic to $k_*^G(S^V;EG^+)$.

(2) The following diagram commutes.

$$\begin{array}{ccc}
k_*(BG^{-W}) & \xrightarrow{\qquad\qquad f_* \qquad\qquad} & k_*(BG^{-V}) \\
\wr\| & & \wr\| \\
k_*^G(S^W;EG^+) & \xrightarrow{(1\wedge e)_*} k_*^G(S^V \wedge S^{W-V};EG^+ \wedge S^{W-V}) \cong & k_*^G(S^V;EG^+)
\end{array}$$

(3) $H^*(BG^{-V})$ is a free $H^*(BG)$-module on one generator ι_V of degree $-\dim V$.

(4) $f^*: H^*(BG^{-V}) \to H^*(BG^{-W})$ is the morphism of $H^*(BG)$-modules determined by $f^*(\iota_V) = \alpha(W-V)\iota_W$, where $\alpha(W-V)$ is the Euler class of $W-V$.

Carlsson constructed such spectra by a kind of double dualization argument based on certain assumed facts about equivariant Spanier-Whitehead duality, proofs of which are given by Adams [2] and by Lewis and myself [17] in different stable contexts. A more conceptual, but also more technically difficult, construction is due to Lewis, Steinberger, and myself. Some years ago, we used our stable category of G-spectra to construct a spectrum level generalization of the familiar twisted half-smash product construction on spaces. The desired spectra may be specified by

$$BG^{-V} = EG \ltimes_G S^{-V}.$$

The required properties are then immediate consequences of spectrum level generalizations of familiar space level properties of twisted half smash products. By use of equivariant Thom spectra, Lewis and I have shown that BG^{-V} can also be described as the Thom spectrum of the virtual bundle $-V$ over BG. With this description, the cohomological properties in Theorem 19 are consequences of the Thom isomorphism.

Properties (1) and (2) of Theorem 19 give that

$$\hat{k}^G_*(EG^+ \wedge \underline{Z}) = \lim \hat{k}_*(BG^{-jZ}).$$

Properties (3) and (4) give that

$$\operatorname{colim} H^*(BG^{-jZ}) = H^*(BG)[L^{-1}],\ L = \alpha(Z).$$

It is true quite generally that passage to inverse limits from an inverse sequence of convergent Adams spectral sequences gives a convergent spectral sequence [20]. In particular, passage to inverse limits from the Adams spectral sequences of the spectra $BG^{-jZ} \wedge k$, where k represents k^*, gives a spectral sequence converging from

$$E_2 = \operatorname{Ext}_A(H^*(BG)[L^{-1}] \otimes H^*(k), Z_p)$$

to $\hat{k}^G_*(EG^+ \wedge \underline{Z})$. For Theorem 13, the localization is zero by the nilpotency of $\alpha(Z)$, hence $E_2 = 0$ and thus $\hat{k}^G_*(EG^+ \wedge \underline{Z}) = 0$. The first statement of Theorem 16 follows by convergence and an easy comparison of spectral sequences argument from the following homological calculation of Adams, Gunawardena, and Miller.

<u>Theorem 20</u>. If $G = (Z_p)^n$ and K is an A-module which is bounded above (and not in general otherwise), then

$$\mathrm{Ext}_A(H^*(BG)[L^{-1}] \otimes K, Z_p) \cong \mathrm{Ext}_A(N_n \otimes K, Z_p)$$

where N_n is a free Z_p-module on $p^{n(n-1)/2}$ generators which is concentrated in degree $-n$ and has trivial A-action.

The general case follows from the case $K = Z_p$ and the fact that $\mathrm{Ext}_A(Q, Z_p) = 0$ implies $\mathrm{Ext}_A(Q \otimes K, Z_p) = 0$ when K is bounded above. We sketch very briefly the key steps in the proof for $K = Z_p$. Singer (and Li) [30,31] introduced a basic construction R_+ on A-modules. For an A-module M, there is an augmentation $\varepsilon: R_+M \to M$. Adams, Gunawardena, and Miller [3,13] proved that $\mathrm{Ext}_A(\varepsilon, 1)$ is always an isomorphism. It follows inductively that there is an Ext isomorphism $(\Sigma^{-1}R_+)^n(Z_p) \to \Sigma^{-n}Z_p$. Singer and Li [30,31] proved that there is an isomorphism of A-modules

$$(\Sigma^{-1}R_+)(Z_p) \cong (H^*(BG)[L^{-1}])^{B_n}.$$

Here the general linear group $GL(n, Z_p)$ acts on the localization, and B_n denotes the Borel subgroup of upper triangular matrices. To obtain Theorem 20, one climbs up from the invariants to the entire localization to obtain an Ext isomorphism $H^*(BG)[L^{-1}] \to N_n$. This last step is carried out by direct inductive calculation up a chain of parabolic subgroups by Adams, Gunawardena, and Miller [3]. A conceptual, but less elementary, argument which highlights the role played by the Steinberg module is given by Priddy and Wilkerson [23]. (I have oversimplified slightly; when $p > 2$, both [3] and [23] replace R_+ by Gunawardena's enlarged analog [13] which is related to $H^*(BZ_p)$ as R_+ is to $H^*(B\Sigma_p)$.)

To close, we return to the Segal conjecture and describe its nonequivariant equivalent and an implied generalization, following Lewis, May, and McClure [16]. We return to general finite groups G. The coefficient groups $\pi_*^G(S^0)$ have been computed by several people [28,12]. The answer is

$$\text{(A)} \qquad \pi_*^G(S^0) \cong \sum_{(H)} \pi_*(BWH^+).$$

Here the sum ranges over conjugacy classes (H) of subgroups H and $WH = NH/H$, where NH is the normalizer of H in G. It is natural to ask for an interpretation of the Segal conjecture in terms of this isomorphism. The connection is best explained in terms of spectra and G-spectra [17]. The groups $\pi_G^*(S^0)$ are the homotopy groups of the fixed point spectrum $(S_G)^G$ of the sphere G-spectrum S_G. The groups $\pi_G^*(EG^+)$ are the homotopy groups of the fixed point spectrum of the dual G-spectrum $D_G(EG^+)$. Of course, $D_G(S^0) = S_G$, and $\varepsilon^+: EG^+ \to S^0$ induces a map of spectra

(B) $$\varepsilon^*: (S_G)^G \longrightarrow D_G(EG^+)^G \simeq D(BG^+),$$

where the equivalence comes from Lemma 4. The Segal conjecture may be viewed as a statement about this map. In particular, when G is a p-group, this map induces an equivalence upon p-adic completion. The isomorphism (A) comes from an equivalence

(C) $$\xi: \bigvee_{(H)} \Sigma^\infty(BWH^+) \longrightarrow (S_G)^G.$$

Tom Dieck's proof of (A) in [12] leads to an explicit description of ξ in terms of which $\varepsilon^* \circ \xi$ can be evaluated; see [16, Thms 1 and 8].

Observe that WH is the group of automorphisms of the G-set G/H, so that $G \times WH$ acts on G/H.

Theorem 21. The $H^{\underline{th}}$ component of the composite

$$\varepsilon^* \circ \xi: \bigvee_{(H)} \Sigma^\infty(BWH^+) \longrightarrow D(BG^+)$$

is the adjoint of the element $\tau(1) \in \pi^0((BG \times BWH)^+)$, where τ is the transfer associated to the natural cover

$$(G/H) \times_{G\times WH}(EG \times EWH) \longrightarrow BG \times BWH.$$

If G is a p-group, then $\varepsilon^* \circ \xi$ induces an equivalence upon completion at p.

In view of Theorem 6 and Proposition 7, the last statement is in fact equivalent to the Segal conjecture, and it is this formulation that was studied in special cases in the papers [18, 13, 27, 3, 21], among others.

Thus the Segal conjecture gives a description of the function spectrum

$$D(BG^+) = F(BG^+,S) = F(BG^+,\Sigma^\infty S^0).$$

It is natural to ask more generally if there is an analogous description of the function spectrum

$$F(BG^+,\Sigma^\infty B\Pi^+) \simeq F(\Sigma^\infty BG^+,\Sigma^\infty B\Pi^+)$$

for finite groups G and Π. The question was raised by Adams and Miller and answered when G and Π are elementary Abelian by Adams, Gunawardena, and Miller [3]. Lewis, McClure, and I [16] proved that the Segal conjecture implies an answer for arbitrary G and Π.

Let $B(G,\Pi)$ be the classifying G-space for principal (G,Π)-bundles. We have the G-prespectrum $S_G B(G,\Pi)^+$ of Example 11. Its cohomology theory is always split and is a split ring theory if Π is Abelian. The following is the main result of [16].

Theorem 22. The Segal conjecture implies the completion conjecture for the G-cohomology theories $(S_G B(G,\Pi)^+)^*$ for all finite groups G and Π.

Of course, the Segal conjecture itself is the case $\Pi = e$.

Again, the coefficient groups have been computed by tom Dieck [12]; indeed, he computes $\pi_*^G(X^+)$ in nonequivariant terms for any G-space X. When $X = B(G,\Pi)$, his result leads to the following description; see [16, Thm 1 and Prop 5].

$$\text{(A')} \qquad \pi_*^G B(G,\Pi)^+ \cong \sum_{(H)} \sum_{[(\rho)]} \pi_*(BW\rho^+).$$

Here the sums run over conjugacy classes (H) of subgroups H of G and WH-orbits $[(\rho)]$ of Π-conjugacy classes (ρ) of homomorphisms $\rho: H \to \Pi$; the groups are

$$W\rho = N_{G\times\Pi}(\Delta\rho)/\Delta\rho, \quad \text{where} \quad \Delta\rho = \{(h,\rho(h)) \mid h \in H\} \subset G \times \Pi.$$

Let Σ_G^∞ denote the suspension G-spectrum functor; $\Sigma_G^\infty X$ is the G-spectrum associated to the G-prespectrum $S_G X$, and Theorem 22 may be viewed as a statement about the map of fixed point spectra

$$\text{(B')} \qquad \varepsilon^*: [\Sigma_G^\infty B(G,\Pi)^+]^G \longrightarrow F(EG^+, \Sigma_G^\infty B(G,\Pi)^+)^G \simeq F(BG^+, \Sigma^\infty B\Pi^+),$$

where the equivalence again comes from Lemma 4. The isomorphism (A') comes from an equivalence

$$\text{(C')} \qquad \xi: \bigvee_{(H)} \bigvee_{[(\rho)]} \Sigma^\infty(BW\rho^+) \longrightarrow [\Sigma_G^\infty B(G,\Pi)^+]^G.$$

Theorem 21 generalizes as follows. See [16, Thms 1 and 8].

