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On the K-theory of local fields

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Contents

Introduction

In this paper we establish a connection between the Quillen K-theory of certain local fields and the de Rham-Witt complex of their rings of integers with logarithmic poles at the maximal ideal. The fields K we consider are complete discrete valuation fields of characteristic zero with perfect residue field k of characteristic $p > 2$. When K contains the p^v-th roots of unity, the relationship between the K-theory with $\mathbb{Z}/p^v$-coefficients and the de Rham-Witt complex can be described by a sequence

$$\cdots \to K_*(K,\mathbb{Z}/p^v) \to W\,\omega^*_{(A,M)}\otimes S_{\mathbb{Z}/p^v}(\mu_{p^v}) \xrightarrow{1-F} W\,\omega^*_{(A,M)}\otimes S_{\mathbb{Z}/p^v}(\mu_{p^v}) \xrightarrow{\partial} \cdots$$

which is exact in degrees ≥ 1. Here $A = \mathcal{O}_K$ is the valuation ring and $W\,\omega^*_{(A,M)}$ is the de Rham-Witt complex of A with log poles at the maximal ideal. The factor $S_{\mathbb{Z}/p^v}(\mu_{p^v})$ is the symmetric algebra of μ_{p^v} considered as a $\mathbb{Z}/p^v$-module located in degree two. Using this sequence, we evaluate the K-theory with $\mathbb{Z}/p^v$-coefficients of K. The result, which is valid also if K does not con-

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tain the p^v-th roots of unity, verifies the Lichtenbaum-Quillen conjecture for K, [26], [38]:

Theorem A. *There are natural isomorphisms for $s \geq 1$,*

$$\begin{aligned} K_{2s}(K, \mathbb{Z}/p^v) &= H^0(K, \mu_{p^v}^{\otimes s}) \oplus H^2(K, \mu_{p^v}^{\otimes (s+1)}), \\ K_{2s-1}(K, \mathbb{Z}/p^v) &= H^1(K, \mu_{p^v}^{\otimes s}). \end{aligned}$$

The Galois cohomology on the right can be effectively calculated when k is finite, or equivalently, when K is a finite extension of $\mathbb{Q}_p$, [42]. For m prime to p,

$$K_i(K, \mathbb{Z}/m) = K_i(k, \mathbb{Z}/m) \oplus K_{i-1}(k, \mathbb{Z}/m)$$

by Gabber-Suslin, [44], and for k finite, the K-groups on the right are known by Quillen, [36].

For any linear category with cofibrations and weak equivalences in the sense of [48], one has the cyclotomic trace

$$\mathrm{tr}\colon K(\mathcal{C}) \to \mathrm{TC}(\mathcal{C}; p)$$

from K-theory to topological cyclic homology, [7]. It coincides in the case of the exact category of finitely generated projective modules over a ring with the original definition in [3]. The exact sequence above and Theorem A are based upon calculations of $\mathrm{TC}_*(\mathcal{C}; p, \mathbb{Z}/p^v)$ for certain categories associated with the field K. Let $A = \mathcal{O}_K$ be the valuation ring in K, and let $\mathcal{P}_A$ be the category of finitely generated projective A-modules. We consider three categories with cofibrations and weak equivalences: the category $C_z^b(\mathcal{P}_A)$ of bounded complexes in $\mathcal{P}_A$ with homology isomorphisms as weak equivalences, the subcategory with cofibrations and weak equivalences $C_z^b(\mathcal{P}_A)^q$ of complexes whose homology is torsion, and the category $C_q^b(\mathcal{P}_A)$ of bounded complexes in $\mathcal{P}_A$ with rational homology isomorphisms as weak equivalences. One then has a cofibration sequence of K-theory spectra

$$K(C_z^b(\mathcal{P}_A)^q) \xrightarrow{i^!} K(C_z^b(\mathcal{P}_A)) \xrightarrow{j} K(C_q^b(\mathcal{P}_A)) \xrightarrow{\partial} \Sigma K(C_z^b(\mathcal{P}_A)^q),$$

and by Waldhausen's approximation theorem, the terms in this sequence may be identified with the K-theory of the exact categories $\mathcal{P}_k$, $\mathcal{P}_A$ and $\mathcal{P}_K$. The associated long-exact sequence of homotopy groups is the localization sequence of [37],

$$\dots \to K_i(k) \xrightarrow{i^!} K_i(A) \xrightarrow{j_*} K_i(K) \xrightarrow{\partial} K_{i-1}(k) \to \dots$$

The map ∂ is a split surjection by [15]. We show in Section 1.5 below that one has a similar cofibration sequence of topological cyclic homology spectra

$$\mathrm{TC}(C_z^b(\mathcal{P}_A)^q; p) \xrightarrow{i^!} \mathrm{TC}(C_z^b(\mathcal{P}_A); p) \xrightarrow{j} \mathrm{TC}(C_q^b(\mathcal{P}_A); p) \xrightarrow{\partial} \Sigma\,\mathrm{TC}(C_z^b(\mathcal{P}_A)^q; p),$$

and again Waldhausen's approximation theorem allows us to identify the first two terms on the left with the topological cyclic homology of the exact categories $\mathcal{P}_k$ and $\mathcal{P}_A$. But the third term is different from the topological cyclic homology of $\mathcal{P}_K$. We write

$$\mathrm{TC}(A|K;p) = \mathrm{TC}(C_q^b(\mathcal{P}_A);p),$$

and we then have a map of cofibration sequences

$$\begin{array}{ccccccc}
K(k) & \xrightarrow{i!} & K(A) & \xrightarrow{j_*} & K(K) & \xrightarrow{\partial} & \Sigma K(k) \\
\big\downarrow \mathrm{tr} & & \big\downarrow \mathrm{tr} & & \big\downarrow \mathrm{tr} & & \downarrow \mathrm{tr} \\
\mathrm{TC}(k;p) & \xrightarrow{i!} & \mathrm{TC}(A;p) & \xrightarrow{j_*} & \mathrm{TC}(A|K;p) & \xrightarrow{\partial} & \Sigma\,\mathrm{TC}(k;p).
\end{array}$$

By [19, Th. D], the first two vertical maps from the left induce isomorphisms of homotopy groups with $\mathbb{Z}/p^v$-coefficients in degrees ≥ 0. It follows that the remaining two vertical maps induce isomorphisms of homotopy groups with $\mathbb{Z}/p^v$-coefficients in degrees ≥ 1,

$$\mathrm{tr}\colon K_i(K,\mathbb{Z}/p^v) \xrightarrow{\sim} \mathrm{TC}_i(A|K;p,\mathbb{Z}/p^v), \quad i \geq 1.$$

It is the right-hand side we evaluate.

The spectrum $\mathrm{TC}(\mathcal{C};p)$ is defined as the homotopy fixed points of an operator called Frobenius on another spectrum $\mathrm{TR}(\mathcal{C};p)$; so there is a natural cofibration sequence

$$\mathrm{TC}(\mathcal{C};p) \to \mathrm{TR}(\mathcal{C};p) \xrightarrow{1-F} \mathrm{TR}(\mathcal{C};p) \to \Sigma\,\mathrm{TC}(\mathcal{C};p).$$

The spectrum $\mathrm{TR}(\mathcal{C};p)$, in turn, is the homotopy limit of a pro-spectrum $\mathrm{TR}^{\boldsymbol{\cdot}}(\mathcal{C};p)$, its homotopy groups given by the Milnor sequence

$$0 \to \varprojlim_R{}^1 \mathrm{TR}^{\boldsymbol{\cdot}}_{s+1}(\mathcal{C};p) \to \mathrm{TR}_s(\mathcal{C};p) \to \varprojlim_R \mathrm{TR}^{\boldsymbol{\cdot}}_s(\mathcal{C};p) \to 0,$$

and there are maps of pro-spectra

$$\begin{aligned}
F\colon \mathrm{TR}^n(\mathcal{C};p) &\to \mathrm{TR}^{n-1}(\mathcal{C};p),\\
V\colon \mathrm{TR}^{n-1}(\mathcal{C};p) &\to \mathrm{TR}^n(\mathcal{C};p).
\end{aligned}$$

The spectrum $\mathrm{TR}^1(\mathcal{C};p)$ is the topological Hochschild homology $T(\mathcal{C})$. It has an action by the circle group $\mathbb{T}$ and the higher levels in the pro-system by definition are the fixed sets of the cyclic subgroups of $\mathbb{T}$ of p-power order,

$$\mathrm{TR}^n(\mathcal{C};p) = T(\mathcal{C})^{C_{p^{n-1}}}.$$

The map F is the obvious inclusion and V is the accompanying transfer. The structure map R in the pro-system is harder to define and uses the so-called cyclotomic structure of $T(\mathcal{C})$; see Section 1.1 below.

The homotopy groups $\mathrm{TR}^{\bullet}_*(A|K;p)$ of this pro-spectrum with its various operators have a rich algebraic structure which we now describe. The description involves the notion of a log differential graded ring from [24]. A log ring (R,M) is a ring R with a pre-log structure, defined as a map of monoids

$$\alpha\colon M \to (R,\,\cdot\,),$$

and a log differential graded ring (E^*,M) is a differential graded ring E^*, a pre-log structure $\alpha\colon M\to E^0$ and a map of monoids $d\log\colon M \to (E^1,+)$ which satisfies $d\circ d\log = 0$ and $d\alpha(a) = \alpha(a)d\log a$ for all $a\in M$. There is a universal log differential graded ring with underlying log ring (R,M): the de Rham complex with log poles $\omega^*_{(R,M)}$.

The groups $\mathrm{TR}^1_*(A|K;p)$ form a log differential graded ring whose underlying log ring is $A=\mathcal{O}_K$ with the *canonical* pre-log structure given by the inclusion

$$\alpha\colon M = A\cap K^{\times} \to A.$$

We show that the canonical map

$$\omega^*_{(A,M)} \to \mathrm{TR}^1_*(A|K;p)$$

is an isomorphism in degrees ≤ 2 and that the left-hand side is uniquely divisible in degrees ≥ 2. We do not know a *natural* description of the higher homotopy groups, but we do for the homotopy groups with $\mathbb{Z}/p$-coefficients. The Bockstein

$$\mathrm{TR}^1_2(A|K;p,\mathbb{Z}/p) \xrightarrow{\sim} {}_p\mathrm{TR}^1_1(A|K;p)$$

is an isomorphism, and we let κ be the element on the left which corresponds to the class $d\log(-p)$ on the right. The abstract structure of the groups $\mathrm{TR}^1_*(A;p)$ was determined in [27]. We use this calculation in Section 2 below to show:

Theorem B. *There is a natural isomorphism of log differential graded rings*

$$\omega^*_{(A,M)} \otimes_{\mathbb{Z}} S_{\mathbb{F}_p}\{\kappa\} \xrightarrow{\sim} \mathrm{TR}^1_*(A|K;p,\mathbb{Z}/p),$$

where $d\kappa = \kappa d\log(-p)$.

The higher levels $\mathrm{TR}^n_*(A|K;p)$ are also log differential graded rings. The underlying log ring is the ring of Witt vectors $W_n(A)$ with the pre-log structure

$$M \xrightarrow{\alpha} A \to W_n(A),$$

where the right-hand map is the multiplicative section $\underline{a}_n = (a,0,\dots,0)$. The maps R, F and V extend the restriction, Frobenius and Verschiebung of Witt vectors. Moreover,

$$F\colon \mathrm{TR}^n_*(A|K;p) \to \mathrm{TR}^{n-1}_*(A|K;p)$$

is a map of pro-log graded rings, which satisfies

$$\begin{aligned} F d\log_n a &= d\log_{n-1} a, && \text{for all } a \in M = A \cap K^{\times}, \\ F d\underline{a}_n &= \underline{a}_{n-1}^{p-1} d\underline{a}_{n-1}, && \text{for all } a \in A, \end{aligned}$$

and V is a map of pro-graded modules over the pro-graded ring $\mathrm{TR}^{\boldsymbol{\cdot}}_*(A|K;p)$,

$$V\colon F^* \,\mathrm{TR}^{n-1}_*(A|K;p) \to \mathrm{TR}^n_*(A|K;p).$$

Finally,

$$FdV = d, \qquad FV = p.$$

The algebraic structure described here makes sense for any log ring (R,M), and we show that there exists a universal example: the de Rham-Witt pro-complex with log poles $W_{\boldsymbol{\cdot}}\,\omega^*_{(R,M)}$. For log rings of characteristic $p > 0$, a different construction has been given by Hyodo-Kato, [23].

We show in Section 3 below that the canonical map

$$W_{\boldsymbol{\cdot}}\,\omega^*_{(A,M)} \to \mathrm{TR}^{\boldsymbol{\cdot}}_*(A|K;p)$$

is an isomorphism in degrees ≤ 2 and that the left-hand side is uniquely divisible in degrees ≥ 2. Suppose that $\mu_{p^v} \subset K$. We then have a map

$$S_{\mathbb{Z}/p^v}(\mu_{p^v}) \to \mathrm{TR}^{\boldsymbol{\cdot}}_*(A|K;p,\mathbb{Z}/p^v)$$

which takes $\zeta \in \mu_{p^v}$ to the associated Bott element defined as the unique element with image $d\log_{\boldsymbol{\cdot}}\zeta$ under the Bockstein

$$\mathrm{TR}^{\boldsymbol{\cdot}}_2(A|K;p,\mathbb{Z}/p^v) \xrightarrow{\sim} {}_{p^v}\mathrm{TR}^{\boldsymbol{\cdot}}_1(A|K;p).$$

The following is the main theorem of this paper.

Theorem C. *Suppose that $\mu_{p^v} \subset K$. Then the canonical map*

$$W_{\boldsymbol{\cdot}}\,\omega^*_{(A,M)} \otimes_{\mathbb{Z}} S_{\mathbb{Z}/p^v}(\mu_{p^v}) \xrightarrow{\sim} \mathrm{TR}^{\boldsymbol{\cdot}}_*(A|K;p,\mathbb{Z}/p^v)$$

is a pro-isomorphism.

We explain the structure of the groups in the theorem for $v = 1$; the structure for $v > 1$ is unknown. Let $E^*_{\boldsymbol{\cdot}}$ stand for either side of the statement above. The group E^i_n has a natural descending filtration of length n given by

$$\mathrm{Fil}^s E^i_n = V^s E^i_{n-s} + dV^s E^{i-1}_{n-s} \subset E^i_n, \quad 0 \leq s < n.$$

There is a natural k-vector space structure on E^i_n, and for all $0 \leq s < n$ and all $i \geq 0$,

$$\dim_k \mathrm{gr}^s E^i_n = e_K,$$

the absolute ramification index of K. In particular, the domain and range of the map in the statement are abstractly isomorphic.

The main theorem implies that for $s \geq 0$,

$$\begin{aligned} \mathrm{TC}_{2s}(A|K;p,\mathbb{Z}/p^v) &= H^0(K,\mu_{p^v}^{\otimes s}) \oplus H^2(K,\mu_{p^v}^{\otimes(s+1)}), \\ \mathrm{TC}_{2s+1}(A|K;p,\mathbb{Z}/p^v) &= H^1(K,\mu_{p^v}^{\otimes(s+1)}), \end{aligned}$$

and thus, in turn, Theorem A.

It is also easy to see that the canonical map

$$K_*(K,\mathbb{Z}/p^v) \to K_*^{\text{ét}}(K,\mathbb{Z}/p^v)$$

is an isomorphism in degrees ≥ 1. Here the right-hand side is the Dwyer-Friedlander étale K-theory of K with $\mathbb{Z}/p^v$-coefficients. This may be defined as the homotopy groups with $\mathbb{Z}/p^v$-coefficients of the spectrum

$$K^{\text{ét}}(K) = \underset{L/K}{\underrightarrow{\mathrm{holim}}}\, \mathbb{H}^{\bullet}(G_{L/K}, K(L)),$$

where the homotopy colimit runs over the finite Galois extensions L/K contained in an algebraic closure $\bar{K}/K$, and where the spectrum $\mathbb{H}^{\bullet}(G_{L/K},K(L))$ is the group cohomology spectrum or homotopy fixed point spectrum of $G_{L/K}$ acting on $K(L)$. There is a spectral sequence

$$E^2_{s,t} = H^{-s}(K,\mu_{p^v}^{\otimes(t/2)}) \Rightarrow K^{\text{ét}}_{s+t}(K,\mathbb{Z}/p^v),$$

where the identification of the E^2-term is a consequence of the celebrated theorem of Suslin, [43], that

$$K_t(\bar{K},\mathbb{Z}/p^v) = \mu_{p^v}^{\otimes(t/2)}.$$

For K a finite extension of $\mathbb{Q}_p$, the p-adic homotopy type of the $K^{\text{ét}}(K)$ is known by [45] and [8]. Let $F\Psi^r$ be the homotopy fiber

$$F\Psi^r \to \mathbb{Z}\times BU \xrightarrow{\Psi^r - 1} BU.$$

It follows from this calculation and from the isomorphism above that:

Theorem D. *If K is a finite extension of $\mathbb{Q}_p$, then after p-completion*

$$\mathbb{Z}\times BGL(K)^+ \simeq F\Psi^{g^{p^{a-1}d}} \times BF\Psi^{g^{p^{a-1}d}} \times U^{|K:\mathbb{Q}_p|},$$

where $d = (p-1)/|K(\mu_p):K|$, $a = \max\{v \mid \mu_{p^v} \subset K(\mu_p)\}$, and where $g \in \mathbb{Z}_p^\times$ is a topological generator.

The proof of theorem C is given in Section 6 below. It is based on the calculation in Section 5 of the Tate spectra for the cyclic groups C_{p^n} acting on the topological Hochschild spectrum $T(A|K)$: Given a finite group G and

G-spectrum X, one has the Tate spectrum $\hat{\mathbb{H}}(G, X)$ of [11], [12]. Its homotopy groups are approximated by a spectral sequence

$$E^2_{s,t} = \hat{H}^{-s}(G, \pi_t X) \Rightarrow \pi_{s+t}\hat{\mathbb{H}}(G, X),$$

which converges conditionally in the sense of [1]. In Section 4 below we give a slightly different construction of this spectral sequence which is better suited for studying multiplicative properties. The cyclotomic structure of $T(A|K)$ gives rise to a map

$$\hat{\Gamma}_K\colon \mathrm{TR}^n(A|K; p) \to \hat{\mathbb{H}}(C_{p^n}, T(A|K)),$$

and we show in Section 5 that this map induces an isomorphism of homotopy groups with $\mathbb{Z}/p^v$-coefficients in degrees ≥ 0. We then evaluate the Tate spectral sequence for the right-hand side.

Throughout this paper, A will be a complete discrete valuation ring with field of fractions K of characteristic zero and perfect residue field k of characteristic $p > 2$. All rings are assumed commutative and unital without further notice. Occasionally, we will write $\bar{\pi}_*(-)$ for homotopy groups with $\mathbb{Z}/p$-coefficients.

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1. Topological Hochschild homology and localization

1.1. This section contains the construction of $\mathrm{TR}^n(A|K; p)$. The main result is the localization sequence of Theorem 1.5.6, which relates this spectrum to $\mathrm{TR}^n(A; p)$ and $\mathrm{TR}^n(k; p)$. We make extensive use of the machinery developed by Waldhausen in [48] and some familiarity with this material is assumed.

The stable homotopy category is a triangulated category and a closed symmetric monoidal category, and the two structures are compatible; see e.g. [22, Appendix]. By a spectrum we will mean an object in this category, and by a ring spectrum we will mean a monoid in this category. The purpose of this section is to produce the following. Let $\mathcal{C}$ be a *linear* category with cofibrations and weak equivalences in the sense of [48, §1.2]. We define a pro-spectrum $\mathrm{TR}^{\boldsymbol{\cdot}}(\mathcal{C};p)$ together with maps of pro-spectra

$$\begin{aligned} F\colon \mathrm{TR}^n(\mathcal{C};p) &\to \mathrm{TR}^{n-1}(\mathcal{C};p), \\ V\colon \mathrm{TR}^{n-1}(\mathcal{C};p) &\to \mathrm{TR}^n(\mathcal{C};p), \\ \mu\colon S^1_+ \wedge \mathrm{TR}^n(\mathcal{C};p) &\to \mathrm{TR}^n(\mathcal{C};p). \end{aligned}$$

The spectrum $\mathrm{TR}^1(\mathcal{C};p)$ is the topological Hochschild spectrum of $\mathcal{C}$. The cyclotomic trace is a map of pro-spectra

$$\mathrm{tr}\colon K(\mathcal{C}) \to \mathrm{TR}^{\boldsymbol{\cdot}}(\mathcal{C};p),$$

where the algebraic K-theory spectrum on the left is regarded as a constant pro-spectrum.

Suppose that the category $\mathcal{C}$ has a strict symmetric monoidal structure such that the tensor product is bi-exact. Then there is a natural product on $\mathrm{TR}^{\boldsymbol{\cdot}}(\mathcal{C};p)$ which makes it a commutative pro-ring spectrum. Similarly, $K(\mathcal{C})$ is naturally a commutative ring spectrum and the maps F and tr are maps of ring-spectra.

The pro-spectrum $\mathrm{TR}^{\boldsymbol{\cdot}}(\mathcal{C};p)$ has a preferred homotopy limit $\mathrm{TR}(\mathcal{C};p)$, and there are preferred lifts to the homotopy limit of the maps F, V and μ. Its homotopy groups are related to those of the pro-system by the Milnor sequence

$$0 \to \varprojlim_R{}^1 \mathrm{TR}^{\boldsymbol{\cdot}}_{s+1}(\mathcal{C};p) \to \mathrm{TR}_s(\mathcal{C};p) \to \varprojlim_R \mathrm{TR}^{\boldsymbol{\cdot}}_s(\mathcal{C};p) \to 0.$$

There is a natural cofibration sequence

$$\mathrm{TC}(\mathcal{C};p) \to \mathrm{TR}(\mathcal{C};p) \xrightarrow{R-F} \mathrm{TR}(\mathcal{C};p) \to \Sigma\,\mathrm{TC}(\mathcal{C};p),$$

where $\mathrm{TC}(\mathcal{C};p)$ is the topological cyclic homology spectrum of $\mathcal{C}$. The cyclotomic trace has a preferred lift to a map

$$\mathrm{tr}\colon K(\mathcal{C}) \to \mathrm{TC}(\mathcal{C};p),$$

and in the case where $\mathcal{C}$ has a bi-exact strict symmetric monoidal product, the natural product on $\mathrm{TR}^{\boldsymbol{\cdot}}(\mathcal{C};p)$ have preferred lifts to natural products on $\mathrm{TR}(\mathcal{C};p)$ and $\mathrm{TC}(\mathcal{C};p)$, and the maps F and tr are ring maps.

Let G be a compact Lie group. One then has the G-stable category which is a triangulated category with a compatible closed symmetric monoidal structure. The objects of this category are called G-spectra, and the monoids for the smash product are called ring G-spectra. Let $H \subset G$ be a closed subgroup and let $W_H G = N_G H/H$ be the Weyl group. There is a forgetful functor which to a G-spectrum X assigns the underlying H-spectrum $U_H X$. We also write $|X|$ for $U_{\{1\}}X$. It comes with a natural map of spectra

$$\mu_X\colon G_+ \wedge |X| \to |X|.$$

One also has the H-fixed point functor which to a G-spectrum X assigns the $W_H G$-spectrum X^H. If $H \subset K \subset G$ are two closed subgroups, there is a map of spectra

$$\iota_H^K \colon |X^K| \to |X^H|,$$

and if $|K:H|$ is finite, a map in the opposite direction

$$\tau_H^K \colon |X^H| \to |X^K|.$$

If X is a ring G-spectrum then $U_H X$ is a ring H-spectrum and X^H is a ring $W_G H$-spectrum.

Let $\mathbb{T}$ be the circle group, and let $C_r \subset \mathbb{T}$ be the cyclic subgroup of order r. We then have the canonical isomorphism of groups

$$\rho_r \colon \mathbb{T} \xrightarrow{\sim} \mathbb{T}/C_r = W_{\mathbb{T}} C_r$$

given by the r-th root. It induces an isomorphism of the $\mathbb{T}/C_r$-stable category and of the $\mathbb{T}$-stable category by assigning to a $\mathbb{T}/C_r$-spectrum Y the $\mathbb{T}$-spectrum $\rho_r^* Y$. Moreover, there is a transitive system of natural isomorphisms of spectra

$$\varphi_r \colon |\rho_r^* Y| \xrightarrow{\sim} |Y|,$$

and the following diagram commutes

$$\begin{array}{ccc}
\mathbb{T}_+ \wedge |\rho_r^* Y| & \xrightarrow{\mu} & |\rho_r^* Y| \\
\big\downarrow{\scriptstyle \rho \wedge \varphi_r} & & \big\downarrow{\scriptstyle \varphi_r} \\
\mathbb{T}/C_{r+} \wedge |Y| & \xrightarrow{\mu} & |Y|.
\end{array}$$

We will define a $\mathbb{T}$-spectrum $T(\mathcal{C})$ such that

$$\mathrm{TR}^n(\mathcal{C};p) = |\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}|$$

with the maps F and V given by the composites

$$F = \varphi_{p^{n-2}}^{-1} \iota_{C_{p^{n-2}}}^{C_{p^{n-1}}} \varphi_{p^{n-1}} \colon |\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}| \to |\rho_{p^{n-2}}^* T(\mathcal{C})^{C_{p^{n-2}}}|,$$
$$V = \varphi_{p^{n-1}}^{-1} \tau_{C_{p^{n-2}}}^{C_{p^{n-1}}} \varphi_{p^{n-2}} \colon |\rho_{p^{n-2}}^* T(\mathcal{C})^{C_{p^{n-2}}}| \to |\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}|,$$

and the map μ given by

$$\mu = \mu_{\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}} \colon \mathbb{T}_+ \wedge |\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}| \to |\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}|.$$

There is a natural map

$$K(\mathcal{C}) \to T(\mathcal{C})^{\mathbb{T}},$$

and the cyclotomic trace is then the composite of this map and $\varphi_{p^{n-1}}^{-1} \iota_{C_{p^{n-1}}}^{\mathbb{T}}$. The definition of the structure maps in the pro-system $\mathrm{TR}^{\boldsymbol{\cdot}}(\mathcal{C};p)$ is more complicated and uses the *cyclotomic* structure on $T(\mathcal{C})$ which we now explain.

There is a cofibration sequence of $\mathbb{T}$-CW-complexes

$$E_+ \to S^0 \to \tilde{E} \to \Sigma E_+,$$

where E is a free contractible $\mathbb{T}$-space, and where the left-hand map collapses E to the nonbase point of S^0. It induces, upon smashing with a $\mathbb{T}$-spectrum T, a cofibration sequence of $\mathbb{T}$-spectra

$$E_+ \wedge T \to T \to \tilde{E} \wedge T \to \Sigma E_+ \wedge T,$$

and hence the following basic cofibration sequence of spectra

$$|\rho_{p^n}^*(E_+ \wedge T)^{C_{p^n}}| \to |\rho_{p^n}^* T^{C_{p^n}}| \to |\rho_{p^n}^*(\tilde{E} \wedge T)^{C_{p^n}}| \to \Sigma|\rho_{p^n}^*(E_+ \wedge T)^{C_{p^n}}|,$$

natural in T. The left-hand term is written $\mathbb{H}_{\bullet}(C_{p^n}, T)$ and called the group homology spectrum or Borel spectrum. Its homotopy groups are approximated by a strongly convergent first quadrant homology type spectral sequence

$$E^2_{s,t} = H_s(C_{p^n}, \pi_t T) \Rightarrow \pi_{s+t}\mathbb{H}_{\bullet}(C_{p^n}, T).$$

The cyclotomic structure on $T(\mathcal{C})$ means that there is a natural map of $\mathbb{T}$-spectra

$$r\colon \rho_p^*(\tilde{E} \wedge T(\mathcal{C}))^{C_p} \to T(\mathcal{C})$$

such that $U_{C_{p^s}} r$ is an isomorphism of C_{p^s}-spectra, for all $s \geq 0$. More generally, since

$$\rho_{p^n}^*(\tilde{E} \wedge T(\mathcal{C}))^{C_{p^n}} = \rho_{p^{n-1}}^*(\rho_p^*(\tilde{E} \wedge T(\mathcal{C}))^{C_p})^{C_{p^{n-1}}},$$

the map r induces a map of $\mathbb{T}$-spectra

$$r_{n+1}\colon \rho_{p^n}^*(\tilde{E} \wedge T(\mathcal{C}))^{C_{p^n}} \to \rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}$$

such that $U_{C_{p^s}} r_{n+1}$ is an isomorphism of C_{p^s}-spectra, for all $s \geq 0$. The map

$$R\colon \mathrm{TR}^n(\mathcal{C}; p) \to \mathrm{TR}^{n-1}(\mathcal{C}; p)$$

is then defined as the composite

$$|\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}| \to |\rho_{p^{n-1}}^*(\tilde{E} \wedge T(\mathcal{C}))^{C_{p^{n-1}}}| \xrightarrow[\sim]{r_n} |\rho_{p^{n-2}}^* T(\mathcal{C})^{C_{p^{n-2}}}|,$$

where the left-hand map is the middle map in the cofibration sequence above. We thus have a natural cofibration sequence of spectra

$$\mathbb{H}_{\bullet}(C_{p^{n-1}}, T(\mathcal{C})) \xrightarrow{N} \mathrm{TR}^n(\mathcal{C}; p) \xrightarrow{R} \mathrm{TR}^{n-1}(\mathcal{C}; p) \xrightarrow{\partial} \Sigma\mathbb{H}_{\bullet}(C_{p^{n-1}}, T(\mathcal{C})).$$

When $\mathcal{C}$ has a bi-exact strict symmetric monoidal product, the map r is a map of ring $\mathbb{T}$-spectra, and hence R is a map of ring spectra. The cofibration sequence above is a sequence of $\mathrm{TR}^n(\mathcal{C}; p)$-module spectra and maps.

For any $\mathbb{T}$-spectrum X, one has the function spectrum $F(E_+, X)$, and the projection $E_+ \to S^0$ defines a natural map

$$\gamma\colon X \to F(E_+, X).$$

This map induces an isomorphism of group homology spectra. One defines the group cohomology spectrum and the Tate spectrum,

$$\begin{aligned}\mathbb{H}^{\bullet}(C_{p^n}, X) &= |\rho_{p^n}^* F(E_+, X)^{C_{p^n}}|,\\ \hat{\mathbb{H}}\,(C_{p^n}, X) &= |\rho_{p^n}^*(\tilde{E} \wedge F(E_+, X))^{C_{p^n}}|.\end{aligned}$$

Their homotopy groups are approximated by homology type spectral sequences

$$\begin{aligned}&E^2_{s,t} = H^{-s}(C_{p^n}, \pi_t X) \Rightarrow \pi_{s+t}\mathbb{H}^{\bullet}(C_{p^n}, X),\\ &\hat{E}^2_{s,t} = \hat{H}^{-s}(C_{p^n}, \pi_t X) \Rightarrow \pi_{s+t}\hat{\mathbb{H}}\,(C_{p^n}, X),\end{aligned}$$

both of which converge conditionally in the sense of [1, Def. 5.10]. The latter sequence, called the Tate spectral sequence, will be considered in great detail in Section 4 below. Taking $T = F(E_+, X)$ in the basic cofibration sequence above, we get the Tate cofibration sequence of spectra

$$\mathbb{H}_{\bullet}(C_{p^n}, X) \xrightarrow{N^h} \mathbb{H}^{\bullet}(C_{p^n}, X) \xrightarrow{R^h} \hat{\mathbb{H}}(C_{p^n}, X) \xrightarrow{\partial^h} \Sigma\mathbb{H}_{\bullet}(C_{p^n}, X).$$

Finally, if $X = T(\mathcal{C})$, the map

$$\gamma\colon T(\mathcal{C}) \to F(E_+, T(\mathcal{C}))$$

induces a map of cofibration sequences

$$\begin{array}{ccccccc}\mathbb{H}_{\bullet}(C_{p^n}, T(\mathcal{C})) & \xrightarrow{N} & \mathrm{TR}^{n+1}(\mathcal{C};p) & \xrightarrow{R} & \mathrm{TR}^{n}(\mathcal{C};p) & \xrightarrow{\partial} & \Sigma\mathbb{H}_{\bullet}(C_{p^n}, T(\mathcal{C}))\\ \| & & \downarrow{\scriptstyle\Gamma} & & \downarrow{\scriptstyle\hat{\Gamma}} & & \|\\ \mathbb{H}_{\bullet}(C_{p^n}, T(\mathcal{C})) & \xrightarrow{N^h} & \mathbb{H}^{\bullet}(C_{p^n}, T(\mathcal{C})) & \xrightarrow{R^h} & \hat{\mathbb{H}}(C_{p^n}, T(\mathcal{C})) & \xrightarrow{\partial^h} & \Sigma\mathbb{H}_{\bullet}(C_{p^n}, T(\mathcal{C})),\end{array}$$

in which all maps commute with the action maps μ. Moreover, if $\mathcal{C}$ is strict symmetric monoidal with bi-exact tensor product, the four spectra in the middle square are all ring spectra and R, R^h, Γ and $\hat{\Gamma}$ are maps of ring spectra. In this case, the diagram is a diagram of $\mathrm{TR}^{n+1}(\mathcal{C};p)$-module spectra, [19, pp. 71–72].

1.2. In order to construct the $\mathbb{T}$-spectrum $T(\mathcal{C})$ we need a model category for the $\mathbb{T}$-stable category. The model category we use is the category of symmetric spectra of orthogonal $\mathbb{T}$-spectra, see [31] and [21, Th. 5.10]. We first recall the topological Hochschild space $\mathrm{THH}(\mathcal{C})$. See [7], [10] and [19] for more details.

A linear category $\mathcal{C}$ is naturally enriched over the symmetric monoidal category of symmetric spectra. The symmetric spectrum of maps from c to d, $\underline{\mathrm{Hom}}_{\mathcal{C}}(c, d)$, is the Eilenberg-MacLane spectrum for the abelian group $\mathrm{Hom}_{\mathcal{C}}(c, d)$ concentrated in degree zero. In more detail, if X is a pointed simplicial set, then

$$\mathbb{Z}(X) = \mathbb{Z}\{X\}/\mathbb{Z}\{x_0\}$$

is a simplicial abelian group whose homology is the reduced singular homology of X. Here $\mathbb{Z}\{X\}$ denotes the degree-wise free abelian group generated by X. Let S^i be the i-fold smash product of the standard simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$. Then the spaces $\{|\mathbb{Z}(S^i)|\}_{i\geq 0}$ is a symmetric ring spectrum with the homotopy type of an Eilenberg-MacLane spectrum for $\mathbb{Z}$ concentrated in degree zero, and we define

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(c,d)_i = |\operatorname{Hom}_{\mathcal{C}}(c,d)\otimes\mathbb{Z}(S^i)|.$$

Let I be the category with objects the finite sets

$$\underline{i} = \{1,2,\dots,i\}, \qquad i\geq 1,$$

and the empty set $\underline{0}$, and morphisms all injective maps. It is a strict monoidal category under concatenation of sets and maps. There is a functor $V_k(\mathcal{C};X)$ from I^{k+1} to the category of pointed spaces which on objects is given by

$$V_k(\mathcal{C};X)(\underline{i_0},\dots,\underline{i_k}) = \bigvee_{c_0,\dots,c_k\in\operatorname{ob}\mathcal{C}} \underline{\mathrm{Hom}}_{\mathcal{C}}(c_0,c_k)_{i_0}\wedge\dots\wedge\underline{\mathrm{Hom}}_{\mathcal{C}}(c_k,c_{k-1})_{i_k}\wedge X.$$

It induces a functor $G_k(\mathcal{C};X)$ from I^{k+1} to pointed spaces with

$$G_k(C;X)(\underline{i_0},\dots,\underline{i_k}) = F(S^{i_0}\wedge\dots\wedge S^{i_k}, V_k(\mathcal{C};X)(\underline{i_0},\dots,\underline{i_k})),$$

and we define

$$\mathrm{THH}_k(\mathcal{C}) = \operatorname*{holim}_{\overrightarrow{I^{k+1}}} G_k(\mathcal{C};S^0).$$

This is naturally the space of k-simplices in a cyclic space and, by definition,

$$\mathrm{THH}(\mathcal{C}) = |[k]\mapsto\mathrm{THH}_k(\mathcal{C})|.$$

It is a $\mathbb{T}$-space by Connes' theory of cyclic spaces, [28, 7.1.9].

More generally, let (n) be the finite ordered set $\{1,2,\dots,n\}$ and let (0) be the empty set. The product category $I^{(n)}$ is a strict monoidal category under component-wise concatenation of sets and maps. Concatenation of sets and maps according to the ordering of (n) also defines a functor

$$\sqcup_n\colon I^{(n)}\to I,$$

but this does *not* preserve the monoidal structure. By convention $I^{(0)}$ is the category with one object and one morphism, and $\sqcup_0$ includes this category as the full subcategory on the object $\underline{0}$. We let $G_k^{(n)}(\mathcal{C};X)$ be the functor from $(I^{(n)})^{k+1}$ to the category of pointed spaces given by

$$G_k^{(n)}(\mathcal{C};X) = G_k(\mathcal{C};X)\circ(\sqcup_n)^{k+1},$$

and define

$$\mathrm{THH}_k^{(n)}(\mathcal{C};X) = \operatorname*{holim}_{\overrightarrow{(I^{(n)})^{k+1}}} G_k^{(n)}(\mathcal{C};X).$$

In particular, $\mathrm{THH}_k^{(0)}(\mathcal{C};X) = N_k^{\mathrm{cy}}(\mathcal{C}) \wedge X$, where

$$N_k^{\mathrm{cy}}(\mathcal{C}) = \bigvee_{c_0,\dots,c_k \in \mathrm{ob}\,\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(c_0,c_k) \wedge \dots \wedge \mathrm{Hom}_{\mathcal{C}}(c_k,c_{k-1})$$

is the cyclic bar construction of $\mathcal{C}$. Again this is the space of k-simplices in a cyclic space, and hence we have the $\Sigma_n \times \mathbb{T}$-space

$$\mathrm{THH}^{(n)}(\mathcal{C};X) = |[k] \mapsto \mathrm{THH}_k^{(n)}(\mathcal{C};X)|.$$

There is a natural product

$$\mathrm{THH}^{(m)}(\mathcal{C};X) \wedge \mathrm{THH}^{(n)}(\mathcal{D};Y) \to \mathrm{THH}^{(m+n)}(\mathcal{C} \otimes \mathcal{D}; X \wedge Y),$$

which is $\Sigma_m \times \Sigma_n \times \mathbb{T}$-equivariant if $\mathbb{T}$ acts diagonally on the left. Here the category $\mathcal{C} \otimes \mathcal{D}$ has as objects all pairs (c,d) with $c \in \mathrm{ob}\,\mathcal{C}$ and $d \in \mathcal{D}$, and

$$\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}((c,d),(c',d')) = \mathrm{Hom}_{\mathcal{C}}(c,c') \otimes \mathrm{Hom}_{\mathcal{D}}(d,d').$$

For any category $\mathcal{C}$, the nerve category $\mathbf{N}_{\bullet}\mathcal{C}$ is the simplicial category with k-simplicies the functor category

$$\mathbf{N}_k\mathcal{C} = \mathcal{C}^{[k]},$$

where the partially ordered set $[k] = \{0,1,\dots,k\}$ is viewed as a category. An order-preserving map $\theta\colon [k] \to [l]$ may be viewed as a functor and hence induces a functor

$$\theta^*\colon \mathbf{N}_l\mathcal{C} \to \mathbf{N}_k\mathcal{C}.$$

The objects of $\mathbf{N}_{\bullet}\mathcal{C}$ comprise the nerve of $\mathcal{C}$, $N_{\bullet}\mathcal{C}$. Clearly, the nerve category is a functor from categories to simplicial categories.

Suppose now that $\mathcal{C}$ is a category with cofibrations and weak equivalences in the sense of [48, §1.2]. We then define

$$\mathbf{N}^w_{\bullet}\mathcal{C} \subset \mathbf{N}_{\bullet}\mathcal{C}$$

to be the *full* simplicial subcategory with

$$\mathrm{ob}\,\mathbf{N}^w_{\bullet}\mathcal{C} = N_{\bullet}w\mathcal{C}.$$

There is a natural structure of simplicial categories with cofibrations and weak equivalences on $\mathbf{N}^w_{\bullet}\mathcal{C}$: $\mathrm{co}\,\mathbf{N}^w_{\bullet}\mathcal{C}$ and $w\mathbf{N}^w_{\bullet}\mathcal{C}$ are the simplicial subcategories which contain all objects but where morphisms are natural transformations through cofibrations and weak equivalences in $\mathcal{C}$, respectively. With these definitions there is a natural isomorphism of bi-simplicial categories with cofibrations and weak equivalences

$$\mathbf{N}_{\bullet} S_{\bullet}\mathcal{C} \cong S_{\bullet}\mathbf{N}_{\bullet}\mathcal{C}, \tag{1.2.1}$$

where $S_{\bullet}\mathcal{C}$ is Waldhausen's construction, [48, §1.3].

Let V be a finite-dimensional orthogonal $\mathbb{T}$-representation. We define the (n, V)-th space in the symmetric orthogonal $\mathbb{T}$-spectrum $T(\mathcal{C})$ by

$$T(\mathcal{C})_{n,V} = |\,\mathrm{THH}^{(n)}(\mathbf{N}^w_{\bullet} S^{(n)}_{\bullet}\mathcal{C}; S^V)|. \tag{1.2.2}$$

There are two $\mathbb{T}$-actions on this space: one which comes from the topological Hochschild space, and another induced from the $\mathbb{T}$-action on S^V. We give $T(\mathcal{C})_{n,V}$ the diagonal $\mathbb{T}$-action. There are also two Σ_n-actions: one which comes from the Σ_n-action on the topological Hochschild space, and another induced from the permutation of the simplicial directions in the n-simplicial category $S^{(n)}_{\bullet}\mathcal{C}$; compare [10, 6.1]. We also give $T(\mathcal{C})_{n,V}$ the diagonal Σ_n-action. In particular, the $(0, 0)$-th space is the cyclic bar construction

$$T(\mathcal{C})_{0,0} = |N^{\mathrm{cy}}_{\bullet}(\mathbf{N}^w_{\bullet}\mathcal{C})|.$$

In general, the $\mathbb{T}$-fixed set of the realization of a cyclic space $X_{\bullet}$ is given by

$$|X_{\bullet}|^{\mathbb{T}} = \{x \in X_0 \,|\, s_0(x) = t_1 s_0(x)\},$$

and hence, we have a canonical map

$$|\,\mathrm{ob}\,\mathbf{N}^w_{\bullet} S^{(n)}_{\bullet}\mathcal{C} \wedge S^{V^{\mathbb{T}}}| \to (T(\mathcal{C})_{n,V})^{\mathbb{T}}.$$

The space on the left is the $(n, V^{\mathbb{T}})$-th space of a symmetric orthogonal spectrum, which represents the spectrum $K(\mathcal{C})$ in the stable homotopy category, and the map above defines the cyclotomic trace. Moreover, by a construction similar to that of [19, §2], there are $\mathbb{T}$-equivariant maps

$$\rho_p^*(T(\mathcal{C})_{n,V})^{C_p} \to T(\mathcal{C})_{n,\rho_p^* V^{C_p}},$$

and one can prove that for fixed n, the object of the $\mathbb{T}$-stable category defined by the orthogonal spectrum $V \mapsto T(\mathcal{C})_{n,V}$ has a cyclotomic structure.

Suppose that $\mathcal{C}$ is a strict symmetric monoidal category and that the tensor product is bi-exact. There is then an induced $\Sigma_m \times \Sigma_n$-equivariant product

$$S^{(m)}_{\bullet}\mathcal{C} \otimes S^{(n)}_{\bullet}\mathcal{C} \to S^{(m+n)}_{\bullet}\mathcal{C},$$

and hence

$$T(\mathcal{C})_{m,V} \wedge T(\mathcal{C})_{n,W} \to T(\mathcal{C})_{m+n,V\oplus W}.$$

This product makes $T(\mathcal{C})$ a monoid in the symmetric monoidal category of symmetric orthogonal $\mathbb{T}$-spectra.

1.3. We need to recall some of the properties of this construction. It is convenient to work in a more general setting.

Let Φ be a functor from a category of categories with cofibrations and weak equivalences to the category of pointed spaces. If $\mathcal{C}_\bullet$ is a simplicial category with cofibrations and weak equivalences, we define

$$\Phi(\mathcal{C}_\bullet) = |[n] \mapsto \Phi(\mathcal{C}_n)|.$$

We shall assume that Φ satisfies the following axioms:

(i) The trivial category with cofibrations and weak equivalences is mapped to a one-point space.

(ii) For any pair $\mathcal{C}$ and $\mathcal{D}$ of categories with cofibrations and weak equivalences, the canonical map

$$\Phi(\mathcal{C} \times \mathcal{D}) \xrightarrow{\sim} \Phi(\mathcal{C}) \times \Phi(\mathcal{D})$$

is a weak equivalence.

(iii) If $f_\bullet : \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ is a map of simplicial categories with cofibrations and weak equivalences, and if for all n, $\Phi(f_n): \Phi(\mathcal{C}_n) \to \Phi(\mathcal{D}_n)$ is a weak equivalence, then

$$\Phi(f_\bullet): \Phi(\mathcal{C}_\bullet) \to \Phi(\mathcal{D}_\bullet)$$

is a weak equivalence.

In [48], Φ is the functor which to a category assigns the set of objects. Here our main concern is the functor THH and variations thereof.

We next recall some generalities. Let

$$f, g: \mathcal{C}_\bullet \to \mathcal{D}_\bullet$$

be two exact simplicial functors. An *exact* simplicial homotopy from f to g is an exact simplicial functor

$$h: \Delta[1]_\bullet \times \mathcal{C}_\bullet \to \mathcal{D}_\bullet$$

such that $h \circ (d^1 \times \mathrm{id}) = f$ and $h \circ (d^0 \times \mathrm{id}) = g$. Here $\Delta[n]_\bullet$ is viewed as a discrete simplicial category with its unique structure of a simplicial category with cofibrations and weak equivalences. An exact simplicial functor $f: \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ is an exact simplicial homotopy equivalence if there exists an exact simplicial functor $g: \mathcal{D}_\bullet \to \mathcal{C}_\bullet$ and exact simplicial homotopies of the two composites to the respective identity simplicial functors.

Lemma 1.3.1. *An exact simplicial homotopy $\Delta[1]_\bullet \times \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ induces a homotopy*

$$\Delta[1] \times \Phi(\mathcal{C}_\bullet) \to \Phi(\mathcal{D}_\bullet).$$

Hence Φ takes exact simplicial homotopy equivalences to homotopy equivalences.

Proof. There is a natural transformation

$$\Delta[1]_k \times \Phi(\mathcal{C}_k) \to \Phi(\Delta[1]_k \times \mathcal{C}_k).$$

Indeed, $\Delta[1]_k \times \Phi(\mathcal{C}_k)$ and $\Delta[1]_k \times \mathcal{C}_k$ are coproducts in the category of spaces and the category of categories with cofibrations and weak equivalences, respectively, indexed by the set $\Delta[1]_k$. The map exists by the universal property of coproducts. $\square$

LEMMA 1.3.2. *An exact functor of categories with cofibrations and weak equivalences $f: \mathcal{C} \to \mathcal{D}$ induces an exact simplicial functor $\mathbf{N}^w_\cdot f: \mathbf{N}^w_\cdot \mathcal{C} \to \mathbf{N}^w_\cdot \mathcal{D}$. A natural transformation through weak equivalences of $\mathcal{D}$ between two such functors f and g induces an exact simplicial homotopy between $\mathbf{N}^w_\cdot f$ and $\mathbf{N}^w_\cdot g$.*

Proof. The first statement is clear. We view the partially ordered set $[1]$ as a category with cofibrations and weak equivalences where the nonidentity map is a weak equivalence but not a cofibration. Then the natural transformation defines an exact functor $[1] \times \mathcal{C} \to \mathcal{D}$, and the required exact simplicial homotopy is given by the composite

$$\Delta[1]_\cdot \times \mathbf{N}^w_\cdot \mathcal{C} \to \mathbf{N}^w_\cdot [1] \times \mathbf{N}^w_\cdot \mathcal{C} \to \mathbf{N}^w_\cdot ([1] \times \mathcal{C}) \to \mathbf{N}^w_\cdot \mathcal{D},$$

where the first and the middle arrow are the canonical simplicial functors, and the last is induced from the natural transformation. (Note that $\mathbf{N}^w_\cdot [n]$ is not a discrete category.) $\square$

LEMMA 1.3.3 ([48, Lemma 1.4.1]). *Let $f, g: \mathcal{C} \to \mathcal{D}$ be a pair of exact functors of categories with cofibrations. A natural isomorphism from f to g induces an exact simplicial homotopy*

$$\Delta[1]_\cdot \times S_\cdot \mathcal{C} \to S_\cdot \mathcal{D}$$

from $S_\cdot f$ to $S_\cdot g$.

