INVERTIBLE SPECTRA IN THE $E(n)$-LOCAL STABLE HOMOTOPY CATEGORY

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Introduction

Suppose $\mathcal{C}$ is a category with a symmetric monoidal structure, which we will refer to as the smash product. Then the *Picard category* is the full subcategory of objects which have an inverse under the smash product in $\mathcal{C}$, and the *Picard group* $\operatorname{Pic}(\mathcal{C})$ is the collection of isomorphism classes of such invertible objects. The Picard group need not be a set in general, but if it is then it is an abelian group canonically associated with $\mathcal{C}$.

There are many examples of symmetric monoidal categories in stable homotopy theory. In particular, one could take the whole stable homotopy category $\mathcal{S}$. In this case, it was proved by Hopkins that the Picard group is just $\mathbf{Z}$, where a representative for n can be taken to be simply the n-sphere S^n [HMS94, Str92]. It is more interesting to consider Picard groups of the E-local category, for various spectra E (all of which will be p-local for some fixed prime p in this paper). Here the smash product of two E-local spectra need not be E-local, so one must relocalize the result by applying the Bousfield localization functor L_E. The most well-known case is $E = K(n)$, the nth Morava K-theory, considered in [HMS94].

In this paper we study the case $E = E(n)$, where $E(n)$ is the Johnson-Wilson spectrum. In this case the E-localization functor is universally denoted L_n, and we denote the category of E-local spectra by $\mathcal{L}$. Our main theorem is the following result.

Theorem A. *Suppose* $2p-2 > n^2+n$. *Then* $\operatorname{Pic}(\mathcal{L}) \cong \mathbf{Z}$.

In order to prove this theorem, we need several results which are of some independent interest. In particular, recall that Ravenel [Rav84, p. 353] asked whether $E(n)$ is a summand in $v_n^{-1}BP$ after applying some completion functor. Baker and Würgler proved that the Artinian completion of $v_n^{-1}BP$ is a product of copies of $L_{K(n)}E(n)$ in [BW89].

Theorem B. *We have isomorphisms of spectra*

$$L_{K(n)}BP \cong L_{K(n)}\Big(\bigvee_I \Sigma^{r(I)} E(n)\Big) \cong L_{K(n)}\Big(\bigvee_I \Sigma^{r(I)} L_{K(n)}E(n)\Big)$$

where I runs through all sequences of nonnegative integers $(i_1, i_2, \dots)$ all but finitely many of which are 0, and where $r(I) = \sum_k 2p^n i_k(p^k-1)$.

Note that the $K(n)$-localization of BP is the completion of $v_n^{-1}BP$ at the ideal $I_n = (p, v_1, \dots, v_{n-1})$, as is proved in [Hov95]. Theorem B was originally proved as

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part of work on [AMS]. It is included here since [AMS] has not yet been published. That such a splitting should exist was conjectured by the first author in [Hov94, Conjecture 2.3.5]. Note that $L_{K(n)}$ does not commute with coproducts, so $L_{K(n)}BP$ is not the obvious wedge of suspensions of $L_{K(n)}E(n)$. For our purposes here we need a similar splitting for the $K(n)$-localization of $E(m)$ when $m \geq n$.

We use Theorem B to give a simple proof of a generalization of the Miller-Ravenel change of rings theorem [MR77]. We also present another proof of this change of rings theorem using an algebraic result of Hopkins that deserves to be more widely known. We prove that every spectrum in $\mathcal{L}$ is $E(n)$-nilpotent. These facts give us some control over the E_2-term of the $E(n)$-based Adams spectral sequence (Theorem 5.1) and its convergence (Theorem 5.3) respectively. These properties are used in [Dev96].

We also investigate what happens when $p = 2$ and $n = 1$, the first case not covered by Theorem A. We find $\mathrm{Pic}(\mathcal{L}) \cong \mathbf{Z} \oplus \mathbf{Z}/2$ in this case.

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1. Basic results

We begin with some recollections about the category $\mathcal{L}$. Fix a prime p. All spectra we consider will be p-local. The ring spectrum $E(n)$ was first studied in [JW75]. It is characterized by its homotopy ring $E(n)_* = \mathbf{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n, v_n^{-1}]$ and the isomorphism

$$E(n)_*X \cong E(n)_* \otimes_{BP_*} BP_*X$$

where BP is of course the Brown-Peterson spectrum, and the v_i denote, for concreteness, the Hazewinkel generators.

The category $\mathcal{L}$ is the full subcategory of the stable homotopy category $\mathcal{S}$ consisting of $E(n)$-local spectra. A spectrum X is $E(n)$-local if and only if $[W, X] = 0$ for all spectra W such that $E(n)_*W = 0$. There is a Bousfield localization functor $L_n : \mathcal{S} \longrightarrow \mathcal{L}$ [Bou79] adjoint to the inclusion functor $\mathcal{L} \longrightarrow \mathcal{S}$.

By the smashing theorem of Hopkins-Ravenel [Rav92, Chapter 8], $L_nX \cong L_nS^0 \wedge X$. Thus $\mathcal{L}$ is closed under the smash product in $\mathcal{S}$. Furthermore, in the terminology of [HPS95], $\mathcal{L}$ is a monogenic Brown category. That is, $\mathcal{L}$ has almost all of the same formal properties as the ordinary stable homotopy category $\mathcal{S}$, where the unit of the smash product is no longer S^0 but L_nS^0.

Definition 1.1. A spectrum X in $\mathcal{L}$ is *invertible* if there is a spectrum Y in $\mathcal{L}$ such that $X \wedge Y \cong L_nS^0$. The *Picard group* of $\mathcal{L}$, $\mathrm{Pic}(\mathcal{L})$, is the set of isomorphism classes of invertible spectra in $\mathcal{L}$, given a group structure by the smash product.

Implicit in this definition is the claim that $\mathrm{Pic}(\mathcal{L})$ is a set. Because $\mathcal{L}$ is such a well-behaved category, this is automatic from the results of [HPS95], as we will see after the following definition. One can also verify this by using generalized Adams spectral sequence arguments following the similar result about Pic_n in [HMS94].

Recall that a full subcategory in a triangulated category such as $\mathcal{L}$ is called *thick* if it is closed under suspensions, retracts, and the operation of taking the cofiber of a map. A thick subcategory is called *localizing* if it is also closed under arbitrary coproducts.

Definition 1.2. A spectrum X in $\mathcal{L}$ is *finite* if it is in the thick subcategory of $\mathcal{L}$ generated by L_nS^0.

Note that if X is a finite spectrum in the usual sense, then L_nX is finite in $\mathcal{L}$. There may, however, be other spectra that are finite in $\mathcal{L}$.

Lemma 1.3. *Suppose X is finite in $\mathcal{L}$. Then X is a retract of L_nY for some ordinary finite spectrum Y.*

Proof. The proof of this lemma relies rather heavily on [HPS95]. We write X as a minimal weak colimit in $\mathcal{S}$ of $\Lambda(X)$, the ordinary finite spectra mapping to X, as in [HPS95, Theorem 4.2.4]. It follows from [HPS95, Proposition 2.2.3(e)] that $X = L_nS^0 \wedge X$ is the minimal weak colimit in $\mathcal{S}$ of $L_n\Lambda(X)$. The minimal weak colimit taken in $\mathcal{L}$ is the same. Since X is finite in $\mathcal{L}$, the functor that takes Y to $[X, Y]$ is a homology functor on $\mathcal{L}$. Thus, since homology functors commute with minimal weak colimits, we have $[X, X] = \varinjlim_{Y\in\Lambda(X)}[X, L_nY]$. Thus the identity map $1_X \in [X, X]$ factors through L_nY for some finite Y, so X is a retract of L_nY. □

We do not know of an explicit finite X in $\mathcal{L}$ that is not actually equal to L_nY for an ordinary finite spectrum Y.

Proposition 1.4. *Any spectrum in* $\mathrm{Pic}(\mathcal{L})$ *is finite in* $\mathcal{L}$. *In particular,* $\mathrm{Pic}(\mathcal{L})$ *is a set.*

Proof. This is immediate from [HPS95, Proposition A.2.8], [HPS95, Theorem 2.1.3], and [HPS95, Corollary 2.3.6]. □

Of course, L_nS^m is obviously in $\mathrm{Pic}(\mathcal{L})$ for all integers m. This observation leads to the following lemma.

Lemma 1.5. *There is a natural splitting* $\mathrm{Pic}(\mathcal{L}) \simeq \mathbf{Z} \times \mathrm{Pic}(\mathcal{L})^0$.

