CHAPTER 4: THE IMAGE OF J

1. Introduction

The study of the canonical map $J: O \longrightarrow F$ inducing $J_*: \pi_* O \longrightarrow \pi_*^S$, called the J-homomorphism, was initiated by Adams in a series of four papers [3], [4], [5], [6]. He determined the image of J_* modulo the "Adams conjecture", and the determination was completed with the solution of this conjecture by Quillen [53]. Mahowald [34] has constructed a spectrum J and a map of spectra $j: S \longrightarrow J$ such that the induced map $j_*: \pi_*^S \longrightarrow \pi_* J$ is a split epimorphism. Moreover, j_* is an isomorphsim from ImJ to $\pi_* J$ where ImJ equals Image J_* direct sum with an additional family of Z_2s.

In Section 2, we collect the known facts about $\pi_* J$ which we will use to simplify our computation. In addition, we use the Adams spectral sequence to derive several relations and Toda brackets involving elements in ImJ. The relevance of ImJ to our computation is that all differentials in our spectral sequence which originate on the 0 row land in ImJ $\otimes$ H_*BP. In Section 3, we prove that this theorem is true through degree 66 which suffices for the computation of the first 64 stable stems. In Section 4, we give the computer printout of the computation of the cokernels of these differentials through degree 70. This computation is one of the essential ingredients in our inductive determination of the first 64 stable stems. The computer program itself is discussed in Appendix 5.

2. ImJ and the Adams Spectral Sequence

Mahowald [34] defined the spectrum J as the fiber of a map bo $\longrightarrow \Sigma^4$bsp. The homotopy of J is periodic with period eight and is given by the table in Figure 4.2.1.

DEGREE	$\pi_N J$
1	$Z_2\alpha_0$
8N+1 (N ≥ 1)	$Z_2\alpha_N \oplus Z_2\eta^2\gamma_{N-1}$
8N+2	$Z_2\eta\cdot\alpha_N$
8N+3	$Z_8\beta_N$, $4\beta_N = \eta^2\alpha_N$
8N+4	0
8N+5	0
8N+6	0
8N+7	$Z_{C(N)}\gamma_N$
8N+8	$Z_2\eta\cdot\gamma_N$

FIGURE 4.2.1: The Homotopy of J

In the above table $C(N) = 2^{\mathscr{C}(N)}$ denotes the largest power of two which divides 16N+16. In each row of this table we take $N \geq 0$ except in the second row where $N \geq 1$. These elements of π_*J include the elements of Hopf invariant one: $\eta = \alpha_0$, $\nu = \beta_0$ and $\sigma = \gamma_0$. We apologize for the new notation, but this notation is very convenient. The following theorem describes the how π_*J is related to Image J_*, and how $j:S \longrightarrow J$ induces a split epimorphism j_*.

THEOREM 4.2.1 (a) The map $j:S \longrightarrow J$ induces an isomorphsim j_* from Image $[J_*:\pi_*O \longrightarrow \pi_*^S]$ to the subgroup of π_*J generated by $\{\alpha_N, \eta\alpha_N, \beta_N, \gamma_N, \eta\gamma_N \mid N \geq 0\}$.

(b) Let $\bar{\gamma}_N = J_*[(J_*\circ j_*)^{-1}(\gamma_N)]$. Then the map $j:S \longrightarrow J$ induces an isomorphism between π_*J and

$$\mathrm{ImJ} = \text{Image } [J_*:\pi_*O \longrightarrow \pi_*^S] \oplus Z_2\{\eta^2\bar{\gamma}_N \mid N \geq 0\}.$$

(c) ImJ is a direct summand of π_*^S.

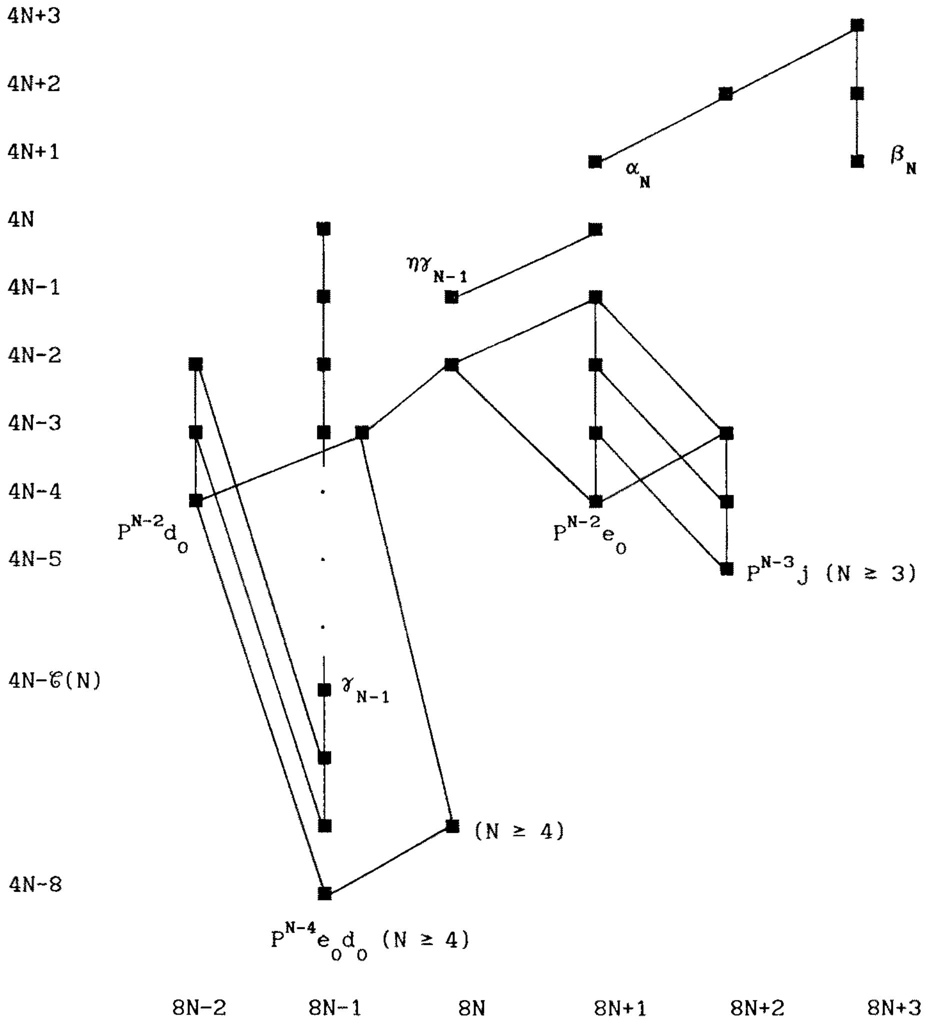
Let $\bar{\alpha}_N = J_*[(J_* \circ j_*)^{-1}(\alpha_N)]$ and let $\bar{\beta}_N = J_*[(J_* \circ j_*)^{-1}(\beta_N)]$. From now on we will abuse our notation by denoting $\bar{\alpha}_N$ by α_N, $\bar{\beta}_N$ by β_N and $\bar{\gamma}_N$ by γ_N. Thus, the table in Figure 4.2.1 can be thought of as giving a description of ImJ. Define CokJ as Kernel j_*. Then we have the direct sum decomposition:

$$\pi_*^S = \mathrm{ImJ}_* \oplus \mathrm{CokJ}_* \qquad [4.2.1]$$

Consider the classical mod two Admas spectral sequence [1]:

$$E_2^{s,t} = \mathrm{Ext}_{\mathfrak{A}}^t(Z_2 . Z_2)_s \Longrightarrow {}_2\pi_s^S \qquad [4.2.2]$$

Here $\mathfrak{A}$ denotes the mod two Steendrod algebra and ${}_2\pi_*^S$ denotes π_*^S modulo the subgroup of all elements of odd degree. This spectral sequence has a vanishing line [8], and directly below the vanishing line are the elements depicted in Figure 4.2.2 which survive to ImJ as well as several "self-destructing families", i.e. elements which are nonzero in E_2 but only zero survives from them to E_5. Below these families is the self-destructing wedge [38]. In this fiugre, vertical lines denote multiplication by h_0, lines of positive slope denote multiplication by h_1 and lines of negative slope -r denote d^r-differentials.



