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# The Adams–Novikov $E_2$-term for Behrens' spectrum $Q(2)$ at the prime 3

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ABSTRACT

We compute the Adams–Novikov $E_2$-term of a spectrum $Q(2)$ constructed by M. Behrens. The homotopy groups of $Q(2)$ are closely tied to the 3-primary stable homotopy groups of spheres; in particular, they are conjectured to detect the homotopy beta family of Greek letter elements at the prime 3. Our computation leverages techniques used by Behrens to compute the rational homotopy of $Q(2)$, and leads to a conjecture that the Adams–Novikov $E_2$-term for $Q(2)$ detects the algebraic beta family in the $BP$-based Adams–Novikov $E_2$-term for the 3-local sphere.
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## 1. Introduction

Among the central objects of study in stable homotopy theory are the $p$-local stable homotopy groups of spheres for a prime $p$, i.e., the homotopy groups of the $p$-local sphere $S_{(p)}$. The chromatic convergence theorem of Hopkins and Ravenel [1] says that further localizations of $S_{(p)}$ yield successive approximations of $\pi_*S_{(p)}$; more precisely, if $L_n$ is localization with respect to the Johnson–Wilson spectrum $E(n)$ at $p$ [2], then

$$S_{(p)} \simeq \operatorname{holim}(L_0S_{(p)} \leftarrow L_1S_{(p)} \leftarrow L_2S_{(p)} \leftarrow \cdots).$$

For each $n$, $L_nS_{(p)}$ lies in a homotopy fracture square with $L_{K(n)}S_{(p)}$ [3], where $L_{K(n)}$ is localization with respect to the $n$th Morava $K$-theory spectrum $K(n)$ at $p$. This means the groups $\pi_*L_{K(n)}S_{(p)}$ for $n \geqslant 0$ are building blocks for $\pi_*S_{(p)}$. The spectra $L_{K(n)}S_{(p)}$ are the *$K(n)$-local spheres* and $n$ is the *chromatic level*. The spectrum $Q(2)$ that we study in this paper yields information at chromatic level 2.

Indeed, Behrens [4] constructs $Q(2)$ in an effort to reinterpret previous groundbreaking work [5–7] on $\pi_*L_{K(2)}S_{(3)}$-groups which lie at the edge of what is accessible computationally, as very little is known about the homotopy of the $K(n)$-local sphere at any prime for $n \geqslant 3$. The spectrum $Q(2)$ is an $E_\infty$ ring spectrum

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with the property that

$$DL_{K(2)}Q(2) \xrightarrow{D\eta} L_{K(2)}S_{(3)} \xrightarrow{\eta} L_{K(2)}Q(2) \tag{1}$$

is a cofiber sequence, where $\eta$ is the $K(2)$-localized unit map of $Q(2)$ and $D$ is the $K(2)$-local Spanier–Whitehead duality functor. The "2" in the notation reflects the fact that $Q(2)$ is built using degree 2 isogenies of elliptic curves (see Subsection 3.3) and is not a reference to the chromatic level $n = 2$. The sequence (1) implies that $L_{K(2)}S_{(3)}$ is built from $L_{K(2)}Q(2)$ and $DL_{K(2)}Q(2)$ and that their respective homotopy groups lie in a long exact sequence. Behrens ([4], Section 1.4.2) observes that there is a spectral sequence converging to $\pi_*Q(2)$ whose input is the cohomology of the totalization of a double cochain complex $C^{*,*}$:

$$E_2^{s,t}Q(2) := H^{s,t}(\operatorname{Tot} C^{*,*}) \Rightarrow \pi_{2t-s}Q(2). \tag{2}$$

This is the Adams–Novikov spectral sequence for $Q(2)$. In particular, the Adams–Novikov $E_2$-term for $Q(2)$ is itself computable via a double complex spectral sequence.

In this paper we compute the double complex spectral sequence converging to the Adams–Novikov $E_2$-term for $Q(2)$, thereby obtaining explicit descriptions of the elements in this $E_2$-term up to an ambiguity in two torsion $\mathbb{Z}_{(3)}$-submodules which we denote $U^{1,*} \subset E_2^{1,*}Q(2)$ and $U^{2,*} \subset E_2^{2,*}Q(2)$. The double complex $C^{*,*}$ is built from the cobar resolution of an elliptic curve Hopf algebroid $(B, \Gamma)$ over $\mathbb{Z}_{(3)}$ to be defined in Section 3. Throughout this paper, $\operatorname{Ext}^{*,*}$ (or just $\operatorname{Ext}^*$) will denote the Hopf algebroid cohomology of $(B, \Gamma)$, i.e.,

$$\operatorname{Ext}^{*,*} := \operatorname{Ext}_\Gamma^{*,*}(B, B)$$

in the category of $\Gamma$-comodules, and $\nu_p(x)$ will denote the $p$-adic valuation of a ($p$-local) integer $x$. The following is our main theorem.

**Theorem 1.** *The Adams–Novikov $E_2$-term for $Q(2)$ is given by*

$$E_2^{0,t}Q(2) = \begin{cases} \mathbb{Z}_{(3)}, & t = 0, \\ 0, & t \neq 0, \end{cases}$$

$$E_2^{1,t}Q(2) = \begin{cases} \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{(3)}, & t = 0, \\ \mathbb{Z}/(3) \oplus \mathbb{Z}/(3), & t = 4, \\ \mathbb{Z}/(3^{\nu_3(3m)}), & t = 4m, m \geqslant 2, \\ U^{1,t}, & t = 4m+2, m \geqslant 1, m \equiv 13 \bmod 27, \\ \mathbb{Z}/(3^{\nu_3(6m+3)}), & t = 4m+2, m \geqslant 1, m \not\equiv 13 \bmod 27, \\ 0, & \textit{otherwise}, \end{cases}$$

$$E_2^{2,t}Q(2) = \operatorname{Ext}^{2,t} \oplus \operatorname{Ext}^{1,t} \oplus M$$

*where*

$$M = \begin{cases} \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/3^{\nu_3(6m+3)}, & t = 4m+2, m \leqslant -1, \\ U^{2,t} \oplus \left( \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/3^{\nu_3(6m+3)} \right), & t = 4m+2, m \geqslant 1, m \equiv 13 \bmod 27, \\ \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/3^{\nu_3(6m+3)}, & t = 4m+2, m \geqslant 1, m \not\equiv 13 \bmod 27, \\ 0, & \textit{otherwise}, \end{cases}$$

*and* $E_2^{s,t}Q(2) = \operatorname{Ext}^{s,t} \oplus \operatorname{Ext}^{s-1,t}$ *for* $s \geqslant 3$.

The cohomology groups $\mathrm{Ext}^*$ have been computed by Hopkins and Miller [8] and they appear as summands in the Adams–Novikov $E_2$-term for $Q(2)$ by virtue of how $Q(2)$ is constructed (see Section 3).

In addition to being a concrete computational tool for accessing $\pi_*Q(2)$, Theorem 1 also sheds light on a conjectured relationship between $Q(2)$ and the beta family in the 3-primary stable stems, as follows. The spectrum $Q(2)$ is a special case of a more general object $Q(N)$, an $E(2)$-local ring spectrum at the prime $p$ built from degree $N$ isogenies, that exists as long as $p$ does not divide $N$. Behrens ([9], Theorem 12.1) proves that for $p > 3$ and $N$ a topological generator of $\mathbb{Z}_p^\times$, nontrivial homotopy divided beta family elements $\beta^h_{i/j,k} \in \pi_*S_{(p)}$ (which are seen in $\pi_*L_2S_{(p)}$) are detected by the homomorphism $(\eta_{E(2)})_*$ induced by the $E(2)$-localized unit map $\eta_{E(2)} : L_2S_{(p)} \to Q(N)$. Behrens conjectures that this holds for $p = 3$ and all corresponding $N$ ([9], Section 1). The case $p = 2$ is addressed by Behrens and Ormsby in [10].

The algebraic divided beta family lives on the 2-line of the $BP$-based Adams–Novikov $E_2$-term for the $p$-local sphere and comprises elements

$$\beta^a_{i/j,k} \in \mathrm{Ext}^{2,2i(p^2-1)-2j(p-1)}_{BP_*BP}(BP_*, BP_*)$$

for certain $i, j, k \in \mathbb{Z}$ [11]. These elements also appear in the $E_2$-term for $L_2S_{(p)}$. Our computation yields evidence for an algebraic version of Behrens' conjecture in the case $p = 3$ and $N = 2$.

**Conjecture 1.** *The elements $\beta^a_{i/j,k}$ have nontrivial image under the map of Adams–Novikov $E_2$-terms induced by $\eta_{E(2)} : L_2S_{(3)} \to Q(2)$.*

**Remark 1.** The statement of Theorem 1 reveals that the undetermined submodules $U^{1,*}$ and $U^{2,*}$ together constitute a small sliver of the Adams–Novikov $E_2$-term for $Q(2)$. In particular, these submodules could not possibly contain all of the algebraic divided beta family.

In Section 2 we outline our main results that lead to Theorem 1. In Section 3 we recall the construction of $Q(2)$ and the algebraic underpinnings of the double complex spectral sequence for $C^{*,*}$. Sections 4, 5, and 6 are the technical heart of the paper, where we prove the results stated in Section 2. We conclude in Section 7 with evidence that Conjecture 1 holds.

## 2. Statement of main results

In this section we state the results that constitute our proof of Theorem 1 (largely suppressing the $t$-degree throughout for readability). Our approach is based on previous work of Behrens on $\pi_*Q(2) \otimes \mathbb{Q}$ [12].

The first of our results reduces the double complex spectral sequence for $C^{*,*}$ to the cohomology of a singly-graded three-term cochain complex and the computation of one additional nontrivial differential. Recall from Section 1 that $\mathrm{Ext}^*$ is the cohomology of the elliptic curve Hopf algebroid $(B, \Gamma)$.

**Proposition 1.** *In the double complex spectral sequence for $C^{*,*}$, there are only two nontrivial $E_1$-page differentials given by $\mathbb{Z}_{(3)}$-module maps*

$$\mathrm{Ext}^0 \xrightarrow{\Phi} \mathrm{Ext}^0 \oplus B \xrightarrow{\Psi} B, \tag{3}$$

*there is only one nontrivial $E_2$-page differential*

$$\widetilde{d} : \mathrm{Ext}^1 \to \operatorname{coker} \Psi,$$

*and $E_3 = E_\infty$. Moreover,*

$$H^s(\operatorname{Tot} C^{*,*}) = \begin{cases} H^0 C^{*,0}, & s = 0 \\ H^1 C^{*,0} \oplus \ker \widetilde{d}, & s = 1 \\ \operatorname{coker} \widetilde{d} \oplus \operatorname{Ext}^2 \oplus \operatorname{Ext}^1, & s = 2 \\ \operatorname{Ext}^s \oplus \operatorname{Ext}^{s-1}, & s \geqslant 3. \end{cases}$$

By a slight abuse of notation, we will denote the cochain complex (3) by $C^*$ (so that $C^* = C^{*,0}$). The following proposition describes a two-stage filtration that we use to compute $H^*C^*$.

**Proposition 2.** *There is a filtration $C^* = F^0 \supset F^1 \supset F^2$ of $C^*$ inducing a short exact sequence $0 \to C' \to C^* \to C'' \to 0$, where*

$$C' = (0 \to B \xrightarrow{h} B), \quad C'' = (\operatorname{Ext}^0 \xrightarrow{g} \operatorname{Ext}^0 \to 0)$$

*The resulting long exact sequence in cohomology is*

$$0 \to H^0 C^* \to \ker g \xrightarrow{\delta^0} \ker h \to H^1 C^* \to \operatorname{coker} g \xrightarrow{\delta^1} \operatorname{coker} h \to H^2 C^* \to 0$$

*so that $H^0C^* = \ker \delta^0$, $H^2C^* = \operatorname{coker} \delta^1$, and $H^1C^*$ lies in the short exact sequence*

$$0 \to \operatorname{coker} \delta^0 \to H^1 C^* \to \ker \delta^1 \to 0. \tag{4}$$

Proposition 2 shows that computing $H^*C^*$ via this two-stage filtration starts with the kernels and cokernels of the maps $g : \operatorname{Ext}^0 \to \operatorname{Ext}^0$ and $h : B \to B$. We compute these kernels and cokernels using judicious choices of bases for $B$ and $\operatorname{Ext}^0$ as $\mathbb{Z}_{(3)}$-modules.