Theorem 23. The ρ^{th} component of the composite

$$\varepsilon^* \circ \xi: \bigvee_{(H)} \bigvee_{[(\rho)]} \Sigma^\infty(BW\rho^+) \longrightarrow F(BG^+, \Sigma^\infty B\Pi^+)$$

is the adjoint of the following composite.

$$\Sigma^\infty(BG \times BW\rho)^+ \xrightarrow{\ \tau\ } \Sigma^\infty(B\rho^+) \xrightarrow{\Sigma^\infty \mu^+} \Sigma^\infty(B\Pi^+)$$

Here $B\rho = E\rho/\Pi$, where $E\rho = [(G \times \Pi)/\Delta\rho] \times_{G\times W\rho} (EG \times EW\rho)$, μ is the classifying map of the Π-bundle $E\rho \to B\rho$, and τ is the transfer associated to the cover

$B\rho \to BG \times BW\rho$. If G is a p-group, then $\varepsilon^* \circ \xi$ induces an equivalence upon completion at p.

On the π_0-level, Theorems 22 and 23 lead to a complete description of the group of stable maps $[\Sigma^\infty BG^+, \Sigma^\infty B\Pi^+]$ in terms of purely algebraic Burnside ring level information. Let $A(G,\Pi)$ be the Grothendieck group of Π-free finite $(G \times \Pi)$-sets. It is free Abelian on the set of transitive $(G \times \Pi)$-sets $S = G \times \Pi/\Delta\rho$, $\rho\colon H \to \Pi$, appearing in the previous theorem, and we associate to S the stable map

$$\alpha(S)\colon \Sigma^\infty(BG^+) \xrightarrow{\tau} \Sigma^\infty(BH^+) \xrightarrow{\Sigma^\infty(B\rho^+)} \Sigma^\infty(B\Pi^+).$$

If $\iota\colon BG \to BG \times BW\rho$ is obtained by choosing a basepoint in $BW\rho$, then $\alpha(S) = \Sigma^\infty\mu^+ \circ \tau \circ \Sigma^\infty\iota^+$ by an easy verification. Observe that $A(G,\Pi)$ is an $A(G)$-module and let $\hat{A}(G,\Pi)$ be its completion with respect to the topology given by the augmentation ideal of $A(G)$.

Corollary 24. There is a natural isomorphism

$$\alpha\colon \hat{A}(G,\Pi) \longrightarrow [\Sigma^\infty BG^+, \Sigma^\infty B\Pi^+].$$

When G is a p-group, we may use completion at p provided we also complete the right-hand side. As observed by Nishida [22], this has the following striking (and easy) consequence.

Corollary 25. If G and Π are finite groups such that BG and $B\Pi$ are stably p-equivalent, then G and Π have isomorphic p-Sylow subgroups.

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EQUIVARIANT LOCALIZATION AND LIMITS OF THOM SPECTRA

E. OSSA

1. Introduction

In a talk at this conference, J. Jones presented applications of the Segal conjecture to "Limits of Stable Homotopy and Cohomotopy Groups" [6] obtained in collaboration with S. Wegmann. As a first step to their results, Jones and Wegmann investigate an interesting direct system of spectra:
Denoting by $D_2(Y) = (S^\infty)^+ \wedge_{\mathbb{Z}/2} (Y \wedge Y)$ the quadratic construction on Y, they define natural stable maps $S^{-k}D_2(S^kY) \to S^{-k-1}D_2(S^{k+1}Y)$. They show that these make sense even for negative integers k and proceed to take an inverse limit as $k \to -\infty$. Among other things, they prove then that applying 2-adically completed homotopy groups leads to an isomorphism $\varprojlim \hat{\pi}_*(S^{-k}D_2(S^kY)) \cong \hat{\pi}_*Y$.

Our aim is to offer a natural explanation of these and related phenomena through an equivariant localization theorem. To explain this let us first recall the classical equivariant localization theorem (originally due to Atiyah and Segal) for a multiplicative equivariant cohomology theory h^*_G (see e.g. [9,4]). The most simple version states that for a finite G-CW-complex X the inclusion of the fixed point set X^G in X induces an isomorphism of localized groups $E^{-1}h^*_G(X) \xrightarrow{\cong} E^{-1}h^*_G(X^G)$, where E is the set of Euler-classes of fixed point free representations of G. In § 2 we describe a modified version of this localization theorem which is suited to our applications. The essential difference is that we concentrate on properties of the functor $E^{-1}h^*_G$. The proof, however, follows closely the classical arguments.

In § 3 we consider, for a finite G-CW-complex X, a directed system $\dots \to S_k(X) \to S_{k+1}(X) \to \dots$ of Thom spectra $S_i(X)$ corresponding to a sequence of (virtual) fixed point free representations of G. In the case $G = \mathbb{Z}/2$ and $X = Y \wedge Y$ this is precisely the directed system $(S^{-k}D_2(S^kY))$ of Jones and Wegmann mentioned above. We then go on to show that certain limits $L(X)$ associated to the system $(S_i(X))$ fulfil the requirements of the localization theorem, thus yielding isomorphisms $L(X) \cong L(X^G)$ which generalize (and explain) the results of Jones and Wegmann.

Our main result is stated as proposition (3.2). We do not treat here further consequences and corollaries. A point which could be worth further study might be the relation to the work of W. Singer (see [10] and [6]). Another puzzling question is how much information on the homotopy type of the inverse limit of Thom spectra over BG can be derived from the formal properties stated in (3.2); for G cyclic some information on this homotopy type is given in [8], for $G = \mathbb{Z}/p$ it is a completed sphere by the results of [7] and [5].

This work was done while I spent a free term at the IHES. I would like to take this opportunity to thank Prof. N.H. Kuiper and the staff of the IHES for their kind and generous hospitality.

2. The localization theorem.

Let G be a compact Lie group.

We denote by $\mathcal{C}_o(G)$ the category of finite G-CW-complexes . Let $\mathcal{D}$ be either the category of CW-spectra or the category of graded abelian groups. Then in $\mathcal{D}$ we have a notion of cofibration sequences, where in the case of graded abelian groups a cofibration sequence $(A_i) \to (B_i) \to (C_i)$ is a long exact sequence $.. \to A_i \to B_i \to C_i \to A_{i-1} \to ..$ (to achieve full generality we should probably postulate that $\mathcal{D}$ is a triangulated category).

Consider functors $T : \mathcal{C}_o(G) \to \mathcal{D}$. It does not matter which variance T has, but let us assume that T is covariant. We call T homological if it has the following two properties :

(H) T is a homotopy functor, that is if $f,g : X \to Y$ are G-homotopic, then $T(f) = T(g)$.

(CF) T preserves cofibrations, that is if $X \to Y \to Z$ is a G-cofibration in $C_o(G)$, then $T(X) \to T(Y) \to T(Z)$ is a cofibration in D.

Now let $\mathbb{V}$ be a set of representations of G. For example $\mathbb{V}$ might be the set $\mathbb{F}(G)$ of all fixed point free representations of G (here V is called fixed point free if $V^G = 0$). For a representation V we denote by V^+ the one point compactification of V as a G-space.

We say that the functor $T : C_o(G) \to D$ is $\mathbb{V}$-localizing if it has the following property :

(LO) For any $X \in C_o(G)$ and any $V \in \mathbb{V}$ the inclusion $X = X \wedge S^o \to X \wedge V^+$ induces an isomorphism $T(X) \to T(X \wedge V^+)$.

Note that in the classical situation, where T is a cohomology functor, this condition is enforced by making the Euler-class of V invertible.

Now we can state the localization theorem for a homological $\mathbb{V}$-localizing functor. A G-space Z will be called $\mathbb{V}$-adapted if for some representation W, which is sum of representations in $\mathbb{V}$, there exists an equivariant map $f : Z \to W - \{0\}$.

(2.1) Localization Theorem : Let $T : C_o(G) \to D$ be homological and $\mathbb{V}$-localizing. Let (X,Y) be a pair in $C_o(G)$ such that $X - Y$ is $\mathbb{V}$-adapted. Then the inclusion $Y \to X$ induces an isomorphism $T(Y) \xrightarrow{\cong} T(X)$.
It is well known that in the case $\mathbb{V} = \mathbb{F}(G)$ the pair $(X,Y) = (X,X^G)$ satisfies the requirements of the theorem. Thus if T is homological and $\mathbb{F}(G)$-localizing, then the inclusion $X^G \to X$ induced an isomorphism $T(X^G) \cong T(X)$.

The proof of (2.1) proceeds in two steps.

First assume that Z is a finite G-CW-complex which is $\mathbb{V}$-adapted. Let $f : Z \to W - \{0\}$ be G-equivariant where W is sum of representations in $\mathbb{V}$. Then the inclusion $Z^+ = Z^+ \wedge S^o \to Z^+ \wedge W^+$ is homotopic to zero by the homotopy $f_t(z) = z \wedge t \cdot f(z)$, $0 \leq t \leq \infty$. By (H) the induced map $T(Z^+) \to T(Z^+ \wedge W^+)$ is zero. On the other hand (LO) guarantees that this induced map is an isomorphism. Hence the claim of the theorem is true for the pair $(X,Y) = (Z^+,*)$.

As the second step we treat the general pair (X,Y). Observe that there are finite subcomplexes $A,B\subset X$ with the following properties : (a) $X = A\cup B$, (b) $B\subset X-Y$, (c) the inclusion $Y\to A$ is a G-homotopy equivalence. Put $C = X/B^+$. Then we have cofibrations $B^+\to X\to C$ and $(A\cap B)^+\to A\to C$ which, by the result of the first step, induce isomorphisms $T(X)\cong T(C)$ and $T(A)\cong T(C)$. Since $T(Y)\cong T(A)$, the claim follows.

3. Limits of Thom spectra.

Let EG be a universal principal G-bundle. For any finite G-CW-complex X put $B_G(X) = EG\times_G X$; in particular $B_G(*) = BG$ is the classifying space of G . If V is a representation of G we can construct the vectorbundle $B_G(X;V) = EG\times_G(X\times V)$ over $B_G(X)$, and if $\alpha = V-W$ is a virtual representation we obtain the virtual vectorbundle $B_G(X;V) - B_G(X;W)$. Finally let $M_G(X;\alpha)$ be the Thom spectrum of $B_G(X;\alpha)$, and if X is a based G-CW-complex with base point $*$, let $\bar{M}_G(X;\alpha)$ be the cofibre of $M_G(*,\alpha)\to M_G(X;\alpha)$.