In addition, $h_2\alpha_N = 0$, $h_2\beta_{2N+1} = 0$, $h_2\beta_{2N} = h_0^2P^{N-2}d_0$ and $h_2\gamma_N = 0$.

FIGURE 4.2.2: ImJ in the Adams Spectral Sequence ($N \geq 2$)

The following theorem gives the structure of ImJ as a module over the subring of π_*^S generated by η, ν and σ.

THEOREM 4.2.2 (a) $\nu\alpha_N = 0$;

(b) $\sigma\alpha_N = k\eta\gamma_N$;

(c) $\eta\beta_N = 0$;

(d) $\nu\beta_N = 0$ for $N \geq 1$;

(e) $\sigma\beta_N = 0$;

(f) $\nu\gamma_N = 0$;

(g) $\sigma\gamma_N = 0$ for $N \geq 1$.

PROOF. (a), (c), (d), (g) There are no possible nonzero elements of E_2 of the Adams spectral sequence to represent any of these products with the exception of $\nu\beta_{2N}$ which can only be the boundary $h_0^2P^{2N-1}d_0$.

(b) From Figure 4.2.2, we see that the only possibility for $\sigma\alpha_N$ to be nonzero is if it equals $\eta\gamma_N$. Thus, $\sigma\alpha_N = k\eta\gamma_N$.

(e) From Figure 4.2.2, we see that the only possibility for $\sigma\beta_N$ to be nonzero is if it equals $\eta\alpha_{N+1}$. However, $\eta\sigma\beta_N = 0$ while $\eta^2\alpha_{N+1} \neq 0$.

(f) From Figure 4.2.2, we see that the only possibility for $\nu\gamma_N$ to be nonzero is if it equals $\eta\alpha_{N+1}$. However, $\eta\nu\gamma_N = 0$ while $\eta^2\alpha_{N+1} \neq 0$. ∎

We will use the following theorem to identify certain Toda brackets in the Adams spectral sequence (*).

THEOREM 4.2.3 (a) Let A', B', C' be an element of E_2 of the Admas spectral sequence (*) which converges to A, B, C, respectively. Assume that $\langle A,B,C\rangle$ is defined in π_*^S and $\langle A',B',C'\rangle$ is defined in E_r where $r \geq 2$. Let (p,q), (s,t) be the bidegree of the product A'B', B'C', respectively. Assume that the only d_u-boundary in $E_{N+r}^{p,q+N}$ and $E_{N+r-1}^{s,t+N}$, $N \geq 1$, $u \geq N + r$, is zero. Then there is an element X of $\langle A',B',C'\rangle$ in E_r of (*) which is an infinite cycle and represents an element of $\langle A,B,C\rangle$.

(b) Let A', B', C', D' be an element of E_2 of the Adams spectral sequence (*) which converegeges to A, B, C, D, respectively. Assume that $\langle A,B,C,D\rangle$ is defined in π_*^S, and $\langle A',B',C',D'\rangle$ is defined in E_r where $r \geq 2$. Let (p_1,q_1),

(p_2,q_2), (p_3,q_3), (p_4,q_4), (p_5,q_5) be the bidegree of A'B', B'C', C'D', <A',B',C'>, <B',C',D'>, respectively. Asume that

(i) $E_{N+r}^{p_i,q_i+N} = 0$, $N \geq 1$, if $4 \leq i \leq 5$ and

(ii) the only d_u-boundary in $E_{N+r}^{p_i,q_i+N}$, $N \geq 1$, $u \geq N + r$, is zero if $1 \leq i \leq 3$.

Then there is an element X of <A',B',C',D'> in E_r of (*) which is an infinite cycle and represents an element of <A,B,C,D>.

The first part of this theorem is due to Moss [47]. (Our technical hypothesis is equivalent to his, i.e. there are no "crossing differentials".) The second part of this theorem is a variation of Lawrence's generalization [33] of Moss's theorem. However, we do not assume that <A,B,C,D> is strictly defined, and we replace two of the five cases of the technical hypothesis of no crossing differentials by the more stringent hypothesis that the target groups of such differentials are zero. Part (b) of this theorem can be proved either by the methods of Moss and Lawrence or by using bordism chains as in [28].

Observe that the conclusion in (a) or (b) does not preclude the possibility that X is zero in $E_\infty^{a,b}$ and is the projection into E_∞ of a nonzero element ξ of the Toda bracket which lies in $F^{b+1}\pi_a^S$. However, if $E_\infty^{a,c} = 0$ for all $c > b$ then X = 0 implies that 0 is an elements of the Toda bracket. This explains why we require the stronger hypothesis in (b) when i equals 4 or 5.

THEOREM 4.2.4 (a) $\langle \eta, \nu, \alpha_N \rangle$ contains 0 for $N \geq 1$;

(b) $\langle \eta, \nu, \beta_N \rangle$ contains 0 for $N \geq 1$;

(c) $\langle \nu, \eta, \beta_N \rangle$ contains 0 for $N \geq 1$;

(d) $\langle \eta, \nu, \gamma_N \rangle$ contains 0;

(e) $\langle \nu, \eta, \eta^2\gamma_N \rangle$ contains 0;

(f) $\langle \nu, \eta, \beta_N, \eta \rangle$ contains 0 when N is odd, 2, 4, or 6;

(g) $\langle\nu,\eta,\eta^2\gamma_N,\eta\rangle$ contains 0;

(h) $\langle\eta,\nu,\alpha_N,\nu\rangle$ contains 0 for $N \geq 1$;

(i) $\langle\eta,\nu,\beta_N,\nu\rangle$ contains 0 for $N \geq 1$;

(j) $\langle\eta,\nu,\gamma_N,\nu\rangle$ contains 0 for $N \geq 1$;

(k) $\langle\eta,\beta_N,\sigma\rangle$ contains 0;

(l) $\langle\eta,\eta^2\gamma_N,\sigma\rangle$ contains 0 for $N \geq 1$.

PROOF. (a), (c), (d), (e), (k), (l) From Figure 4.2.2, we see that Theorem 4.2.3(a) with r = 2 applies to show that each of these Toda brackets has an element which projects to zero in $E_\infty^{8N+6,4N+2}$, $E_\infty^{8N+8,4N+2}$, $E_\infty^{8N+12,4N-\mathscr{E}(N)+5}$, $E_\infty^{8N+14,4N+5}$, $E_\infty^{8N+12,4N+2}$, $E_\infty^{8N+18,4N+5}$ in case (a), (c), (d), (e), (k), (l), respectively. With two exceptions, E_∞ is zero in each of these degrees in higher filtration degrees. The first exception is that possibly $\eta\gamma_{N+1}$ might be an element of $\langle\nu,\eta,\beta_N\rangle$. However, $\eta^2\gamma_{N+1} \neq 0$ and $\eta\langle\nu,\eta,\beta_{N+1}\rangle = \langle\eta,\nu,\eta\rangle\beta_{N+1} = \nu^2\beta_{N+1} = 0$. It follows that each of these Toda brackets contains 0. The second exception is that $\eta\alpha_{N+2}$ might be an element of $\langle\eta,\eta^2\gamma_N,\sigma\rangle$. However, $\eta\alpha_{N+2}$ is in the indeterminacy of $\langle\eta,\eta^2\gamma_N,\sigma\rangle$. Thus, $0 \in \langle\eta,\eta^2\gamma_N,\sigma\rangle$ in all cases.