**Proposition 3.** *As modules over $\mathbb{Z}_{(3)}$, $\ker g = \ker h = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{(3)}$,*

$$\operatorname{coker} g = \left(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{(3)}\right) \oplus \left(\bigoplus_{x\in\mathcal{B}_{MF}^{\neq 0}} \mathbb{Z}/(3^{\nu_3(\deg x)+1})\right)$$

*where $\mathcal{B}_{MF}^{\neq 0}$ is a basis for the submodule of $\operatorname{Ext}^{0,*}$ of elements of nonzero $t$-degree (see Definition 5), and*

$$\operatorname{coker} h = \bigoplus_{i<j\in\mathbb{Z}} \left(\mathbb{Z}/(3^{\nu_3(i+j)+1}) \oplus \mathbb{Z}/(3^{\nu_3(2i+2j+1)+1})\right).$$

The following theorem describes $H^*C^*$. We prove this result by computing the connecting homomorphisms $\delta^0$ and $\delta^1$ from Proposition 2. As in Proposition 3, the proof is based on judicious choices of generators for the sources and targets.

**Theorem 2.**

(a) $H^0C^* = \mathbb{Z}_{(3)}$

(b) $H^1C^* = \left(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{(3)}\right) \oplus \left(\bigoplus_{m>0} \mathbb{Z}/(3^{\nu_3(3m)})\right) \left(\bigoplus_{\substack{m>0 \\ m\not\equiv 13 \bmod 27}} \mathbb{Z}/(3^{\nu_3(6m+3)})\right) \oplus U^1$

(c) $H^2C^* = \left(\bigoplus_{m\neq 0} \bigoplus_{n\in\mathbb{N}} \mathbb{Z}/(3^{\nu_3(6m+3)})\right) \oplus U^2$

Finally, we compute the differential $\widetilde{d}$ described in Proposition 1.

**Theorem 3.** $\ker \widetilde{d} = \mathbb{Z}/3$, *and* $\operatorname{coker} \widetilde{d} = H^2C^* \Big/ \left( \bigoplus_{m \in 3\mathbb{Z}} \mathbb{Z}/(3^{\nu_3(6m+3)}) \right)$.

Our computation will reveal that generators of $\mathbb{Z}_{(3)}$ summands in $E_2^{s,t}Q(2)$ lie in $t$-degree 0, generators of $\mathbb{Z}/(3^{\nu_3(3m)})$ summands lie in $t$-degree $4m$, and generators of $\mathbb{Z}/(3^{\nu_3(6m+3)})$ summands lie in $t$-degree $4m+2$ (see, e.g., Remark 4). With this, Proposition 1 and Theorems 2 and 3 piece together to yield Theorem 1.

**Remark 2.** As we shall see in Section 5, the summand

$$\bigoplus_{\substack{m>0 \\ m \not\equiv 13 \bmod 27}} \mathbb{Z}/(3^{\nu_3(6m+3)}) \tag{5}$$

of $E_2^{1,*}Q(2)$ is a submodule of $\ker \delta^1$. The source and target of $(\delta^1)_{4m+2}$ (the restriction of $\delta^1$ to elements of $t$-degree $4m+2$ in $\operatorname{coker} g$) are infinite direct sums of copies of $\mathbb{Z}/(3^{\nu_3(6m+3)})$. If $m \not\equiv 13 \bmod 27$ then $\mathbb{Z}/(3^{\nu_3(6m+3)}) \cong \mathbb{Z}/(3)$, $\mathbb{Z}/(9)$, or $\mathbb{Z}/(27)$, making the kernel and cokernel of $(\delta^1)_{4m+2}$ explicitly computable in those cases. This is not true for $m \equiv 13 \bmod 27$, for which $\nu_3(6m+3) \geqslant 4$. To demonstrate this, below we have the first few columns of matrix representations of $(\delta^1)_{4m+2}$ for a general $m \not\equiv 13 \bmod 27$, and $m = 13$ (so that $\mathbb{Z}/(3^{\nu_3(6m+3)}) \cong \mathbb{Z}/(81)$), respectively:

$$\begin{bmatrix} \vdots & \vdots & & & & \\ u_0 & * & & & & \\ 0 & u_1 & & \vdots & \boldsymbol{\vdots} & \\ \vdots & 0 & \ddots & * & \mathbf{0} & \\ & \vdots & & u_y & \mathbf{0} & \cdots \\ & & & 0 & \mathbf{0} & \\ & & & \vdots & \boldsymbol{\vdots} & \end{bmatrix}, \quad \begin{bmatrix} 0 & 78 & 31 & 39 & \mathbf{0} & \\ 8 & 21 & 17 & 72 & \mathbf{0} & \\ 80 & 5 & 79 & 6 & \mathbf{0} & \\ 0 & 62 & 56 & 19 & \mathbf{27} & \\ 0 & 0 & 44 & 72 & \mathbf{0} & \\ 0 & 0 & 0 & 26 & \mathbf{54} & \cdots \\ 0 & 0 & 0 & 0 & \mathbf{0} & \\ \vdots & \vdots & \vdots & \vdots & \boldsymbol{\vdots} & \end{bmatrix}$$

(see Eq. (31) in Subsection 5.2). The entries $u_0, \dots, u_y$ are units. The bolded zero column on the left yields the summand (5) and also makes the cokernel a direct sum of cyclic $\mathbb{Z}_{(3)}$-modules when $m \not\equiv 13 \bmod 27$. The corresponding column for $m \equiv 13 \bmod 27$ is always *nonzero* (see Lemma 12(a)), as in the $m = 13$ example shown above on the right. This causes complications, including relations in $\operatorname{coker} \delta^1$ that we cannot compute in general. It is precisely these unknown parts of $\ker \delta^1$ and $\operatorname{coker} \delta^1$ that constitute $U^1$ and $U^2$, respectively (see Definition 10). In this $m = 13$ example, the ordered bases for the source and target are $\{D_0^{13}, D_1^{13}, D_2^{13}, \dots\}$ and $\{B_0^{13}, B_1^{13}, B_2^{13}, \dots\}$ (see Definitions 4 and 7), the kernel is $\mathbb{Z}/(81)$ generated by $-27D_3^{13} + D_4^{13}$, and the cokernel is

$$\left(\mathbb{Z}/(81)\{B_0^{13}, B_1^{13}\}/(B_0^{13} - 3B_1^{13} = 0)\right) \oplus \mathbb{Z}/(81)\{B_6^{13}\} \oplus \mathbb{Z}/(81)\{B_7^{13}, B_9^{13}, B_{11}^{13}, \dots\}. \tag{6}$$

This computation will follow from the proof of Proposition 9(e). Example 2 in Section 7 suggests that the algebraic beta family elements $\beta^a_{9/9,1}$ and $\beta^a_{7/1,1}$ (the former being related to the 3-primary Kervaire invariant problem) may be detected in the submodule (6).

## 3. The Adams–Novikov $E_2$-term for $Q(2)$

The spectrum $Q(2)$ is the homotopy inverse limit of a semi-cosimplicial diagram of the form

$$TMF \rightrightarrows TMF \vee TMF_0(2) \Rrightarrow TMF_0(2), \tag{7}$$

where $TMF$ and $TMF_0(2)$ are both 3-local variants of the spectrum of topological modular forms [13]. The diagram (7) can be viewed as a more efficient version of a tower of spectra used by the authors of [6] in their study of the 3-primary $K(2)$-local sphere. In this section, we describe the Adams–Novikov $E_2$-term for $Q(2)$ in terms of the data in (7) and we set up the spectral sequence we will use to compute it. In particular, we will prove Propositions 1 and 2.

### 3.1. Setup of the double complex spectral sequence for $C^{*,*}$

Our starting point is the definition of the elliptic curve Hopf algebroid $(B, \Gamma)$ introduced in Section 1.

**Definition 1.** The graded $\mathbb{Z}_{(3)}$-algebras $B$ and $\Gamma$ are defined as follows:

$$B = \mathbb{Z}_{(3)}[q_2, q_4, \Delta^{-1}]/(\Delta = q_4^2(16q_2^2 - 64q_4))$$

with $\deg(q_2) = 2$ and $\deg(q_4) = 4$ (hence $\deg(\Delta) = 12$), and

$$\Gamma = B[r]/(r^3 + q_2r^2 + q_4r)$$

with $\deg(r) = 2$.

The groups $\mathrm{Ext}^*$ are encoded as the cohomology groups of the cobar resolution $C^*(\Gamma)$ for $(B, \Gamma)$ ([14], A1.2.11), a cochain complex of the form

$$B \xrightarrow{d} \Gamma \xrightarrow{d} \Gamma \otimes \Gamma \xrightarrow{d} \Gamma \otimes \Gamma \otimes \Gamma \xrightarrow{d} \cdots$$

where the differentials $d$ are defined in terms of the structure maps of $(B, \Gamma)$ (the coproduct, the right and left units, etc.). Formulas for these structure maps are given in [8].

The Hopf algebroid $(B, \Gamma)$ is connected to $TMF$ via elliptic curves. Any Hopf algebroid co-represents the objects and morphisms of a groupoid; in the case of $(B, \Gamma)$, the corresponding groupoid is that of non-singular elliptic curves with Weierstrass equation

$$y^2 = 4x(x^2 + q_2x + q_4) \tag{8}$$

and isomorphisms $x \mapsto x + r$ that preserve this Weierstrass form. If $\mathcal{M}$ is the moduli stack of such elliptic curves over $\mathbb{Z}_{(3)}$, the Goerss–Hopkins–Miller theorem [15] gives a sheaf $\mathcal{O}_{ell}$ of $E_\infty$ ring spectra on $\mathcal{M}$, and $TMF$ is defined as the global sections of this sheaf, i.e., $TMF = \mathcal{O}_{ell}(\mathcal{M})$. As a result, there is a spectral sequence

$$E_2^{*,*} = \mathrm{Ext}^{*,*} = \mathrm{Ext}_\Gamma^{*,*}(B, B) \Rightarrow \pi_* TMF$$

whose $E_2$-term is the cohomology of $(B, \Gamma)$. This the Adams–Novikov spectral sequence for $TMF$.

To recover $TMF_0(2)$, consider the groupoid whose objects are elliptic curves as in (8) but with the additional datum of a $\Gamma_0(2)$ structure (i.e., a choice of order 2 subgroup). There are no nontrivial structure-preserving isomorphisms $x \mapsto x+r$ in this case, so the underlying Hopf algebroid is the trivial Hopf algebroid $(B, B)$. If $\mathcal{M}_0(2)$ is the moduli stack of such elliptic curves over $\mathbb{Z}_{(3)}$, the Goerss–Hopkins–Miller theorem once again gives a sheaf of $E_\infty$ ring spectra lying over it, and we obtain $TMF_0(2)$ by taking global sections. The Adams–Novikov $E_2$-term for $TMF_0(2)$ is therefore $\mathrm{Ext}_B^*(B, B) = B$. The spectral sequence collapses at

$E_2$ and yields

$$\pi_{2k}\, TMF_0(2) = B_k$$

where $B_k$ denotes the elements of $B$ of degree $k$.

The following proposition gives the Adams–Novikov spectral sequence converging to $\pi_* Q(2)$, whose $E_2$-term is stitched together from the Adams–Novikov $E_2$-terms for $TMF$ and $TMF_0(2)$ according to the maps in (7).