Proof. Suppose that $X \in \mathrm{Pic}(\mathcal{L})$, say $X \wedge Y = L_nS^0$. Then $H\mathbf{Q}_*X \otimes H\mathbf{Q}_*Y \simeq H\mathbf{Q}_*L_nS^0 = \mathbf{Q}$, so that $H\mathbf{Q}_*X$ must be concentrated in a single degree $d(X)$. It is clear that $d : \mathrm{Pic}(\mathcal{L}) \longrightarrow \mathbf{Z}$ is a homomorphism, and that $k \mapsto L^nS^k$ is a splitting. Thus $\mathrm{Pic}(\mathcal{L}) \simeq \mathbf{Z} \times \mathrm{Pic}(\mathcal{L})^0$, where $\mathrm{Pic}(\mathcal{L})^0 = \ker(d)$. □

2. Certain $E(n)_*E(n)$-comodules

In this section, we investigate the structure of $E(n)_*X$, when X is in $\mathrm{Pic}(\mathcal{L})$. We will use the fact that $E(n)$ is Landweber exact. Recall from [Lan76] that this means that the functor F that takes a BP_*BP-comodule M to the $E(n)_*$-module $E(n)_* \otimes_{BP_*} M$ is exact. We also need to recall that $F(M)$ in fact admits a natural structure as an $E(n)_*E(n)$-comodule. Indeed, Landweber exactness gives an isomorphism $E(n)_*E(n) \cong E(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n)_*$. Thus we have the structure map

$$\begin{aligned} E(n)_* \otimes_{BP_*} M &\xrightarrow{1\otimes\psi} E(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} M \\ &\to (E(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n)_*) \otimes_{E(n)_*} (E(n)_* \otimes_{BP_*} M). \end{aligned}$$

It also follows from these considerations that $E(n)_*E(n)$ is flat over $E(n)_*$ (see [MR77, Remark 3.7]).

Now, let $\mathcal{C}$ denote the category of all $E(n)_*E(n)$-comodules, which is abelian since $E(n)_*E(n)$ is flat over $E(n)_*$, and let $\mathcal{C}_0$ denote the full subcategory of comodules which are finitely generated as $E(n)_*$-modules. Because $E(n)_*$ is Noetherian, $\mathcal{C}_0$ is an abelian subcategory of $\mathcal{C}$. Finally, define $\mathcal{D}$ to be the full subcategory of $\mathcal{C}_0$ consisting of all M which can be written as $F(N)$ for some *bounded below* BP_*BP-comodule N.

We will prove a Landweber filtration theorem for objects of $\mathcal{D}$. To do so, recall that I_j, for $0 \leq j \leq \infty$, usually denotes the ideal of BP_* generated by $p, v_1, \dots, v_{j-1}$. We will use this notation, but will also use I_j to refer to the analogous ideal of $E(n)_*$. The ideals I_j are invariant, so there is a natural comodule structure on BP_*/I_j, and we have $F(BP_*/I_j) = E(n)_*/I_j$. Of course, $E(n)_*/I_j = 0$ for $j > n$.

Theorem 2.1. *Any object $M \in \mathcal{D}$ has a finite filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$$

by subcomodules M_i such that each filtration quotient M_i/M_{i+1} is isomorphic as a comodule to a suspension of $E(n)_*/I_j$ for some j depending on i.

Proof. Write $M = F(N)$ for some bounded below BP_*BP-comodule N. We can assume that N has no v_n-torsion, since in any case the v_n-torsion forms a subcomodule, so we can divide out by it. If $N \neq 0$, choose a non-zero element x of lowest degree in N. Then x is primitive. We follow an inductive procedure, as follows. If x is p-torsion, multiply x by p until the resulting primitive x_1 is killed by p. If x_1 is v_1-torsion, mutiply x_1 by v_1 until the result x_2 is killed by v_1. The class x_2 is still primitive since v_1 is primitive mod p. Continue in this fashion until we reach a primitive x_r that is killed by $p, v_1, \dots, v_{r-1}$ but is not v_r-torsion. Since N has no v_n-torsion, we have $r \leq n$.

The annihilator I of x_r is then precisely I_r. Indeed, I is an invariant ideal containing I_r. If I properly contains I_r, any nonzero element of I not in I_r of lowest possible dimension is a primitive in $I/I_r \subseteq BP_*/I_r$. But the primitives in BP_*/I_r are well-known to be $\mathbf{F}_p[v_r]$ [Rav86, Theorem 4.3.2]. Thus if I properly contains I_r, then $v_r^k \in I$ for some k, so x_r is v_r-torsion. This contradiction implies that $I = I_r$.

We thus find an inclusion of comodules $\Sigma^l BP_*/I_r \xrightarrow{i} N$ for some l and some $r \leq n$. Let P^1 denote the cokernel of i, and let N^1 denote the quotient of P^1 by the subcomodule of v_n-torsion elements. Iterating our construction, we find a sequence of surjections of comodules

$$N \longrightarrow P^1 \longrightarrow N^1 \longrightarrow P^2 \longrightarrow N^2 \longrightarrow \dots$$

Of course, this sequence may stop at some finite stage because $N^k = 0$ for some k. In that case we let $N^i = P^i = 0$ for all $i \geq k$. Let P_i denote the kernel of $N \longrightarrow P^i$, and let N_i denote the kernel of $N \longrightarrow N^i$. Then we have $P_i \subseteq N_i \subseteq P_{i+1} \subseteq N$, and N_i/P_i is v_n-torsion, whereas P_{i+1}/N_i is isomorphic to a suspension of BP_*/I_j for some $0 \leq j \leq n$ depending on i (or $P_{i+1}/N_i = 0$ if the sequence stops at k and $i \geq k$).

Now we apply the exact functor F. Let $M_i = F(N_i) \subseteq M$. Since N_i/P_i is v_n-torsion, $M_i = F(P_i)$ as well. Thus the quotient M_{i+1}/M_i is isomorphic to a suspension of $F(BP_*/I_j) = E(n)_*/I_j$ for some $0 \leq j \leq n$ depending on i. Since M is a finitely generated module over the Noetherian ring $E(n)_*$, there is a k such that $M_l = M_k$ for all $l \geq k$. Thus $F(P_{k+1}/N_k) = 0$ for all $i \geq k$, so $P_{k+1}/N_k = 0$, and thus $P^{k+1} = N^k$. This can only happen if $N^k = 0$, and hence $M_k = M$. $\square$

In order to apply the filtration theorem, we need to know when $E(n)_*X \in \mathcal{D}$.

Lemma 2.2. *Suppose X is in $\mathcal{L}$ and $E(n)_*X$ is a finitely generated $E(n)_*$-module. This will hold in particular if X is finite in $\mathcal{L}$. Then $E(n)_*X \in \mathcal{D}$.*

Proof. Because $E(n)_*$ is Noetherian, the full subcategory of all X such that $E(n)_*X$ is finitely generated is thick. Since it contains L_nS^0 it also contains X for all finite X in $\mathcal{L}$. Now suppose X is in $\mathcal{L}$ and $E(n)_*X$ is finitely generated. Then $(p^{-1}E(n))_*X$ is finitely generated over $p^{-1}E(n)_*$. Since $(p^{-1}E(n))_*X \cong H\mathbf{Q}_*X \otimes_{\mathbf{Q}} p^{-1}E(n)_*$, it follows that HQ_*X is a finite-dimensional vector space. In particular, there is an N such that $\pi_iX \otimes \mathbf{Q} = 0$ for $i < N$. Let $Y = X[N, \dots, \infty]$ be the Postnikov cover of X starting in degree N. Then there is a cofiber sequence $Y \longrightarrow X \longrightarrow Z$, where Z is bounded above and has no rational homotopy. Any such Z is the direct limit of spectra with finite homotopy, as one can see from its Postnikov tower, so $E(n)_*(Z) = 0$. Hence $E(n)_*X \cong E(n)_*Y$. Since Y is bounded below, we have $E(n)_*Y \in \mathcal{D}$. $\square$

Now we consider the Landweber exact spectrum E_n [HMS94], which we will just denote by E. Recall $E_* = W(\mathbf{F}_{p^n})[[u_1, \dots, u_{n-1}]][u, u^{-1}]$, where $W(\mathbf{F}_{p^n})$ denotes the Witt vectors of $\mathbf{F}_{p^n}$, the degree of u_i is 0 and the degree of u is -2. The ring E_* is flat over $E(n)_*$ via the map that takes v_i to $u^{1-p^i}u_i$ if $i < n$ and takes v_n to u^{1-p^n}. When considering $K(n)$-local questions, it is convenient to consider $\mathcal{K}_{n,*}(X) = \pi_*L_{K(n)}(E \wedge X)$, a sort of completion of E_*X known as the Morava module of X, considered in [HMS94].