(b) Since $h_2\beta_{2N+1} = 0$ in E_2, the preceeding argument applies to $\langle\eta,\nu,\beta_{2N+1}\rangle$ which is defined in E_2. It shows that $0 \in \langle\eta,\nu,\beta_{2N+1}\rangle$. However, $h_2\beta_{2N} = h_0^2P^{N-2}d_0$ which is a d^3-boundary. Now the analogue of the preceeding argument with r = 3 applies to $\langle\eta,\nu,\beta_{2N}\rangle$ to show that it contains zero.

(f), (g), (h), (i), (j) Note that by Theorem 2.2.7(g) all of the four-fold Toda brackets in this theorem are defined: In (h), $\eta\langle\nu,\alpha_N,\nu\rangle = \langle\eta,\nu,\alpha_N\rangle\nu$ which contains 0. Since $\eta\langle\nu,\alpha_N,\nu\rangle$ is a singleton, it must equal {0}. Thus, $\eta\gamma_N$ can not be an element of $\langle\nu,\alpha_N,\nu\rangle$ because $\eta^2\gamma_N \neq 0$. From Figure 4.2.2, we see that there is now no possibility for $\langle\nu,\alpha_N,\nu\rangle$ to contain a nonzero element. In (i), $\eta\langle\nu,\beta_N,\nu\rangle = \langle\eta,\nu,\beta_N\rangle\nu = \{0\}$ while $\eta(\eta\alpha_{N+1}) \neq 0$. Hence $\eta\alpha_{N+1}$ can not be an element of $\langle\nu,\beta_N,\nu\rangle$. Thus, we see from Figure 4.2.2 that there is now no possibility for $\langle\nu,\beta_N,\nu\rangle$ to contain a nonzero element. All the other triple products in (f) - (j) contain zero by (a) - (e) or must equal zero by Theorem 4.2.3(a) and Figure 4.2.2.

Assume that N is odd in case (f). Let r = 2 in cases (f), (g), (j) and in case (i) when N is odd. Let r = 3 in case (h) and in case (i) when N is even. From Figure 4.2.2, we see that Theorem 4.2.3(b) applies to show that each of these Toda brackets has an element which projects to zero in $E_\infty^{8N+10,4N+2}$, $E_\infty^{8N+16,4N+1}$, $E_\infty^{8N+10,4N}$, $E_\infty^{8N+12,4N+2}$, $E_\infty^{8N+12,4N}$, $E_\infty^{8N+16,4N-\mathscr{C}(N)+1}$ in case (f), (g), (h), (i) with N odd, (i) with N even, (j), respectively. With four exceptions, E_∞ is zero in each of these degrees in higher filtration degrees: $\eta\alpha_{N+1}$ could be in $\langle\nu,\eta,\beta_N,\eta\rangle$, $\eta\gamma_{N+1}$ could be in $\langle\nu,\eta,\eta^2\gamma_N,\eta\rangle$, $\eta\alpha_{N+1}$ could be in $\langle\eta,\nu,\alpha_N,\nu\rangle$ and $\eta\gamma_{N+1}$ could be in $\langle\eta,\nu,\gamma_N,\nu\rangle$. However, $\eta\alpha_{N+1}$, $\eta\gamma_{N+1}$, $\eta\alpha_{N+1}$, $\eta\gamma_{N+1}$ is in the indeterminacy of $\langle\nu,\eta,\beta_N,\eta\rangle$, $\langle\nu,\eta,\eta^2\gamma_N,\eta\rangle$, $\langle\eta,\nu,\alpha_N,\nu\rangle$, $\langle\eta,\nu,\gamma_N,\nu\rangle$, respectively. Thus, each of these Toda brackets contains 0.

Now consider (f) when N is 2, 4 or 6. We shall see that there are only two elements of π_{26}^S, π_{42}^S and π_{58}^S that are not contained in (η,ν), and (η,ν) is contained in the indetermincacy of $\langle\nu,\eta,\beta_{2N},\eta\rangle$. The two exceptions are C[42] and 2C[42]. Thus, $\langle\nu,\eta,\beta_2,\eta\rangle$ and $\langle\nu,\eta,\beta_6,\eta\rangle$ contain 0. Now $2\langle\nu,\eta,\beta_4,\eta\rangle \subset \langle\nu,\eta,\langle\beta_4,\eta,2\rangle\rangle = \langle\nu,\eta,0\rangle$ because $\langle\beta_4,\eta,2\rangle \in \pi_{37}^S$, $\eta\langle\beta_4,\eta,2\rangle = \beta_4\langle\eta,2,\eta\rangle = 2\nu\beta_4 = 0$, $\nu\langle\beta_4,\eta,2\rangle = 2\langle\nu,\beta_4,\eta\rangle$ and only 0 in π_{37}^S has these properties. Thus, $2\langle\nu,\eta,\beta_4,\eta\rangle = (\nu)$ which does not contain 2C[42] or 4C[42]. Therefore, $\langle\nu,\eta,\beta_4,\eta\rangle$ contains 0.∎

The reader should not worry that we may get involved in circular reasoning when we use the facts that $\langle\eta,\nu,\beta_2,\nu\rangle$, $\langle\eta,\nu,\beta_4,\nu\rangle$ and $\langle\eta,\nu,\beta_6,\nu\rangle$ contain 0. We will only use these facts to show that no leader XV can be of one of the following forms:

(i) $X \in \mathrm{CokJ}_{17}$, $V \in H_{12}BP$ and XV is in the image of d^{18};

(ii) $X \in \mathrm{CokJ}_{33}$, $V \in H_{12}BP$ and XV is in the image of d^{34};

(iii) $X \in \mathrm{CokJ}_{49}$, $V \in H_{12}BP$ and XV is in the image of d^{50}.

The only leader with such a bidegree is $\nu A[30]M_1^6$. Since twice $d^{12}(\nu A[30]M_1^6)$ equals $\sigma^2 A[30] = 4C[44] \neq 0$, $\nu A[30]M_1^6$ can not bound. Therefore, we have an alternate proof that there is no leader XV as in (i), (ii) or (iii).

3. Differentials Originating on the 0 Row - Theory

The material in this section is divided into two parts. First, we study the map of Atiyah-Hirzebruch spectral sequences induced by j. We deduce that j induces an isomorphism j_∞ on $E^\infty_{*,0}$. Second, we prove that j_r induces an isomorphism on $E^r_{N,0}$ for $r \geq 1$ and $N \leq 66$. That is, all differentials on the 0 row of our spectral sequence land in $\mathrm{ImJ} \otimes H_*BP$. Consider the Atiyah-Hirzebruch spectral sequence:

$$'E^2_{N,t} = H_N BP \otimes \pi_t J \Longrightarrow J_{N+t}BP \qquad [4.3.1]$$

The map of spectra $j: S \longrightarrow J$ induces a map of spectral sequences

$$j_r: E^r_{N,t} \longrightarrow {}'E^r_{N,t}, \quad 2 \leq r \leq \infty. \qquad [4.3.2]$$

Moreover, $j_2: H_*BP \otimes \pi^S_* \longrightarrow H_*BP \otimes \pi_*J$ equals $1 \otimes j_*$. Since H_NBP is zero when N is odd, we have the following result:

$$E^r_{N,t} = 0 \text{ if N is odd,}$$
$$d^r = 0 \text{ if r is odd and}$$
$$E^{2r+1} = E^{2r+2} \text{ for } r \geq 1.$$

The following simple theorem is the basis for many of the results of this section.