**Proposition 4.** *(See [4, Section 1.4.2].) The Adams–Novikov $E_2$-term for $Q(2)$ is the cohomology of the totalization of the double complex $C^{*,*}$ given by*

$$C^*(\Gamma) \xrightarrow{\Phi} \overline{C}^*(\Gamma) \oplus B \xrightarrow{\Psi} B \to 0 \tag{9}$$

*where $\overline{C}^*(\Gamma)$ is obtained from $C^*(\Gamma)$ by multiplying its differentials by $-1$, $B$ is viewed as a cochain complex concentrated in Ext-degree 0, and the cochain complex maps $\Phi$, $\Psi$ are induced by the corresponding maps of spectra in (7).*

### *3.2. Algebraic properties of $B$ and* $\mathrm{Ext}^0$

In this subsection we lay the algebraic groundwork for our computation by examining the ring $B$ and the subring $\mathrm{Ext}^0 \subset B$. The latter is called the *ring of invariants* of the Hopf algebroid $(B, \Gamma)$; it is the set of elements that are fixed by the right unit structure map $\eta_R : B \to \Gamma$.

Following [12], we begin by defining a new element $\mu \in B$:

$$\mu := 16q_2^2 - 64q_4,$$

an element of degree 4. For computational convenience, we will replace $q_4$ and $\mu$ by scalar multiples of themselves, namely

$$\begin{aligned} s &:= 8q_4, \\ t &:= \mu/8, \end{aligned} \tag{10}$$

thus $\deg(s) = \deg(t) = 4$. [Note: While we also use $s$ and $t$ to refer to the bidegrees $(s,t)$ in $E_2^{*,*}Q(2)$, we believe their meanings will always be clear from the context.]

**Lemma 1.** *As a $\mathbb{Z}_{(3)}$-algebra,*

$$B = \mathbb{Z}_{(3)}[q_2, q_4, q_4^{-1}, \mu^{-1}]/(\mu = 16q_2^2 - 64q_4)$$

*and thus $\{s^i t^j q_2^\epsilon : i, j \in \mathbb{Z}, \epsilon = 0 \text{ or } 1\}$ is a basis for $B$ as a $\mathbb{Z}_{(3)}$-module.*

**Proof.** Since $\Delta = q_4^2\mu$, inverting $\Delta$ is equivalent to inverting $q_4$ and $\mu$, which proves the first statement. The second statement follows from (10) and the relation $q_2^2 = (\mu + 64q_4)/16$. □

**Lemma 2.** $B_0 = \mathbb{Z}_{(3)}[j_B, j_B^{-1}]$*, where*

$$j_B := s/t. \tag{11}$$

**Proof.** The only elements $s^i t^j q_2^\epsilon$ in $B_0$ are those with $i = -j$ and $\epsilon = 0$. $\square$

**Definition 2.** Given $i \leqslant j \in \mathbb{Z}$ and $\epsilon = 0$ or 1, define submodules

$$V_{i,j,\epsilon} := \mathbb{Z}_{(3)}\{s^i t^j q_2^\epsilon, s^j t^i q_2^\epsilon\} \subset B$$

free of rank 1 if $i = j$, and free of rank 2 otherwise.

**Lemma 3.** *As a $\mathbb{Z}_{(3)}$-module,* $B = \displaystyle\bigoplus_{i \leqslant j,\ \epsilon=0,1} V_{i,j,\epsilon}$.

**Proof.** This follows from Lemma 1. $\square$

We will see in Subsection 4.3 that the following elements form a basis of eigenvectors for $B$ with respect to the map $h : B \to B$ from Proposition 2.

**Definition 3.** For $i < j \in \mathbb{Z}$,

$$a_{i,j} := s^i t^j - s^j t^i, \quad \bar{a}_{i,j} := s^i t^j + s^j t^i, \quad b_{i,j} := a_{i,j} q_2, \quad \bar{b}_{i,j} := \bar{a}_{i,j} q_2$$

and for $\epsilon = 0$ or 1, $c_i^\epsilon := s^i t^i q_2^\epsilon$.

The elements $\{a_{i,j}\}$ and $\{b_{i,j}\}$ from Definition 3 will be key in Section 5 when we compute the connecting homomorphisms in the long exact sequence from Proposition 2. The following definition gives a convenient enumeration of these elements for our study of $\delta^1$ in Subsection 5.2.

**Definition 4.** For $0 \leqslant v \in \mathbb{Z}$ and $m \in \mathbb{Z}$,

$$A_v^m := a_{\lfloor \frac{m-1}{2} \rfloor - v, \lceil \frac{m+1}{2} \rceil + v}, \quad B_v^m := b_{\lfloor \frac{m-1}{2} \rfloor - v, \lceil \frac{m+1}{2} \rceil + v}.$$

Hereafter we denote the ring of invariants $\mathrm{Ext}^0$ by $MF$. The following proposition is an explicit description of $MF$ proven in [16].

**Proposition 5.** *If*

$$\begin{aligned} c_4 &:= \mu + 16 q_4 = 2s + 8t, \\ c_6 &:= 4q_2(8q_4 - \mu) = 4q_2(s - 8t), \end{aligned} \tag{12}$$

*then*

$$MF = \mathbb{Z}_{(3)}[c_4, c_6, \Delta, \Delta^{-1}]/(1728\Delta = c_4^3 - c_6^2)$$

*where* $\Delta = q_4^2 \mu = s^2 t/8$ *as before,* $\deg(c_4) = 4$, *and* $\deg(c_6) = 6$.

**Remark 3.** The notation "$MF$" stands for "modular forms." Indeed, the ring $MF \otimes \mathbb{C}$ is the ring of modular forms over $\mathbb{C}$ for the full modular group $\mathrm{SL}(2, \mathbb{Z})$. Note also that $B \otimes \mathbb{C}$ is the ring of modular forms over $\mathbb{C}$ for the congruence subgroup $\Gamma_0(2) \subset \mathrm{SL}(2, \mathbb{Z})$.

In the following definition we identify bases for $MF$ and some of its $\mathbb{Z}_{(3)}$-submodules that will prove useful for our computations in Sections 4 and 5. Note that $\mathcal{B}_{MF}^{\neq 0}$ defined below appears in Proposition 3 (see Section 1).

**Definition 5.** Let

$$\mathcal{B}_{MF} := \{c_4^n c_6^\epsilon \Delta^\ell : n \geqslant 0, \ell \in \mathbb{Z}, \epsilon = 0 \text{ or } 1\},$$
$$\mathcal{B}_{MF}^{\neq 0} := \{x \in \mathcal{B}_{MF} : \deg(x) \neq 0\} \subset \mathcal{B}_{MF}$$

and, for any $m \in \mathbb{Z}$ and $\epsilon = 0$ or 1,

$$\mathcal{B}_{MF}^{\epsilon,m} := \{c_4^n c_6^\epsilon \Delta^\ell : n + 3\ell + \epsilon = m\} \subset \mathcal{B}_{MF}.$$

**Lemma 4.** *The set $\mathcal{B}_{MF}$ is a basis for $MF$ as a $\mathbb{Z}_{(3)}$-module.*

**Proof.** This follows from the relation $c_6^2 = c_4^3 - 1728\Delta$ in $MF$. $\square$

**Lemma 5.** $MF_0 = \mathbb{Z}_{(3)}[j_{MF}]$, *where*

$$j_{MF} := c_4^3/\Delta \tag{13}$$

*is the $j$-invariant ([17], Section III.1).*

**Proof.** First note that $c_4^3 - c_6^2$ is irreducible in $MF$. To see this, we temporarily put $X = c_4$ and $Y = c_6$, in which case it suffices to show that $Y^2 - X^3 = -(c_4^3 - c_6^2)$ is irreducible. Suppose not. Then we may write

$$Y^2 - X^3 = (Y + f(X))(Y + g(X))$$

where $f$ and $g$ are polynomials in $X$, $f(X)g(X) = -X^3$ and $f(X) = -g(X)$. In particular, $[g(X)]^2 = X^3$, which is impossible.

Since $c_4^3 - c_6^2$ is irreducible, the only basis elements $c_4^n c_6^\epsilon \Delta^\ell$ in $MF_0$ are those with $3\ell = -n$ and $\epsilon = 0$. $\square$

We now give notation for the submodules of $MF$ spanned by the sets $\mathcal{B}_{MF}^{\epsilon,m}$ in Definition 5.

**Definition 6.** Given $m \in \mathbb{Z}$ and $\epsilon = 0$ or 1, define submodules

$$W_{\epsilon,m} := \mathbb{Z}_{(3)}\{\mathcal{B}_{MF}^{\epsilon,m}\} \subset MF.$$

**Lemma 6.** *As a $\mathbb{Z}_{(3)}$-module,*

$$MF = W_{0,0} \oplus \left( \bigoplus_{\substack{m\in\mathbb{Z},\epsilon=0,1,\\ (\epsilon,m)\neq(0,0)}} W_{\epsilon,m} \right).$$

**Proof.** By degree counting, $MF_0 = W_{0,0}$. The result then follows from Lemmas 4 and 5 and the union decomposition

$$\mathcal{B}_{MF}^{\neq 0} = \bigcup_{\substack{m\in\mathbb{Z},\epsilon=0,1,\\ (\epsilon,m)\neq(0,0)}} \mathcal{B}_{MF}^{\epsilon,m}. \qquad \square$$

For an element $c_4^n \Delta^\ell \in \mathcal{B}_{MF}^{0,m}$, the largest possible value of $\ell$ is

$$\ell_0^m := \left\lfloor \frac{m}{3} \right\rfloor,$$

while for $c_4^n c_6 \Delta^\ell \in \mathcal{B}_{MF}^{1,m}$ it is

$$\ell_1^m := \left\lfloor \frac{m-1}{3} \right\rfloor.$$

This allows us to give the following enumeration of the elements in $\mathcal{B}_{MF}^{\neq 0}$, convenient for our study of $\delta^1$ in Subsection 5.2.

**Definition 7.** For $0 \leqslant v \in \mathbb{Z}$ and $m \in \mathbb{Z}$,

$$C_v^m := c_4^{m-3\ell_0^m+3v} \Delta^{\ell_0^m - v}, \quad D_v^m := c_4^{m-3\ell_1^m-1+3v} c_6 \Delta^{\ell_1^m - v}$$

so that $\mathcal{B}_{MF}^{0,m} = \{C_0^m, C_1^m, C_2^m, \ldots\}$ and $\mathcal{B}_{MF}^{1,m} = \{D_0^m, D_1^m, D_2^m, \ldots\}$.

**Remark 4.** The enumerations in Definitions 4 and 7 are analogous in terms of how the integer $m$ compares with the polynomial degree. Specifically,

$$\deg(A_v^m) = \deg(C_v^m) = 4m, \quad \deg(B_v^m) = \deg(D_v^m) = 4m + 2. \tag{14}$$

*3.3. Maps of the double complex*

In this subsection we describe four Hopf algebroid maps, denoted $\psi_d$, $\phi_f$, $\phi_q$, and $\psi_{[2]}$, that assemble to give $\Phi$ and $\Psi$ as follows:

$$\begin{aligned}\Phi &= (\psi_{[2]} \oplus \phi_q) - (1_\Gamma \oplus \phi_f),\\ \Psi &= \psi_d - \phi_f + 1_B.\end{aligned} \tag{15}$$

This yields the diagram

$$\Gamma \xrightarrow{\Phi} \Gamma \oplus B \xrightarrow{\Psi} B \to 0$$

of $\mathbb{Z}_{(3)}$-modules inducing the double cochain complex (9) in Proposition 4.

Each of $\psi_d$, $\phi_f$, $\phi_q$, and $\psi_{[2]}$ corresponds to a maneuver with elliptic curves (see Remark 5 below) and is defined by the effect of the maneuver on Weierstrass equations, as computed in Section 1.5 of [4] (where they are denoted $\psi_d^*$, $\phi_f^*$, $\phi_q^*$, and $\psi_{[2]}^*$, respectively). We briefly summarize those computations here. Since each map is a Hopf algebroid morphism, those with source $(B, B)$ are determined by their values on $q_2$ and $q_4$, while those with source $(B, \Gamma)$ are determined by their values on $q_2$, $q_4$, and $r$.