Corollary 2.3. *If X is a finite object of $\mathcal{L}$, then the natural map $E(n)_*X \longrightarrow E_*X$ is injective. Furthermore, the natural map $E_*X \longrightarrow \mathcal{K}_{n,*}(X)$ is an isomorphism.*

Proof. It is clear that the map $E(n)_*/I_j \longrightarrow E_*/I_j$ is injective for all $0 \leq j \leq n$. We now use flatness, Theorem 2.1, and induction to conclude that the map $M \longrightarrow E_* \otimes_{E(n)_*} M$ is injective for all $M \in \mathcal{D}$.

It was pointed out in [HMS94, Section 7] that the natural map $E_*X \longrightarrow \mathcal{K}_{n,*}(X)$ is an isomorphism for all finite X. Since every finite Y in $\mathcal{L}$ is a retract of L_nY for some ordinary finite X by Lemma 1.3, this map is an isomorphism for such Y as well. $\square$

The next theorem puts considerable restrictions on objects of $\mathrm{Pic}(\mathcal{L})$.

Theorem 2.4. *Suppose X is in $\mathrm{Pic}^0(\mathcal{L})$. Then $E(n)_*X \cong E(n)_*$ as an $E(n)_*E(n)$-comodule.*

Proof. Suppose $X \in \mathrm{Pic}(\mathcal{L})$. Then that $L_{K(n)}X \in \mathrm{Pic}_n$. Recall from [HMS94] that this means $\mathcal{K}_{n,*}(X) \cong E_*$ as an E_*-module. Since X is finite, this means $E_*X \cong E_*$ as an E_*-module. Since $E(n)_*X \subseteq E_*X$, we find that $E(n)_*X$ is torsion-free.

Consider a Landweber filtration $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k = E(n)_*X$ given by Theorem 2.1. Here we have $M_j/M_{j-1} \cong \Sigma^s E(n)_*/I_r$ for some r and s depending on j. We will show that $M_j \cong \Sigma^t E(n)_*$ for some t by induction on j. This is obvious when $j = 1$ since $E(n)_*X$ is torsion-free. Now suppose $M_{j-1} \cong \Sigma^t E(n)_*$ on a primitive generator α_1, and M_j/M_{j-1} is non-zero. Write $\alpha_1 = p^v\beta$ for some β in $M_j \subseteq E_*$ which is not divisible by p. Then β must also be primitive in M_j, since M_j is torsion-free. The class β then defines an injective map of comodules $\Sigma^t E(n)_* \longrightarrow M_j$. We will show this map is an isomorphism.

To see this, note that M_j/M_{j-1} cannot be torsion-free, for then M_j would be a rank 2 free submodule of $E(n)_*X$. Then $M_j \otimes_{E(n)_*} E_*$ would be a rank 2 free submodule of $E_*X \cong E_*$, which is impossible. Denote a preimage in M_j of a generator of M_j/M_{j-1} by α_2. By the above remarks, we have

$$p\alpha_2 = x\alpha_1 = xp^v\beta$$

for some $x \in E(n)_*$. This is an equation in the ring E_* since $M_j \subseteq E(n)_*X \subseteq E_*$. If $v = 0$, then we must have $x = py$ for some y in E_* (and hence in $E(n)_*$) since (p) is a prime ideal of E_*. Since E_* is torsion-free, this implies that $\alpha_2 = y\alpha_1$, contradicting the fact that M_j/M_{j-1} is non-zero. Thus we must have $v > 0$, from which we deduce that $\alpha_2 = xp^{v-1}\beta$. Therefore the injective map $\Sigma^t E(n)_* \longrightarrow M_j$ defined by β is also surjective, as required. □

Theorem 2.4 is critical to the proof of our main application, Theorem 5.4.

The converse of this theorem is true under the additional hypothesis that X is finite in $\mathcal{L}$, and can be proved in a similar fashion to [HMS94, Theorem 1.3]. We do not know if the finiteness hypothesis is necessary.

Recall that there is a homomorphism $\alpha : \mathrm{Pic}_n \longrightarrow H^1(S_n; E_0^\times)$ known as algebraic approximation [HMS94, Section 7]. The kernel of α is sometimes denoted κ_n.

Corollary 2.5. *The image of* $\mathrm{Pic}^0(\mathcal{L})$ *in* Pic_n *lies in* κ_n.

Proof. By the above theorem, if $X \in \mathrm{Pic}(\mathcal{L})$, the E_*E comodule structure on E_*X is the same as that on the sphere. Since the S_n action is derived from this comodule structure, it too is the same as the action on the sphere. □

3. A change of rings theorem

We saw above that if $X \in \mathrm{Pic}^0(\mathcal{L})$, then $E(n)_*X \cong E(n)_*$ as a comodule over $E(n)_*E(n)$. This suggests considering the Adams spectral sequence based on $E(n)$-homology. The E_2 term of this spectral sequence is $\mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*X)$ [Ada74], since $E(n)_*E(n)$ is flat over $E(n)_*$. The differential d_r follows the usual Adams pattern, lowering the $t - s$ degree by 1 and raising the filtration s by r.

In order to understand this E_2-term, we prove the following generalization of the Miller-Ravenel change of rings theorem [MR77].

Theorem 3.1. *Suppose M is a BP_*BP-comodule, on which v_j acts isomorphically, and $n \geq j$. Then the map*

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M) \longrightarrow \mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_* \otimes_{BP_*} M)$$

is an isomorphism.

Note that if $n = j$ we recover the Miller-Ravenel change of rings theorem. We will offer two different proofs of this theorem, one using a general change of rings theorem for Hopf algebroids and a standard fact about formal group laws, and the other using a splitting of $L_{K(n)}BP$ and the Adams spectral sequence. Both of these methods rely on the following lemma.

Lemma 3.2. *Suppose that the map*

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, M) \longrightarrow \mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_* \otimes_{BP_*} M)$$

is an isomorphism for either

(a) *$M = v_j^{-1}BP_*/I_j$; or*

(b) *$M = v_j^{-1}BP_*/I$ for a collection of ideals I of the form $(p^{i_0}, v_1^{i_1}, \dots, v_{j-1}^{i_{j-1}})$ with each $i_k > 0$ and whose intersection is empty.*

Then Theorem 3.1 holds.

Proof. We fist prove that $(b) \Rightarrow (a)$. Indeed, given (b), we can take direct limits to show that the change of rings map is an isomorphism for the module

$$v_j^{-1}BP_*/(p^\infty, v_1^\infty, \dots, v_{j-1}^\infty).$$

We can then use short exact sequences to deduce the isomorphism for the module $v_j^{-1}BP_*/(p, v_1^\infty, \dots, v_{j-1}^\infty)$ as it is the kernel of the comodule map p on M. The map v_1 is then a comodule map, so we deduce the isomorphism for its kernel $v_j^{-1}BP_*/(p, v_1, v_2^\infty, \dots, v_{j-1}^\infty)$. Continue in this fashion to $v_j^{-1}BP_*/I_j$.

The proof that (a) imples Theorem 3.1 is just like the analogous argument in [MR77, Theorem 3.10]. □

We now give our first proof of the change-of-rings theorem, relying on the results of the next section.