COROLLARY 1.3.4. *Let $\mathcal{C}$ be a category with cofibrations, and let $i\mathcal{C}$ be the subcategory of isomorphisms. Then the map induced from the degeneracies in the nerve direction induces a weak equivalence*

$$\Phi(S_\cdot \mathcal{C}) \xrightarrow{\sim} \Phi(\mathbf{N}^i_\cdot S_\cdot \mathcal{C}).$$

Proof. For each k, the iterated degeneracy functor

$$s: \mathcal{C} = \mathbf{N}^i_0 \mathcal{C} \to \mathbf{N}^i_k \mathcal{C},$$

has the retraction

$$\theta^*: \mathbf{N}^i_k \mathcal{C} \to \mathcal{C},$$

where $\theta: [0] \to [k]$ is given by $\theta(0) = 0$. Moreover, there is a natural isomorphism $\mathrm{id} \xrightarrow{\sim} \theta^*$, and hence by Lemma 1.3.3,

$$S_{\bullet}s\colon S_{\bullet}\mathcal{C} \to S_{\bullet}\mathbf{N}^i_k\mathcal{C} = \mathbf{N}^i_k S_{\bullet}\mathcal{C}$$

is an exact simplicial homotopy equivalence. The corollary follows from Lemma 1.3.1 and from property (iii) above. □

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories with cofibrations and weak equivalences and suppose that $\mathcal{A}$ and $\mathcal{B}$ are subcategories of $\mathcal{C}$ and that the inclusion functors are exact. Following [48, p. 335], let $E(\mathcal{A},\mathcal{C},\mathcal{B})$ be the category with cofibrations and weak equivalences given by the pull-back diagram

$$\begin{array}{ccc} E(\mathcal{A},\mathcal{C},\mathcal{B}) & \xrightarrow{(s,t,q)} & \mathcal{A}\times\mathcal{C}\times\mathcal{B} \\ \downarrow & & \downarrow \\ S_2\mathcal{C} & \xrightarrow{(d_2,d_1,d_0)} & \mathcal{C}\times\mathcal{C}\times\mathcal{C}. \end{array}$$

In other words, $E(\mathcal{A},\mathcal{C},\mathcal{B})$ is the category of cofibration sequences in $\mathcal{C}$ of the form

$$A \rightarrowtail C \twoheadrightarrow B, \quad A \in \mathcal{A},\ B \in \mathcal{B}.$$

The exact functors s, t and q take this sequence to A, C and B, respectively. The extension of the additivity theorem to the present situation is due to McCarthy, [34]. Indeed, the proof given there for Φ the cyclic nerve functor generalizes mutatis mutandis to prove the statement (1) below. The equivalence of the four statements follows from [48, Prop. 1.3.2].

Theorem 1.3.5 (Additivity theorem). *The following equivalent assertions hold*:

(1) *The exact functors s and q induce a weak equivalence*

$$\Phi(\mathbf{N}^w_{\bullet}S_{\bullet}E(\mathcal{A},\mathcal{C},\mathcal{B})) \xrightarrow{\sim} \Phi(\mathbf{N}^w_{\bullet}S_{\bullet}\mathcal{A}) \times \Phi(\mathbf{N}^w_{\bullet}S_{\bullet}\mathcal{B}).$$

(2) *The exact functors s and q induce a weak equivalence*

$$\Phi(\mathbf{N}^w_{\bullet}S_{\bullet}E(\mathcal{C},\mathcal{C},\mathcal{C})) \xrightarrow{\sim} \Phi(\mathbf{N}^w_{\bullet}S_{\bullet}\mathcal{C}) \times \Phi(\mathbf{N}^w_{\bullet}S_{\bullet}\mathcal{C}).$$

(3) *The functors t and $s \vee q$ induce homotopic maps*

$$\Phi(\mathbf{N}^w_{\bullet}S_{\bullet}E(\mathcal{C},\mathcal{C},\mathcal{C})) \to \Phi(\mathbf{N}^w_{\bullet}S_{\bullet}\mathcal{C}).$$

(4) *Let $F' \rightarrowtail F \twoheadrightarrow F''$ be a cofibration sequence of exact functors $\mathcal{C} \to \mathcal{D}$. Then the exact functors F and $F' \vee F''$ induce homotopic maps*

$$\Phi(\mathbf{N}^w_{\bullet}S_{\bullet}\mathcal{C}) \to \Phi(\mathbf{N}^w_{\bullet}S_{\bullet}\mathcal{D}).$$

Let $f\colon \mathcal{C} \to \mathcal{D}$ be an exact functor and let $S_\bullet(f\colon \mathcal{C} \to \mathcal{D})$ be Waldhausen's relative construction, [48, Def. 1.5.4]. Then the commutative square

$$\begin{array}{ccc} \Phi(\mathbf{N}^w_\bullet S_\bullet \mathcal{C}) & \longrightarrow & \Phi(\mathbf{N}^w_\bullet S_\bullet S_\bullet(\mathrm{id}\colon \mathcal{C} \to \mathcal{C})) \\ \downarrow & & \downarrow \\ \Phi(\mathbf{N}^w_\bullet S_\bullet \mathcal{D}) & \longrightarrow & \Phi(\mathbf{N}^w_\bullet S_\bullet S_\bullet(f\colon \mathcal{C} \to \mathcal{D})) \end{array} \tag{1.3.6}$$

is homotopy cartesian, and there is a canonical contraction of the upper right-hand term. In particular, if we let $\mathcal{D}$ be the category with one object and one morphism, this shows that the canonical map

$$\Phi(\mathbf{N}^w_\bullet S_\bullet \mathcal{C}) \xrightarrow{\sim} \Omega\Phi(\mathbf{N}^w_\bullet S_\bullet S_\bullet \mathcal{C})$$

is a weak equivalence.

Definition 1.3.7. A map $f\colon X \to Y$ of $\mathbb{T}$-spaces is called an *$\mathcal{F}$-equivalence* if for all $r \geq 1$ the induced map of C_r-fixed points is a weak equivalence of spaces.

PROPOSITION 1.3.8. *Let $\mathcal{C}$ be a linear category with cofibrations and weak equivalences, and let $T(\mathcal{C})$ be the topological Hochschild spectrum. Then for all orthogonal $\mathbb{T}$-representations W and V, the spectrum structure maps*

$$T(\mathcal{C})_{n,V} \xrightarrow{\sim} F(S^m \wedge S^W, T(\mathcal{C})_{m+n,W\oplus V})$$

are $\mathcal{F}$-equivalences, provided that $n \geq 1$.

Proof. We factor the map in the statement as

$$T(\mathcal{C})_{n,V} \to F(S^m, T(\mathcal{C})_{m+n,V}) \to F(S^m, F(S^W, T(\mathcal{C})_{m+n,W\oplus V})).$$

Since S^m is C_r-fixed the map of C_r-fixed sets induced from the first map may be identified with the map

$$(T(\mathcal{C})_{n,V})^{C_r} \to \Omega^m (T(\mathcal{C})_{m+n,V})^{C_r},$$

and by definition, this is the map

$$\mathrm{THH}^{(n)}(\mathbf{N}^w_\bullet S^{(n)}_\bullet \mathcal{C}; S^V)^{C_r} \to \Omega^m \,\mathrm{THH}^{(m+n)}(\mathbf{N}^w_\bullet S^{(m+n)}_\bullet \mathcal{C}; S^V)^{C_r}.$$

By the approximation lemma, [2, Th. 1.6] or [30, Lemma 2.3.7], we can replace the functor $\mathrm{THH}^{(k)}(-;-)$ by the common functor $\mathrm{THH}(-;-)$, and the claim now follows from (1.3.6) applied to the functor

$$\Phi(\mathcal{C}) = \mathrm{THH}(\mathcal{C}; S^V)^{C_r}.$$

Finally, it follows from the proof of [19, Prop. 2.4] that

$$(T(\mathcal{C})_{m+n,V})^{C_r} \to F(S^W, T(\mathcal{C})_{m+n,W\oplus V}))^{C_r}$$

is a weak equivalence. □

We next extend Waldhausen's fibration theorem to the present situation. We follow the original proof in [48, §1.6], where also the notion of a cylinder functor is defined.

LEMMA 1.3.9. *Suppose that $\mathcal{C}$ has a cylinder functor, and that $w\mathcal{C}$ satisfies the cylinder axiom and the saturation axiom. Then*

$$\Phi(\mathbf{N}^{\bar{w}}_{\bullet}\mathcal{C}) \xrightarrow{\sim} \Phi(\mathbf{N}^{w}_{\bullet}\mathcal{C})$$

is a weak equivalence. Here $\bar{w}\mathcal{C} = w\mathcal{C} \cap \operatorname{co}\mathcal{C}$.

Proof. The proof is analogous to the proof of [48, Lemma 1.6.3], but we need the proof of [37, Th. A] and not just the statement. We consider the bi-simplicial category $\mathbf{T}(\mathcal{C})$ whose category of (p, q)-simplices has, as objects, pairs of diagrams in $\mathcal{C}$ of the form

$$(A_q \to \cdots \to A_0, A_0 \to B_0 \to \cdots \to B_p),$$

and morphisms, all natural transformations of such pairs of diagrams. We let

$$\mathbf{T}^{\bar{w},w}(\mathcal{C}) \subset \mathbf{T}(\mathcal{C})$$

be the *full* subcategory with objects the pairs of diagrams with the left-hand diagram in $\bar{w}\mathcal{C}$ and the right-hand diagram in $w\mathcal{C}$. There are bi-simplicial functors

$$\mathbf{N}^{\bar{w}}(\mathcal{C}^{\mathrm{op}})R \xleftarrow{p_1} \mathbf{T}^{\bar{w},w}(\mathcal{C}) \xrightarrow{p_2} \mathbf{N}^{w}(\mathcal{C})L,$$

where for a simplicial object X, the bi-simplicial objects XL and XR are obtained by precomposing X with projections pr_1 and pr_2 from $\mathbf{\Delta} \times \mathbf{\Delta}$ to $\mathbf{\Delta}$, respectively. Applying Φ in each bi-simplicial degree, we get corresponding maps of bi-simplicial spaces. We show that both maps induce weak equivalences after realization.

For fixed q, the simplicial functor

$$p_1 \colon \mathbf{T}^{\bar{w},w}_{\bullet,q}(\mathcal{C}) \to \mathbf{N}^{\bar{w}}_{q}(\mathcal{C}^{\mathrm{op}})$$

is a simplicial homotopy equivalence, and hence induces a homotopy equivalence upon realization. It follows that

$$\Phi(p_1) \colon \Phi(\mathbf{T}^{\bar{w},w}(\mathcal{C})) \xrightarrow{\sim} \Phi(\mathbf{N}^{\bar{w}}_{\bullet}(\mathcal{C}^{\mathrm{op}}))$$

is a weak equivalence of spaces.

Similarly, we claim that for fixed p, the simplicial functor

$$p_2 \colon \mathbf{T}^{\bar{w},w}_{p,\bullet}(\mathcal{C}) \to \mathbf{N}^{w}_{p}(\mathcal{C})$$

is a simplicial homotopy equivalence. The homotopy inverse σ maps

$$(B_0 \to \cdots \to B_p) \mapsto (B_0 \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} B_0, B_0 \xrightarrow{\mathrm{id}} B_0 \to \cdots \to B_p).$$

Following the proof of [48, Lemma 1.6.3] we consider the simplicial functor

$$t\colon \mathbf{T}^{\bar{w},w}_{p,\bullet}(\mathcal{C}) \to \mathbf{T}^{\bar{w},w}_{p,\bullet}(\mathcal{C})$$

which maps

$$\begin{aligned} &(A_q \to \dots \to A_0, A_0 \to B_0 \to \dots B_p) \\ \mapsto &(T(A_q \to B_0) \to \dots \to T(A_0 \to B_0), T(A_0 \to B_0) \xrightarrow{p} B_0 \to \dots \to B_p), \end{aligned}$$

where T is the cylinder functor. There are exact simplicial homotopies from $\sigma \circ p_2$ to t and from the identity functor to t. Hence

$$\Phi(p_2)\colon \Phi(\mathbf{T}^{\bar{w},w}(\mathcal{C})) \xrightarrow{\sim} \Phi(\mathbf{N}^{w}(\mathcal{C}))$$

is a weak equivalence of spaces.

Finally, consider the diagram of bi-simplicial categories

$$\begin{array}{ccccc} \mathbf{N}^{\bar{w}}(\mathcal{C}^{\mathrm{op}})R & \xleftarrow{p_1} & \mathbf{T}^{\bar{w},w}(\mathcal{C}) & \xrightarrow{p_2} & \mathbf{N}^{w}(\mathcal{C})L \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} & & \| \\ \mathbf{N}^{w}(\mathcal{C}^{\mathrm{op}})R & \xleftarrow{p_1} & \mathbf{T}^{w,w}(\mathcal{C}) & \xrightarrow{p_2} & \mathbf{N}^{w}(\mathcal{C})L, \end{array}$$

where i' is the obvious inclusion functor. Applying Φ, we see that the horizontal functors all induce weak equivalences. The lemma follows. □

Let $\mathcal{C}$ be a category with cofibrations and two categories of weak equivalences $v\mathcal{C}$ and $w\mathcal{C}$, and write

$$\mathbf{N}^{v,w}\mathcal{C} = \mathbf{N}^{v}_{\bullet}(\mathbf{N}^{w}_{\bullet}\mathcal{C}) \cong \mathbf{N}^{w}_{\bullet}(\mathbf{N}^{v}_{\bullet}\mathcal{C}).$$

This is a bi-simplicial category with cofibrations which again has two categories of weak equivalences.

Lemma 1.3.10 (Swallowing lemma). *If $v\mathcal{C} \subset w\mathcal{C}$ then*

$$\Phi(\mathbf{N}^{w}_{\bullet}\mathcal{C}) = \Phi((\mathbf{N}^{w}\mathcal{C})R) \xrightarrow{\sim} \Phi(\mathbf{N}^{v,w}\mathcal{C})$$

is a homotopy equivalence with a canonical homotopy inverse.

Proof. We claim that for fixed m, the iterated degeneracy in the v-direction,

$$\mathbf{N}^{w}_{\bullet}\mathcal{C} \to \mathbf{N}^{w}_{\bullet}(\mathbf{N}^{v}_{m}\mathcal{C}),$$

is an exact simplicial homotopy equivalence. Given this, the lemma follows from Lemma 1.3.1 and from property (iii). The iterated degeneracy above is induced from the (exact) iterated degeneracy map $\mathcal{C} \to \mathbf{N}^{v}_{m}\mathcal{C}$ in the simplicial category $\mathbf{N}^{v}_{\bullet}\mathcal{C}$. This map has a retraction given by the (exact) iterated face map which takes $c_0 \to \dots \to c_m$ to c_0. The other composite takes

$c_0 \to \cdots \to c_m$ to the appropriate sequence of identity maps on c_0. There is a natural transformation from this functor to the identity functor, given by

$$\begin{array}{ccccccc}
c_0 & = & c_0 & = \cdots = & c_0 \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_m \circ \ldots \circ f_1} \\
c_0 & \xrightarrow{f_1} & c_1 & \xrightarrow{f_2} \ldots \xrightarrow{f_m} & c_m.
\end{array}$$

The natural transformation is through arrows in $v\mathcal{C}$, and hence in $w\mathcal{C}$. The claim now follows from Lemma 1.3.2. □

The proof of [48, Th. 1.6.4] now gives:

THEOREM 1.3.11 (Fibration theorem). *Let $\mathcal{C}$ be a category with cofibrations equipped and two categories of weak equivalences $v\mathcal{C} \subset w\mathcal{C}$, and let $\mathcal{C}^w$ be the subcategory with cofibrations of $\mathcal{C}$ given by the objects A such that $* \to A$ is in $w\mathcal{C}$. Suppose that $\mathcal{C}$ has a cylinder functor, and that $w\mathcal{C}$ satisfies the cylinder axiom, the saturation axiom, and the extension axiom. Then*

$$\begin{array}{ccc}
\Phi(\mathbf{N}^v_{\bullet} S_{\bullet} \mathcal{C}^w) & \longrightarrow & \Phi(\mathbf{N}^w_{\bullet} S_{\bullet} \mathcal{C}^w) \\
\downarrow & & \downarrow \\
\Phi(\mathbf{N}^v_{\bullet} S_{\bullet} \mathcal{C}) & \longrightarrow & \Phi(\mathbf{N}^w_{\bullet} S_{\bullet} \mathcal{C})
\end{array}$$

is a homotopy cartesian square of pointed spaces, and there is a canonical contraction of the upper right-hand term.

1.4. Let $\mathcal{A}$ be an abelian category. We view $\mathcal{A}$ as a category with cofibrations and weak equivalences by choosing a null-object and taking the monomorphisms as the cofibrations and the isomorphisms as the weak equivalences. Let $\mathcal{E}$ be an additive category embedded as a full subcategory of $\mathcal{A}$, and assume that for every exact sequence in $\mathcal{A}$,

$$0 \to A' \to A \to A'' \to 0,$$

if A' and A'' are in $\mathcal{E}$ then A is in $\mathcal{E}$, and if A and A'' are in $\mathcal{E}$ then A' is in $\mathcal{E}$. We then view $\mathcal{E}$ as a subcategory with cofibrations and weak equivalences of $\mathcal{A}$ in the sense of [48, §1.1].

The category $C^b(\mathcal{A})$ of bounded complexes in $\mathcal{A}$ is a category with cofibrations and weak equivalences, where the cofibrations are the degree-wise monomorphisms and the weak equivalences $zC^b(\mathcal{A})$ are the quasi-isomorphisms. We view the category $C^b(\mathcal{E})$ of bounded complexes in $\mathcal{E}$ as a subcategory with cofibrations and weak equivalences of $C^b(\mathcal{A})$. The inclusion $\mathcal{E} \to C^b(\mathcal{E})$ of $\mathcal{E}$ as the subcategory of complexes concentrated in degree zero, is an exact functor. The assumptions of the fibration Theorem 1.3.11 are satisfied for $C^b(\mathcal{E})$.

THEOREM 1.4.1. *With $\mathcal{E}$ as above, the inclusion induces an equivalence*

$$\Phi(\mathbf{N}^i_{\bullet} S_{\bullet} \mathcal{E}) \xrightarrow{\sim} \Phi(\mathbf{N}^z_{\bullet} S_{\bullet} C^b(\mathcal{E})).$$

Proof. We follow the proof of [46, Th. 1.11.7]. Since the category $C^b(\mathcal{E})$ has a cylinder functor which satisfies the cylinder axiom with respect to quasi-isomorphisms, the fibration theorem shows that the right-hand square in the diagram

$$\begin{array}{ccccc} \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{E}^i) & \longrightarrow & \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}C^b(\mathcal{E})^z) & \longrightarrow & \Phi(\mathbf{N}^z_{\bullet}S_{\bullet}C^b(\mathcal{E})^z) \\ \downarrow & & \downarrow & & \downarrow \\ \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{E}) & \longrightarrow & \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}C^b(\mathcal{E})) & \longrightarrow & \Phi(\mathbf{N}^z_{\bullet}S_{\bullet}C^b(\mathcal{E})) \end{array}$$

is homotopy cartesian. Moreover, the composite of the maps in the lower row is equal to the map of the statement, and the upper left-hand and upper right-hand terms are contractible. Hence the theorem is equivalent to the statement that the left-hand square, and thus the outer square, are homotopy cartesian.

Let $\mathcal{C}^b_a$ be the full subcategory of $C^b(\mathcal{E})$ consisting of the complexes E_* with $E_i = 0$ for $i > b$ and $i < a$. Then $C^b(\mathcal{E})$ is the colimit of the categories $\mathcal{C}^b_a$ as a and b tend to $-\infty$ and $+\infty$, respectively. We consider $\mathcal{C}^b_a$ as a subcategory with cofibrations of $C^b(\mathcal{E})$. We first show that there is a weak equivalence

$$\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}^b_a) \to \prod_{a\le s\le b} \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{E}), \qquad E_* \mapsto (E_b, E_{b-1}, \dots, E_a).$$

The map is an isomorphism for $b = a$. If $b > a$, the functor

$$e\colon \mathcal{C}^b_a \to E(\mathcal{C}^a_a, \mathcal{C}^b_a, \mathcal{C}^b_{a+1}),$$

which takes E_* to the extension

$$\sigma_{\le a}E_* \rightarrowtail E_* \twoheadrightarrow \sigma_{>a}E_*,$$

is an exact equivalence of categories. Here $\sigma_{\le n}E_*$ is the brutal truncation, [49, 1.2.7]. The inverse, given by the total-object functor, is also exact. Hence, the induced map

$$\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}^b_a) \xrightarrow{\sim} \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}E(\mathcal{C}^a_a, \mathcal{C}^b_a, \mathcal{C}^b_{a+1})),$$

is a homotopy equivalence by Lemma 1.3.2. The additivity Theorem 1.3.5 then shows that

$$(s,q)\colon \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}E(\mathcal{C}^a_a, \mathcal{C}^b_a, \mathcal{C}^b_{a+1})) \xrightarrow{\sim} \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}^a_a) \times \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}^b_{a+1});$$

thus, we have a weak equivalence

$$\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}^b_a) \xrightarrow{\sim} \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{E}) \times \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}^b_{a+1}), \qquad E_* \mapsto (E_a, \sigma_{>a}E_*).$$

It now follows by easy induction that the map in question is a weak equivalence.

Next, we claim that the map

$$\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}^{bz}_a) \to \prod_{a\le s<b} \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{E}), \qquad E_* \mapsto (B_{b-1}, B_{b-2}, \dots, B_a),$$

where $B_i \subset E_i$ are the boundaries, is a weak equivalence. Note that the exactness of the functors $E_* \mapsto B_i$ uses the fact that the complex E_* is acyclic. If $a = b-1$ the functor $E_* \mapsto B_{b-1}$ is an equivalence of categories with exact inverse functor. Therefore, in this case, the claim follows from Lemma 1.3.2. If $b-1 > a$, we consider the functor

$$\mathcal{C}_a^{bz} \to E(\mathcal{C}_{b-1}^{bz}, \mathcal{C}_a^{bz}, \mathcal{C}_a^{(b-1)z}),$$

which takes the acyclic complex E_* to the extension

$$\tau_{\geq b-1}E_* \rightarrowtail E_* \twoheadrightarrow \tau_{<b-1}E_*,$$

where $\tau_{\geq n}E_*$ is the good truncation, [49, 1.2.7]. The functor is exact, since we only consider acyclic complexes, and it is an equivalence of categories with exact inverse given by the total-object functor. Hence the induced map

$$\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}_a^{bz}) \xrightarrow{\sim} \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}E(\mathcal{C}_{b-1}^{bz}, \mathcal{C}_a^{bz}, \mathcal{C}_a^{(b-1)z}))$$

is a homotopy equivalence by Lemma 1.3.2. The additivity theorem now shows that

$$\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}_a^{bz}) \xrightarrow{\sim} \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{E}) \times \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}_a^{b-1}), \qquad E_* \mapsto (B_{b-1}, \tau_{<b-1}E_*),$$

is a weak equivalence, and the claim follows by induction.

Statement (4) of the additivity theorem shows that there is a homotopy commutative diagram

$$\begin{array}{ccc}
\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}_a^{bz}) & \xrightarrow{\sim} & \prod_{a\leq s<b}\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{E}) \\
\downarrow & & \downarrow \\
\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}_a^{b}) & \xrightarrow{\sim} & \prod_{a\leq s\leq b}\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{E})
\end{array}$$

where the horizontal maps are the equivalences established above, and where the right-hand vertical map takes (x_s) to $(x_s + x_{s-1})$. It follows that the diagram

$$\begin{array}{ccc}
\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}_0^{0z}) & \longrightarrow & \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}_a^{bz}) \\
\downarrow & & \downarrow \\
\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}_0^{0}) & \longrightarrow & \Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{C}_a^{b})),
\end{array}$$

where the maps are induced by the canonical inclusions, is homotopy cartesian. Indeed, the map of horizontal homotopy fibers may be identified with the map

$$\prod_{a\leq s<b}\Omega\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{E}) \to \prod_{a\leq s\leq b, s\neq 0}\Omega\Phi(\mathbf{N}^i_{\bullet}S_{\bullet}\mathcal{E}),$$

which takes (x_s) to $(x_s + x_{s-1})$, and this, clearly, is a homotopy equivalence. Taking the homotopy colimit over a and b, we see that the left-hand square in the diagram at the beginning of the proof is homotopy cartesian. $\square$

1.5. In the remainder of this section, A will be a discrete valuation ring with quotient field K and residue field k. The main result is Theorem 1.5.2 below. It seems unlikely that this result is valid in the generality of the previous section. Indeed, the proof of the corresponding result for K-theory uses the approximation theorem [48, Th. 1.6.7], and this fails for general Φ, topological Hochschild homology included. Our proof of Theorem 1.5.2 uses the equivalence criterion of Dundas-McCarthy for topological Hochschild homology, which we now recall.

If $\mathcal{C}$ is a category and $n \geq 0$ an integer, we let $\mathbf{End}_n(\mathcal{C})$ be the category where an object is a tuple $(c; v_1, \dots, v_n)$ with c an object of $\mathcal{C}$ and $v_1, \dots, v_n$ endomorphisms of c, and where a morphism from $(c; v_1, \dots, v_n)$ to $(d; w_1, \dots, w_n)$ is a morphism $f\colon c \to d$ in $\mathcal{C}$ such that $fv_i = w_i f$, for $1 \leq i \leq n$. We note that $\mathbf{End}_0(\mathcal{C}) = \mathcal{C}$.

Proposition 1.5.1 ([7, Prop. 2.3.3]). *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor of linear categories with cofibrations and weak equivalences, and suppose that for all $n \geq 0$, the map $|\operatorname{ob} \mathbf{N}^w_{\boldsymbol{\cdot}} S_{\boldsymbol{\cdot}} \mathbf{End}_n(F)|$ is an equivalence. Then*

$$F_*\colon \mathrm{THH}(\mathbf{N}^w_{\boldsymbol{\cdot}} S_{\boldsymbol{\cdot}} \mathcal{C}) \xrightarrow{\sim} \mathrm{THH}(\mathbf{N}^w_{\boldsymbol{\cdot}} S_{\boldsymbol{\cdot}} \mathcal{D})$$

is an $\mathcal{F}$-equivalence (see Def. 1.3.7).

Let $\mathcal{M}_A$ be the category of finitely generated A-modules. We consider two categories with cofibrations and weak equivalences, $C^b_z(\mathcal{M}_A)$ and $C^b_q(\mathcal{M}_A)$, both of which have the category of bounded complexes in $\mathcal{M}_A$ with degree-wise monomorphisms as their underlying category with cofibrations. The weak equivalences are the categories $zC^b(\mathcal{M}_A)$ of quasi-isomorphisms and $qC^b(\mathcal{M}_A)$ of chain maps which become quasi-isomorphisms in $C^b(\mathcal{M}_K)$, respectively. We note that $C^b(\mathcal{M}^q_A)$ and $C^b(\mathcal{M}_A)^q$ are the categories of bounded complexes of finitely generated torsion A-modules and bounded complexes of finitely generated A-modules with torsion homology, respectively.

Theorem 1.5.2. *The inclusion functor induces an $\mathcal{F}$-equivalence*

$$\mathrm{THH}(\mathbf{N}^z_{\boldsymbol{\cdot}} S_{\boldsymbol{\cdot}} \mathcal{C}^b(\mathcal{M}^q_A)) \xrightarrow{\sim} \mathrm{THH}(\mathbf{N}^z_{\boldsymbol{\cdot}} S_{\boldsymbol{\cdot}} \mathcal{C}^b(\mathcal{M}_A)^q).$$

Proof. We show that the assumptions of Proposition 1.5.1 are satisfied. The proof relies on Waldhausen's approximation theorem, [48, Th. 1.6.7], but in a formulation due to Thomason, [46, Th. 1.9.8], which is particularly well suited to the situation at hand.

For $n \geq 0$, let A_n be the ring of polynomials in n noncommuting variables with coefficients in A, and let $\mathcal{M}_{A,n} \subset \mathcal{M}_{A_n}$ be the category of A_n-modules which are finitely generated as A-modules. Then the category $\mathbf{End}_n(C^b(\mathcal{M}_A))$ (resp. $\mathbf{End}_n(C^b(\mathcal{M}_A))^q$, resp. $\mathbf{End}_n(C^b(\mathcal{M}^q_A))$) is canonically isomorphic to the category $C^b(\mathcal{M}_{A,n})$ (resp. $C^b(\mathcal{M}_{A,n})^q$, resp. $C^b(\mathcal{M}^q_{A,n})$). Here $C^b(\mathcal{M}_{A,n})^q \subset$

$C^b(\mathcal{M}_{A,n})$ is the full subcategory of complexes whose image under the forgetful functor $C^b(\mathcal{M}_{A,n}) \to C^b(\mathcal{M}_A)$ lies in $C^b(\mathcal{M}_A)^q$, and similarly for $\mathcal{M}^q_{A,n}$. We must show that the inclusion functor induces a weak equivalence

$$|\,\mathrm{ob}\,\mathbf{N}^z_{\boldsymbol{\cdot}} S_{\boldsymbol{\cdot}} C^b(\mathcal{M}^q_{A,n})| \xrightarrow{\sim} |\,\mathrm{ob}\,\mathbf{N}^z_{\boldsymbol{\cdot}} S_{\boldsymbol{\cdot}} C^b(\mathcal{M}_{A,n})^q|,$$

for which we use [46, Th. 1.9.8]. The categories $C^b(\mathcal{M}^q_{A,n})$ and $C^b(\mathcal{M}_{A,n})^q$ are both complicial bi-Waldhausen categories in the sense of [46, 1.2.4], which are closed under the formation of canonical homotopy pushouts and homotopy pullbacks in the sense of [46, 1.9.6]. The inclusion functor

$$F\colon C^b(\mathcal{M}^q_{A,n}) \to C^b(\mathcal{M}_{A,n})^q$$

is a complicial exact functor in the sense of [46, 1.2.16]. We must verify the conditions [46, 1.9.7.0–1.9.7.3]. These conditions are easily verified with the exception of condition 1.9.7.1 which reads: for every object B of $C^b(\mathcal{M}_{A,n})^q$, there exist an object A of $C^b(\mathcal{M}^q_{A,n})$ and a map $FA \xrightarrow{\sim} B$ in $zC^b(\mathcal{M}_{A,n})^q$. This follows from Lemma 1.5.3 below. $\square$

Lemma 1.5.3. *Let A be a commutative noetherian ring, and let B be a not necessarily commutative A-algebra. Let C_* be a bounded complex of left B-modules which as A-modules are finitely generated and suppose that the homology of C_* is annihilated by some power of an ideal $I \subset A$. Then there exists a quasi-isomorphism*

$$C_* \xrightarrow{\sim} D_*$$

with D_* a bounded complex of left B-modules which as A-modules are finitely generated and annihilated by some power of I.

Proof. Let n be an integer such that for all $i \geq n$, C_i is annihilated by some power of I. We construct a quasi-isomorphism $C \xrightarrow{\sim} C''$ to a bounded complex C'' of left B-modules which as A-modules are finitely generated and such that for all $i \geq n-1$, C''_i is annihilated by some power of I. The lemma follows by easy induction. To begin we note that the exact sequences

$$0 \to Z_n \to C_n \xrightarrow{d} B_{n-1} \to 0,$$
$$0 \to B_{n-1} \to Z_{n-1} \to H_{n-1} \to 0,$$

show that Z_{n-1} is annihilated by some power of I, say, by I^r. As an A-module Z_{n-1} is finitely generated because C_{n-1} is a finitely generated A-module and because A is noetherian. Hence, by the Artin-Rees lemma, [32, Th. 8.5], we can find $s \geq 1$ such that $Z_{n-1} \cap I^s C_{n-1} \subset I^r Z_{n-1} = 0$. We now define C'' to be the complex with $C''_i = C_i$, if $\neq n-1, n-2$, with $C''_{n-1} = C_{n-1}/I^s C_{n-1}$, and with C''_{n-2} given by the pushout square

$$\begin{array}{ccc} C_{n-1} & \xrightarrow{d} & C_{n-2} \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow \\ C''_{n-1} & \longrightarrow & C''_{n-2}. \end{array}$$

There is a unique differential on C'' such that the canonical projection $C \to C''$ is a map of complexes. The kernel complex C' is concentrated in degrees $n-1$ and $n-2$. The differential $C'_{n-1} \to C'_{n-2}$ is injective, since $Z_{n-1} \cap I^s C_{n-1}$ is zero, and surjective, since the square is a pushout. Hence, the homology sequence associated with the short exact sequence of complexes

$$0 \to C' \to C \to C'' \to 0$$

shows that $C \to C''$ is a quasi-isomorphism. And by construction, some power of I annihilates C''_i, if $i \geq n-1$. □

We thank Thomas Geisser and Stefan Schwede for help with the argument above.

Let $C^b_z(\mathcal{P}_A)$ and $C^b_q(\mathcal{P}_A)$ be the category of bounded complexes of finitely generated projective A-modules considered as a subcategory with cofibrations and weak equivalences of $C^b_z(\mathcal{M}_A)$ and $C^b_q(\mathcal{M}_A)$, respectively.

Proposition 1.5.4. *The inclusion functor induces an $\mathcal{F}$-equivalence*

$$\mathrm{THH}(\mathbf{N}^z_{\boldsymbol{\cdot}} S_{\boldsymbol{\cdot}} C^b(\mathcal{P}_A)^q) \xrightarrow{\sim} \mathrm{THH}(\mathbf{N}^z_{\boldsymbol{\cdot}} S_{\boldsymbol{\cdot}} C^b(\mathcal{M}_A)^q).$$

Proof. Let A_n and $\mathcal{M}_{A,n}$ be as in the proof of Theorem 1.5.2, and let $\mathcal{P}_{A,n}$ be the full subcategory of $\mathcal{M}_{A,n}$ consisting of the A_n-modules which as A-modules are finitely generated projective. Then $\mathbf{End}_n(C^b(\mathcal{M}_A))^q$ and $\mathbf{End}_n(C^b(\mathcal{P}_A))^q$ are canonically isomorphic to $C^b(\mathcal{M}_{A,n})^q$ and $C^b(\mathcal{P}_{A,n})^q$, respectively, and we must show that the inclusion functor induces a weak equivalence

$$|\operatorname{ob} \mathbf{N}^z_{\boldsymbol{\cdot}} S_{\boldsymbol{\cdot}} C^b(\mathcal{P}_{A,n})^q| \xrightarrow{\sim} |\operatorname{ob} \mathbf{N}^z_{\boldsymbol{\cdot}} S_{\boldsymbol{\cdot}} C^b(\mathcal{M}_{A,n})^q|.$$

Again, we use [46, Th. 1.9.8], where the nontrivial thing to check is condition 1.9.7.1: for every object C_* of $C^b(\mathcal{M}_{A,n})^q$, there exists an object P_* of $C^b(\mathcal{P}_{A,n})^q$ and a map $P_* \xrightarrow{\sim} C_*$ in $zC^b(\mathcal{M}_{A,n})^q$. But this follows from [5, Chap. XVII, Prop. 1.2]. Indeed, let $\varepsilon\colon P_{*,*} \to C_*$ be a projective resolution of C_* regarded as a complex of A-modules. We may assume that each $P_{i,j}$ is a finitely generated A-module, and since A is regular, that $P_{i,j}$ is zero for all but finitely many (i,j). Furthermore, it is proved in *loc.cit.* that there exists an A_n-module structure on $P_{*,*}$ such that ε is A_n-linear. Hence, the total complex $P_* = \mathrm{Tot}(P_{*,*})$ is in $C^b(\mathcal{P}_{A,n})$ and $\mathrm{Tot}(\varepsilon)\colon P_* \xrightarrow{\sim} C_*$ is in $zC^b(\mathcal{M}_{A,n})$. It follows that P_* is in $C^b(\mathcal{P}_{A,n})^q$ as desired. □

Definition 1.5.5. We define ring $\mathbb{T}$-spectra

$$T(A|K) = T(C^b_q(\mathcal{P}_A)), \quad T(A) = T(C^b_z(\mathcal{P}_A)), \quad T(k) = T(C^b_z(\mathcal{P}_A)^q)$$

and let $\mathrm{TR}^n(A|K;p)$, $\mathrm{TR}^n(A;p)$, and $\mathrm{TR}^n(k;p)$ be the associated $C_{p^{n-1}}$-fixed point ring spectra.

We show that the definition of the spectra $\mathrm{TR}^n(A;p)$ and $\mathrm{TR}^n(k;p)$ given here agrees with the usual definition. By Morita invariance, [7, Prop. 2.1.5], it suffices to show that there are canonical isomorphisms of spectra

$$\mathrm{TR}^n(A;p) \simeq \mathrm{TR}^n(\mathcal{P}_A;p), \quad \mathrm{TR}^n(k;p) \simeq \mathrm{TR}^n(\mathcal{P}_k;p),$$

compatible with the maps R, F, V, and μ. Here the exact category $\mathcal{P}_R$ is considered a category with cofibrations and weak equivalences in the usual way. It follows from Theorem 1.4.1, applied to the functor $\Phi(\mathcal{C}) = \mathrm{THH}(\mathcal{C})^{C_r}$, and Proposition 1.3.8 that the map induced by the inclusion functor

$$T(\mathcal{P}_A) \to T(C^b_z(\mathcal{P}_A)) = T(A)$$

is an $\mathcal{F}$-equivalence. This gives the first of the stated isomorphisms of spectra. A similar argument shows that the inclusion functor induces an $\mathcal{F}$-equivalence

$$T(\mathcal{P}_k) = T(\mathcal{M}_k) \to T(C^b_z(\mathcal{M}_k)).$$

By devisage, [6, Th. 1], the same is true for

$$T(C^b_z(\mathcal{M}_k)) \to T(C^b_z(\mathcal{M}^q_A)).$$

Finally, Theorem 1.5.2 and Proposition 1.5.4 show that the maps induced from the inclusion functors

$$T(C^b_z(\mathcal{M}^q_A)) \xrightarrow{\sim} T(C^b_z(\mathcal{M}_A)^q) \xleftarrow{\sim} T(C^b_z(\mathcal{P}_A)^q) = T(k)$$

are both $\mathcal{F}$-equivalences. This establishes the second of the stated isomorphisms of spectra. Let

$$i_*\colon \mathrm{TR}^n(A;p) \to \mathrm{TR}^n(k;p)$$

be the map induced from the reduction.

THEOREM 1.5.6. *For all $n \geq 1$, there is a natural cofibration sequence of spectra*

$$\mathrm{TR}^n(k;p) \xrightarrow{i^!} \mathrm{TR}^n(A;p) \xrightarrow{j_*} \mathrm{TR}^n(A|K;p) \xrightarrow{\partial} \Sigma\,\mathrm{TR}^n(k;p),$$

and all maps in the sequence commute with the maps R, F, V, and μ. The map j_* is a map of ring spectra, and the maps $i^!$ and ∂ are maps of $\mathrm{TR}^n(A;p)$-module spectra. Here $\mathrm{TR}^n(k;p)$ is considered a $\mathrm{TR}^n(A;p)$-module spectrum via the map i_*. Moreover, the preferred homotopy limits form a cofibration sequence of spectra.

Proof. We have a commutative square of symmetric orthogonal $\mathbb{T}$-spectra

$$\begin{array}{ccc} T(C_z^b(\mathcal{P}_A)^q) & \longrightarrow & T(C_q^b(\mathcal{P}_A)^q) \\ \downarrow & & \downarrow \\ T(C_z^b(\mathcal{P}_A)) & \longrightarrow & T(C_q^b(\mathcal{P}_A)), \end{array}$$

and the fibration Theorem 1.3.11 applied to the functor $\Phi(\mathcal{C}) = \mathrm{THH}(\mathcal{C})^{C_r}$ shows that the corresponding square of C_r-fixed point spectra is homotopy cartesian. It follows that there is natural cofibration sequence of spectra

$$\mathrm{TR}^n(k;p) \xrightarrow{i^!} \mathrm{TR}^n(A;p) \xrightarrow{j_*} \mathrm{TR}^n(A|K;p) \xrightarrow{\partial} \Sigma\,\mathrm{TR}^n(k;p),$$

compatible with R, F, V and μ. It is clear that this is a sequence of $\mathrm{TR}^n(A;p)$-module spectra. $\square$

Addendum 1.5.7. *There is a natural map of cofibration sequences*

$$\begin{array}{ccccccc} K(k) & \xrightarrow{i^!} & K(A) & \xrightarrow{j_*} & K(K) & \xrightarrow{\partial} & \Sigma K(k) \\ \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \mathrm{tr}} \\ \mathrm{TC}(k;p) & \xrightarrow{i^!} & \mathrm{TC}(A;p) & \xrightarrow{j_*} & \mathrm{TC}(A|K;p) & \xrightarrow{\partial} & \Sigma\,\mathrm{TC}(k;p) \end{array}$$

and the vertical maps are all maps of ring spectra.

Remark 1.5.8. Let X be a regular affine scheme and let $i\colon Y \hookrightarrow X$ be a closed subscheme with open complement $j\colon U \hookrightarrow X$. Then, more generally, the proof of Theorem 1.5.6 gives a cofibration sequence of spectra

$$\mathrm{TR}'^n(Y;p) \xrightarrow{i^!} \mathrm{TR}^n(X;p) \xrightarrow{j_*} \mathrm{TR}^n(X|U;p) \xrightarrow{\partial} \Sigma\,\mathrm{TR}'^n(Y;p),$$

where the three terms are as in Definition 1.5.5 with $\mathcal{P}_A$ replaced by the category $\mathcal{P}_X$ of locally free $\mathcal{O}_X$-modules of finite rank. The weak equivalences are the quasi-isomorphisms, $zC^b(\mathcal{P}_X)$, and the chain maps which become quasi-isomorphisms after restriction to U, $qC^b(\mathcal{P}_X)$, respectively. Similarly, the argument following Definition 1.5.5 gives canonical isomorphisms of spectra

$$\mathrm{TR}^n(X;p) \simeq \mathrm{TR}^n(\mathcal{P}_X;p), \quad \mathrm{TR}'^n(Y;p) \simeq \mathrm{TR}^n(\mathcal{M}_Y;p),$$

where $\mathcal{M}_Y$ is the category of coherent $\mathcal{O}_Y$-modules. Moreover, if Y is regular, the resolution theorem, [7, prop. 2.2.3], shows that $\mathrm{TR}^n(\mathcal{M}_Y;p)$ is canonically isomorphic to $\mathrm{TR}^n(\mathcal{P}_Y;p)$.

2. The homotopy groups of $T(A|K)$

2.1. In this section we evaluate the homotopy groups with $\mathbb{Z}/p$-coefficients of the topological Hochschild spectrum $T(A|K)$. We first fix some conventions.

Let G be a finite group and let k be a commutative ring. The category of chain complexes of left kG-modules and chain homotopy classes of chain maps is a triangulated category and a closed symmetric monoidal category, and the two structures are compatible. The same is true for the category of G-CW-spectra and homotopy classes of cellular maps. We fix our choices for the triangulated and closed symmetric monoidal structures in such a way that the cellular chain functor preserves our choices.

We first consider complexes. If $f\colon X \to Y$ is a chain map, we define the mapping cone C_f to be the complex

$$(C_f)_n = Y_n \oplus X_{n-1}, \quad d(y,x) = (dy - f(x), -dx),$$

and the suspension ΣX to be the cokernel of the inclusion $\iota\colon Y \to C_f$ of the first summand. More explicitly,

$$(\Sigma X)_n = X_{n-1}, \quad d_{\Sigma X}(x) = -d_X(x).$$

Then, by definition, a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle or a cofibration sequence if it isomorphic to the *distinguished* triangle

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{\partial} \Sigma X,$$

where ∂ is the canonical projection. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a short exact sequence of complexes then the projection $p\colon C_f \to Z$, $p(y,x) = g(y)$, is a quasi-isomorphism and the composite

$$H_n Z \underset{\sim}{\xleftarrow{p_*}} H_n C_f \xrightarrow{\partial_*} H_n \Sigma X = H_{n-1} X$$

is equal to the connecting homomorphism.

Let X and Y be two complexes. We define the tensor product complex by

$$(X \otimes Y)_n = \bigoplus_{s+t=n} X_s \otimes Y_t; \quad d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy,$$

and the complex of (k-linear) homomorphisms by

$$\operatorname{Hom}(X,Y)_n = \prod_{s\in\mathbb{Z}} \operatorname{Hom}(X_s, Y_{n+s}); \quad d(f(x)) = (df)(x) + (-1)^{|f|} f(dx).$$

We note that $Z_0 \operatorname{Hom}(X,Y)$ is equal to the set of chain maps from X to Y and that $H_0 \operatorname{Hom}(X,Y)$ is equal to the set of chain homotopy classes of chain maps from X to Y. The adjunction and twist isomorphisms are given by

$$\phi\colon \operatorname{Hom}(X \otimes Y, Z) \to \operatorname{Hom}(X, \operatorname{Hom}(Y,Z)), \quad \phi(f)(x)(y) = f(x \otimes y),$$
$$\gamma\colon X \otimes Y \to Y \otimes X, \quad \gamma(x \otimes y) = (-1)^{|x||y|} y \otimes x.$$

The triangulated and closed symmetric monoidal structures are compatible in the sense that

$$\Sigma(X \otimes Y) = (\Sigma X) \otimes Y$$

and that if W is a complex and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle, then so is

$$X \otimes W \xrightarrow{f\otimes 1} Y \otimes W \xrightarrow{g\otimes 1} Z \otimes W \xrightarrow{h\otimes 1} \Sigma X \otimes W.$$

Indeed, the isomorphism

$$\rho \colon C_f \otimes W \xrightarrow{\sim} C_{f\otimes W}, \qquad \rho((y,x)\otimes w) = (y \otimes w, x \otimes w),$$

and the identity map of $X \otimes W$, $Y \otimes W$, and $\Sigma X \otimes W$ define an isomorphism of the appropriate distinguished triangles.

We define the homology of X with $\mathbb{Z}/m$-coefficients by

$$H_*(X, \mathbb{Z}/m) = H_*(M_m \otimes X),$$

where M_m is the Moore complex given by the distinguished triangle

$$k \xrightarrow{m} k \xrightarrow{\iota} M_m \xrightarrow{\beta} \Sigma k.$$

Suppose that X is m-torsion free such that $X \xrightarrow{m} X \xrightarrow{\mathrm{pr}} X/mX$ is a short-exact sequence of complexes. Then the composite

$$H_n(X/mX) \xleftarrow[\sim]{p_*} H_n(C_m) \xleftarrow[\sim]{\rho_*} H_n(M_m \otimes X) \xrightarrow{\beta} H_n(\Sigma X) = H_{n-1}(X)$$

is equal to the connecting homomorphism.

We next consider the category of G-CW-spectra and homotopy classes of cellular maps, see [25, Chap. I, §5]. This category, we recall, is equivalent to the G-stable category. In one direction, the equivalence associates to a G-CW-spectrum X the underlying G-spectrum UX. In the other direction, we choose a functorial G-CW-replacement ΓX such that $U\Gamma X \xrightarrow{\sim} X$.

If X and Y are two G-CW-spectra, the smash product $UX \wedge UY$ has a canonical G-CW-structure. But the function spectrum $F(UX, UY)$ usually does not. Instead we consider $\Gamma F(UX, UY)$. This defines the closed symmetric monoidal structure.