Given an elliptic curve $C$ over $\mathbb{Z}_{(3)}$ with Weierstrass equation as in (8) and an order 2 subgroup $H$, $\psi_d : (B, B) \to (B, B)$ records the effect on $q_2$ and $q_4$ when $C$ is replaced by its quotient $C/H$, or equivalently, when the degree 2 isogeny $C \to C/H$ is replaced by its dual isogeny $C/H \to C$. The effect is

$$\begin{aligned}\psi_d : q_2 &\mapsto -2q_2,\\ q_4 &\mapsto q_2^2 - 4q_4.\end{aligned}$$

If $C$ is an elliptic curve as before, then $\phi_f : (B, \Gamma) \to (B, B)$ forgets the choice of order 2 subgroup $H \subset C$. Simply forgetting this extra structure does not impact the coefficients $q_2$ and $q_4$ but it does impact which elliptic curve morphisms are allowed. Since there are no transformations $x \mapsto x + r$ that preserve $H$, $\phi_f$ is given by

$$\begin{aligned}\phi_f : q_2 &\mapsto q_2,\\ q_4 &\mapsto q_4,\\ r &\mapsto 0.\end{aligned}$$

For $\phi_q : (B, \Gamma) \to (B, B)$, the relation $\phi_q = \psi_d \circ \phi_f$ imposed by the semi-cosimplicial structure of (7) implies

$$\begin{aligned}\phi_q : q_2 &\mapsto -2q_2,\\ q_4 &\mapsto q_2^2 - 4q_4,\\ r &\mapsto 0.\end{aligned}$$

The map $\psi_{[2]}$ can be viewed either as a self-map of $(B, \Gamma)$ or as a self-map of $(B, B)$ ([4], Section 1.1). In either case, $\psi_{[2]}$ corresponds to taking the quotient of $C$ by its subgroup $C[2]$ of points of order 2. The standard elliptic curve addition formulas show that, on the level of Weierstrass equations, this corresponds to replacing $q_2$ by $2^2 q_2$ and $q_4$ by $2^4 q_4$. Moreover, the allowable transformations in this case are of the form $x \mapsto x + 2^2 r$. Thus, as a self-map of $(B, \Gamma)$,

$$\begin{aligned}\psi_{[2]} : q_2 &\mapsto 4q_2,\\ q_4 &\mapsto 16q_4,\\ r &\mapsto 4r\end{aligned}$$

and restriction yields the corresponding self-map of $(B, B)$.

Combined with (15) the above formulas yield

$$\begin{aligned}\Phi : q_2 &\mapsto (3q_2, -3q_2),\\ q_4 &\mapsto (15q_4, q_2^2 - 5q_4),\\ r &\mapsto (3r, 0)\end{aligned} \tag{16}$$

and $\Psi : (x, y) \mapsto \psi_d(y) - \phi_f(x) + y$ for $(x, y) \in \Gamma \oplus B$.

**Remark 5.** The semi-cosimplicial diagram (7) underlying $Q(2)$ is the topological realization of a semi-simplicial diagram of stacks

$$\mathcal{M} \Leftarrow \mathcal{M} \coprod \mathcal{M}_0(2) \Lleftarrow \mathcal{M}_0(2). \tag{17}$$

The stacks $\mathcal{M}$ and $\mathcal{M}_0(2)$ are categories fibered in groupoids over the category of $\mathbb{Z}_{(3)}$-affine schemes. Given a $\mathbb{Z}_{(3)}$-algebra $T$, the groupoid lying over $\mathrm{Spec}(T)$ in $\mathcal{M}$ is the one co-represented by $(B, \Gamma)$, while the groupoid lying over $\mathrm{Spec}(T)$ in $\mathcal{M}_0(2)$ is the one co-represented by $(B, B)$. The Hopf algebroid maps defined in this subsection correspond to the morphisms of stacks in (17), and the elliptic curve maneuvers can be interpreted as descriptions of what these stack morphisms do on the level of $T$-points.

### *3.4. Proof of Proposition 1*

Recall from Section 3.1 that the Adams–Novikov $E_2$-term for $Q(2)$ is the target of the double complex spectral sequence for $C^{*,*}$, which is a first quadrant double complex of the form

$$\begin{tikzcd}
C^*(\Gamma) \arrow[r, "\Phi"] \arrow[d, equal] & \overline{C}^*(\Gamma) \oplus B \arrow[r, "\Psi"] \arrow[d, equal] & B \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
C^{0,*} & C^{1,*} & C^{2,*} & C^{3,*}
\end{tikzcd}$$

The vertical differentials are induced by the cobar complex differentials for $(B, \Gamma)$ and are formally the $d_0$-differentials of the double complex spectral sequence; the horizontal maps $\Phi$ and $\Psi$ were defined in the previous section and will induce the $d_1$-differentials.

Expanding the three nontrivial columns of $C^{*,*}$ gives

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
\Gamma \otimes \Gamma \arrow[r, "\Phi"] \arrow[u] & \Gamma \otimes \Gamma \arrow[r] \arrow[u] & 0 \arrow[u] \\
\Gamma \arrow[r, "\Phi"] \arrow[u, "d"] & \Gamma \arrow[r] \arrow[u, "-d"'] & 0 \arrow[u] \\
B \arrow[r, "\Phi"] \arrow[u, "d"] & B \oplus B \arrow[r, "\Psi"] \arrow[u, "-d\oplus 0"'] & B \arrow[u]
\end{tikzcd} \tag{18}$$

and turning to the $E_1$-page yields the following result.

**Lemma 7.** *Taking cohomology with respect to the vertical differentials in (18) gives*

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
\mathrm{Ext}^2 \arrow[r, "0"] \arrow[u] & \mathrm{Ext}^2 \arrow[r] \arrow[u] & 0 \arrow[u] \\
\mathrm{Ext}^1 \arrow[r, "0"] \arrow[u] & \mathrm{Ext}^1 \arrow[r] \arrow[u] & 0 \arrow[u] \\
MF \arrow[r, "\Phi"] \arrow[u] & MF \oplus B \arrow[r, "\Psi"] \arrow[u] & B \arrow[u]
\end{tikzcd} \tag{19}$$

**Proof.** We know $\mathrm{Ext}^0 = MF$ by Proposition 5. The computation of the homotopy groups of $TMF$ by Hopkins and Miller [8] shows that $\mathrm{Ext}^n$ is entirely 3-torsion for $n \geqslant 1$. Eq. (16) therefore implies $\Phi : \mathrm{Ext}^n \to \mathrm{Ext}^n$ must be identically zero for $n \geqslant 1$. $\square$

Lemma 7 shows that the double complex spectral sequence for $C^{*,*}$ has only two potentially nontrivial differentials on its $E_1$-page: $\Phi$ and $\Psi$. By sparseness, the only potentially nontrivial differential on the $E_2$-page is a map $\mathrm{Ext}^1 \to \operatorname{coker}\Psi$ (denoted $\widetilde{d}$ in Proposition 1) and $E_3 = E_\infty$. Therefore, to obtain the $E_\infty$-page from (19) we need only replace the 0th row by

$$H^0C^* \to H^1C^* \to \operatorname{coker}\widetilde{d}$$

and replace $\mathrm{Ext}^1$ in the 0th column by $\ker \widetilde{d}$. This completes the proof of Proposition 1.

### 3.5. Proof of Proposition 2

In this subsection we define the two-stage filtration of $C^*$ we shall use to compute $H^*C^*$ and we prove Proposition 2. For ease of notation, we will henceforth denote by 1 the maps $1_B$, $1_\Gamma$, and any maps they induce; the meaning should be clear from the context.

If $F^0 = C^*$, $F^1 = (MF \xrightarrow{\psi_{[2]}-1} MF \to 0)$, and $F^2$ is the trivial complex, then $F^0 \supset F^1 \supset F^2$ is our filtration. It induces a short exact sequence

$$0 \to C' \to C^* \to C'' \to 0 \tag{20}$$

of chain complexes, given by

$$\begin{tikzcd}
C' : & 0 \arrow[r] \arrow[d] & B \arrow[r, "\psi_d+1"] \arrow[d, hook] & B \arrow[d, equal] \\
C^* : & MF \arrow[r, "\Phi"] \arrow[d, equal] & B \oplus MF \arrow[r, "\Psi"] \arrow[d] & B \arrow[d] \\
C'' : & MF \arrow[r, "\psi_{[2]}-1"] & MF \arrow[r] & 0
\end{tikzcd}$$

**Definition 8.** $g := \psi_{[2]} - 1 : MF \to MF, \quad h := \psi_d + 1 : B \to B.$

Proposition 2 follows from standard homological algebra (see, e.g., Section 1.3 of [18]). The map $\delta^0$ is the restriction of $\phi_q - \phi_f$ to $\ker g$, while the map $\delta^1$ is the map induced by $-\phi_f$ on $\operatorname{coker} g$.

## 4. Computation of the maps $g$ and $h$

In this section we initiate our computation of the Adams–Novikov $E_2$-term for $Q(2)$ by computing the kernel and cokernel of the maps $g : MF \to MF$ and $h : B \to B$ defined in Section 3.

### 4.1. A 3-divisibility result

The following result in 3-adic analysis is one we shall leverage numerous times throughout the remainder of this paper.

**Lemma 8.**

(a) *If $n$ is a nonzero even integer, then*

$$\nu_3(4^n - 1) = \nu_3(n) + 1.$$

(b) *If $n$ is an odd integer, then*

$$\nu_3(2^n + 1) = \nu_3(n) + 1.$$

**Proof.** Let $|\cdot|$ denote 3-adic absolute value. Fix an even integer $n > 1$ (the case $n < -1$ will follow immediately), and let

$$f(x) = (1 + x)^n - 1.$$

Recall that the (3-adic) functions $e^x$ and $\log(1+x)$ converge for $|x| \leqslant |3|$. Moreover, $|\log(1+x)| = |x|$, $|e^x| = 1$, and $|1 - e^x| = |x|$ for any $|x| \leqslant |3|$. But since

$$f(x) = (1+x)^n - 1 = e^{n\log(1+x)} - 1$$

this implies that for $|x| \leqslant |3|$,

$$|f(x)| = |e^{n\log(1+x)} - 1| = |n\log(1+x)| = |n||\log(1+x)| = |n||x|.$$

In particular, setting $x = 3$ yields $|f(3)| = |4^n - 1| = |n||3|$, which proves (a).

To prove (b), we need only slightly alter the above argument. Fix an odd integer $n > 0$. Replacing $x$ by $-x$ in the definition of $f(x)$ yields a new function

$$g(x) = (1-x)^n - 1 = e^{n\log(1-x)} - 1$$

and a similar analysis shows that if $|x| \leqslant |3|$, then $|g(x)| = |n||x|$. Setting $x = 3$ as before yields $|g(3)| = |(-2)^n - 1| = |n||3|$. But since $n$ is odd, this implies $|2^n + 1| = |n||3|$. □

### 4.2. Kernel and cokernel of $g : MF \to MF$

If $x \in MF$, the formulas for $\psi_{[2]}$ in Subsection 3.3 imply

$$g(x) = (2^{\deg(x)} - 1)x. \tag{21}$$

Since $2^{\deg(x)} - 1 = 0$ if and only if $\deg(x) = 0$, Lemma 5 implies

$$\ker g = MF_0 = \mathbb{Z}_{(3)}[j_{MF}] = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{(3)}.$$

Now suppose $x \in \mathcal{B}_{MF}^{\neq 0}$. The degree of $x$ must be even, say $\deg(x) = 2k$. By (21), $g(x) = (2^{\deg(x)} - 1)x$, and Lemma 8(a) implies

$$\nu_3(2^{\deg(x)} - 1) = \nu_3(4^k - 1) = \nu_3(k) + 1 = \nu_3(\deg(x)) + 1.$$

Thus

$$\operatorname{im} g = \bigoplus_{x\in\mathcal{B}_{MF}^{\neq 0}} 3^{\nu_3(\deg(x))+1}\mathbb{Z}_{(3)}$$

and the result for $\operatorname{coker} g$ in Proposition 3 follows.