First proof of Theorem 3.1. It is proved in the next section (Theorems 4.1 and 4.7) that there is a $K(j)$-equivalence from BP to a wedge of suspensions of $E(j)$ and a similar $K(j)$-equivalence from $E(n)$ to a different wedge of suspensions of $E(j)$ when $n \geq j$. Let X be a type j finite spectrum with v_j-self map v, as in [Rav92]. It follows that there is a $K(j)$-equivalence from $BP \wedge v^{-1}X$ to a wedge of suspensions of $E(j) \wedge v^{-1}X$ and a similar $K(j)$-equivalence from $E(n) \wedge v^{-1}X$ to a wedge of suspensions of $E(j) \wedge v^{-1}X$. These maps must actually be isomorphisms, since smashing with $v^{-1}X$ makes any BP-module $K(j)$-local. It follows easily from this that the canonical BP or $E(n)$ Adams resolution of $v^{-1}X$ is also an $E(j)$-Adams resolution, though not the canonical one, in the sense of [Rav86, Definition 2.2.1]. Indeed, in the notation of Ravenel, K_s is obviously a retract of $E(j) \wedge K_s$, since K_s is an $E(j)$-module spectrum. To see that $E(j) \wedge X_s$ is a retract of $E(j) \wedge K_s$, we use the fact that the map $X_s \longrightarrow K_s$ is induced by the unit of BP or $E(n)$, and that the splitting preserves the unit. Since the E_2-term of the Adams spectral sequence is independent of the Adams resolution, we find isomorphisms

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*(v^{-1}X)) \cong \mathrm{Ext}^{*,*}_{E(j)_*E(j)}(E(j)_*, E(j)_*(v^{-1}X))$$

and

$$\mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*(v^{-1}X)) \cong \mathrm{Ext}^{*,*}_{E(j)_*E(j)}(E(j)_*, E(j)_*(v^{-1}X)).$$

Combining these and taking X to be a generalized Moore space of type j, so that $BP_*(v^{-1}X) \cong v_j^{-1}BP_*/(p^{i_0}, v_1^{i_1}, \dots, v_{j-1}^{i_{j-1}})$, we find that condition (b) of Lemma 3.2 holds, completing the proof of Theorem 3.1. □

We now give the algebraic proof of the change-of-rings theorem. In [Hop95], Hopkins considers the following general situation. Suppose (A, Γ) is a Hopf algebroid over a commutative ring K such that Γ is flat as a left (and hence also as a right) A-module. Let $f : A \longrightarrow B$ be a map of K-algebras. Then we define Γ_B to be $B \otimes_A \Gamma \otimes_A B$, where we have used the A-bimodule structure on Γ coming from the left and right units $\eta_L, \eta_R : A \longrightarrow \Gamma$ to form the tensor product. There is a natural Hopf algebroid structure on (B, Γ_B) and a map of Hopf algebroids $(A, \Gamma) \longrightarrow (B, \Gamma_B)$. The map $\Gamma \xrightarrow{\overline{f}} \Gamma_B$ sends x to $1 \otimes x \otimes 1$.

Hopkins proves the following theorem.

Theorem 3.3 ([Hop95]). *Suppose (A, Γ) is a Hopf algebroid over a ring K, and suppose $f : A \longrightarrow B$ is a map of K-algebras. Form the Hopf algebroid (B, Γ_B) as above. Suppose we have a diagram of K-algebras*

$$A \xrightarrow{1 \otimes \eta_R} B \otimes_A \Gamma \xrightarrow{g} C$$

such that $g \circ (1 \otimes \eta_R)$ is a faithfully flat extension of A. Then the induced map

$$\operatorname{Ext}^*_\Gamma(A, A) \longrightarrow \operatorname{Ext}^*_{\Gamma_B}(B, B)$$

is an isomorphism.

In the statement of Theorem 3.3, the map denoted $1 \otimes \eta_R$ takes x to $1 \otimes \eta_R(x)$. The proof we give makes it clear that we could use the map $1 \otimes h : A \longrightarrow B \otimes D$ for any commutative A-bimodule algebra D and right A-algebra map $h : A \longrightarrow D$ instead of $1 \otimes \eta_R$.

We will prove this theorem for the reader's convenience. Our proof is a translation of the scheme-theoretic proof in [Hop95] into more conventional homological algebra language.

Proof. Suppose M is a left Γ-comodule. Then we can form the cosimplicial A-module $N(A, \Gamma) \otimes M$, whose k-simplices are $\Gamma^{\otimes k} \otimes M$. Here the tensor product is always taken over A unless notated otherwise, and we use the bimodule structure on Γ in forming it. The ring $\Gamma^{\otimes 0}$ is to be interpreted as A. The coface map d^0 is induced by η_R, the coface maps d^i are induced by the diagonal in the ith tensor factor, and the coface map d^{n+1} is induced by the coaction of M. The codegeneracy map s^i is induced by the counit of the ith tensor factor of Γ. There is an associated cochain complex obtained by taking the alternating sum of the coface maps, which we also denote $N(A, \Gamma) \otimes M$. Then one can see by flatness that a short exact sequence of comodules induces a short exact sequence of the associated cochain complexes, and hence a long exact sequence in cohomology. Since $H^0(N(A, \Gamma) \otimes M) \cong \operatorname{Hom}_\Gamma(A, M)$, it follows that $H^*(N(A, \Gamma) \otimes M) \cong \operatorname{Ext}^*_\Gamma(A, M)$. We could have done this with a right Γ-comodule as well, forming the simplicial A-module $M \otimes N(A, \Gamma)$ instead. Similar remarks hold for (B, Γ_B), though of course we must tensor over B instead of A.

The plan of the proof is to form a double complex which interpolates between $N(A, \Gamma)$ and $N(B, \Gamma_B)$. The double complex in question is associated to the bicosimplicial ring R, where $R^{m,n} = \Gamma^{\otimes m+1} \otimes \Gamma_B^{\otimes_B n}$, where we interpret $\Gamma_B^{\otimes_B 0} = B$. We refer to the cosimplicial structure maps that change the first coordinate but not the second as horizontal structure maps, and the other ones as vertical structure maps. The horizontal coface map d^0_H is induced by the right unit, and the horizontal coface maps d^i_H for $1 \leq i \leq m+1$ are induced by the diagonal on the ith tensor factor. The 0th vertical coface map d^0_V is induced by taking the diagonal on the last tensor factor of Γ and then applying the map $\overline{f} : \Gamma \longrightarrow \Gamma_B$ to the resulting last factor of Γ. The vertical coface maps d^i_V for $1 \leq i \leq n$ are induced by the diagonal on the ith tensor factor of Γ_B. The last vertical coface map d^{n+1}_V is induced by the left unit of Γ_B. The vertical and horizontal codegeneracy maps are induced by the counit. We leave it to the reader to check that this does define a bicosimplicial ring, and hence an associated double complex.

There are then two different spectral sequences associated to this double complex, both converging to the total cohomology. The E_2 term of one of them first takes the horizontal cohomology and then the vertical cohomology of the result, whereas the E_2 term of the other reverses the order. We will show that both these spectral sequences collapse, so their E_2 terms must be isomorphic.

Consider first the horizontal cohomology, so that we fix n and consider the complex $R^{*,n}$. This complex is isomorphic to the complex $N(A,\Gamma)\otimes(\Gamma\otimes\Gamma_B^{\otimes_B n})$. Since $\Gamma\otimes\Gamma_B^{\otimes_B n}$ is an injective comodule, we find that $H^sR^{*,n}\cong\Gamma_B^{\otimes_B n}$ when $s=0$ and is 0 for positive s. Hence the E_2 term of the associated spectral sequence is $\operatorname{Ext}_{\Gamma_B}(B,B)$ concentrated in filtration 0. The spectral sequence then must collapse as claimed.

Now we consider the vertical cohomology, so that we fix m and consider the complex $R^{m,*}$. We have an isomorphism of complexes

$$R^{m,*}\cong(\Gamma^{\otimes m}\otimes\Gamma\otimes B)\otimes_B N(B,\Gamma_B)$$

where we give $\Gamma\otimes B$ the right Γ_B-comodule structure induced by the diagonal of Γ. That is, the comodule structure takes $x\otimes b$ to $\sum(x_i'\otimes 1)\otimes_B(1\otimes x_i''\otimes b)$ with the usual notation for the diagonal. We would like to show that the map

$$\Gamma^{\otimes m}\xrightarrow{g}R^{m,*}$$

induced by including the first m copies of Γ is a cohomology isomorphism. We can easily show this after tensoring on the left with $B\otimes\Gamma$. Indeed, we have

$$\begin{aligned}B\otimes\Gamma\otimes R^{m,*}&\cong(B\otimes\Gamma^{\otimes m+1}\otimes\Gamma\otimes B)\otimes_B N(B,\Gamma_B)\\&\cong(\Gamma^{\otimes m+1}\otimes B\otimes\Gamma\otimes B)\otimes_B N(B,\Gamma_B)\\&\cong((\Gamma^{\otimes m+1}\otimes B)\otimes_B\Gamma_B)\otimes_B N(B,\Gamma_B)\end{aligned}$$

For the second step, even though the bimodule tensor product is not in general commutative, it is for symmetric bimodules such as B; we have an isomorphism of A-bimodules $\Gamma\otimes B\to B\otimes\Gamma$ by $x\otimes b\mapsto b\otimes\chi(x)$. Hence we have

$$H^0(B\otimes\Gamma\otimes R^{m,*})\cong B\otimes\Gamma\otimes\Gamma^{\otimes m}$$

and the higher cohomology is 0. In particular, the map $B\otimes\Gamma\otimes g$ is an isomorphism. By tensoring over $B\otimes\Gamma$ with C, we find that $C\otimes g$ is also an isomorphism. Since C is faithfully flat, we conclude that g must have been an isomorphism.