The mapping cone of a celluar map $f \colon X \to Y$ is defined by

$$C_f = Y \cup_X ([0,1] \wedge X),$$

where we use 1 as the base point for the smash product. The interval is given the usual CW-structure with a single 1-cell oriented from 0 to 1, and the mapping cone is given the induced G-CW-structure. Collapsing the image of the canonical inclusion $i \colon Y \to C_f$ to the base point defines the map

$$\partial \colon C_f \to S^1 \wedge X = \Sigma X,$$

where $S^1 = [0,1]/\partial[0,1]$ with the induced CW-structure. We then define the distinguished triangles to be sequences of the form

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{\partial} \Sigma X.$$

Again, the triangulated and the closed symmetric monoidal structures are compatible. Indeed, the associativity isomorphism, which is part of the monoical structure, gives rise to canonical isomorphisms

$$\alpha\colon \Sigma(X \wedge W) \xrightarrow{\sim} (\Sigma X) \wedge W, \qquad \rho\colon C_f \wedge W \xrightarrow{\sim} C_{f\wedge W}.$$

The choices made above are preserved by the cellular chain functor. To be more precise, if X (resp. $f\colon X \to Y$) is a G-CW-spectrum (resp. a cellular map), then the suspension isomorphism gives rise to a canonical isomorphism of complexes $\Sigma C_*(X;k) \xrightarrow{\sim} C_*(\Sigma X;k)$ (resp. $C_*(C_f;k) \xrightarrow{\sim} C_{f_*}$). Under these identifications, the cellular chain functor carries the distinguished triangles of G-CW-spectra to the distinguished triangles of complexes of left kG-modules. Similarly, if X and Y are two G-CW-complexes, then the Künneth isomorphism gives a canonical isomorphism $C_*(X;k) \otimes C_*(Y;k) \xrightarrow{\sim} C_*(X \wedge Y;k)$.

We define the homotopy groups of X with $\mathbb{Z}/m$-coefficients by

$$\pi_*(X, \mathbb{Z}/m) = \pi_*(M_m \wedge X),$$

where M_m is the Moore spectrum given by the distinguished triangle

$$S^0 \xrightarrow{m} S^0 \xrightarrow{\iota} M_m \xrightarrow{\beta} S^1,$$

and the homotopy groups with $\mathbb{Z}_p$-coefficients by

$$\pi_*(X, \mathbb{Z}_p) = \pi_*(\operatorname{holim}_{\overleftarrow{v}}(M_{p^v} \wedge X)).$$

The latter are related to the former by the Milnor sequence

$$0 \to \varprojlim_{v}{}^1 \pi_{q+1}(X, \mathbb{Z}/p^v) \to \pi_q(X, \mathbb{Z}_p) \to \varprojlim_{v} \pi_q(X, \mathbb{Z}/p^v) \to 0.$$

We shall often abbreviate $\pi_q(X, \mathbb{Z}/p)$ and write $\bar{\pi}_q(X)$. Let $H\mathbb{Z}/m$ be the Eilenberg-MacLane spectrum for $\mathbb{Z}/m$. It is a ring spectrum, and we let $\varepsilon \in \pi_1(H\mathbb{Z}/m, \mathbb{Z}/m)$ be the unique element such that $\beta(\varepsilon) = 1$. Then for left $H\mathbb{Z}/m$-module spectra X, we have a natural sum-diagram

$$X \underset{r}{\overset{\iota\wedge\mathrm{id}}{\rightleftarrows}} M_m \wedge X \underset{s}{\overset{\beta\wedge\mathrm{id}}{\rightleftarrows}} \Sigma X, \tag{2.1.1}$$

where s is the composite

$$S^1 \wedge X \xrightarrow{\varepsilon\wedge\mathrm{id}} M_m \wedge H\mathbb{Z}/m \wedge X \xrightarrow{\mathrm{id}\wedge\mu} M_m \wedge X,$$

and where r is determined by the requirement that $r \circ \iota = \mathrm{id}$ and $r \circ s = 0$.

We recall Connes' operator. Let $\mathbb{T}$ be the space $S(\mathbb{C})$ of complex numbers of length 1 considered as a group under multiplication. We give $\mathbb{T}$ the orientation induced from the standard orientation of the complex plane, and let

$[\mathbb{T}] \in H_1(\mathbb{T})$ be the corresponding fundamental class. The reduced homology of a $\mathbb{T}$-space X has a natural differential given by the composite

$$d\colon \tilde{H}_q(X) \xrightarrow{[\mathbb{T}]} \tilde{H}_{q+1}(\mathbb{T}_+ \wedge X) \xrightarrow{\mu} \tilde{H}_{q+1}(X),$$

where the left-hand map is given by the Künneth isomorphism and the right-hand map is induced by the action map. There is a sum-diagram

$$\mathbb{Z}/2 \cdot \eta = \pi_1^S(S^0) \underset{c}{\overset{e}{\rightleftarrows}} \pi_1^S(\mathbb{T}_+) \underset{\sigma}{\overset{h}{\rightleftarrows}} H_1(\mathbb{T}) = \mathbb{Z} \cdot [\mathbb{T}]$$

where h is the Hurewitz homomorphism, e is induced from the map $S^0 \to \mathbb{T}_+$ which takes the nonbase-point of S^0 to $1 \in \mathbb{T}$, c is induced from the map $\mathbb{T}_+ \to S^0$ which collapses $\mathbb{T}$ to the nonbase-point of S^0, and σ is determined by $h\sigma = \mathrm{id}$ and $c\sigma = 0$. Let T be a $\mathbb{T}$-spectrum. Then Connes' operator is the map

$$d\colon \pi_q(T) \xrightarrow{[\mathbb{T}]\wedge-} \pi_{q+1}(\mathbb{T}_+ \wedge T) \xrightarrow{\mu_T} \pi_{q+1}(T). \tag{2.1.2}$$

If $T = \mathrm{HH}(A)$ is the Hochschild spectrum of a ring A, then this definition agrees with Connes' original definition, [16, Prop. 1.4.6]. We recall from *op. cit.*, Lemma 1.4.2, that, in general, $dd = d\eta = \eta d$. Hence, d is a differential, provided that multiplication by η is trivial on $\pi_*(T)$. This is the case, for instance, if multiplication by 2 on $\pi_*(T)$ is an isomorphism.

2.2. We next recall the notion of differentials with logarithmic poles. The standard reference for this material is [24]. A pre-log structure on a ring R is a map of monoids

$$\alpha\colon M \to R,$$

where R is considered a monoid under multiplication. By a log ring we mean a ring with a pre-log structure. A derivation of a log ring (R, M) into an R-module E is a pair of maps

$$(D, D\log)\colon (R, M) \to E,$$

where $D\colon R \to E$ is a derivation and $D\log\colon M \to E$ a map of monoids, such that for all $a \in M$,

$$\alpha(a) D\log a = D\alpha(a).$$

A log differential graded ring (E^*, M) consists of a differential graded ring E^*, a pre-log structure $\alpha\colon M \to E^0$, and a derivation $(D, D\log)\colon (E^0, M) \to E^1$ such that D is equal to the differential $d\colon E^0 \to E^1$ and such that $d \circ D\log = 0$.

There is a universal example of a derivation of a log ring (R, M) given by the R-module

$$\omega^1_{(R,M)} = (\Omega^1_R \oplus (R \otimes_{\mathbb{Z}} M^{\mathrm{gp}}))/\langle d\alpha(a) - \alpha(a) \otimes a \mid a \in M\rangle,$$

where M^{gp} is the group completion (or Grothendieck group) of M and $\langle \ldots \rangle$ denotes the submodule generated by the indicated elements. The structure maps are

$$\begin{aligned} d\colon R &\to \omega^1_{(R,M)}, & da &= da \oplus 0, \\ d\log\colon M &\to \omega^1_{(R,M)}, & d\log a &= 0 \oplus (1 \otimes a). \end{aligned}$$

The exterior algebra

$$\omega^*_{(R,M)} = \Lambda^*_R(\omega^1_{(R,M)})$$

endowed with the usual differential is the universal log differential graded ring whose underlying log ring is (R, M). We stress that here and throughout we use Ω^1_R to mean the *absolute* differentials.

Let A be a complete discrete valuation ring with quotient field K and *perfect* residue field k of mixed characteristic $(0, p)$. We recall the structure of A from [40, §5, Th. 4]. Let $W(k)$ be the ring of Witt vectors in k, and let K_0 be the quotient field of $W(k)$. There is a unique ring homomorphism

$$f\colon W(k) \to A$$

such that the induced map of residue fields is the identity homomorphism. We will always view A as an algebra over $W(k)$ via the map f. Moreover, if π_K is a generator of the maximal ideal $\mathfrak{m}_K \subset A$, then

$$A = W(k)[\pi_K]/(\phi_K(\pi_K)), \tag{2.2.1}$$

and the minimal polynomial takes the form

$$\phi_K(x) = x^{e_K} + p\theta_K(x),$$

where $e_K = |K : K_0|$ is the ramification index and where $\theta_K(x)$ is a polynomial of degree less that e_K such that $\theta_K(0)$ is a unit in $W(k)$. It follows that $\theta_K(\pi_K)$ is a unit and that

$$-p = \pi_K^{e_K} \theta_K(\pi_K)^{-1}.$$

We will use this formula on numerous occasions in the following. The valuation ring A has a canonical pre-log structure given by the inclusion

$$\alpha\colon M = A \cap K^\times \hookrightarrow A.$$

Let $v_K\colon K^\times \to \mathbb{Z}$ be the valuation.

Proposition 2.2.2. *There is a natural short exact sequence*

$$0 \to \Omega^1_A \to \omega^1_{(A,M)} \xrightarrow{\mathrm{res}} k \to 0,$$

where $\mathrm{res}(a\,d\log b) = av_K(b) + \mathfrak{m}_K$.

Proof. If $a \in A \cap K^\times$ then $av_K(a) \in \mathfrak{m}_K$, and hence, the composition of the two maps in the statement is zero. Only the exactness in the middle needs proof. Let $ad\log b$ be an element of $\omega^1_{(A,M)}$ and write $b = \pi_K^i u$ with $u \in A^\times$. Then

$$ad\log b = iad\log \pi_K + au^{-1}du.$$

Suppose that $\operatorname{res}(ad\log b) = ia + \mathfrak{m}_K$ is trivial. Then $ia \in \mathfrak{m}_K$, which implies that $ia\pi_K^{-1} \in A$, and hence, $iad\log \pi_K = ia\pi_K^{-1} d\pi_K$. □

We define the module of relative differentials

$$\omega^1_{(A,M)/W(k)} = \left(\Omega^1_{A/W(k)} \oplus (A \otimes_{\mathbb{Z}} K^\times)\right) \Big/ \left\langle da - a \otimes a \;\Big|\; a \in A \cap K^\times \right\rangle.$$

Again, there is a natural exact sequence

$$0 \to \Omega^1_{A/W(k)} \to \omega^1_{(A,M)/W(k)} \xrightarrow{\text{res}} k \to 0.$$

LEMMA 2.2.3. *Let $\pi_K \in A$ be a uniformizer with minimal polynomial $\phi_K(x)$. Then the element $d\log \pi_K$ generates the A-module $\omega^1_{(A,M)/W(k)}$, and its annihilator is the ideal generated by $\phi'_K(\pi_K)\pi_K$. This ideal contains p.*

Proof. Since every element of $K^\times$ can be written as a product $\pi_K^i u$ with $i \in \mathbb{Z}$ and $u \in A^\times$, the formula

$$d\log(\pi_K^i u) = id\log \pi_K + u^{-1}du$$

shows that $\omega^1_{(A,M)/W(k)}$ is generated by $d\log \pi_K$. The relation identifies

$$\phi'_K(\pi_K)\pi_K d\log \pi_K = d(\phi_K(\pi_K)) = 0,$$

so the annihilator ideal is generated $\phi'_K(\pi_K)\pi_K$. □

LEMMA 2.2.4. *For all $i > 0$, there is a natural exact sequence*

$$A \otimes_{W(k)} \Omega^i_{W(k)} \to \omega^i_{(A,M)} \to \omega^i_{(A,M)/W(k)} \to 0,$$

and the left-hand group is uniquely divisible.

Proof. The stated sequence for $i = 1$ follows from the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^1_A & \longrightarrow & \omega^1_{(A,M)} & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega^1_{A/W(k)} & \longrightarrow & \omega^1_{(A,M)/W(k)} & \longrightarrow & k & \longrightarrow & 0
\end{array}$$

with horizontal exact sequences and from the standard exact sequence

$$A \otimes_{W(k)} \otimes \Omega^1_{W(k)} \to \Omega^1_A \to \Omega^1_{A/W(k)} \to 0.$$

We show that the group $\Omega^1_{W(k)} \xrightarrow{\sim} \mathrm{HH}_1(W(k))$ is a uniquely divisible group or, more generally, that $\mathrm{HH}_i(W(k))$ is uniquely divisible, for all $i > 0$. Since $W(k)$ is torsion-free and since $W(k)/p = k$, the coefficient sequence takes the form

$$\cdots \to \mathrm{HH}_{i+1}(k) \to \mathrm{HH}_i(W(k)) \xrightarrow{p} \mathrm{HH}_i(W(k)) \to \mathrm{HH}_i(k) \to \cdots .$$

But $\mathrm{HH}_i(k) = 0$, for $i > 0$, since k is perfect, [19, Lemma 5.5]. This proves the lemma for $i = 1$. In particular, the maximal divisible sub-A-module of $\omega^1_{(A,M)}$ is equal to the image of $A \otimes_{W(k)} \Omega^1_{W(k)}$, and $\omega^1_{(A,M)}$ is the sum of this divisible module D and the cyclic torsion A-module $\omega^1_{(A,M)/W(k)}$. It follows that for $i > 1$, $\omega^i_{(A,M)} = \Lambda^i_A D$, and this in turn is the image of the left-hand map of the statement. $\square$

COROLLARY 2.2.5. *The p-torsion submodule of $\omega^1_{(A,M)}$ is*

$${}_p\omega^1_{(A,M)} = A/p \cdot d\log(-p).$$

Proof. It follows from Lemma 2.2.4 that the canonical map

$${}_p\omega^1_{(A,M)} \xrightarrow{\sim} {}_p\omega^1_{(A,M)/W(k)}$$

is an isomorphism. Let π_K be a uniformizer with minimal polynomial $\phi_K(x)$. Then by Lemma 2.2.3,

$$\omega^1_{(A,M)/W(k)} = A/(\pi_K \phi'_K(\pi_K)) \cdot d\log \pi_K.$$

We write $\phi_K(x) = x^{e_K} + p\theta_K(x)$ such that $-p = \pi_K^{e_K}\theta_K(\pi_K)^{-1}$. Hence, on the one hand, we have

$$\pi_K\phi'_K(\pi_K) = e_K\pi_K^{e_K} + p\pi_K\theta'_K(\pi_K) = (e_K - \pi_K\theta'_K(\pi_K)\theta_K(\pi_K)^{-1})\pi_K^{e_K},$$

and on the other hand,

$$d\log(-p) = d\log(\pi_K^{e_K}\theta_K(\pi_K)^{-1}) = (e_K - \pi_K\theta'_K(\pi_K)\theta_K(\pi_K)^{-1})d\log \pi_K.$$

The claim follows. $\square$

Let L be a finite extension of K, let B be the integral closure of A in L, and let $e_{L/K} = e_L/e_K$ be the ramification index of L/K. Then the following diagram commutes

$$\begin{array}{ccc} \omega^1_{(A,M_A)/W(k)} & \xrightarrow{\mathrm{res}_A} & A/\mathfrak{m}_K \\ \big\downarrow{\scriptstyle i_*} & & \big\downarrow{\scriptstyle e_{L/K}\cdot i} \\ \omega^1_{(B,M_B)/W(k)} & \xrightarrow{\mathrm{res}_B} & B/\mathfrak{m}_L. \end{array}$$

Recall that $B \otimes_A \Omega^1_{A/W(k)} \to \Omega^1_{B/W(k)}$ is an isomorphism if and only if $e_{L/K} = 1$.

LEMMA 2.2.6. *The canonical map*

$$B \otimes_A \omega^1_{(A,M_A)/W(k)} \to \omega^1_{(B,M_B)/W(k)}$$

is an isomorphism if and only if p does not divide $e_{L/K}$.

Proof. Suppose that p does not divide $e_{L/K}$. If $e_{L/K} = 1$ the lemma follows from the natural exact sequence

$$0 \to \Omega^1_{A/W(k)} \to \omega^1_{(A,M)/W(k)} \to A/\mathfrak{m}_K \to 0$$

and from the isomorphism mentioned before the lemma. Thus, replacing K by the maximal subfield of L which is unramified over K, we may assume that the extension is totally ramified. Then there exists $\pi_K \in A$ such that

$$L = K\left(\pi_K^{1/e_{L/K}}\right).$$

Indeed, if π_K and π_L are uniformizers of A and B over $W(k)$, then $\pi_K = u\pi_L^{e_{L/K}}$, where $u \in B^\times$ is a unit. But the sequence

$$1 \to U^1_B \to B^\times \xrightarrow{r} k^\times \to 1$$

is split by the composition of the Teichmüller character $\tau\colon k^\times \to W(k)^\times$ and the inclusion $W(k)^\times \hookrightarrow B^\times$. Therefore, replacing π_K by $\tau(r(u))^{-1}\pi_K$, we can assume that the unit u lies in the subgroup U^1_B of units in B which are congruent to 1 mod $\mathfrak{m}_L$. But every element of U^1_B has an $e_{L/K}$-th root, so replacing π_L by $u^{1/e_{L/K}}\pi_L$ we may assume that $u = 1$.

Let π_K and π_L be uniformizers of A and B over $W(k)$ such that $\pi_K = \pi_L^{e_{L/K}}$, and let $\phi_K(x)$ be the minimal polynomial of π_K. Then

$$\phi_L(x) = \phi_K(x^{e_{L/K}})$$

is the minimal polynomial of π_L. The A-module $\omega^1_{(A,M_A)/W(k)}$ is generated by $d\log\pi_K$ with annihilator $(\phi'_K(\pi_K)\pi_K)$, and similarly, the B-module $\omega^1_{(B,M_B)/W(k)}$ is generated by $d\log\pi_L$ with annihilator $(\phi'_L(\pi_L)\pi_L)$. But

$$d\log\pi_K = d\log(\pi_L^{e_{L/K}}) = e_{L/K}d\log\pi_L$$

and

$$\phi'_L(\pi_L)\pi_L = \phi'_K(\pi_L^{e_{L/K}}) \cdot e_{L/K}\pi_L^{e_{L/K}} = e_{L/K}\phi'_K(\pi_K)\pi_K;$$

so the claim follows since $e_{L/K}$ is a unit. It is also clear from this argument that the map of the statement cannot be an isomorphism if the extension L/K is wildly ramified. □

2.3. In this section we show that the homotopy groups $(\pi_*T(A|K), M)$ form a log differential graded ring. In effect, we prove the more general statement:

PROPOSITION 2.3.1. *The homotopy groups* $(\mathrm{TR}^n_*(A|K;p), M)$ *form a log differential graded ring, if* p *is odd or* $n=1$.

The homotopy groups $\mathrm{TR}^n_*(A|K;p)$ form a graded-commutative differential graded ring with the differential given by Connes' operator (2.1.2), [16, §1]. It remains to define the maps

$$\alpha_n\colon M \to \mathrm{TR}^n_0(A|K;p), \qquad d\log_n\colon M \to \mathrm{TR}^n_1(A|K;p) \tag{2.3.2}$$

and to verify the relation $\alpha_n(a)d\log_n a = d\alpha_n(a)$. We define α_n as the composite of the inclusion $M = A\cap K^\times \hookrightarrow A$ and the multiplicative map

$$\underline{}_n\colon A \to \mathrm{TR}^n_0(A|K;p).$$

This, we recall, is the map of components induced from the composite

$$A \xrightarrow{i} |N^{\mathrm{cy}}_\bullet(\mathbf{N}^q_\bullet\mathcal{C})| \xrightarrow[\sim]{D_r\circ\Delta_r} |N^{\mathrm{cy}}_\bullet(\mathbf{N}^q_\bullet\mathcal{C})|^{C_r} = \mathrm{TR}^n(A|K;p)_{0,0},$$

where $\mathcal{C} = C^b_q(\mathcal{P}_A)$ of Definition 1.5.5, $i(a)$ is the 0-simplex $A \xrightarrow{a} A$, and $r = p^{n-1}$. We refer the reader to [3, §1] for the definition of the maps Δ_r and D_r.

In general, if $\mathcal{C}$ is a category with cofibrations and weak equivalences and if X is an object of $\mathcal{C}$, there is a natural map in the stable category

$$\widetilde{\det}\colon \Sigma^\infty B\operatorname{Aut}(X) \to K(\mathcal{C}),$$

where $\operatorname{Aut}(X)$ is the monoid of endomorphisms of X in the category $w\mathcal{C}$ of weak equivalences. The inclusion of $\operatorname{Aut}(X)$ as a full subcategory of $w\mathcal{C}$ induces

$$B\operatorname{Aut}(X) = |N_\bullet \operatorname{Aut}(X)| \to |N_\bullet w\mathcal{C}| = K(\mathcal{C})_0,$$

but this map does not preserve the basepoint (unless X is the chosen null object). However, we still get a map of symmetric spectra

$$\det\colon \Sigma^\infty B\operatorname{Aut}(X)_+ \to K(\mathcal{C}).$$

To get the map $\widetilde{\det}$, we use the fact that for every pointed space B, there is a natural isomorphism $S^0\vee\Sigma^\infty B \xrightarrow{\sim} \Sigma^\infty B_+$ in the stable category. The inverse is induced from the map which collapses B to the nonbase point in S^0 and the map which identifies the extra base point with the base point in B.

We again let $\mathcal{C} = C^b_q(\mathcal{P}_A)$ and view A as a complex concentrated in degree zero. Then $\operatorname{Aut}(A) = A\cap K^\times = M$ such that we have a map of monoids

$$M \to \pi_1 BM \xrightarrow{\widetilde{\det}_*} \pi_1 K(\mathcal{C}),$$

and we define $d\log_n$ to be the composite of this map and the cyclotomic trace. Spelling out the definition, we see that $d\log_n$ is given by the composite

$$\begin{aligned} S^{l+1}\wedge M_+ &\xrightarrow{\sigma\wedge\mathrm{id}} S^l\wedge\mathbb{T}_+\wedge M_+ \xrightarrow{\mathrm{id}\wedge j} S^l\wedge|N^{\mathrm{cy}}_\bullet(\mathbf{N}^q_\bullet\mathcal{C})| \\ &\xrightarrow[\sim]{D_r\circ\Delta_r} S^l\wedge|N^{\mathrm{cy}}_\bullet(\mathbf{N}^q_\bullet\mathcal{C})|^{C_r} \xrightarrow{\lambda_{l,0}} \mathrm{TR}^n(A|K;p)_{l,0}, \end{aligned}$$

where the map j, when restricted to $\mathbb{T} \times \{a\}$, traces out the loop in the realization given by the 1-simplex (in the diagonal simplicial set):

$$\begin{array}{ccccc} A & \xrightarrow{1} & A & \xrightarrow{1} & A \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} \\ A & \xrightarrow{1} & A & \xrightarrow{1} & A. \end{array}$$

LEMMA 2.3.3. *For all $a \in M$, $d\alpha_n(a) = \alpha_n(a) d\log_n a$.*

Proof. Spelling out the definitions, one readily recognizes that it will suffice to show that the following diagram homotopy-commutes:

$$\begin{tikzcd} \mathbb{T}_+ \wedge M_+ \arrow[r, "\mathrm{id}\wedge\Delta"] \arrow[d, "\mathrm{id}\wedge i"] & \mathbb{T}_+ \wedge M_+ \wedge M_+ \arrow[r, "j\wedge\mathrm{id}"] & {|N^{\mathrm{cy}}_{\bullet}(\mathbf{N}^q_{\bullet}\mathcal{C})| \wedge |N^{\mathrm{cy}}_{\bullet}(\mathbf{N}^q_{\bullet}\mathcal{C})|} \arrow[d, "\mu_{0,0}"] \\ \mathbb{T}_+ \wedge |N^{\mathrm{cy}}_{\bullet}(\mathbf{N}^q_{\bullet}\mathcal{C})| \arrow[rr, "\mu"] & & {|N^{\mathrm{cy}}_{\bullet}(\mathbf{N}^q_{\bullet}\mathcal{C})|.} \end{tikzcd}$$

Since M is discrete, we may check this separately for each $a \in M$. The composite of the upper horizontal maps and the right-hand vertical map, when restricted to $\mathbb{T} \times \{a\}$, traces out the loop in the realization given by the 1-simplex (in the diagonal simplicial set) on the left below. Similarly, the composite of the left-hand vertical map and the lower horizontal map, when restricted to $\mathbb{T} \times \{a\}$, traces out the loop given by the 1-simplex on the right below:

$$\begin{array}{ccccc} A & \xrightarrow{a} & A & \xrightarrow{1} & A \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} \\ A & \xrightarrow{a} & A & \xrightarrow{1} & A \end{array} \quad , \quad \begin{array}{ccccc} A & \xrightarrow{1} & A & \xrightarrow{a} & A \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{1} & A & \xrightarrow{a} & A. \end{array}$$

Note that both loops are based at the vertex $A \xrightarrow{a} A$. We must show that the two loops are homotopic through loops based at $A \xrightarrow{a} A$. To this end, we consider the 2-simplices

$$\begin{array}{ccccccc} A & \xrightarrow{a} & A & \xrightarrow{1} & A & \xrightarrow{1} & A \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{a} & A & \xrightarrow{1} & A & \xrightarrow{1} & A \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} \\ A & \xrightarrow{1} & A & \xrightarrow{a} & A & \xrightarrow{1} & A \end{array} \quad , \quad \begin{array}{ccccccc} A & \xrightarrow{1} & A & \xrightarrow{1} & A & \xrightarrow{a} & A \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{1} & A & \xrightarrow{a} & A & \xrightarrow{1} & A \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ A & \xrightarrow{1} & A & \xrightarrow{a} & A & \xrightarrow{1} & A. \end{array}$$

The 2-simplex on the left gives a homotopy through loops based at $A \xrightarrow{a} A$ between the loop given by the left-hand 1-simplex above and the loop given by the 1-simplex

$$\begin{array}{ccccc} A & \xrightarrow{a} & A & \xrightarrow{1} & A \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle a} \\ A & \xrightarrow{1} & A & \xrightarrow{a} & A. \end{array}$$

Similarly, the 2-simplex on the right gives a homotopy through loops based at $A \xrightarrow{a} A$ between this loop and the loop given by the right-hand 1-simplex above. $\square$

Proposition 2.3.4. *The canonical map*

$$\omega^q_{(A,M)} \to \pi_q T(A|K)$$

is an isomorphism, for $q \leq 2$, and a rational isomorphism, for all $q \geq 0$.

Proof. We consider the long exact sequence of homotopy groups associated with the sequence of Theorem 1.5.6,

$$T(k) \xrightarrow{i^!} T(A) \xrightarrow{j_*} T(A|K) \xrightarrow{\partial} \Sigma T(k),$$

and note that $i^!\colon \pi_q T(k) \to \pi_q T(A)$ is zero, if $q = 0, 1$. Indeed, for $q = 0$ this is a map from a torsion group to a torsion-free group, and for $q = 1$ the domain is isomorphic to the group Ω^1_k which vanishes since k is a perfect, [19, Lemma 5.5]. This proves the statement for $q = 0$. It also shows that the top sequence in the following diagram of A-modules and A-linear maps,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1 T(A) & \xrightarrow{j_*} & \pi_1 T(A|K) & \xrightarrow{\partial_*} & \pi_0 T(k) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Omega^1_A & \longrightarrow & \omega^1_{(A,M)} & \xrightarrow{\mathrm{res}} & k & \longrightarrow & 0, \end{array}$$

is exact. The lower sequence is the exact sequence of Proposition 2.2.2 and the vertical maps are the canonical maps. The left-hand square commutes since j_* preserves the differential. The commutativity of the right-hand square is equivalent to the statement that $\partial_*(d\log x) = v_K(x)$, for all $x \in M$. But this follows from the definition of the map $d\log$ in (2.3.2) and from the commutativity of the right-hand square in Addendum 1.5.7. Since the left- and right-hand vertical maps in the diagram are isomorphisms, so is the middle vertical map. This proves the statement for $q = 1$.

We next argue that the map of the statement is a rational isomorphism, for all $q \geq 0$. Since $\pi_* T(k)$ is torsion the long exact sequence associated with the cofibration sequence above shows that

$$j_*\colon \pi_* T(A) \otimes \mathbb{Q} \xrightarrow{\sim} \pi_* T(A|K) \otimes \mathbb{Q}$$

is an isomorphism. Moreover, the linearization map induces an isomorphism

$$l\colon \pi_* T(A) \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{HH}_*(A) \otimes \mathbb{Q},$$

and the right-hand side is canonically isomorphic to $\mathrm{HH}_*(K)$. It thus remains to prove that the canonical map $\Omega^*_K \to \mathrm{HH}_*(K)$ is an isomorphism. This in turn follows from [20] and the fact that K can be written as a filtered colimit of smooth $\mathbb{Q}$-algebras, [13, IV.17.5.1].

It remains to show that $\pi_2 T(A|K)$ is uniquely divisible. The structure of the p-adic homotopy groups $\pi_*(T(A),\mathbb{Z}_p)$ is known from [27, Th. 5.1]. (The assumption that the residue field is finite is not needed. For *op. cit.*, Propositions 5.3 and 5.4 and [1, Th. 7.1] show that the Bockstein spectral sequence converges strongly.) The result is that for $m>0$, $\pi_{2m}(T(A),\mathbb{Z}_p)$ vanishes and $\pi_{2m-1}(T(A),\mathbb{Z}_p)$ is isomorphic to $A/(m\phi_K'(\pi_K))$. The latter is a torsion group of bounded exponent. It follows that for $m>0$, $\pi_{2m}T(A)$ is a uniquely divisible group and $\pi_{2m-1}T(A)$ is the sum of a uniquely divisible group and the torsion group $\pi_{2m-1}(T(A),\mathbb{Z}_p)$. Since $\pi_1 T(k)$ is trivial, we see that $\pi_2 T(A|K)$ is uniquely divisible as stated. □

2.4. It follows from Proposition 2.3.1 that the homotopy groups with $\mathbb{Z}/p$-coefficients $\bar{\pi}_* T(A|K)$ form a log differential graded k-algebra. We now evaluate this log differential graded k-algebra and prove Theorem B of the introduction.

The proof of Theorem B is based on the calculation in [27, Ths. 4.4, 4.6] of the graded k-algebra $\bar{\pi}_*T(A)=\pi_*(T(A),\mathbb{Z}/p)$. The result, which we now recall, depends on whether p divides e_K or not. We consider the graded k-algebra

$$B = A/p\otimes\Lambda\{\alpha_1\}\otimes S\{\alpha_2\}$$

with the generators in the indicated degrees. Let $C\subset B$ be the subalgebra generated by all elements $a\alpha_1^{\varepsilon}\alpha_2^m$ for which $a\in\mathfrak{m}_K/pA$ or $\varepsilon=1$ or p divides m, and let $I\subset C$ be the ideal generated by all elements $a\alpha_1\alpha_2^{m-1}$ for which $a\in\mathfrak{m}_K^{e_K-1}/pA$ and m is prime to p. Then as graded k-algebras,

$$\bar{\pi}_*T(A)\cong\begin{cases} B, & \text{if } p \text{ divides } e_K,\\ C/I, & \text{if } p \text{ does not divide } e_K.\end{cases}$$

We note that, in the former case, the dimension of the k-vector space $\bar{\pi}_qT(A)$ is equal to e_K, for all $q\geq 0$. In the latter case, this dimension is equal to e_K, if q is congruent to either -1 or 0 modulo $2p$, and is equal to e_K-1, otherwise.

We also recall from [19, Th. 5.2, Cor. 5.5] that as a graded k-algebra,

$$\bar{\pi}_*T(k)=\Lambda\{\varepsilon\}\otimes S\{\sigma\},$$

with the generators ε and σ characterized as follows: the Bockstein takes ε to 1 and Connes' operator (2.1.2) takes ε to σ. It follows from the proof of [27, Ths. 4.4, 4.6] that the reduction map

$$i_*\colon \bar{\pi}_*T(A)\to\bar{\pi}_*T(k)$$

is induced from a k-algebra map $B \to \bar{\pi}_* T(k)$, which in degree zero is given by the reduction $A/p \to k$, and which takes the generators α_1 and α_2 to zero and a unit times σ, respectively.

Since the group $\pi_2 T(A|K)$ is uniquely divisible, by Proposition 2.3.4, the integral Bockstein induces an isomorphism

$$\beta\colon \bar{\pi}_2 T(A|K) \xrightarrow{\sim} {}_p\pi_1 T(A|K).$$

We define $\kappa \in \bar{\pi}_2 T(A|K)$ to be the class which corresponds to the generator $d\log(-p)$ on the right. (Note that $\kappa \in \bar{\pi}_2 T(\mathbb{Z}_p|\mathbb{Q}_p)$.) We now prove Theorem B of the introduction:

THEOREM 2.4.1. *There is a natural isomorphism of log differential graded rings*

$$\omega^*_{(A,M)} \otimes_{\mathbb{Z}} S_{\mathbb{Z}/p}\{\kappa\} \xrightarrow{\sim} \bar{\pi}_* T(A|K),$$

where $d\kappa = \kappa d\log(-p)$.

Proof. It is clear that there is a map of graded k-algebras as stated. We show that this is an isomorphism.

Suppose first that p divides e_K. We know from Proposition 2.3.4 that the map of the statement is an isomorphism in degrees $q \leq 1$. So it suffices to show that multiplication by κ induces an isomorphism

$$\kappa\colon \bar{\pi}_q T(A|K) \xrightarrow{\sim} \bar{\pi}_{q+2} T(A|K).$$

To this end, we consider the long-exact sequence associated with the cofibration sequence of Theorem 1.5.6,

$$\cdots \to \bar{\pi}_q T(k) \xrightarrow{i^!} \bar{\pi}_q T(A) \xrightarrow{j_*} \bar{\pi}_q T(A|K) \xrightarrow{\partial} \bar{\pi}_{q-1} T(k) \to \cdots.$$

This is a sequence of graded $\bar{\pi}_* T(A)$-modules, where $\bar{\pi}_* T(A|K)$ (resp. $\bar{\pi}_* T(k)$) is viewed as a graded $\bar{\pi}_* T(A)$-module via the map j_* (resp. i_*). We claim that the map j_* is an isomorphism for $q = 2$. Granting this for the moment, there exists $\tilde{\kappa} \in \bar{\pi}_2 T(A)$ such that $\kappa = j_*(\tilde{\kappa})$. And since $\bar{\pi}_2 T(A)$ and $\bar{\pi}_2 T(A|K)$ are both free A/p-modules of rank one, the class $\tilde{\kappa}$ is necessarily a generator. It follows that in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \bar{\pi}_q T(k) \arrow[r, "i^!"] \arrow[d, "\tilde{\kappa}", "\sim"'] & \bar{\pi}_q T(A) \arrow[r, "j_*"] \arrow[d, "\tilde{\kappa}", "\sim"'] & \bar{\pi}_q T(A|K) \arrow[r, "\partial"] \arrow[d, "\tilde{\kappa}"] & \bar{\pi}_{q-1} T(k) \arrow[r] \arrow[d, "\tilde{\kappa}", "\sim"'] & \cdots \\
\cdots \arrow[r] & \bar{\pi}_{q+2} T(k) \arrow[r, "i^!"] & \bar{\pi}_{q+2} T(A) \arrow[r, "j_*"] & \bar{\pi}_{q+2} T(A|K) \arrow[r, "\partial"] & \bar{\pi}_{q+1} T(k) \arrow[r] & \cdots,
\end{tikzcd}$$

two out of three of the vertical maps are isomorphisms. Hence, so are the remaining vertical maps. To prove the claim, we consider the diagram of

A/p-modules

$$\begin{array}{ccccc}
\bar{\pi}_2 T(A) & \xrightarrow[\sim]{\beta} & {}_p\pi_1 T(A) & \xleftarrow{\sim} & {}_p\Omega^1_A \\
\downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\
\bar{\pi}_2 T(A|K) & \xrightarrow[\sim]{\beta} & {}_p\pi_1 T(A|K) & \xleftarrow{\sim} & {}_p\omega^1_{(A,M)}.
\end{array}$$

The left-hand horizontal maps are isomorphisms since $\pi_2 T(A)$ and $\pi_2 T(A|K)$ are (uniquely) divisible. It follows from Proposition 2.3.4 that the right-hand horizontal maps are isomorphisms and that the right-hand vertical map is a monomorphism. But the domain and range of the latter are both k-vector spaces of dimension e_K. Hence, this map is an isomorphism. This proves the claim.

Suppose now that p does not divide e_K. Let L/K be a totally ramified extension such that p divides $e_{L/K}$, and let B be the integral closure of A in L. Then we have a commutative diagram

$$\begin{array}{ccc}
\omega^*_{(A,M_A)} \otimes S\{\kappa\} & \longrightarrow & \bar{\pi}_* T(A|K) \\
\downarrow & & \downarrow \\
\omega^*_{(B,M_B)} \otimes S\{\kappa\} & \xrightarrow{\sim} & \bar{\pi}_* T(B|\,L),
\end{array}$$

and the lower horizontal map is an isomorphism. It is easy to see that there exists L/K for which the left-hand vertical map is a monomorphism. For example, one can take $L = K[\pi_L]/(\pi_L^{e_{L/K}} + \pi_K(\pi_L + 1))$. Hence, the upper horizontal map is a monomorphism. The domain and range of this map are graded k-vector spaces concentrated in nonnegative degrees. The dimension of the domain is equal to e_K in each degree. Hence the dimension of the range is at least e_K in each degree. We can estimate the dimension of the range further by means of the exact sequence of k-vector spaces

$$\cdots \to \bar{\pi}_q T(k) \xrightarrow{i^!} \bar{\pi}_q T(A) \xrightarrow{j_*} \bar{\pi}_q T(A|K) \xrightarrow{\partial} \bar{\pi}_{q-1} T(k) \to \cdots .$$

The dimension of $\bar{\pi}_q T(A)$ is equal to e_K, if $q \equiv -1, 0 \pmod{2p}$, and is equal to $e_K - 1$, otherwise. The dimension of $\bar{\pi}_q T(k)$ is equal to one, for all $q \geq 0$. It follows that the dimension of $\bar{\pi}_q T(A|K)$ is equal to either e_K or $e_K + 1$, if $q \equiv -1, 0 \pmod{2p}$, and is equal to e_K otherwise. We argue that for $q \equiv -1, 0$, the dimension of $\bar{\pi}_q T(A|K)$ is equal to e_K. This happens if and only if for all $s \geq 0$, the map

$$i^! \colon \bar{\pi}_{2ps-1} T(k) \to \bar{\pi}_{2ps-1} T(A)$$

is nonzero. We show that the class $\pi_K^{e_K-1}\alpha_1\alpha_2^{ps-1}$ on the right is in the image of $i^!$, or equivalently, that it maps to zero under j_*. If $e_K > 1$, we can write

$$\pi_K^{e_K-1}\alpha_1\alpha_2^{ps-1} = \pi_K^{e_K-2}\alpha_1 \cdot \pi_K\alpha_2^{ps-1}.$$

The image of this class under j_* is equal to a unit in A/p times the class

$$\pi_K^{e_K-1} d\log \pi_K \cdot \pi_K \kappa^{ps-1}.$$

But this class is in the image of the ring homomorphism

$$\omega^*_{(A,M)} \otimes_{\mathbb{Z}} S_{\mathbb{Z}/p}\{\kappa\} \to \bar{\pi}_* T(A|K)$$

and the product is equal to zero on the left. Hence $j_*(\pi_K^{e_K-1}\alpha_1\alpha_2^{ps-1})$ is equal to zero. Finally, in the unramified case we choose a totally ramified extension K/K_0 such that p does not divide e_K and consider the diagram

$$\begin{array}{ccc}
\bar{\pi}_{2ps-1}T(k) & \xrightarrow{i^!} & \bar{\pi}_{2ps-1}T(W(k)) \\
\Big\downarrow e_K & & \Big\downarrow \\
\bar{\pi}_{2ps-1}T(k) & \xrightarrow{i^!} & \bar{\pi}_{2ps-1}T(A).
\end{array}$$

We have just proved that the lower horizontal map is a monomorphism, for all $s \geq 0$. And the left-hand vertical map is an isomorphism since e_K is prime to p. Hence the top horizontal map is a monomorphism. We have proved that the map of the statement is an isomorphism of graded k-algebras for all K. In particular, the class $d\kappa$ on the right is the image of an element on the left. To determine this element, we may assume that $K = \mathbb{Q}_p$. In the diagram

$$\begin{array}{ccc}
\bar{\pi}_3 T(\mathbb{Z}_p|\mathbb{Q}_p) & \xrightarrow{\partial} & \bar{\pi}_2 T(\mathbb{F}_p) \\
\Big\uparrow d & & \Big\uparrow d \\
\bar{\pi}_2 T(\mathbb{Z}_p|\mathbb{Q}_p) & \xrightarrow{\partial} & \bar{\pi}_1 T(\mathbb{F}_p)
\end{array}$$

the horizontal maps and the right-hand vertical map are isomorphisms. Hence also the left-hand vertical map is an isomorphism. This shows that $d\kappa = u\kappa d\log(-p)$ with $u \in \mathbb{F}_p^\times$. We show in remark 5.3.3 below that in fact $u = 1$. □

Remark 2.4.2. An argument similar to [27, §5] shows that for $m > 0$, there exists a noncanonical isomorphism

$$\pi_{2m-1}(T(A|K), \mathbb{Z}_p) \cong A/(m\pi_K \phi_K'(\pi_K))$$

and that $\pi_{2m}(T(A|K), \mathbb{Z}_p)$ vanishes. It would be interesting to give a functorial description of the left-hand group analogous to Proposition 2.3.4.

Let L/K be a Galois extension with Galois group $G_{L/K}$. The descent problem for topological Hochschild homology asks under what conditions the canonical map

$$T(A|K) \to \mathbb{H}^{\bullet}(G_{L/K}, T(B|\,L))$$

is a weak equivalence. It is not hard to see from Theorem 2.4.1 that this is false in general, e.g. for a cyclotomic extension $\mathbb{Q}_p(\mu_{p^n})/\mathbb{Q}_p$ with $n > 1$. However:

THEOREM 2.4.3. *Let L/K be a finite and tamely ramified Galois extension. Then the canonical map induces an isomorphism*

$$\bar{\pi}_* T(A|K) \xrightarrow{\sim} \bar{\pi}_* \mathbb{H}^\bullet(G_{L/K}, T(B|\, L)).$$

Proof. It will suffice to show that for all $t \geq 0$, the $G_{L/K}$-module $\bar{\pi}_t T(B|\, L)$ is isomorphic to B/p. Indeed, a classical theorem of Noether, [9, I.3, Th. 3], states that B is isomorphic to $A[G_{L/K}]$ as a $G_{L/K}$-module, if and only if L/K is tamely ramified. Hence, the spectral sequence

$$E^2_{s,t} = H^{-s}(G_{L/K}, \bar{\pi}_t T(B|\, L)) \Rightarrow \bar{\pi}_{s+t} \mathbb{H}^\bullet(G_{L/K}, T(B|\, L))$$

collapses to yield the isomorphism of the statement.

We use Theorem 2.4.1 to get the natural isomorphisms

$$\kappa^i \colon \bar{\pi}_\varepsilon T(B|\, L) \xrightarrow{\sim} \bar{\pi}_{2i+\varepsilon} T(B|\, L).$$

Hence, we only need to consider $\bar{\pi}_0 T(B|\, L)$ and $\bar{\pi}_1 T(B|\, L)$. The former is naturally isomorphic to B/p regardless of whether L/K is tamely ramified or not, and the latter is naturally isomorphic to $\omega^1_{(B,M_B)}/p$. We have from Lemma 2.2.3 that

$$\omega^1_{(A,M_A)}/p = A/p \cdot d\log \pi_K,$$

and since L/K is tamely ramified, Lemma 2.2.6 shows that

$$\omega^1_{(B,M_B)}/p = B/p \cdot d\log \pi_K.$$

Hence, also $\omega^1_{(B,M_B)}/p$ is is isomorphic to B/p as a $G_{L/K}$-module. □

3. The de Rham-Witt complex and $\mathrm{TR}^\bullet_*(A|K;p)$

3.1. In this paragraph, we evaluate the integral homotopy groups $\mathrm{TR}^\bullet_i(A|K;p)$, for $i \leq 2$. We first consider Witt vectors, see e.g. [35, Appendix].

The ring $W_n(R)$ of Witt vectors of length n in R is the set of n-tuples in R but with a new ring structure characterized by the requirement that the "ghost" map

$$w \colon W_n(R) \to R^n,$$

which to the vector $(a_0, , \dots, a_{n-1})$ associates the sequence $(w_0, \dots, w_{n-1})$ with

$$w_s = a_0^{p^s} + p a_1^{p^{s-1}} + \dots + p^s a_s,$$

be a natural transformation of functors from rings to rings. If R has no p-torsion then the ghost map is injective. If, in addition, there exists a ring endomorphism $\phi \colon R \to R$ such that $a^p \equiv \phi(a) \pmod{pR}$, then a sequence $(w_0, \dots, w_{n-1})$ is in the image if and only if $w_s \equiv \phi(w_{s-1}) \pmod{p^s R}$, for all

$0 < s < n$. If $R = \mathbb{Z}[X_\alpha]$, the ring homomorphism which maps X_α to X_α^p is such an endomorphism. Let

$$\underline{}_n\colon R \to W_n(R)$$

be the multiplicative section given by $\underline{a}_n = (a, 0, \dots, 0)$.

Lemma 3.1.1. *If p is odd then $V(1) \equiv \underline{-p}_n$ and $\underline{-1}_n \equiv -1$ modulo $pW_n(R)$.*

Proof. By naturality, we may assume that $R = \mathbb{Z}$. Now,

$$w(\underline{p}_n + V(\underline{1})) = p(1, 1 + p^{p-1}, 1 + p^{p^2-1}, 1 + p^{p^3-1}, \dots),$$

and therefore it is enough to show that the sequence

$$(1, 1 + p^{p-1}, 1 + p^{p^2-1}, \dots, 1 + p^{p^{n-1}-1})$$

is in the image of the ghost map. The identity $\phi\colon \mathbb{Z} \to \mathbb{Z}$ has the property that $a^p \equiv \phi(a) \pmod{p\mathbb{Z}}$. Hence, this sequence is in the image of the ghost map if and only if for all $1 < s < n$,

$$1 + p^{p^s-1} \equiv 1 + p^{p^{s-1}-1} \pmod{p^s}.$$

This is true, if p is odd, but fails for $p = 2$ and $s = 2$. The second congruence of the statement is proved in a similar manner. □

In general, $\underline{(x+y)}_n$ and $\underline{x}_n + \underline{y}_n$ are not equivalent modulo $pW_n(A)$. However, we have the following:

Lemma 3.1.2. *For all $x, y \in R$,*

$$\underline{(x+y)}_n^p \equiv (\underline{x}_n + \underline{y}_n)^p \equiv \underline{x}_n^p + \underline{y}_n^p$$

modulo $pW_n(R)$.

Proof. The right-hand congruence is valid in any ring. To prove the left-hand congruence, we place ourselves in the universal case $R = \mathbb{Z}[x, y]$. The ghost map

$$w\colon W_n(R) \to R^n$$

is an injection and maps the vector $\underline{x}_n^p + \underline{y}_n^p - (\underline{x+y}_n)^p$ to the sequence

$$(x^p + y^p - (x+y)^p, \dots, x^{p^n} + y^{p^n} - (x+y)^{p^n}).$$

As an element of R^n this is divisible by p. We must show that the quotient is in the image of the ghost map. By the criterion recalled above, we must show that

$$(x^{p^{s+1}} + y^{p^{s+1}} - (x+y)^{p^{s+1}})/p \equiv (x^{p^{s+1}} + y^{p^{s+1}} - (x^p + y^p)^{p^s})/p \pmod{p^s},$$

or equivalently, that

$$(x+y)^{p^{s+1}} \equiv (x^p+y^p)^{p^s} \pmod{p^{s+1}}.$$

But this follows from

$$(x+y)^p \equiv x^p + y^p \pmod{p}$$

and from the fact, valid in any commutative ring, that $a \equiv b \pmod{p}$ implies $a^{p^s} \equiv b^{p^s} \pmod{p^{s+1}}$. Indeed, one easily sees that $a \equiv b \pmod{p^k}$ implies that $a^p \equiv b^p \pmod{p^{k+1}}$, and the desired formula then follows by simple induction. □

Now, from Lemma 3.1.2, for every ring R, the map

$$R \to \bar{W}_n(R) = W_n(R)/p,$$

which takes x to the class of $\underline{x}_n^p$, is a ring homomorphism. Let A be a complete discrete valuation ring with quotient field K and perfect residue field k of mixed characteristic $(0,p)$. We recall from (2.2.1) that there is a unique ring homomorphism $f\colon W(k) \to A$ such that the induced map of residue fields is the identity homomorphism. Hence, we have a ring homomorphism

$$\rho_n\colon k \to \bar{W}_n(A) \tag{3.1.3}$$

which to x assigns $\underline{f(\widetilde{x^{1/p}})}_n^{\,p} + pW_n(A)$. Here $\widetilde{x^{1/p}} \in W(k)$ is any element whose residue class modulo p is the unique p-th root of x. We will always view $\bar{W}_n(A)$ as a k-algebra via the map ρ_n. We note that

$$R(\rho_n(x)) = \rho_{n-1}(x), \qquad F(\rho_n(x)) = \rho_{n-1}(x^p).$$

Let $\pi = \pi_K$ be a uniformizer with minimal polynomial $x^{e_K} + p\theta_K(x)$. We introduce the modified Verschiebung

$$V_\pi\colon \bar{W}_{n-1}(A) \to \bar{W}_n(A), \quad V_\pi(a) = \theta_K(\underline{\pi}_n)V(a), \tag{3.1.4}$$

where $\theta_K(\underline{\pi}_n)$ is the image of $\theta_K(x)$ under the k-algebra map $k[x] \to \bar{W}_n(A)$ which to x assigns the class of $\underline{\pi}_n$. The composite FV_π is zero modulo p.