### 4.3. Kernel and cokernel of $h : B \to B$

We begin by studying $h$ on the submodules $V_{i,j,\epsilon} \subset B$ from Definition 2.

**Proposition 6.** *Each $V_{i,j,\epsilon}$ is invariant under $h$, and $h\big|_{V_{i,j,\epsilon}}$ has a matrix representation with respect to $\{s^i t^j q_2^\epsilon, s^j t^i q_2^\epsilon\}$ depending on $i, j, \epsilon$ as follows:*

(a) *If $i < j$ and $\epsilon = 0$,*

$$h\big|_{V_{i,j,\epsilon}} = \begin{bmatrix} 1 & 4^{i+j} \\ 4^{i+j} & 1 \end{bmatrix}$$

*with eigenvectors $\bar{a}_{i,j}$, $a_{i,j}$ (see Definition 3) and corresponding eigenvalues*

$$\bar{\lambda}_{i,j} := 1 + 4^{i+j} \in \mathbb{Z}_{(3)}^{\times}, \quad \lambda_{i,j} := 1 - 4^{i+j} \notin \mathbb{Z}_{(3)}^{\times}.$$

(b) *If $i < j$ and $\epsilon = 1$,*

$$h\big|_{V_{i,j,\epsilon}} = \begin{bmatrix} 1 & -2^{2i+2j+1} \\ -2^{2i+2j+1} & 1 \end{bmatrix}$$

*with eigenvectors $b_{i,j}$, $\bar{b}_{i,j}$ (see Definition 3) and corresponding eigenvalues*

$$\rho_{i,j} := 1 + 2^{2i+2j+1} \notin \mathbb{Z}_{(3)}^{\times}, \quad \bar{\rho}_{i,j} := 1 - 2^{2i+2j+1} \in \mathbb{Z}_{(3)}^{\times}.$$

(c) $h\big|_{V_{i,i,\epsilon}}$ *is multiplication by $16^i(-2)^\epsilon + 1 \in \mathbb{Z}_{(3)}^{\times}$.*

**Proof.** The formulas for $\psi_d$ in Section 3.3 imply that

$$h(s^i t^j q_2^\epsilon) = 4^{i+j}(-2)^\epsilon s^j t^i q_2^\epsilon + s^i t^j q_2^\epsilon \tag{22}$$

which proves invariance and gives the matrix for $h\big|_{V_{i,j,\epsilon}}$ in all three cases (the matrix in case (c) being $1 \times 1$). The eigenvectors and eigenvalues can be found by a direct computation. The eigenvalues $\bar{\lambda}_{i,j}$, $\bar{\rho}_{i,j}$, and $16^i(-2)^\epsilon + 1$ are congruent to 2 modulo 3 and therefore are invertible in $\mathbb{Z}_{(3)}$. Lemma 8(a) implies $\lambda_{i,j}$ is 3-divisible, and Lemma 8(b) implies $\rho_{i,j}$ is 3-divisible. □

Lemma 3 and Proposition 6 show that the set

$$\{a_{i,j}; \bar{a}_{i,j}; b_{i,j}; \bar{b}_{i,j}; c_i^\epsilon : i < j \in \mathbb{Z}, \epsilon = 0, 1\}$$

is a $\mathbb{Z}_{(3)}$-basis of eigenvectors for $B$ relative to $h$. The generators $\bar{a}_{i,j}$, $\bar{b}_{i,j}$, and $c_i^\epsilon$ all map to unit multiples of themselves under $h$ by Proposition 6, and hence are not contained in the kernel. Since $\rho_{i,j} \neq 0$ for all $i < j \in \mathbb{Z}$, the generators $b_{i,j}$ are also not in the kernel. Finally, since $\lambda_{i,j} = 0$ if and only if $i = -j$, the only generators of the form $a_{i,j}$ that lie in the kernel of $h$ are $\{a_{-i,i} : i \geqslant 1\}$. Thus

$$\ker h = \mathbb{Z}_{(3)}\{a_{-i,i} : i \geqslant 1\} = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}_{(3)}.$$

Lemma 8(a) implies

$$\nu_3(\lambda_{i,j}) = \nu_3(1 - 4^{i+j}) = \nu_3(i+j) + 1$$

and similarly, Lemma 8(b) implies

$$\nu_3(\rho_{i,j}) = \nu_3(1 + 2^{2i+2j+1}) = \nu_3(2i+2j+1) + 1.$$

These results, together with Proposition 6, imply that

$$\operatorname{im} h = \left( \bigoplus_{x \in \{\bar{a}_{i,j};\bar{b}_{i,j};c_i^{\epsilon}\}} \mathbb{Z}_{(3)} \right) \oplus \left( \bigoplus_{i<j\in\mathbb{Z}} \left( 3^{\nu_3(i+j)+1}\mathbb{Z}_{(3)} \oplus 3^{\nu_3(2i+2j+1)+1}\mathbb{Z}_{(3)} \right) \right).$$

The result for $\operatorname{coker} h$ in Proposition 3 follows if we take $m = i+j$. This completes the proof of Proposition 3.

## 5. Computation of the connecting homomorphisms $\delta^0$ and $\delta^1$

In this section we compute the kernel and cokernel of $\delta^0$ and $\delta^1$. Using these computations, we prove Theorem 2.

### 5.1. *The kernel and cokernel of $\delta^0 : \ker g \to \ker h$*

From the results of Section 4,

$$\delta^0 = \phi_q - \phi_f : \mathbb{Z}_{(3)}[j_{MF}] \to \mathbb{Z}_{(3)}\{a_{-i,i} : i \geqslant 1\}.$$

**Proposition 7.** $\ker \delta^0 = \mathbb{Z}_{(3)}\{1_{MF}\}$ *and* $\operatorname{coker} \delta^0 = \mathbb{Z}_{(3)}\{a_{-i,i} : i \geqslant 1,\ \textit{odd}\}$.

**Proof.** The map $\delta^0$ is completely determined by where it sends nonnegative powers of $j_{MF}$. The formula for $\phi_q$ in Section 3.3 implies

$$\begin{aligned} \phi_q : q_4 &\mapsto \mu/16, \\ \mu &\mapsto 256 q_4. \end{aligned} \tag{23}$$

Combining (23) with the formula for $\phi_f$, the formulas (10) for $s$ and $t$, and Definition 3, yields

$$\begin{aligned} \delta^0(j_{MF}^k) &= (\phi_q - \phi_f)(c_4^{3k}\Delta^{-k}) \\ &= (256q_4 + \mu)^{3k}\mu^{-2k}q_4^{-k} - (\mu + 16q_4)^{3k}q_4^{-2k}\mu^{-k} \\ &= \frac{2^{6k}(4s+t)^{3k}}{t^{2k}s^k} - \frac{2^{6k}(4t+s)^{3k}}{s^{2k}t^k} \\ &= \sum_{r=0}^{3k}\binom{3k}{r}2^{12k-2r}(s^{2k-r}t^{r-2k} - s^{r-2k}t^{2k-r}) \\ &= \sum_{r=0}^{3k}\binom{3k}{r}2^{12k-2r}a_{2k-r,r-2k} \\ &= 2^{8k}\sum_{v=1}^{k}\left(\binom{3k}{2k+v}4^{-v} - \binom{3k}{2k-v}4^{v}\right)a_{-v,v} - 2^{8k}\sum_{v=k+1}^{2k}\binom{3k}{2k-v}4^{v}a_{-v,v}. \end{aligned} \tag{24}$$

Thus, with respect to $\{1, j_{MF}, j_{MF}^2, \ldots\}$ and $\{a_{-1,1}, a_{-2,2}, \ldots\}$,

$$\delta^0 = \begin{bmatrix} 0 & * & * & * & \\ \vdots & u_1 & * & * & \\ & 0 & * & * & \\ & \vdots & u_2 & * & \\ & & 0 & * & \\ & & \vdots & u_3 & \\ & & & 0 & \ddots \end{bmatrix} \tag{25}$$

where $u_k = -2^{12k} \in \mathbb{Z}_{(3)}^{\times}$ for $k \geqslant 1$. $\square$

*5.2. The connecting map $\delta^1 : \operatorname{coker} g \to \operatorname{coker} h$*

From the results of Section 4,

$$\delta^1 = -\phi_f : \left( \bigoplus_{x \in \mathcal{B}_{MF}^{\neq 0}} \mathbb{Z}/(3^{\nu_3(\deg(x))+1}) \right) \oplus \mathbb{Z}_{(3)}[j_{MF}] \to \bigoplus_{i<j\in\mathbb{Z}} \left( \mathbb{Z}/(3^{\mu_3(3i+3j)}) \oplus \mathbb{Z}/(3^{\mu_3(6i+6j+3)}) \right)$$

where $\mathbb{Z}/(3^{\mu_3(3i+3j)})$ is generated by $a_{i,j}$ and $\mathbb{Z}/(3^{\mu_3(6i+6j+3)})$ is generated by $b_{i,j}$. In particular, $\delta^1$ is completely determined by where it sends nonnegative powers of $j_{MF}$ and the elements of $\mathcal{B}_{MF}^{\neq 0}$.

**Lemma 9.** *For all $i < j \in \mathbb{Z}$, $s^i t^j = a_{i,j}/2$ and $s^i t^j q_2 = b_{i,j}/2$ in $\operatorname{coker} h$.*

**Proof.** From Definition 3,

$$s^i t^j = \frac{a_{i,j} + \bar{a}_{i,j}}{2}, \quad s^i t^j q_2 = \frac{b_{i,j} + \bar{b}_{i,j}}{2}$$

and $\bar{a}_{i,j} = \bar{b}_{i,j} = 0$ in $\operatorname{coker} h$. $\square$

**Proposition 8.** $\ker\left(\delta^1|_{\mathbb{Z}_{(3)}[j_{MF}]}\right) = \mathbb{Z}_{(3)}\{1_{MF}\}$ *and* $\operatorname{coker}\left(\delta^1|_{\mathbb{Z}_{(3)}[j_{MF}]}\right) = \mathbb{Z}_{(3)}\{a_{-i,i} : i \geqslant 1,\ odd\}$.

**Proof.** By Lemma 9 and the formula for $\phi_f$ from Section 3.3, a computation similar to (24) in Proposition 7 gives

$$\begin{aligned} \delta^1(j_{MF}^k) &= 2^{8k-1} \sum_{v=1}^{k} \left( \binom{3k}{2k+v} 4^{-v} - \binom{3k}{2k-v} 4^{v} \right) a_{-v,v} - 2^{8k-1} \sum_{v=k+1}^{2k} \binom{3k}{2k-v} 4^{v} a_{-v,v} \\ &= \frac{1}{2} \delta^0(j_{MF}^k) \end{aligned} \tag{26}$$

so $\ker\left(\delta^1|_{\mathbb{Z}_{(3)}[j_{MF}]}\right) = \ker \delta^0$ and $\operatorname{coker}\left(\delta^1|_{\mathbb{Z}_{(3)}[j_{MF}]}\right) = \operatorname{coker} \delta^0$. $\square$

We now study $\delta^1|_{W_{\epsilon,m}}$ for $m \in \mathbb{Z}$ and $\epsilon = 0$ or $1$ by finding matrix representations, as we did with $\delta^0$ in (25). The set $\mathcal{B}_{MF}^{\epsilon,m}$ is an ordered basis for the source. Degree counting shows that an ordered basis for the target is given by $\{A_0^m, A_1^m, A_2^m, \ldots\}$ if $\epsilon = 0$, and $\{B_0^m, B_1^m, B_2^m, \ldots\}$ if $\epsilon = 1$.

