manuscripta math. 18, 273 - 278 (1976)

A REMARK ON MACKEY-FUNCTORS

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In the following note we characterize the category of Mackey-functors from a category $\underline{C}$, satisfying a few assumptions, to a category $\underline{D}$ as the category of functors from $Sp(\underline{C})$, the category of "spans" in $\underline{C}$, to $\underline{D}$ which preserve finite products. This caracterization permits to apply all results on categories of functors preserving a given class of limits to the case of Mackey-functors.

We recall (cf. [3], §6) the definition of a Mackey-functor:

1. DEFINITION: Let $\underline{C}$ and $\underline{D}$ be categories. A pair of functors $M^* : \underline{C} \longrightarrow \underline{D}$ and $M_* : \underline{C}^o \longrightarrow \underline{D}$ ($\underline{C}^o$ the dual category) is called a Mackey-functor (from $\underline{C}$ to $\underline{D}$) iff

(i) For every object $A \in |\underline{C}|$: $M^* A = M_* A$ (=: MA)

(ii) If (1) is a pullback diagram in $\underline{C}$, then the diagram (2) commutes:

(1)
$$\begin{array}{ccc} P & \xrightarrow{b} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

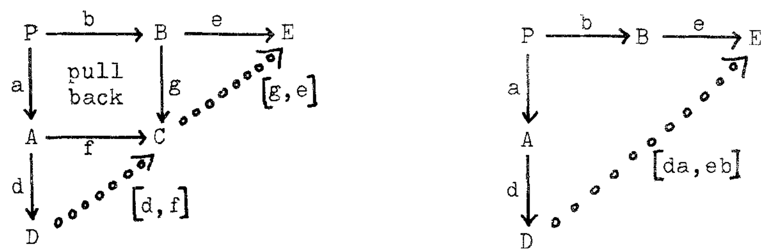
(2)
$$\begin{array}{ccc} MP & \xrightarrow{M^* b} & MB \\ {\scriptstyle M_* a}\uparrow & & \uparrow{\scriptstyle M_* g} \\ MA & \xrightarrow[M^* f]{} & MC \end{array}$$

(iii) $M_* : \underline{C}^o \longrightarrow \underline{D}$ preserves finite products.

If only (i) and (ii) are satisfied, we will call (M^*, M_*) a P-functor (P for pullback-property or pre-Mackey-functor). If (M^*, M_*) and (N^*, N_*) are both Mackey-functors (or both P-functors) from $\underline{C}$ to $\underline{D}$, a natural transformation from (M^*,M_*) to (N^*,N_*) consists of a family $\alpha = \{\alpha_A : MA \longrightarrow NA \mid A \in |\underline{C}|\}$ such that α is both a natural transformation from M^* to N^* and from M_* to N_*. Hence there is a (illegitimate) category of Mackey-functors (resp. P-functors) from $\underline{C}$ to $\underline{D}$. Furthermore, if $(M^*,M_*) : \underline{C} \longrightarrow \underline{D}$ is a P-functor and $G : \underline{D} \longrightarrow \underline{E}$ is any functor, the composition $G(M^*,M_*) := (GM^*,GM_*)$ is a P-functor from $\underline{C}$ to $\underline{E}$. We obtain therefore a 2-functor $\Pi_{\underline{C}} : \underline{Cat} \longrightarrow \underline{CAT}$ ($\underline{Cat}$ denotes the 2-category of $\mathfrak{U}$-categories, $\mathfrak{U}$ a fixed universe, $\underline{CAT}$ denotes the 2-category of <u>small</u> $\mathfrak{W}$-categories, $\mathfrak{W}$ a universe such that $\mathfrak{U} \in \mathfrak{W}$, hence the illegitimate $\mathfrak{U}$-categories are in $\underline{CAT}$ (cf. [6] 3.5, 3.6)), mapping a category $\underline{D}$ to the category of P-functors from $\underline{C}$ to $\underline{D}$.

<u>2. THEOREM: Let $\underline{C}$ be a category with pullbacks. The 2-functor $\Pi_{\underline{C}} : \underline{Cat} \longrightarrow \underline{CAT}$ is 2-representable.</u>