PROPOSITION 3.1.5. *Suppose that p is odd. Then the k-algebra $\bar{W}_n(A)$ is generated by the elements $V_\pi^s(\underline{\pi}^i)$ with $0 \le s < n$ and $i \ge 0$ subject to the relations*

$$\begin{aligned} V_\pi^s(\underline{\pi}^i)\cdot V_\pi^t(\underline{\pi}^j) &= \begin{cases} V_\pi^t(\underline{\pi}^{p^t i + j}), & \text{if } 0 = s \le t < n, \\ 0, & \text{if } 0 < s \le t < n, \end{cases} \\ V_\pi^s(\underline{\pi}^{e_K + i}) &= V_\pi^{s+1}(\underline{\pi}^{pi}). \end{aligned}$$

Proof. The k-vector space $\bar{W}_n(A)$ is generated by $V^s(\underline{\pi}^i)$ with $0 \le s < n$ and $i \ge 0$. Indeed, write $a \in A$ as $a = x_d\pi^d + \cdots + x_0$ with $x_i \in W(k)$. Then

$$V^s(\underline{a}) \equiv V^s(\underline{x_d\pi^d}) + \cdots + V^s(\underline{x_0}) \equiv V^s(\rho_{n-s}(\bar{x}_d)\underline{\pi}^d) + \cdots + V^s(\rho_{n-s}(\bar{x}_0))$$

modulo $V^{s+1}\bar{W}_n(A)$, and

$$V^s(\rho_{n-s}(\bar{x}_i)\underline{\pi}^i) = \rho_n(\bar{x}^{1/p^s})V^s(\underline{\pi}^i).$$

Since $\theta_K(\underline{\pi})$ is a unit, we instead can use the elements $V_\pi^s(\underline{\pi}^i)$ as generators. In general, for $s \le t$,

$$V_\pi^s(\underline{\pi}^i) \cdot V_\pi^t(\underline{\pi}^j) = V_\pi^t(F^t V_\pi^s(\underline{\pi}^i) \cdot \underline{\pi}^j)$$

from which the first relation follows. Next, Lemmas 3.1.1 and 3.1.2 show that

$$\begin{aligned} \underline{\pi}^{e_K} &= \underline{-p} \cdot \underline{\theta_K(\pi)} \equiv V(1)\underline{\theta_K(\pi)} = V((\underline{\theta_K(\pi)})^p) \\ &\equiv V(\theta_K^{(1)}(\underline{\pi}^p)) = V(1)\theta_K(\underline{\pi}) = V_\pi(1), \end{aligned}$$

where $\theta_K^{(1)}(x)$ denotes the image of $\theta_K(x)$ under the automorphism of $k[x]$ induced by the Frobenius of k. The second relation is an immediate consequence. It remains to prove that there are no further relations. The sequences

$$0 \to A/p \xrightarrow{V^{n-1}} \bar{W}_n(A) \xrightarrow{R} \bar{W}_{n-1}(A) \to 0$$

are exact, since $W_n(A)$ is torsion-free, and show that $\bar{W}_n(A)$ is an ne_K-dimensional k-vector space. The relations of the statement imply that

$$\mathrm{gr}_V^s \bar{W}_n(A) = k\Big\{V_\pi^s(\underline{\pi}^i) \;\Big|\; 0 \le i < e_K\Big\},$$

which is an e_K-dimensional k-vector space. Thus there can be no further relations among the $V_\pi^s(\underline{\pi}^i)$. □

3.2. A pre-log structure $\alpha\colon M \to R$ on a ring R induces one on $W_n(R)$ upon composition with the multiplicative section $\underline{}_n\colon R \to W_n(R)$. We write $(W_n(R), M)$ for this log ring. We now assume that p is odd and that R is a $\mathbb{Z}_{(p)}$-algebra.

Definition 3.2.1. A log Witt complex over (R, M) consists of:

(i) a pro-log differential graded ring $(E_\bullet^*, M_E)$ together with a map of pro-log rings $\lambda\colon (W_\bullet(R), M) \to (E_\bullet^0, M_E)$;

(ii) a map of pro-log graded rings

$$F\colon E_n^* \to E_{n-1}^*,$$

such that $\lambda F = F\lambda$ and such that

$$\begin{aligned} F d\log_n a &= d\log_{n-1} a, && \text{for all } a \in M, \\ F d\underline{a}_n &= \underline{a}_{n-1}^{p-1} d\underline{a}_{n-1}, && \text{for all } a \in R; \end{aligned}$$

(iii) a map of pro-graded modules over the pro-graded ring $E^*_{\boldsymbol{\cdot}}$,

$$V\colon F^*E^*_n \to E^*_{n+1},$$

such that $\lambda V = V\lambda$, $FV = p$ and $FdV = d$.

A map of log Witt complexes over (R, M) is a map of pro-log differential graded rings which commutes with the maps λ, F and V.

The following relations are valid in any log Witt complex:

$$dF = pFd, \quad Vd = pdV, \quad V(xdy) = V(x)dV(y).$$

Indeed, $V(xdy) = V(xFdV(y)) = V(x)dV(y)$, and

$$\begin{aligned} dF(x) &= FdVF(x) = Fd(V(1)x) = FdV(1)F(x) + FV(1)F(dx) \\ &= d(1)F(x) + pFd(x) = pFd(x), \\ Vd(x) &= V(1)dV(x) = d(V(1)V(x)) - dV(1)V(x) \\ &= dV(xFV(1) - V(xd(1))) = pdV(x). \end{aligned}$$

Proposition 3.2.2. *The category of log Witt complexes over (R, M) has an initial object $W_{\boldsymbol{\cdot}}\,\omega^*_{(R,M)}$. Moreover, the canonical map is surjective*:

$$\lambda\colon \omega^*_{(W_{\boldsymbol{\cdot}}(R),M)} \twoheadrightarrow W_{\boldsymbol{\cdot}}\,\omega^*_{(R,M)}.$$

Proof. This is a fairly straightforward application of the Freyd adjoint functor theorem, [29, p. 116]. For a detailed proof, we refer the reader to [17, §1]. □

We note that $W_{\boldsymbol{\cdot}}\,\omega^0_{(R,M)} = W_{\boldsymbol{\cdot}}(R)$. For we may consider $(W_{\boldsymbol{\cdot}}(R), M)$ a log Witt complex concentrated in degree zero. Moreover, from [17, Th. D] we have:

Addendum 3.2.3. *The canonical map is an isomorphism*:

$$\lambda\colon \omega^*_{(R,M)} \xrightarrow{\sim} W_1\,\omega^*_{(R,M)}.$$

The filtration of a log Witt complex by the differential graded ideals

$$\operatorname{Fil}^s E^i_n = V^s E^i_{n-s} + dV^s E^{i-1}_{n-s} \subset E^i_n$$

is called the standard filtration. It satisfies

$$\begin{aligned} F(\operatorname{Fil}^s E^i_n) &\subset \operatorname{Fil}^{s-1} E^i_{n-1}, \\ V(\operatorname{Fil}^s E^i_n) &\subset \operatorname{Fil}^{s+1} E^i_{n+1}, \end{aligned}$$

but in general is not multiplicative.

Lemma 3.2.4. *The restriction map induces an isomorphism*

$$W_n\,\omega^i_{(R,M)}/\operatorname{Fil}^s W_n\,\omega^i_{(R,M)} \xrightarrow{\sim} W_s\,\omega^i_{(R,M)}.$$

Proof. For a fixed value of $n - s$, the filtration quotients

$$'W_s\,\omega^i_{(R,M)} = W_n\,\omega^i_{(R,M)}/\operatorname{Fil}^s W_n\,\omega^i_{(R,M)}$$

form a log Witt complex over (R, M). We show that it has the universal property. Let $(E^*_\bullet, M_E)$ be a log Witt complex over (R, M). Then there exists a map of log Witt complexes over (R, M):

$$'W_\bullet\,\omega^*_{(R,M)} \to E^*_\bullet.$$

Indeed, the standard filtration is natural, so we have maps

$$W_n\,\omega^i_{(R,M)}/\operatorname{Fil}^s W_n\,\omega^i_{(R,M)} \to E^i_n/\operatorname{Fil}^s E^i_n \to E^i_s,$$

where the right-hand map is induced from the restriction maps in $E^*_\bullet$. We must show that this map of log Witt complexes is unique. To prove this, it will suffice to show that the canonical map

$$\omega^i_{(W_s(R),M)} \to {}'W_n\,\omega^i_{(R,M)}$$

is surjective. But this follows from the commutativity of the diagram

$$\begin{array}{ccc} \omega^i_{(W_n(R),M)} & \twoheadrightarrow & W_n\,\omega^i_{(R,M)} \\ \big\downarrow\!\!\downarrow & & \big\downarrow\!\!\downarrow \\ \omega^i_{(W_s(R),M)} & \longrightarrow & 'W_s\,\omega^i_{(R,M)} \end{array}$$

since the top horizontal and right-hand vertical maps are surjective. □

We define a map $F^{n-1}d\colon W_n(R) \to \omega^1_{(R,M)}$ by the formula

$$F^{n-1}d(a) = a_0^{p^{n-1}-1}da_0 + a_1^{p^{n-2}-1}da_1 + \cdots + da_{n-1},$$

where $a = (a_0, \ldots, a_{n-1})$. One easily verifies that $F^{n-1}d$ is a derivation of $W_n(R)$ into the $W_n(R)$-module $(F^{n-1})^*\omega^1_{(R,M)}$ and that the following relation holds:

$$dF^{n-1} = p^{n-1}F^{n-1}d.$$

It follows immediately from the derivation property that the formula

$$a \cdot (\omega_1, \omega_2) = (F^{n-1}(a)\omega_1, F^{n-1}(a)\omega_2 - F^{n-1}da \cdot \omega_1)$$

defines a $W_n(R)$-module structure on $\omega^{i-1}_{(R,M)} \oplus \omega^i_{(R,M)}$. And the relation shows that

$$(F^{n-1})^*\omega^{i-1}_{(R,M)} \to \omega^{i-1}_{(R,M)} \oplus \omega^i_{(R,M)}, \quad \omega \mapsto (p^{n-1}\omega, -d\omega),$$

is a map of $W_n(R)$-modules. We let ${}_hW_n\,\omega^i_{(R,M)}$ be the quotient $W_n(R)$-module. This definition is motivated by Lemma 3.3.3 below.

Lemma 3.2.5. *There is a natural exact sequence of $W_n(R)$-modules*

$$\begin{aligned}&(F^{n-1})^*{}_{p^{n-1}}\omega^{i-1}_{(R,M)} \xrightarrow{d} (F^{n-1})^*\omega^i_{(R,M)}\\ &\xrightarrow{\iota_2} {}_hW_n\,\omega^i_{(R,M)} \xrightarrow{\mathrm{pr}_1} (F^{n-1})^*(\omega^{i-1}_{(R,M)}/p^{n-1}) \to 0.\end{aligned}$$

Proof. Indeed, as an abelian group, ${}_hW_n\,\omega^i_{(R,M)}$ is equal to the push out

$$\begin{array}{ccc}\omega^{i-1}_{(R,M)} & \xrightarrow{d} & \omega^i_{(R,M)}\\ \big\downarrow{\scriptstyle p^{n-1}} & & \big\downarrow{\scriptstyle \iota_2}\\ \omega^{i-1}_{(R,M)} & \xrightarrow{\iota_1} & {}_hW_n\,\omega^i_{(R,M)}\end{array}$$

so the underlying sequence of abelian groups is exact. One readily verifies that the various maps are $W_n(R)$-linear. □

Proposition 3.2.6. *For any log ring (R,M), there is a natural exact sequence of $W_n(R)$-modules,*

$${}_hW_n\,\omega^i_{(R,M)} \xrightarrow{N} W_n\,\omega^i_{(R,M)} \xrightarrow{R} W_{n-1}\,\omega^i_{(R,M)} \to 0,$$

where $N(\omega_1,\omega_2) = dV^{n-1}\lambda(\omega_1) + V^{n-1}\lambda(\omega_2)$.

Proof. It follows immediately from Definition 3.2.1 that for all $a \in W_n(R)$,

$$\lambda(F^{n-1}da) = F^{n-1}d\lambda(a),$$

and hence N is $W_n(R)$-linear. Since the image of N is equal to $\mathrm{Fil}^{n-1}\,W_n\,\omega^i_{(R,M)}$, the statement follows from Lemma 3.2.4. □

Corollary 3.2.7. *Let A be a complete discrete valuation ring of mixed characteristic $(0,p)$ with perfect residue field, and let $\alpha\colon M \to A$ be the canonical log structure. Then for all $n \geq 1$ and $i \geq 2$, $W_n\,\omega^i_{(A,M)}$ is a uniquely divisible group.*

Proof. Lemma 2.2.4 shows that $\omega^i_{(A,M)}$ is uniquely divisible, if $i \geq 2$. It follows that ${}_hW_n\,\omega^i_{(A,M)}$ is uniquely divisible, if $i \geq 3$, and an induction argument based on Proposition 3.2.6 shows that so is $W_n\,\omega^i_{(A,M)}$. The group ${}_hW_n\,\omega^2_{(A,M)}$ is a direct sum of a uniquely divisible group and the group $\omega^1_{(A,M)}/p^{n-1}$. Hence $W_n\,\omega^2_{(A,M)}$ is a direct sum of a uniquely divisible group and a finitely generated torsion $W(k)$-module. It is therefore enough to show that the modulo p reduction $\bar{W}_n\,\omega^2_{(A,M)}$ is trivial. Inductively, it suffices to show that the map

$$dV^{n-1}\colon \bar{\omega}^1_{(A,M)} \to \bar{W}_n\omega^2_{(A,M)}$$

is trivial. The map is k-linear, and the domain is generated as a k-vector space by the elements $\pi_K^i d\log\pi_K$ with $0 \le i < e_K$. Now the relation

$$\underline{\pi}_n^{e_K} + \theta_K(\underline{\pi}_n)V(1),$$

valid in $\bar{W}_n(A)$, shows that $V^{n-1}(\pi_K^i d\log\pi_K) = V^{n-1}(\pi_K^i)d\log_n\pi_K$ is either trivial or contained in the span of elements of the form $\underline{\pi_K}_n^j\, d\log_n\pi_K$. But these elements have vanishing differential. □

3.3. We refer the reader to [17, §2] for a fuller discussion of the following result.

PROPOSITION 3.3.1. *The homotopy groups* $\mathrm{TR}_*^{\bullet}(A|K;p)$ *form a log Witt complex over* (A, M), *provided that* p *is odd. In particular*, *there is a canonical map*

$$W_{\bullet}\,\omega_{(A,M)}^* \to \mathrm{TR}_*^{\bullet}(A|K;p).$$

Proof. We recall from Proposition 2.3.1 above that for all $n \ge 1$, the homotopy groups $\mathrm{TR}_*^n(A|K;p)$ form a log differential graded ring whose underlying log ring is $(W_n(R), M)$. The relation that for all $a \in M$,

$$F d\log_n a = d\log_{n-1} a,$$

is immediate from the definition of the maps F and $d\log_n$, and the remaining relations are proved in [19, Lemma 3.3] and [16, Lemmas 1.5.1 and 1.5.6]. □

The homotopy groups of the homotopy orbit spectra,

$${}_h\mathrm{TR}_*^n(A|K;p) = \pi_*(\mathbb{H}_{\bullet}(C_{p^{n-1}}, T(A|K))),$$

are differential graded modules over $\mathrm{TR}_*^n(A|K;p)$, and there are $\mathrm{TR}_*^n(A|K;p)$-linear maps

$$\begin{aligned} F&: {}_h\mathrm{TR}_*^n(A|K;p) \to F^*({}_h\mathrm{TR}_*^{n-1}(A|K;p)),\\ V&: F^*({}_h\mathrm{TR}_*^{n-1}(A|K;p)) \to {}_h\mathrm{TR}_*^n(A|K;p), \end{aligned}$$

which satisfy that $FdV = d$ and $FV = p$. Moreover, there is a natural spectral sequence of $W_n(A)$-modules,

$$E_{s,t}^2 = H_s(C_{p^{n-1}}, (F^{n-1})^*\,\pi_t T(A|K)) \Rightarrow {}_h\mathrm{TR}_{s+t}^n(A|K;p). \tag{3.3.2}$$

The reader is referred to [16, §1] and [19, §5] for proofs of these statements.

LEMMA 3.3.3. *Let* $\iota\colon \omega_{(A,M)}^i \to \pi_i T(A|K)$ *be the canonical map. Then the map*

$$\begin{array}{rcl} {}_hW_n\,\omega_{(A,M)}^i & \to & {}_h\mathrm{TR}_i^n(A|K;p),\\ (\omega_1,\omega_2) & \mapsto & dV^{n-1}\iota(\omega_1) + V^{n-1}\iota(\omega_2), \end{array}$$

is a map of $W_n(A)$-modules. It is an isomorphism for $i \leq 1$, and for $i = 2$, there is an exact sequence

$$(F^{n-1})^*(A/p^{n-1}) \to {}_hW_n\,\omega^2_{(A,M)} \to {}_h\mathrm{TR}^n_2(A|K;p) \to 0,$$

where the map on the left takes a to $(da, 0)$.

Proof. If $a \in W_n(A)$, $\omega_1 \in \omega^{i-1}_{(A,M)}$ and $\omega_2 \in \omega^i_{(A,M)}$, then

$$\begin{aligned} a \cdot dV^{n-1}\iota(\omega_1) &= d(a \cdot V^{n-1}\iota(\omega_1)) - da \cdot V^{n-1}\iota(\omega_1) \\ &= dV^{n-1}(F^{n-1}a \cdot \iota(\omega_1)) - V^{n-1}(F^{n-1}da \cdot \iota(\omega_1)) \\ &= dV^{n-1}\iota(F^{n-1}a \cdot \omega_1) - V^{n-1}\iota(F^{n-1}da \cdot \omega_1)), \\ a \cdot V^{n-1}\iota(\omega_2) &= V^{n-1}(F^{n-1}a \cdot \iota(\omega_2)) \\ &= V^{n-1}\iota(F^{n-1}a \cdot \omega_2), \end{aligned}$$

which shows that the map of the statement is indeed a map of $W_n(A)$-modules. The map ι is an isomorphism for $i \leq 2$. So the spectral sequence gives an isomorphism of $W_n(A)$-modules

$$\iota_0 \colon (F^{n-1})^*A \xrightarrow{\sim} {}_h\mathrm{TR}^n_0(A|K;p)$$

and a natural exact sequence of $W_n(A)$-modules

$$0 \to (F^{n-1})^*\omega^1_{(A,M)} \xrightarrow{\iota_1} {}_h\mathrm{TR}^n_1(A|K;p) \to (F^{n-1})^*(A/p^{n-1}) \to 0.$$

The sequence of Lemma 3.2.5 maps to the sequence above, and the map of the left-hand terms is an isomorphism. It remains to show that the same holds for the map of the right-hand terms. This map is induced from the composite

$$A \to {}_hW_n\,\omega^1_{(A,M)} \to {}_h\,\mathrm{TR}^n_1(A|K;p) \to A/p^{n-1}$$

which in turn may be identified with the map

$$H_0(C_{p^{n-1}}, A) \to H_1(C_{p^{n-1}}, A)$$

given by multiplication by the fundamental class $[\mathbb{T}/C_{p^{n-1}}]$. This map is an epimorphism with kernel $p^{n-1}A$, and the lemma follows for $i = 1$. The statement for $i = 2$ is proved in a similar manner, using the spectral sequence in total degree ≤ 3 and Proposition 4.4.3 below. □

Remark 3.3.4. For $i \leq 1$, the proof above does not use the fact that A is a discrete valuation ring beyond the definition of $T(A|K)$. In effect, the same proof gives an isomorphism

$${}_hW_n\,\Omega^1_R \xrightarrow{\sim} \pi_1\mathbb{H}_\bullet(C_{p^{n-1}}, T(R)),$$

for any $\mathbb{Z}_{(p)}$-algebra R.

Since $\omega^2_{(A,M)}$ is a uniquely divisible group, by Lemma 2.2.4, the spectral sequence (3.3.2) gives an exact sequence of $W_n(A)$-modules

$$(F^{n-1})^*(A/p^{n-1}) \xrightarrow{d} (F^{n-1})^*(\omega^1_{(A,M)}/p^{n-1}) \to {}_h\mathrm{TR}^n_2(A|K;p,\mathbb{Z}_p) \to 0,$$

and d is $W_n(A)$-linear since $dF^{n-1} = p^{n-1}F^{n-1}d$. If π_K is a uniformizer, then $d\log\pi_K$ represents a class in the cokernel. We denote this class by $[d\log\pi_K]_n$.

LEMMA 3.3.5. *The map of $W_n(A)$-modules*

$$F\colon {}_h\mathrm{TR}^n_2(A|K;p,\mathbb{Z}_p) \to {}_h\mathrm{TR}^{n-1}_2(A|K;p,\mathbb{Z}_p)$$

is a surjection whose kernel is generated by $p^{n-2}[d\log\pi_K]_n$.

Proof. The exact sequence above shows that the map of the statement is a surjection and that the kernel is a quotient of the cokernel of the following map:

$$(F^{n-1})^*(p^{n-2}A/p^{n-1}A) \xrightarrow{d} (F^{n-1})^*(p^{n-2}\omega^1_{(A,M)}/p^{n-1}\omega^1_{(A,M)}).$$

Hence, it suffices to show that this cokernel is generated by $p^{n-2}[d\log\pi_K]_n$. We consider the polynomial ring $P = W(k)[x]$ with the pre-log structure $\alpha\colon \mathbb{N}_0 \to P$ given by $\alpha(i) = x^i$. The map of $W(k)$-algebras $\varepsilon\colon P \to A$, $\varepsilon(x) = \pi_K$, preserves the pre-log structure and induces a surjection $\omega^1_{(P,\mathbb{N}_0)} \twoheadrightarrow \omega^1_{(A,M)}$. It follows that the map $p^i\omega^1_{(P,\mathbb{N}_0)} \twoheadrightarrow p^i\omega^1_{(A,M)}$ is a surjection for $i \geq 0$, and therefore it will be enough to show that the cokernel of the map

$$(F^{n-1})^*(p^{n-2}P/p^{n-1}P) \xrightarrow{d} (F^{n-1})^*(p^{n-2}\omega^1_{(P,\mathbb{N}_0)}/p^{n-1}\omega^1_{(P,\mathbb{N}_0)})$$

is generated as a $W_n(P)$-module by the canonical image of $p^{n-2}d\log x$. Now as a P-module, the quotient $p^{n-2}\omega^1_{(P,\mathbb{N}_0)}/p^{n-1}\omega^1_{(P,\mathbb{N}_0)}$ is generated by $p^{n-2}d\log x$, and hence the $W_n(P)$-module $(F^{n-1})^*(p^{n-2}\omega^1_{(P,\mathbb{N}_0)}/p^{n-1}\omega^1_{(P,\mathbb{N}_0)})$ is generated by the elements $p^{n-2}d\log x$ and $p^{n-2}x^{p^i}d\log x$, $0 \leq i < n-1$. But the last $n-1$ generators are all in the image of the map d:

$$p^{n-2}x^{p^i}d\log x = p^{n-2-i}d(x^{p^i}).$$

Hence the cokernel of d is generated by $p^{n-2}d\log x$, and the lemma follows. $\square$

PROPOSITION 3.3.6. *The sequences*

$$0 \to {}_h\mathrm{TR}^n_i(A|K;p) \xrightarrow{N} \mathrm{TR}^n_i(A|K;p) \xrightarrow{R} \mathrm{TR}^{n-1}_i(A|K;p) \to 0$$

are exact for $i \leq 1$, and $\mathrm{TR}^n_2(A|K;p)$ is uniquely divisible.

Proof. The statement for $i = 0$ is [19, Prop. 3.3] and for $i = 1$ is equivalent to the statement that the norm map is injective. The corresponding sequence of maximal uniquely divisible subgroups is exact, since $F^{n-1} \circ N$ is injective on this part. Hence, it suffices to show that $\mathrm{TR}^{n-1}_2(A|K;p)$ is uniquely divisible.

We show by induction on $m \geq 1$ that $\mathrm{TR}_2^m(A|K;p)$ is uniquely divisible, or equivalently, that $\mathrm{TR}_2^m(A|K;p,\mathbb{Z}_p)$ vanishes. The basic case $m=1$ follows from Proposition 2.3.4 and Lemma 2.2.4. In the induction step, we show that

$$\partial_{K,m}\colon \mathrm{TR}_3^{m-1}(A|K;p,\mathbb{Z}_p) \to {}_h\mathrm{TR}_2^m(A|K;p,\mathbb{Z}_p)$$

is surjective. We first consider the case $m=2$. In the diagram of $W_2(A)$-modules

$$\begin{array}{ccc}
\mathrm{TR}_3^1(A|K;p,\mathbb{Z}_p) & \xrightarrow{\partial_{K,2}} & {}_h\mathrm{TR}_2^2(A|K;p,\mathbb{Z}_p) \\
\big\downarrow \delta & & \big\downarrow \delta \\
\mathrm{TR}_2^1(k;p) & \xrightarrow{\partial_k} & {}_h\mathrm{TR}_1^2(k;p),
\end{array}$$

the lower horizontal map and the left-hand vertical map are both surjections. Indeed, for the former, this was proved in [19, Th. 5.5], and for the latter, it follows from the fact, proved in [27], that $\mathrm{TR}_2^1(A;p,\mathbb{Z}_p)$ is trivial. The upper right-hand group Q is a quotient of the $W_2(A)$-module $M = F^*(\omega^1_{(A,M)}/p)$. We claim that M is annihilated by the ideal $I = VW_2(A) + pW_2(A)$. Indeed, as an abelian group M is p-torsion and $FV = p$. It follows that also Q is annihilated by I, and we can therefore view it as a module over the quotient ring $W_2(A)/I$. This ring is isomorphic to A/p, the isomorphism given by

$$W_2(A)/I \xrightarrow{\sim} A/p, \qquad a + I \mapsto R(a) + pA,$$

and we let $g\colon A/p \to W_2(A)/I$ denote the inverse. The A/p-module g^*Q is generated by the class $[d\log \pi_K]_2$. The image of this class under the right-hand vertical map is a generator ι_1 of the $W_2(A)$-module ${}_h\mathrm{TR}_1^2(k;p)$, which is isomorphic to k. We now pick $\alpha \in \mathrm{TR}_3^1(A|K;p,\mathbb{Z}_p)$ such that $\delta(\partial_{K,2}(\alpha)) = \iota_1$. The difference $\beta = \partial_{K,2}(\alpha) - [d\log \pi_K]_2$ is in the kernel of δ, and therefore,

$$\beta = g(x\pi_K)\cdot [d\log \pi_K]_2,$$

for some $x \in A/p$. We then have

$$g(1 + x\pi_K)\cdot [d\log \pi_K]_2 = \partial_{K,2}(\alpha),$$

and since $(1 + x\pi_K) \in (A/p)^\times$,

$$[d\log \pi_K]_2 = (g(1+x\pi_K)^{-1})\cdot \partial_{K,2}(\alpha).$$

We would like to know that the map of units

$$W_2(A)^\times \to (W_2(A)/I)^\times$$

is a surjection. This will follow if we know that the I-adic topology on $W_2(A)$ is complete and separated. But the formula

$$V(x)\cdot V(y) = V(FV(x)y) = V(pxy) = pV(xy)$$

implies that the I-adic and p-adic topologies on $W_2(A)$ coincide, and the p-adic topology is complete and separated. So we can find a unit $u \in W_2(A)^\times$ such that $u + I = g(1 + x\pi_K)$. Since $\partial_{K,2}$ is $W_2(A)$-linear,

$$[d\log \pi_K]_2 = u^{-1}\partial_{K,2}(\alpha) = \partial_{K,2}(u^{-1}\alpha),$$

which concludes the proof for $m = 2$.

We now proceed inductively, and consider the diagram

$$\begin{tikzcd}
\mathrm{TR}_3^{m-1}(A|K;p,\mathbb{Z}_p) \arrow[r, "\partial_{K,m}"] \arrow[d, two heads, "F"] & {}_h\mathrm{TR}_2^{m}(A|K;p,\mathbb{Z}_p) \arrow[r, "N"] \arrow[d, two heads, "F"] & \mathrm{TR}_2^{m}(A|K;p,\mathbb{Z}_p) \arrow[d, two heads, "F"] \\
\mathrm{TR}_3^{m-2}(A|K;p,\mathbb{Z}_p) \arrow[r, "\partial_{K,m-1}"] & {}_h\mathrm{TR}_2^{m-1}(A|K;p,\mathbb{Z}_p) \arrow[r, "N"] & \mathrm{TR}_2^{m-1}(A|K;p,\mathbb{Z}_p).
\end{tikzcd}$$

Inductively, the map $\partial_{K,m-1}$ is surjective, and the left-hand vertical map F is surjective by Lemma 5.6.1. Moreover, the kernel of the middle vertical map is generated as a $W_m(A)$-module by the class $p^{m-2}[d\log \pi_K]_m$. It therefore suffices to show that this class is in the image of $\partial_{K,m}$ in the top row, and this in turn will follow if we show that the class $[d\log \pi_K]_m$ is in the image of $\partial_{K,m}$. To see this, we pick $\alpha \in \mathrm{TR}_3^{m-1}(A|K;p),\mathbb{Z}_p)$ such that $\partial_{K,m-1}(F(\alpha)) = [d\log \pi_K]_{m-1}$. Then $\beta = \partial_{K,m}(\alpha) - [d\log \pi_K]_m$ is in the kernel of the middle vertical map, so we can write $\beta = x \cdot p^{m-2} d\log \pi_K$, for some $x \in W_m(A)$. But then

$$(1 + p^{m-2}x)[d\log \pi_K]_m = \partial_{K,m}(\alpha),$$

and hence

$$[d\log \pi_K]_m = (1 + p^{m-2}x)^{-1}\partial_{K,m}(\alpha) = \partial_{K,m}((1 + p^{m-2}x)^{-1}\alpha),$$

where the inverse exists since the p-adic topology on $W_m(A)$ is complete and separated. □

ADDENDUM 3.3.7. *The group* $\mathrm{TR}_2^n(A;p)$ *is uniquely divisible for all* n.

Proof. It suffices to show that $\mathrm{TR}_2^n(A;p,\mathbb{Z}_p)$ is trivial. We prove this by induction, and refer to the proof of Proposition 2.3.4 for the case $n = 1$. Since $\mathrm{TR}_2^n(A|K;p,\mathbb{Z}_p)$ vanishes, there is an exact sequence

$$\mathrm{TR}_3^n(A|K;p,\mathbb{Z}_p) \xrightarrow{\delta_n} \mathrm{TR}_2^n(k;p) \to \mathrm{TR}_2^n(A;p,\mathbb{Z}_p) \to 0,$$

and we must prove that the map δ_n is surjective. We consider the diagram

$$\begin{tikzcd}
\mathrm{TR}_3^n(A|K;p,\mathbb{Z}_p) \arrow[r, "\delta_n"] \arrow[d, two heads, "F"] & \mathrm{TR}_2^n(k;p) \arrow[d, two heads, "F"] \\
\mathrm{TR}_3^{n-1}(A|K;p,\mathbb{Z}_p) \arrow[r, two heads, "\delta_{n-1}"] & \mathrm{TR}_2^{n-1}(k;p).
\end{tikzcd}$$

The map δ_{n-1} is surjective by induction, and the left-hand vertical map is surjective by Lemma 5.6.1. Moreover, it was proved in [19, Th. 5.5] that the right-hand vertical map F is a surjection whose kernel is equal to the image of the map

$$V\colon \mathrm{TR}_2^{n-1}(k;p) \to \mathrm{TR}_2^n(k;p).$$

Since the square

$$\begin{array}{ccc}
\mathrm{TR}_3^{n-1}(A|K;p,\mathbb{Z}_p) & \xrightarrow{\delta_{n-1}} & \mathrm{TR}_2^{n-1}(k;p) \\
\downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} \\
\mathrm{TR}_3^{n}(A|K;p,\mathbb{Z}_p) & \xrightarrow{\delta_{n}} & \mathrm{TR}_2^{n}(k;p)
\end{array}$$

commutes and the top horizontal map is a surjection, the proof of the induction step is complete. □

THEOREM 3.3.8. *The canonical map*

$$W_n\,\omega^q_{(A,M)} \to \mathrm{TR}_q^n(A|K;p)$$

is an isomorphism, for $q \le 2$, and a rational isomorphism, for all $q \ge 0$.

Proof. The proof is by induction on n starting from Proposition 2.3.4. In the induction step, we use the exact sequences of Lemma 3.2.6 and Proposition 3.3.6,

$$\begin{array}{ccccccccc}
 & & {}_hW_n\,\omega^q_{(A,M)} & \longrightarrow & W_n\,\omega^q_{(A,M)} & \xrightarrow{R} & W_{n-1}\,\omega^q_{(A,M)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\sim} & & \\
0 & \longrightarrow & {}_h\mathrm{TR}_q^n(A|K;p) & \xrightarrow{N} & \mathrm{TR}_q^n(A|K;p) & \xrightarrow{R} & \mathrm{TR}_q^{n-1}(A|K;p) & \longrightarrow & 0,
\end{array}$$

where the lower sequence is exact, for $q \le 1$, and exact modulo torsion, for all q. If $q \le 1$, the left-hand vertical map is an isomorphism by Lemma 3.3.3, and hence the statement follows in this case. When $q = 2$, the left-hand vertical map is an epimorphism with torsion kernel. Since the domain and range of the middle and right-hand vertical maps are both divisible groups, the statement follows. □

In the proof of Proposition 3.3.6, Addendum 3.3.7, and Theorem 3.3.8 above for $n > 3$ we have used Lemma 5.6.1 below. However, the lemma is not needed to prove these statements for $n \le 3$. In particular, the proof of the following result does not use Lemma 5.6.1.

ADDENDUM 3.3.9. *The connecting homomorphism*

$$\partial\colon \mathrm{TR}_2^1(A|K;p,\mathbb{Z}/p) \to {}_h\mathrm{TR}_1^2(A|K;p,\mathbb{Z}/p)$$

maps κ to $dV(1) - V(d\log(-p))$.

Proof. To prove the statement, we apply Lemma 3.3.10 below to the 3×3-diagram obtained from the smash product of the coefficient cofibration sequence

$$S^0 \xrightarrow{p} S^0 \xrightarrow{i} M_p \xrightarrow{\beta} S^1$$

and the fundamental cofibration sequence

$${}_h\mathrm{TR}^n(A|K;p) \xrightarrow{N} \mathrm{TR}^n(A|K;p) \xrightarrow{R} \mathrm{TR}^{n-1}(A|K;p) \xrightarrow{\partial} \Sigma({}_h\mathrm{TR}^n(A|K;p)).$$

Since $\mathrm{TR}_2(A|K;p)$ is uniquely divisible and $\mathrm{TR}_0(A|K;p)$ torsion-free, the lemma shows that the connecting homomorphism of the statement is equal to the opposite of the connecting homomorphism associated with the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}_h\,\mathrm{TR}_1^2(A|K;p) & \xrightarrow{N} & \mathrm{TR}_1^2(A|K;p) & \xrightarrow{R} & \mathrm{TR}_1^1(A|K;p) & \longrightarrow & 0 \\
 & & \big\downarrow p & & \big\downarrow p & & \big\downarrow p & & \\
0 & \longrightarrow & {}_h\,\mathrm{TR}_1^2(A|K;p) & \xrightarrow{N} & \mathrm{TR}_1^2(A|K;p) & \xrightarrow{R} & \mathrm{TR}_1^1(A|K;p) & \longrightarrow & 0.
\end{array}$$

And by Theorem 3.3.8, this diagram is canonically isomorphic to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}_hW_2\omega^1_{(A,M)} & \xrightarrow{N} & W_2\omega^1_{(A,M)} & \xrightarrow{R} & W_1\omega^1_{(A,M)} & \longrightarrow & 0 \\
 & & \big\downarrow p & & \big\downarrow p & & \big\downarrow p & & \\
0 & \longrightarrow & {}_hW_2\,\omega^1_{(A,M)} & \xrightarrow{N} & W_2\,\omega^1_{(A,M)} & \xrightarrow{R} & W_1\,\omega^1_{(A,M)} & \longrightarrow & 0.
\end{array}$$

The Bockstein maps κ to $d\log(-p) \in W_1\,\omega^1_{(A,M)}$, which is the image by the restriction of $d\log_2(-p) \in W_2\,\omega^1_{(A,M)}$. To evaluate $pd\log_2(-p)$ we use the formula

$$-\underline{(-p)}_2 + V(1) = p(1 + p^{p-2}V(1)),$$

which one readily verifies using the ghost map. Differentiating, we find

$$-d\underline{(-p)}_2 + dV(1) = p^{p-2}dV(1) = 0,$$

and if we multiply by $d\log_2(-p)$, we get

$$-d\underline{(-p)}_2 + V(d\log(-p)) = pd\log_2(-p) + p^{p-2}V(d\log(-p)) = pd\log_2(-p).$$

This shows that $pd\log_2(-p) = V(d\log(-p)) - dV(1)$ as desired. $\square$

Lemma 3.3.10. *Given a 3×3-diagram of cofibration sequences*

$$\begin{array}{ccccccc}
E_{11} & \xrightarrow{f_{11}} & E_{12} & \xrightarrow{f_{12}} & E_{13} & \xrightarrow{f_{13}} & \Sigma E_{11} \\
\downarrow{\scriptstyle g_{11}} & & \downarrow{\scriptstyle g_{12}} & & \downarrow{\scriptstyle g_{13}} & & \downarrow{\scriptstyle \Sigma g_{11}} \\
E_{21} & \xrightarrow{f_{21}} & E_{22} & \xrightarrow{f_{22}} & E_{23} & \xrightarrow{f_{23}} & \Sigma E_{21} \\
\downarrow{\scriptstyle g_{21}} & & \downarrow{\scriptstyle g_{22}} & & \downarrow{\scriptstyle g_{23}} & & \downarrow{\scriptstyle \Sigma g_{21}} \\
E_{31} & \xrightarrow{f_{31}} & E_{32} & \xrightarrow{f_{32}} & E_{33} & \xrightarrow{f_{33}} & \Sigma E_{31} \\
\downarrow{\scriptstyle g_{31}} & & \downarrow{\scriptstyle g_{32}} & & \downarrow{\scriptstyle g_{33}} & (-1) & \downarrow{\scriptstyle -\Sigma g_{11}} \\
\Sigma E_{11} & \xrightarrow{\Sigma f_{11}} & \Sigma E_{12} & \xrightarrow{\Sigma f_{12}} & \Sigma E_{13} & \xrightarrow{-\Sigma f_{13}} & \Sigma^2 E_{11}
\end{array}$$

and classes $e_{ij} \in \pi_* E_{ij}$ such that $g_{33}(e_{33}) = \Sigma f_{12}(e_{12})$ and $f_{33}(e_{33}) = \Sigma g_{21}(e_{21})$. Then the sum $f_{21}(e_{21}) + g_{12}(e_{12})$ is in the image of $\pi_* E_{11} \to \pi_* E_{22}$.

3.4. The k-algebra $\bar{W}_n(A)$ was evaluated in Proposition 3.1.5 above. We now evaluate the differential graded k-algebra $\bar{W}_n\,\omega^*_{(A,M)}$. Let $\pi = \pi_K$ be a uniformizer. Then the modified Verschiebung from (3.1.4) satisfies

$$F dV_\pi(a) = \theta_K(\underline{\pi})^p da.$$

Let $r = r(i, e_K) = v_p(i - pe_K/(p-1))$.

Proposition 3.4.1. *The differential graded k-algebra $E^* = \bar{W}_n\,\omega^*_{(A,M)}$ is concentrated in degrees 0 and 1 and satisfies:*

(i) *A k-basis for E^1_n is given by the elements $V^s_\pi(\underline{\pi}^i d\log\pi)$, where $0 \le i < e_K$ and $0 \le s \le r$, and $dV^s_\pi(\underline{\pi}^i)$, where $0 \le i < e_K$ and $r < s < n$. Moreover, $V^s_\pi(\underline{\pi}^i d\log\pi)$ vanishes, if $s > r$, $dV^s_\pi(\underline{\pi}^i)$ vanishes, if $s < r$, and*

$$dV^r_\pi(\underline{\pi}^i) = p^{-r}(i - pe_K/(p-1)) \cdot V^r_\pi(\underline{\pi}^i d\log\pi).$$

(ii) *The E^0_n-module structure on E^1_n is given by*

$$V^s_\pi(\underline{\pi}^i) dV^t_\pi(\underline{\pi}^j) = \begin{cases} dV^t_\pi(\underline{\pi}^{p^t i + j}) - iV^t_\pi(\underline{\pi}^{p^t i+j} d\log\pi) & \text{if } 0 = s \le t, \\ -iV^t_\pi(\theta_K(\underline{\pi})^{p^{t-s}(\frac{p^{s+1}-1}{p-1}-1)} \underline{\pi}^{p^{t-s}i+j} d\log\pi) & \text{if } 0 < s \le t, \\ jV^s_\pi(\theta_K(\underline{\pi})^{p^{s-t}(\frac{p^{t+1}-1}{p-1}-1)} \underline{\pi}^{i+p^{s-t}j} d\log\pi) & \text{if } s \ge t, \end{cases}$$

$$V^s_\pi(\underline{\pi}^i) V^t_\pi(\underline{\pi}^j d\log\pi) = \begin{cases} V^t_\pi(\underline{\pi}^{p^t i+j} d\log\pi) & \text{if } s = 0, \\ V^s_\pi(\underline{\pi}^{i+p^s j} d\log\pi) & \text{if } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It follows from Propositions 3.1.5 and 3.2.2 that E^*_n is generated, as a graded k-vector space, by the monomials in the variables $V^s_\pi(\underline{\pi}^i)$, $dV^s_\pi(\underline{\pi}^i)$, $V^s_\pi(\underline{\pi}^i d\log\pi)$, and $dV^s_\pi(\underline{\pi}^i d\log\pi)$ with $0 \le s < n$ and $i \ge 0$. Theorem 3.3.8

and Corollary 3.2.7 show that E_n^q vanishes, for $q \geq 2$. In particular, the latter generators, which are of degree two, must vanish.

We verify the relations in (i). If $s \leq r$ then $p^{-s}(i + pe_K(p^s - 1)/(p-1))$ is an integer, and iterated use of the second relation in Proposition 3.1.5 shows that

$$V_\pi^s(\underline{\pi}^i) = \underline{\pi}^{p^{-s}(i + pe_K \frac{p^s - 1}{p-1})}.$$

It follows that $dV_\pi^s(\underline{\pi}^i)$ vanishes, if $s < r$, and that $dV_\pi^r(\underline{\pi}^i)$ and $V_\pi^r(\underline{\pi}^i d\log \pi)$ are related as stated. And since $V_\pi d$ is the zero homomorphism, this also shows that for $s > r$, $V_\pi^s(\underline{\pi}^i d\log\pi) = V_\pi^{s-r} V_\pi^r(\underline{\pi}^i d\log \pi)$ vanishes.

The formulas in (ii) are readily obtained by differentiating the first set of relations in Proposition 3.1.5. If, for instance, $0 < s \leq t < n$, we find that

$$\begin{aligned} V_\pi^s(\underline{\pi}^i) dV_\pi^t(\underline{\pi}^j) &= -dV_\pi^s(\underline{\pi}^i) V_\pi^t(\underline{\pi}^j) = -V_\pi^t(F^t dV_\pi^s(\underline{\pi}^i)\underline{\pi}^j) \\ &= -iV_\pi^t(\theta_K(\underline{\pi})^{p^{t-s}(\frac{p^{s+1}-1}{p-1} - 1)} \underline{\pi}^{p^{t-s}i + j} d\log\pi), \end{aligned}$$

and the remaining formulas are verified in a similar manner. It remains to prove that this gives all relations in E_n^1. This is the case if and only if E_n^1 is an ne_K-dimensional k-vector space. We prove in Proposition 6.1.1 below that this is indeed the case, and hence there can be no further relations. $\square$

4. Tate cohomology and the Tate spectrum

4.1. Let G be a finite group and let k be a commutative ring. The norm element $N_G \in kG$ is defined as the sum of all the elements of G. If M is a left kG-module, multiplication by N_G defines a map

$$N_G \colon M_G \to M^G$$

from the coinvariants $M_G = k \otimes_{kG} M$ to the invariants $M^G = \mathrm{Hom}_{kG}(k, M)$. We note that for left kG-modules M and N, there are canonical isomorphisms

$$(M \otimes N)_G \cong c^* M \otimes_{kG} N, \qquad \mathrm{Hom}(M, N)^G \cong \mathrm{Hom}_{kG}(M, N),$$

where c^*M denotes the right kG-module with $m \cdot g = g^{-1} m$.

Let $\varepsilon \colon P \to k$ be a projective resolution and let $\tilde{P}$ be the mapping cone of ε such that there is a distinguished triangle (see §2.1 above)

$$P \xrightarrow{\varepsilon} k \xrightarrow{\iota} \tilde{P} \xrightarrow{\partial} \Sigma P.$$

Definition 4.1.1. Let M be a left kG-module. The *Tate cohomology of G with coefficients* in M is given by

$$\hat{H}^*(G, M) = H_{-*}((\tilde{P} \otimes \mathrm{Hom}(P, M))^G).$$

It is clear that the Tate cohomology groups are well-defined up to canonical isomorphism. We show that the definition given here agrees with the usual definition in terms of complete resolutions, [5, Chap. XII, §3].

LEMMA 4.1.2. *The following maps are quasi-isomorphisms*:

$$(P \otimes M)_G \xrightarrow[\sim]{N} (P \otimes M)^G \xrightarrow[\sim]{\mathrm{id} \otimes \varepsilon^*} (P \otimes \mathrm{Hom}(P, M))^G.$$

Proof. We first show that the norm map is an isomorphism of complexes. It will suffice to show that the norm map

$$(kG \otimes M)_G \xrightarrow{N} (kG \otimes M)^G$$

is an isomorphism, for both sides commute with the formation of arbitrary direct sums. Let $\eta\colon k \to kG$ and $\varepsilon\colon kG \to k$ be the unit and co-unit of the Hopf algebra kG, respectively. Then we have an isomorphism of left kG-modules

$$\xi\colon kG \otimes \varepsilon^*\eta^* M \xrightarrow{\sim} kG \otimes M, \quad \xi(g \otimes x) = g \otimes gx.$$

The left-hand side is isomorphic to a direct sum indexed by the elements of G of copies of M, and G acts by permuting the summands. Hence N is an isomorphism.