This construction is natural in α : if U is another representation of G the inclusion $\alpha\hookrightarrow\alpha\oplus U$ of virtual representations induces maps of spectra $\bar{M}_G(X;\alpha)\to\bar{M}_G(X;\alpha\oplus U)$.

This construction has also good naturality properties in the variable X : if $X\to Y\to Z$ is a cofibration sequence in $C_o(G)$, then $\bar{M}_G(X;\alpha)\to\bar{M}_G(Y;\alpha)\to\bar{M}_G(Z;\alpha)$ is a cofibration sequence of spectra.

Now suppose that we are given a sequence $\mathbb{V} = (V_i)$ of representations of G which is directed in the sense that the following condition is fulfilled :

(D) $V_i\subset V_{i+1}$ and $V_i\oplus V_i\subset V_{i+k}$ for some $k = k(i)$.

In particular the inclusions $V_i\subset V_{i+1}$ lead to a directed system of spectra

$$\text{(3.1)}\qquad \dots\to\bar{M}_G(X;-V_{i+1})\to\bar{M}_G(X;-V_i)\to\bar{M}_G(X;-V_{i-1})\to\dots$$

We can now state the main result of this note :

(3.2) Proposition : Assume that $\mathbb{V} = (V_i)$ is directed (D) . Let (X,Y) be a pair in $C_o(G)$ such that $X-Y$ is $\mathbb{V}$-adapted. Then the inclusion $Y \to X$ induces isomorphisms as follows :

(a) $$\lim_{\overrightarrow{i}} \mathrm{Map}(\bar{M}_G(X;-V_i),Z) \xrightarrow{\cong} \lim_{\overrightarrow{i}} \mathrm{Map}(\bar{M}_G(Y;-V_i),Z)$$

for any spectrum Z , where $\mathrm{Map}(F,Z)$ denotes the function spectrum of maps from F to Z [2] , defined by $[-, \mathrm{Map}(F,Z)] := [- \wedge F, Z]$.

(b) $$\operatorname*{holim}_{\overleftarrow{i}} (\bar{M}_G(Y;-V_i) \wedge Z)^{\wedge} \xrightarrow{\cong} \operatorname*{holim}_{\overleftarrow{i}} (\bar{M}_G(X;-V_i) \wedge Z)^{\wedge}$$

for any connected spectrum Z of finite type, where $\hat{}$ stands for profinite completion.

Of course we then have corresponding isomorphisms of homotopy groups. Perhaps we should remark that the homotopy limit of a sequence $.. \to E_i \xrightarrow{f_i} E_{i+1} \to ..$ of spectra can be defined through the cofibration sequence $\operatorname*{holim}_{\overleftarrow{i}} E_i \to \prod_i E_i \xrightarrow{d} \prod_i E_i$ where $d = \prod \mathrm{id}_{E_i} - \prod f_i$ [3] . Thus the homotopy groups of $\operatorname*{holim}_{\overleftarrow{i}} E_i$ fit into the Milnor-sequence for the $\pi_*(E_i)$ (see [3] or [1]).

It is perhaps worthwhile to point out a special case of (3.2). Let $\mathbb{F}(G) = (V_i)$ be a family of representations such that any fixed point free representation of G is contained in some V_i .

(3.3) Corollary : Let $(V_i) = \mathbb{F}(G)$, and put $L_G = \operatorname*{holim}_{\overleftarrow{i}} \bar{M}_G(S^o;-V_i)^{\wedge}$. Then for any finite G-CW-complex we have a homotopy equivalence

$$\operatorname*{holim}_{\overleftarrow{i}} \bar{M}_G(X;-V_i)^{\wedge} \simeq X^G \wedge L_G$$

Note in particular, that for a cyclic group G of prime order p , we may identify L_G by Lin's theorem [7,5] with the p-adic completion of the (-1)-sphere.

We turn now to the proof of (3.2).

Let L be any functor defined on sequences of spectra of finite type $\dots \to E_{-i-1} \to E_{-i} \to E_{-i+1} \to \dots$ (indexed by the negative integers) with the following two properties :

(i) if $E_{-i} \to F_{-i} \to G_{-i}$ are cofibrations, then also $L(E) \to L(F) \to L(G)$ is a cofibration.

(ii) if $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function and $F_{-i} = E_{-f(i)}$, then the obvious maps $F_{-i} \to E_{-i}$ induce an isomorphism $L(F) \xrightarrow{\cong} L(E)$.

Examples for such functors are clearly the functors which enter in (3.2) : $\varinjlim_i \mathrm{Map}(E_{-i},Z)$ and $\operatorname{holim}_{\overleftarrow{i}} (E_{-i}\wedge Z)^{\wedge}$. Hence, to prove (3.2), it suffices to refer to (2.1) and to prove the following :

(3.4) <u>Proposition</u> : Let L satisfy conditions (i), (ii) above, and let $\mathbb{V} = (V_i)$ be a directed sequence of representations of G . For $X \in C_o(G)$ put $S_{-i}(X) = \bar{M}_G(X;-V_i)$, and define $L_{\mathbb{V}}(X) = L(\ldots \to S_{-i-1}(X) \to S_{-i}(X) \to \ldots)$.

Then the functor $X \to L_{\mathbb{V}}(X)$ is homological and $\mathbb{V}$-localizing.

But the fact that $L_{\mathbb{V}}(X)$ is homological is obvious from condition (ii) and the fact that the functor $\bar{M}_G(\ldots;-V_i)$ preserves cofibrations. The fact that $L_{\mathbb{V}}(X)$ is $\mathbb{V}$-localizing is routine, using (ii) : let V be in $\mathbb{V}$ and define $\mathbb{W} = (W_i)$ by $W_i = V_i + V$. Then we have inclusions $V_i \subset W_i \subset V_{f(i)}$ with $f : \mathbb{N} \to \mathbb{N}$ strictly increasing; these induce (already using condition (ii)) natural transformations $L_{\mathbb{V}}(X) \leftrightarrows L_{\mathbb{W}}(X)$ which, again by (ii), must be isomorphisms. This proves (3.4).

Finally we point out the connection with [6]. Let $G = \mathbb{Z}/2$ and $X = Y \wedge Y$ with the obvious G-action. Let V be the non-trivial one-dimensional representation of G . Then the quadratic construction $D_2(S^kY)$ coincides with $\bar{M}_G(X;k(V\oplus\mathbb{R})) = S^k\bar{M}_G(X;kV)$, and the maps constructed in [6] are induced by inclusions of representations.

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UPPER BOUNDS FOR THE TORAL SYMMETRY OF CERTAIN HOMOTOPY SPHERES

Reinhard Schultz

Given a smooth manifold M^n, its <u>degree of (compact) symmetry</u> $N(M^n)$ is the maximum dimension of the compact Lie groups which act smoothly and effectively on M. A classical theorem of differential geometry states that $N(M) \leq n(n+1)/2$ with equality if and only if $M^n = S^n$ or $\mathbb{R}P^n$ (compare [6,7]). One can define an analogous toral degree of symmetry $N_T(M^n)$ by considering the rank of the largest torus that acts smoothly and effectively on M. A fairly elementary argument shows that $N_T \leq n$ with equality if and only if $M^n = T^n$. Using cohomological methods one can show that

$$N_T(S^n) = [\tfrac{n+1}{2}]$$

("[ ]" denotes the greatest integer function),

the upper limit being realized by the maximal torus in SO_{n+1} (see [11] for stronger results).

Suppose now that M^n is an integral cohomology n-sphere and T^n acts smoothly and effectively with $r = [(n+1)/2]$. In this case it is known that M^n bounds a contractible manifold (e.g., combine [8] with weight considerations from [4]; see also [5]). In particular, if M is a homotopy sphere and $n \geq 5$, then M is diffeomorphic to S^n.

Motivated by results on the degree of symmetry of exotic spheres (compare [14]), one suspects that $N_T(M^n)$ is always <u>significantly</u> less than $N_T(S^n)$ if M^n is a homotopy sphere. In fact, a conjecture of L. Mann [13] states that $N_T(M^n)$ should be at most $[(n+3)/4]$. It is known that the Kervaire sphere of dimension $4k+1$ admits such an action by virtue of its presentation as a Brieskorn variety. Although some general statements can be made about $N_T(M^n)$, at present they are much weaker than Mann's conjecture. In this paper we study $N_T(\Sigma^n)$ for a specific class of examples; namely, for each odd prime p we consider the first exotic sphere that has order p in the Kervaire-Milnor group Θ_n but does not bound a parallelizable manifold. From [6] and [14] one expects that the bound in Mann's conjecture is far from being optimal. Our main result confirms this:

THEOREM. <u>Let</u> p <u>be an odd prime, and let</u> Σ^n <u>be a homotopy</u> $(2p^2-2p-2)$-<u>sphere whose Pontrjagin-Thom invariant in</u> $\Sigma_{n(p)}$ <u>has order</u> p (i.e., <u>it is a nontrivial multiple of Toda's</u> β_1). <u>Then</u>

$$N_T(\Sigma^n) \leq \frac{p+3}{2}.$$

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It seems likely that this estimate is not best possible. However, it should be added that $N_T \geq 1$ for the "basic" examples $\sum^n(p)$ which have order p in Θ_n [15].

Note. If $p \equiv 3 \bmod 4$, then $2p^2 - 2p - 2 \equiv 2 \bmod 8$. In this case we may take the connected sum of $\sum(p)$ with some $\sum'$ not bounding a spin manifold. By results of Lawson and Yau [10] we know $N(\sum\#\sum') = N_T(\sum\#\sum')$; from this one can show that $N(\sum\#\sum') \leq \sqrt{\dim \sum}/2$. In this case we also know $N(\sum\#\sum') \geq 1$.