Hence the cohomology of $R^{m,*}$ is isomorphic to $\Gamma^{\otimes m}$ in degree 0 and is 0 in higher degree. The horizontal cohomology of this is of course $\operatorname{Ext}^*_\Gamma(A,A)$. This is the E_2 term of the other spectral sequence, and it too is concentrated in filtration 0, so the spectral sequence must collapse. Since the E_∞ terms of our two collapsing spectral sequences are isomorphic, the E_2 terms must be as well, completing the proof of the theorem. □

To prove our change of rings theorem, we apply Theorem 3.3 to the case

$$(A,\Gamma)=(v_j^{-1}BP_*/I_j,v_j^{-1}BP_*BP/I_j)$$

and $B=v_j^{-1}E(n)_*/I_j$, with f being the obvious map. To do so, we need the following theorem, proved (though not explicitly) by Lazard in [Laz55, pp. 269–271] and pointed out to us by Neil Strickland.

Theorem 3.4. *Suppose F, G are two p-typical formal group laws over a ring R classified by $\theta_F, \theta_G : BP_* \to R$. Suppose $\theta_F(v_i) = \theta_G(v_i) = 0$ for all $i < j$, and suppose $\theta_F(v_j) = \theta_G(v_j) = u$ is a unit in R. We say that F and G have* strict height j*. Then there is a faithfully flat extension of R over which F and G become strictly isomorphic.*

Lazard is working in a separably closed field, so he can solve equations necessary to construct a strict isomorphism. One works inductively, and finds that one must solve an equation of the form

$$t_k^{p^r} - t_k a = b$$

where a is a unit. If instead we introduce a root to this equation, we are making a finite free extension, which is faithfully flat. Taking the direct limit of these extensions, we find the required faithfully flat extension.

We can now give our second proof of Theorem 3.1.

Second proof of Theorem 3.1. By a slight extension of [MR77, Proposition 1.3], we have isomorphisms

$$\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, v_j^{-1}BP_*/I_j) \to \mathrm{Ext}^{*,*}_{v_j^{-1}BP_*BP/I_j}(v_j^{-1}BP_*/I_j, v_j^{-1}BP_*/I_j)$$

and

$$\mathrm{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, v_j^{-1}E(n)_*/I_j) \to$$
$$\mathrm{Ext}^{*,*}_{v_j^{-1}E(n)_*E(n)/I_j}(v_j^{-1}E(n)_*/I_j, v_j^{-1}E(n)_*/I_j).$$

Now take $A = v_j^{-1}BP_*/I_j$, $\Gamma = v_j^{-1}BP_*BP/I_j$ and $B = v_j^{-1}E(n)_*/I_j$, with the evident ring homomorphism $f : A \to B$. The ring A has a p-typical formal group law F of strict height j over it coming from the evident homomorphism $BP_* \to A$. This is in fact the universal p-typical formal group law of strict height j. Similarly, B has a formal group law G of strict height j over it, which is the pushforward of F through the map f. We are looking for a diagram of the form

$$A \xrightarrow{1\otimes\eta_R} B \otimes_A \Gamma \xrightarrow{h} R$$

where $A \to R$ is faithfully flat. A map $B \otimes_A \Gamma \xrightarrow{h} R$ is the same thing as a strict isomorphism φ of strict height j formal group laws over R whose source is a pushforward of G. The composite $h \circ (1 \otimes \eta_R)$ is the classifying map of the target of φ.

Consider first the ring $C = A \otimes_{\mathbf{F}_p[v_j, v_j^{-1}]} B$. The obvious ring map $A \to C$ which sends x to $x \otimes 1$ classifies a formal group law of strict height j over C, which we will also call F. Similarly, the obvious ring map $B \to C$ which sends x to $1 \otimes x$ classifies a formal group law of strict height j over C which we also call G. Furthermore, we have $\theta_F(v_j) = \theta_G(v_j)$. Thus there is a faithfully flat extension $C \to R$ and a strict isomorphism whose source is the pushforward of G and whose target is the pushforward of F. This defines a map $B \otimes_A \Gamma \to R$ such that the composite

$$A \xrightarrow{1\otimes\eta_R} B \otimes_A \Gamma \to R$$

is the classifying map of the pushforward of F, which is the composite $A \to C \to R$. Since R is faithfully flat over C and C is obviously faithfully flat over A, R is faithfully flat over A. Theorem 3.3 and part (a) of Lemma 3.2 completes the proof. $\square$

4. The $K(j)$-localization of BP and $E(n)$

We still owe the reader a proof of the splitting of the $K(j)$-localizations of BP and $E(n)$, used in our first proof of the change of rings theorem 3.1. The results in this section were first proved as part of [AMS].

Suppose R is a BP-algebra spectrum, so that there is a map $BP \to R$ of ring spectra. We will always denote the unit of a ring spectrum R by $\eta : S^0 \to R$. Then there is a natural isomorphism $BP \wedge R \cong R[t_1, t_2, \dots]$. We explain this as follows. We have a natural isomorphism $R_*BP \cong R_*[t_1, t_2, \dots]$ as in [Rav86, Lemma 4.1.7]. Adapting notation from [JW75] when $E = (e_1, e_2, \dots)$ is a sequence of nonnegative integers, all but finitely many of which are zero, we let $t^E = t_1^{e_1} t_2^{e_2} \cdots$ and likewise for v^E. We let $|E| = |t^E| = \sum 2e_i(p^i - 1)$. Then the class $t^E \in R_*BP$ gives rise to a map $S^{|E|} \to BP \wedge R$. By smashing with R and using the multiplication of R, we get a map $\Sigma^{|E|} R \to BP \wedge R$. Putting these maps together gives us the desired isomorphism

$$R[t_1, t_2, \dots] = \bigvee_E \Sigma^{|E|} R \cong BP \wedge R$$

If we take $R = BP$, the map

$$BP \xrightarrow{\eta \wedge 1} BP \wedge BP \longrightarrow \Sigma^{|E|} BP$$

induced by this splitting is the Landweber-Novikov operation r_E studied in [JW75]. On homotopy, $r_E(x)$ is the coefficient of t^E in $\eta_R(x)$, the right unit of x.

Given $j > 0$, let $\mathcal{E}$ denote the set of exponent sequences E such that $e_i = 0$ for all $i < j$. Give an $E \in \mathcal{E}$, let σE be the sequence $(p^j e_j, p^j e_{j+1}, \dots)$. Let q denote the composite

$$BP \xrightarrow{\eta \wedge 1} BP \wedge BP \xrightarrow{p} \bigvee_{E \in \mathcal{E}} \Sigma^{|\sigma E|} BP \longrightarrow \bigvee_{E \in \mathcal{E}} \Sigma^{|\sigma E|} BP\langle j \rangle$$

Here the map p is the projection map onto only some of the summands, and the last map is induced by the canonical ring spectrum map $BP \to BP\langle j \rangle$. Recall that $BP\langle j \rangle_* \cong \mathbf{Z}_{(p)}[v_1, \dots, v_j]$. On homotopy, q first takes the right unit of x, then projects onto $BP_*[t_1^{p^j}, t_2^{p^j}, \dots]$, then reduces mod v_i for $i > j$.

Theorem 4.1. *Let Z be a type j finite spectrum, with v_j-self map v, as in* [Rav92]. *Then the map*

$$v^{-1}Z \wedge q : v^{-1}Z \wedge BP \longrightarrow v^{-1}Z \wedge \bigvee_{E \in \mathcal{E}} \Sigma^{|\sigma E|} BP\langle j \rangle \cong \bigvee_{E \in \mathcal{E}} \Sigma^{|\sigma E|} Z \wedge E(j)$$

is an homotopy equivalence. In particular, q induces an isomorphism on $K(j)$-homology.

Note that Theorem B is follows immediately from Theorem 4.1, since $L_{K(n)}(q)$ is then an isomorphism.

Proof. Note that the second statement follows from the first and the Kunneth theorem for $K(n)$-homology. For the first statement, it suffices to verify it for any specific finite type j spectrum by the thick subcategory theorem. So take Z to be a generalized Moore space $M(I)$, where $I = (p^{i_0}, v_1^{i_1}, \dots, v_{j-1}^{i_{j-1}})$. Recall that such Z exist by iterating the nilpotence theorem, and we have

$$BP_*(v^{-1}Z) \cong v_j^{-1} BP_*/I.$$

We must therefore show that the map

$$q_*: v_j^{-1}BP_*/I \to E(j)_*/I[t_1^{p^j}, t_2^{p^j}, \dots]$$

induced by the right unit and projection is an isomorphism for

$$I = (p^{i_0}, v_1^{i_1}, \dots, v_{j-1}^{i_{j-1}}).$$

We first show this for $I = I_j = (p, v_1, \dots, v_{j-1})$, though of course $M(I_j)$ probably does not exist. We will actually show that the map

$$BP_*/I_j \xrightarrow{\overline{\eta}_R} \mathbf{F}_p[v_j][t_1^{p^j}, t_2^{p^j}, \dots]$$

induced by the right unit is an injection with cokernel which is v_j-torsion. Inverting v_j then gives the desired isomorphism.