In order to show that the right-hand map of the statement is a quasi-isomorphism, we filter the double complex on the right after the first tensor factor. This gives, by [1, Th. 6.1], a strongly convergent fourth quadrant spectral sequence

$$E^1_{s,t} = H_t((P_s \otimes \mathrm{Hom}(P, M))^G) \Rightarrow H_{s+t}((P \otimes \mathrm{Hom}(P, M))^G),$$

and hence, it suffices to show that for all $s \geq 0$, the map

$$(P_s \otimes M)^G \xrightarrow{\mathrm{id} \otimes \varepsilon^*} (P_s \otimes \mathrm{Hom}(P, M))^G$$

is a quasi-isomorphism. Since both sides commute with filtered colimits in the first tensor factor, we can further assume that the projective kG-module P_s is finitely generated. In this case, the dual $DP_s = \mathrm{Hom}(P_s, k)$ again is a (finitely generated) projective kG-module, and there is a commutative diagram

$$\begin{tikzcd}
(P_s \otimes M)^G \arrow[r, "\mathrm{id} \otimes \varepsilon^*"] \arrow[d, "\sim"] & (P_s \otimes \mathrm{Hom}(P, M))^G \arrow[d, "\sim"] \\
\mathrm{Hom}(DP_s, M)^G \arrow[r, "(\varepsilon \otimes \mathrm{id})^*"] & \mathrm{Hom}(P \otimes DP_s, M)^G,
\end{tikzcd}$$

with the vertical maps isomorphisms. The map

$$\varepsilon \otimes \mathrm{id}\colon P \otimes DP_s \xrightarrow{\sim} DP_s$$

is a quasi-isomorphism between bounded below complexes of projective kG-modules. Therefore, it is a chain homotopy equivalence, and hence, so is the lower horizontal map in the diagram above. The lemma follows. $\square$

Remark 4.1.4. The triangle preceding Definition 4.1.1 and Lemma 4.1.2 gives rise to natural isomorphisms

$$\hat{H}^i(G,M) \cong \begin{cases} H^i(G,M) & \text{if } i \geq 1 \\ H_{-i-1}(G,M) & \text{if } i \leq -1 \end{cases}$$

and to a natural exact sequence

$$0 \to \hat{H}^{-1}(G,M) \xrightarrow{\partial} H_0(G,M) \xrightarrow{N} H^0(G,M) \xrightarrow{i} \hat{H}^0(G,M) \to 0.$$

Hence, the definition of Tate cohomology given here agrees with the original one in terms of complete resolutions, [5, Chap. XII, §3]. This can also be seen more directly as follows. Let $\varepsilon\colon \hat{P} \to k$ be a complete resolution in the sense of *loc. cit.*, and let P and P^- be the complexes whose nonzero terms are $P_i = \hat{P}_i$, if $i \geq 0$, and $P_i^- = \hat{P}_i$, if $i < 0$, respectively. Then $\varepsilon\colon P \to k$ is a resolution of k by finitely generated projective left kG-modules and there is a canonical triangle

$$P^- \to \hat{P} \to P \to \Sigma P^-.$$

An argument similar to the proof of Lemma 4.1.2 shows that the canonical maps

$$\operatorname{Hom}(\hat{P},M)^G \xrightarrow{\sim} (\tilde{P} \otimes \operatorname{Hom}(\hat{P},M))^G \xrightarrow{\sim} (\tilde{P} \otimes \operatorname{Hom}(P,M))^G$$

are quasi-isomorphisms.

Definition 4.1.5. The *cup product*

$$\hat{H}^*(G,M) \otimes \hat{H}^*(G,M') \to \hat{H}^*(G,M \otimes M')$$

is the map on homology induced by the composite

$$\begin{aligned} (\tilde{P} \otimes \operatorname{Hom}(P,M))^G \ \otimes \ & (\tilde{P} \otimes \operatorname{Hom}(P,M'))^G \\ \to \ & (\tilde{P} \otimes \tilde{P} \otimes \operatorname{Hom}(P \otimes P, M \otimes M'))^G \\ \to \ & (\tilde{P} \otimes \operatorname{Hom}(P, M \otimes M'))^G, \end{aligned}$$

where the first map is the canonical map, and the second map is induced from a choice of chain maps $P \to P \otimes P$ and $\tilde{P} \otimes \tilde{P} \to \tilde{P}$ compatible with the canonical isomorphisms $k \to k \otimes k$ and $k \otimes k \to k$, respectively.

It is well-known that the chain map $P \to P \otimes P$ exists and is unique up to chain homotopy. The analogous statement for the map $\tilde{P} \otimes \tilde{P} \to \tilde{P}$ is proved in a similar manner. Hence, the cup product is well-defined. It makes $\hat{H}^*(G,k)$ a graded commutative graded ring and $\hat{H}^*(G,M)$ a graded module over this ring.

4.2. Let C be a cyclic group of order r and let $g \in C$ be a generator. We let $\varepsilon\colon W \to k$ be the standard resolution which in degree $s \geq 0$ is a free kC-module on a single generator x_s with differential

$$dx_s = \begin{cases} Nx_{s-1}, & s \text{ even}, \\ (g-1)x_{s-1}, & s \text{ odd}, \end{cases}$$

and with augmentation $\varepsilon(x_0) = 1$. Then $\tilde{W}$ is the complex which in degree $s > 0$ is a free kC-module on the generator $y_s = (0, x_{s-1})$ and in degree $s = 0$ is a trivial kC-module on the generator $y_0 = (1, 0)$. The differential is

$$dy_s = \begin{cases} -(g-1)y_{s-1}, & s \text{ even}, \\ -Ny_{s-1}, & s > 1 \text{ odd}, \\ -y_0 & s = 1. \end{cases}$$

The dual of x_s is the element $x_s^* \in DW_s = \mathrm{Hom}(W_s, k)$ given by $x_s^*(g^i x_s) = \delta_{i,0}$. Note that $g^i \cdot x_n^* = (g^i x_n)^*$ and that the map $(g^i)^*\colon DW_s \to DW_s$ maps $x_s^* \mapsto g^{-i} x_s^*$. Thus

$$dx_s^* = \begin{cases} (g^{-1}-1)x_{s+1}^*, & s \text{ even}, \\ Nx_{s+1}^*, & s \text{ odd}. \end{cases}$$

Lemma 4.2.1. *Suppose that the order of C is odd and congruent to zero in k. Then as a graded k-algebra*

$$\hat{H}^*(C, k) = \Lambda\{u\} \otimes S\{t^{\pm 1}\}$$

where t and u are the classes of $y_0 \otimes Nx_2^*$ and $y_0 \otimes Nx_1^*$, respectively. Moreover, the classes 1, ut^{-1} and t^{-1} are represented by the elements $y_0 \otimes Nx_0^*$, $-Ny_1 \otimes Nx_0^*$ and $Ny_2 \otimes Nx_0^*$, respectively.

Proof. We first evaluate the homology of the complex

$$(\tilde{W} \otimes \mathrm{Hom}(W, k))^C = (\tilde{W} \otimes DW)^C.$$

This is the total complex of a double complex, and the filtration after the first tensor factor gives rise to a fourth quadrant homology type spectral sequence which converges strongly to the homology of the total complex, [1, Th. 6.1]. We have

$$E_{s,t}^1 = H_{s+t}(\tilde{W}_s \otimes DW)^C \xrightarrow{\sim} H_{s+t}(\mathrm{Hom}(W, \tilde{W}_s)^C),$$

which vanishes if both s and t are nonzero. Hence $E_{s,t}^2 = E_{s,t}^\infty$ and it is easy to see that if either s or t is zero, this is a free k-module of rank one generated by the classes of $y_0 \otimes Nx_{-t}^*$ and $Ny_s \otimes Nx_0^*$, respectively. We note that these elements are also cycles in the total complex.

To evaluate the multiplicative structure, we choose liftings

$$\Psi\colon W \to W \otimes W,$$
$$\Phi\colon \tilde{W} \otimes \tilde{W} \to \tilde{W}$$

of the canonical maps $k \to k \otimes k$ and $k \otimes k \to k$, respectively:

$$\Psi_{m,n}(g^s x_{m+n}) = \begin{cases} \sum_{s \leq p < q < s} g^p x_m \otimes g^q x_n, & m \text{ and } n \text{ odd} \\ g^s x_m \otimes g^{s+1} x_n, & m \text{ odd, } n \text{ even} \\ g^s x_m \otimes g^s x_n, & m \text{ even} \end{cases}$$

and

$$\Phi_{m,n}(g^p y_m \otimes g^q y_n) = \begin{cases} \sum_{p \leq s < q < p} g^s y_{m+n}, & m \text{ and } n \text{ odd} \\ \delta_{p,q+1} g^p y_{m+n}, & m \text{ odd, } n \text{ even} \\ \delta_{p,q} g^p y_{m+n}, & m \text{ even,} \end{cases}$$

where in the first line the sum ranges over the g^s between g^p and g^{q-1}, both included, in the cyclic ordering of C specified by the generator g. The sum is zero if and only if $p = q$. The map Ψ induces a product map on the dual DW given by the composite

$$\Psi^* \colon DW \otimes DW \xrightarrow{\nu} D(W \otimes W) \xrightarrow{D\Psi} DW,$$

or

$$\Psi^*_{m,n}(g^{-p} x^*_m \otimes g^{-q} x^*_n) = \begin{cases} -\sum_{p \leq s < q < p} g^{-s} x^*_{m+n}, & m \text{ and } n \text{ odd} \\ \delta_{p,q+1} g^{-p} x^*_{m+n}, & m \text{ odd, } n \text{ even} \\ \delta_{p,q} g^{-p} x^*_{m+n}, & m \text{ even.} \end{cases}$$

We find that

$$(y_0 \otimes N x^*_m) \cdot (y_0 \otimes N x^*_n) = \begin{cases} -\frac{r(r-1)}{2} \, y_0 \otimes N x^*_{m+n}, & m \text{ and } n \text{ odd} \\ y_0 \otimes N x^*_{m+n} & \text{otherwise} \end{cases}$$

and

$$(N y_m \otimes N x^*_0) \cdot (N y_n \otimes N x^*_0) = \begin{cases} \frac{r(r-1)}{2} \, N y_{m+n} \otimes N x^*_0, & m \text{ and } n \text{ odd} \\ N y_{m+n} \otimes N x^*_0 & \text{otherwise.} \end{cases}$$

Moreover, the product

$$(y_0 \otimes N x^*_2) \cdot (N y_2 \otimes N x^*_0) = N y_2 \otimes N x^*_2$$

is homologous to $y_0 \otimes N x^*_0$, which represents the multiplicative unit in the cohomology ring. Indeed, with $\Delta(N) = \sum_{0 \leq s < n} g^s \otimes g^s$

$$d(\Delta(N)(y_1 \otimes x^*_0) + \Delta(N)(y_2 \otimes x^*_1)) = -y_0 \otimes N x^*_0 + N y_2 \otimes N x^*_2.$$

Hence $N y_2 \otimes N x^*_0$ represents the class t^{-1}. Finally, for any element $\alpha \in kC$,

$$(1 \otimes \alpha)\Delta(N) = (\bar{\alpha} \otimes 1)\Delta(N),$$

where $\bar{\alpha} = c(\alpha)$ is the antipode. Therefore, if $\alpha \in kC$ is such that $(g-1)\alpha = r - N$ (for example $\alpha = 1 + 2g + \cdots + rg^{r-1}$ is such an element), then

$$\begin{aligned} &d((\alpha \otimes 1)\Delta(N)(y_2 \otimes x_0^*)) \\ &\qquad = \quad -((g-1) \otimes 1)(\alpha \otimes 1)\Delta(N)(y_1 \otimes x_0^*) \\ &\qquad \qquad -(1 \otimes (\bar{g}-1))(1 \otimes \bar{\alpha})\Delta(N)(y_2 \otimes x_1^*) \\ &\qquad = \quad Ny_1 \otimes Nx_0^* + Ny_2 \otimes Nx_1^* - r\Delta(N)(y_1 \otimes x_0^* + y_2 \otimes x_1^*), \end{aligned}$$

and hence, $Ny_1 \otimes Nx_0^*$ represents the class $-ut^{-1}$ in the cohomology ring. □

Addendum 4.2.2. *The boundary map* $\partial \colon \hat{H}^{-1}(C,k) \to H_0(C,k)$ *takes* ut^{-1} *to the class of* -1.

Proof. The boundary map, by definition, is induced by the composite

$$(\tilde{W} \otimes \mathrm{Hom}(W,k))^C \xrightarrow{\partial \otimes \mathrm{id}} (\Sigma W \otimes \mathrm{Hom}(W,k))^C \xleftarrow[\sim]{\mathrm{id} \otimes \varepsilon^*} (\Sigma W \otimes k)^C$$

$$\xleftarrow[\sim]{N} (\Sigma W \otimes k)_C \xrightarrow{\varepsilon \otimes 1} \Sigma k_C.$$

The class ut^{-1} is represented by the element $-Ny_1 \otimes Nx_0^*$ whose image under $\partial \otimes \mathrm{id}$ is $-Nx_0 \otimes Nx_0^*$. This element is equal to $(\mathrm{id} \otimes \varepsilon^*)(-Nx_0 \otimes 1)$ and $-Nx_0 \otimes 1 = N(-x_0 \otimes 1)$. Finally $(\varepsilon \otimes \mathrm{id})(-x_0 \otimes 1)$ is equal to the class of -1. □

4.3. We recall that for spectra X and Y, there are natural maps

$$\begin{aligned} &\wedge \colon \pi_s X \otimes \pi_t Y \to \pi_{s+t}(X \wedge Y), \\ &\vee \colon \pi_{s+t} F(X,Y) \to \mathrm{Hom}(\pi_{-s}X, \pi_t Y), \end{aligned} \tag{4.3.1}$$

where $\wedge$ is the external product and $\vee$ is the adjoint of the composite

$$\pi_{s+t}F(X,Y) \otimes \pi_{-s}X \xrightarrow{\wedge} \pi_t(F(X,Y) \wedge X) \xrightarrow{\mathrm{ev}} \pi_t Y.$$

Let X be a G-CW-spectrum with an increasing filtration $\{X_s\}$ by sub-G-CW-spectra. Then the exact couple

$$D_{s-1,t+1} \xrightarrow{i} D_{s,t} \xrightarrow{j} E_{s,t} \xrightarrow{\partial} D_{s-1,t}$$

with

$$\begin{aligned} D_{s,t}(X) &= \pi_{s+t}((X_s)^G) \\ E_{s,t}(X) &= \pi_{s+t}((X_s/X_{s-1})^G) \end{aligned} \tag{4.3.2}$$

gives rise to a spectral sequence which abuts the homotopy groups of X^G. The spectral sequence converges conditionally in the sense of [1, Def. 5.10], provided that $\bigcup X_s = X$ and $\mathrm{holim}_{\longleftarrow}(X_s)^G$ is contractible.

If X and X' are two G-CW-spectra with such filtrations, we give the smash product $X \wedge X'$ the usual product filtration

$$(X \wedge X')_n = \bigcup_{s+s'=n} X_s \wedge X'_{s'}.$$

with filtration quotients

$$(X \wedge X')_n/(X \wedge X')_{n-1} = \bigvee_{s+s'=n} X_s/X_{s-1} \wedge X_{s'}/X_{s'-1}.$$

The external product (4.3.1) and the inclusions

$$\begin{aligned} X_s \wedge X'_{s'} &\to (X \wedge X')_{s+s'}, \\ X_s/X_{s-1} \wedge X'_{s'}/X'_{s'-1} &\to (X \wedge X')_{s+s'}/(X \wedge X')_{s+s'-1} \end{aligned}$$

then give rise to pairings

$$\begin{aligned} D_{s,t}(X) \otimes D_{s',t'}(X') &\to D_{s+s',t+t'}(X \wedge X'), \\ E_{s,t}(X) \otimes E_{s',t'}(X') &\to E_{s+s',t+t'}(X \wedge X'). \end{aligned}$$

These, in turn, give rise to an external pairing of the associated spectral sequences, that is, pairings

$$E^r_{s,t}(X) \otimes E^r_{s',t'}(X') \to E^r_{s+s',t+t'}(X \wedge X'),$$

for all $r \geq 1$, which satisfies the Leibnitz rule

$$d^r(xx') = d^r xx' + (-1)^{|x|} x d^r x'.$$

Here $|x|$ is the total degree of x. A filtration-preserving product map $X \wedge X \to X$ induces a map of the associated spectral sequences which, pre-composed by the external product, give an internal product on the spectral sequence $E^*(X)$. The differentials act as derivations for this product, and if the product on X is associative, commutative or unital, the same holds for the internal product in the spectral sequence. Commutativity in the spectral sequence is up to the usual sign.

Let G be a finite group and let E be a free contractible G-CW-complex. Let $\tilde{E}$ be the mapping cone of the projection $\mathrm{pr}\colon E_+ \to S^0$ which collapses E to the nonbase point of S^0. The associated suspension-G-CW-spectra (we make no change in notation) form a distinguished triangle

$$E_+ \xrightarrow{\mathrm{pr}} S^0 \to \tilde{E} \xrightarrow{\partial} \Sigma E_+.$$

Let P and $\tilde{P}$ be the cellular complexes of E_+ and $\tilde{E}$ with coefficients in a commutative ring k. We then have a distinguished triangle

$$P \xrightarrow{\mathrm{pr}_*} k \to \tilde{P} \to \Sigma P$$

in the category of chain complexes.

The Tate spectrum of a G-spectrum T is defined by

$$\hat{\mathbb{H}}(G,T) = (\tilde{E} \wedge \Gamma F(E_+,T))^G,$$

where $\Gamma X \xrightarrow{\sim} X$ is a functorial G-CW-substitute. If T and T' are two G-spectra, we define a pairing

$$\hat{\mathbb{H}}(G,T) \wedge \hat{\mathbb{H}}(G,T') \to \hat{\mathbb{H}}(G,T \wedge T') \tag{4.3.3}$$

as follows. By elementary obstruction theory, there are cellular G-homotopy equivalences $E_+ \to E_+ \wedge E_+$ and $\tilde{E} \wedge \tilde{E} \to \tilde{E}$ compatible with the canonical isomorphisms $S^0 \to S^0 \wedge S^0$ and $S^0 \wedge S^0 \to S^0$, respectively, and any two such equivalences are G-homotopic. The pairing then is given by

$$\begin{aligned}(\tilde{E} \wedge \Gamma F(E_+,T))^G \wedge (\tilde{E} \wedge \Gamma F(E_+,T))^G &\to (\tilde{E} \wedge \tilde{E} \wedge \Gamma F(E_+ \wedge E_+, T \wedge T'))^G \\ &\to (\tilde{E} \wedge \Gamma F(E_+, T \wedge T'))^G,\end{aligned}$$

where the first map is the canonical map and the second is induced from the chosen G-equivalences. If T is a G-ring spectrum, the composition of the external product with the map of Tate spectra induced from the product map on T, makes $\hat{\mathbb{H}}(G,T)$ a ring spectrum. This ring spectrum is associative, commutative or unital if the G-ring spectrum T is associative, commutative or unital, respectively.

The CW-filtrations of E and $\tilde{E}$ give rise to a double filtration of the Tate spectrum. In more detail, we define

$$\begin{aligned} X_{r,s} &= \tilde{E}_r \wedge \Gamma F(E/E_{-s-1},T), \\ Y_{r,s} &= \tilde{E}_r/\tilde{E}_{r-1} \wedge \Gamma F(E/E_{-s-1},T), \\ Z_{r,s} &= \tilde{E}_r \wedge \Gamma F(E_{-s}/E_{-s-1},T)), \\ W_{r,s} &= \tilde{E}_r/\tilde{E}_{r-1} \wedge \Gamma F(E_{-s}/E_{-s-1},T)). \end{aligned}$$

To get an honest filtration by sub-G-CW-spectra, we let

$$\bar{X}_{r,s} = \operatorname{holim}_{\longrightarrow} X_{r',s'},$$

where the homotopy colimit runs over all $0 \le r' \le r$ and $s' \le s \le 0$. There is a canonical homotopy equivalence $\bar{X}_{r,s} \xrightarrow{\sim} X_{r,s}$ and $\bar{X}_{r,s}$ is a sub-G-CW-spectrum of the G-CW-spectrum $\bar{X} = \bar{X}_{\infty,0}$. We also let

$$\begin{aligned} \bar{Y}_{r,s} &= \bar{X}_{r,s}/\bar{X}_{r-1,s} \\ \bar{Z}_{r,s} &= \bar{X}_{r,s}/\bar{X}_{r,s-1} \\ \bar{W}_{r,s} &= \bar{X}_{r,s}/\bar{X}_{r-1,s} \cup \bar{X}_{r,s-1} \end{aligned}$$

and define

$$\bar{X}_n = \bigcup_{r+s \le n} \bar{X}_{r,s} \subset \bar{X}.$$

The exact couple (4.3.2) associated with the filtration $\{\bar{X}_n\}$ gives rise to a conditionally convergent spectral sequence

$$\hat{E}^*(G,T) = E^*(\bar{X}) \Rightarrow \pi_*(\hat{\mathbb{H}}(G,T)).$$

LEMMA 4.3.4. *There is a canonical isomorphism of complexes*

$$\hat{E}^1_{*,t}(G,T) \cong (\tilde{P} \otimes \mathrm{Hom}(P, \pi_t T))^G$$

and hence $\hat{E}^2_{s,t}(G,T) \cong \hat{H}^s(G, \pi_t T)$.

Proof. The inclusions $\bar{X}_{r,s} \to \bar{X}_{r+s}$ induce an isomorphism

$$\bigvee_{r+s=n} \bar{W}_{r,s} \xrightarrow{\sim} \bar{X}_n/\bar{X}_{n-1}$$

such that the boundary map

$$\bar{X}_n/\bar{X}_{n-1} \to \Sigma\bar{X}_{n-1} \to \Sigma(\bar{X}_{n-1}/\bar{X}_{n-2})$$

maps the summand $\bar{W}_{r,s}$ to the summands $\Sigma\bar{W}_{r-1,s}$ and $\Sigma\bar{W}_{r,s-1}$ by the maps

$$\partial' : \bar{W}_{r,s} \to \Sigma\bar{Y}_{r,s-1} \to \Sigma\bar{W}_{r,s-1},$$
$$\partial'' : \bar{W}_{r,s} \to \Sigma\bar{Z}_{r-1,s} \to \Sigma\bar{W}_{r-1,s},$$

respectively. We identify

$$\pi_{r+s+t}((\bar{W}_{r,s})^G) \cong (\tilde{P}_r \otimes \mathrm{Hom}(P_{-s}, \pi_t T))^G$$

as follows: If X and Y are two G-spectra, we have the canonical map

$$\pi_*((X \wedge Y)^G) \to (\pi_*(X \wedge Y))^G.$$

This is an isomorphism, for example, if X is a wedge of free G-cells. The desired isomorphism is the composition of the inverse of this map with $X = \tilde{E}_r/\tilde{E}_{r-1}$ and $Y = \Gamma F(E_{-s}/E_{-s-1}, T)$ and the map of G-fixed sets induced by

$$\begin{aligned}
\pi_{r+s+t}&(\tilde{E}_r/\tilde{E}_{r-1} \wedge \Gamma F(E_{-s}/E_{-s-1}, T)) \\
&\underset{\sim}{\xleftarrow{\wedge}} \pi_r(\tilde{E}_r/\tilde{E}_{r-1}) \otimes \pi_{s+t}\Gamma F(E_{-s}/E_{-s-1}, T) \\
&\xrightarrow{\sim} \pi_r(\tilde{E}_r/\tilde{E}_{r-1}) \otimes \pi_{s+t} F(E_{-s}/E_{-s-1}, T) \\
&\underset{\sim}{\xrightarrow{h\otimes\vee}} H_r(\tilde{E}_r/\tilde{E}_{r-1}) \otimes \mathrm{Hom}(\pi_{-s}(E_{-s}/E_{-s-1}), \pi_t T) \\
&\underset{\sim}{\xleftarrow{1\otimes h^*}} H_r(\tilde{E}_r/\tilde{E}_{r-1}) \otimes \mathrm{Hom}(H_{-s}(E_{-s}/E_{-s-1}), \pi_t T).
\end{aligned}$$

Here h is the Hurewitz homomorphism. One readily shows that under this identification, $\pi_*(\partial')$ and $\pi_*(\partial'')$ correspond to the differentials in the algebraic double complex. □

The pairing (4.3.3) induces a pairing $\bar{X}(T)\wedge\bar{X}(T') \to \bar{X}(T\wedge T')$, and since the equivalences $E_+ \to E_+ \wedge E_+$ and $\tilde{E} \wedge \tilde{E} \to \tilde{E}$ were chosen to be cellular, this pairing preserves the filtration by the sub-CW-spectra $\{\bar{X}_n\}$. Hence, we get an induced pairing of the associated spectral sequences.

Proposition 4.3.5. *Let T and T' be two G-spectra. Then the pairing of Tate spectra* (4.3.3) *induces a pairing of the associated spectral sequences. On E^2-terms, this pairing corresponds to the pairing on Tate cohomology*

$$\hat{H}^*(G, \pi_* T) \otimes \hat{H}^*(G, \pi_* T') \to \hat{H}^*(G, \pi_*(T \wedge T'))$$

under the isomorphism of Lemma 4.3.4. *In particular, if T is an associative G-ring spectrum, then $E^2 \cong \hat{H}^*(G, \pi_* T)$ as a bi-graded ring.*

Proof. The equivalences $E_+ \to E_+ \wedge E_+$ and $\tilde{E} \wedge \tilde{E} \to \tilde{E}$ induces chain maps $P \to P \otimes P$ and $\tilde{P} \otimes \tilde{P} \to \tilde{P}$ which lift the canonical maps $k \to k \otimes k$ and $k \otimes k \to k$, respectively. Now suppose T and T' are two G-spectra and consider the spectral sequences corresponding to the filtrations $\{(\bar{X}(T) \wedge \bar{X}(T'))_n\}$ and $\{\bar{X}(T \wedge T')_n\}$. An argument analogous to the proof of Lemma 4.3.4 identifies the E^1-terms of the associated spectral sequences with the complexes

$$(\tilde{P} \otimes \operatorname{Hom}(P, \pi_* T) \otimes \tilde{P} \otimes \operatorname{Hom}(P, \pi_* T'))^G$$

and

$$(\tilde{P} \otimes \operatorname{Hom}(P, \pi_*(T \wedge T')))^G,$$

respectively. We claim that under these identifications, the pairing

$$\bar{X}(T) \wedge \bar{X}(T') \to \bar{X}(T \wedge T')$$

corresponds to the composition

$$\begin{aligned}&(\tilde{P} \otimes \operatorname{Hom}(P, \pi_* T))^G \otimes (\tilde{P} \otimes \operatorname{Hom}(P, \pi_* T'))^G \\ &\quad \to (\tilde{P} \otimes \tilde{P} \otimes \operatorname{Hom}(P \otimes P, \pi_* T \otimes \pi_* T'))^G \to (\tilde{P} \otimes \operatorname{Hom}(P, \pi_*(T \otimes T')))^G,\end{aligned}$$

where the first map is the canonical map of chain complexes (which involves sign changes) and the second map is induced from the maps $P \to P \otimes P$ and $\tilde{P} \otimes \tilde{P} \to \tilde{P}$ and from the exterior product (4.3.1). This is straightforward to check. Similarly, under the isomorphism of Lemma 4.3.4 and the analogous isomorphism above, the external pairing corresponds to the canonical map (no sign changes)

$$\begin{aligned}&(\tilde{P} \otimes \operatorname{Hom}(P, \pi_* T))^G \otimes (\tilde{P} \otimes \operatorname{Hom}(P, \pi_* T'))^G \\ &\qquad\qquad \to (\tilde{P} \otimes \operatorname{Hom}(P, \pi_* T) \otimes \tilde{P} \otimes \operatorname{Hom}(P, \pi_* T'))^G.\end{aligned}$$

But this was our definition of the pairing in Tate cohomology; see (4.1.5). $\square$

Remark 4.3.6. We show that the spectral sequence $\hat{E}^*(G,T)$ considered here is canonically isomorphic to the spectral sequence obtained from Greenlees' $\mathbb{Z}$-graded 'filtration' of $\tilde{E}$, [11], [12]. This is the sequence of G-CW-spectra,

$$\cdots \to \tilde{E}_{r-1} \to \tilde{E}_r \to \tilde{E}_{r+1} \to \cdots,$$

where, for $r \geq 0$, $\tilde{E}_r$ is the suspension G-spectrum of the r-skeleton of $\tilde{E}$, and for $r < 0$, $\tilde{E}_r$ is the dual $D(\tilde{E}_{-r}) = \Gamma F(\tilde{E}_{-r}, S^0)$. In particular, $\tilde{E}_0 = S^0$ is the sphere G-spectrum. The maps $\tilde{E}_{r-1} \to \tilde{E}_r$ are induced from the canonical inclusions, and for $r = 0$, from the canonical map $D(S^0) \xrightarrow{\sim} S^0$. In the definition of the G-CW-spectra $\bar{X}_{r,s}$ and $\bar{X}_n$, we now may vary r over all integers. Let $\bar{X}'_{r,s}$ and $\bar{X}'_n$ denote the G-CW-spectra so obtained. Then, for $r \geq 0$, the canonical inclusion $\bar{X}_{r,s} \xrightarrow{\sim} \bar{X}'_{r,s}$ is a homotopy equivalence. We have maps of filtrations

$$\{\bar{X}_n\}_{n\in\mathbb{Z}} \to \{\bar{X}'_n\}_{n\in\mathbb{Z}} \leftarrow \{\bar{X}'_{r,0}\}_{r\in\mathbb{Z}},$$

and the filtration on the right is Greenlees' filtration. We show that both maps induce isomorphisms of the E^2-terms of the associated spectral sequences. In order to identify the E^1-terms, let $\varepsilon\colon \hat{P} \to k$ be the complete resolution, where

$$(\Sigma\hat{P})_s = H_s(\tilde{E}_s \cup C\tilde{E}_{s-1}; k)$$

with differential

$$H_s(\tilde{E}_s \cup C\tilde{E}_{s-1}) \xrightarrow{\partial_*} H_s(\Sigma\tilde{E}_{s-1}) \xleftarrow[\sim]{\text{susp}} H_{s-1}(\tilde{E}_{s-1}) \xrightarrow{i_*} H_{s-1}(E_{s-1} \cup C\tilde{E}_{s-2})$$

and with structure map

$$\varepsilon\colon \hat{P}_0 = H_1(\tilde{E}_1 \cup C\tilde{E}_0) \xrightarrow{\partial_*} H_1(\Sigma E_0) \xleftarrow[\sim]{\text{susp}} H_0(E_0) = k.$$

The map of distinguished triangles

$$\begin{array}{ccccccc}
P & \xrightarrow{\varepsilon} & k & \longrightarrow & \tilde{P} & \longrightarrow & \Sigma P \\
\| & & \downarrow{\scriptstyle \varepsilon^*} & & \downarrow & & \| \\
P & \longrightarrow & \Sigma P^- & \longrightarrow & \Sigma\hat{P} & \longrightarrow & \Sigma P
\end{array}$$

defines a quasi-isomorphism of the mapping cones of the two middle vertical maps. (See remark 4.1.4 for the definition of the lower triangle.) Now an argument similar to the proof of Lemma 4.3.4 identifies the maps of E^1-terms induced from the above maps of filtrations with the canonical maps

$$(\tilde{P} \otimes \mathrm{Hom}(P,M))^G \to (\Sigma\hat{P} \otimes \mathrm{Hom}(P,M))^G \leftarrow (\Sigma\hat{P} \otimes \mathrm{Hom}(k,M))^G.$$

Finally, an argument similar to the proof of Lemma 4.1.2 shows that both maps are quasi-isomorphisms.

4.4. Again let C be a cyclic group of order r and let g be a generator. As our model for E, we choose the unit sphere

$$E = S(\mathbb{C}^\infty),$$

where the generator g acts on $\mathbb{C}$ by multiplication by $e^{2\pi i/r}$. We give E the usual C-CW-structure with one free cell in each dimension. The skeletons are

$$E_n = \begin{cases} S(\mathbb{C}^d) & n = 2d-1 \\ S(\mathbb{C}^d) * (C \cdot 1) & n = 2d, \end{cases}$$

where in the latter case, we identify the join with its image under the canonical homeomorphism $S(\mathbb{C}^n) * S(\mathbb{C}) \cong S(\mathbb{C}^n \oplus \mathbb{C})$. The attaching maps

$$\alpha_n \colon D^n \times C \to E_n$$

are defined in even dimensions by the composite

$$D^{2d} \times C \xrightarrow{\xi} D(\mathbb{C}^d) \times C \xrightarrow{\pi} S(\mathbb{C}^d) * (C \cdot 1),$$

where $\xi(z, g^s) = (g^s \cdot z, g^s)$ and π is the canonical projection. We define

$$\alpha_1(x, g^s) = g^s \cdot e^{\pi i(x+1)/r}$$

and let α_{2d+1} be the composite

$$D^{2d} \times D^1 \times C \xrightarrow{\xi} D(\mathbb{C}^d) \times D^1 \times C \xrightarrow{1\times\alpha_1} D(\mathbb{C}^d) \times S(\mathbb{C}) \xrightarrow{\pi} S(\mathbb{C}^d) * S(\mathbb{C}).$$

We give $D(\mathbb{C}^d)$ the complex orientation and $D^1 = D(\mathbb{R}) = [-1,1]$ the standard orientation from -1 to 1. We may then identify the cellular complex of E with the standard complex W by the isomorphism

$$W \xrightarrow{\sim} C_*(E; k)$$

which maps the generator $x_n \in W_n$ to the image of the fundamental class under the composite

$$H_n(D^n, S^{n-1}) \xrightarrow{\iota_0} H_n(D^n \times C, S^{n-1} \times C) \xrightarrow{\alpha_n} H_n(E_n, E_{n-1}).$$

Here $\iota_0 \colon D^n \to D^n \times C$ maps z to $(z, 1)$.

The C-CW-structure on E induces one on $\tilde{E}$ and the isomorphism above induces an isomorphism of chain complexes

$$\tilde{W} \xrightarrow{\sim} \tilde{C}_*(\tilde{E}; k).$$

We identify $\tilde{E}$ with $S^{\mathbb{C}^\infty}$ by the homeomorphism

$$CS(\mathbb{C}^\infty)_+ \cup S^0 \xrightarrow{\sim} D(\mathbb{C}^\infty)/S(\mathbb{C}^\infty)$$

which maps $t \wedge z \mapsto tz$. Note that under this homeomorphism, the orientation of the cells in $\tilde{E}$ corresponds to the complex orientation of $S^{\mathbb{C}^\infty}$. In particular, the composite

$$H_2(S^{\mathbb{C}}) \xleftarrow{\sim} H_2(\tilde{E}_2) \xrightarrow{\mathrm{pr}_*} H_2(\tilde{E}_2, \tilde{E}_1) \xleftarrow{\sim} \tilde{W}_2$$

maps the fundamental class $[S^{\mathbb{C}}]$ to the class Ny_2.

Let $C \subset \mathbb{T}$ be the subgroup of order r. We give $\mathbb{T}$ the C-CW-structure of $S(\mathbb{C}) = E_1$. Then the multiplication is cellular, and hence, the cellular complex

$$\Lambda = C_*(\mathbb{T}; k)$$

is naturally a differential graded Hopf algebra with unit $1 = x_0$. The differential maps x_1 to $(g-1) \cdot x_0$, x_1 is primitive, the coproduct on g is $g \otimes g$, and the antipode is given by $c(x_1) = -x_1$. We note that x_1 represents the fundamental class $[\mathbb{T}]$. The C-action on $E = S(\mathbb{C}^\infty)$ naturally extends to a $\mathbb{T}$-action, and the action map

$$\mu\colon \mathbb{T} \times E \to E$$

is cellular. The induced action on $\tilde{E}$,

$$\tilde{\mu}\colon \mathbb{T}_+ \wedge \tilde{E} = \mathbb{T}_+ \wedge C_{\mathrm{pr}} \xrightarrow{\rho} C_{\mathbb{T}_+ \wedge \mathrm{pr}} \xrightarrow{C_\mu} C_{\mathrm{pr}} = \tilde{E},$$

again is cellular. The induced left Λ-module structures on the cellular complexes W and $\tilde{W}$ are given by

$$x_1 \cdot x_s = \begin{cases} x_{s+1} & s \text{ even} \\ 0 & s \text{ odd} \end{cases}, \qquad x_1 \cdot y_s = \begin{cases} 0 & s \text{ even} \\ -y_{s+1} & s \text{ odd.} \end{cases}$$

Let T be a $\mathbb{T}$-spectrum and let $\bar{X} = \bar{X}(T)$ be the filtered $\mathbb{T}$-CW-spectrum, which gives rise to the spectral sequence $\hat{E}^*(C, T)$. We give $\mathbb{T}/C$ the skeleton filtration such that $\Lambda_C = C_*(\mathbb{T}/C; k)$. Then the $\mathbb{T}$-actions on E, $\tilde{E}$, and T induce a filtration-preserving map

$$\omega\colon \mathbb{T}/C_+ \wedge \bar{X}^C \to \bar{X}^C.$$

An argument similar to the proof of Lemma 4.3.4 identifies the induced map of E^1-terms of the associated spectral sequences with the map

$$\Lambda_C \otimes (\tilde{W} \otimes \mathrm{Hom}(W, \pi_* T))^C \to (\tilde{W} \otimes \mathrm{Hom}(W, \pi_* T))^C$$

given by the composite

$$\begin{aligned} \Lambda_C \otimes (\tilde{W} \otimes \mathrm{Hom}(W, \pi_* T))^C &\xrightarrow[\sim]{N \otimes \mathrm{id}} \Lambda^C \otimes (\tilde{W} \otimes \mathrm{Hom}(W, \pi_* T))^C \\ &\to (\Lambda \otimes \tilde{W} \otimes \mathrm{Hom}(W, \pi_* T))^C \xrightarrow{\omega_*} (\tilde{W} \otimes \mathrm{Hom}(W, \pi_* T))^C. \end{aligned}$$

PROPOSITION 4.4.1. *Let T be a $\mathbb{T}$-spectrum. Then $\hat{E}^*(C, T)$ is a spectral sequence of left Λ_C-modules. Moreover, if the class $a \in \pi_*(\hat{\mathbb{H}}(C, T))$ is represented by the infinite cycle $z \in E^1_{s,t}$, and if $x_1 \cdot z \in E^1_{s+1,t}$ is nonzero, then $x_1 \cdot z$ is an infinite cycle and represents the class of $da \in \pi_*(\hat{\mathbb{H}}(C, T))$.*

Let k be a perfect field of odd characteristic p and let $T(k)$ be the topological Hochschild spectrum of k. Then as a differential graded k-algebra,

$$\pi_*(T(k), \mathbb{Z}/p) = \Lambda\{\varepsilon\} \otimes S\{\sigma\}$$

with the classes $\varepsilon \in \pi_1(T(k), \mathbb{Z}/p)$ and $\sigma \in \pi_2(T(k), \mathbb{Z}/p)$ characterized by $\beta(\varepsilon) = 1$ and $d(\varepsilon) = \sigma$. The Tate spectral sequence takes the form

$$\hat{E}^2(C_p, M_p \wedge T(k)) = \Lambda\{u_1, \varepsilon\} \otimes S\{t^{\pm 1}, \sigma\} \Rightarrow \pi_*(\hat{\mathbb{H}}(C_p, T(k)), \mathbb{Z}/p),$$

where $u_1 = u$ and t are the generators of $\hat{H}^*(C_p, k)$ from Lemma 4.2.1. The nonzero differentials are multiplicatively generated from $d^2(\varepsilon) = t\sigma$.

Corollary 4.4.2. *The image of the classes ε and σ under the map induced from*

$$\hat{\Gamma}_k : T(k) \to \hat{\mathbb{H}}(C_p, T(k))$$

are represented by the infinite cycles ut^{-1} and t^{-1}, respectively.

Proof. We recall from Section 1.1 that $\hat{\Gamma}_k$ is defined as the composite

$$T(k) \underset{\sim}{\overset{r}{\leftarrow}} \rho^*_{C_p}(\tilde{E} \wedge T)^{C_p} \to \rho^*_{C_p}(\tilde{E} \wedge F(E_+, T))^{C_p}.$$

Both maps are $\mathbb{T}$-equivariant, so $\hat{\Gamma}$ commutes with Connes' operator. It also commutes with the Bockstein operator. Hence, it suffices to show that $ut^{-1} \otimes 1$ represents the unique class whose Bockstein is the multiplicative unit 1, and that $t^{-1} \otimes 1$ represents the image under Connes' operator of this class. To this end, we recall from Lemma 4.2.1 that the classes 1, ut^{-1} and t^{-1} in $\hat{H}^*(C_p, \mathbb{F}_p)$ are represented by the elements $y_0 \otimes Nx_0^*$, $-Ny_1 \otimes Nx_0^*$ and $Ny_2 \otimes Nx_0^*$, respectively. We recall from Section 2.1 above that the Bockstein

$$\beta : \hat{H}^*(C_p, \mathbb{F}_p) \to \hat{H}^{*+1}(C_p, \mathbb{Z})$$

is equal to the connecting homomorphism associated with the exact sequence

$$0 \to (\tilde{W} \otimes \mathrm{Hom}(W, \mathbb{Z}))^{C_p} \xrightarrow{p} (\tilde{W} \otimes \mathrm{Hom}(W, \mathbb{Z}))^{C_p} \xrightarrow{\mathrm{pr}} (\tilde{W} \otimes \mathrm{Hom}(W, \mathbb{F}_p))^{C_p} \to 0.$$

This takes $-Ny_1 \otimes Nx_0^*$ to $y_0 \otimes Nx_0^*$, and hence $\beta(ut^{-1}) = 1$. Next,

$$x_1 \cdot (-Ny_1 \otimes Nx_0^*) = -N(x_1 \cdot y_1) \otimes Nx_0^* + Ny_1 \otimes N(x_1 \cdot x_0^*) = Ny_2 \otimes Nx_0^*,$$

and so by Proposition 4.4.1, the image under Connes' operator of the class represented by $ut^{-1} \otimes 1$ is represented by $t^{-1} \otimes 1$. $\square$

Finally, for a $\mathbb{T}$-spectrum T, we will also consider the $\mathbb{T}$-Tate spectrum

$$\hat{\mathbb{H}}(\mathbb{T}, T) = (\tilde{E} \wedge \Gamma F(E_+, T))^{\mathbb{T}},$$

where again $E = S(\mathbb{C}^\infty)$. The filtration of E by the odd skeletons E_{2d-1}, $d \geq 1$, and the associated filtration of $\tilde{E}$ both are preserved by the $\mathbb{T}$-action.

The induced filtration of the Tate spectrum gives a conditionally convergent spectral sequence

$$\hat{E}^2(\mathbb{T}, T) = S\{t^{\pm 1}\} \otimes \pi_*(T) \Rightarrow \pi_*(\hat{\mathbb{H}}(\mathbb{T}, T)),$$

with the generator t in bi-degree $(-2, 0)$. Let $C \subset \mathbb{T}$ be a subgroup. Then the canonical inclusion $\hat{\mathbb{H}}(\mathbb{T}, T) \to \hat{\mathbb{H}}(C, T)$ induces a map of spectral sequences

$$\hat{E}^*(\mathbb{T}, T) \to \hat{E}^*(C, T).$$

If the order of C is odd and annihilates $\pi_*(T)$ then, on E^2-terms, this map is the canonical inclusion which maps t to the generator t of Lemma 4.2.1. This is the case, for instance, if $T = M_p \wedge T(A|K)$ and $C = C_{p^n}$.

PROPOSITION 4.4.3. *Let T be a $\mathbb{T}$-spectrum and let $C \subset \mathbb{T}$ be a subgroup whose order r is odd and annihilates $\pi_*(T)$. Then the d^2-differential in*

$$\hat{E}^2(C, T) = \hat{H}^*(C, \mathbb{Z}/r) \otimes \pi_*(T) \Rightarrow \pi_*(\hat{\mathbb{H}}(C, T))$$

is given by $d^2(\gamma \otimes \tau) = \gamma t \otimes d\tau$, where d is Connes' operator.

Proof. It was proved in [16, Lemma 1.4.2] that in the $\mathbb{T}$-Tate spectral sequence, the d^2-differential is given by the formula of the statement. Moreover, every C-spectrum T is a module C-spectrum over the sphere C-spectrum S^0. Hence, it suffices to show that the class u is a d^2-cycle in the spectral sequence $\hat{E}^*(C, S^0)$. But $\pi_1(S^0, \mathbb{Z}/r)$ vanishes since r is odd. □

5. The Tate spectral sequence for $T(A|K)$

5.1. The Tate spectral sequence $\hat{E}^*(C_{p^n}, M_p \wedge T(A|K))$ is a spectral sequence of bi-graded k-algebras in a canonical way, which we now explain. (We will abuse notation and write $\hat{E}^*(C_{p^n}, T(A|K))$ for this spectral sequence.) For every C_{p^n}-ring spectrum T, $\hat{\mathbb{H}}(C_{p^n}, T)$ is a $T^{C_{p^n}}$-algebra spectrum, and the Tate spectral sequence is one of bi-graded $\bar{\pi}_*(T^{C_{p^n}})$-algebras,

$$\hat{E}^2(C_{p^n}, T) = \hat{H}^{-*}(C_{p^n}, F^{n*}\bar{\pi}_*(T)) \Rightarrow \bar{\pi}_*(\hat{\mathbb{H}}(C_{p^n}, T)),$$

where $F^n \colon T^{C_{p^n}} \to T$ is the natural inclusion. Here $F^{n*}\bar{\pi}_*T$ denotes the graded ring $\bar{\pi}_*T = \pi_*(T, \mathbb{Z}/p)$ considered as a $\bar{\pi}_*(T^{C_{p^n}})$-algebra via the ring homomorphism induced by F^n. In the case at hand, we consider this a spectral sequence of bi-graded k-algebras via the ring homomorphism (3.1.3),

$$\rho_{n+1} \colon k \to \bar{W}_{n+1}(A) = \bar{\pi}_0(T(A|K)^{C_{p^n}}).$$

We recall that $F^n \circ \rho_{n+1} = \rho_1 \circ \varphi^n$, where $\varphi \colon k \to k$ is the Frobenius. The latter is an automorphism, by our assumption that k is perfect, and hence

$$\hat{E}^2(C_{p^n}, T(A|K)) = \Lambda\{u_n, d\log \pi_K\} \otimes S\{\pi_K, \kappa, t^{\pm 1}\}/(\pi_K^{e_K}),$$

where u_n and t are the canonical generators from Lemma 4.2.1. The differential structure of this spectral sequence is evaluated in this section. We briefly outline the argument.

The d^2-differential in $\hat{E}^*(C_{p^n}, T(A|K))$ is given by Proposition 4.4.3 in terms of Connes' operator (2.1.2) on $\bar{\pi}_* T(A|K)$. Hence, by Theorem 2.4.1,

$$d^2 \pi_K = t d \log \pi_K \cdot \pi_K, \qquad d^2 \kappa = t d \log(-p) \cdot \kappa,$$

and we can use the equation $-p = \pi_K^{e_K} \theta_K(\pi_K)^{-1}$ to express $d\log(-p)$ as a polynomial in π_K times $d\log \pi_K$. In Section 5.2, we replace κ by a new generator α_K, defined as a certain linear combination of the elements $\pi_K^i \kappa$ with $0 \le i < e_K$, which satisfies that $d\alpha_K = e_K d\log \pi_K \cdot \alpha_K$. In particular, α_K is a d^2-cycle, if p divides e_K. We also replace t by a new generator τ_K defined in a similar manner.

The key results that make it possible to completely evaluate the spectral sequence are consequences of the map

$$\hat{\Gamma}_{A|K} \colon T(A|K)^{C_{p^{n-1}}} \to \hat{\mathbb{H}}(C_{p^n}, T(A|K)),$$

and of the unit map of the ring spectrum on the right,

$$\ell \colon S^0 \to \hat{\mathbb{H}}(C_{p^n}, T(A|K)).$$

We show in Section 5.3 that for $n < v_p(e_K)$, $\pi_K^{p^n}$ and $-\tau_K \alpha_K$ are infinite cycles which represent the classes $\hat{\Gamma}_{A|K}(\underline{\pi_K}_n)$ and $\hat{\Gamma}_{A|K}(\underline{\pi_K}_n^{e_K/p^n})$, respectively. We also show that $-\tau_K \alpha_K^p$ is always an infinite cycle which represents the image by the unit map of the canonical generator $v_1 \in \bar{\pi}_{2p-2}(S^0)$. Given these infinite cycles together with the value of the differentials on the p-powers of π_K, which we examine by a universal example in Section 5.5, one can evaluate the spectral sequence, if $n < v_p(e_K)$. The final part of the argument consists of a somewhat complicated induction argument, which we present at the end of Section 5.5. The key for this part is naturality, going back and forth between $T(A|K)$ and $T(B|L)$ for suitable ramified extensions L/K.

The handling of the spectral sequences is algebraically somewhat complex. To ease the presentation we first consider in Section 5.4 the case of $\hat{E}^*(C_p, T(A|K))$. This section also contains the proof that the map $\hat{\Gamma}_{A|K}$ induces an isomorphism of homotopy groups with $\mathbb{Z}/p$-coefficients in nonnegative degrees.