The idea behind the proof is fairly transparent. Given a smooth $\mathbf{Z}/p$ action on a homotopy sphere one can define a knot invariant as in [16,17]; this invariant will lie in a quotient of the homotopy group $\pi_k(F_{\mathbf{Z}/p})_{(p)}$, where k is the dimension of the fixed point set and $F_{\mathbf{Z}/p}$ is to the limit of the spaces of equivariant self-maps of the unit spheres $S(V)$ in free orthogonal representations V of $\mathbf{Z}/p$. By [3] this homotopy group is the stable homotopy of $(B(\mathbf{Z}/p) \vee S^0)$. For all actions on $\sum(p)$ as above, the knot invariant is known to be nontrivial (compare (3.2) below). If the $\mathbf{Z}/p$ action extends to (say) an action of the torus T^r, this knot invariant turns out to lie in the image of some stable homotopy "transfer map"

$$(0.1) \qquad (f^!)_*: \pi_k^s(X(T^r))_{(p)} \to \pi_k^s(B(\mathbf{Z}/p) \vee S^0),$$

where $X(T^r)$ is some complex whose precise definition depends on some detailed data that need not concern us yet (see Section 1). One uses the Atiyah-Hirzebruch spectral sequence to study $f^!$ (see Section 2). This is possible in part because the stable stems $\pi_{j(p)}$ are mostly zero for $j \leq 2p^2 - 2p - 2$; a second important factor is that the S-map $f^!$ factors as an r-fold composite

$$(0.2) \qquad X(T^r) \to X(T^{r-1}) \to \cdots \to X(S^1) \to B(\mathbf{Z}/p) \vee S^0,$$

and all factors except the last one have an unusual property: They cause the Atiyah-Hirzebruch filtration of an element to decrease by at least 1. Since there are only finitely many filtrations that are possibly nonzero, this says that $(f^!)_*$ is zero if r is a moderately large number depending on k (exactly how large must be determined). If r is so large that $(f^!)_*$ is zero regardless of k, then the knot invariant for any $\mathbf{Z}/p$ action on $\sum$ that extends to T^r must be zero. Since the knot invariant of the $\mathbf{Z}/p$ action must be nonzero, there a contradiction which implies no T^r action exists. Little additional work is needed to determine the possible values for r associated to each k (see Section 3). Specifically, these conclusions exclude T^r actions for $r \geq p$. In order to halve this estimate to $r > (p+3)/2$ we must use more specific conclusions that follow from this argument together with an analysis of the possible local representations of the induced $(\mathbf{Z}/p)^r$ action at one of its fixed points. This is done in the final Section 4.

The principal new tool is a concept of knot invariant for torus actions that justifies the assertion

(Knot invariant of $\mathbb{Z}/p$-action which extends to T^r) $\in$

Image $(f^!)_*$.

($f^!$ defined as in (0.1)).

The previous methods to define knot invariants relied on the fact that certain subspaces of a G-manifold had free G-actions. In Section 1 we indicate how one can avoid this problem to some extend by using the product of a subspace with the universal free G-space (or some finite approximation to it); in some sense this was inspired by the notion of "completed bundle data" in work of T. Petrie and others. The "completed knot invariants" we define also prove to be useful in other connections, so we have merely summarized what is needed here.

One other innovation deserves an acknowledgment. The argument involving (0.2) and filtration-lowering properties was suggested by previous arguments of Kh. Knapp [9], S. Stolz [22] and others along parallel lines.

Further results on $N_T(\Sigma)$ based on the ideas presented here, in [19], and in [20] will appear in subsequent papers.

1. Knot invariants arising from toral actions

Suppose that G is a compact Lie group acting smoothly on a homotopy sphere Σ^n, and assume that F^k is the fixed point set of the (closed) normal subgroup H. Assume also that the fixed point set of G is nonempty and connected. Suppose that F is an R-homology sphere, where R is a suitably chosen subring of the rationals. If V denotes the normal representation of G at a fixed point and $S(V)$ is its unit sphere, then the composite

(1.0) $$S(V) \subseteq S(\nu_F) \subseteq \Sigma - F$$

(ν_F denotes the equivariant normal bundle of F)

is an R-local homotopy equivalence if $\dim V > 2$ and an R-homotopy retract otherwise. Furthermore, the composite (1.0) is obviously equivariant. The knot invariant construction from [16,17] depends on the fact that the R-localization of (1.0) has an R-homotopy inverse (if $\dim V \leq 2$, a canonical one-sided inverse).

In the cases considered it was fairly routine to construct the inverses, one key point being that G always acted freely on $\Sigma - F$. However, once one looks beyond a special (but important) class of groups, such freeness assumptions become increasingly restrictive; in fact, it is never possible for a torus of rank > 1 to act freely on $\Sigma - F$ (this follows from the Borel formulas [4]). Suppose however that we cross (1.0) with a highly connected free G-space E. Then $S(V) \times E \to (\Sigma - F) \times E$ will again be an R-homotopy equivalence or retraction, and an obstruction-theoretic argument as in [16,17] will imply that the composite

$$(1.1) \qquad S(V) \times E \xrightarrow{proj} S(V) \xrightarrow{\lambda} S(V)_R$$

(λ = equivariant localization in, say, the sense of [12])

has a canonical equivariant extension to $\hat{\rho}:(\sum - F) \times \vec{E} \to S(V)_R$ (unique, in fact, up to homotopy). Therefore, we define the "homotopy class" of the pair

$$(\nu_F, \hat{\rho} \mid S(\nu_F) \times E)$$

to be the (formally) completed knot invariant of the pair $(\sum, F = \sum^H)$ with respect to the ring R. In other words, the knot invariant data consists of

(i) A G-vector bundle ξ over F.

(ii) An equivariant map ϕ which is a equivariant fiber homotopy trivialization for the R-localization of $S(\hat{\xi})$, where $\hat{\xi}$ denotes the "completed bundle" $\xi \times E$ over $F \times E$.

(The term "completed bundle" is taken from Petrie's terminology, which in turn alludes to the Atiyah-Segal completion theorem [1]).

The correct choice of E poses some problems. It would be nice to say E is contractible, but this would force E to be infinite dimensional, and some of the necessary homotopy-theoretic constructions become at best questionable with this condition. For our purposes it will be best to make the following assumption: E is chosen in advance to be a very highly connected closed π-manifold, the connectivity far exceeding that of any of the fixed objects of primary interest. For example, if we are looking at a homotopy sphere $\sum^n$, the connectivity of E will exceed n by a great deal. However, if we talk about stabilizing (ξ,ϕ) by "adding a trivial bundle", we shall allow the dimension of the trivial bundle to exceed the connectivity of E; the stable bundle being added is not a "fixed object" in this contest.

It is time to give a more formal definition:

DEFINITION. Let E be a highly connected free G-manifold (G compact Lie) such that E/G is stably parallelisable, let ℓ be a set of primes, let V be a G-module, and let X be an invariantly based GCW-complex (i.e., X has a basepoint x_0 which is fixed under G). The set

$$(1.1) \qquad (E,F)/O_{G,V,\ell}(X,x_0)$$

consists of equivalence classes of triples (ξ,j,ϕ) where ξ is a G-vector bundle over X, a G-isomorphism $V \to \xi_{x_0}$ (fiber), and a G-map

$$\phi: S(\xi) \times E \to S(V)_\ell$$

(here "S( )" denotes unit sphere bundles and "ℓ" denotes equivariant localization [12]).

such that $\phi \circ [j \times id(E)] \simeq_G \lambda$, where λ is the composite $E \times S(V) \to S(V) \to S(V)_\ell$. Isomorphisms of such triples are defined in the standard fashion, and two triples are said to be equivalent (concordant) if there is a triple over $X \times I$ whose restrictions to $X \times \{0\}$ and $X \times \{1\}$ are isomorphic to the original triples.

With this definition and the previous construction we can define a knot invariant $\omega_E(\Sigma,G,H)$ as an element of the set

$$(E,F)/O_{G,V,\ell}(\Sigma^H).$$

If Γ is a group between H and G, then by construction the knot invariant $\omega_E(\Sigma, \ ,H)$ is just the image of $\omega_E(\Sigma,G,H)$ under the forgetful map from G-objects to Γ-objects.

It is immediate that (1.1) gives a contravariant G-homotopy functor that is representable. Furthermore, it also follows that one has well defined and natural direct sum pairings

$$(1.2) \qquad (E,F)/O_{G,V,\ell} \times (E,F)/O_{G,W,\ell} \to (E,F)/O_{G,V\oplus W,\ell}$$

with familiar homotopy associativity and commutativity properties (we shall not need any higher order properties of these sorts, however).

One can also pass to a limit under stabilization by representations within some fixed family $\mathcal{F}$ that may be more or less arbitrary at this stage. The resulting functor $(E,F)/O_{G,\ell}$ will be abelian monoid valued.

In order to work with $(E,F)/O_{G,\ell}$ effectively, we must trap it between two representable functors that we can work with more easily. Fortunately, there is a complete analogy with the standard homotopy exact sequence $F \to F/O \to BO$.

PROPOSITION 1.2. *Let* $Vect_{G,V}(Y,B)$ *be given by G-vector bundles* ξ *over* Y *with trivializations* $\xi|B \cong B \times V$ (*for* V *a G-subcomplex of the GCW complex* Y). *Let* $(E,F)_{G,V,\ell}(X,x_0)$ *be the set of G-homotopy classes of G-maps*

$$f\colon Y \times S(V) \times E \to S(V)_\ell$$

such that $f|\{x_0\} \times S(V) \times E = \lambda$ (*as above*). *Then the following sequence of representable G-homotopy functors is exact*:

$$(E,F)_{G,V,\ell}(X,x_0) \xrightarrow{\alpha} (E,F)/O_{G,V,\ell}(X,x_0) \xrightarrow{\beta} Vect_{G,V}(E \times X, E\times\{x_0\}).$$

A similar statement holds for functors stabilized over some family of representations.

Notation. The map α is obtained by setting ξ = trivial bundle with fiber V, j = standard inclusion, and $\phi = f$. The map β is formed by taking the product of ξ and E.

The proof of 1.2 is a direct generalization of the arguments used for $F \to F/O \to BO$ in the nonequivariant case and for other, previously studied, equivariant analogs [17].

The functors $\mathrm{Vect}_{G,V}$ can be handled by more or less standard vector bundle theory, and the functors $(E,F)_G$ are essentially just equivariant function spaces (very similar to the space of G-maps from E to F). If $X = S^W$ for W a G-module, then the methods of [3] (see also [17]) can be modified to give partial but adequate information on $(E,F)_{G,\ell}(X,x_0)$. The maps $E \times X \times S(V) \to S(V)_\ell$ correspond to ex-maps $(E \times_G X) \times S^0 \to E \times X \times S(V) \times S(V)_\ell/G$, which in turn map by fiberwise suspension to the stable cohomotopy of the Thom space of

$$(1.4) \qquad E \times X \times (\nu_{S(V)} - \mathrm{ad}_G)/G.$$

The map from $(e,F)_G$ to this cohomotopy group need <u>not</u> be injective or surjective in general. Despite this, we can still proceed to take S-duals. Since (E/G) is a π-manifold, the S-dualization simplifies and we find that the construction of [3] maps $(E,F)_G(S^W,\infty)$ into

$$(1.5) \qquad \pi^S_{\dim W}(E/G)^{\mathrm{ad}+\dim W-W})_\ell$$

and via the universal map $E/G \to BG$ into

$$(1.6) \qquad \pi^S_{\dim W}(BG^{\mathrm{ad}+\dim W-W})_\ell.$$

The latter group already arose in the discussion of knot invariants for ultrasemifree actions. Let Λ denote the induced homomorphism from $(E,F)_G(S^W,\infty)$ to the group considered in (1.6).