THEOREM 4.3.3 (a) If $j_2 \circ d^2(X) \neq 0$ modulo CokJ then X is an element of $Z_{(2)}\{M_1, \alpha_N M_1, \eta\alpha_N M_1, \gamma_N M_1, \eta\gamma_N M_1\} \otimes B\langle 2\rangle$ which reduces to a nonzero element modulo (2).

(b) If $j_{2r} \circ d^{2r}(X) \neq 0$ and $r \geq 2$ then $X \in E^{2r}_{*,0}$.

PROOF. (a) By Theorem 3.2.3, $E^2 = [Z_2 \oplus Z_2M_1] \otimes \pi_*^S \otimes B\langle 2\rangle$ and d^2 is the homomorphsim of $\pi_*^S \otimes B\langle 2\rangle$ - modules given by $d^2(1) = 0$ and $d^2(M_1) = \eta$. Hence if $d^2(X) \neq 0$ then $X = \xi M_1 B$ where $\xi \in \pi_*^S$, $B \in B\langle 2\rangle$ and $\eta\xi \neq 0$. Thus, if $j^2 \circ d^2(X) \neq 0$ then $\eta\xi$ is a nonzero element of ImJ. This part of the theorem now follows from the table in Figure 4.2.1.

(b) By Theorem 3.2.3:

$$'E^4 = [Z_2\alpha_K \oplus Z_4\beta_N \oplus Z_8\beta_N M_1 \oplus Z_{C(N)}\gamma_N \oplus Z_{C(N)/2}\gamma_N M_1 \oplus Z_2\eta^2\gamma_N M_1 \mid K \geq 1,\ N \geq 0] \otimes B\langle 2\rangle.$$

Thus, $'E^{2r}_{2N,2t} = 0$ for all N, $r \geq 2$ and $t > 0$. Therefore for all $r \geq 4$, $d^{2r}: {}'E^{2r}_{2N,2t} \longrightarrow {}'E^{2r}_{2N-2r,2t+2r-1}$ is zero if $t \neq 0$. That is, for $r \geq 4$ the only nonzero differentials in $'E^{2r}$ originate on the 0 row. Thus, if $0 \neq j^{2r} \circ d^{2r}(X) = d^{2r} \circ j^{2r}(X)$ then $j^{2r}(X)$ and hence X must be on the 0 row.∎

We deduce that all the elements of $ImJ/(\eta \cdot ImJ)$ are hit by nonzero transgressions originating on the 0 row.

COROLLARY 4.3.4 All of the α_N, β_N, γ_N, $\beta_N M_1$ and $\eta^2\gamma_N M_1$ are hit by differentials which originate on the 0 row.

PROOF. All of the elements listed in this corollary are nonzero in E^4 and map to a nonzero element of $'E^4$. Of course, they are zero in E^∞ and thus must be hit by differentials. By Theorem 4.3.3(b), they can only be hit by differentials which originate on the 0 row.∎

The next theorem specifies bounds on when the nonzero differentials must occur on the 0 row to turn $E^2_{*,0} = H_*BP$ into $E^\infty_{*,0} = \pi_*BP$. Let $U_N = V_N/2$, $N \geq 1$, be polynomial generators of H_*BP.

THEOREM 4.3.5 Let F_t denote the $Z_{(2)}$-submodule of H_*BP generated by all $U_{N_1}\cdots U_{N_q}$ with $q < t$.

(a) No element of $2^{4t}U_{N_1}\cdots U_{N_{4t+1}} + F_{4t+1}$ survives to $E^{8t+4}_{*,0}$.

(b) No element of $2^{4t+1}U_{N_1}\cdots U_{N_{4t+2}} + F_{4t+2}$ or of

$2^{4t+2}U_{N_1}\cdots U_{N_{4t+3}} + F_{4t+3}$ survives to $E^{8t+6}_{*,0}$.

(c) No element of $2^{4t-1}U_{N_1}\cdots U_{N_{4t}} + F_{4t}$ survives to $E^{8t+2}_{*,0}$.

PROOF. We use induction on t to prove that there is some $X_t \in F_t$ such that:

(i) no element of $2^{t-1}U_{N_1}\cdots U_{N_t} + F_t$ survives to the $E^r_{*,0}$ specified above;

(ii) $d^{2t}(2^{t-1}U_1^t + X_t) = \alpha_k$ if $t = 4k+1$;

(iii) $d^{2t}(2^{t-1}U_1^t + X_t) = 2\beta_k$ if $t = 4k+2$;

(iv) $d^{2t-2}(2^{t-1}U_1^t + X_t) = 4\beta_k M_1$ if $t = 4k+3$;

(v) $d^{2t}(2^{t-1}U_1^t + X_t) = (C(k)/2)\gamma_k$ if $t=4k+4$.

When $t = 1$, we know that $d^2(U_1) = \alpha_1$ and that no U_N survives to $E^4_{*,0}$. Assume that the above five conditions are true for $t < T$. Let $\varepsilon = 2$ if $T \equiv 3 \bmod 4$ and let $\varepsilon = 0$ otherwise. Note that $2^qU_{N_1}\cdots U_{N_q} = V_{N_1}\cdots V_{N_q} \in \pi_*BP$ and is thus an infinite cycle. By the induction hypothesis, $E^{2T-\varepsilon}_{2T,0} \equiv Z_{(2)}(2^kU_1^T + X)$ modulo $\pi_{2T}BP$ for some $X \in F_T$ and some $k \le T$. It follows from Corollary 4.3.4 that $k \le T-1$ and $d^{2T-\varepsilon}(2^{T-1}U_1^T + 2^{T-k-1}X)$ is α_k, $2\beta_k$, $4\beta_k M_1$, $(C(k)/2)\gamma_k$ if T is 4k+1, 4k+2, 4k+3, 4k+4, respectively. This proves whichever of conditions (ii) - (v) is relevant. Now assume that $Y = 2^{T-1}U_{N_1}\cdots U_{N_T} + f$ for some $f \in F_T$ survives to $E_{*,0}$. Let $I = 2\Delta_{N_1-1} + \cdots + 2\Delta_{N_T-1}$. Then $N_1+\cdots+N_T > T$ and degree $r_I > 0$. Observe that $U_N \equiv M_N$ modulo 2. Thus in $E^{2T-\varepsilon}_{*,0}$, $r_I(Y) = 2^{T-1}U_1^T + f'$ for some $f' \in F_T + (2^T)$. Thus $r_I \circ d^{2T-\varepsilon}(Y) \neq 0$ and Y does not survive to $E^{2T-\varepsilon+2}_{*,0}$, a contradiction. This verifies condition (i) and completes the proof of the induction step.∎

We can now specify the elements on the 0 row which transgress to hit the elements of $\mathrm{Im}J/(\eta\cdot\mathrm{Im}J)$.