Since v_j is invariant mod I_j, the map $\overline{\eta}_R$ is an $\mathbf{F}_p[v_j]$-module map. Furthermore, we have the formula, for $E \in \mathcal{E}$,

$$\eta_R(v^E) \equiv v_j^m t^{\sigma E} \tag{4.1}$$

modulo I_j and terms of strictly lower t-degree [JW75, Corollary 1.8]. Here m is the sum of the e_i in E.

Then $\overline{\eta}_R$ is obviously injective from this formula. Let 1_E denote the generator of $\Sigma^{|\sigma E|}\mathbf{F}_p[v_j]$, corresponding to t^E. We will show that $v_j^r 1_E$ is hit by $\overline{\eta}_R$ for some r by induction on $|E|$. If $|E| = 0$, then $E = 0$, and $\overline{\eta}_R 1 = 1_E$. Now suppose $|E| > 0$. By induction, there is an s such that $v_j^s 1_F$ is in the image of $\overline{\eta}_R$ for all $F \in \mathcal{E}$ of lower degree. Then the Johnson-Wilson formula (4.1) shows that $v_j^{s+m} 1_E$ is in the image of $\overline{\eta}_R$ as well, completing the proof that q_* is an isomorphism when $I = I_j$.

Now, if $I = (p^{i_0}, v_1^{i_1}, \dots, v_{j-1}^{i_{j-1}})$, then $v_j^{-1}BP_*/I$ is related to $v_j^{-1}BP_*/I$ by a collection of exact sequences of comodules. Since q_* is a natural transformation of exact functors on comodules (this is easy to see directly for the comodules we are interested in, but it also follows from Landweber exactness), it must also be an isomorphism for any I of this form. $\square$

Note that the splitting of $L_{K(j)}BP$ given above is compatible with the unit of BP and $E(j)$, though it is not a splitting of ring spectra.

Remark 4.2. Given an ideal $I = (p^{i_0}, v_1^{i_1}, \dots, v_{j-1}^{i_{j-1}})$, one can form the spectrum $v_j^{-1}BP/I$ even when there is no finite spectrum $M(I)$ with $BP_*M(I) \cong BP_*/I$. In this case, it is still possible to prove that $v_j^{-1}BP/I$ splits as a wedge of suspensions of $E(j)/I$ using the methods of [JW75] and [Wür76]. If $I = I_n$, this is the result in [Wür76] that $B(n)$ splits additively into a wedge of suspensions of $K(n)$.

We would like to prove an analogous splitting for $L_{K(n)}E(n)$ when $n \geq j$. We begin with the spectrum $BP\langle n\rangle$. Let $\mathcal{E}_n$ denote the exponent sequences with $e_i = 0$ for $i < j$ and $e_i = 0$ for $i > n$. Then $\sigma\mathcal{E}_n$ consists of exponent sequences with $e_i = 0$ for $i > n - j$ and each e_i divisible by p^j. We let q_n denote the composite

$$BP\langle n\rangle \xrightarrow{\eta\wedge 1} BP \wedge BP\langle n\rangle \xrightarrow{p} \bigvee_{E\in\mathcal{E}_n} \Sigma^{|\sigma E|}BP\langle n\rangle \to \bigvee_{E\in\mathcal{E}_n} \Sigma^{|\sigma E|}BP\langle j\rangle$$

where p is the projection map and the last map is induced by the canonical map of ring spectra $BP\langle n\rangle \to BP\langle j\rangle$.

Theorem 4.3. *Suppose Z is a type j finite spectrum, with v_j-self map v, and $n \geq j$. Then the map*

$$v^{-1}Z \wedge q_n : v^{-1}Z \wedge BP\langle n\rangle \to \bigvee_{E \in \mathcal{E}_n} \Sigma^{|\sigma E|} Z \wedge E(j)$$

is an isomorphism. In particular, q_n induces an isomorphism on $K(j)$-homology, and thus we have an isomorphism

$$L_{K(j)} BP\langle n\rangle \cong L_{K(j)} \bigvee_{E \in \mathcal{E}_n} \Sigma^{|\sigma E|} E(j)$$

Proof. The proof is the same as that of Theorem 4.1. Indeed, there is a commutative diagram

$$\begin{array}{ccc} BP & \xrightarrow{q} & BP\langle j\rangle[t_1^{p^j}, t_2^{p^j}, \dots] \\ \downarrow & & {\scriptstyle p}\downarrow \\ BP\langle n\rangle & \xrightarrow{q_n} & BP\langle j\rangle[t_1^{p^j}, \dots, t_{n-j}^{p^j}] \end{array}$$

where p is the obvious projection map. Using the evident section $BP\langle n\rangle_* \to BP_*$, we find that the effect of q_n on homotopy is also to take the right unit and then project off onto certain terms. The Johnson-Wilson formula (4.1) still applies: one just has fewer exponent sequences to keep track of. One cannot use Landweber exactness anymore to deduce that proving it for $I = I_j$ is enough, but these ideals are so simple that we can see directly that tensoring with $E(j)_*$ preserves exactness for them. □

Note that the splitting of $L_{K(j)}BP\langle n\rangle$ given above is also compatible with the units of $BP\langle n\rangle$ and $E(j)$, though again it is not multiplicative.

We would like to deduce from this an analogous splitting for $L_{K(j)}E(n)$ when $n \geq j$. The crucial fact that allows us to do this is the following corollary.

Corollary 4.4. *Suppose $n \geq j$. Then the map*

$$v_n : \Sigma^{|v_n|} L_{K(j)} BP\langle n\rangle \to L_{K(j)} BP\langle n\rangle$$

is a split monomorphism of spectra.

Proof. The cofiber of v_n is the map $L_{K(j)}BP\langle n\rangle \to L_{K(j)}BP\langle n-1\rangle$. Applying the compatible isomorphisms $L_{K(j)}q_n$ and $L_{K(j)}q_{n-1}$, we find that this map is a split epimorphism, as required. □

The spectrum $E(n)$ is of course the telescope of the self-map v_n of $BP\langle n\rangle$. The functor $L_{K(j)}$ does not commute with taking the telescope of a map, but we do have the following lemma.

Lemma 4.5. *Suppose R and X are spectra, and $v : \Sigma^d X \to X$ is a self-map of X. Then we have an isomorphism $L_R v^{-1} L_R X \cong L_R(v^{-1}X)$.*

Proof. Recall that $v^{-1}X$ is the sequential colimit of the $\Sigma^{-md}X$ under the self-map v, so that we have a sequence of compatible maps $\Sigma^{-md}X \to v^{-1}X$. These induce a sequence of compatible maps $\Sigma^{-md}L_R X \to L_R(v^{-1}X)$, and so a not necessarily unique map $v^{-1}L_R X \to L_R(v^{-1}X)$. This map is an isomorphism on R-homology, since

$$R_*(v^{-1}L_R X) \cong (R_* v)^{-1} R_* X \cong R_*(L_R v^{-1} X)$$

Hence we get the required isomorphism $L_R v^{-1} L_R X \cong L_R(v^{-1}X)$. □

Lemma 4.5 reduces us to considering $v_n^{-1} L_{K(j)} BP\langle n\rangle$, and then relocalizing it. For this case, we have the following lemma.

Lemma 4.6. *Suppose $v : \Sigma^d X \longrightarrow X$ is a self-map that is also a split monomorphism of spectra. Let $X \longrightarrow Y$ denote the cofiber of v with splitting $i : Y \longrightarrow X$. Then there is an isomorphism*

$$X \vee \bigvee_m \Sigma^{-md} Y \xrightarrow[\cong]{f} v^{-1}X$$

Proof. The map f is induced by the canonical map $X \longrightarrow v^{-1}X$ and the composite $\Sigma^{-md}Y \xrightarrow{i} \Sigma^{-md}X \longrightarrow v^{-1}X$. One can see by an easy induction that the map

$$X \vee \bigvee_{k \leq m} \Sigma^{-kd} Y \xrightarrow{f_m} \Sigma^{-md} X$$

is an isomorphism. Since every homotopy class in $v^{-1}X$ is in the image of the homotopy of $\Sigma^{-md}X$ for some m, it follows that f is surjective on homotopy. Conversely, every element in $\pi_*(X \vee \bigvee_m \Sigma^{-md}Y)$ lies in the homotopy of some finite wedge, and so in the homotopy of $\Sigma^{-md}X$ for some m. Since v is a split monomorphism, the induced map $\pi_*\Sigma^{-md}X \longrightarrow \pi_*(v^{-1}X)$ is injective. Thus $\pi_* f$ is injective as well, completing the proof. □

Combining Theorem 4.3, Corollary 4.4, Lemma 4.5, and Lemma 4.6 gives the following theorem.