The following is a routine verification that follows by checking the definitions, and using the G-equivalence $Y \simeq Y \times E(\infty)$ (which holds if Y is free and $E(\infty)$ is free and contractible), and observing that the connectivity of E is much greater than $\dim W + \dim G$.

(1.7) <u>Let</u> V <u>be a free G-module</u>. <u>If</u>

$$\lambda\colon F_{G,V,\ell}(S^W) \to \pi^S_{\dim W}(BG^{\mathrm{ad}+\dim W-W})$$

<u>is defined as in</u> [3, §3] <u>and</u>

$$\pi^*\colon F_{G,V,\ell} \to (E,G)_{G,V,\ell}$$

<u>is defined by crossing with</u> E, <u>then</u> $\lambda = \lambda\pi^*$. <u>A similar statement holds for stabilized functors</u>. ∎

In order to use these results on equivariant stable homotopy, we must be able to compare the functors in (1.2) for S^W and $\sum^H$, where G acts on $\sum$ as in the beginning of this section. As usual, this is done by the equivariant degree 1 map $f\colon \Sigma^H \to S^W$ which collapses everything outside a linear neighborhood of a fixed point. Now assume G is a torus, Σ is a closed smooth $\mathbb{Z}/p$ homology sphere, and H is a p-subgroup or subtorus; the possibility $p = 0$ with H a subtorus is allowed, and in this case there is no localization of homotopy groups or spaces (in other words, all "(p)"'s should be deleted in this case). By Smith theory we know that Σ^H is also a $\mathbb{Z}/p$ homology sphere. It is therefore immediate that $f \times \mathrm{id}_E$ is an equivariant p-local homology equivalence, and because of this one can prove the following result by an obstruction theoretic argument.

PROPOSITION 1.8. *The maps*

$$f^*\colon (E,F)_{G,V,(p)}(S^W) \to (E,F)_{G,V,(p)}(\Sigma^H)$$

$$f^*\colon (E,F)_{G,(p)}(S^W) \to (E,F)_{G,(p)}(\Sigma^H)$$

are isomorphisms. ■

Here is where we now stand: We know that the stabilized knot invariant $s_*\omega_E(\Sigma,G,H)$ lies in $(E,F)/O_{G,(p)}(\Sigma^H)$ and that the latter is trapped between the groups $KO_G(E \times \Sigma^H, E \times \{x_0\})$ and $(E,F)_{G,(p)}(\Sigma^H) = (E,F)_{G,(p)}(S^W)$. We would like to show that the image of $s_*\omega_E$ comes from a class in the latter group; actually, for our purposes it is enough to show this is true after a suitable localization at p (in fact, the conclusion might be false without such a localization). We proceed as follows: Each of the functors in the exact sequence

$$(1.9)\qquad (E,F)_{G,(p)} \xrightarrow{\alpha} (E,F)/O_{G,(p)} \xrightarrow{\beta} KO_G(E\times\cdots E\times\{\text{basept.}\})$$

has a natural abelian monoid structure induced by direct sum; in addition natural transformations α and β are homomorphisms with respect to these binary operations. Therefore by [12] the **GCW** complexes representing these functors are equivariant Hopf spaces[1] with good equivariant localizations. The localization of $KO_G(\cdots)$ at p is merely the algebraic localization $KO_G(\cdots)_{(p)}$, (compare [17, (4.2), p. 269]) and therefore the assertion

$$\mathrm{Loc}(p)s_*\omega_E(\Sigma,H,G) \in \text{Image } \alpha_{(p)}$$

reduces to the proof of the following result:

THEOREM 1.10. *Let* p *be odd. Then the image of* $s_*\omega_{E(p)}$ *in* $KO_G(E\times\Sigma^H, E\times\{x_0\})_{(p)}$ *is zero.*

PROOF. In fact, we claim the image of $s_*\omega_{E(p)}$ in $KO_G(E_\infty\times\Sigma^H, E\times\{x_0\})_{(p)} \subseteq K_G(E_\infty\times\Sigma^H, E_\infty\times\{x_0\})_{(p)}$ is zero (since we are localizing away from 2, complexification is split monic). Since f is a mod p homology equivalence and **W** is the sum of a

[1] When dealing with group actions it seems preferable to avoid talking about "equivariant H-spaces".

complex G-module with a trivial G-module, we know that $K_G(E_\infty \times (\Sigma^H,\{x_0\}))_{(p)}$ is 0 if dim W is odd, and if dim W is even it is isomorphic to

$$K_G(BG^W,\{\infty\})_{(p)} \cong R(G)^\wedge_{(p)}.$$

SUBLEMMA. <u>If</u> T <u>is a torus acting smoothly on a closed mod</u> p <u>homology sphere</u> Q^{2q} <u>with local representation</u> U, <u>then the image of the tangent bundle</u> τ_Q <u>in</u>

$$K_T(Q\times E_\infty,\{q_0\}\times E_\infty)_{(p)} \cong R(T)^\wedge_{(p)}$$

<u>is zero</u>.

PROOF OF SUBLEMMA. Since $R(T)^\wedge_{(p)}$ is detected by restrictions to all circle subgroups, it suffices to consider the case where $T = S^1$. But if Q' denotes the fixed point set of S^1 in Q, then the diagram below commutes:

$$\begin{array}{ccc} K_T(Q\times E_\infty,\{q\}\times E_\infty)_{(p)} & \longrightarrow & K_T(Q'\times E_\infty,\{q\}\times E_\infty)_{(p)} \\ \downarrow & & \downarrow \\ R(T)^\wedge_{(p)} & \xrightarrow{\cdot\chi} & R(T)^\wedge_{(p)} \end{array}$$

Here $\chi \in R(T)^\wedge_{(p)}$ is the product of terms of the form $t^d - 1$ (specifically, $\chi = \Pi(t^{dj}-1)$ where the normal representation to Q' is $\sum t^{dj}$). Since $R(T)^\wedge_{(p)}$ is an integral domain, it suffices to check that $\tau_Q|Q'$ maps to zero. But $\tau_Q|Q' = \tau_{Q'} \oplus \nu_{(Q',Q)}$, where $\tau_{Q'}$ goes to zero because Q' is a mod p homology sphere [21], and $\nu_{(Q',Q)}$ is detected by the rational Chern classes of the complex eigenbundles into which $\nu_{(Q',Q)}$ splits. But it is well-known that these classes are all zero (compare [18, Thm. 6.1]; one can also obtain this directly from [2,§6-7]) ■

PROOF OF (1.10) CONCLUDED. The bundle ξ may be expressed as

$$\tau(\Sigma)|\Sigma^H - \tau(\Sigma^H).$$

The sublemma implies that both of the latter terms vanish (observe that G/H is a torus acting smoothly on Σ^H). ■

Let us summarize what we have established:

THEOREM 1.11. <u>Let</u> G <u>be a torus acting smoothly on a</u> mod p <u>homology sphere</u> Σ, <u>and let</u> H <u>be a</u> p-<u>subgroup or subtorus</u>. <u>Then there is a knot invariant</u>

$$\omega_E(\Sigma,G,H) \in (E,F)/O_{G,V,(p)}(\Sigma^H)$$

<u>that is natural with restriction to a group</u> Γ <u>lying between</u> H <u>and</u> G. <u>Furthermore, there is a class</u> $\omega' \in (E,F)_{G,(p)}(S^W)_{(p)}$ <u>such that the images of</u> ω_E <u>and</u> ω' <u>under the composites into</u> $(E,F)/O_{G,(p)}(\Sigma^H)_{(p)}$ <u>are equal</u>.

If we assume now that H acts freely on $\Sigma - \Sigma^H$ (for example, let H have order p), then we have an important addition to the statement above.

THEOREM 1.12. <u>Suppose</u> H <u>acts freely on</u> $\Sigma - \Sigma^H$. <u>Let</u> $\omega(\Sigma,H)$ <u>denote the stabilized localized knot invariant of the</u> H-<u>action in the sense of</u> [16] <u>or</u> [17], <u>with value in the group</u>

$$F/O_{H,\mathrm{free}}(S^W)_{(p)}.$$

<u>Then</u> $\omega(\sum,H)$ <u>is the image of</u> ω' <u>under the composite below</u>:

(1.13)
$$\begin{tikzcd}
(E,F)_G(S^W)_{(p)} \arrow[d, "\Lambda_{(p)}"] & \\
\pi^S_{\dim W}(BG^{\mathrm{ad}\,H\,\dim W-W})_{(p)} \arrow[r, "\mathrm{Umkehr}"] & \pi^S_{\dim W}(BH^{\mathrm{ad}\,H+\dim W-W})_{(p)} \arrow[d, "\lambda^{-1}_{(p)}"] \\
F/O_{H,\mathrm{free}}(S^W)_{(p)} & F_{H,\mathrm{free}}(S^W)_{(p)} \arrow[l]
\end{tikzcd}$$

PROOF. By (1.8) and naturality of knot invariants under restriction, we know that $\omega(\sum,H)$ is the image of ω' under the composite below:

(1.14)
$$\begin{tikzcd}
(E,F)_G(S^W)_{(p)} \arrow[r, "\text{forgetful}", "\text{map}"'] & (E,F)_H(S^W)_{(p)} \arrow[d, "\Lambda_{(p)}"] \\
& \pi^S_{\dim W}(BH^{\mathrm{ad}\,H+\dim W-W})_{(p)} \arrow[d, "\lambda^{-1}_{(p)}"] \\
F/O_{H,\mathrm{free}}(S^W)_{(p)} & F_{H,\mathrm{free}}(S^W)_{(p)}. \arrow[l]
\end{tikzcd}$$

Therefore the result reduces to a more general statement of independent interest:

PROPOSITION 1.15. <u>The following diagram commutes for arbitrary</u> $H \subseteq \Gamma \subseteq G$:

$$\begin{tikzcd}
(E,F)_G(S^W)_{(p)} \arrow[r, "\text{forgetful}", "\text{map}"'] \arrow[d, "\Lambda_{(p)}"] & (E,F)_\Gamma(S^W)_{(p)} \arrow[d, "\Lambda_{(p)}"] \\
\pi^S_{\dim W}(BG^{\mathrm{ad}\,G+\dim W-W}) \arrow[r, "\mathrm{Umkehr}"] & \pi^S_{\dim W}(B\Gamma^{\mathrm{ad}\,\Gamma+\dim W-W}).
\end{tikzcd}$$

This result generalizes [3, (5.16), p. 11] and [17, §3]; the proof given in [3] goes through with minimal changes.