COROLLARY 4.3.6 (a) For all $t \geq 0$, there is $A_t \in F_{4t+1}$ such that $2^{4t}M_1^{4t+1} + A_t$ survives to $E^{8t+2}_{4t+2,0}$ and $d^{8t+2}(2^{4t}M_1^{4t+1} + A_t) = \alpha_t$.
(b) For all $t \geq 0$, there are B_t and C_t in F_{4t+2} such that $2^{4t}M_1^{4t+2} + B_t$ and $2^{4t}M_1^{4t+3} + C_t$ survive to $E^{8t+4}_{*,0}$, $d^{8t+4}(2^{4t}M_1^{4t+2} + B_t) = \beta_t$ and $d^{8t+4}(2^{4t}M_1^{4t+3} + C_t) = 3\beta_t M_1$.
(c) For all $t \geq 1$, there is $D_t \in F_{4t}$ such that $(2^{4t}/C(t))M_1^{4t} + D_t$ survives to $E^{8t}_{8t,0}$ and $d^{8t}((2^{4t}/C(t))M_1^{4t} + D_t) = \gamma_{t-1}$.
(d) For all $t \geq 1$, there is $G_t \in F_{4t+2}$ such that $2^{4t-1}M_1^{4t+2} + G_t$ survives to $E^{8t+2}_{8t+4,0}$ and $d^{8t+4}(2^{4t-1}M_1^{4t+2} + G_t) = \eta^2\gamma_{t-1}M_1$.

PROOF. Parts (a), (b) and (c) follow from the proof of Theorem 4.3.5. Note that we can choose C_t in F_{4t+2} because $F_{4t+2} \cap E^{8t+4}_{8t+6,0} = F_{4t+3} \cap E^{8t+4}_{8t+6,0}$. Also observe that $r_{\Delta_1}(M_1^{4t+3}) = (4t+3)M_1^{4t+2}$ and therefore $d^{8t+4}(2^{4t}M_1^{4t+3} + C_t)$ is either $3\beta_t M_1$ or $7\beta_t M_1$. Define β_t so that $d^{8t+4}(2^{4t}M_1^{4t+3} + C_t) = 3\beta_t$. By Corollary 4.3.4, $\eta^2\gamma_{t-1}M_1$ must bound from the 0 row. Thus the reasoning used to prove Theorem 4.3.5 also applies to prove (d). ∎

As a consequence of our theorem we have another proof of the famous theorem of Adams [2] of the nonexistence of elements of Hopf invariant one in degrees 2^k-1 for $k \geq 4$.

COROLLARY 4.3.7 If $\xi \in \pi^S_{2^N-1}$ has Hopf invariant one then either $N = 0$, $\xi = 2$ or $N = 1$, $\xi = \eta$ or $N = 2$, $\xi = \nu$ or $N = 3$, $\xi = \sigma$.

PROOF. Recall that ξ has Hopf invariant one if and only if Sq^{2^N} is nonzero in the mapping cone C_ξ of ξ. In that case there must be an element

$X \in Z_{(2)}[M_1,\ldots,M_{N-1}]_{2^N}$ such that the coefficient of $M_1^{2^N}$ in X is odd and X transgresses to ξ. By Corollary 4.3.6(c), N must be 0, 1 or 2.▮

COROLLARY 4.3.8 The following Hurewicz homomorphisms have the same image:

$$\text{Image } [h: \pi_*BP \longrightarrow H_*BP] = \text{Image } [h: J_*BP \longrightarrow H_*BP].$$

PROOF. Recall that for a generalized homology theory F and a spectrum X, the image of the Hurewicz homomorphism h is given by $E^{\infty}_{*,0}$ of the Atiyah-Hirzebruch spectral sequence for F_*X. Moreover, the Hurewicz homomorphism for BP is a monomorphism, and thus $E^{\infty}_{*,0}$ in the spectral sequence for π_*BP equals π_*BP. The proof of Theorem 4.3.5 is valid in the Atiyah-Hirzebruch spectral sequence for J_*BP. In that context it says that in the spectral sequence for J_*BP, the intersection of the kernels of all differentials from the 0 row that land in $\pi_*J \otimes H_*BP$, equals $Z_{(2)}\{2^q U_{N_1} \cdots U_{N_q} \mid 0 \le q \text{ and } 1 \le N_1 \le \cdots \le N_q\}$ $= Z_{(2)}\{V_{N_1} \cdots V_{N_q} \mid 0 \le q \text{ and } 1 \le N_1 \le \cdots \le N_q\} = h(\pi_*BP)$.▮

By Theorem 4.3.5, if none of the differentials in our spectral sequence have image in CokJ ⊗ H_*BP then there would be enough image to these differentials so that $E^{\infty}_{*,0}$, the intersection of their kernels, would be π_*BP. It remains to show that this is indeed the case A differential which originates on the 0 row and lands in CokJ ⊗ H_*BP is what we called in Definition 1.3.7 a hidden differential originating on the 0 row. The proof of the following theorem can only be understood after reading Chapters 5, 6 and 7. There is no circular reasoning: we compute π^S_N for $N \le 64$ assuming Theorem 4.3.9 is true. The proof of Theorem 4.3.9 demonstrates that considering the leaders that occurred in the computation, no error could have been made through the assumption that Theorem 4.3.9 is true.

THEOREM 4.3.9 All differentials on $E^r_{N,0}$ have image in ImJ $\otimes$ H_*BP when $N \le 66$.

PROOF. Assume that $d^r(V) = XU$ is a hidden differential originating on the 0 row where $U \in H_{2M}BP$, $V \in H_{2N}BP$ and $X \in$ CokJ. Applying Landweber-Novikov operations shows that XU must be a leader. As a consequence of this hidden differential there is an element ζW, $\zeta \in$ ImJ and $W \in H_*BP$ which we incorrectly thought was hit by V, but in fact can not bound. Note that:

$W \ne 1$ lest ζ would nonzero in E^∞;

$W \ne M_1$ lest $\eta\zeta \ne 0$ in π_*^S while $\eta\zeta = 0$ in π_*ImJ, contradicting Theorem 4.2.2;

$W \ne M_1^2$ lest $\nu\zeta \ne 0$ in π_*^S while $\nu\zeta = 0$ in π_*ImJ, contradicting Theorem 4.2.2;

$W \ne M_2$ lest $0 \notin \langle\nu,\eta,\zeta\rangle$ in π_*^S while $0 \in \langle\nu,\eta,\zeta\rangle$ in π_*ImJ, contradicting Theorem 4.2.4(c),(e);

$W \ne \bar{M}_2$ lest $0 \notin \langle\eta,\nu,\zeta\rangle$ in π_*^S while $0 \in \langle\eta,\nu,\zeta\rangle$ in π_*ImJ, contradicting Theorem 4.2.4(a),(b),(d);

$W \ne M_1^4$ lest $\sigma\zeta \ne 0$ in π_*^S while $\sigma\zeta = 0$ in π_*ImJ, contradicting Theorem 4.2.2 (if $\zeta = \alpha_N$ and $\sigma\alpha_N = \eta\gamma_N$ then $\alpha_N M_1^4$ must bound because it transgresses to $\sigma\alpha_N = \eta\gamma_N$ which is zero in $E^8_{0,8N+8}$);

$W \ne M_1\bar{M}_2$ lest $0 \notin \langle\nu,\eta,\zeta,\eta\rangle$ in π_*^S while $0 \in \langle\nu,\eta,\zeta,\eta\rangle$ in π_*ImJ, contradicting Theorem 4.2.4(f),(g);

$W \ne M_1^2\bar{M}_2$ lest $0 \notin \langle\eta,\nu,\zeta,\nu\rangle$ in π_*^S while $0 \in \langle\eta,\nu,\zeta,\nu\rangle$ in π_*ImJ, contradicting Theorem 4.2.4(h),(i),(j);

$W \ne M_1^5$ lest $0 \notin \langle\eta,\zeta,\sigma\rangle$ in π_*^S while $0 \in \langle\eta,\zeta,\sigma\rangle$ in π_*ImJ, contradicting Theorem 4.2.4(k),(l).