Theorem 4.7. *Suppose $n \geq j$. Then we have an isomorphism*

$$L_{K(j)}E(n) \cong L_{K(j)}(BP\langle n\rangle \vee \bigvee_m \Sigma^{-m|v_n|} BP\langle n-1\rangle)$$

which is in turn isomorphic to the $K(j)$-localization of a wedge of suspensions of $E(j)$.

Note that this splitting is compatible with the unit as well.

5. The $E(n)$ Adams spectral sequence

In this section we apply the change of rings theorem 3.1 to the calculation of the E_2-term of the $E(n)$ Adams spectral sequence. We also prove that this spectral sequence converges to the homotopy of $L_n X$ for all X. This allows us to prove Theorem A.

Recall that the change of rings theorem is about comodules on which v_j acts isomorphically for some j. We must therefore relate a general comodule to such comodules. The standard tool for doing this is the chromatic spectral sequence [Rav86, Chapter 5]. For $0 \leq s \leq n$, let M^s denote the $E(n)_*E(n)$-comodule

$$v_s^{-1} E(n)_*/(p^\infty, v_1^\infty, \ldots, v_{s-1}^\infty).$$

If $s = 0$, we interpret this to mean $M^0 = E(n)_* \otimes \mathbf{Q}$. Then we have an exact sequence of $E(n)_*E(n)$-comodules

$$E(n)_* \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \ldots \longrightarrow M^n \longrightarrow 0.$$

Associated to this exact sequence, we have the chromatic spectral sequence much as in [Rav86, Chapter 5]. The E_1 term of the chromatic spectral sequence is

$$E_1^{s,r,*} \cong \operatorname{Ext}^{r,*}_{E(n)_*E(n)}(E(n)_*, M^s),$$

and the spectral sequence converges to $\operatorname{Ext}^{r+s,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*)$.

Since v_s acts invertibly on M^s, the change of rings theorem implies the following theorem.

Theorem 5.1. *In the chromatic spectral sequence for* $\operatorname{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*)$, *we have*

$$E_1^{s,r,*} \cong \begin{cases} \operatorname{Ext}^{r,*}_{E(s)_*E(s)}(E(s)_*, E(s)_*/(p^\infty, \dots, v_{s-1}^\infty)) & s \le n \\ 0 & s > n. \end{cases}$$

In particular, if $p-1$ does not divide s, we have $E_1^{s,r,*} = 0$ for $r > s^2$. Thus, if $p > n+1$,

$$\operatorname{Ext}^{s,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*) = 0$$

for $s > n^2 + n$.

Proof. Apply the change of rings theorem 3.1 to the BP_*BP-comodule

$$v_s^{-1}BP_*/(p^\infty, \dots, v_{s-1}^\infty)$$

with $j = s$ and $j = n$. Now if $p-1$ does not divide s, we have

$$\operatorname{Ext}^{r,*}_{E(s)_*E(s)}(E(s)_*, E(s)_*/(p, v_1, \dots, v_{s-1})) = 0$$

for $r > s^2$ by Morava's vanishing theorem [Rav86, Theorem 6.2.10]. It follows using standard exact sequences that

$$\operatorname{Ext}^{r,*}_{E(s)_*E(s)}(E(s)_*, E(s)_*/(p, \dots, v_{s-2}, v_{s-1}^k)) = 0$$

for $r > s^2$, and hence taking direct limits that

$$\operatorname{Ext}^{r,*}_{E(s)_*E(s)}(E(s)_*, E(s)_*/(p, \dots, v_{s-2}, v_{s-1}^\infty)) = 0$$

for $r > s^2$. Repeating this same process for v_{s-2}, v_{s-3} and so on, we find

$$\operatorname{Ext}^{r,*}_{E(s)_*E(s)}(E(s)_*, E(s)_*/(p^\infty, \dots, v_{s-1}^\infty)) = 0$$

for $r > s^2$ as well. □

Remark 5.2. Note that Theorem 5.1 shows precisely that the chromatic spectral sequence converging to $\operatorname{Ext}^{*,*}_{E(n)_*E(n)}(E(n)_*, E(n)_*)$ is the truncation at E_1 of the chromatic spectral sequence converging to $\operatorname{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ obtained by setting $E_1^{s,r,*} = 0$ when $s > n$.

We now turn our attention to the convergence of the $E(n)$ Adams spectral sequence. Our proof of convergence is very similar to the proof of the smashing conjecture in [Rav92, Chapter 8]. Recall Hopkins and Ravenel construct a finite spectrum X whose integral homology is torsion-free and such that

$$\operatorname{Ext}^{s,*}_{BP_*BP}(v_m^{-1}BP_*/I_m, BP_*X) = 0$$

for all $s > s_0$ and $0 \le m \le n$.

Theorem 5.3. *The spectrum X above is $E(n)$-prenilpotent. Thus, every spectrum is $E(n)$-prenilpotent, and the $E(n)$ Adams spectral sequence converges to $\pi_* L_n X$ for all X.*

Proof. The proof is very similar to that of [Rav92, Lemma 8.3.1]. First note that it suffices to show $M_m X$ is $E(n)$-nilpotent for all $m \leq n$, where $M_m X$ is the fiber of the natural map $L_m X \longrightarrow L_{m-1} X$. By Bousfield's convergence criterion [Bou79], stated in [Rav92, Corollary 8.2.7], it suffices to show that

$$\mathrm{Ext}^{s,*}_{E(n)_* E(n)}(E(n)_*, E(n)_*(M_m X \wedge Y)) = 0$$

for all $s > s_0$ and all finite spectra Y. Now we have

$$BP_*(M_m X \wedge Y) = M_m BP_*(X \wedge Y)$$

by the localization theorem [Rav92, Theorem 7.5.2]. Now $M_m BP$ is a BP-module spectrum on which v_m acts invertibly, in the sense that the map

$$\Sigma^{2(p^m-1)} M_m BP \xrightarrow{\times v_m} M_m BP$$

is an isomorphism. Thus v_m acts invertibly on $M_m BP_*(X \wedge Y)$ as well. Hence our change of rings theorem 3.1 applies, and we have

$$\mathrm{Ext}^{s,*}_{E(n)_* E(n)}(E(n)_*, E(n)_*(M_m X \wedge Y)) = \mathrm{Ext}^{s,*}_{BP_* BP}(BP_*, BP_*(M_m X \wedge Y))$$

Then Lemma 8.3.1 of [Rav92] shows that this latter group is 0 for $s > s_0$. Thus X is $E(n)$-prenilpotent. Since the full subcategory of $E(n)$-prenilpotent finite spectra is thick, it follows from the thick subcategory theorem that S^0 is $E(n)$-prenilpotent. Just as in [Rav92, Chapter 8], we then find that every spectrum is $E(n)$-prenilpotent, so by Bousfield's criterion again [Bou79], we find that the $E(n)$ based Adams spectral sequence converges to $\pi_* L_n X$ for all X. □

We now restate our main application and prove it.

Theorem 5.4. $\mathrm{Pic}^0(\mathcal{L}) = 0$ *when* $2p - 1 > n^2 + n$.

Proof. First note that $2p - 2 > n^2 + n$ implies $p > n + 1$. Theorem 2.4 and Theorem 5.1 then show that the E_2 term of the $E(n)$ based Adams spectral sequence is zero in filtration $> n^2 + n$. Since the usual sparseness holds, the only possible nonzero differentials raise filtration by a multiple of $2p - 1$. Thus there is no room for any differentials, or, again by sparseness, any extensions. Thus the class in the E_2 term corresponding to the primitive 1 survives to give a map $S^0 \longrightarrow L_n X$ which is an isomorphism on $E(n)$ homology. Thus $L_n S^0 \cong L_n X$. □

6. An example

In this section, we will compute the group $\mathrm{Pic}(\mathcal{L})$ for $n = 1$ and $p = 2$. Notice that this is the first case not covered by Theorem 5.4.