2. Images of Umkehr mappings

The results at the end of Section 1 may be extended as follows:

THEOREM 2.1. <u>Suppose that</u> G <u>is a torus of rank</u> r <u>and</u> H <u>lies inside a circle subgroup</u>. <u>Let</u> G <u>act on</u> $\sum$ <u>as in</u> (1.14). <u>Then the knot invariant</u> $\omega(\sum,H)$ <u>lies in the image of the composite below</u>:

$$\pi^S_{\dim W}(BT^{r(r+\dim W-W)})_{(p)} \xrightarrow{\text{Umkehr}} \pi^S_{\dim W}(BT^{r-1(r-1+\dim W-W)})_{(p)}$$

$$\cdots \to \pi^S_{\dim W}(BT^{1(1+\dim W-W)})_{(p)} \longrightarrow \pi^S_{\dim W}(BH^{(\dim H+\dim W-W})_{(p)}$$

$$\longrightarrow F/O_{H,\mathrm{free}}(S^W)_{(p)}.$$

This follows from (1.12) and the factorization of the Umkehr homomorphism for the chain of inclusions $H \subseteq S^1 \subseteq \cdots \subseteq T^{r-1} \subseteq T^r$.

The following result may be viewed as an elementary analog of Kh. Knapp's result on the Adams-Novikov filtration of Umkehr maps [9]:

PROPOSITION 2.2. Let α be a virtual representation of T^k, and let $T' \subseteq T^k$ be a corank 1 subgroup. Let

$$g: BT^{k(k+\alpha)} \longrightarrow BT'^{(k-1+\alpha)}$$

be the corresponding Umkehr S-map. Then for every element $0 \neq x \in \pi^S_*(BT^{k(k+\alpha)})$, the filtration of $g_*(x)$ in the Atiyah-Hirzebruch spectral sequence

$$H_*(Y;\pi^S_*) \Rightarrow \pi^S_*(Y)$$

is STRICTLY LESS than that of x.

PROOF. The homology of $BT^{k(k+\alpha)}$ is concentrated in dimensions congruent to $k + \dim \alpha \bmod 2$, while the homology of $BT'^{(k-1+\alpha)}$ is concentrated in dimensions congruent to $k + \dim \alpha - 1$. Hence the induced map g_* in ordinary homology must be zero; therefore the induced map on E_∞ is also zero, a fact which implies that g_* must decrease the filtrations of nonzero elements. ∎

The goal of this section is to give some technical conditions on r and $q = \dim W$ for which (2.2) implies that the composite

$$\pi_q(BT^{r(r+\dim W-W)})_{(p)} \longrightarrow \pi_q(BH^{(\dim H+\dim W-W)})_{(p)} \tag{2.3}$$

is zero. Since $\pi_{*(p)}$ is sparse in low degrees, there are very few nonzero filtrations in E_∞ for q relatively small, and accordingly (2.3) must be trivial for r greater than the number of possibly nonzero filtrations. It is only necessary to be explicit about the numbers of possibilities for appropriate values of q. The following restriction on q may look very unreasonable, but we shall justify it in Section 3.

HYPOTHESIS 2.4. p is an odd prime, and $q_j = 2p - 2 + 2jp$ for $0 \leq j \leq p - 3$.

The values of q are precisely the positive integers for which $(2p^2 - 2p - 2) - q$ is a positive multiple of $2p$.

PROPOSITION 2.5. The groups $\pi_{t(p)}$ are nonzero for exactly $(j+2)$ values of t satisfying $0 \leq t \leq q_j$.

PROOF. For $t \leq 2p^2 - 2p - 2$, the group $\pi_{t(p)}$ is nonzero if and only if $t = 0$ or $t = 2k(p-1) - 1$ for some $k \geq 1$ (i.e., $t = 0$ or $t = 2kp - (2k + 1)$, see [23]).

Thus the only values with $\pi_{t(p)} \neq 0$ for $t < q_j$ will be $0, 2p-3, 4p-5, \cdots,$ $2(j+1)p-(2j+3)$ provided $2(j+2)p-(2j+5) > 2(j+1)p-2$. This inequality reduces to $j \leq p-2$, so there is no problem. ■

COROLLARY 2.6. *The Umkehr map*

$$g_*: \pi_{q_j}(BT^{r(r+\dim \alpha)})_{(p)} \longrightarrow \pi_{q_j}(BH^{(\dim H+\dim \alpha)})_{(p)}$$

is zero for $r \geq j+3$.

PROOF. There are at most $(j+2)$ nontrivial filtrations in the Atiyah-Hirzebruch spectral sequence by (2.5), but by (2.2) we know that there are at least $(j+2)$ drops in the filtration of a nonzero element under the map g_*. Thus for all x the class of g_*x in E^∞ must lie below all filtrations containing nonzero elements. Therefore g_*x must be zero. ■

The following is now a consequence of the results in the first two Sections:

THEOREM 2.7. *Let* T^r *act smoothly on a homotopy sphere* Σ, *let* $H \cong \mathbb{Z}/p$ *be a subgroup of* T^r, *and assume* $\dim \Sigma^H = q_j$. *Then* $\omega(\Sigma, H) = 0$ *if* $r \geq j+3$. ■

This result does not look particularly interesting, but it does summarize an important part of the proof of the main result.

3. Preliminary upper estimates

As noted before, the nonzero groups in the sequence $\pi_{t(p)}$ are sparse for low values of p. The first value of t for which $\pi_{t(p)} \neq \text{Image } J_{(p)}$ is $t = 2p^2-2p-2$, and $\pi_{2p^2-2p-2(p)}$ is generated by a class β_1 of order p (compare [23]). Standard results on homotopy spheres imply that β_1 is represented as a framed bordism class by an exotic sphere $\Sigma(\beta_1)$ that does not bound a parallelizable manifold. The following result is a consequence of [16, (4.18)].

(3.1) *If* $\mathbb{Z}/p$ *acts smoothly and effectively on a homotopy* $(2p^2-2p-2)$-*sphere* $\Sigma(\beta_1)$ *with Pontrjagin-Thom invariant* β_1, *then the dimension of the fixed point set is* q_j (*defined as in* (2.7)) *for some* j *with* $0 \leq j \leq p-3$. ■

In fact, $\Sigma(\beta_1)$ admits smooth $\mathbb{Z}/p$ actions with fixed point sets of all such dimensions. We do not need this here, but we do need one important feature of any such action on $\Sigma(\beta_1)$:

PROPOSITION 3.2. *If* $\mathbb{Z}/p$ *acts smoothly and effectively on* $\Sigma(\beta_1)$, *then the knot invariant of the action is a stably nonzero element of*

$$F/O_{\mathbb{Z}/p,\text{free}}(S^q)_{(p)} = \pi_q(F_{\mathbb{Z}/p}/C_{\mathbb{Z}/p})_{(p)}.$$

PROOF. Suppose the knot invariant is stably zero. Let V be the local normal representation at the fixed point set, and let W be a 2-dimensional free $\mathbb{Z}/p$ module; denote the corresponding lens spaces by $L(V\oplus W)$ and $L(V)$. Then $L(V\oplus W)/L(V) \simeq S^k \vee S^{k+1}$ where $q+k = 2p^2-2p-2$, and by [16, Thm. 3.2] and [16, (2.6)] the image of β_1 under the composite

$$j(\beta_1) \in \pi_{2p^2-2p-2}(F/O)_{(p)} \xrightarrow{\text{inclusion}} [S^{2p^2-2p-1} \vee S^{2p^2-2p-2}, F/O]_{(p)}$$

$$\cong$$

$$[S^q(L(V\oplus W)/L(V)), F/O]_{(p)}$$

$$\oplus$$

$$[S^q(L(V\oplus W)), F/O]_{(p)}$$

would have to be zero. By the coexactness of the sequence

$$L(V) \to L(V\oplus W) \to S^k \vee S^{k+1},$$

this means that $j(\beta_1)$ would lie in the image of the map

(3.3) $\quad \phi^*: [S^{q+1}L(V), F/O]_{(p)} \to \pi_{2p^2-2p-2}(F/O)_{(p)},$

which is induced by the covering map $S^{k-1} \to L(V)$.

It therefore suffices to show that ϕ^* is trivial. To see this, consider the splitting of $F/O_{(p)}$ as $BSO_{(p)} \times \mathrm{Cok}\, J_{(p)}$. The class β_1 in fact lies in $\pi_*(\mathrm{Cok}\, J_{(p)})$, so that we need only consider the induced map

$$\phi^*: [S^{q+1}L(V), \mathrm{Cok}\, J]_{(p)} \to \pi_{2p^2-2p-p}(\mathrm{Cok}\, J)_{(p)}.$$

The space $\mathrm{Cok}\, J_{(p)}$ is $(2p^2-2p-3)$-connected, so the collapsing map

$$S^{q+1}L(V) \to S^{q+1}(S^{k-1} = L(V)/(k-2)\text{-skel.})$$

induces an isomorphism on the functor $[\cdot, \mathrm{Cok}\, J]_{(p)}$. Hence the image of ϕ^* is the image of ψ^* where ψ is the $(q+1)$-fold suspension of the map

$$S^{k-1} \to L(V) \to L(V)/(k-2)\text{-skel} \simeq S^{k-1}.$$

The latter map has degree p, so Image ϕ^* = Image $\psi^* = p\pi_{2p^2-2p-2}(\mathrm{Cok}\, J)_{(p)} = 0.$ ∎

If we combine these results with the results of Section 2, we obtain the following conclusions:

THEOREM 3.4. *Let T^r act smoothly and effectively on $\sum(\beta_1)$, and assume the fixed point set of some $\mathbb{Z}/p$ subgroup is q_j-dimensional. Then $r \leq j+2$.*

THEOREM 3.5. *If T^r acts effectively on $\sum(\beta_1)$, then $r \leq p-1$.*

PROOF. We first deduce (3.5) from (3.4). By (3.1), a $\mathbf{Z}/p$ subgroup of T^r has fixed point set of some dimension q_j, $0 \leq j \leq p-3$. By (3.4), we know $r \leq j+2$, since $j \leq p-3$, this implies $r \leq p-1$.