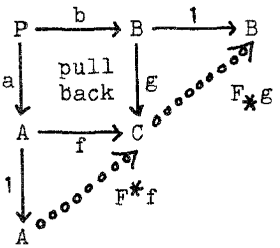
<u>PROOF:</u> A representing object for the 2-functor $\Pi_{\underline{C}}$ is the following (illegitimate) category $Sp(\underline{C})$ ($Sp(\underline{C})$ is the "classifying category" (cf. [2], 7.2) of the bicategory of "spans" in $\underline{C}$ (cf. [2], 2.6)): The objects of $Sp(\underline{C})$ are the objects of $\underline{C}$. The morphisms in $Sp(\underline{C})$ from A to B are the equivalence classes of the following equivalence relation on the set (actually a $\mathfrak{W}$-set) $\coprod_{P \in |\underline{C}|} \underline{C}(P,A) \sqcap \underline{C}(P,B)$: $(A \xleftarrow{a} P \xrightarrow{b} B) \sim (A \xleftarrow{a'} P' \xrightarrow{b'} B)$ iff there is an isomorphism $i : P \longrightarrow P'$ such that $a'i = a$ and $b'i = b$. We denote the equivalence class of (a,b) by $[a,b]$. The composition in $Sp(\underline{C})$ is defined as follows:

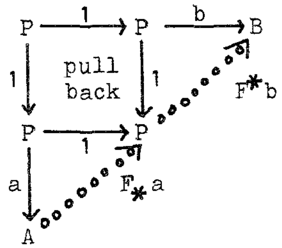


We now define two functors $F^*: \underline{C} \longrightarrow Sp(\underline{C})$ and F_* : $\underline{C}^o \longrightarrow Sp(\underline{C})$ by requiring (for $f :: A \longrightarrow C$)

(3) $F^*(f) := [1_A, f]$ $F_*(f) := [f, 1_A]$

This forces (F^*, F_*) to satisfy the condition 1(i). The condition 1(ii) is also satisfied, since $(F_*g)(F^*f) = [a,b] = (F^*b)(F_*a)$:



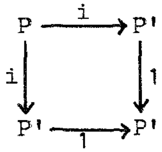


Now let $(M^*, M_*) : \underline{C} \longrightarrow \underline{D}$ be a P-functor. If $H : Sp(\underline{C}) \longrightarrow \underline{D}$ is any functor such that $HF^* = M^*$ and $HF_* = M_*$, then the following equation holds for all $a : P \longrightarrow A$ and $b : P \longrightarrow B$ in $\underline{C}$:

(4) $H([a,b]) = H((F^*b)(F_*a)) = (HF^*b)(HF_*a) = (M^*b)(M_*a)$

The right hand side of (4) does not depend on the particular choiche of the representing element (a,b) of the equivalence class $[a,b]$. In fact, if $(a,b) \sim (a',b')$, i.e. there exists an isomorphism $i : P \longrightarrow P'$ satisfying $a'i = a$ and $b'i = b$, then $(M^*b)(M_*a) = (M^*b')(M^*i)(M_*i)(M_*a')$, but $(M^*i)(M_*i) = 1_{MP'}$ as can be seen by applying 1(ii) to the pullback (5). Therefore,

(5)



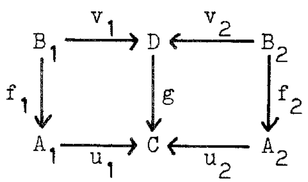
any such functor $H : Sp(\underline{C}) \longrightarrow \underline{D}$ is uniquely determined. On the other hand, given a P-functor $(M^*, M_*) : \underline{C} \longrightarrow \underline{D}$, we define $H : Sp(\underline{C}) \longrightarrow \underline{D}$ by (4). Using 1(ii), we can easily prove that H is in fact a functor. Furthermore, (4) clearly implies $HF^* = M^*$ and $HF_* = M_*$.

Finally, let $\alpha : (M^*, M_*) \longrightarrow (N^*, N_*) : \underline{C} \longrightarrow \underline{D}$ be a natural transformation (of P-functors). If H and I, resp., denote the corresponding functors from $Sp(\underline{C})$ to $\underline{D}$, then α is a natural transformation from H to I and vice versa. This completes the proof.

In order to prove a corresponding theorem for Mackey-functors we first consider a lemma:

<u>3. LEMMA: Let $\underline{C}$ be a category with pullbacks and finite coproducts. Let the initial object of $\underline{C}$ be strictly initial, and assume, for any commutative diagram (6) in $\underline{C}$ such that the bottom row is a coproduct diagram, the two squares are pullbacks if and only if the top row is a coproduct diagram. Then $F_* : \underline{C}^o \longrightarrow Sp(\underline{C})$ preserves finite products.</u>

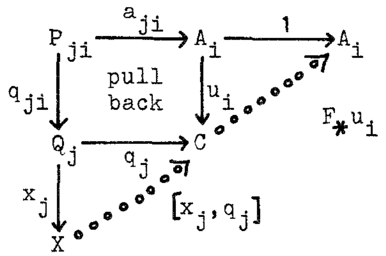
(6)

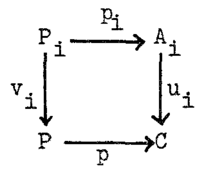


Before proving the lemma, we remark that the hypotheses of the lemma are satisfied in all the situations where Mackey-functors have been considered, e.g. if $\underline{C}$ is the category of all functors from a small category $\underline{B}$ (e.g. a <u>group</u>) to the category of sets (cf. [3], lemma 6.4).