5.2. Let L be a finite and totally ramified extension of K, and let B be the integral closure of A in L. Then B is a complete discrete valuation ring with quotient field L and residue field k. Let π_K and π_L be uniformizers of A and B, respectively. The minimal polynomial of π_L over K has the form

$$\phi_{L/K}(x) = x^{e_{L/K}} + \pi_K \theta_{L/K}(x),$$

where $\theta_{L/K}(x)$ is a polynomial over A of degree $< e_{L/K}$ and $\theta_{L/K}(0) \in A^\times$. Moreover, the canonical map

$$A[\pi_L]/(\phi_{L/K}(\pi_L)) \xrightarrow{\sim} B$$

is an isomorphism. When $K = K_0$ is the quotient field of $W(k)$, we will always use $\pi_{K_0} = p$ and write $\theta_L(x)$ instead of $\theta_{L/K_0}(x)$.

LEMMA 5.2.1. *Suppose that $\mu_p \subset K$. Then a choice of a generator $\zeta \in \mu_p$ and a uniformizer $\pi_K \in A$ determines a polynomial $u_K(x) \in W(k)[x]$ of degree $< e_K$ such that $u_K(\pi_K)^{p-1} = \theta_K(\pi_K)$. Moreover, in $\omega^1_{(A,M)}$,*

$$d\log\zeta = -\pi_K^{e_K/(p-1)} u_K(\pi_K)^{-1} d\log(-p).$$

Proof. Consider the power series $f(x) = px + x^p$ and $g(x) = (1+x)^p - 1$ and recall from [39, §3, Prop. 3] that there exists a *unique* power series $\varphi(x)$ such that $f(\varphi(x)) = \varphi(g(x))$ and $\varphi(x) \equiv x$ modulo (x^2). Hence, if $\zeta \in \mu_p$ is a generator then $\varphi(\zeta - 1)$ is a $(p-1)$st root of $-p$. We define $u_K(x)$ to be the unique polynomial of degree $< e_K$ such that

$$u_K(\pi_K) = \pi_K^{e_K/(p-1)} \varphi(\zeta - 1)^{-1}.$$

To prove the second statement, we first note that

$$\begin{aligned} d\varphi(\zeta - 1) &= \varphi(\zeta - 1) d\log\varphi(\zeta - 1) \\ &= \pi_K^{e_K/(p-1)} u_K(\pi_K)^{-1} \cdot (p-1)^{-1} d\log(-p) \\ &= -\pi_K^{e_K/(p-1)} u_K(\pi_K)^{-1} \cdot d\log(-p), \end{aligned}$$

where the last equality uses that $d\log(-p)$ is p-torsion. Hence, it suffices to show that $d\varphi(\zeta - 1) = d\log\zeta$. We may assume that $K = \mathbb{Q}_p(\mu_p)$, where as a uniformizer, we take $\pi_K = \zeta - 1$. Then $\omega^1_{(A,M)}$ is annihilated by π_K^{p-1}, and since $d\varphi(\zeta - 1) = \varphi'(\zeta - 1)\zeta d\log\zeta$, it remains to show that $\varphi'(x) \equiv (1+x)^{-1}$ modulo (x^{p-1}), or equivalently, that $\varphi(x) \equiv \log(1+x)$ modulo (x^p). But this follows from the uniqueness of $\varphi(x)$ and from the calculation in $\mathbb{Z}_p[x]/(x^p)$:

$$\log(1 + g(x)) = \log((1+x)^p) = p\log(1+x) = f(\log(1+x)). \qquad \square$$

ADDENDUM 5.2.2. *Let L/K be a finite and totally ramified extension. Then the inclusion of valuation rings, $\iota\colon A \to B$, maps*

$$\iota(u_K(\pi_K)) = (-\theta_{L/K}(\pi_L))^{-e_K/(p-1)} u_L(\pi_L).$$

Proof. We can write the $\iota(\varphi(\zeta - 1)) = \varphi(\zeta - 1)$ as

$$\iota(\pi_K^{e_K/(p-1)} u_K(\pi_K)^{-1}) = \pi_L^{e_L/(p-1)} u_L(\pi_L)^{-1}.$$

Since $\iota(\pi_K) = \theta_{L/K}(\pi_L)^{-1}\pi_L^{e_{L/K}}$, the left-hand side also is equal to

$$(-\theta_{L/K}(\pi_L)^{-1}\pi_L^{e_{L/K}})^{e_K/(p-1)}\iota(u_K(\pi_K)^{-1}).$$

The formula follows since $e_{L/K}e_K = e_L$ and since π_L is a nonzero divisor. $\square$

Suppose that $\mu_p \subset K$. We choose a generator $\zeta \in \mu_p$ and a uniformizer $\pi_K \in A$ and let $u_K(x)$ be the polynomial from Lemma 5.2.1. Let $\kappa \in \bar{\pi}_2 T(A|K)$ be the unique class with $\beta(\kappa) = d\log(-p)$ and define $\alpha_K = u_K(\pi_K)^{-1}\kappa$.

PROPOSITION 5.2.3. *As a differential graded k-algebra*

$$\bar{\pi}_* T(A|K) = \Lambda\{d\log\pi_K\} \otimes S\{\alpha_K, \pi_K\}/(\pi_K^{e_K})$$

with $d\pi_K = \pi_K d\log\pi_K$ and $d\alpha_K = e_K\alpha_K d\log\pi_K$.

Proof. It follows from Theorem 2.4.1 and Lemma 2.2.3 that as a differential graded k-algebra

$$\bar{\pi}_* T(A|K) = \Lambda\{d\log\pi_K\} \otimes S\{\kappa, \pi_K\}/(\pi_K^{e_K})$$

with the differential given by $d\pi_K = \pi_K d\log\pi_K$ and $d\kappa = \kappa d\log(-p)$. Moreover, differentiating the equation $-p = \pi_K^{e_K}\theta_K(\pi_K)^{-1}$, we find

$$d\log(-p) = (e_K d\log\pi_K - d\log\theta_K(\pi_K)).$$

Finally, $\theta_K(\pi_K) = u_K(\pi_K)^{p-1}$, and hence

$$\begin{aligned} d(\alpha_K) &= -u_K(\pi_K)^{-1} d\log u_K(\pi_K)\cdot\kappa + u_K(\pi_K)^{-1}\cdot\kappa d\log(-p) \\ &= -\alpha_K d\log u_K(\pi_K) + \alpha_K(e_K d\log\pi_K - (p-1)d\log u_K(\pi_K)) \\ &= e_K\alpha_K d\log\pi_K \end{aligned}$$

as stated. $\square$

We recall the Bott element. Since p is odd, the Bockstein is an isomorphism,

$$\bar{\pi}_2(\Sigma^\infty B\mu_{p+}) \xrightarrow{\sim} {}_p\pi_1(\Sigma^\infty B\mu_{p+}) \xleftarrow{\sim} \mu_p,$$

and by definition, the Bott element $b = b_\zeta$ is the class on the left which corresponds to the chosen generator ζ on the right. The spectrum $\Sigma^\infty B\mu_{p+}$ is a ring spectrum and the $(p-1)$st power b^{p-1}, which is independent of the choice of generator, is the image by the unit map of a generator v_1 in $\bar{\pi}_{2p-2}(S^0)$. If $\mu_p \subset K$, we have the maps of ring spectra

$$\Sigma^\infty B\mu_{p+} \xrightarrow{\det} K(K) \xrightarrow{\mathrm{tr}} T(A|K)^{C_{p^{n-1}}},$$

and let $b_n = b_{n,\zeta}$ be the image of the Bott element in $\bar{\pi}_2(T(A|K)^{C_{p^{n-1}}})$. We note that $\beta(b_n) = d\log_n\zeta$ and that since $\pi_2(T(A|K)^{C_{p^{n-1}}})$ is uniquely divisible, this equation characterizes b_n. In particular, the calculation

$$\beta(b_1) = d\log\zeta = -\pi_K^{e_K/(p-1)}u_K(\pi_K)^{-1}d\log(-p) = \beta(-\pi_K^{e_K/(p-1)}\alpha_K)$$

shows that

$$b_1 = -\pi_K^{e_K/(p-1)}\alpha_K. \tag{5.2.4}$$

The elements b_n for $n > 1$, however, are not well understood.

Let L/K be a finite and totally ramified extension, and let $\iota\colon A \to B$ be the inclusion of valuation rings. Then the map

$$\iota_*\colon \bar{\pi}_* T(A|K) \to \bar{\pi}_* T(B|L) \tag{5.2.5}$$

is given by

$$\begin{aligned} \iota_*(\pi_K) &= -\theta_{L/K}(\pi_L)^{-1}\pi_L^{e_{L/K}}, \\ \iota_*(d\log\pi_K) &= e_{L/K} d\log\pi_K - d\log\theta_{L/K}(\pi_L), \\ \iota_*(\alpha_K) &= (-\theta_{L/K}(\pi_L))^{e_K/(p-1)}\alpha_L. \end{aligned}$$

The first two equalities follow immediately from the definition of $\theta_{L/K}(\pi_L)$, and the last equality follows from Addendum 5.2.2.

Let $f(x) \in k[x]$ and let n be an integer. We write $f^{(n)}(x)$ for the image of $f(x)$ under the automorphism $\varphi^n[x]\colon k[x] \to k[x]$, which applies φ^n to the coefficients of a polynomial. If R is a k-algebra and if $\pi \in R$ then, as usual, $f(\pi)$ denotes the image of $f(x)$ by the unique k-algebra homomorphism $k[x] \to R$ which takes x to π. We note that $f^{(-n)}(\pi) \in \varphi^{n*}R$ and $f(\pi) \in R$ is the same element.

Suppose either $\mu_p \subset K$ or $K = K_0$. In the former case, let π_K be a uniformizer, let $\zeta \in \mu_p$ be a generator and let $u_K(x)$ be the polynomial from Lemma 5.2.1. In the latter case, let $u_{K_0}(x) = 1$. Then as a bi-graded k-algebra,

$$\hat{E}^2(C_{p^n}, T(A|K)) = \Lambda\{u_n, d\log\pi_K\} \otimes S\{\pi_K, \alpha_K, \tau_K^{\pm 1}\}/(\pi^{e_K}), \tag{5.2.6}$$

with the new generators given by

$$\alpha_K = u_K^{(-n)}(\pi_K)^{-1}\kappa, \qquad \tau_K = u_K^{(-n)}(\pi_K)^p\, t.$$

We note the relations $\tau_K\alpha_K = \theta_K^{(-n)}(\pi_K)t\kappa$ and $\tau_K\alpha_K^p = t\kappa^p$.

It will be important to know how these new generators behave under extensions. For integers a, r, d with $0 \le r < e_K$ and $d \ge 0$, we define

$$\{a, r, d\}_K = (pa - d)e_K/(p-1) + r.$$

If $\mu_p \subset K$ then $p-1$ divides e_K such that $\{a, r, d\}_K$ is an integer. Let L/K be a finite and totally ramified extension, and let $\iota\colon A \to B$ be the inclusion of valuation rings. Then $\{a, e_{L/K}r, d\}_L = e_{L/K}\{a, r, d\}_K$ and

$$\iota_*\colon \hat{E}^2(C_{p^n}, T(A|K)) \to \hat{E}^2(C_{p^n}, T(B|L)), \tag{5.2.7}$$

is given by

$$\begin{aligned} \iota_*(\tau_K^a \pi_K^r \alpha_K^d) &= (-\theta_{L/K}^{(-n)}(\pi_L))^{-\{a,r,d\}_K} \tau_L^a \pi_L^{e_{L/K} r} \alpha_L^d, \\ \iota_*(d\log \pi_K) &= (e_{L/K} - \frac{\theta_{L/K}^{(-n)\prime}(\pi_L)\pi_L}{\theta_{L/K}^{(-n)}(\pi_L)}) d\log \pi_L. \end{aligned}$$

5.3. In this section, we produce a number of infinite cycles in the spectral sequence $\hat{E}^*(C_{p^n}, T(A|K))$. This uses the maps of differential graded k-algebras

$$\bar{\pi}_* T(A|K) \xleftarrow{j_*} \bar{\pi}_* T(A) \xrightarrow{\rho_*} \bar{\pi}_* T(A/p),$$

where the right-hand map is induced from the reduction. We evaluate these maps assuming that $v_p(e_K) > 0$. The left map may be identified with the map of graded k-algebras

$$j_*\colon \Lambda\{d\pi_K\} \otimes S\{\tilde{\kappa}, \pi_K\}/(\pi_K^{e_K}) \to \Lambda\{d\log \pi_K\} \otimes S\{\kappa, \pi_K\}/(\pi_K^{e_K}),$$

which takes π_K to π_K, $d\pi_K$ to $\pi_K d\log \pi_K$ and $\tilde{\kappa}$ to κ. (See the discussion preceding Theorem 2.4.1.) The group $\pi_2 T(A)$ is uniquely divisible so the Bockstein induces an isomorphism $\beta\colon \bar{\pi}_2 T(A) \xrightarrow{\sim} {}_p\pi_1 T(A)$ and the class $\tilde{\kappa}$ corresponds to the generator

$$d\log(-p) = -((e_K/p)\pi_K^{e_K-1} + \theta_K'(\pi_K))\theta_K(\pi_K)^{-1} d\pi_K$$

on the right. The differential graded k-algebra $\bar{\pi}_* T(A/p)$ is evaluated in Proposition A.1.4 of the appendix. We refer to *loc. cit.* for the notation.

PROPOSITION 5.3.1. *If $v_p(e_K) > 0$ the map $\rho_*\colon \bar{\pi}_* T(A) \to \bar{\pi}_* T(A/p)$ may be identified with the inclusion of differential graded k-algebras*

$$\rho_*\colon \Lambda\{d\pi_K\} \otimes S\{\pi_K, \tilde{\kappa}\}/(\pi_K^{e_K}) \hookrightarrow \Lambda\{d\bar{\pi}_K, \varepsilon\} \otimes S\{\sigma, \bar{\pi}_K\}/(\bar{\pi}_K^{e_K}) \otimes \Gamma\{\bar{c}_2\}$$

which takes π_K to $\bar{\pi}_K$ and $\tilde{\kappa}$ to the class

$$\sigma - \theta_K(\bar{\pi}_K)^{-1}\bar{c}_2 - \varepsilon \cdot ((e_K/p)\bar{\pi}_K^{e_K-1} + \theta_K'(\bar{\pi}_K))\theta_K(\bar{\pi}_K)^{-1} d\bar{\pi}_K.$$

Proof. Only the formula for $\rho_*(\tilde{\kappa})$ requires proof. Consider the diagram

$$\begin{tikzcd} T(A) \arrow[r, "p"] \arrow[d, "T(\rho)"] & T(A) \arrow[r, "i"] \arrow[d, "T(\rho)"] & M_p \wedge T(A) \arrow[r, "\beta"] \arrow[d, "M_p \wedge T(\rho)"] & \Sigma T(A) \arrow[d, "\Sigma T(\rho)"] \\ T(A/p) \arrow[r, "p=0"] & T(A/p) \arrow[r, shift left, "i"] & M_p \wedge T(A/p) \arrow[l, shift left, "r"] \arrow[r, shift left, "\beta"] & \Sigma T(A/p), \arrow[l, shift left, "s"] \end{tikzcd}$$

with horizontal triangles, the lower triangle split by the maps r and s of Section 2.1 above. It shows that

$$\rho_*(\tilde{\kappa}) = \varepsilon \cdot ((\Sigma T(\rho))_* \circ \beta_*)(\tilde{\kappa}) + (i_* \circ r_* \circ (M_p \wedge T(\rho))_*)(\tilde{\kappa}).$$

The value of the first summand is easily determined from the diagram

$$\begin{array}{ccccccc}
\pi_2(M_p \wedge T(A)) & \xrightarrow{\beta_*} & \pi_2(\Sigma T(A)) & \xleftarrow[\sim]{\text{susp}} & \pi_1(T(A)) & \xleftarrow{\sim} & \Omega^1_A \\
\big\downarrow{\scriptstyle (M_p\wedge T(\rho))_*} & & \big\downarrow{\scriptstyle (\Sigma T(\rho))_*} & & \big\downarrow{\scriptstyle T(\rho)_*} & & \big\downarrow{\scriptstyle \rho_*} \\
\pi_2(M_p \wedge T(A/p)) & \xrightarrow{\beta_*} & \pi_2(\Sigma T(A/p)) & \xleftarrow[\sim]{\text{susp}} & \pi_1(T(A/p)) & \xleftarrow{\sim} & \Omega^1_{A/p}
\end{array}$$

and the formula for the Bockstein of $\tilde{\kappa}$ above:

$$((\Sigma T(\rho))_* \circ \beta_*)(\tilde{\kappa}) = -((e_K/p)\bar{\pi}_K^{e_K-1} + \theta'_K(\bar{\pi}_K))\theta_K(\bar{\pi}_K)^{-1} d\bar{\pi}_K.$$

It remains to show that

$$(r_* \circ (M_p \wedge T(\rho))_*)(\tilde{\kappa}) = \sigma - \theta_K(\bar{\pi}_K)^{-1}\bar{c}_2.$$

We first show that the linearization $l_*\colon \pi_* T(A/p) \to \pi_* \mathrm{HH}(A/p)$ takes this class to $-\theta_K(\bar{\pi}_K)^{-1}\bar{c}_2$. The following diagram

$$\begin{array}{ccccc}
\pi_*(M_p \wedge T(A)) & \xrightarrow{\rho_*} & \pi_*(M_p \wedge T(A/p)) & \xrightarrow{r_*} & \pi_* T(A/p) \\
\big\downarrow{\scriptstyle l_*} & & \big\downarrow{\scriptstyle l_*} & & \big\downarrow{\scriptstyle l_*} \\
\pi_*(M_p \wedge \mathrm{HH}(A)) & \xrightarrow{\rho_*} & \pi_*(M_p \wedge \mathrm{HH}(A/p)) & \xrightarrow{r_*} & \pi_* \mathrm{HH}(A/p)
\end{array}$$

commutes and the composite of the lower horizontal maps is an isomorphism. Let $c_1, c_2^{[d]} \in \pi_*(M_p \wedge \mathrm{HH}(A))$ be the classes which correspond to $\bar{c}_1, \bar{c}_2^{[d]} \in \pi_* \mathrm{HH}(A/p)$ under this isomorphism. We claim that $c_2 = -\theta_K(\pi_K)\tilde{\kappa}$. The diagram

$$\begin{array}{ccc}
\pi_2(M_p \wedge T(A)) & \xrightarrow[\sim]{\beta} & {}_p\pi_1 T(A) \\
\sim\big\downarrow{\scriptstyle l_*} & & \sim\big\downarrow{\scriptstyle l_*} \\
\pi_2(M_p \wedge \mathrm{HH}(A)) & \xrightarrow[\sim]{\beta} & {}_p\pi_1 \mathrm{HH}(A),
\end{array}$$

where all maps are isomorphisms, shows that to prove this, it will suffice to show that the lower Bockstein takes c_2 to $((e/p)\pi_K^{e_K-1} + \theta'_K(\pi_K))d\pi_K$. This Bockstein, in turn, may be identified with the connecting homomorphism in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A \cdot c_2 & \xrightarrow{p} & A \cdot c_2 & \xrightarrow{\rho} & A/p \cdot \bar{c}_2 & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \phi'_K(\pi_K)} & & \big\downarrow{\scriptstyle \phi'_K(\pi_K)} & & \big\downarrow{\scriptstyle \phi'_K(\bar{\pi}_K)=0} & & \\
0 & \longrightarrow & A \cdot c_2 & \xrightarrow{p} & A \cdot c_2 & \xrightarrow{\rho} & A/p \cdot \bar{c}_2 & \longrightarrow & 0
\end{array}$$

and the claim follows; compare Section A.1 below. We consider again the diagram from the beginning of the proof. This may be further refined to a diagram of horizontal triangles

$$\begin{array}{ccccccc}
T(A;A) & \xrightarrow{p} & T(A;A) & \xrightarrow{i} & M_p \wedge T(A;A) & \xrightarrow{\beta} & \Sigma T(A;A) \\
\big\downarrow{\scriptstyle T(A;\rho)} & & \big\downarrow{\scriptstyle T(A;\rho)} & & \big\downarrow{\scriptstyle M_p\wedge T(A;\rho)} & & \big\downarrow{\scriptstyle \Sigma T(A;\rho)} \\
T(A;A/p) & \xrightarrow{p} & T(A;A/p) & \underset{r}{\overset{i}{\rightleftarrows}} & M_p \wedge T(A;A/p) & \underset{s}{\overset{\beta}{\rightleftarrows}} & \Sigma T(A;A/p) \\
\big\downarrow{\scriptstyle T(\rho;A/p)} & & \big\downarrow{\scriptstyle T(\rho;A/p)} & & \big\downarrow{\scriptstyle M_p\wedge T(\rho;A/p)} & & \big\downarrow{\scriptstyle \Sigma T(\rho;A/p)} \\
T(A/p;A/p) & \xrightarrow{p=0} & T(A/p;A/p) & \underset{r}{\overset{i}{\rightleftarrows}} & M_p \wedge T(A/p;A/p) & \underset{r}{\overset{\beta}{\rightleftarrows}} & \Sigma T(A/p;A/p),
\end{array}$$

where for an A-A-bimodule M, $T(A;M)$ is the topological Hochschild spectrum of A with coefficients in M. It shows that

$$(r_* \circ (M_p \wedge T(\rho))_*)(\tilde{\kappa}) = (T(\rho;A/p)_* \circ r_* \circ (M_p \wedge T(A;\rho))_*)(\tilde{\kappa}).$$

The map $T(\rho;A/p)_*$ is equal to the edge homomorphism of the spectral sequence

$$E^2_{s,t} = \pi_s T(A/p, \operatorname{Tor}^A_t(A/p,A/p)) \Rightarrow \pi_{s+t} T(A;A/p)$$

considered in [27]. Hence, *loc.cit.*, Proposition 4.3, shows that there is a unique class in the image of

$$T(\rho;A/p)_*\colon \pi_2 T(A;A/p) \to \pi_2 T(A/p;A/p) = \pi_2 T(A/p)$$

whose image under $l_*\colon \pi_2 T(A/p) \to \pi_2 \operatorname{HH}(A/p)$ is $-\theta_K(\bar{\pi}_K)^{-1}\bar{c}_2$ and that this class has the form $\lambda\cdot\sigma - \theta_K(\bar{\pi}_K)^{-1}\bar{c}_2$, where $\lambda \in (\mathbb{Z}/p)^\times$ is a unit. Finally, the following lemma shows that $\lambda = 1$ (or equivalently, that the class σ of *loc.cit.* agrees with our class σ). $\square$

Lemma 5.3.2. *The reduction $i_*\colon \bar{\pi}_* T(A) \to \bar{\pi}_* T(k)$ maps $\tilde{\kappa}$ to σ.*

Proof. We proved in Addendum 3.3.9 that in the diagram

$$\begin{array}{ccccc}
\bar{\pi}_2 T(A|K) & \xleftarrow{j_*} & \bar{\pi}_2 T(A) & \xrightarrow{i_*} & \bar{\pi}_2 T(k) \\
\big\downarrow{\scriptstyle \partial_{A|K}} & & \big\downarrow{\scriptstyle \partial_A} & & \big\downarrow{\scriptstyle \partial_k} \\
\bar{\pi}_1 \mathbb{H}_\bullet(C_p, T(A|K)) & \xleftarrow{j_*} & \bar{\pi}_1 \mathbb{H}_\bullet(C_p, T(A)) & \xrightarrow{i_*} & \bar{\pi}_1 \mathbb{H}_\bullet(C_p, T(k))
\end{array}$$

the left-hand vertical map takes κ to $dV(1) - V(d\log(-p))$. It follows that the middle vertical map takes $\tilde{\kappa}$ to $dV(1) - V(d\log(-p)) + aV(\pi_K^{e_K-1}d\pi_K)$ for some $a \in A$. Since Ω^1_k vanishes, we conclude that the right-hand vertical map takes $i_*(\tilde{\kappa})$ to $dV(1)$, and since $\bar{\pi}_2 T(k)$ is a one-dimensional k-vector space, it thus suffices to show also that $\partial_k(\sigma) = dV(1)$. To this end, we consider the diagram

$$\begin{array}{ccccc}
\bar{\pi}_2 T(k) & \xleftarrow{d} & \bar{\pi}_1 T(k) & \xrightarrow{\hat{\Gamma}} & \bar{\pi}_1 \hat{\mathbb{H}}(C_p, T(k)) \\
\downarrow{\scriptstyle \partial_k} & (-1) & \downarrow{\scriptstyle \partial_k} & & \downarrow{\scriptstyle \partial_k^h} \\
\bar{\pi}_1 \mathbb{H}_{\bullet}(C_p, T(k)) & \xleftarrow{d} & \bar{\pi}_0 \mathbb{H}_{\bullet}(C_p, T(k)) & = & \bar{\pi}_0 \mathbb{H}_{\bullet}(C_p, T(k)),
\end{array}$$

where the left-hand square anti-commutes by our conventions from Section 2.1 above. The class σ, by definition, is the image of ε under the top differential, and the bottom differential takes $V(1)$ to $dV(1)$. Hence, it suffices to show that $\partial_k(\varepsilon) = -V(1)$. We recall from corollary 4.4.2 that the class $\hat{\Gamma}(\varepsilon)$ is represented in the spectral sequence $\hat{E}^*(C_p, T(k))$ by the infinite cycle $u_1 t^{-1}$. Hence, Addendum 4.2.2 shows that the image of this class by the right-hand vertical map is $-V(1)$. □

Remark 5.3.3. It follows from Propositions 5.3.1 and A.1.4 that in $\bar{\pi}_* T(A)$,

$$d\tilde{\kappa} = -\theta'_K(\pi_K)\theta_K(\pi_K)^{-1} d\pi_K \cdot \tilde{\kappa}.$$

This implies that $d\kappa = \kappa d\log(-p)$ in $\bar{\pi}_* T(A|K)$ as stated in Theorem 2.4.1.

We construct a number of infinite cycles. Recall the map of ring spectra

$$\hat{\Gamma}_{A|K} \colon T(A|K)^{C_{p^{n-1}}} \to \hat{\mathbb{H}}(C_{p^n}, T(A|K)).$$

PROPOSITION 5.3.4. *For all K, the element $d\log \pi_K \in \hat{E}^2(C_{p^n}, T(A|K))$ is an infinite cycle and represents the homotopy class $\hat{\Gamma}_{A|K}(d\log_n \pi_K)$.*

Proof. We consider the diagram

$$\begin{array}{ccc}
T(A|K)^{C_{p^n}} & \xrightarrow{R} & T(A|K)^{C_{p^{n-1}}} \\
\downarrow{\scriptstyle \Gamma_{A|K}} & & \downarrow{\scriptstyle \hat{\Gamma}_{A|K}} \\
\mathbb{H}^{\bullet}(C_{p^n}, T(A|K)) & \xrightarrow{R^h} & \hat{\mathbb{H}}(C_{p^n}, T(A|K)).
\end{array}$$

In the spectral sequence

$$\begin{aligned}
E^2(C_{p^n}, T(A|K)) &= \Lambda\{u_n, d\log \pi_K\} \otimes S\{\pi_K, t, \kappa\}/(\pi_K^{e_K}) \\
&\Rightarrow \bar{\pi}_*(\mathbb{H}^{\bullet}(C_{p^n}, T(A|K))),
\end{aligned}$$

the element $d\log \pi_K$ is an infinite cycle and represents $\Gamma_{A|K}(d\log_{n+1} \pi_K)$. Indeed, if we compose $\Gamma_{A|K}$ and the edge-homomorphism of this spectral sequence, we get the map $F^n \colon \bar{\pi}_* T(A|K)^{C_{p^n}} \to \bar{\pi}_* T(A|K)$ which takes $d\log_{n+1} \pi_K$ to $d\log \pi_K$. The map R^h induces the obvious inclusion on E^2-terms. Hence, the element $d\log \pi_K$ of $\hat{E}^*(C_{p^n}, T(A|K))$ is an infinite cycle and, since it is not a boundary, represents $R^h(\Gamma_{A|K}(d\log_{n+1} \pi_K)) = \hat{\Gamma}_{A|K}(R(d\log_{n+1} \pi_K)) = \hat{\Gamma}_{A|K}(d\log_n \pi_K)$. □

Let $\alpha_A = u_K^{(-n)}(\pi_K)^{-1}\tilde{\kappa}$ and $\tau_A = u_K^{(-n)}(\pi_K)^p t$ such that $j_*(\alpha_A) = \alpha_K$ and $j_*(\tau_A) = \tau_K$.

LEMMA 5.3.5. *Suppose that $\mu_p \subset K$ and let $n < v_p(e_K)$. Then the elements $\pi_K^{p^n}$ and $-\tau_A\alpha_A$ of $\hat{E}^2(C_{p^n}, T(A))$ are infinite cycles and represent the homotopy classes $\hat{\Gamma}_A(\underline{\pi_K}_n)$ and $\hat{\Gamma}_A(\underline{\pi_K}_n^{e_K/p^n})$, respectively.*

Proof. We consider the diagram

$$\begin{array}{ccc}
\bar{\pi}_*(T(A)^{C_{p^{n-1}}}) & \xrightarrow{\Gamma_A} & \bar{\pi}_*(\hat{\mathbb{H}}(C_{p^n}, T(A))) \\
\Big\downarrow{\scriptstyle\rho_*} & & \Big\downarrow{\scriptstyle\rho_*} \\
\bar{\pi}_*(T(A/p)^{C_{p^{n-1}}}) & \xrightarrow{\Gamma_{A/p}} & \bar{\pi}_*(\hat{\mathbb{H}}(C_{p^n}, T(A))),
\end{array}$$

with the vertical maps induced from the reduction $\rho: A \to A/p$. The lower horizontal map is studied in the appendix. By Addendum A.1.6,

$$\begin{aligned}
\hat{E}^2(C_{p^n}, T(A/p)) &= \Lambda\{u_n, d\pi_K, \varepsilon\} \otimes S\{t^{\pm 1}, \pi_K, \sigma\}/(\pi_K^{e_K}) \otimes \Gamma\{\bar{c}_2\}, \\
\hat{E}^3(C_{p^n}, T(A/p)) &= \Lambda\{u_n, d\pi_K\} \otimes S\{t^{\pm 1}, \pi_K^p\}/(\pi_K^{e_K}) \otimes \Gamma\{\bar{c}_2\},
\end{aligned}$$

and $\hat{E}^3(C_{p^n}, T(A/p)) = \hat{E}^\infty(C_{p^n}, T(A/p))$. We compare this to

$$\begin{aligned}
\hat{E}^2(C_{p^n}, T(A)) &= \Lambda\{u_n, d\pi_K\} \otimes S\{\tau_A^{\pm 1}, \alpha_A, \pi_K\}/(\pi_K^{e_K}), \\
\hat{E}^3(C_{p^n}, T(A)) &= \Lambda\{u_n, \pi_K^{p-1}d\pi_K\} \otimes S\{\tau_A^{\pm 1}, \alpha_A, \pi_K^p\}/(\pi_K^{e_K}).
\end{aligned}$$

The map $\rho_*: \bar{\pi}_* T(A) \to \bar{\pi}_* T(A/p)$ was evaluated above. The induced map

$$\hat{E}^3(C_{p^n}, T(A)) \hookrightarrow \hat{E}^3(C_{p^n}, T(A/p))$$

is the monomorphism which takes $\tau_A\alpha_A$ to $-t\bar{c}_2$. Indeed, the map of E^2-terms takes the element $\tau_A\alpha_A$ to $-t\bar{c}_2 + \theta_K(\pi_K)t\sigma - t\varepsilon \cdot \theta'_K(\pi_K)d\pi_K$, and the last two summands are equal to the image by the d^2-differential of $\varepsilon \cdot \theta_K(\pi_K)$. For $0 \le s \le 1$, we have the diagram

$$\begin{array}{ccc}
\hat{E}^3_{-s,s}(C_{p^n}, T(A)) & \hookrightarrow & \hat{E}^3_{-s,s}(C_{p^n}, T(A/p)) \\
\Big\uparrow & & \Big\uparrow{\scriptstyle\sim} \\
\hat{E}^\infty_{-s,s}(C_{p^n}, T(A)) & \to & \hat{E}^\infty_{-s,s}(C_{p^n}, T(A/p))
\end{array}$$

and we conclude that the lower horizontal map is a monomorphism. We show in Proposition A.1.7 that the classes $\hat{\Gamma}_{A/p}(\underline{\pi_K}_n)$ and $\hat{\Gamma}_{A/p}(\underline{\pi_K}_n^{e_K/p^n})$ are represented in the spectral sequence $\hat{E}^*(C_{p^n}, T(A/p))$ by the infinite cycles $\pi_K^{p^n}$ and $t\bar{c}_2$, respectively. It follows immediately that $\hat{\Gamma}_A(\underline{\pi_K}_n)$ is represented by $\pi_K^{p^n}$ as stated. To conclude that $\hat{\Gamma}_A(\underline{\pi_K}_n^{e_K/p^n})$ is represented by $-\tau_A\alpha_A$ we must rule out that an element of $\hat{E}^2_{-s,s}(C_{p^n}, T(A))$ with $0 \le s \le 1$ represent this class. But this follows from the injectivity of the lower horizontal map in the diagram above. $\square$

PROPOSITION 5.3.6. *Suppose that $\mu_p \subset K$ and let $n < v_p(e_K)$. Then the elements $\pi_K^{p^n}$ and $-\tau_K\alpha_K$ of $\hat{E}^2(C_{p^n}, T(A|K))$ are infinite cycles and represent the homotopy classes $\hat{\Gamma}_{A|K}(\underline{\pi_K}_n)$ and $\hat{\Gamma}_{A|K}(\underline{\pi_K}_n^{e_K/p^n})$, respectively.*

Proof. The map $j_*: \hat{E}^2(C_{p^n}, T(A)) \to \hat{E}^2(C_{p^n}, T(A|K))$ takes the infinite cycle $\pi_K^{p^n}$ (resp. $-\tau_A\alpha_A$) to the element $\pi_K^{p^n}$ (resp. $-\tau_K\alpha_K$) which therefore is an infinite cycle. It follows $\pi_K^{p^n}$ (resp. $-\tau_K\alpha_K$) either represents $\hat{\Gamma}_{A|K}(\underline{\pi_K}_n)$ (resp. $\hat{\Gamma}_{A|K}(\underline{\pi_K}_n^{e_K/p^n})$) or else it is a boundary. The element $\pi_K^{p^n}$ cannot be a boundary, but we must check that $-\tau_K\alpha_K$ is not a d^3-boundary. To this end we consider the diagram

$$\begin{tikzcd} \hat{E}^3_{1,0}(C_{p^n}, T(A|K)) \arrow[r, "d^3"] & \hat{E}^3_{-2,2}(C_{p^n}, T(A|K)) \\ \hat{E}^3_{1,0}(C_{p^n}, T(A)) \arrow[r, "d^3"] \arrow[u, "\sim"', "j_*"] & \hat{E}^3_{-2,2}(C_{p^n}, T(A)) \arrow[u, "\sim"', "j_*"] \end{tikzcd}$$

with vertical isomorphisms. The right-hand vertical map takes $-\tau_A\alpha_A$ to $-\tau_K\alpha_K$, and since $-\tau_A\alpha_A$ is not a d^3-boundary, neither is $-\tau_K\alpha_K$. □

Let $\ell: S^0 \to \hat{\mathbb{H}}(C_{p^n}, T(A|K))$ be the unit map and let $v_1 \in \bar{\pi}_{2(p-1)}(S^0)$ be the canonical generator. If $\mu_p \subset K$, then $\ell_*(v_1) = \hat{\Gamma}_{A|K}(b_n)^{p-1}$.

ADDENDUM 5.3.7. *The elements $-t\kappa^p$ and $(-t\kappa)^{p^n}$ of $\hat{E}^2(C_{p^n}, T(A|K))$ are infinite cycles which, if not boundaries, represent the homotopy classes $\ell_*(v_1)$ and $V(1)$, respectively.*

Proof. The elements $-t\kappa^p$ and $(-t\kappa)^{p^n}$ are in the image of

$$\iota_*: \hat{E}^2(C_{p^n}, T(W(k)|K_0)) \to \hat{E}^2(C_{p^n}, T(A|K))$$

and so the statement, if valid for some K, is valid for all K. Now, suppose that $\mu_p \subset K$ and that $v_p(e_K) > n$. We may argue as in the proof of Proposition 5.3.4 that $\hat{\Gamma}_{A|K}(b_n)$ is represented by the infinite cycle $-\pi_K^{e_K/(p-1)}\alpha_K$. Indeed, $b_n = R(b_{n+1})$ and $F^n(b_{n+1}) = b_1$ and from (5.2.4) we know that $b_1 = -\pi_K^{e_K/(p-1)}\alpha_K$. Now

$$-\pi_K^{e_K/(p-1)}\alpha_K = -(\pi_K^{p^n})^{e_K/p^n(p-1)}\alpha_K,$$

and it follows from Proposition 5.3.6 that $\pi_K^{e_K/(p-1)}$ is an infinite cycle and represents $\hat{\Gamma}_{A|K}(\underline{\pi_K}_n^{e_K/p^n(p-1)})$. Hence, also α_K is an infinite cycle, and since it is not a boundary, represents a homotopy class, say, $\tilde{\alpha}_K$. Since the classes $\hat{\Gamma}_{A|K}(b_n)$ and $-\hat{\Gamma}_{A|K}(\underline{\pi_K}_n^{e_K/p^n(p-1)})\tilde{\alpha}_K$ are represented in the spectral sequence by the same element so are their $(p-1)$st powers. We know from Proposition 5.3.6 that

$$\hat{\Gamma}_{A|K}(\underline{\pi_K}_n^{e_K/p^n(p-1)})^{p-1} = \hat{\Gamma}_{A|K}(\underline{\pi_K}_n^{e_K/p^n})$$

is represented by $-\tau_K\alpha_K$. And α_K^{p-1}, if not a boundary, represents $\tilde{\alpha}_K^{p-1}$. It follows that $-\tau_K\alpha_K^p$, if not a boundary, represents $\hat{\Gamma}_{A|K}(b_n^{p-1}) = \ell_*(v_1)$.

We recall from Lemmas 3.1.1 and 3.1.2 that in the Witt ring $\bar{W}_n(A)$,

$$V(1) = \theta_K(\underline{\pi_K}_n)^{-1}\underline{\pi_K}_n^{e_K},$$

and hence, in $\bar{\pi}_*\hat{\mathbb{H}}(C_{p^n}, T(A|K))$,

$$\begin{aligned} V(1) &= V(\hat{\Gamma}_{A|K}(1)) = \hat{\Gamma}_{A|K}(V(1)) = \hat{\Gamma}_{A|K}(\theta_K(\underline{\pi_K}_n)^{-1}\underline{\pi_K}_n^{e_K}) \\ &= \hat{\Gamma}_{A|K}(\theta_K(\underline{\pi_K}_n))^{-1}\cdot\hat{\Gamma}_{A|K}(\underline{\pi_K}_n^{e_K/p^n})^{p^n}. \end{aligned}$$

It follows that the infinite cycle

$$\theta_K(\pi_K^{p^n})^{-1}(-\tau_K\alpha_K)^{p^n} = (-\theta_K^{(-n)}(\pi_K)^{-1}\tau_K\alpha_K)^{p^n} = (-t\kappa)^{p^n},$$

if not a boundary, represents $V(1)$ as stated. □

5.4. In this section we evaluate the spectral sequence $\hat{E}^*(C_p, T(A|K))$.

Lemma 5.4.1. *Let k be a field of characteristic $p > 0$, let $f(x)$ be a power series over k with nonzero constant term, and let*

$$\frac{f'(x)x}{f(x)} = a_1x + a_2x^2 + \dots$$

be the logarithmic derivative. Then $a_{pi} = a_i^p$, for all $i \geq 1$.

Proof. We may assume that $f(x)$ is a polynomial with $f(0) \in k^\times$. Moreover, replacing k by a splitting field for $f(x)$, we can assume that $f(x)$ splits as a product of linear factors. And since the logarithmic derivative takes products of power series to sums, we are reduced to the case of a linear polyonomial. The result in this case is readily verified by computation. □

Proposition 5.4.2. *Suppose either $\mu_p \subset K$ or $K = K_0$. Then, up to a unit, the nonzero differentials in the spectral sequence $\hat{E}^2(C_p, T(A|K))$ are generated from*

$$\begin{aligned} d^2(\tau_K^a\pi_K^r\alpha_K^d) &= \tau_K\, d\log\pi_K\cdot\tau_K^a\pi_K^r\alpha_K^d, \qquad \text{if } v_p\{a,r,d\}_K = 0, \\ d^{2p+1}(u_1) &= (\tau_K\alpha_K)^p\tau_K \end{aligned}$$

and from the fact that $d\log\pi_K$ is an infinite cycle.

Proof. The d^2-differential follows from Proposition 4.4.3. If $K = K_0$, we have

$$\hat{E}^3(C_p, T(W(k)|K_0)) = \Lambda\{u_1, d\log(-p)\}\otimes S\{t^{\pm1}, \kappa^p\},$$

and for degree reasons, the first possible differential is d^{2p+1}. The canonical map

$$\hat{E}^{2p+1}(\mathbb{T}, T(W(k)|K_0)) \hookrightarrow \hat{E}^{2p+1}(C_p, T(W(k)|K_0))$$

may be identified with the inclusion of the subalgebra generated by t, κ^p, and $d\log(-p)$. The d^{2p+1}-differential on these elements in the left-hand spectral sequence are zero for degree reasons. Hence, the d^{2p+1}-differential on these classes in the right-hand spectral sequence are zero as well. We claim that, up to a unit,

$$d^{2p+1}u_1 = t^{p+1}\kappa^p.$$

For if not, $t\kappa^p$ would survive the spectral sequence and represent the homotopy class $-v_1 \cdot 1$. But $\hat{\mathbb{H}}(C_p, T(W|K_0))$ is a module spectrum over the generalized Eilenberg-MacLane spectrum $T(W)$, and therefore, is itself a generalized Eilenberg-MacLane spectrum. Hence, multiplication by v_1 on $\bar{\pi}_*\hat{\mathbb{H}}(C_p, T(W|K_0))$ is identically zero. All further differentials must vanish for degree reasons.

If $\mu_p \subset K$ and $v_p(e_K) > 1$,

$$\hat{E}^3(C_p, T(A|K)) = \Lambda\{u_1, d\log\pi_K\} \otimes S\{\pi_K^p, \alpha_K, \tau_K^{\pm 1}\}/(\pi_K^{e_K}),$$

and by Proposition 5.3.6, π_K^p and $\tau_K\alpha_K$ are infinite cycles. From the previous case, we know that t is an infinite cycle, and hence, so is $\tau_K = u_K(\pi_K^p)\,t$. It follows that α_K also is an infinite cycle. Hence, the remaining nonzero differentials are generated from the differential on u_1. Again all further differentials vanish for degree reasons.

Finally suppose that $\mu_p \subset K$, but with no restriction on $v_p(e_K)$. Then

$$\hat{E}^3(C_p, T(A|K)) = \Lambda\{u_1, d\log\pi_K\} \otimes k\{\tau_K^a\pi_K^r\alpha_K^d \mid v_p\{a,r,d\}_K \geq 1\},$$

and we need to show that the elements $\tau_K^a\pi_K^r\alpha_K^d$ with $v_p\{a,r,d\}_K \geq 1$ are d^q-cycles, for $3 \leq q \leq 2p+1$. To this end, we let L/K be a totally ramified extension such that $v_p(e_L) > 1$ and consider

$$\iota_*\colon \hat{E}^q(C_p, T(A|K)) \to \hat{E}^q(C_p, T(B|L)).$$

We have from (5.2.7) and Lemma 5.4.1 that

$$\begin{aligned}
\iota_*(\tau_K^a\pi_K^r\alpha_K^d) &= (\theta_{L/K}(\pi_L^p))^{-\{a,r,d\}_K/p}\tau_L^a\pi_L^{e_{L/K}r}\alpha_L^d,\\
\iota_*(d\log\pi_K) &= (e_{L/K} - \frac{\theta'_{L/K}(\pi_L^p)\pi_L^p}{\theta_{L/K}(\pi_L^p)})d\log\pi_L,
\end{aligned}$$

where in the first line, $v_p\{a,r,d\}_K \geq 1$. We know from the previous case that the d^q-differential on $\iota_*(\tau_K^a\pi_K^r\alpha_K^d)$ vanishes, and hence, it will suffice to show that we can find L/K for which the map ι_* is injective.

If $v_p(e_{L/K}) > 1$ and $\theta_{L/K}(x) = x-1$ then, up to a unit,

$$\iota_*(u_1^\varepsilon\tau_K^a\pi_K^r\alpha_K^d d\log\pi_K) = u_1^\varepsilon\tau_L^a\pi_L^{e_{L/K}r+p}\alpha_L^r d\log\pi_L,$$

and hence, for ι_* to be injective, we need that $e_{L/K}r+p < e_L$. Since $r \leq e_K - 1$ and $e_L = e_{L/K}e_K$, this is equivalent to the requirement that $e_{L/K} \geq p$. We also need $v_p(e_L) > 1$. The extension L with $e_{L/K} = p^2$ and $\theta_{L/K}(x) = x-1$ satisfies both requirements. It follows that the d^q-differentials vanish, if $3 \le q \le 2p$, and that the nonzero d^{2p+1}-differentials are generated from the differential on u_1. All further differentials vanish for degree reasons. □

THEOREM 5.4.3. *For all K, and for $i \ge 0$, the map*

$$\hat{\Gamma}_{A|K}\colon \bar{\pi}_i T(A|K) \xrightarrow{\sim} \bar{\pi}_i \hat{\mathbb{H}}(C_p, T(A|K))$$

is an isomorphism.

Proof. If we let $L = K(\mu_p)$, then in the diagram

$$\begin{array}{ccc} \bar{\pi}_* T(A|K) & \xrightarrow{\hat{\Gamma}_{A|K}} & \bar{\pi}_* \hat{\mathbb{H}}(C_p, T(A|K)) \\ \downarrow{\sim} & & \downarrow{\sim} \\ \bar{\pi}_* T(B|L)^{G_{L/K}} & \xrightarrow{\hat{\Gamma}_{B|L}} & \bar{\pi}_* \hat{\mathbb{H}}(C_p, T(B|L))^{G_{L/K}}, \end{array}$$

the vertical maps are isomorphisms. Indeed, this follows from Theorem 2.4.3 and from the Tate spectral sequence, since the order of $G_{L/K}$ is prime to p. Hence, we can assume that $\mu_p \subset K$.

If $\mu_p \subset K$ and $v_p(e_K) > 1$ or if $K = K_0$, then

$$\hat{E}^{\infty}(C_p, T(A|K)) = \Lambda\{d\log \pi_K\} \otimes S\{\pi_K^p, \alpha_K, \tau_K^{\pm 1}\}/(\pi_K^e, \alpha_K^p),$$

and by Proposition 5.3.6, there is a multiplicative extension $(\pi_K^p)^{e/p} = -\tau_K \alpha_K$ in the passage to the actual homotopy groups. Hence, as a k-algebra

$$\bar{\pi}_* \hat{\mathbb{H}}(C_p, T(A|K)) = \Lambda\{\hat{\Gamma}_{A|K}(d\log \pi_K)\} \otimes S\{\hat{\Gamma}_{A|K}(\pi_K), \tilde{\tau}_K^{\pm 1}\}/(\hat{\Gamma}_{A|K}(\pi_K)^{e_K}),$$

where the class $\tilde{\tau}_K$ is a lifting of τ_K. It follows that $\bar{\pi}_* T(A|K)$ and the nonnegatively graded part of $\bar{\pi}_* \hat{\mathbb{H}}(C_p, T(A|K))$ are abstractly isomorphic k-algebras, and that the map $\hat{\Gamma}_{A|K}$ is an isomorphism for $i = 0$ and $i = 1$. To show that $\hat{\Gamma}_{A|K}$ is an isomorphism, for $i \ge 0$, it will therefore suffice to show that

$$\hat{\Gamma}_{W(k)|K_0}\colon \bar{\pi}_2 T(W|K_0) \xrightarrow{\sim} \bar{\pi}_2 \hat{\mathbb{H}}(C_p, T(W|K_0))$$

is an isomorphism. To this end, we consider the diagram

$$\begin{array}{ccc} \bar{\pi}_2 T(W|K_0) & \underset{\sim}{\xrightarrow{\beta_1}} & \bar{\pi}_1 T(W|K_0) \\ \downarrow{\hat{\Gamma}_{W(k)|K_0}} & & \sim\downarrow{\hat{\Gamma}_{W(k)|K_0}} \\ \bar{\pi}_2 \hat{\mathbb{H}}(C_p, T(W|K_0)) & \xrightarrow{\beta_1} & \bar{\pi}_1 \hat{\mathbb{H}}(C_p, T(W|K_0)), \end{array}$$

where the upper horizontal map and right-hand vertical maps are isomorphisms. Since all groups in the diagram are one-dimensional k-vector spaces, the left-hand vertical map and lower horizontal map must also be isomorphisms. This shows that the map of the statement is an isomorphism if $\mu_p \subset K$ and $v_p(e_K) > 1$ or if $K = K_0$.