We now prove (3.4). If T^r acts on $\sum(\beta_1)$ with q_j-dimensional fixed point set for subgroup $H \cong \mathbf{Z}/p$, then the knot invariant of the H-action must be nontrivial by (3.2). But by (2.7) this implies $r \leq j+2$. ∎

4. Weight systems and the improved estimate

Given the low estimates for the toral symmetry in (3.5), the next question is to determine the best possible upper estimates. It is known that S^1 acts smoothly on $\sum(\beta_1)$ [15], but this is about the limit to the positive information we have. In this section we shall show that the bound in (3.5) can be roughly cut in half.

THEOREM 4.1. *If* T^r *acts smoothly and effectively on* $\sum(\beta_1)$, *then* $r \leq (p+3)/2$.

PROOF. The key to doing this is the Borel dimension formula applied to the induced actions of $(\mathbf{Z}/p)^k$ for $1 \leq k \leq r$ [4]. We state this in a form that is particularly useful here:

(4.2) *Let* G *be a finite elementary abelian* p-*group* (p *odd*) *acting on a closed* $(\mathbf{Z}/p)$-*homology sphere* $\sum$. *Then*

$$\operatorname{codim} \operatorname{Fix}(G) = \sum_H (\dim \operatorname{Fix}(H) - \dim \operatorname{Fix}(G)),$$

the sum running over all subgroups H *of index* p. ■

Consider the case $G = \mathbf{Z}/p^2$. Then $\dim \operatorname{Fix}(H) = q_j$ for some j. This, (4.2), and some elementary but tedious arithmetic lead to the following conclusion:

(4.3) *If* $\mathbf{Z}/p \times \mathbf{Z}/p$ *acts smoothly and effectively on* $\sum(\beta_1)$, *then the codimension of the fixed point set of* $\mathbf{Z}/p \times \mathbf{Z}/p$ *is one of the following numbers*:

$$2(p+1),\ 4p,\ 4p+2,\ 4p+4,\ 6p,\ 6p+2,\ 6p+4,\ 6p+6, \ldots,$$
$$2jp,\ 2jp+2, \ldots\ 2jp+2j, \ldots$$
$$(2 \leq j \leq p-2).$$

Furthermore, if the codimension is at least $2jp$, *then some cyclic* $\mathbf{Z}/p$ *subgroup has a fixed point set of dimension* $\leq q_{p-2-j}$. *Conversely, if the codimension is less than* $2jp$, *then all* $\mathbf{Z}/p$-*subgroups have fixed point sets of dimension greater than* q_{p-2-j}.

This is much less concise than the list of possibilities for $\dim \operatorname{Fix}(\mathbf{Z}/p)$. It turns out that no new possibilities for $\dim \operatorname{Fix}(G)$ occur in higher ranks.

(4.4) *If* $G = (\mathbf{Z}/p)^r$, $r \geq 3$, *acts smoothly and effectively on* $\sum(\beta_1)$, *then* $\dim \operatorname{Fix}(G) = \dim \operatorname{Fix}(K)$ *for some subgroup* K *of order* p^2.

PROOF. We shall show that if $r \geq 3$ then G has a subgroup H of index p such that $\dim \operatorname{Fix}(G) = \dim \operatorname{Fix}(H)$. (4.4) will follow from this statement and induction on r.

If the statement in the preceding paragraph is false, then by (4.2) we have

$$\operatorname{codim} \operatorname{Fix}(G) \geq 2(\#\text{ subgroups of index } p).$$

But this number of subgroups is well-known to be $1 + p + \cdots + p^{r-1}$, so we obtain

$$2p^2 - 2p - 2 \geq \operatorname{codim} \operatorname{Fix}(G) \geq 2(1 + \cdots + p^{r-1}).$$

This is patently false for $r - 1 \geq 2$. ■

In contrast to (4.4), we clearly have $\dim \mathrm{Fix}(G) < \dim \mathrm{Fix}(H)$ for some H of index p because the right hand side of (4.2) is positive. This yields the following conclusion: If $(\mathbf{Z}/p)^r$ acts and q_k is the minimum dimension of a fixed point set, then

$$(4.5) \qquad 2p(p-2-k) + 2(r-2) \geq \dim \mathrm{Fix}(\mathbf{Z}/p)^r$$
$$\geq 2p(p-2-k) + 2(p-2-k).$$

In order for the left side of this inequality to be greater than the right side, the numbers r and k must satisfy $r \leq p - k$. On the other hand, since some $(\mathbf{Z}/p)$ subgroup of $(\mathbf{Z}/p)^r$ has a q_k-dimensional fixed point set, by (3.4) we know $r \leq k + 2$. If $r \geq (p+5)/2$ were true, then

$$\frac{p+5}{2} \leq r \leq p - k \quad \text{would imply} \quad k \leq \frac{p-3}{2}.$$

On the other hand,

$$r \leq k + 2 \quad \text{would then imply} \quad r \leq \frac{p+1}{2},$$

which contradicts $r \geq (p+5)/2$. Therefore r must be at most $(p+3)/2$. ∎

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ON STEENROD'S PROBLEM FOR NON ABELIAN FINITE GROUPS

Pierre Vogel

Let M be a module over a group G and let k be a positive integer. A G-space X is called a G-Moore space of type (M,k) if X is a pointed topological space endowed with a based G-action and satisfying the following conditions:

X is k-1-connected

$H_k(X) = M$ as G-module

$\tilde{H}_i(X) = 0$ for any $i \neq k$

If such a G-space exists, the G-module M will be called (G,k)-realizable.

In 1960 N. Steenrod posed the following problem: Given a module over a finite group G, is it (G,k)-realizable by a finite G-complex? In 1969 R. Swan (10) found a counterexample to this problem. He proved that $\mathbf{Z}_{47}$ endowed with some $\mathbf{Z}_{23}$-action is not realizable by a finite complex. Nevertheless this module is $(\mathbf{Z}_{23},k)$-realizable for some k (by an infinite complex). In 1977 J. Arnold (2) solved the problem for any cyclic p-group and proved that any finitely generated module over a cyclic p-group G is (G,k)-realizable for some k. Actually Arnold's theorem and the obstruction theory of G. Cooke (6) imply that any finitely generated G-module is (G,k)-realizable for some k if G is cyclic. In 1980 G. Carlsson (4) found for the group $\mathbf{Z}_p \times \mathbf{Z}_p$ the first counterexample to the general Steenrod's problem (without finiteness condition). More recently P. Kahn (7) and J. Smith (9) found other counterexamples for many groups.

Now it is interesting to consider the following problem: Let G be a group. Under what condition is any G-module (G,k)-realizable for some k ?

In this paper we give a partial answer to this problem. We prove the following:

Theorem. Let G be a finite group whose order has no square factor, then any G-module is (G,k)-realizable for any $k > 2$.

This theorem is a first step in view of the following conjecture proposed by A. Assadi:

Conjecture. Let G be a finite group. Then any G-module is (G,k)-realizable for some k if and only if G has periodic cohomology.

§1 Steenrod's problem and cohomology equivalence.

Let G be a finite group. Let f: $M \longrightarrow M'$ be a G-linear map between two G-modules. We will said that f is a G-equivalence if f induces an isomorphism from $\hat{H}^*(S,M)$ to $\hat{H}^*(S,M')$ for any Sylow-subgroup S of G.

Proposition 1.1

Let $f: M \longrightarrow M'$ be a G- equivalence and let $k \geq 2$ be an integer. Suppose that M is (G,k)-realizable, then M' is (G,k)-realizable too.

Moreover if M is $\mathbb{Z}$-free, this holds also for $k = 1$.

Proof:

Let $L \longrightarrow M'$ be an epimorphism from a free G-module L onto M'. We get an epimorphism $f': M \oplus L \longrightarrow M'$ which is a G-equivalence. Then the kernel of f' is cohomologically trivial and there exists an exact sequence (5):

$$0 \longrightarrow L'' \longrightarrow L' \longrightarrow M \oplus L \longrightarrow M' \longrightarrow 0$$

where L' and L'' are projective G-modules and L'' is trivial if M is $\mathbb{Z}$-free. If L is choose to be big enough we may as well suppose that L' and L'' are free.

Let X be a G-Moore space of type (M,k). By adding trivial free G-cells we get a G-Moore space X' of type $(M \oplus L,k)$, and by attaching free G-cells to X' along a basis of L' we get a G-space Y with the following homology groups:

$$\tilde{H}_i(Y) = \begin{cases} M' & \text{if } i = k \\ L'' & \text{if } i = k + 1 \\ 0 & \text{otherwise} \end{cases}$$

If $k = 1$ L'' is trivial and Y is a G-Moore space of type (M',k). If $k \geq 2$ Y is simply connected and the Hurewicz homomorphism is onto in dimension $k + 1$. So we can kill $H_{k+1}(Y)$ by attaching free G-cells to Y along a basis of L'' and we get a G-Moore space of type (M',k).

Proposition 1.2

Let $f: M \longrightarrow M'$ be a G-monomorphism with $\mathbb{Q}$-local cokernel and let $k \geq 2$ be an integer.Suppose that M' is (G,k)-realizable, then M is (G,k)-realizable too.

Proof:

Let us consider the following diagram:

$$\begin{array}{ccc} M & \longrightarrow & M' \\ \downarrow & & \downarrow \\ M \otimes \mathbb{Q} & \longrightarrow & M' \otimes \mathbb{Q} \end{array}$$

By assumption this diagram is cartesian and cocartesian.

Let X' be a G-Moore space of type (M',k). Up to homotopy type we may suppose that X' has only free G-cells outside of the base point.

On the other hand the trivial maps $0 \longrightarrow M \otimes \mathbb{Q}$ and $0 \longrightarrow M' \otimes \mathbb{Q}$ are G-equivalences, and by proposition 1.1 there exist G-Moore spaces Y and Y' of type $(M \otimes \mathbb{Q},k)$ and $(M' \otimes \mathbb{Q},k)$ and Y has only free G-cells outside of the base point.