By Lemma 9 and the formula for $\phi_f$,

$$\begin{aligned}\delta^1(c_4^n\Delta^\ell) &= -\phi_f(c_4^n\Delta^\ell) = -(\mu+16q_4)^n(q_4^2\mu)^\ell \\ &= -\frac{s^{2\ell}t^\ell(2s+8t)^n}{8^\ell} = -8^{n-\ell}\sum_{r=0}^{n}\binom{n}{r}4^{-r}s^{2\ell+r}t^{n+\ell-r} \\ &= -2^{3n-3\ell-1}\sum_{r=0}^{n}\binom{n}{r}4^{-r}a_{2\ell+r,n+\ell-r}\end{aligned} \tag{27}$$

and

$$\begin{aligned}\delta^1(c_4^nc_6\Delta^\ell) &= -\phi_f(c_4^nc_6\Delta^\ell) = -(\mu+16q_4)^n(q_4^2\mu)^\ell(4q_2(8q_4-\mu)) \\ &= -\frac{q_2s^{2\ell}t^\ell(2s+8t)^n(s-8t)}{2^{3\ell-2}} \\ &= -4^{2n-2\ell+1}\sum_{r=0}^{n}\binom{n}{r}4^{-r}q_2\left(s^{2\ell+r+1}t^{n+\ell-r}-8s^{2\ell+r}t^{n+\ell-r+1}\right) \\ &= -2^{4n-4\ell+1}\sum_{r=0}^{n}\binom{n}{r}4^{-r}\left(b_{2\ell+r+1,n+\ell-r}-8b_{2\ell+r,n+\ell-r+1}\right).\end{aligned} \tag{28}$$

**Remark 6.** To obtain matrix representations of $\delta^1|_{W_{\epsilon,m}}$, the right-hand sums of (27) (resp. (28)) must be put solely in terms of the generators $a_{i,j}$ (resp. $b_{i,j}$) with $i<j$, because of the identities

$$a_{i,j} = -a_{j,i}, \quad b_{i,j} = -b_{j,i}. \tag{29}$$

Note that this was done implicitly in the proofs of Propositions 7 and 8.

**Proposition 9.**

(a) $\ker\left(\delta^1|_{W_{0,m}}\right) = \begin{cases} 0, & m<0, \\ \mathbb{Z}/(3^{\nu_3(m)+1}), & m>0. \end{cases}$

(b) *For all* $0\neq m\in\mathbb{Z}$, $\operatorname{coker}\left(\delta^1|_{W_{0,m}}\right) = \bigoplus_{n\in\mathbb{N}}\mathbb{Z}/(3^{\nu_3(m)+1})$.

(c) *For* $m\leqslant 0$, $\ker\left(\delta^1|_{W_{1,m}}\right)=0$ *and* $\operatorname{coker}\left(\delta^1|_{W_{1,m}}\right) = \bigoplus_{n\in\mathbb{N}}\mathbb{Z}/(3^{\nu_3(2m+1)+1})$.

(d) *For* $m>0$ *and* $m\not\equiv 13 \bmod 27$, $\ker\left(\delta^1|_{W_{1,m}}\right) = \mathbb{Z}/(3^{\nu_3(2m+1)+1})$ *and*

$$\operatorname{coker}\left(\delta^1|_{W_{1,m}}\right) = \bigoplus_{n\in\mathbb{N}}\mathbb{Z}/(3^{\nu_3(2m+1)+1}).$$

(e) *For* $m>0$ *and* $m\equiv 13 \bmod 27$, $\operatorname{coker}\left(\delta^1|_{W_{1,m}}\right)$ *has, as a direct summand,* $\bigoplus_{n\in\mathbb{N}}\mathbb{Z}/(3^{\nu_3(2m+1)+1})$.

We now establish the following convenient notational conventions.

**Definition 9.**

(a) If $c_4^n\Delta^\ell\in\mathcal{B}_{MF}^{0,m}$ (resp. $c_4^nc_6\Delta^\ell\in\mathcal{B}_{MF}^{1,m}$), the $a_{i,m-i}$ term of $\delta^1(c_4^n\Delta^\ell)$ (resp. the $b_{i,m-i}$ term of $\delta^1(c_4^nc_6\Delta^\ell)$) with the *least* first subscript $i$ will be denoted the *leading term*, and the remaining terms will be denoted *higher order terms*.

(b) The symbol $\doteq$ will denote equality up to multiplication by a unit in $\mathbb{Z}_{(3)}$.
(c) If $M$ is a matrix with columns $M_1, \ldots, M_v$ and $N$ is a matrix with (possibly infinitely many) columns $N_1, N_2, \ldots$, then $M \boxplus N$ will denote the matrix with columns $M_1, \ldots, M_v, N_1, N_2, \ldots$.

To prove Proposition 9, we will need the following four lemmas.

**Lemma 10.** *If $\ell \neq 0$,*

$$\delta^1(c_4^n \Delta^\ell) \doteq \begin{cases} a_{\ell,n+2\ell} + \textit{higher order terms}, & \ell > 0 \\ a_{2\ell,n+\ell} + \textit{higher order terms}, & \ell < 0. \end{cases}$$

**Proof.** We use (27) and (29) as in the proof of the previous lemma. For $-\ell \geqslant n$,

$$\delta^1(c_4^n \Delta^\ell) = -2^{3n-\ell-1} \sum_{r=2\ell}^{2\ell+n} \binom{n}{r-2\ell} 4^{-r} a_{r,n+3\ell-r}$$

with leading term $-2^{3n-5\ell-1} a_{2\ell,n+\ell}$. For $\ell \geqslant n$,

$$\delta^1(c_4^n \Delta^\ell) = 2^{n-5\ell-1} \sum_{r=\ell}^{\ell+n} \binom{n}{n+\ell-r} 4^r a_{r,n+3\ell-r}$$

with leading term $2^{n-3\ell-1} a_{\ell,n+2\ell}$. For $|\ell| < n$,

$$\delta^1(c_4^n \Delta^\ell) = 2^{n-5\ell-1} \sum_{r=\ell}^{\left\lfloor \frac{n+3\ell-1}{2} \right\rfloor} \binom{n}{n+\ell-r} 4^r a_{r,n+3\ell-r} - 2^{3n+\ell-1} \sum_{r=2\ell}^{\left\lfloor \frac{n+3\ell-1}{2} \right\rfloor} \binom{n}{r-2\ell} 4^{-r} a_{r,n+3\ell-r}$$

with leading term $2^{n-3\ell-1} a_{\ell,n+2\ell}$ if $\ell > 0$ and $-2^{3n-3\ell-1} a_{2\ell,n+\ell}$ if $\ell < 0$. $\square$

**Lemma 11.** *When $\ell = 0$,*

$$\begin{aligned} \delta^1(c_4^n c_6) \doteq{}& 4 \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{r-1} 4^{-r} b_{r,n+1-r} - 4^{-n} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{r} 4^r b_{r,n+1-r} \\ &- 8 \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{r} 4^{-r} b_{r,n+1-r} + 2^{-2n+1} \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{r-1} 4^r b_{r,n+1-r} \end{aligned}$$

*and the leading term $(-4^{-n} - 8) b_{0,n+1}$ is zero in* $\operatorname{coker} h$.

**Proof.** The formula is obtained from (28) by setting $\ell = 0$. Each $b_{r,n+1-r}$ generates a copy of $\mathbb{Z}/(3^{\nu_3(2n+3)+1})$ in $\operatorname{coker} h$, and since Lemma 8(b) implies

$$\nu_3(-4^{-n} - 8) = \nu_3(2^{-2n-3} + 1) = \nu_3(-2n-3) + 1,$$

the leading term does indeed vanish. $\square$

**Lemma 12.**

(a) $\delta^1(c_4^n) = 0$ *for $n \geqslant 1$.*
(b) $\delta^1(c_4^n c_6) = 0$ *except when $n+1 \equiv 13 \bmod 27$. In these exceptional cases, the $b_{1,n}$ term is always nonzero.*

**Proof.** We begin with (a). By (27) and (29),

$$-2^{1-3n}\delta^1(c_4^n) = \sum_{r=0}^{n}\binom{n}{r}4^{-r}a_{r,n-r} = \sum_{r=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{r}\left(4^{-r}-4^{r-n}\right)a_{r,n-r}$$

and since $a_{r,n-r}$ generates $\mathbb{Z}/(3^{\mu_3(n)+1})$ in $\operatorname{coker} h$, it suffices to show that the coefficient of $a_{r,n-r}$ is divisible by $3n$ for all $n$. But Lemma 8(a) implies

$$4^{-r}-4^{r-n} = \frac{1-4^{2r-n}}{4^r}$$

is a multiple of $3n-6r$, so it is enough to show that $\binom{n}{r}(n-2r)$ is divisible by $n$. This is clear if $r=0$. For $0<r\leqslant\lfloor(n-1)/2\rfloor$,

$$\begin{aligned}\binom{n}{r}(n-2r) &= \frac{n}{r}\binom{n-1}{r-1}(n-2r)\\ &= n\left(\frac{(n-1)(n-2)\cdots(n-r+1)}{(r-1)!}\right)\left(\frac{n-r}{r}-1\right)\\ &= n\left(\binom{n-1}{r}-\binom{n-1}{r-1}\right).\end{aligned}$$

Next, we prove (b). Let $m=n+1$. Collecting terms in the formula from Lemma 11 (see Remark 6) yields

$$\delta^1(c_4^{m-1}c_6) \doteq \sum_{r=1}^{\lfloor\frac{m-1}{2}\rfloor}\frac{4^{1-r}}{r}\binom{m-1}{r-1}(3r(1+2^{4r-2m})-2m(1+2^{4r-2m-1}))b_{r,m-r}.$$

Suppose first that $m\not\equiv 13 \bmod 27$. Let $f(r,m)$ be the coefficient on $b_{r,m-r}$ in the above formula. Each $b_{r,m-r}$ generates a copy of $\mathbb{Z}/(3^{\nu_3(2m+1)+1}) = \mathbb{Z}_{(3)}/(6m+3)$ in $\operatorname{coker} h$, and the condition on $m$ implies $\nu_3(6m+3)\leqslant 3$. Thus, to prove the first claim it will suffice to show that each $f(r,m)$ is a multiple of $3^3$ modulo $6m+3$. This is true for $r=1$ since

$$\begin{aligned}f(1,m) &= 3(1+2^{4-2m})-m(2+2^{4-2m})\\ &= 3(1+32\cdot 2^{-2m-1})-m(2+32\cdot 2^{-2m-1})\\ &\equiv 3(1-32)-m(2-32)\\ &\equiv -4\cdot 3^3 \bmod (6m+3).\end{aligned}$$

For $r>1$,

$$\begin{aligned}4^{r-1}f(r,m) &= \binom{m-1}{r-1}(3(1+2^{4r-2m}))-\binom{m}{r}(2+2^{4r-2m})\\ &\equiv \binom{m-1}{r-1}(3(1-2^{4r+1}))-\binom{m}{r}(2-2^{4r+1})\\ &\equiv \binom{m-1}{r-1}\left(3(1-2^{4r+1})+\frac{1-2^{4r}}{r}\right) \bmod (6m+3).\end{aligned}$$

Let $A(r)=3(1-2^{4r+1})+(1-2^{4r})/r$. If 3 does not divide $r$, then $A(r)$ is divisible by $3^3$, and so $f(r,m)b_{r,m-r} = 0$ in those cases. If 3 divides $r$, then $A(r)$ is only divisible by $3^2$, and this is sufficient to annihilate $b_{r,m-r}$

except when $m \equiv 4$ or $22 \bmod 27$. However in those cases the binomial coefficient $\binom{m-1}{r-1}$ contributes the additional power of 3 that is needed.