If we incorrectly assumed that this hidden differential $d^r(V) = XU$ did not occur then we would have drawn one of the following types of false conclusions:

(a) $d^{2M}(XU) = \xi$ is a nonzero element of π^S_{2M} ; or

(b) $d^{2Q}(Y) = XU$ where we had correctly proved that XU must be a bounding leader but we had incorrectly thought that Y was the only possible leader that could hit XU.

From the above observations it follows that 2M = degree U ≥ 14. Checking the leaders of odd degree we see that the only possibilities for such an XU are $\eta A[30]M_1M_2^2$ and $\sigma C[44]M_1^4\bar{M}_2$. Since $d^{14}(\eta A[30]M_1M_2^2) = C[44]$ and $4C[44] = \sigma^2 A[30] \neq 0$, $\eta A[30]M_1M_2^2$ can not bound. If $\sigma C[44]M_1^4\bar{M}_2$ transgresses then $d^{14}(\sigma C[44]M_1^4\bar{M}_2) = B[64]$ and $2B[64] = \eta^2 B[62,1] \neq 0$. Thus, $\sigma C[44]M_1^4\bar{M}_2$ can not bound. Therefore in degrees less than 67, there is no leader XU as in (a) or (b), and there is no possibility for a hidden differential originating on the 0 row. ∎

4. Differentials which Originate on the 0 Row - Computation

In this section we reproduce the computer printout of the "cokernels" of the differentials which originate on the zero row. We can not actually compute the cokernels at this point. We do know E^4 of the spectral sequence but for each element $\xi \in ImJ_{2r-1}$ of order k we do not know which differentials d^{2s}, $r \geq s \geq 2$, originate from elements with representatives in $Z_k\xi \otimes H_*BP$. By Theorem 4.3.9, the only differentials d^{2s}, $s \geq 2$, which can hit an element with a representative in $Z_k\xi \otimes H_*BP$ must originate on the 0 row. Therefore, $\{X \in E^{2r}_{*,2r-1} \mid X \text{ has a representative in } Z_k\xi \otimes H_*BP\}$ is a subgroup of $(Z_k\xi \otimes H_*BP)/(\text{Image } d^2)$. Let $\pi_r : E^2 \longrightarrow E^r$ denote the canonical projection. Then Cokernel $[d^{2r}: E^{2r}_{*,0} \longrightarrow \pi_{2r}(ImJ_{2r-1} \otimes H_*BP)]$ is a subgroup of $\pi_4(ImJ_{2r-1} \otimes H_*BP)$ / Image $[d^{2r}: E^{2r}_{*,0} \longrightarrow E^{2r}_{*,2r-1}]$. We thus make the following definition.

DEFINITION 4.4.1 Let "Cokernel $[d^{2r}: E^{2r}_{*,0} \longrightarrow E^{2r}_{*,2r-1}]$"

$$= \pi_4(ImJ_{2r-1} \otimes H_*BP) \ / \ \text{Image } [d^{2r}: E^{2r}_{*,0} \longrightarrow E^{2r}_{*,2r-1}].$$

We order the data in this section by rows for convenience. The computer program which produced this data is discussed in Appendix 5. We will use the following notation.

1. When the range of the differentials is a Z_2-subspace of $Z_2\xi \otimes H_*BP$ then an entry "a b c d" on the list means that $\xi M_1^a \bar{M}_2^b \bar{M}_3^c \bar{M}_4^d$ is an element of the Z_2-basis of the "cokernel" of the differential.
2. The following will be used to denote a direct summand

$$Z_k(M_1^{e(1,1)}\bar{M}_2^{e(1,2)}\bar{M}_3^{e(1,3)}\bar{M}_4^{e(1,4)} + \cdots + M_1^{e(N,1)}\bar{M}_2^{e(N,2)}\bar{M}_3^{e(N,3)}\bar{M}_4^{e(N,4)}):$$

Z_k e(1,1) e(1,2) e(1,3) e(1,4)

.
.
.

e(N,1) e(N,2) e(N,3) e(N,4).

3. The following will be used to denote a direct summand

$$Z_k(A_1\cdot M_1^{e(1,1)}\bar{M}_2^{e(1,2)}\bar{M}_3^{e(1,3)}\bar{M}_4^{e(1,4)} + \cdots + A_N\cdot M_1^{e(N,1)}\bar{M}_2^{e(N,2)}\bar{M}_3^{e(N,3)}\bar{M}_4^{e(N,4)}):$$

Z_k A_1 / e(1,1) e(1,2) e(1,3) e(1,4)

.
.
.

A_N / e(N,1) e(N,2) e(N,3) e(N,4).

We begin by listing the "cokernels" of the d^{10}-differentials from the 0 row to the 9 row. Note that $E^4_{*,9} = [Z_2\alpha_1 \oplus Z_2\eta^2\sigma M_1] \otimes B\langle 2\rangle$. Thus, the monomials below with an odd power of M_1 have coefficient $\eta^2\sigma$ and the monomials with an even power of M_1 have coefficient α_1.

(4.4.2) "COKERNEL $[d^{10}: E^{10}_{*,0} \longrightarrow E^{10}_{*,9}]$":

DEGREE	BASIS	DEGREE	BASIS	DEGREE	BASIS
(18,9)	6 1 0 0	(20,9)	7 1 0 0	(22,9)	11 0 0 0
(24,9)	5 0 1 0		9 1 0 0	(26,9)	7 2 0 0

	6 0 1 0
	7 0 1 0
	6 3 0 0
	6 1 1 0
	14 1 0 0
	3 5 0 0
(38,9)	13 2 0 0
	6 2 1 0
	13 0 1 0
(42,9)	11 1 1 0
	5 3 1 0
(44,9)	15 0 1 0
	7 5 0 0
	6 3 1 0
	13 1 1 0
	6 1 2 0
	5 4 1 0
	9 5 0 0
	6 1 0 1
	7 1 0 1
	11 0 2 0
	6 4 1 0
	11 0 0 1
	1 6 1 0
	5 0 3 0
	7 4 1 0
	21 2 0 0
	9 1 0 1

(28,9)	11 1 0 0
(30,9)	15 0 0 0
(32,9)	13 1 0 0
(34,9)	11 2 0 0
(36,9)	11 0 1 0
	5 2 1 0
	3 3 1 0
(40,9)	7 2 1 0
	5 5 0 0
	15 2 0 0
	14 0 1 0
	5 1 2 0
	9 2 1 0
(46,9)	5 1 0 1
	5 6 0 0
	14 3 0 0
	7 1 2 0
	11 2 1 0
	14 1 1 0
	7 6 0 0
	3 5 1 0
	22 1 0 0
	11 5 0 0
	3 3 2 0
	5 2 0 1
(54,9)	15 4 0 0
	27 0 0 0
	3 3 0 1

	5 3 0 0
	5 1 1 0
	7 3 0 0
	7 1 1 0
	15 1 0 0
	9 3 0 0
	9 1 1 0
	11 3 0 0
	4 3 1 0
	7 0 2 0
	6 5 0 0
	7 0 0 1
	13 3 0 0
	11 4 0 0
	7 3 1 0
(48,9)	3 7 0 0
	3 2 0 1
	15 3 0 0
(50,9)	5 2 2 0
	9 3 1 0
	15 1 1 0
(52,9)	9 1 2 0
	13 2 1 0
	23 1 0 0
	5 7 0 0
	5 5 1 0
	7 2 2 0
	11 3 1 0