Since $M' \otimes \mathbb{Q}$ is $\mathbb{Q}$-local, Y' is a $\mathbb{Q}$-local space and the homotopy groups of Y' are $\mathbb{Q}$-local. Then there is no obstruction to construct G-maps from Y and X' to Y' inducing the maps $M \otimes \mathbb{Q} \longrightarrow M' \otimes \mathbb{Q}$ and $M' \longrightarrow M' \otimes \mathbb{Q}$ on homology.

Let X be the homotopy pull-back of Y and X' over Y':

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

Since the map $X'\longrightarrow Y'$ is a $\mathbb{Q}$-homology equivalence, the homotopy fiber F of $X\longrightarrow Y$ and $X'\longrightarrow Y'$ is $\mathbb{Q}$-acyclic. But Y and Y' are $\mathbb{Q}$-local spaces. Hence the spectral sequences of the fibrations $X\longrightarrow Y$ and $X'\longrightarrow Y'$ collapse to long exact sequences:

$$\ldots \longrightarrow \tilde{H}_i(F) \longrightarrow \tilde{H}_i(X) \longrightarrow \tilde{H}_i(Y) \longrightarrow \tilde{H}_{i-1}(F) \longrightarrow \ldots$$

$$\ldots \longrightarrow \tilde{H}_i(F) \longrightarrow \tilde{H}_i(X') \longrightarrow \tilde{H}_i(Y') \longrightarrow \tilde{H}_{i-1}(F) \longrightarrow \ldots$$

and we have:

$$H_*(Y,X) \simeq \tilde{H}_{*-1}(F) \simeq H_*(Y',X')$$

This implies that X is a G-Moore space of type (M,k).

§2 Steenrod's problem for $\hat{\mathbb{Z}}_p[G]$-modules.

Let p be a prime dividing the order of a finite group G, and suppose that the p-Sylow subgroup S of G is cyclic of order p (i.e. the order of G is not divisible by p^2) . Let N be the normalizer of S in G and let W be the quotient N/S. We will denote by $\hat{\mathbb{Z}}_p$ the ring of p-adic integers.

Let M be a G-module. Then $\hat{H}^i(S,M)$ has a canonical $\mathbb{F}_p[W]$-module structure.

Lemma 2.1

Let M be a $\hat{\mathbb{Z}}_p[G]$-module and let A be a finitely generated $\mathbb{F}_p[W]$-module. Let f be a map from A to $\hat{H}^0(S,M)$, and let $k \geq 1$ be an integer.

Then there exist a $\hat{\mathbb{Z}}_p[G]$-module and a morphism g from B to M such that:

i) B is $\hat{\mathbb{Z}}_p$-free and (G,k)-realizable

ii) $\hat{H}^1(S,B) = 0$

iii) $\hat{H}^0(S,B) = A$ and f is induced by g

Proof:

Let C be a free $\hat{\mathbb{Z}}_p$-module such that $C \otimes \mathbb{F}_p$ is isomorphic to A. Choose such an isomorphism. So we get an epimorphism from Aut(C) to Aut(A) and the kernel of this morphism is a profinite p-group. Hence the group homomorphism from W to Aut(A) given by the W-module structure of A lift through Aut(C) and C becommes a $\hat{\mathbb{Z}}_p[W]$-module such that $C \otimes \mathbb{F}_p$ is isomorphic to A as $\mathbb{F}_p[W]$-module. Moreover C is projective.

Let $M^S \subset M$ be the group $H^0(S,M)$. The group M^S is a $\hat{\mathbb{Z}}_p[W]$-module and the map $M^S \longrightarrow \hat{H}^0(S,M)$ is W-linear. Since C is projective the composite map:

$$C \longrightarrow C \otimes \mathbb{F}_p \longrightarrow A \xrightarrow{f} \hat{H}^0(S,M)$$

lift through M^S and we get a $\hat{\mathbb{Z}}_p[N]$-linear map h from C to M.

It is not difficult to prove the following:

i) C is $\hat{\mathbb{Z}}_p$-free

ii) $\hat{H}^1(S,C) = 0$

iii) $\hat{H}^0(S,C) = A$ and f is induced by h

On the other hand C is a $\hat{\mathbb{Z}}_p[W]$-module and the map $0 \longrightarrow C$ is a W-equivalence. By proposition 1.1 there exists a W-Moore space Y of type (C,k), and Y can be considered as a N-Moore space.

Let G_+ be the set G with an extra base point, and let X be the G-space $Y \wedge_N G_+$. It is easy to see that X is a G-Moore space of type $(C \otimes_N \mathbb{Z}[G],k)$.

Let B be the $\hat{\mathbb{Z}}_p[G]$-module $C \otimes_N \mathbb{Z}[G]$ and let $g\colon B \longrightarrow M$ be the G-linear map induced by h. One can see that $B/_C$ is $\hat{\mathbb{Z}}_p[S]$-free and that implies the following:

i) B is $\hat{\mathbb{Z}}_p$-free and (G,k)-realizable

ii) $\hat{H}^1(S,B) = \hat{H}^1(S,C) = 0$

iii) $\hat{H}^0(S,B) = \hat{H}^0(S,C) = A$ and f is induced by g

Corollary 2.2

Let M be a $\hat{\mathbb{Z}}_p[G]$-module and let $k \geq 1$ be an integer. Then there exist a $\hat{\mathbb{Z}}_p[G]$-module M' and a map $f\colon M' \longrightarrow M$ with the following properties:

i) M' is $\hat{\mathbb{Z}}_p$-free and (G,k)-realizable

ii) $\hat{H}^1(S,M') = 0$

iii) $\hat{H}^0(S,M') \xrightarrow{\simeq} \hat{H}^0(S,M)$

Proof:

Since $\mathbb{F}_p[W]$ is semi-simple $\hat{H}^0(S,M)$ is a direct sum of simple modules A_i. By lemma 2.1 there exist $\hat{\mathbb{Z}}_p[G]$-modules B_i and maps $f_i\colon B_i \longrightarrow M$ such that B_i is $\hat{\mathbb{Z}}_p$-free and (G,k)-realizable, $\hat{H}^1(S,B_i)$ vanishes and $\hat{H}^0(S,B_i)$ is isomorphic to A_i via f_i. Then the module $M' = \oplus B_i$ and the map $f = \oplus f_i$ satisfy the desired conditions.

Lemma 2.3

Let M be a $\mathbb{Z}_p[G]$-module and let $k \geq 2$ be an integer. Then there exist a $\hat{\mathbb{Z}}_p[G]$-module M' and a map $g\colon M'' \longrightarrow M$ with the following properties:

i) M'' is $\hat{\mathbb{Z}}_p$-free and (G,k)-realizable

ii) $\hat{H}^0(S,M'') = 0$

iii) $\hat{H}^1(S,M'') \xrightarrow{\simeq} \hat{H}^1(S,M)$

Proof:

Let I be the injective hull of M. We have an exact sequence:

$$0 \longrightarrow M \longrightarrow I \longrightarrow J \longrightarrow 0$$

and $H^*(S,I)$ vanishes. By corollary 2.2 there exist a $\hat{\mathbb{Z}}_p[G]$-module J' and a map $f\colon J' \longrightarrow J$ such that:

i) J' is $\hat{\mathbf{Z}}_p$-free and $(G,k-1)$-realizable

ii) $\hat{H}^1(S,J') = 0$

iii) $\hat{H}^0(S,J') \xrightarrow{\simeq} \hat{H}^0(S,J)$

Consider the module $J' \otimes \mathbf{Z}[G]$ equipped with the diagonal action. Since J' is $\hat{\mathbf{Z}}_p$-free $J' \otimes \mathbf{Z}[G]$ is $\hat{\mathbf{Z}}_p[G]$-free. Let M' be the kernel of the augmentation map $J' \otimes \mathbf{Z}[G] \longrightarrow J'$. We can construct a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & J' \otimes \mathbf{Z}[G] & \longrightarrow & J' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & I & \longrightarrow & J & \longrightarrow & 0 \end{array}$$

Clearly we have the following:

$$\hat{H}^0(S,M') = \hat{H}^{-1}(S,J') = \hat{H}^1(S,J') = 0$$

$$g_* : \hat{H}^1(S,M') \xrightarrow{\simeq} \hat{H}^1(S,M)$$

and M' is $\hat{\mathbf{Z}}_p$-free.

Let Y be a G-Moore space of type $(J',k-1)$. Then $Y \wedge G_+$ endowed with the diagonal action is a G-Moore space of type $(J' \otimes \mathbf{Z}[G],k-1)$. The G-map $Y \wedge G_+ \longrightarrow Y \wedge S^0 = Y$ induces the augmentation from $J' \otimes \mathbf{Z}[G]$ to J' on homology and its mapping cone is a G-Moore space of type (M',k). So M' is (G,k)-realizable.

Corollary 2.4

Any $\hat{\mathbf{Z}}_p[G]$-module is (G,k)-realizable for any $k \geq 2$.

Proof:

Let M be a $\hat{\mathbf{Z}}_p[G]$-module and $k \geq 2$ be an integer. By lemmas 2.2 and 2.3 we can construct a $\hat{\mathbf{Z}}_p[G]$-module M'' and a G-equivalence from M'' to M, M'' being (G,k)-realizable. By proposition 1.1 M is (G,k)-realizable.

§3 The main theorem

Theorem

Let G be a finite group whose order has no square factor. Then ,for any G-module M and any $k \geq 2$ there exists a G-Moore space of type (M,k).

Proof:

Let q be the order of G. Denote by $\hat{\mathbf{Z}}_q = \prod_{p|q} \hat{\mathbf{Z}}_p$ the ring of q-adic integers and by $\hat{\mathbf{Z}} = \prod_p \hat{\mathbf{Z}}_p$ the profinite completion of $\mathbf{Z}$.

Let M be a G-module and let $k \geq 2$ be an integer. By corollary 2.4 $M \otimes \hat{\mathbf{Z}}_p$ is (G,k)-realizable for any $p|q$ and $M \otimes \hat{\mathbf{Z}}_q$ is (G,k)-realizable. But the map $M \otimes \hat{\mathbf{Z}}_q \longrightarrow M \otimes \hat{\mathbf{Z}}$ is a G-equivalence, so by proposition 1.1 $M \otimes \hat{\mathbf{Z}}$ is (G,k)-realizable.

On the other hand the completion map $M \longrightarrow M \otimes \hat{\mathbf{Z}}$ is a monomorphism with $\mathbf{Q}$-local cokernel. Then by proposition 1.2 M is (G,k)-realizable.

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