Finally, if $m \equiv 13 \bmod 27$, then $\nu_3(6m+3) > 3$, so the calculation of $f(1,m)$ above shows that $f(1,m)b_{1,m-1} \neq 0$ in $\mathbb{Z}_{(3)}/(6m+3)$. Hence $\delta^1(c_4^{m-1}c_6)$ is nonzero in $\operatorname{coker} h$. $\square$

**Lemma 13.** *If $\ell \neq 0$,*

$$\delta^1(c_4^n c_6 \Delta^\ell) \doteq \begin{cases} b_{\ell,n+2\ell+1} + \textit{higher order terms}, & \ell > 0 \\ b_{2\ell,n+\ell+1} + \textit{higher order terms}, & \ell < 0. \end{cases}$$

**Proof.** Assume $\ell \neq 0$ throughout. For $-\ell \geqslant n+1$,

$$\begin{aligned} -2^{-4n+4\ell-1}\delta^1(c_4^n c_6 \Delta^\ell) = {} & 2^{4\ell+2} \sum_{r=2\ell+1}^{n+2\ell+1} \binom{n}{r-2\ell-1} 4^{-r} b_{r,n+3\ell+1-r} \\ & - 2^{4\ell+3} \sum_{r=2\ell}^{n+2\ell} \binom{n}{r-2\ell} 4^{-r} b_{r,n+3\ell+1-r} \end{aligned}$$

with leading term $-8b_{2\ell,n+\ell+1}$. For $\ell \geqslant n+1$,

$$\begin{aligned} -2^{-4n+4\ell-1}\delta^1(c_4^n c_6 \Delta^\ell) = {} & -2^{-2n-2\ell} \sum_{r=\ell}^{n+\ell} \binom{n}{r-\ell} 4^{r} b_{r,n+3\ell+1-r} \\ & + 2^{-2n-2\ell+1} \sum_{r=\ell+1}^{n+\ell+1} \binom{n}{r-\ell-1} 4^{r} b_{r,n+3\ell+1-r} \end{aligned}$$

with leading term $-2^{-2n}b_{\ell,n+2\ell+1}$.

For $|\ell| < n+1$,

$$\begin{aligned} -2^{-4n+4\ell-1}\delta^1(c_4^n c_6 \Delta^\ell) = {} & 2^{4\ell+2} \sum_{r=2\ell+1}^{\lceil \frac{n+3\ell-1}{2} \rceil} \binom{n}{r-2\ell-1} 4^{-r} b_{r,n+3\ell+1-r} \\ & - 2^{-2n-2\ell} \sum_{r=\ell}^{\lceil \frac{n+3\ell-1}{2} \rceil} \binom{n}{r-\ell} 4^{r} b_{r,n+3\ell+1-r} \\ & - 2^{4\ell+3} \sum_{r=2\ell}^{\lceil \frac{n+3\ell-1}{2} \rceil} \binom{n}{r-2\ell} 4^{-r} b_{r,n+3\ell+1-r} \\ & + 2^{-2n-2\ell+1} \sum_{r=\ell+1}^{\lceil \frac{n+3\ell-1}{2} \rceil} \binom{n}{r-\ell-1} 4^{r} b_{r,n+3\ell+1-r} \end{aligned}$$

with leading term $-2^{-2n}b_{\ell,n+2\ell+1}$ if $\ell > 0$ and $-8b_{2\ell,n+\ell+1}$ if $\ell < 0$. $\square$

**Proof of Proposition 9.** Suppose first that $\epsilon = 0$. If $m < 0$, then $\ell_0^m < 0$, and

$$\delta^1(C_v^m) \doteq A^m_{\lfloor \frac{m-1}{2} \rfloor - 2\ell_0^m + 2v} + \text{higher order terms}$$

for $v \geqslant 0$, by Lemma 10. Thus,

$$\delta^1\big|_{W_{0,m}} = \begin{bmatrix} \vdots & \vdots & & \\ u_0 & * & & \\ 0 & * & \vdots & \\ \vdots & u_1 & * & \\ & 0 & * & \\ & \vdots & u_2 & \\ & & 0 & \ddots \end{bmatrix} \tag{30}$$

where $u_0, u_1, u_2, \ldots \in \mathbb{Z}_{(3)}^{\times}$ and $u_0$ is in the row corresponding to $A^m_{\lfloor \frac{m-1}{2} \rfloor - 2\ell_0^m}$. By (30), $\delta^1\big|_{W_{0,m}}$ has trivial kernel, and has cokernel generated by

$$\left\{ A_0^m, \ldots, A^m_{\lfloor \frac{m-1}{2} \rfloor - 2\ell_0^m - 1} \right\} \cup \left\{ A^m_{\lfloor \frac{m-1}{2} \rfloor - 2\ell_0^m + i} : i \geqslant 1, \text{ odd} \right\}.$$

Each $A_v^m$ generates $\mathbb{Z}/(3^{\nu_3(m)+1})$ in $\operatorname{coker} h$. This proves parts (a) and (b) for $m < 0$.

If $m > 0$, then $\ell_0^m \geqslant 0$. By Lemmas 10 and 12(a),

$$\delta^1(C_v^m) \doteq \begin{cases} A^m_{\lfloor \frac{m-1}{2} \rfloor - \ell_0^m + v} + \text{higher order terms}, & 0 \leqslant v < \ell_0^m \\ 0, & v = \ell_0^m \\ A^m_{\lfloor \frac{m-1}{2} \rfloor - 2\ell_0^m + 2v} + \text{higher order terms}, & v > \ell_0^m. \end{cases}$$

Thus,

$$\delta^1\big|_{W_{0,m}} = \begin{bmatrix} \vdots & \vdots & & & \\ u_0 & * & & & \\ 0 & u_1 & & \vdots & \boldsymbol{\vdots} \\ \vdots & 0 & \ddots & * & \mathbf{0} \\ & \vdots & & u_y & \mathbf{0} \\ & & & 0 & \mathbf{0} \\ & & & \vdots & \boldsymbol{\vdots} \end{bmatrix} \boxplus \begin{bmatrix} \vdots & \vdots & & \\ u_{y+1} & * & & \\ 0 & * & \vdots & \\ \vdots & u_{y+2} & * & \\ & 0 & * & \\ & \vdots & u_{y+3} & \\ & & 0 & \ddots \end{bmatrix} \tag{31}$$

where the $u_i$ are units in $\mathbb{Z}_{(3)}$. Here, $u_0$ is in the row corresponding to $A^m_{\lfloor \frac{m-1}{2} \rfloor - \ell_0^m}$, $u_y$ is in the row corresponding to $A^m_{\lfloor \frac{m-1}{2} \rfloor - 1}$, $u_{y+1}$ is in the row corresponding to $A^m_{\lfloor \frac{m-1}{2} \rfloor + 2}$, and the zero column in bold corresponds to $C^m_{\ell_0^m}$. By (31), $\delta^1\big|_{W_{0,m}}$ has kernel generated by $C^m_{\ell_0^m}$, and has cokernel generated by

$$\left\{ A_0^m, A_1^m, \ldots, A^m_{\lfloor \frac{m-1}{2} \rfloor - \ell_0^m - 1} \right\} \cup \left\{ A^m_{\lfloor \frac{m-1}{2} \rfloor} \right\} \cup \left\{ A^m_{\lfloor \frac{m-1}{2} \rfloor + i} : i \geqslant 1, \text{ odd} \right\}.$$

Since $C_v^m$ generates $\mathbb{Z}/(3^{\nu_3(m)+1})$ in $\operatorname{coker} g$ and $A_v^m$ generates $\mathbb{Z}/(3^{\nu_3(m)+1})$ in $\operatorname{coker} h$, this proves parts (a) and (b) for $m > 0$.

Suppose next that $\epsilon = 1$. If $m \leqslant 0$, then $\ell_1^m < 0$. By Lemma 13,

$$\delta^1(D_v^m) \doteq B^m_{\lfloor \frac{m-1}{2} \rfloor - 2\ell_1^m + 2v} + \text{higher order terms}$$

for $v \geqslant 0$. Thus, $\delta^1\big|_{W_{1,m}}$ is represented by a matrix of the form identical to (30), where in this case the unit $u_0$ appears in the row corresponding to $B^m_{\lfloor \frac{m-1}{2} \rfloor - 2\ell_1^m}$. Thus $\delta^1\big|_{W_{1,m}}$ has trivial kernel, and its cokernel is generated by

$$\left\{B_0^m, \ldots, B^m_{\lfloor \frac{m-1}{2} \rfloor - 2\ell_1^m - 1}\right\} \cup \left\{B^m_{\lfloor \frac{m-1}{2} \rfloor - 2\ell_1^m + i} : i \geqslant 1,\ \text{odd}\right\}.$$

Since each $B_v^m$ generates $\mathbb{Z}/(3^{\mu_3(2m+1)+1})$ in $\operatorname{coker} h$, this proves part (c).

If $m > 0$, then $\ell_1^m \geqslant 0$. As long as $m \not\equiv 13 \bmod 27$,

$$\delta^1(D_v^m) \doteq \begin{cases} B^m_{\lfloor \frac{m-1}{2} \rfloor - \ell_1^m + v} + \text{higher order terms}, & 0 \leqslant v < \ell_1^m \\ 0, & v = \ell_1^m \\ B^m_{\lfloor \frac{m-1}{2} \rfloor - 2\ell_0^m + 2v} + \text{higher order terms}, & v > \ell_1^m \end{cases}$$

by Lemmas 12(b) and 13. The matrix representation in this case is of the form identical to (31) above, where in this case $u_0$ is in the row corresponding to $B^m_{\lfloor \frac{m-1}{2} \rfloor - \ell_1^m}$, $u_y$ is in the row corresponding to $B^m_{\lfloor \frac{m-1}{2} \rfloor - 1}$, and $u_{y+1}$ is in the row corresponding to $B^m_{\lfloor \frac{m-1}{2} \rfloor + 2}$. Thus $\delta^1|_{W_{1,m}}$ has kernel generated by $D^m_{\ell_1^m}$, and has cokernel generated by

$$\left\{B_0^m, B_1^m, \ldots, B^m_{\lfloor \frac{m-1}{2} \rfloor - \ell_1^m - 1}\right\} \cup \left\{B^m_{\lfloor \frac{m-1}{2} \rfloor}\right\} \cup \left\{B^m_{\lfloor \frac{m-1}{2} \rfloor + i} : i \geqslant 1,\ \text{odd}\right\}.$$

Since each $D_v^m$ generates $\mathbb{Z}/(3^{\mu_3(2m+1)+1})$ in $\operatorname{coker} g$ and each $B_v^m$ generates $\mathbb{Z}/(3^{\mu_3(2m+1)+1})$ in $\operatorname{coker} h$, this proves part (d).

If $m > 0$ and $m \equiv 13 \bmod 27$, Lemma 12(b) implies $\delta^1|_{W_{1,m}}$ has matrix representation identical in form to (31) except for the column in bold; it is not a column of zeros in this case. Rather, it has at least one nonzero entry in and above the row containing $u_y$ by Lemma 12(a). This makes the kernel and cokernel less straightforward to compute (see Remark 2). What we can conclude, however, is that the cokernel of $\delta^1|_{W_{1,m}}$ contains copies of $\mathbb{Z}/(3^{\mu_3(2m+1)+1})$ generated by

$$\left\{B^m_{\lfloor \frac{m-1}{2} \rfloor}\right\} \cup \left\{B^m_{\lfloor \frac{m-1}{2} \rfloor + i} : i \geqslant 1,\ \text{odd}\right\}$$

which proves part (e). □

**Definition 10.** Let

$$U^1 := \ker \left( \delta^1 \Big|_{\bigoplus_{0<m\equiv 13 \bmod 27} W_{1,m}} \right)$$

and define $U^2$ via the direct sum decomposition

$$\operatorname{coker} \left( \delta^1 \Big|_{\bigoplus_{0<m\equiv 13 \bmod 27} W_{1,m}} \right) = \bigoplus_{n\in\mathbb{N}} \left( \bigoplus_{0<m\equiv 13 \bmod 27} \mathbb{Z}/(3^{\nu_3(6m+3)}) \right) \oplus U^2.$$

**Proof of Theorem 2.** By Propositions 2 and 7, $H^0C^* = \mathbb{Z}_{(3)}$. Proposition 7 also implies $\operatorname{coker} \delta^0 = \bigoplus_{\mathbb{N}} \mathbb{Z}_{(3)}$ concentrated in degree zero. By Proposition 8, the degree zero part of $\ker \delta^1$ is a copy of $\mathbb{Z}_{(3)}$ generated by $1_{MF}$. Thus, the short exact sequence (4) in Proposition 2 implies that $H^1C^*$ is a countable direct sum of copies of $\mathbb{Z}_{(3)}$ in degree zero, and is isomorphic to $\ker \delta^1$ in positive degrees. The result for $H^1C^*$ then follows from Proposition 9. The result for $H^2C^*$ follows from Propositions 2 and 9 and Definition 10. □

## 6. Differential on the $E_2$-page

In this section we compute $\widetilde{d} : \mathrm{Ext}^1 \to \operatorname{coker} \Psi$ and prove Theorem 3. The computation of $\widetilde{d}$ amounts to a diagram chase with the maps

$$\begin{tikzcd} \Gamma \arrow[r, "\Phi"] & \Gamma & \\ & B \oplus B \arrow[u, "-d\oplus 0"] \arrow[r, "\Psi"] & B \end{tikzcd}$$

of $C^{*,*}$. Since

$$\mathrm{Ext}^1 = \mathbb{Z}_{(3)}\{\Delta^k \alpha : k \in \mathbb{Z}\} = \bigoplus_{k\in\mathbb{Z}} \mathbb{Z}/(3)$$

where $\alpha$ is represented by $r \in \Gamma$ [8], it suffices to compute $\widetilde{d}(\Delta^k\alpha)$ for all $k \in \mathbb{Z}$.