	5 0 1 1		6 0 3 0	6 7 0 0
	14 2 1 0	(56,9)	11 1 2 0	3 6 1 0
	13 5 0 0		15 2 1 0	5 3 2 0
	21 0 1 0		25 1 0 0	7 0 3 0
	7 2 0 1		7 7 0 0	3 1 1 1
	12 3 1 0		6 5 1 0	6 0 1 1
(58,9)	5 3 0 1		1 7 1 0	23 2 0 0
	7 0 1 1		7 5 1 0	5 1 3 0
	11 1 0 1		11 6 0 0	13 3 1 0
	15 0 2 0		14 5 0 0	22 0 1 0
	6 3 2 0	(60,9)	13 1 2 0	15 0 0 1
	15 5 0 0		5 1 1 1	21 3 0 0
	23 0 1 0		27 1 0 0	5 6 1 0
	7 3 2 0		3 4 0 1	3 2 3 0
	9 7 0 0		11 4 1 0	14 3 1 0
	6 1 3 0		6 3 0 1	

Next we list the elements in the "cokernels" of the d^{12}-differentials from the 0 row to the 11 row. Recall that $E^4_{*,11} = [Z_4\beta_1 \oplus Z_8\beta_1 M_1] \otimes B\langle 2\rangle$. Thus, all the monomials below have coefficient β_1.

(4.4.3) "COKERNEL $[d^{12}: E^{12}_{*,0} \longrightarrow E^{12}_{*,11}]$":

DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR
(20,11)	Z_2	10 0 0 0	(22,11)	Z_4	8 1 0 0	(24,11)	Z_2	9 1 0 0
	Z_2	3/ 6 2 0 0	(26,11)	Z_2	1/ 6 0 1 0 2/ 10 1 0 0		Z_4	10 1 0 0
(28,11)	Z_2	2/ 11 1 0 0 1/ 4 1 1 0 3/ 14 0 0 0		Z_2	2/ 11 1 0 0 3/ 14 0 0 0		Z_8	11 1 0 0