If $\mu_p \subset K$, but there are no restrictions on $v_p(e_K)$,

$$\hat{E}^\infty(C_p, T(A|K)) = \Lambda\{d\log \pi_K\} \otimes k\{\tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \geq 1,\ d < p\},$$

where $0 \leq r < e_K$, $d \in \mathbb{N}_0$ and $a \in \mathbb{Z}$. Again, the domain and range of $\hat{\Gamma}_{A|K}$ are abstractly isomorphic k-vector spaces. We choose an extension L/K such that $v_p(e_L) > 1$ and such that $\iota\colon \bar{\pi}_* T(A|K) \to \bar{\pi}_* T(B|L)$ is a monomorphism. Since $\hat{\Gamma}_{B|L}$ is an isomorphism in nonnegative degrees, $\hat{\Gamma}_{A|K}$ is a monomorphism, and hence an isomorphism, in nonnegative degrees. □

ADDENDUM 5.4.4. *For all K, for all $n, v \geq 1$, and for all $i \geq 0$, the maps*

$$\begin{aligned} \hat{\Gamma}_{A|K}\colon \pi_i(T(A|K)^{C_{p^{n-1}}}, \mathbb{Z}/p^v) &\xrightarrow{\sim} \pi_i(\hat{\mathbb{H}}(C_{p^n}, T(A|K)), \mathbb{Z}/p^v), \\ \Gamma_{A|K}\colon \pi_i(T(A|K)^{C_{p^n}}, \mathbb{Z}/p^v) &\xrightarrow{\sim} \pi_i(\mathbb{H}^{\bullet}(C_{p^n}, T(A|K)), \mathbb{Z}/p^v), \end{aligned}$$

are isomorphisms.

Proof. If $v = 1$ this follows from Theorem 5.4.3 and the main theorem of [47], and the general case follows by easy induction based on the Bockstein sequence. □

5.5. We now evaluate the spectral sequences $\hat{E}^*(C_{p^n}, T(A|K))$.

THEOREM 5.5.1. *Suppose either $\mu_p \subset K$ or $K = K_0$. Then the nonzero differentials in the spectral sequence*

$$\begin{aligned} \hat{E}^2(C_{p^n}, T(A|K)) &= \Lambda\{u_n, d\log \pi_K\} \otimes S\{\pi_K, \alpha_K, \tau_K^{\pm 1}\}/(\pi_K^{e_K}) \\ &\Rightarrow \bar{\pi}_*(\hat{\mathbb{H}}(C_{p^n}, T(A|K))) \end{aligned}$$

are multiplicatively generated from

$$\begin{aligned} d^{2(\frac{p^{v+1}-1}{p-1})}(\tau_K^a \pi_K^r \alpha_K^d) &= \lambda \cdot (\tau_K \alpha_K)^{\frac{p^{v+1}-1}{p-1} - 1} \tau_K\, d\log \pi_K \cdot \tau_K^a \pi_K^r \alpha_K^d, \\ d^{2(\frac{p^{n+1}-1}{p-1})-1}(u_n) &= \mu \cdot (\tau_K \alpha_K)^{\frac{p^{n+1}-1}{p-1} - 1} \tau_K, \end{aligned}$$

and from $d\log \pi_K$ being an infinite cycle. Here λ and μ are units in A/p and in the first line $v = v_p\{a,r,d\}_K$, where $\{a,r,d\}_K = (pa - d)e_K/(p-1) + r$.

Remark 5.5.2. We show that the units λ and μ above are given by

$$\lambda = -\lambda_v \cdot p^{-v}\{a,r,d\}_K \cdot u_K^{(v-n)}(\pi_K^{p^v})^{-p}, \qquad \mu = \mu_n \cdot u_K(\pi_K^{p^n})^{-p},$$

where λ_v and μ_n are units in $\mathbb{F}_p$ independent of K.

The proof of Theorem 5.5.1 is similar to the proof of Proposition 5.4.2 above, but the individual steps are more involved. It will be necessary to know the structure of the E^r-terms, given the differential structure.

LEMMA 5.5.3. *Suppose $\mu_p \subset K$ or $K = K_0$, and assume that Theorem* 5.5.1 *is true for K. Let $\hat{E}^q = \hat{E}^q(C_{p^n}, T(A|K))$. Then for $0 \leq s < n$ and $2(p^s - 1)/(p-1) < q \leq 2(p^{s+1} - 1)/(p-1)$,*

$$\begin{aligned}\hat{E}^q \;=\; & \bigoplus_{v=1}^{s-1} \Lambda\{u_n\} \otimes\; k\Big\{\tau_K^a \pi_K^r \alpha_K^d d\log \pi_K \mid v_p\{a,r,d\}_K = v,\, d < \tfrac{p^{v+1}-1}{p-1} - 1\Big\} \\ & \oplus \Lambda\{u_n, d\log \pi_K\} \otimes k\Big\{\tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \geq s\Big\},\end{aligned}$$

$$\begin{aligned}\hat{E}^\infty \;=\; & \bigoplus_{v=1}^{n-1} \Lambda\{u_n\} \otimes k\{\tau_K^a \pi_K^r \alpha_K^d d\log \pi_K \mid v_p\{a,r,d\}_K = v,\, d < \tfrac{p^{v+1}-1}{p-1} - 1\} \\ & \oplus \Lambda\{d\log \pi_K\} \otimes k\{\tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \geq n,\, d < \tfrac{p^{n+1}-1}{p-1} - 1\},\end{aligned}$$

where $0 \leq r < e_K$, $d \in \mathbb{N}_0$ and $a \in \mathbb{Z}$.

Proof. When we assume the result for $s < n-1$, Theorem 5.5.1 implies that

$$\hat{E}^{2(\frac{p^{s+2}-1}{p-1})} = \hat{E}^{2(\frac{p^{s+1}-1}{p-1})+1},$$

and inductively, $\hat{E}^{2(\frac{p^s-1}{p-1})}$ is given by the statement of the lemma. Indeed, this is clear in the basic case $s = 0$. The differential $d^{2(p^{s+1}-1)/(p-1)}$ only affects the last summand on the right-hand side of the statement and does not involve the tensor factor $\Lambda\{u_n\}$. If we rewrite

$$\begin{aligned}&\Lambda\{d\log \pi_K\} \otimes k\{\tau_K^a \pi_K^r \alpha_K^d \,|\, v_p\{a,r,d\}_K \geq s\} \\ &\qquad = k\{\tau_K^a \pi_K^r \alpha_K^d \,|\, v_p\{a,r,d\} = s\} \\ &\qquad\quad \oplus k\{\tau_K^a \pi_K^r \alpha_K^d d\log \pi_K |\, v_p\{a,r,d\}_K = s,\, d \geq \tfrac{p^{s+1}-1}{p-1} - 1\} \\ &\qquad\quad \oplus k\{\tau_K^a \pi_K^r \alpha_K^d d\log \pi_K |\, v_p\{a,r,d\}_K = s,\, d < \tfrac{p^{s+1}-1}{p-1} - 1\} \\ &\qquad\quad \oplus \Lambda\{d\log \pi_K\} \otimes k\{\tau_K^a \pi_K^r \alpha_K^d |\, v_p\{a,r,d\}_K \geq s+1\},\end{aligned}$$

the differential $d^{2(p^{s+1}-1)/(p-1)}$ vanishes on the last two summands and maps the first summand isomorphically onto the second. Indeed,

$$\{a + \tfrac{p^{s+1}-1}{p-1}, r, d + \tfrac{p^{s+1}-1}{p-1} - 1\}_K = \{a,r,d\}_K + \tfrac{p^{s+1}}{p-1}.$$

Assuming that Theorem 5.5.1 holds for K, we have

$$\hat{E}^{2(\frac{p^n-1}{p-1})+1} = \hat{E}^{2(\frac{p^{n+1}-1}{p-1})-1},$$

and the common value has already been determined. Up to a unit,

$$d^{2(\frac{p^{n+1}-1}{p-1})-1} u_n = (\tau_K \alpha_K)^{\frac{p^{n+1}-1}{p-1} - 1} \tau_K.$$

If we rewrite

$$\begin{aligned}
&\Lambda\{u_n, d\log\pi_K\} \otimes k\{\tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \geq n\} \\
&\qquad = \Lambda\{d\log\pi_K\} \otimes k\{\tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \geq n,\ d \geq \tfrac{p^{n+1}-1}{p-1} - 1\} \\
&\qquad\quad \oplus \Lambda\{d\log\pi_K\} \otimes k\{u_n \tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \geq n\} \\
&\qquad\quad \oplus \Lambda\{d\log\pi_K\} \otimes k\{\tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \geq n,\ d < \tfrac{p^{n+1}-1}{p-1} - 1\},
\end{aligned}$$

the differential $d^{2(p^{n+1}-1)/(p-1)-1}$ maps the second summand isomorphically onto the first summand and leaves the last summand unchanged. □

Proposition 5.5.4. *Let $T = T(W(k)|K_0)$. In the spectral sequence*

$$\hat{E}^2(C_{p^n}, T) = \Lambda\{u_n, d\log(-p)\} \otimes S\{t^{\pm 1}, \kappa\} \Rightarrow \bar{\pi}_*(\hat{\mathbb{H}}(C_{p^n}, T)),$$

the higher differentials are multiplicatively generated from

$$\begin{aligned}
d^{2(\frac{p^{v+1}-1}{p-1})}(t^{p^{v-1}}) &= \lambda_v \cdot (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} t d\log(-p) \cdot t^{p^{v-1}}, \quad 1 \leq v < n, \\
d^{2(\frac{p^{n+1}-1}{p-1})-1}(u_n) &= \mu_n \cdot (t\kappa)^{\frac{p^{n+1}-1}{p-1}-1} t,
\end{aligned}$$

where $\lambda_v, \mu_n \in \mathbb{F}_p$ are units, and from $t\kappa^p$ and $d\log(-p)$ being infinite cycles. Moreover, the infinite cycles $(-t\kappa)^{p^{s+1}} d\log(-p)$, $1 \leq s < n$, represent $dV^{n-s}(1)$.

Proof. The proof is by induction on n and is similar to the proof in [4] of the differential structure of the spectral sequences $\hat{E}^*(C_{p^n}, T(W(k)))$. The basic case $n = 1$ was proved in Proposition 5.4.2. So assume the statement for $n-1$.

We first argue that in $\bar{\pi}_*(\hat{\mathbb{H}}(C_{p^n}, T))$, the class v_1^m is nonzero if and only if $m < (p^n - 1)/(p-1)$. By Addendum 5.4.4, the maps

$$\bar{\pi}_*(\hat{\mathbb{H}}(C_{p^n}, T)) \xleftarrow{\hat{\Gamma}} \bar{\pi}_*(T^{C_{p^{n-1}}}) \xrightarrow{\hat{\Gamma}} \bar{\pi}_*(\mathbb{H}^{\bullet}(C_{p^{n-1}}, T))$$

are isomorphisms in nonnegative degrees, and hence, we may instead consider the class v_1^m in $\bar{\pi}_*(\mathbb{H}^{\bullet}(C_{p^{n-1}}, T))$. To this end, we use the spectral sequence

$$E^2(C_{p^{n-1}}, T) = \Lambda\{u_{n-1}, d\log(-p)\} \otimes S\{t, \kappa\} \Rightarrow \pi_*(\mathbb{H}^{\bullet}(C_{p^{n-1}}, T))$$

whose differential structure is determined by the statement for $n-1$. We evaluate the E^r-term by an argument similar to the proof of Lemma 5.5.3. To state the result, let $P(a, d, v)$ be the statement

$$\text{“}a < \tfrac{p^{v+1}-1}{p-1} \text{ or } d < \tfrac{p^{v+1}-1}{p-1} - 1, \text{ or both”}.$$

Then for $0 \le s < n-1$,

$$E^{2(\frac{p^{s+1}-1}{p-1})} = \bigoplus_{v=0}^{s-1} \Lambda\{u_{n-1}\} \otimes k\Big\{t^a\kappa^d d\log(-p) \mid v_p(pa-d) = v \text{ and } P(a,d,v)\Big\}$$
$$\oplus\, \Lambda\{u_{n-1}, d\log(-p)\} \otimes k\Big\{t^a\kappa^d \mid v_p(pa-d) \ge s\Big\},$$

$$E^{\infty} = \bigoplus_{v=0}^{n-2} \Lambda\{u_{n-1}\} \otimes k\Big\{t^a\kappa^d d\log(-p) \mid v_p(pa-d) = v \text{ and } P(a,d,v)\Big\}$$
$$\oplus\, \Lambda\{d\log(-p)\} \otimes k\{t^a\kappa^d \mid v_p(pa-d) \ge n-1 \text{ and } P(a,d,n-1)\Big\}.$$

We know from Addendum 5.3.7 that the infinite cycle $(-t\kappa^p)^m$, if not a boundary, represents the class v_1^m. The smallest power m_0 such that $(-t\kappa^p)^{m_0}$ is a boundary is $m_0 = (p^n-1)/(p-1)$, $(-t\kappa^p)^{m_0} = d^{2m_0-1}(u_{n-1}\kappa^{p^n})$. Hence, v_1^m is nonzero, if $m < m_0$, and $v_1^{m_0}$ is represented by an element of $E^{\infty}_{s,2(p-1)m_0-s}$ with $s < -2m_0$. But these groups are all zero, and therefore, so is $v_1^{m_0}$.

We next show that in $\hat{E}^*(C_{p^n}, T)$, $(-t\kappa)^{p^{s+1}} d\log(-p)$, $1 \le s < n-1$, represents $dV^{n-s}(1)$, and that $(-t\kappa)^{p^n} d\log(-p)$, if not a boundary, represents $dV(1)$. The latter follows from Proposition 5.3.4 and Addendum 5.3.7, since, by Lemma 3.1.1,

$$dV(1) = d(\underline{-p}_n) = \underline{-p}_n d\log_n(-p) = V(1) d\log_n(-p).$$

To prove the former, we consider the map

$$F\colon \bar{\pi}_1(\hat{\mathbb{H}}(C_{p^n}, T)) \to \bar{\pi}_1(\hat{\mathbb{H}}(C_{p^{n-1}}, T)),$$

which, by Lemma 3.3.3 and Proposition 3.4.1, is a surjection whose kernel is generated by $dV(1)$. Moreover, it takes $dV^{n-s}(1)$ to $dV^{n-1-s}(1)$ and the induced map of spectral sequences

$$F\colon \hat{E}^*(C_{p^n}, T) \to \hat{E}^*(C_{p^{n-1}}, T)$$

takes $(-t\kappa)^{p^{s+1}} d\log(-p)$ to $(-t\kappa)^{p^{s+1}} d\log(-p)$. The claim follows, inductively, since the generator $dV(1)$ of the kernel of F is represented by an element of $\hat{E}^*_{m,1-m}(C_{p^n}, T)$ with $m \le -2p^n$.

We now begin the proof of the statement of the proposition for n. Suppose first that $2 < r < 2(p^n-1)/(p-1)$. The statement for $n-1$ implies that in the spectral sequence

$$\hat{E}^2(\mathbb{T}, T) = \Lambda\{d\log(-p)\} \otimes S\{t^{\pm 1}, \kappa\} \Rightarrow \bar{\pi}_*\hat{\mathbb{H}}(\mathbb{T}, T),$$

the d^r-differential is multiplicatively generated from the stated differentials on $t^{p^{v-1}}$ and from $d\log(-p)$ and $t\kappa^p$ being infinite cycles. Indeed, one shows inductively that the canonical map induces an isomorphism

$$\gamma_{n-1}\colon \Lambda\{u_{n-1}\} \otimes \hat{E}^r(\mathbb{T}, T) \xrightarrow{\sim} \hat{E}^r(C_{p^{n-1}}, T).$$

We claim that for r in the stated range, $d^r(u_n)$ is zero. To see this, we consider the map of spectral sequences induced from $V\colon \bar{\pi}_*(\hat{\mathbb{H}}(C_{p^{n-1}},T)) \to \bar{\pi}_*(\hat{\mathbb{H}}(C_{p^n},T))$,

$$V\colon \hat{E}^r(C_{p^{n-1}},T) \to \hat{E}^r(C_{p^n},T).$$

The map of E^2-terms is given by the transfer map in Tate cohomology. It follows that $u_n = V(u_{n-1})$, and hence, $d^r(u_n) = V(d^r(u_{n-1}))$, which is zero for r in the stated range. We now conclude, by induction on r, that

$$\gamma_n\colon \Lambda\{u_n\} \otimes \hat{E}^r(\mathbb{T},T) \xrightarrow{\sim} \hat{E}^r(C_{p^n},T)$$

is an isomorphism and that the d^r-differential is as stated. Before we proceed, we note that in $\hat{E}^*(\mathbb{T},T)$, and hence in $\hat{E}^*(C_{p^n},T)$, the elements $t^i\kappa^j d\log(-p)$ are infinite cycles. This follows, by arguments similar to [19, §5.3], from the fact that the homotopy groups of T with $\mathbb{Z}_p$-coefficients are concentrated in degree zero and in odd positive degree.

If $r = 2(p^n-1)/(p-1)$, the possible nonzero differentials are generated from

$$\begin{aligned} d^r(t^{p^{n-2}}) &= \lambda_{n-1}\cdot(t\kappa)^{\frac{p^n-1}{p-1}-1} t d\log(-p)\cdot t^{p^{n-2}}, \\ d^r(u_n) &= \nu_n\cdot(t\kappa)^{\frac{p^n-1}{p-1}-1} t d\log(-p)\cdot u_n, \end{aligned}$$

where $\lambda_{n-1},\nu_n \in \mathbb{F}_p$. We first show that λ_{n-1} is a unit. If $n=2$, the k-vector space $\bar{\pi}_1(\hat{\mathbb{H}}(C_{p^2},T))$ is generated by the classes $d\log_2(-p)$ and $dV(1)$. The former is represented by $d\log(-p)$ and the latter by an element of $\hat{E}^*_{m,1-m}(C_{p^2},T)$ with $m \le -2p^2$. Hence, the infinite cycle $(t\kappa)^p d\log(-p)$ must be hit by a differential, and this can happen only if λ_1 is a unit. If $n>2$, we consider $dV^2(1) \in \bar{\pi}_1(\hat{\mathbb{H}}(C_{p^n},T))$ which is represented by $(-t\kappa)^{p^{n-1}} d\log(-p)$. We know that $v_1^{(p^{n-2}-1)/(p-1)}$ annihilates $1 \in \bar{\pi}_*(\hat{\mathbb{H}}(C_{p^{n-2}},T))$ and hence also $dV^2(1) \in \bar{\pi}_1(\hat{\mathbb{H}}(C_{p^n},T))$. Therefore,

$$(-t\kappa^p)^{(p^{n-2}-1)/(p-1)}\cdot(-t\kappa)^{p^{n-1}} d\log(-p) = (-t\kappa)^{\frac{p^{n-1}-1}{p-1}-1} t d\log(-p)\cdot t^{-p^{n-2}}$$

must be hit by a differential, and this can happen only if λ_{n-1} is a unit.

We next show that ν_n is zero. If not, then $d^r(u_n t^{p^{n-2}c}) = 0$, for some $0<c<p$, and for degree reasons, the next possible nonzero differential is

$$d^{2(\frac{p^n-1}{p-1}+p^{n-1}c)-1}(u_n t^{p^{n-2}c}) = \xi_n\cdot(t\kappa^p)^{\frac{p^{n-1}-1}{p-1}+p^{n-2}c}\cdot t^{p^{n-1}(c+1)}.$$

But this must be zero, or else $v_1^{(p^{n-1}-1)/(p-1)+p^{n-2}c}$ would be zero. For degree reasons, the next possible nonzero differential is $d^{2(p^{n+1}-1)/(p-1)}$. In particular, no differential can hit $v_1^{(p^n-1)/(p-1)}$. So we must have $\nu_n = 0$.

The next possible differential is the stated one on u_n, and since $v_1^{(p^n-1)/(p-1)}$ is zero, μ_n must be a unit. For degree reasons, all further differentials vanish. □

We next prove Theorem 5.5.1, when $\mu_p \subset K$ and $n < v_p(e_K)$.

PROPOSITION 5.5.5. *If $\mu_p \subset K$ and if $n < v_p(e_K)$, the nonzero differentials in the spectral sequence $\hat{E}^*(C_{p^n}, T(A|K))$ are multiplicatively generated from*

$$\begin{aligned} d^{2(\frac{p^{v+1}-1}{p-1})}(\pi_K^{p^v}) &= -\lambda_v \cdot (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} t\, d\log \pi_K \cdot \pi_K^{p^v}, \quad 0 \le v < n, \\ d^{2(\frac{p^{n+1}-1}{p-1})-1}(u_n) &= \mu_n \cdot (t\kappa)^{\frac{p^{n+1}-1}{p-1}-1} t, \end{aligned}$$

and from τ_K, α_K, and $d\log \pi_K$ being infinite cycles.

Proof. Since $n < v_p(e_K)$, Proposition 5.3.6 and Addendum 5.3.7 show that $\tau_K \alpha_K$ and $\tau_K \alpha_K^p$ are infinite cycles. Hence if $d^r \alpha_K$ is nontrivial then so is $d^r(\alpha_K^p)$ contradicting the fact that d^r is a derivation. It follows that both α_K and τ_K are infinite cycles, and $d\log \pi_K$ is an infinite cycle by Proposition 5.3.4. Hence, Theorem 5.5.1 amounts to the statement above.

Suppose first that $u_K'(0)$ is a unit. We prove the stated formula for $d^r(\pi_K^{p^v})$ by induction on $0 \le v < n$. The basic case $v = 0$ follows from Proposition 4.4.3. Now, assume that the d^r-differential is as stated, for $2 \le r \le 2(p^v - 1)/(p-1)$, and consider $2(p^v - 1)/(p - 1) < r \le 2(p^{v+1} - 1)/(p - 1)$. We note that $\pi_K^i (t\kappa^p)^j d\log \pi_K$ is equal to zero in $\hat{E}^r(C_{p^n}, T(A|K))$, if $v_p(i) = s < v$ and $j \ge (p^s - 1)/(p - 1)$.

By definition, $\tau_K = u_K^{(-n)}(\pi_K)^p\, t$, so that

$$t^{p^{v-1}} = u_K^{(v-n)}(\pi_K^{p^v})^{-1} \tau_K^{p^{v-1}},$$

and since τ_K is an infinite cycle, we find

$$d^r(t^{p^{v-1}}) = -\frac{u_K^{(v-n)\prime}(\pi_K^{p^v})}{u_K^{(v-n)}(\pi_K^{p^v})} t^{p^{v-1}} \cdot d^r(\pi_K^{p^v}).$$

The first factor on the right is a unit in $\hat{E}^r$, for r in the stated range, and hence, we can evaluate $d^r(\pi_K^{p^v})$ from the value of $d^r(t^{p^{v-1}})$, which is known by Proposition 5.5.4. It follows that $d^r(\pi_K^{p^v})$ is equal to zero, if $r < 2(p^{v+1} - 1)/(p - 1)$. If $r = 2(p^{v+1} - 1)/(p - 1)$,

$$\begin{aligned} d^r(t^{p^{v-1}}) &= \lambda_v \cdot (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} t d\log(-p) \cdot t^{p^{v-1}} \\ &= \lambda_v \cdot \frac{u_K^{(-n)\prime}(\pi_K)\pi_K}{u_K^{(-n)}(\pi_K)} (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} t d\log \pi_K \cdot t^{p^{v-1}} \\ &= \lambda_v \cdot \frac{u_K^{(v-n)\prime}(\pi_K^{p^v})\pi_K^{p^v}}{u_K^{(v-n)}(\pi_K^{p^v})} (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} t d\log \pi_K \cdot t^{p^{v-1}}. \end{aligned}$$

The second equation uses $-p = \pi_K^{e_K} \theta_K(\pi_K)^{-1}$ and $\theta_K(\pi_K) = u_K(\pi_K)^{p-1}$, and the third follows from Lemma 5.4.1 since, as noted above, $\pi_K^i (t\kappa^p)^j d\log \pi_K$ is equal to zero in $\hat{E}^r(C_{p^n}, T(A|K))$, if $v_p(i) = s < v$ and $j \geq (p^{s+1} - 1)/(p-1)$. The stated formula for $d^r(\pi_K^{p^v})$ follows. Similarly, we see that $d^r(\pi_K^{p^n})$ is equal to zero, if $r < 2(p^{n+1}-1)/(p-1)$, and the differential on u_n follows from Proposition 5.5.4. For degree reasons, all further differentials are zero.

To treat the general case, let Π be the pointed monoid $\{0, 1, \pi, \pi^2, \dots\}$ with base point 0. The choice of uniformizer π_K determines a map of $\mathbb{T}$-spectra

$$\rho_K \colon T(W(k)|K_0) \wedge |N_{\bullet}^{\mathrm{cy}}(\Pi)| \to T(A|K),$$

which is multiplicative with component-wise multiplication on the left; compare Section A.1 below. As a differential graded k-algebra,

$$\bar{\pi}_*(T(W(k)|K_0) \wedge |N_{\bullet}^{\mathrm{cy}}(\Pi)|) = \Lambda\{d\log(-p), d\pi\} \otimes S\{\kappa, \pi\},$$

and the map of homotopy groups with $\mathbb{Z}/p$-coefficients induced from ρ_K is the unique map of differential graded k-algebras that is $\bar{\pi}_* T(W(k)|K_0)$-linear and takes π to π_K. We claim that in the spectral sequence

$$\hat{E}^*(\Pi) = \hat{E}^*(C_{p^n}, T(W(k)|K_0) \wedge |N_{\bullet}^{\mathrm{cy}}(\Pi)|),$$

the nonzero differentials are generated multiplicatively from

$$d^{2(\frac{p^{v+1}-1}{p-1})}(\pi^{p^v}) = -\lambda_v \cdot (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} t \cdot \pi^{p^v-1} d\pi, \qquad 0 \leq v < n,$$

from the differentials on the $t^{p^{v-1}}$, $1 \leq v < n$, and the differential on u_n given by Proposition 5.5.4, and from $t\kappa^p$, $d\log(-p)$, π^{p^n} and $\pi^{p^n-1}d\pi$ being infinite cycles. This proves the proposition since $\hat{E}^*(C_{p^n}, T(A|K))$ is a module spectral sequence over $\hat{E}^*(\Pi)$.

To prove the claim, we choose a totally ramified extension K/K_0 with $\mu_p \subset K$ such that $n < v_p(e_K)$ and $u_K'(0)$ is a unit. The proposition already has been established for $\hat{E}^*(C_{p^n}, T(A|K))$. As cyclic sets

$$N_{\bullet}^{\mathrm{cy}}(\Pi) = \bigvee_{s \geq 0} N_{\bullet}^{\mathrm{cy}}(\Pi; s),$$

where the s-th summand has n-simplices $(\pi^{i_0}, \dots, \pi^{i_n})$ with $i_0 + \dots i_n = s$, and the spectral sequence $\hat{E}^*(\Pi)$ decomposes accordingly. It will suffice to show that for $0 \leq v \leq n$, the differentials in the p^v-th summand spectral sequence,

$$\begin{aligned} \hat{E}^2(\Pi, p^v) &= \Lambda\{u_n, d\log(-p)\} \otimes S\{t^{\pm 1}, \kappa\} \otimes k\{\pi^{p^v}, \pi^{p^v-1}d\pi\} \\ &\Rightarrow \bar{\pi}_*(\hat{\mathbb{H}}(C_{p^n}, T(W(k)|K_0) \wedge |N_{\bullet}^{\mathrm{cy}}(\Pi, p^v)|), \end{aligned}$$

are multiplicatively generated from the stated differentials on π^{p^v}, the differentials on the p-powers of t, and the differential on u_n, and from $d\log(-p)$

and $\pi^{p^v-1}d\pi$ being infinite cycles. We note that the map $\rho_{K*}\colon \hat{E}^2(\Pi, p^v) \to \hat{E}^2(C_{p^n}, T(A|K))$ is a monomorphism. Indeed, $\pi_K^{p^v}$ and

$$d\log(-p) = \frac{u_K^{(-n)\prime}(\pi_K)\pi_K}{u_K^{(-n)}(\pi_K)} d\log\pi_K$$

are nonzero, since $p^v < e_K$ and since $u_K'(0)$ is a unit, respectively. It follows, by induction on r, that $\rho_{K*}\colon \hat{E}^r(\Pi, p^v) \to \hat{E}^r(C_{p^n}, T(A|K))$ is a monomorphism and that the d^r-differential is as stated. For instance, $\pi^{p^v-1}d\pi$ is an infinite cycle because $\rho_{K*}(\pi^{p^v-1}d\pi) = \pi_K^{p^v} d\log\pi_K$ is. □

Proof of Theorem 5.5.1. Let $n \geq 1$ and K be given. We prove by induction on q that the d^q-differential in $\hat{E}^*(C_{p^n}, T(A|K))$ is as stated. The basic case $q = 2$ follows from Propositions 4.4.3 and 5.2.3. So assume the statement for $q-1$ and suppose first that $2(p^s-1)/(p-1) < q \leq 2(p^{s+1}-1)/(p-1)$ with $s < n$. We recall from Lemma 5.5.3 that $\hat{E}^q = \hat{E}^q(C_{p^n}, T(A|K))$ is given by

$$\begin{aligned} \hat{E}^q \quad &= \quad \bigoplus_{v=1}^{s-1} \Lambda\{u_n\} \otimes k\Big\{\tau_K^a \pi_K^r \alpha_K^d d\log\pi_K \mid v_p\{a,r,d\}_K = v, d < \tfrac{p^{v+1}-1}{p-1} - 1\Big\} \\ &\quad\; \oplus \Lambda\{u_n, d\log\pi_K\} \otimes k\Big\{\tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \geq s\Big\}. \end{aligned}$$

Since the elements $\tau_K^a \pi_K^r \alpha^d d\log\pi_K$ are infinite cycles, and since $d^q(u_n)$ is zero by Proposition 5.5.4, it suffices to evaluate $d^q(\tau_K^a \pi_K^r \alpha_K^d)$ with $v_p\{a,r,d\}_K \geq s$. To this end, we find a totally ramified extension

$$L = K[\pi_L]/(\pi_L^{e_{L/K}} + \pi_K \theta_{L/K}(\pi_L))$$

such that $n < v_p(e_L)$ and such that the map

$$\iota_*\colon \hat{E}^q_{*,t}(C_{p^n}, T(A|K)) \to \hat{E}^q_{*,t}(C_{p^n}, T(B|L))$$

is a monomorphism, for $t \geq q-1$. Since the differential structure of the right-hand spectral sequence is known from Proposition 5.5.5, this allows us to evaluate the d^q-differential in the spectral sequence on the left. We consider the extension L/K with $e_{L/K} = p^{n+1}$ and $\theta_{L/K}(x) = x - 1$, and recall from (5.2.7) that the map of E^2-terms is given by

$$\begin{aligned} \iota_*(\tau_K^a \pi_K^r \alpha_K^d) \quad &= \quad (1-\pi_L)^{-\{a,r,d\}_K} \tau_L^a \pi_L^{e_{L/K} r} \alpha_L^d, \\ \iota_*(d\log\pi_K) \quad &= \quad \pi_L(1-\pi_L)^{-1} d\log\pi_L. \end{aligned}$$

Hence, the induced map of E^q-terms takes $\tau_K^a \pi_K^r \alpha_K^d$ with $v_p\{a,r,d\}_K \geq s$ to

$$(1-\pi_L^{p^s})^{-p^{-s}\{a,r,d\}_K} \cdot \tau_L^a \pi_L^{e_{L/K} r} \alpha_L^d,$$

and $\tau_K^a \pi_K^r \alpha_K^d d\log\pi_K$ with $v_p\{a,r,d\}_K \geq s$ and $d \geq (p^s-1)/(p-1) - 1$ to

$$(1-\pi_L^{p^s})^{-p^{-s}\{a,r,d\}_K} \pi_L^{p^s} (1-\pi_L^{p^s})^{-1} \cdot \tau_L^a \pi_L^{e_{L/K} r} \alpha_L^d d\log\pi_L,$$

where the latter statement uses Lemma 5.4.1 and that $\pi_L^i \cdot \tau_L^a \pi_L^{e_{L/K}r} \alpha_L^d d\log \pi_L$ is equal to zero in $\hat{E}^q(C_{p^n}, T(B|L))$, if $v_p(i) < s$. It is clear that this map is a monomorphism in the stated range. Indeed, $r \le e_K - 1$ and $e_L = e_{L/K}e_K$, and therefore, $e_{L/K}r + p^s \le e_L - p^{n+1} + p^s \le e_L - 1$. It follows immediately from Proposition 5.5.5 that $d^q(\iota_*(\tau_K^a \pi_K^r \alpha_K^d))$ vanishes, if $q < 2(p^{s+1} - 1)/(p-1)$, and a straightforward calculation shows that

$$d^q(\iota_*(\tau_K^a \pi_K^r \alpha_K^d)) = \iota_*(-\lambda_s \cdot p^{-s}\{a,r,d\}_K \cdot (t\kappa)^{\frac{p^{s+1}-1}{p-1}-1} t d\log \pi_K \cdot \tau_K^a \pi_K^r \alpha_K^d),$$

if $q = 2(p^{s+1} - 1)/(p-1)$. Since ι_* is a monomorphism, we conclude that

$$\begin{aligned} d^q(\tau_K^a \pi_K^r \alpha_K^d) &= -\lambda_s \cdot p^{-s}\{a,r,d\}_K \cdot (t\kappa)^{\frac{p^{s+1}-1}{p-1}-1} t d\log \pi_K \cdot \tau_K^a \pi_K^r \alpha_K^d \\ &= -\lambda_s \cdot p^{-s}\{a,r,d\}_K \cdot u_K^{(s-n)} (\pi_K^{p^s})^{-p} \\ &\quad \cdot (\tau_K \alpha_K)^{\frac{p^{s+1}-1}{p-1}-1} \tau_K d\log \pi_K \cdot \tau_K^a \pi_K^r \alpha_K^d \end{aligned}$$

as desired. Finally, an analogous argument shows that $d^q(\tau_K^a \pi_K^r \alpha_K^d)$ is equal to zero, if $2(p^n - 1)/(p-1) < q < 2(p^{n+1} - 1)/(p-1)$, and the stated differential on u_n follows from Proposition 5.5.4. All further differentials vanish for degree reasons. ∎

5.6. We conclude this section with a proof of the following result, which was used in the proof of Proposition 3.3.6 above for $n > 3$.

Lemma 5.6.1. *For all $i \ge 0$, the Frobenius induces a surjection,*

$$F\colon \mathrm{TR}^n_{2i+1}(A|K;p) \twoheadrightarrow \mathrm{TR}^{n-1}_{2i+1}(A|K;p).$$

Proof. For $i > 0$, the group $\mathrm{TR}^n_i(A|K;p)$ is a sum of a uniquely divisible group and a p-torsion group of bounded height. Indeed, this is true when $n = 1$, and the general case follows inductively from the cofibration sequence

$${}_h\mathrm{TR}^n(A|K;p) \xrightarrow{N} \mathrm{TR}^n(A|K;p) \xrightarrow{R} \mathrm{TR}^{n-1}(A|K;p)$$

and the spectral sequence (3.3.2). Since $FV = p$, the Frobenius induces a surjection of uniquely divisible summands. Hence, it suffices to prove that the statement of the lemma holds after p-completion. And, by Addendum 5.4.4, we may instead show that the canonical map

$$\gamma_n\colon \pi_{2i+1}(\mathbb{H}^{\boldsymbol{\cdot}}(\mathbb{T}, T(A|K)), \mathbb{Z}_p) \to \pi_{2i+1}(\mathbb{H}^{\boldsymbol{\cdot}}(C_{p^n}, T(A|K)), \mathbb{Z}_p)$$

is surjective. To this end, we consider the spectral sequences

$$\begin{aligned} E^2_{s,t}(\mathbb{T}) &= H^{-s}(BS^1, \pi_t(T(A|K), \mathbb{Z}_p)) \Rightarrow \pi_{s+t}(\mathbb{H}^{\boldsymbol{\cdot}}(\mathbb{T}, T(A|K)), \mathbb{Z}_p), \\ E^2_{s,t}(C_{p^n}) &= H^{-s}(BC_{p^n}, \pi_t(T(A|K), \mathbb{Z}_p)) \Rightarrow \pi_{s+t}(\mathbb{H}^{\boldsymbol{\cdot}}(C_{p^n}, T(A|K)), \mathbb{Z}_p), \end{aligned}$$

both of which are strongly convergent second quadrant spectral sequences. This means that the filtration of $\pi_*(\mathbb{H}^{\boldsymbol{\cdot}}(G, T(A|K)), \mathbb{Z}_p)$ associated with the spectral sequence $E^*(G)$ is complete and separated and that there is a canonical isomorphism

$$\mathrm{gr}^s\,\pi_{s+t}(\mathbb{H}^{\boldsymbol{\cdot}}(G,T(A|K)),\mathbb{Z}_p)\cong E^{\infty}_{s,t}(G).$$

It will therefore suffice to show that

$$\mathrm{gr}^s(\gamma_n)\colon \mathrm{gr}^s\,\pi_{2i+1}(\mathbb{H}^{\boldsymbol{\cdot}}(\mathbb{T},T(A|K)),\mathbb{Z}_p)\to \mathrm{gr}^s\,\pi_{2i+1}(\mathbb{H}^{\boldsymbol{\cdot}}(C_{p^n},T(A|K)),\mathbb{Z}_p)$$

is a surjection for all $s\leq 0$ and $i\geq 0$. The induced map of E^2-terms is given by the map on cohomology induced from the inclusion $C_{p^n}\to\mathbb{T}$, and hence, is surjective for s even. Moreover, by Remark 2.4.2, $\pi_*(T(A|K),\mathbb{Z}_p)$ is concentrated in odd degrees with the exception of $\pi_0(T(A|K),\mathbb{Z}_p)$, and hence, the nonzero differentials in the spectral sequence $E^r(\mathbb{T})$ must originate on the line $t=0$. It follows that for s even and $t>0$, the map

$$\gamma_{n*}\colon E^r_{s,t}(\mathbb{T})\to E^r_{s,t}(C_{p^n})$$

is surjective for all $2\leq r\leq\infty$. (Since these groups do not support nonzero differentials, they are stable for $r>s$.) Since only the groups $E^r_{s,t}(C_{p^n})$ with s even and $t>0$ can contribute to $\pi_{2i+1}(\mathbb{H}^{\boldsymbol{\cdot}}(C_{p^n},T(A|K)),\mathbb{Z}_p)$, this shows that the map $\mathrm{gr}^s(\gamma_n)$ is indeed surjective. $\square$

6. The pro-system $\mathrm{TR}^{\boldsymbol{\cdot}}_*(A|K;p,\mathbb{Z}/p^v)$

6.1. In this section, we prove the main theorem of this work. Suppose that $\mu_{p^v}\subset K$ such that we have the maps

$$\Sigma^\infty B\mu_{p^v+}\xrightarrow{\mathrm{det}}K(K)\xrightarrow{\mathrm{tr}}\mathrm{TR}^n(A|K;p).$$

Since p is odd, the Bockstein gives an isomorphism

$$\pi_2(\Sigma^\infty B\mu_{p^v+},\mathbb{Z}/p^v)\xrightarrow{\sim}{}_{p^v}\pi_1(\Sigma^\infty B\mu_{p^v},\mathbb{Z}/p^v)\xleftarrow{\sim}\mu_{p^v},$$

and hence, these maps induce

$$\mu_{p^v}\to K_2(K,\mathbb{Z}/p^v)\xrightarrow{\mathrm{tr}}\mathrm{TR}^n_2(A|K;p,\mathbb{Z}/p^v)=\pi_2(\mathrm{TR}^n(A|K;p),\mathbb{Z}/p^v).$$

It follows that there is a canonical map of log Witt complexes

$$W_{\boldsymbol{\cdot}}\,\omega^*_{(A,M)}\otimes S_{\mathbb{Z}/p^v}(\mu_{p^v})\to\mathrm{TR}^{\boldsymbol{\cdot}}_*(A|K;p,\mathbb{Z}/p^v),$$

where on the second tensor factor on the left, the maps R, F and V act as the identity and the differential d acts as zero. We recall from Theorem 3.3.8 that this map is an isomorphism in degrees 0 and 1.

By Addendum 5.4.4 the map

$$\hat{\Gamma}_{A|K}\colon\mathrm{TR}^n_*(A|K;p,\mathbb{Z}/p^v)\to\pi_*(\hat{\mathbb{H}}(C_{p^n},T(A|K)),\mathbb{Z}/p^v)$$

is an isomorphism in nonnegative degrees. The groups on the right, for $v = 1$, are given by the spectral sequence $\hat{E}^* = \hat{E}^*(C_{p^n}, T(A|K))$, which we evaluated in Theorem 5.5.1 above. The result is that

$$\begin{aligned}\hat{E}^\infty \quad = \quad & \bigoplus_{v=1}^{n-1} k\Big\{u_n^\varepsilon \tau_K^a \pi_K^r \alpha_K^d d\log\pi_K \Big| \, v_p\{a,r,d\}_K = v, \, d < \tfrac{p^{v+1}-1}{p-1} - 1\Big\} \\ & \oplus k\Big\{\tau_K^a \pi_K^r \alpha_K^d (d\log\pi_K)^\varepsilon \Big| \, v_p\{a,r,d\}_K \geq n, \, d < \tfrac{p^{n+1}-1}{p-1} - 1\Big\},\end{aligned}$$

where $a \in \mathbb{Z}$, $d \in \mathbb{N}_0$, $\varepsilon \in \{0,1\}$, and $0 \leq r < e_K$, and where

$$\{a,r,d\}_K = (pa-d)e_K/(p-1) + r.$$

We call the basis of $\hat{E}^\infty$ as a k-vector space exhibited here the *standard* basis.

PROPOSITION 6.1.1. *If $\mu_p \subset K$ or if $K = K_0$ then $\mathrm{TR}_q^n(A|K; p, \mathbb{Z}/p)$ is an ne_K-dimensional k-vector space, for all $q \geq 0$.*

Proof. We fix a total degree q and evaluate the cardinality of the standard basis of $\hat{E}^\infty(C_{p^n}, T(A|K))$. An element of the standard basis is in total degree $q = 2m + \varepsilon$ if and only if $d - a = m$. We let $v = v_p\{a,r,d\}_K$ and note that

$$\{a,r,d\}_K = de_K + r - pe_K m/(p-1).$$

Hence, the elements of the standard basis of $\hat{E}^\infty(C_{p^n}, T(A|K))$ in total degree q are indexed by integers $1 \leq v \leq n$, $0 \leq r < e_K$ and $d \geq 0$ such that either $1 \leq v < n$ and $v_p(de_K + r - pe_K m/(p-1)) = v$ and $0 \leq de_K + r < (\frac{p^{v+1}-1}{p-1} - 1)e_K$ or $v = n$ and $v_p(de_K + r - pe_K m/(p-1)) \geq v$ and $0 \leq de_K + r < (\frac{p^{n+1}-1}{p-1} - 1)e_K$. But these requirements are equivalent to the requirement that for all $1 \leq v \leq n$, $de_K + r$ is congruent to $pe_K m/(p-1)$ modulo p^v and

$$(\tfrac{p^v-1}{p-1} - 1)e_K \leq de_K + r < (\tfrac{p^{v+1}-1}{p-1} - 1)e_K = (\tfrac{p^v-1}{p-1} - 1)e_K + p^v e_K.$$

It is clear that for each value of $1 \leq v \leq n$, there are e_K pairs (d, r) which satisfy this requirement. Hence, the dimension is equal to ne_K as stated. $\square$

LEMMA 6.1.2. *Suppose that the class $\xi \in \pi_*(\hat{\mathbb{H}}(C_{p^n}, T(A|K)))$ is represented in $\hat{E}^\infty(C_{p^n}, T(A|K))$ by the element $u_n^\varepsilon \tau_K^a \pi_K^r \alpha_K^d (d\log\pi_K)^\delta$. Then the product $b_n \cdot \xi$ is represented by $\pm u_n^\varepsilon \tau_K^{a+a'} \pi_K^{r'} \alpha_K^{d+a'+1} (d\log\pi_K)^\delta$, where $r + e_K/(p-1) = a'e_K + r'$ and $0 \leq r' < e_K$.*

Proof. We show that the map induced from multiplication by b_n,

$$b_n \colon \hat{E}^3(C_{p^n}, T(A|K)) \to \hat{E}^3(C_{p^n}, T(A|K)),$$

is given by the stated formula. It suffices to consider the case $n = 1$. Indeed,

$$\begin{aligned}F^{n-1} &\colon \hat{E}^3_{s,t}(C_{p^n}, T(A|K)) \to \hat{E}^3_{s,t}(C_p, T(A|K)), \\ V^{n-1} &\colon \hat{E}^3_{s,t}(C_p, T(A|K)) \to \hat{E}^3_{s,t}(C_{p^n}, T(A|K)),\end{aligned}$$

are isomorphisms for s even and odd, respectively, and commute with multiplication by the Bott element, since $F^{n-1}(b_n) = b_1$. Suppose first that $v_p(e_K) > 1$ such that

$$\hat{E}^3(C_p, T(A|K)) = \Lambda\{u_1, d\log\pi_K\} \otimes S\{\tau_K^{\pm 1}, \pi_K^p, \alpha_K\}/(\pi_K^{e_K}).$$

It will suffice to prove that $b_1 \cdot \pi_K^r$ is equal to $\pm\tau_K^{a'}\pi_K^{r'}\alpha_K^{a'+1}$. This follows from the "multiplicative extension" $\pi_K^{e_K} = -\tau_K\alpha_K$. More precisely, Proposition 5.3.6 shows that the elements π_K^p and $-\tau_K\alpha_K$ represent the classes $\hat{\Gamma}_{A|K}(\pi_K)$ and $\hat{\Gamma}_{A|K}(\pi_K^{e_K/p})$, respectively. We also recall from (5.2.4) that the element $-\pi_K^{e_K/(p-1)}\alpha_K$ represents the Bott element b_1. But α_K survives the spectral sequence and represents a homotopy class, say, $\tilde{\alpha}_K$. Hence, $-\pi_K^{e_K/(p-1)}\alpha_K$ also represents the class $-\hat{\Gamma}_{A|K}(\pi_K^{e_K/p(p-1)})\tilde{\alpha}_K$. The claim follows, if $v_p(e_K) > 1$. In general, we pick a totally ramified extension L/K such that $v_p(e_L) > 1$ and such that the map

$$\iota_*\colon \hat{E}^3(C_p, T(A|K)) \to \hat{E}^3(C_p, T(B|L))$$

is a monomorphism. $\square$

We note that multiplication by b_n preserves the symbol

$$\{a, r, d\}_K = \{a + a', r', d + a' + 1\}_K,$$

and that the class b_n^q is represented by $\pm\tau_K^{q_1}\pi_K^{q_0 e_K/(p-1)}\alpha_K^{q_1+q}$ with $q = q_1(p-1) + q_0$ and $0 \le q_0 < p-1$.

Lemma 6.1.3. *An element of the standard basis of $\hat{E}^\infty(C_{p^n}, T(A|K))$ represents a homotopy class in the image of the composite*

$$W_n\,\omega^*_{(A,M)} \otimes S_{\mathbb{Z}/p}(\mu_p) \to \mathrm{TR}^n_*(A|K; p, \mathbb{Z}/p) \to \bar{\pi}_*\hat{\mathbb{H}}(C_{p^n}, T(A|K))$$

if and only if $\{a, r, d\}_K \ge 0$.