Consider first the case $k = 0$. The element $r \in \Gamma$ is mapped to $3r$ under $\Phi$, which in turn must be hit by some element $y \in B \oplus B$ under $-d \oplus 0$; in fact $y = (-q_2, q_2)$ works. Since

$$\Psi(y) = (\psi_d + 1)(-q_2) - \phi_f(q_2) = q_2 - q_2 = 0,$$

$\widetilde{d}(\alpha) = 0$.

Next, suppose $k > 0$. The element $\Delta^k\alpha \in \mathrm{Ext}^1$ is represented by $\Delta^k r \in \Gamma$. Under $\Phi$, $\Delta^k r$ maps to $(2^{12k+2} - 1)\Delta^k r$, which in turn is the image of an element in $B \oplus B$ under $-d \oplus 0$, namely

$$(-d \oplus 0)\left(\frac{1 - 2^{12k+2}}{3}\Delta^k q_2, 0\right) = (2^{12k+2} - 1)\Delta^k r.$$

By Lemma 8(b), $(1 - 2^{12k+2})/3 \in \mathbb{Z}_{(3)}^\times$. Thus, applying $\Psi$ yields

$$\begin{aligned} \Psi\left(\frac{1 - 2^{12k+2}}{3}\Delta^k q_2, 0\right) &= -\phi_f\left(\frac{1 - 2^{12k+2}}{3}\Delta^k q_2\right) \\ &= \frac{2^{12k+2} - 1}{3} q_4^{2k}\mu^k q_2 \\ &\doteq b_{k,2k} - \bar{b}_{k,2k} \end{aligned}$$

and $b_{k,2k} - \bar{b}_{k,2k}$ represents the class $b_{k,2k} = B^{3k}_{\lfloor \frac{3k-1}{2}\rfloor - k} \in \operatorname{coker} \Psi$. Thus

$$\widetilde{d}(\Delta^k\alpha) \doteq B^{3k}_{\lfloor \frac{3k-1}{2}\rfloor - k}.$$

This class is nontrivial by Proposition 9.

If $k < 0$, a similar argument shows

$$\widetilde{d}(\Delta^k\alpha) \doteq B^{3k}_{\lfloor \frac{3k-1}{2}\rfloor - 2k},$$

also nontrivial by Proposition 9.

The preceding arguments show that

$$\ker \widetilde{d} = \mathbb{Z}/(3)$$

generated by $\alpha$, and that $\operatorname{im}\widetilde{d}$ is generated by

$$\left\{B^{3k}_{\left\lfloor\frac{3k-1}{2}\right\rfloor-k} : k>0\right\} \cup \left\{B^{3k}_{\left\lfloor\frac{3k-1}{2}\right\rfloor-2k} : k<0\right\}.$$

Since each $B^m_v$ generates $\mathbb{Z}/(3^{\nu_3(6m+3)})$ in $\operatorname{coker}\Psi$, this proves Theorem 3.

## 7. Detection of Greek letter elements

In this section we use Theorem 1 to give evidence for Conjecture 1. We set up the discussion by first considering the algebraic alpha family element

$$\alpha_1 := \alpha^a_{1/1} \in \operatorname{Ext}^{1,4}_{BP_*BP}(BP_*, BP_*)$$

of order 3 [14].

**Proposition 10.** *$\alpha_1$ is detected by $\alpha = r \in \ker\widetilde{d}$.*

**Proof.** Note first that $\alpha$ is in the correct bidegree. Since the double complex bidegree of any element in $\ker\widetilde{d}$ is $(0,1)$ and $\deg(r) = 2$ in $\Gamma$, the bidegree of $\alpha$ in the Adams–Novikov $E_2$-term for $Q(2)$ is $(s,t) = (0+1, 2)$. But recall that the $E_2$-term for $Q(2)$ is indexed so that $E_2^{s,t} \Rightarrow \pi_{2t-s}Q(2)$ (see Eq. (2)), so the corresponding bidegree in $\operatorname{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ is $(s, 2t) = (1,4)$.

In fact we know $\alpha$ must detect $\alpha_1$ because $\alpha_1$ is detected by $\mathit{TMF}$ by this same element $\alpha = r \in \Gamma$ [8], and the diagram of $E_2$-terms induced by

$$\begin{tikzcd} Q(2) \arrow[r] & \mathit{TMF} \\ & L_2S_{(3)} \arrow[ul] \arrow[u] \end{tikzcd} \tag{32}$$

commutes. $\square$

Although we do not conjecture that the Adams–Novikov $E_2$-term for $Q(2)$ detects the entire algebraic divided alpha family

$$\{\alpha^a_{i/j} \in \operatorname{Ext}^{1,4i}_{BP_*BP}(BP_*, BP_*) : 0 < i \in \mathbb{Z}, j = \nu_3(i)+1\}$$

(where $\alpha^a_{i/j}$ has additive order $3^j$), it does contain elements of the appropriate bidegrees and additive orders that *could* collectively detect it. These elements (other than $\alpha$ discussed above) live not in $\ker\widetilde{d}$, but rather in $H^1C^*$. For example, $\alpha_2 := \alpha^a_{2/1}$ could be detected by the class $C^1_0 = c_4 \in H^1C^*$, an element of order 3 in bidegree $(s,t) = (1,4)$. Another example is $\alpha^a_{3/2}$, which could be detected by $D^1_0 = c_6 \in H^1C^*$, a class in bidegree $(s,t) = (1,6)$ of order 9. In general, the candidate element for detecting $\alpha^a_{i/j}$ is given by

$$\begin{cases} C^{i/2}_{\ell_0^{i/2}}, & \text{if } i \text{ even}, \\ D^{(i-1)/2}_{\ell_1^{(i-1)/2}}, & \text{if } i \text{ odd}. \end{cases}$$

We now turn to the algebraic divided beta family. Since this family lives in $\operatorname{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$, candidate elements for detecting them in $E_2^{s,t}Q(2)$ must be contained in either $\operatorname{Ext}^2$ (with double complex bidegree $(0,2)$), $\operatorname{Ext}^1$ (with double complex bidegree $(1,1)$), or $\operatorname{coker}\widetilde{d}$.

Consider first the element

$$\beta_1 := \beta^a_{1/1,1} \in \mathrm{Ext}^{2,12}_{BP_*BP}(BP_*, BP_*)$$

of order 3. Like $\alpha_1$, we know this element has a nontrivial target in the Adams–Novikov $E_2$-term for $Q(2)$.

**Proposition 11.** *$\beta_1$ is detected by $\beta := r^2 \otimes r - r \otimes r^2 \in \mathrm{Ext}^2$.*

**Proof.** As $\beta_1$ is on the 2-line, $\beta$ must live in $\mathrm{Ext}^2$ with double complex bidegree $(0, 2)$. Using the fact that $\deg(r^2 \otimes r - r \otimes r^2) = 6$ in $\Gamma \otimes \Gamma$, an argument analogous to the proof of Proposition 10 yields the result [8]. □

We now offer three further examples of beta elements and candidate elements for detecting them in the Adams–Novikov $E_2$-term for $Q(2)$.

**Example 1.** The algebraic beta element

$$\beta_{6/3} := \beta^a_{6/3,1} \in \mathrm{Ext}^{2,84}_{BP_*BP}(BP_*, BP_*),$$

itself an element of order 3, is known to be a permanent cycle and represents a nontrivial homotopy element $\beta^h_{6/3,1} \in \pi_{82}S_{(3)}$. Any nontrivial target in the Adams–Novikov $E_2$-term for $Q(2)$ must be in bidegree $(s, t) = (2, 42)$. While there are no such elements in $\mathrm{Ext}^1$, the element

$$\Delta^3\beta \in \mathrm{Ext}^2$$

does have both the required bidegree and additive order to detect $\beta_{6/3}$. By the proof of Proposition 9(d), the other possible targets for this element in the $E_2$-term for $Q(2)$ are the classes

$$3B^{10}_0; 3B^{10}_4; 3B^{10}_5, 3B^{10}_7, 3B^{10}_9, \ldots \in \operatorname{coker} \widetilde{d}.$$

Each $B^{10}_v$ is multiplied by 3 because it generates $\mathbb{Z}/(3^{\nu_3(6\cdot 10+3)}) = \mathbb{Z}/(9)$.

**Example 2.** The Kervaire invariant problem at the prime 3 asks whether the classes

$$\theta_j := \beta_{3^{j-1}/3^{j-1},1} \in \mathrm{Ext}^{2,4\cdot 3^j}_{BP_*BP}(BP_*, BP_*), \quad j \geqslant 1,$$

the so-called *Kervaire classes*, are permanent cycles. For $j = 3$, the corresponding Kervaire class is $\theta_3 = \beta_{9/9,1}$ in bidegree $(2, 108)$, and living in this same bidegree is $\beta_7 := \beta_{7/1,1}$ (itself not a Kervaire class). As in Example 1, there are no elements in $\mathrm{Ext}^1$ to detect either of these classes. But there is a candidate element in $\mathrm{Ext}^2$; in this case it is $\Delta^4\beta$. Potential detecting elements in $\operatorname{coker} \widetilde{d}$ for $\theta_3$ and $\beta_7$ were identified in Remark 2 from Section 1: namely, any element in

$$\begin{aligned}\big(\mathbb{Z}/(81)\{B^{13}_0, B^{13}_1\}/(B^{13}_0 - 3B^{13}_1 = 0)\big) &\oplus \mathbb{Z}/(81)\{B^{13}_6\} \\ &\oplus \mathbb{Z}/(81)\{B^{13}_7, B^{13}_9, B^{13}_{11}, \ldots\}\end{aligned}$$

suitably multiplied by a power of 3 so as to obtain an element of order 3. Along with $\Delta^4\beta$, these are the only elements in $E_2^{2,54}Q(2)$.

**Example 3.** In our final example we consider the algebraic beta element

$$\beta^a_{9/3,2} \in \mathrm{Ext}^{2,132}_{BP_*BP}(BP_*, BP_*),$$

an element of order 9. It is the beta element $\beta^a_{i/j,k}$ of lowest topological degree (which in this case is $t - s = 130$) with $k > 1$. Since there are no elements in $\mathrm{Ext}^2$ or $\mathrm{Ext}^1$ capable of detecting $\beta^a_{9/3,2}$ for degree reasons, we look in $\operatorname{coker} \widetilde{d}$. By Proposition 9(d) there are indeed candidate detecting elements in $\operatorname{coker} \widetilde{d}$ given by

$$B^{16}_0; B^{16}_1; B^{16}_7; B^{16}_8, B^{16}_{10}, B^{16}_{12}, \ldots,$$

themselves classes of order 9 in $E^{2,66}_2 Q(2)$.

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