<u>PROOF:</u> The assumption that the initial object of $\underline{C}$ be strictly initial clearly implies that F_* preserves the terminal object of $\underline{C}^o$. Now let $A_1 \xrightarrow{u_1} C \xleftarrow{u_2} A_2$ be a coproduct diagram in $\underline{C}$. Furthermore, let $[x_j, q_j] : X \longrightarrow C$ $(j = 1,2)$ be two morphisms in $Sp(\underline{C})$ such that $(F_* u_i)[x_1, q_1] = (F_* u_i)[x_2, q_2]$ for $i = 1,2$ (cf. diagram (7), page 5). This implies the existence of isomorphisms $p_i : P_{1i} \longrightarrow P_{2i}$ $(i = 1,2)$ such that:

(8) $$a_{2i}\, p_i = a_{1i}, \qquad x_2\, q_{2i}\, p_i = x_1\, q_{1i} \quad \text{for } i = 1,2.$$

(7) 

(9) 

The hypothesis of the lemma forces $P_{j1} \xrightarrow{q_{j1}} Q_j \xleftarrow{q_{j2}} P_{j2}$ to be a coproduct diagram in $\underline{C}$. Therefore (8) implies $[x_1, q_1] = [x_2, q_2]$.

Finally let $[x_i, p_i] : X \longrightarrow A_i$ $(i = 1,2)$ be two morphisms in $Sp(\underline{C})$ $(x_i : P_i \longrightarrow X,\ p_i : P_i \longrightarrow A_i)$. We choose a coproduct diagram $P_1 \xrightarrow{v_1} P \xleftarrow{v_2} P_2$ in $\underline{C}$ and obtain (unique) morphisms $p : P \longrightarrow C$ and $x : P \longrightarrow X$ such that $p\, v_i = u_i\, p_i$ and $x\, v_i = x_i$ for $i = 1,2$. The hypothesis of the lemma forces (9) to be pullbacks for $i = 1,2$. This clearly implies $(F_* u_i)[x,p] = [x_i, p_i]$ for $i = 1,2$. Hence $A_1 \xleftarrow{F_* u_1} C \xrightarrow{F_* u_2} A_2$ is a product diagram in $Sp(\underline{C})$.

Combining this result with the previous theorem, and taking into account that $|Sp(\underline{C})| = |\underline{C}|$, we obtain as a corollary:

4. THEOREM: Let $\underline{C}$ satisfy the hypotheses of the previous lemma, and let $\underline{D}$ be any category. The category of Mackey-functors from $\underline{C}$ to $\underline{D}$ is canonically isomorphic to the category of all finite-product-preserving-functors from $Sp(\underline{C})$ to $\underline{D}$.

This isomorphism is clearly natural with respect to $\underline{D}$ (and $\underline{C}$); and the theorem can be formulated as the representability of a 2-functor.

This theorem makes it possible to apply the results of [1,4,5] to the category of Mackey-functors from $\underline{C}$ to $\underline{D}$. In particular we note that this category admits an inclusion-functor into the category $[Sp(\underline{C}), \underline{D}]$ of all functors from $Sp(\underline{C})$ to $\underline{D}$ which is an adjoint functor. It inherits therefore completeness and cocompleteness properties from $\underline{D}$.

Finally we remark that the construction of [3], §6, which assigns to every category $\underline{D}$ the category Bi($\underline{D}$) (such that $|Bi(\underline{D})| = |\underline{D}|$ and $Bi(\underline{D})(A,B) = \underline{D}(A,B) \sqcap \underline{D}(B,A)$), is not as useful as Sp($\underline{C}$) in order to characterize Mackey-functors, but it has, however, the following universal property: it provides an adjoint functor "Bi" for the forgetful functor $J : \underline{Cat}^d \longrightarrow \underline{Cat}$ ($\underline{Cat}^d$ is the category of categories $\underline{A}$, equipped with a "duality", i.e. a functor $D : \underline{A}^o \longrightarrow \underline{A}$ such that $D(D^o) = 1_{\underline{A}}$). (This construction can be carried out for categories enriched over a monoidal category which has finite products).

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(Received November 24, 1975)