Proof. The map of the statement is an isomorphism in degrees 0 and 1 by Theorem 3.3.8 and Addendum 5.4.4. Indeed, in these dimensions $\{a, r, d\}_K$ is automatically nonnegative since $a = d$. We must thus show that for all $q \ge 0$ and $\varepsilon = 0, 1$, the map

$$\bigoplus_{s \le 0} \hat{E}^\infty_{s, \varepsilon - s}(C_{p^n}, T(A|K)) \to \bigoplus_{s \le 0} \hat{E}^\infty_{s, 2q+\varepsilon-s}(C_{p^n}, T(A|K))$$

induced by multiplication by the q-th power of the Bott element is a surjection onto the stated subspace. Suppose for example that a homotopy class is represented in the spectral sequence by the element $\tau_K^a\pi_K^r\alpha_K^{a+q}$ and write $r - qe_K/(p-1) = -a_0 e_K + r_0$ with $0 \le r_0 < e_K$. The requirement

$\{a, r, a+q\}_K \geq 0$ is then equivalent to $a_0 \leq a$, and by Lemma 6.1.2

$$b^q \cdot \tau_K^{a-a_0} \pi_K^{r_0} \alpha_K^{a-a_0} = \pm \tau_K^a \pi_K^r \alpha_K^{a+q}.$$

The other elements of the standard basis are treated similarly. □

THEOREM 6.1.4. *Suppose K contains the p-th roots of unity. Then the canonical map is a pro-isomorphism*:

$$W_{\textstyle\cdot}\, \omega^*_{(A,M)} \otimes S_{\mathbb{Z}/p}(\mu_p) \xrightarrow{\sim} \mathrm{TR}^{\textstyle\cdot}_*(A|K; p, \mathbb{Z}/p).$$

Proof. Let $E^*_{\textstyle\cdot}$ denote the pro-system on either side of the map in the statement. The standard filtration, given by

$$\mathrm{Fil}^s E^*_n = V^s E^*_{n-1} + dV^s E^*_{n-1},$$

is a descending filtration with $s \geq 0$. The filtration has length n in level n, i.e. $\mathrm{Fil}^n E^*_n$ is trivial. The map of the statement clearly preserves the filtration. We show that for all $q \geq 0$, there exists $N \geq 1$ such that for all $n \geq 1$ and $0 \leq s < n - N$, the canonical map

$$\mathrm{gr}^s (W_n \, \omega^*_{(A,M)} \otimes S_{\mathbb{Z}/p}(\mu_p))_i \to \mathrm{gr}^s \, \mathrm{TR}^n_i(A|K; p, \mathbb{Z}/p)$$

is an isomorphism when $0 \leq s < n - N$. Since the structure maps in the pro-systems preserve the standard filtration, the theorem follows.

We have already proved that the map of the statement is an isomorphism in degrees 0 and 1. Hence, it suffices to show that for all $q \geq 0$, there exists $N \geq 1$ such that for all $n \geq 1$, $0 \leq s < n - N$ and $\varepsilon = 0, 1$, multiplication by the q-th power of the Bott element induces an isomorphism

$$\mathrm{gr}^s \, \mathrm{TR}^n_\varepsilon(A|K; p, \mathbb{Z}/p) \xrightarrow{\sim} \mathrm{gr}^s \, \mathrm{TR}^n_{2q+\varepsilon}(A|K; p, \mathbb{Z}/p).$$

We claim that any $N \geq 1$ with $p(q+1)e_K/(p-1) < p^N$ will do.

For surjectivity we use Lemma 6.1.3. Consider an element of the standard basis in degree $2q + \varepsilon$ with symbol $\{a, r, d\}_K$. Since $d \geq 0$ and $d = a + q$, we have $a \geq -q$, and hence

$$\begin{aligned} \{a, r, d\}_K &= ae_K - qe_K/(p-1) + r \\ &\geq -pqe_K/(p-1) + r > -p^N. \end{aligned}$$

Therefore, if $v_p\{a, r, d\}_K \geq N$ we have $\{a, r, d\}_K \geq 0$. It follows that multiplication by the q-th power of the Bott element induces a surjection of all summands in $\hat{E}^\infty(C_{p^n}, T(A|K))$ except for the summands with $v < N$. But these summands all represent homotopy classes of filtration greater than or equal to $n - N$. Indeed, by Proposition 4.4.1

$$\begin{aligned} V^s(u_{n-s} \tau_K^a \pi_K^r \alpha_K^d d\log \pi_K) &= u_n \tau_K^a \pi_K^r \alpha_K^d d\log \pi_K, \\ d(u_n \tau_K^a \pi_K^r \alpha_K^d d\log \pi_K) &= \tau_K^a \pi_K^r \alpha_K^d d\log \pi_K. \end{aligned}$$

Thus elements of the standard basis with $\{a,r,d\}_K < N$ are either in the image of V^{n-N} or of dV^{n-N}.

To prove injectivity, we first note that for an element of the standard basis of $\hat{E}^\infty(C_{p^n}, T(A|K))$ in total degree $2q+\varepsilon$, the requirement that

$$0 \le d < \frac{p^{v+1}-1}{p-1} - 1$$

is equivalent to the requirement that

$$r - \frac{pqe_K}{p-1} \le \{a,r,d\}_K < -\frac{pqe_K}{p-1} + e_K\frac{p^{v+1}-1}{p-1} + r - e_K.$$

We show that $v_p\{a,r,d\}_K = v \ge N$ and $\{a,r,d\}_K < e_K(p^{v+1}-1)/(p-1)$ implies that

$$\{a,r,d\}_K < -\frac{pqe_K}{p-1} + e_K\frac{p^{v+1}-1}{p-1} + r - e_K.$$

Indeed, the largest integer which is both congruent to zero modulo p^v and smaller that $e_K(p^{v+1}-1)/(p-1)$ is $e_K p^{v+1}/(p-1) - p^v$. Thus $\{a,r,d\}_K \le e_K p^{v+1}/(p-1) - p^v$, and it suffices to check that

$$e_K p^{v+1}/(p-1) - p^v < -\frac{pqe_K}{p-1} + e_K\frac{p^{v+1}-1}{p-1} + r - e_K.$$

But this is equivalent to the inequality

$$p^v > \frac{p(q+1)e_K}{p-1} - r,$$

which is satisfied for $n < N$. This shows that the map induced by multiplication by the q-th power of the Bott element induces a monomorphism of all summands in $\hat{E}^\infty(C_{p^n}, T(A|K))$ except for the summands with $v < N$. The theorem follows. □

Proof of Theorem C. The proof is by induction on v; the basic case $v=1$ is Theorem 6.1.4. In the induction step, we write $q = 2s+\varepsilon$ with $0 \le \varepsilon \le 1$ and consider the diagram of pro-abelian groups

$$\begin{array}{ccccc}
W_\bullet\, \omega^\varepsilon_{(A,M)} \otimes \mu_{p^{v-1}}^{\otimes s} & \longrightarrow & W_\bullet\, \omega^\varepsilon_{(A,M)} \otimes \mu_{p^v}^{\otimes s} & \longrightarrow & W_\bullet\, \omega^\varepsilon_{(A,M)} \otimes \mu_{p}^{\otimes s} \\
\downarrow\wr & & \downarrow & & \downarrow\wr \\
\mathrm{TR}^\bullet_q(A|K;p,\mathbb{Z}/p^{v-1}) & \longrightarrow & \mathrm{TR}^\bullet_q(A|K;p,\mathbb{Z}/p^v) & \longrightarrow & \mathrm{TR}^\bullet_q(A|K;p,\mathbb{Z}/p),
\end{array}$$

where, inductively, the right- and left-hand vertical maps are pro-isomorphisms. The lower sequence is exact at the middle. Hence, it will suffice to show that the upper horizontal sequence is a short-exact sequence of pro-abelian groups. Clearly, we can assume that $s=0$. If $\varepsilon = 0$, the sequence is exact since $W_n(A)$ is torsion free, for all $n \ge 1$. (This does not use the fact that $\mu_{p^v} \subset K$.) If

$\varepsilon = 1$, only the injectivity of the left-hand map requires proof. To this end, we consider the diagram

$$\begin{array}{ccccc} W_{\bullet}(A) \otimes \mu_p & \longrightarrow & W_{\bullet}\,\omega^1_{(A,M)} \otimes \mathbb{Z}/p^{v-1} & \longrightarrow & W_{\bullet}\,\omega^1_{(A,M)} \otimes \mathbb{Z}/p^{v} \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow \\ \mathrm{TR}^{\bullet}_2(A|K;p,\mathbb{Z}/p) & \xrightarrow{\beta} & \mathrm{TR}^{\bullet}_1(A|K;p,\mathbb{Z}/p^{v-1}) & \longrightarrow & \mathrm{TR}^{\bullet}_1(A|K;p,\mathbb{Z}/p^{v}), \end{array}$$

where the left-hand and middle vertical maps are pro-isomorphisms by induction, and where the lower sequence is exact. It will suffice to show that the upper left-hand horizontal map is zero. But this map takes $x \otimes \zeta$ to $x d\log_{\bullet} \zeta$, and since ζ has a p^{v-1}st root, $d\log_{\bullet} \zeta$ is divisible by p^{v-1}. □

Remark 6.1.5. It follows from Theorem C that if $\mu_{p^v} \subset K$, the map

$$W_{\bullet}(A) \otimes \mu_{p^v} \xrightarrow{\sim} {}_{p^v}W_{\bullet}\,\omega^1_{(A,M)},$$

which takes $x \otimes \zeta$ to $x d\log_{\bullet} \zeta$, is a pro-isomorphism. It would be desirable to have an algebraic proof of this fact.

THEOREM 6.1.6. *There are natural isomorphisms, for $s \geq 0$:*

$$\begin{aligned} \mathrm{TC}_{2s}(A|K;p,\mathbb{Z}/p) &\cong H^0(K,\mu_p^{\otimes s}) \oplus H^2(K,\mu_p^{\otimes(s+1)}), \\ \mathrm{TC}_{2s+1}(A|K;p,\mathbb{Z}/p) &\cong H^1(K,\mu_p^{\otimes(s+1)}). \end{aligned}$$

Proof. Since the extension $K(\mu_p)/K$ is tamely ramified, we may assume that $\mu_p \subset K$. Indeed, Theorem 2.4.3 shows that the canonical map

$$\mathrm{TC}_*(A|K;p,\mathbb{Z}/p) \xrightarrow{\sim} \mathrm{TC}_*(A(\mu_p)|K(\mu_p);p,\mathbb{Z}/p)^{\mathrm{Gal}(K(\mu_p)/K)}$$

is an isomorphism, and the analogous statement holds for $H^*(K,\mu_p^{\otimes s})$. If $\mu_p \subset K$, Theorem 6.1.4 shows that for $s \geq 0$ and $0 \leq \varepsilon \leq 1$, the canonical map

$$\mathrm{TC}_{\varepsilon}(A|K;p,\mathbb{Z}/p) \otimes \mu_p^{\otimes s} \xrightarrow{\sim} \mathrm{TC}_{2s+\varepsilon}(A|K;p,\mathbb{Z}/p)$$

is an isomorphism, and hence, it suffices to prove the statement in degrees 0 and 1.

In degree one, the cyclotomic trace induces an isomorphism

$$K^{\times}/K^{\times p} = K_1(K,\mathbb{Z}/p) \xrightarrow{\sim} \mathrm{TC}_1(A|K;p,\mathbb{Z}/p),$$

and by Kummer theory, the left-hand side is $H^1(K,\mu_p)$, [40, p. 155]. In degree zero, we use the fact that Addendum 1.5.7 gives an exact sequence

$$0 \to \mathrm{TC}_0(A;p,\mathbb{Z}/p) \to \mathrm{TC}_0(A|K;p,\mathbb{Z}/p) \to \mathrm{TC}_{-1}(k;p,\mathbb{Z}/p) \to 0.$$

The left-hand term is naturally isomorphic to $\mathbb{Z}/p = K_0(A,\mathbb{Z}/p)$ by [19, Th. D], and the left-hand map has a natural retraction given by

$$\mathrm{TC}_0(A|K;p,\mathbb{Z}/p) \to \mathrm{TR}_0(A|K;p,\mathbb{Z}/p)^F = \mathbb{Z}/p.$$

It remains to show that the right-hand term in the sequence is naturally isomorphic to $H^2(K,\mu_p)$. We recall from [40, p. 186] the natural short exact sequence

$$0 \to H^2(k,\mu_p) \to H^2(K,\mu_p) \to H^1(k,\mathbb{Z}/p) \to 0.$$

Since k is perfect, the left-hand term vanishes, [40, p. 157]. Let $\bar{k}$ be an algebraic closure of k. The normal basis theorem shows that $H^i(k,\bar{k})$ vanishes for $i > 0$, and hence the cohomology sequence associated with the sequence

$$0 \to \mathbb{Z}/p \to \bar{k} \xrightarrow{\varphi-1} \bar{k} \to 0$$

gives a natural isomorphism $k_\varphi \xrightarrow{\sim} H^1(k,\mathbb{Z}/p)$. Finally, since k is perfect, the restriction induces a natural isomorphism

$$\mathrm{TC}_{-1}(k;p,\mathbb{Z}/p) = W(k)_F/pW(k)_F \xrightarrow{\sim} k_\varphi. \qquad \square$$

Remark 6.1.7. If $\mu_p \subset K$, we can also give the following noncanonical description of the groups $\mathrm{TC}_*(A|K;p,\mathbb{Z}/p)$. Let $\zeta \in \mu_p$ be a generator, let $b = b_\zeta$ be the corresponding Bott element, and let $\pi = \pi_K \in A$ be a uniformizer. Then for $s \geq 0$,

$$\begin{aligned} \mathrm{TC}_{2s}(A|K;p,\mathbb{Z}/p) &= \mathbb{Z}/p \cdot b^s \oplus k_\varphi \cdot \partial(d\log\pi \cdot b^s), \\ \mathrm{TC}_{2s+1}(A|K;p,\mathbb{Z}/p) &= \mathbb{Z}/p \cdot b^s d\log_{\bullet}\pi \oplus k_\varphi \cdot \partial(b^{s+1}) \oplus k^{e_K}, \end{aligned}$$

where k_φ is the cokernel of $1-\varphi\colon k \to k$, e_K is the ramification index, and ∂ is the boundary homomorphism in the long-exact sequence

$$\cdots \xrightarrow{\partial} \mathrm{TC}_q(A|K;p,\mathbb{Z}/p) \to \mathrm{TR}_q(A|K;p,\mathbb{Z}/p) \xrightarrow{1-F} \mathrm{TR}_q(A|K;p,\mathbb{Z}/p) \xrightarrow{\partial} \cdots.$$

The summand k^{e_K} in the second line maps injectively to the kernel of $1-F$, the inclusion

$$\eta\colon k^{e_K} = \bigoplus_{i=0}^{e_K-1} k \to \mathrm{TR}_{2s+1}(A|K;p,\mathbb{Z}/p)$$

given, on the i-th summand, by

$$\eta_i(a) = \sum_{v\geq 0} a^{p^{-v}(\frac{p^{v+1}-1}{p-1})} u_K(\underline{\pi})^{-p} dV_\pi^v(\underline{\pi}^i) \cdot b^s + \sum_{v>0} F^v(a u_K(\underline{\pi})^{-p} d(\underline{\pi}^i)) \cdot b^s.$$

The sum on the right is finite and the sum on the left converges.

We shall need a special case of the Thomason-Godement construction of the hyper-cohomology spectrum associated with a presheaf of spectra on a site, [10, §3]. Suppose that F is a functor which to every finite subextension L/K in an algebraic closure $\bar{K}/K$ assigns a spectrum $F(L)$. For the purpose of this paper, we write

$$F^{\text{ét}}(K) = \underset{L/K}{\underrightarrow{\mathrm{holim}}}\, \mathbb{H}^{\bullet}(G_{L/K}, F(L)). \tag{6.1.8}$$

There is a natural strongly convergent spectral sequence

$$E^2_{s,t} = H^{-s}(K, \varinjlim_{L/K} \pi_t F(L)) \Rightarrow \pi_{s+t} F^{\text{ét}}(K), \tag{6.1.9}$$

which is obtained by passing to the limit from the spectral sequences for the group cohomology spectra

$$E^2_{s,t} = H^{-s}(G_{L/K}, \pi_t F(L)) \Rightarrow \pi_{s+t}\mathbb{H}^{\bullet}(G_{L/K}, F(L)).$$

Indeed, filtered colimits are exact so we get a spectral sequence with abutment

$$\varinjlim_{L/K} \pi_* \mathbb{H}^{\bullet}(G_{L/K}, F(L)) \xrightarrow{\sim} \pi_* F^{\text{ét}}(K),$$

and the identification of the E^2-term follows from the isomorphism

$$\begin{aligned} \varinjlim_{L/K} H^*(G_{L/K}, \pi_* F(L)) \quad &\xrightarrow{\sim} \quad \varinjlim_{L/K} H^*(G_{L/K}, (\varinjlim_{N/L} \pi_* F(N))^{G_L}) \\ &= \quad H^*(K, \varinjlim_{N/K} \pi_* F(N)). \end{aligned}$$

This isomorphism, which can be found in [41, §2 Prop. 8], is a special case of the general fact that on a site with enough points, the Godement construction of a presheaf calculates the sheaf cohomology of the associated sheaf.

THEOREM 6.1.10. *The canonical map is an isomorphism in degrees* ≥ 1:

$$\gamma_K \colon K_*(K, \mathbb{Z}/p^v) \to K_*^{\text{ét}}(K, \mathbb{Z}/p^v).$$

Proof. It suffices to consider the case $v = 1$. In the diagram

$$\begin{array}{ccc} K(K) & \xrightarrow{\gamma_K} & K^{\text{ét}}(K) \\ \big\downarrow{\scriptstyle \mathrm{tr}} & & \big\downarrow{\scriptstyle \mathrm{tr}} \\ \mathrm{TC}(A|K;p) & \xrightarrow{\gamma_K} & \mathrm{TC}^{\text{ét}}(A|K;p), \end{array}$$

the left-hand vertical map induces an isomorphism on homotopy groups with $\mathbb{Z}/p$-coefficients in degrees ≥ 1. This follows from Addendum 1.5.7 and [19, Th. D]. We use Theorem 6.1.6 to prove that the right-hand vertical map induces an isomorphism on homotopy groups with $\mathbb{Z}/p$-coefficients and that the lower horizontal map induces an isomorphism on homotopy groups with $\mathbb{Z}/p$-coefficients in degrees ≥ 0.

We first prove the statement for the map induced from the cyclotomic trace

$$K^{\text{ét}}(K) \to \mathrm{TC}^{\text{ét}}(A|K;p).$$

The spectral sequence (6.1.9) for K-theory with $\mathbb{Z}/p$-coefficients takes the form

$$E^2_{s,t} = H^{-s}(K, \mu_p^{\otimes(t/2)}) \Rightarrow K^{\text{ét}}_{s+t}(K, \mathbb{Z}/p).$$

Indeed, since K-theory commutes with filtered colimits, this follows from

$$K_t(\bar{K}, \mathbb{Z}/p) = \mu_p^{\otimes(t/2)},$$

which is proved in Suslin's celebrated paper [43] or follows from Theorem 6.1.6 above. Similarly, it follows also from Theorem 6.1.6 that the spectral sequence (6.1.9) for topological cyclic homology takes the form

$$E^2_{s,t} = H^{-s}(K, \mu_p^{\otimes(t/2)}) \Rightarrow \mathrm{TC}^{\text{ét}}_{s+t}(A|K; p, \mathbb{Z}/p).$$

Finally, it is clear that the cyclotomic trace induces an isomorphism of E^2-terms.

It remains to show that the map

$$\gamma_K \colon \mathrm{TC}_i(A|K; p, \mathbb{Z}/p) \to \mathrm{TC}^{\text{ét}}_i(A|K; p, \mathbb{Z}/p)$$

is an isomorphism for $i \geq 0$. The domain and range of γ_K are abstractly isomorphic by Theorem 6.1.6. We must show that γ_K is an isomorphism for $i \geq 0$. By theorem 2.4.3 we may assume that $\mu_p \subset K$ and that the residue field k is algebraically closed. When $\mu_p \subset K$, we have a commutative square

$$\begin{array}{ccc}
\mathrm{TC}_\varepsilon(A|K; p, \mathbb{Z}/p) \otimes \mu_p^{\otimes s} & \xrightarrow{\gamma_K \otimes \mathrm{id}} & \mathrm{TC}^{\text{ét}}_\varepsilon(A|K; p, \mathbb{Z}/p) \otimes \mu_p^{\otimes s} \\
\downarrow\sim & & \downarrow\sim \\
\mathrm{TC}_{2s+\varepsilon}(A|K; p, \mathbb{Z}/p) & \xrightarrow{\gamma_K \otimes \mathrm{id}} & \mathrm{TC}^{\text{ét}}_{2s+\varepsilon}(A|K; p, \mathbb{Z}/p),
\end{array}$$

and the vertical maps are isomorphisms for $s \geq 0$ and $0 \leq \varepsilon \leq 1$. Hence, it suffices to show that γ_K is an isomorphism in degrees 0 and 1. And for k algebraically closed, the term $H^2(K, \mu_p) \xrightarrow{\sim} H^1(k, \mathbb{Z}/p)$ in degree zero vanishes. Thus the edge homomorphism of the spectral sequence (6.1.9),

$$\varepsilon_K \colon \mathrm{TC}^{\text{ét}}_0(A|K; p, \mathbb{Z}/p) \to H^0(K, \mathbb{Z}/p),$$

is an isomorphism, and since the composite

$$\mathrm{TC}_0(A|K; p, \mathbb{Z}/p) \xrightarrow{\gamma_K} \mathrm{TC}^{\text{ét}}_0(A|K; p, \mathbb{Z}/p) \xrightarrow{\varepsilon_K} H^0(K, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism, then so is γ_K. In degree one, we use the spectral sequence (6.1.9) for topological cyclic homology with $\mathbb{Q}_p/\mathbb{Z}_p$-coefficients. As a G_K-module

$$\varinjlim_{L/K} \mathrm{TC}_1(B|\, L; p, \mathbb{Q}_p/\mathbb{Z}_p) \xleftarrow{\sim} \varinjlim_{L/K} K_1(L, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\sim} K_1(\bar{K}, \mathbb{Q}_p/\mathbb{Z}_p) = \mu_{p^\infty},$$

and the composite

$$\mathrm{TC}_1(A|K; p, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\gamma_K} \mathrm{TC}^{\text{ét}}_1(A|K; p, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\varepsilon_K} H^0(K, \mu_{p^\infty})$$

is an isomorphism. It follows that γ_K is an isomorphism in degree one. $\square$

Appendix A. Truncated polynomial algebras

A.1. Let $\pi = \pi_K \in A$ be a uniformizer and let $e = e_K$ be the ramification index. Then $A/pA = k[\pi]/(\pi^e)$. The structure of the topological Hochschild spectrum of this k-algebra was examined in [18]. We recall the result.

Let $\Pi = \Pi_e$ be the pointed monoid $\{0, 1, \pi, \dots, \pi^{e-1}\}$ with base-point 0 and with $\pi^e = 0$ such that A/p is the pointed monoid algebra $k(\Pi) = k[\Pi]/k\{0\}$. Then we have from [19, Th. 7.1] a natural $\mathcal{F}$-equivalence of $\mathbb{T}$-spectra

$$T(k) \wedge |N^{\mathrm{cy}}_{\bullet}(\Pi)| \xrightarrow{\sim} T(k(\Pi))$$

defined as follows: Let $C^b(\mathcal{P}_{k(\Pi)})$ be the category of bounded complexes of finitely generated projective $k(\Pi)$-modules and consider Π as a category with a single object and endomorphisms Π. The functor $\Pi \to C^b(\mathcal{P}_{k(\Pi)})$, which takes the unique object to $k(\Pi)$ viewed as a complex concentrated in degree zero and which takes $\pi^i \in \Pi$ (resp. $0 \in \Pi$) to multiplication by $\pi^i \in k(\Pi)$ (resp. $0 \in k(\Pi)$), induces

$$|N^{\mathrm{cy}}_{\bullet}(\Pi)| \to |N^{\mathrm{cy}}_{\bullet}(C^b(\mathcal{P}_{k(\Pi)}))| = T(k(\Pi))_{0,0},$$

and then the desired map is given as the composite

$$T(k) \wedge |N^{\mathrm{cy}}_{\bullet}(\Pi)| \to T(k(\Pi)) \wedge T(k(\Pi)) \xrightarrow{\mu} T(k(\Pi)).$$

Since k and Π are commutative, the equivalence is multiplicative with component-wise multiplication on the left. In particular, the induced map on homotopy groups is an isomorphism of differential graded k-algebras

$$\pi_*(T(k) \wedge |N^{\mathrm{cy}}_{\bullet}(\Pi)|) \xrightarrow{\sim} \pi_* T(k(\Pi)),$$

where the differential is given by Connes' operator (2.1.2). We give the realization $|N^{\mathrm{cy}}_{\bullet}(\Pi)|$ the usual CW-structure, [33, Th. 14.1] (with the simplices Δ^n and the disks D^n identified through a compatible family of orientation-preserving homeomorphisms). Then the skeleton filtration gives a spectral sequence of differential graded k-algebras

$$E^2_{s,t} = \pi_t T(k) \otimes \tilde{H}_s(|N^{\mathrm{cy}}_{\bullet}(\Pi)|; k) \Rightarrow \pi_{s+t}(T(k) \wedge |N^{\mathrm{cy}}_{\bullet}(\Pi)|).$$

The same statements are true for ordinary Hochschild homology. If k is a perfect field of characteristic $p > 0$, $\pi_* \mathrm{HH}(k) = k$ concentrated in degree zero (see e.g. [19, Lemma 5.5]). Hence, the spectral sequence collapses and the edge homomorphism gives an isomorphism of differential graded k-algebras

$$\pi_*(\mathrm{HH}(k) \wedge |N^{\mathrm{cy}}_{\bullet}(\Pi)|) \xrightarrow{\sim} \tilde{H}_*(|N^{\mathrm{cy}}_{\bullet}(\Pi)|; k). \tag{A.1.1}$$

The spectral sequence also collapses for $T(k)$. Indeed, the inclusion of the zero-skeleton gives a map of ring spectra $H(k) \to T(k)$ from the Eilenberg-MacLane spectrum for k, so we have a multiplicative map

$$\pi_*T(k)\otimes\tilde{H}_*(|N^{\mathrm{cy}}_\bullet(\Pi)|;k)\xrightarrow{\sim}\pi_*(T(k)\wedge|N^{\mathrm{cy}}_\bullet(\Pi)|) \tag{A.1.2}$$

given as the composite of the external product

$$\pi_*T(k)\otimes\pi_*(H(k)\wedge|N^{\mathrm{cy}}_\bullet(\Pi)|)\xrightarrow{\wedge}\pi_*(T(k)\wedge H(k)\wedge|N^{\mathrm{cy}}_\bullet(\Pi)|)$$

and the map induced from $\mu\colon T(k)\wedge H(k)\to T(k)$. It follows that the spectral sequence collapses and that this map is an isomorphism of graded k-algebras. However, the map $H(k)\to T(k)$ is not equivariant, so this isomorphism does not preserve the differential.

Let $N_*(k(\Pi))$ be the normalized standard complex, [5, Chap. IX, §7]. The Künneth isomorphism determines an isomorphism of complexes

$$k(\Pi)\otimes_{k(\Pi)^e}N_*(k(\Pi))\xrightarrow{\sim}\tilde{C}_*(|N^{\mathrm{cy}}_\bullet(\Pi)|;k),$$

and since $N_*(k(\Pi))\xrightarrow{\mu}k(\Pi)$ is a resolution of $k(\Pi)$ by free $k(\Pi)^e$-modules, we have a canonical isomorphism of graded k-algebras

$$\mathrm{Tor}^{k(\Pi)^e}_*(k(\Pi),k(\Pi))\xrightarrow{\sim}\tilde{H}_*(|N^{\mathrm{cy}}_\bullet(\Pi)|;k).$$

To evaluate this, we consider instead the resolution $R_*(k(\Pi))\xrightarrow{\varepsilon}k(\Pi)$ of [14],

$$\begin{aligned} R_*(k(\Pi)) &= k(\Pi)^e\otimes\Lambda\{c_1\}\otimes\Gamma\{c_2\},\\ \delta(c_1) &= \pi\otimes 1-1\otimes\pi,\qquad \delta(c_2^{[d]})=\frac{\pi^e\otimes 1-1\otimes\pi^e}{\pi\otimes 1-1\otimes\pi}\cdot c_1c_2^{[d-1]},\end{aligned}$$

where $\Gamma\{c_2\}$ is a divided power algebra and $c_2^{[d]}$ the d-th divided power of c_2. An augmentation-preserving chain map $g\colon R_*(k(\Pi))\to N_*(k(\Pi))$ is given by

$$\begin{aligned} g(c_2^{[d]}) &= \sum 1\otimes x^{k_0}\otimes x\otimes x^{k_1}\otimes\ldots\otimes x\otimes x^{k_d},\\ g(c_1c_2^{[d]}) &= \sum 1\otimes x\otimes x^{k_0}\otimes\ldots\otimes x\otimes x^{k_d},\end{aligned}$$

where both sums run over tuples $(k_0,\dots,k_d)$ with $k_0+\dots+k_d=d(e-1)$ and $0\le k_i<e$. (The summands with some $k_i=0$, for $0\le i<d$, are zero.) Hence, if e annihilates k, we have an isomorphism of differential graded k-algebras

$$k(\Pi)\otimes\Lambda\{c_1\}\otimes\Gamma\{c_2\}\xrightarrow{\sim}\tilde{H}_*(|N^{\mathrm{cy}}_\bullet(\Pi)|;k), \tag{A.1.3}$$

where $d\pi=c_1$ and $dc_2^{[d]}=0$. The value of the differential is readily verified using the standard formula, [16, Prop. 1.4.6].

PROPOSITION A.1.4. *Let k be a perfect field of characteristic $p>0$ and suppose p divides e. Then there is a canonical isomorphism of differential graded k-algebras*

$$S\{\sigma\}\otimes k(\Pi)\otimes\Lambda\{c_1\}\otimes\Gamma\{c_2\}\xrightarrow{\sim}\pi_*T(k(\Pi)),$$

where $d\pi=c_1$ and $d(c_2^{[d+1]})=-(e/p)\pi^{e-1}c_1c_2^{[d]}\sigma$.

Proof. The map of the statement is given by the maps (A.1.2) and (A.1.3). Since both are isomorphisms of graded k-algebras, it remains only to verify the differential structure. The formula for $d\pi$ is clear since the edge homomorphism

$$\pi_q(T(k) \wedge |N^{\mathrm{cy}}_{\bullet}(\Pi)|) \to \tilde{H}_q(|N^{\mathrm{cy}}_{\bullet}(\Pi)|; k)$$

is an isomorphism for $q \leq 1$ and commutes with the differential. But the proof of the formula for $dc_2^{[d]}$ is more involved and uses the calculation in [18, Th. B] of the homotopy type of the $\mathbb{T}$-CW-complex $|N^{\mathrm{cy}}_{\bullet}(\Pi)|$. As cyclic sets

$$N^{\mathrm{cy}}_{\bullet}(\Pi) = \bigvee_{s \geq 0} N^{\mathrm{cy}}_{\bullet}(\Pi; s), \tag{A.1.5}$$

where the s-th summand has n-simplices $(\pi^{i_0}, \dots, \pi^{i_n})$ with $i_0 + \dots i_n = s$, and the realization decomposes accordingly. If we write $s = de + r$ with $0 < r \leq e$ then under the isomorphism of the statement

$$\pi_*(T(k) \wedge |N^{\mathrm{cy}}_{\bullet}(\Pi; s)|) \cong \begin{cases} S\{\sigma\} \otimes k\{\pi^r c_2^{[d]}, \pi^{r-1} c_1 c_2^{[d]}\}, & \text{if } 0 < r < e, \\ S\{\sigma\} \otimes k\{\pi^{e-1} c_1 c_2^{[d]}, c_2^{[d+1]}\}, & \text{if } r = e. \end{cases}$$

The formula we wish to prove involves the case $r = e$. In this case, [18, Th. B] gives a canonical triangle of $\mathbb{T}$-CW-complexes

$$\mathbb{T}/C_{(d+1)+} \wedge S^{V_d} \xrightarrow{\mathrm{pr}} \mathbb{T}/C_{s+} \wedge S^{V_d} \xrightarrow{i} |N^{\mathrm{cy}}_{\bullet}(\Pi; s)| \xrightarrow{\partial} \Sigma\mathbb{T}/C_{(d+1)+} \wedge S^{V_d},$$

where $V_d = \mathbb{C}(1) \oplus \dots \oplus \mathbb{C}(d)$. If we form the smash product with $T(k)$ and take homotopy groups, the triangle gives rise to a long-exact sequence, which we now describe. Let x_0 (resp. y_0) be the class of the 0-cycle C_{d+1}/C_{d+1} (resp. C_s/C_s) and let x_1 (resp. y_1, resp. z_{2d}) be the fundamental class of $\mathbb{T}/C_{d+1}$ (resp. $\mathbb{T}/C_s$, resp. S^{V_d}). Then

$$\pi_*(T(k) \wedge \mathbb{T}/C_{n+} \wedge S^{V_d}) \cong \begin{cases} S\{\sigma\} \otimes k\{x_0 z_{2d}, x_1 z_{2d}\}, & \text{if } n = d+1, \\ S\{\sigma\} \otimes k\{y_0 z_{2d}, y_1 z_{2d}\}, & \text{if } n = s, \end{cases}$$

and the differential is $\pi_* T(k)$-linear and maps

$$\begin{aligned} d(y_0 z_{2d}) &= (d+1) y_1 z_{2d}, & d(y_1 z_{2d}) &= 0, \\ d(x_0 z_{2d}) &= s x_1 z_{2d}, & d(x_1 z_{2d}) &= 0. \end{aligned}$$

The induced maps in the long-exact sequence of homotopy groups associated with the triangle above all are $\pi_* T(k)$-linear and

$$\begin{aligned} \mathrm{pr}_*(y_0 z_{2d}) &= x_0 z_{2d}, & \mathrm{pr}_*(y_1 z_{2d}) &= e x_1 z_{2d}, \\ i_*(x_0 z_{2d}) &= 0, & i_*(x_1 z_{2d}) &= \pi^{e-1} c_1 c_2^{[d]}, \\ \partial_*(\pi^{e-1} d\pi \cdot c_2^{[d]}) &= 0, & \partial_*(c_2^{[d+1]}) &= -y_1 z_{2d}. \end{aligned}$$

The statements for the maps pr_* and i_* are clear from the construction of the triangle in [18]. We verify the statement for the map ∂_*. To this end we first choose a cellular homotopy equivalence

$$\alpha \colon C_{\mathrm{pr}} \xrightarrow{\sim} |N^{\mathrm{cy}}_{\bullet}(\Pi; s)|$$

such that we have a map of triangles from the distinguished triangle given by the map pr to the triangle above. Since the cellular chain functor carries distinguished triangles of CW-complexes to distinguished triangles of chain complexes, we have

$$\begin{aligned}\partial_*(\alpha_*((0, y_1 z_{2d}))) &= y_1 z_{2d},\\ \alpha_*((x_1 z_{2d}, 0)) &= \pi^{e-1} c_1 c_2^{[d]}.\end{aligned}$$

Hence, it suffices to show that $\alpha_*((0, y_1 z_{2d}))$ is homologous to $-c_2^{[d+1]}$. To do this, we consider the diagram

$$\begin{array}{ccc}\tilde{H}_{2d+2}(|N^{\mathrm{cy}}_\bullet(\Pi; s)|; \mathbb{Z}/p) & \overset{\beta}{\hookrightarrow} & \tilde{H}_{2d+1}(|N^{\mathrm{cy}}_\bullet(\Pi; s)|; \mathbb{Z}) \\ {\scriptstyle\alpha_*}\big\uparrow{\scriptstyle\sim} & & {\scriptstyle\alpha_*}\big\uparrow{\scriptstyle\sim} \\ \tilde{H}_{2d+2}(C_{\mathrm{pr}}; \mathbb{Z}/p) & \overset{\beta}{\hookrightarrow} & \tilde{H}_{2d+1}(C_{\mathrm{pr}}; \mathbb{Z})\end{array}$$

with injective horizontal maps. A straightforward calculation shows that (on the level of chains) the top Bockstein takes $c_2^{[d+1]}$ to $(e/p)\pi^{e-1} c_1 c_2^{[d]}$ and the bottom Bockstein takes $(0, y_1 z_{2d})$ to $-(e/p) x_1 z_{2d}$. We have already noted that the right-hand vertical map takes $(x_1 z_{2d}, 0)$ to $\pi^{e-1} c_1 c_2^{[d]}$. This completes the proof of the stated formula for ∂_*.

We now prove the formula for $d(c_2^{[d]})$. First note that we can write

$$d(c_2^{[d]}) = d_1(c_2^{[d]}) + d_2(c_2^{[d]}),$$

where d_1 (resp. d_2) is defined in same way as d but with $\mathbb{T}$ acting in the first (resp. second) smash factor of $T(k) \wedge |N^{\mathrm{cy}}_\bullet(\Pi; s)|$ only. Since the differential d_2 commutes with the isomorphism

$$\pi_* T(k) \otimes \tilde{H}_*(|N^{\mathrm{cy}}_\bullet(\Pi; s)|; k) \xrightarrow{\sim} \pi_*(T(k) \wedge |N^{\mathrm{cy}}_\bullet(\Pi; s)|),$$

we find that $d_2(c_2^{[d]}) = 0$. Hence, we can ignore the $\mathbb{T}$-action on $|N^{\mathrm{cy}}_\bullet(\Pi; s)|$. We have a map of triangles of (nonequivariant) CW-complexes

$$\begin{array}{ccccccc}\mathbb{T}/C_{(d+1)+} \wedge S^{V_d} & \xrightarrow{\mathrm{pr}} & \mathbb{T}/C_{s+} \wedge S^{V_d} & \xrightarrow{i} & C_{\mathrm{pr}} & \xrightarrow{\partial} & \Sigma\mathbb{T}/C_{(d+1)+} \wedge S^{V_d} \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & {\scriptstyle\sim}\big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle \Sigma f} \\ S^{2d+1} & \xrightarrow{\Sigma^{2d+1} e} & S^{2d+1} & \xrightarrow{\Sigma^{2d+1} i} & \Sigma^{2d+1} M_e & \xrightarrow{-\Sigma^{2d+1}\beta} & S^{2d+2},\end{array}$$

such that f_* (resp. g_*) maps $x_1 z_{2d}$ (resp. $y_1 z_{2d}$) to the fundamental class of S^{2d+1}. Hence, it suffices to show that the image of $-h_*((0, y_1 z_{2d})) = 1 \cdot \mathrm{susp}(\varepsilon)$ under

$$d\colon \pi_{2q+2}(T(k) \wedge \Sigma^{2d+1} M_e) \to \pi_{2q+3}(T(k) \wedge \Sigma^{2d+1} M_e)$$

is equal to $-(e/p)\sigma \cdot \operatorname{susp}(1) = -(e/p)h_*((x_1 z_{2d}, 0))$. To this end, we consider the diagram

$$\begin{array}{ccccc}
\pi_1(M_e \wedge T(k)) & \xrightarrow{\text{susp}} & \pi_{2d+2}(\Sigma^{2d+1}M_e \wedge T(k)) & \xrightarrow{\text{tw}_*} & \pi_{2d+2}(T(k) \wedge \Sigma^{2d+1}M_e) \\
\Big\downarrow d & (-1) & \Big\downarrow d & & \Big\downarrow d \\
\pi_2(M_e \wedge T(k)) & \xrightarrow{\text{susp}} & \pi_{2d+3}(\Sigma^{2d+1}M_e \wedge T(k)) & \xrightarrow{\text{tw}_*} & \pi_{2d+3}(T(k) \wedge \Sigma^{2d+1}M_e),
\end{array}$$

which commutes up to the indicated sign. By the definition of the class σ, the left-hand vertical map takes $\varepsilon \cdot 1$ to $(e/p)1 \cdot \sigma$. Hence, the right-hand vertical map takes $1 \cdot \operatorname{susp}(\varepsilon)$ to $-(e/p)\sigma \cdot \operatorname{susp}(1)$. The stated formula for $d(c_2^{[d+1]})$ follows. □

Addendum A.1.6. *The nonzero differentials in the spectral sequence*

$$\begin{aligned}
\hat{E}^2(C_{p^n}, T(k(\Pi))) \quad &= \quad \Lambda\{u_n, c_1, \varepsilon\} \otimes S\{t^{\pm 1}, \sigma, \pi\}/(\pi^e) \otimes \Gamma\{c_2\} \\
&\Rightarrow \quad \bar{\pi}_*(\hat{\mathbb{H}}(C_{p^n}, T(k(\Pi))))
\end{aligned}$$

are generated from $d^2\varepsilon = t\sigma$, $d^2\pi = tc_1$, *and* $d^2 c_2^{[d+1]} = -(e/p)t\pi^{e-1}c_1 c_2^{[d]}\sigma$.

Proof. The d^2-differential is given by Propositions 4.4.3 and A.1.4. It remains only to show that the higher differentials d^r, $r \geq 3$, vanish. The decomposition of cyclic sets (A.1.5) induces one of spectral sequences. And if we write $s = de + r$ with $0 < r \leq e$, then the E^3-term of the s-th summand is concentrated on the lines $E^3_{*,d}$ and $E^3_{*,d+1}$, if $0 < r < e$, and on the lines $E^3_{*,d+1}$ and $E^3_{*,d+2}$, if $r = e$. In either case, all further differentials must be zero for degree reasons. □

Proposition A.1.7. *Let $n \leq v_p(e)$. The images of $\underline{\pi}_n$ and $\underline{\pi}_n^{e/p^n}$ by the map*

$$\hat{\Gamma}\colon \bar{\pi}_*(T(k(\Pi))^{C_{p^{n-1}}}) \to \bar{\pi}_*(\hat{\mathbb{H}}(C_{p^n}, T(k(\Pi))))$$

are represented in the spectral sequence $\hat{E}^*(C_{p^n}, T(k(\Pi)))$ by the infinite cycles π^{p^n} and tc_2, if $v_p(e) > n$, and by π^{p^n} and $-(e/p^n)u_1\pi^{e-1}c_1$, if $v_p(e) = n$.

Proof. The statement only involves the summand $|N^{\text{cy}}_\bullet(\Pi, e)|$. We consider the map of spectral sequences induced from the linearization map,

$$l_*\colon \hat{E}^*(C_{p^n}, T(k) \wedge |N^{\text{cy}}_\bullet(\Pi, e)|) \to \hat{E}^*(C_{p^n}, \mathrm{HH}(k) \wedge |N^{\text{cy}}_\bullet(\Pi, e)|).$$

In the left-hand spectral sequence, $E^3 = E^\infty$, and in the right-hand spectral sequence, $E^2 = E^\infty$. The induced map of E^∞-terms may be identified with the canonical inclusion

$$\begin{aligned}
\Lambda\{u_n\} \otimes S\{t^{\pm 1}\} \otimes k\{\pi^{e-1}c_1, c_2 + \varepsilon \cdot (e/p)\pi^{e-1}c_1\} \\
\hookrightarrow \Lambda\{u_n, \varepsilon\} \otimes S\{t^{\pm 1}\} \otimes k\{\pi^{e-1}c_1, c_2\}.
\end{aligned}$$

Since the map is injective, it suffices to show that $l_*(\hat{\Gamma}(\underline{\pi}_n^{e/p^n}))$ is represented in the sequence on the right by $-u_n\pi^{e-1}c_1$, if $v_p(e) = n$, and by tc_2, if $v_p(e) > n$. In the proof of this, we shall use the notation and results of Sections 4.2 and 4.3 above.

We have from [3, §1] the $\mathbb{T}$-equivariant homeomorphism

$$D\colon |\,\mathrm{sd}_{p^n} N^{\mathrm{cy}}_\bullet(\Pi, e)| \xrightarrow{\sim} |N^{\mathrm{cy}}(\Pi, e)|,$$

where on the left, the action by the subgroup $C_{p^n} \subset \mathbb{T}$ is induced from a simplicial C_{p^n}-action. It follows that this space has a canonical C_{p^n}-CW-structure, and the homeomorphism D then defines a C_{p^n}-CW-structure on $|N^{\mathrm{cy}}_\bullet(\Pi, e)|$. We fix, as in the proof of Proposition A.1.4, a cellular homotopy equivalence

$$\alpha\colon C_{\mathrm{pr}} \xrightarrow{\sim} |N^{\mathrm{cy}}_\bullet(\Pi, e)|$$

with the C_{p^n}-CW-structure on C_{pr} induced from the C_{p^n}-CW-structure of $\mathbb{T} = S(\mathbb{C}) = E_1$ given in Section 4.4 above. The cellular complex $C_* = \tilde{C}_*(C_{\mathrm{pr}}; k)$ is canonically identified with the complex

$$k[C_{p^n}]\cdot(0, x_1) \xrightarrow{\delta} k\cdot(x_1, 0) \oplus k[C_{p^n}]\cdot(0, x_0) \xrightarrow{\delta} k\cdot(x_0, 0),$$

where $\delta((0, x_1)) = -(e/p^n)(x_1, 0) - (g-1)(0, x_0)$, $\delta((x_1, 0)) = 0$, and $\delta((0, x_0)) = -(x_0, 0)$. One shows as in the proof of Proposition A.1.4 that the cycles $\alpha_*((x_1, 0))$ and $\alpha_*(N(0, x_1))$ represent the classes $\pi^{e-1}c_1$ and $-c_2$, respectively.

We now turn to the spectral sequence $\hat{E}^* = \hat{E}^*(C_{p^n}, \mathrm{HH}(k) \wedge C_{\mathrm{pr}})$. There are canonical isomorphisms of complexes

$$\hat{E}^1_{*,t} \cong (\tilde{P} \otimes \mathrm{Hom}(P, \bar{\pi}_t(\mathrm{HH}(k) \wedge C_{\mathrm{pr}})))^{C_{p^n}} \cong (\tilde{P} \otimes \mathrm{Hom}(P, \bar{H}_t(C_*)))^{C_{p^n}}$$

with the left-hand isomorphism given by Lemma 4.3.4 and the right-hand isomorphism by (A.1.1). We claim that in fact

$$\bar{\pi}_*(\hat{\mathbb{H}}(C_{p^n}, \mathrm{HH}(k) \wedge C_{\mathrm{pr}})) \cong \bar{H}_*((\tilde{P} \otimes \mathrm{Hom}(P, C_*))^{C_{p^n}}) \tag{A.1.8}$$

and that the spectral sequence $\hat{E}^*$ is canonically isomorphic to the one associated with the double complex on the right. To see this, we filter M_p, $\tilde{E}$, E, and C_{pr} by the skeletons. We get, as in Section 4.3, a conditionally convergent spectral sequence

$$E^2_{s,t} = \bar{H}_s((\tilde{P} \otimes \mathrm{Hom}(P, \pi_t\,\mathrm{HH}(k) \otimes C_*))^{C_{p^n}}) \Rightarrow \bar{\pi}_{s+t}(\hat{\mathbb{H}}(C_{p^n}, \mathrm{HH}(k) \wedge C_{\mathrm{pr}})),$$

which collapses since $\pi_t\,\mathrm{HH}(k)$ vanishes for $t > 0$. The edge homomorphism gives the desired isomorphism. Moreover, under this isomorphism, the filtration of $\tilde{E}$ and E, which gives rise to the spectral sequence $\hat{E}^*$, corresponds to the filtration of the complexes $\tilde{P}$ and P. Tracing through the definitions, one readily sees that the class $l_*(\hat{\Gamma}(\underline{\pi}_n^{e/p^n}))$ is represented by the element $y_0 \otimes Nx_0^* \otimes (x_0, 0) \in \hat{E}^1_{0,0}$. To finish the proof, we note that in the total complex (A.1.8),

$$\begin{aligned}&\delta(N(y_0 \otimes x_1^* \otimes (0, x_1) - y_0 \otimes x_0^* \otimes (0, x_0)))\\&\qquad = y_0 \otimes Nx_0^* \otimes (x_0, 0) + y_0 \otimes Nx_2^* \otimes N(0, x_1) + (e/p^n) y_0 \otimes Nx_1^* \otimes (x_1, 0),\end{aligned}$$

and in the lower line, the first summand represents $l_*(\hat{\Gamma}(\underline{\pi}_n^{e/p^n}))$, the second $-tc_2$, and the third $(e/p^n)u_n\pi^{e-1}c_1$. The statement follows, since $-tc_2$ and $u_n\pi^{e-1}c_1$ are not boundaries. $\square$

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