Our goal is to prove the following theorem.

Theorem 6.1. *If $p = 2$ and $n = 1$, the Picard group* $\mathrm{Pic}^0(\mathcal{L})$ *is isomorphic to* $\mathbf{Z}/2$, *generated by $L_1 QM$, where QM is the question mark complex.*

Recall that QM fits into a cofiber sequence

$$S^{-1} \xrightarrow{f} \Sigma^{-3} M(2) \longrightarrow QM \longrightarrow S^0$$

where the composite $S^{-1} \xrightarrow{f} \Sigma^{-3}M(2) \longrightarrow S^{-2}$ is the Hopf map η, also denoted α_1. This does not uniquely determine the attaching map, but either choice gives an isomorphic complex QM. To see why QM is called the question mark, draw the action of the Steenrod algebra on its cohomology. It is well-known [Hop86, Corollary 7.7] that $QM \wedge DQM \cong S^0 \vee Z$, where $H^*(Z, \mathbf{F}_2)$ is isomorphic to A_1 as an A_1-module. It follows that $K(0)_*Z = K(1)_*Z = 0$. Hence $L_1(QM \wedge DQM) \cong L_1S^0$, and QM is indeed in $\mathrm{Pic}^0(\mathcal{L}_1)$.

Before we can prove Theorem 6.1, we need to explicitly recall the computation of $\pi_*L_1S^0$ at $p = 2$. This computation, in the form we need it, is due to Ravenel. The results of the computation are given in [Rav84, Theorem 8.15]. Ravenel does not write down $\mathrm{Ext}^{*,*}_{E(1)_*E(1)}(E(1)_*, E(1)_*)$, but it can be calculated easily from the results of [Rav86, Chapter 5]. Before writing down this algebra, we describe it in terms of the usual Adams spectral sequence picture, where an element in bidegree (s, t) is thought of as in the $t-s$-stem and of filtration s. In every even stem except 0 and -2 we have a $\mathbf{Z}/2$ in every positive even filtration. In the 0-stem, we also have a $\mathbf{Z}_{(2)}$ in filtration 0, and in the -2-stem we also have a $\mathbf{Q}/\mathbf{Z}_{(2)}$ in filtration 2. In every odd stem we have a $\mathbf{Z}/2$ in every positive odd filtration except 1. In filtration 1, we have a $\mathbf{Z}/2$ in the $4s+1$-stem, a $\mathbf{Z}/8$ in the $8s+3$-stem, and, when $s \neq 0$, a $\mathbf{Z}/16s$ in the $8s-1$-stem. Note that there is a small typo in [Rav86, Theorem 5.3.6], from which this picture can be read off. The class denoted $v_1^{-1}t_1/2$ in that theorem is equal to the class denoted $j(v_1^{-1}h_0)$.

Of course, this picture does not specify the multiplicative structure. The sufficiently careful reader can verify that the multiplicative structure is given by

$$\mathrm{Ext}^{*,*}_{E(1)_*E(1)}(E(1)_*, E(1)_*) \cong \mathbf{Z}_{(2)}[\rho_{2t+1}, x_{-2,i} : t \neq -1, i > 0]/(R)$$

where ρ_{2t+1} is in bidegree $(1, 2t+2)$, $x_{-2,i}$ is in bidegree $(2, 0)$, and R is an ideal of relations. To describe these relations, let x denote $\rho_7\rho_{-7}$ in bidegree $(2, 2)$. Then R is generated by the following relations:

- $2\rho_{4t+1} = 0$;
- $8\rho_{8t+3} = 0$;
- $16t\rho_{8t-1} = 0$ for all $t \neq 0$;
- $2x_{-2,i} = x_{-2,i-1}$ for $i > 0$ and $2x_{-2,1} = 0$;
- $\rho_{2s+1}\rho_{2t+1} = \rho_1\rho_{2(s+t)+1}$ except in the following cases:
  - $\rho_{8s+1}\rho_{8t+7} = x$ when $s + t = -1$;
  - $\rho_{8s+5}\rho_{8t+3} = x$ when $s + t = -1$;
  - $\rho_{8s+3}\rho_{8t+7} = 0$;
  - $\rho_{8s+7}\rho_{8t+7} = 0$ when $s + t \neq -2$;
  - $\rho_{8s+7}\rho_{8t+7} = x_{-2,k}$ when $s + t = -2$, where 2^k is the order of ρ_{8s+7} (which is also the order of ρ_{8t+7});
  - $\rho_{8s+3}\rho_{8t+3} = 0$ when $s + t \neq -1$; and
  - $\rho_{8s+3}\rho_{8t+3} = x_{-2,3}$ when $s + t = -1$.

Now the first possible nonzero differential in the $E(1)$-based Adams spectral sequence converging to $\pi_*L_1S^0$ is d_3. The behavior of d_3 is determined by $d_3(1) = d_3(\rho_{8s+1}) = d_3(\rho_{8s+7}) = 0$ and $d_3(\rho_3) = \rho_1^2x$, $d_3\rho_{8s+3} = \rho_1^3\rho_{8s-1}$ when $s \neq 0$, and $d_3(\rho_{8s+5}) = \rho_1^3\rho_{8s+1}$. The resulting E_4 term is concentrated in filtrations less than 4, so no further differentials are possible and $E_4 = E_\infty$. There is a nontrivial extension: we have $4(2\rho_{8s+3}) = \rho_1^2\rho_{8s+1}$. Of course, the homotopy element corresponding to ρ_{8s+1} is often denoted α_{4s+1}, and the homotopy element corresponding to $2\rho_{8s+3}$

is often denoted $\alpha_{4s+2/2}$. The reader will note that there are some mistakes in the statement of [Rav84, Theorem 8.15]. Indeed, $\pi_i L_1 S^0 = \mathbf{Z}/16s$ if $i = 8s-1$, not $\mathbf{Z}/2s$. Also, the multiplication map $\pi_{8s+3} \otimes \pi_{-8s-5}$ has kernel $\mathbf{Z}/4$, not $\mathbf{Z}/2$.

Armed with this calculation, we can now prove Theorem 6.1.

Proof of Theorem 6.1. Suppose $Z \in \mathrm{Pic}^0(\mathfrak{L}_1)$. Then the E_2-term of the $E(1)$-based Adams spectral sequence converging to $\pi_* Z$ is a free module over the E_2-term for $L_1 S^0$, generated by some class y, by Theorem 2.4. If $d_3 y = 0$, then the differential d_3 is precisely the same as it is in the spectral sequence for $L_1 S^0$. Thus the spectral sequence collapses at E_4, and the class y corresponds to a homotopy class $L_1 S^0 \longrightarrow Z$ inducing an $E(1)$-equivalence. Thus $Z \cong L_1 S^0$.

We may therefore assume $d_3 y \neq 0$. Then we must have $d_3 y = \rho_1^2 \rho_{-3}$. This then determines d_3, and we find once again that there are no elements in E_4 above filtration 3, so $E_4 = E_\infty$. We can then read off the homotopy groups of Z, and we find $\pi_0 Z \cong \mathbf{Z}_{(2)}$, $\pi_1 Z = \pi_2 Z = 0$, and $\pi_3 Z \cong \mathbf{Z}/8$. Thus an element Y of $\mathrm{Pic}^0(\mathfrak{L})$ is trivial if and only if $\pi_0 Y \cong \mathbf{Z}/2 \oplus \mathbf{Z} \cong \pi_0 L_1 S^0$.

Our strategy is to apply this criterion to $Z \wedge L_1 QM$. Smashing the defining cofiber sequence of QM with Z, we find an exact sequence

$$\pi_1 Z \longrightarrow \pi_0(Z \wedge \Sigma^{-3} M(2)) \longrightarrow \pi_0(Z \wedge QM) \longrightarrow \pi_0 Z \longrightarrow \pi_{-1}(Z \wedge \Sigma^{-3} M(2))$$

From our calculation above, we find $\pi_0(Z \wedge \Sigma^{-3} M(2)) \cong \mathbf{Z}/2$ and $\pi_{-1}(Z \wedge \Sigma^{-3} M(2)) = 0$. Hence $\pi_0(Z \wedge QM) \cong \mathbf{Z}/2 \oplus \mathbf{Z}$. Thus $Z \wedge QM \cong L_1 S^0$, so $Z \cong L_1 DQM$. By taking $Z = QM$, we find $L_1 DQM \cong L_1 QM$, so the theorem is proved. □

We point out that the 2-completion of $L_1 QM$ is the Brown-Comenetz dual of $M_1 S^0$.

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