(30,11) Z_4 6 3 0 0

Z_4 12 1 0 0

(32,11) Z_2 7 3 0 0
6 1 1 0

Z_2 3/ 13 1 0 0
2/ 6 1 1 0

Z_2 2/ 6 1 1 0
1/ 10 2 0 0

Z_4 6 1 1 0

(34,11) Z_2 3/ 7 1 1 0
2/ 14 1 0 0

Z_2 2/ 8 3 0 0
3/ 10 0 1 0
1/ 14 1 0 0

Z_4 8 3 0 0

Z_4 14 1 0 0

(36,11) Z_2 3/ 9 3 0 0
1/ 2 3 1 0
3/ 8 1 1 0
2/ 12 2 0 0

Z_2 2/ 15 1 0 0
3/ 2 3 1 0
3/ 8 1 1 0
1/ 12 2 0 0

Z_2 2/ 15 1 0 0
3/ 12 2 0 0

Z_4 2 3 1 0

Z_8 15 1 0 0

(38,11) Z_2 3/ 9 1 1 0
2/ 6 2 1 0
2/ 12 0 1 0

Z_4 6 2 1 0
10 3 0 0

Z_4 6 2 1 0
12 0 1 0

Z_4 6 2 1 0

(40,11) Z_2 1/ 7 2 1 0
2/ 11 3 0 0
2/ 13 0 1 0
3/ 4 3 1 0
1/ 10 1 1 0
2/ 14 2 0 0

Z_2 2/ 11 3 0 0
3/ 13 0 1 0
2/ 4 3 1 0
2/ 6 0 2 0
3/ 14 2 0 0

Z_2 2/ 11 3 0 0
2/ 4 3 1 0
3/ 6 0 2 0
1/ 14 2 0 0

Z_2 6 0 2 0

Z_4 4 3 1 0

Z_4 10 1 1 0

Z_8 11 3 0 0

(42,11) Z_2 2/ 5 3 1 0
2/ 11 1 1 0
1/ 6 0 0 1
1/ 8 2 1 0
1/ 12 3 0 0
1/ 14 0 1 0

Z_2 2/ 5 3 1 0
2/ 11 1 1 0
3/ 6 0 0 1
2/ 14 0 1 0

Z_4 5 3 1 0
11 1 1 0

Z_4 12 3 0 0
14 0 1 0

Z_4 12 3 0 0

Z_8 5 3 1 0

(44,11) Z_2 1/ 13 3 0 0
3/ 15 0 1 0
1/ 4 6 0 0
1/ 10 4 0 0
3/ 12 1 1 0

Z_2 3/ 4 1 0 1
2/ 6 3 1 0
2/ 12 1 1 0

Z_2 3/ 4 6 0 0
1/ 10 4 0 0

Z_2 2/ 13 3 0 0
1/ 4 6 0 0
2/ 6 3 1 0
2/ 12 1 1 0

Z_4 6 3 1 0
12 1 1 0

Z_4 6 3 1 0

Z_8 13 3 0 0

(46,11) Z_2 3/ 13 1 1 0
1/ 4 4 1 0
1/ 6 1 2 0
2/ 8 5 0 0
2/ 14 3 0 0

Z_2 3/ 4 4 1 0
3/ 6 1 2 0
1/ 10 2 1 0
3/ 14 3 0 0

Z_4 6 1 2 0
8 5 0 0
14 3 0 0

Z_4 4 4 1 0

Z_4 4 4 1 0
14 3 0 0

Z_4 6 1 2 0

Z_8 7 3 1 0

(48,11) Z_2 3/ 7 1 2 0
2/ 9 5 0 0
1/ 11 2 1 0
3/ 15 3 0 0
3/ 4 2 2 0
1/ 6 1 0 1
1/ 6 6 0 0
3/ 8 3 1 0
2/ 14 1 1 0

Z_2 3/ 7 1 2 0
2/ 9 5 0 0
2/ 15 3 0 0
1/ 4 2 2 0
1/ 6 1 0 1
2/ 8 3 1 0
3/ 10 0 2 0

Z_2 3/ 9 5 0 0
1/ 4 2 2 0
2/ 6 1 0 1
3/ 6 6 0 0
2/ 14 1 1 0

Z_2 3/ 4 2 2 0
2/ 6 1 0 1
1/ 6 6 0 0
2/ 8 3 1 0
3/ 10 0 2 0

Z_2 3/ 4 2 2 0
2/ 6 1 0 1
1/ 6 6 0 0
2/ 14 1 1 0

Z_2 2/ 6 1 0 1
1/ 6 6 0 0
2/ 8 3 1 0
2/ 14 1 1 0

Z_4 8 3 1 0

Z_4 8 3 1 0
14 1 1 0

Z_4 6 1 0 1
8 3 1 0

Z_8 15 3 0 0

(50,11) Z_2 1/ 7 1 0 1
2/ 15 1 1 0
3/ 4 2 0 1
2/ 4 7 0 0
2/ 6 4 1 0
1/ 10 0 0 1
2/ 10 5 0 0

Z_2 3/ 9 3 1 0
1/ 4 2 0 1
2/ 4 7 0 0
2/ 6 4 1 0
2/ 8 1 2 0
3/ 10 0 0 1
2/ 10 5 0 0
2/ 12 2 1 0

Z_2 3/ 4 2 0 1
1/ 10 0 0 1
2/ 10 5 0 0

Z_2 1/ 4 2 0 1
1/ 4 7 0 0
2/ 6 4 1 0
3/ 12 2 1 0

Z_2 2/ 15 1 1 0
3/ 4 7 0 0
2/ 8 1 2 0
1/ 10 5 0 0
1/ 12 2 1 0

Z_4 4 2 0 1
4 7 0 0
6 4 1 0

Z_4 8 1 2 0

Z_4 4 7 0 0

Z_4 6 4 1 0

Z_8 15 1 1 0

(52,11) Z_2 3/ 5 7 0 0
1/ 7 4 1 0
3/ 11 5 0 0
2/ 4 0 1 1
1/ 4 5 1 0
1/ 6 2 2 0

Z_2 1/ 9 1 2 0
2/ 11 5 0 0
2/ 13 2 1 0
1/ 2 3 0 1
3/ 4 0 1 1
2/ 4 5 1 0
2/ 6 2 2 0
3/ 8 1 0 1
1/ 14 4 0 0
3/ 26 0 0 0

Z_2 1/ 5 7 0 0
3/ 11 5 0 0
1/ 13 2 1 0
3/ 2 3 0 1
3/ 4 5 1 0
3/ 8 1 0 1
1/ 26 0 0 0

	Z_2 1/ 2 3 0 1 2/ 4 0 1 1 2/ 4 5 1 0 1/ 6 2 2 0 1/ 8 1 0 1 2/ 26 0 0 0	Z_2 3/ 4 0 1 1 2/ 4 5 1 0 3/ 6 2 2 0 3/ 26 0 0 0 Z_2 26 0 0 0	Z_2 2/ 11 5 0 0 2/ 13 2 1 0 2/ 2 3 0 1 1/ 6 2 2 0 1/ 14 4 0 0 1/ 26 0 0 0
	Z_2 2/ 4 5 1 0 3/ 6 2 2 0	Z_4 4 5 1 0	Z_4 2 3 0 1
	Z_4 10 3 1 0	Z_8 11 5 0 0	Z_8 13 2 1 0
(54,11)	Z_2 3/ 5 5 1 0 1/ 9 1 0 1 1/ 4 3 2 0 2/ 6 0 3 0 2/ 6 2 0 1 3/ 6 7 0 0 1/ 10 1 2 0 3/ 12 5 0 0 1/ 20 0 1 0 1/ 24 1 0 0	Z_2 1/ 5 5 1 0 1/ 2 6 1 0 1/ 6 7 0 0 2/ 12 5 0 0 2/ 14 2 1 0 2/ 24 1 0 0	Z_2 1/ 2 6 1 0 3/ 4 3 2 0 1/ 6 2 0 1 3/ 6 7 0 0 1/ 10 1 2 0 3/ 12 5 0 0 1/ 14 2 1 0 3/ 20 0 1 0 3/ 24 1 0 0
	Z_2 2/ 11 3 1 0 1/ 2 6 1 0 3/ 4 3 2 0 2/ 6 0 3 0 2/ 6 7 0 0 1/ 10 1 2 0 1/ 12 5 0 0 2/ 14 2 1 0 1/ 20 0 1 0 3/ 24 1 0 0	Z_2 2/ 4 3 2 0 2/ 6 7 0 0 2/ 14 2 1 0 3/ 20 0 1 0	Z_4 10 1 2 0 12 5 0 0 24 1 0 0 Z_4 4 3 2 0 6 0 3 0 14 2 1 0
	Z_4 6 0 3 0 12 5 0 0 24 1 0 0	Z_4 6 0 3 0 6 7 0 0 14 2 1 0 24 1 0 0	Z_4 6 0 3 0 6 7 0 0
	Z_4 6 7 0 0 24 1 0 0	Z_4 24 1 0 0	Z_8 11 3 1 0
(56,11)	Z_2 3/ 5 3 2 0 1/ 7 0 3 0 1/ 11 1 2 0 2/ 13 5 0 0 3/ 15 2 1 0 1/ 25 1 0 0 1/ 4 1 3 0 1/ 12 3 1 0 2/ 22 2 0 0	Z_2 3/ 13 5 0 0 2/ 15 2 1 0 2/ 25 1 0 0 2/ 0 7 1 0 2/ 4 3 0 1 2/ 6 5 1 0 2/ 10 1 0 1 2/ 10 6 0 0 2/ 14 0 2 0	Z_2 3/ 7 0 3 0 3/ 7 7 0 0 2/ 11 1 2 0 3/ 15 2 1 0 1/ 25 1 0 0 3/ 0 7 1 0 2/ 4 1 3 0 2/ 6 0 1 1 2/ 6 5 1 0 3/ 22 2 0 0

Z_2 1/ 7 0 3 0
2/ 7 7 0 0
2/ 11 1 2 0
2/ 25 1 0 0
3/ 0 7 1 0
1/ 6 5 1 0
3/ 10 6 0 0
3/ 12 3 1 0
2/ 14 0 2 0
2/ 22 2 0 0

Z_2 2/ 7 7 0 0
2/ 11 1 2 0
3/ 0 7 1 0
1/ 4 1 3 0
1/ 6 0 1 1
2/ 10 1 0 1
2/ 10 6 0 0
1/ 12 3 1 0
1/ 14 0 2 0
3/ 22 2 0 0

Z_4 6 0 1 1
10 1 0 1

Z_4 6 0 1 1

Z_4 6 0 1 1
12 3 1 0

Z_2 1/ 25 1 0 0
2/ 0 7 1 0
2/ 4 3 0 1
2/ 6 0 1 1
2/ 6 5 1 0
2/ 10 6 0 0
2/ 12 3 1 0
2/ 14 0 2 0
3/ 22 2 0 0

Z_2 2/ 7 7 0 0
2/ 0 7 1 0
2/ 4 3 0 1
1/ 10 6 0 0
2/ 12 3 1 0
2/ 14 0 2 0

Z_4 0 7 1 0
4 3 0 1
6 0 1 1
12 3 1 0

Z_8 7 7 0 0

Z_2 2/ 7 7 0 0
1/ 0 7 1 0
2/ 4 1 3 0
1/ 4 3 0 1
1/ 6 5 1 0
1/ 10 1 0 1
2/ 10 6 0 0
3/ 12 3 1 0
2/ 14 0 2 0
2/ 22 2 0 0

Z_2 2/ 11 1 2 0
2/ 6 0 1 1
2/ 12 3 1 0
1/ 14 0 2 0

Z_2 2/ 0 7 1 0
2/ 4 3 0 1
2/ 10 1 0 1
2/ 12 3 1 0
3/ 22 2 0 0

Z_4 0 7 1 0
6 0 1 1
12 3 1 0

Z_8 11 1 2 0

(58,11) Z_2 1/ 7 0 1 1
2/ 13 3 1 0
2/ 4 1 1 1
2/ 12 1 2 0
1/ 14 0 0 1
1/ 14 5 0 0

Z_2 1/ 5 3 0 1
3/ 7 5 1 0
1/ 11 1 0 1
1/ 13 3 1 0
1/ 4 1 1 1
1/ 4 6 1 0
1/ 8 7 0 0
2/ 14 0 0 1
2/ 14 5 0 0
3/ 22 0 1 0
1/ 26 1 0 0

Z_4 5 3 0 1
11 1 0 1
4 1 1 1
4 6 1 0
8 7 0 0
14 5 0 0

Z_2 2/ 7 5 1 0
2/ 8 7 0 0
3/ 10 4 1 0
1/ 14 5 0 0
1/ 20 3 0 0
1/ 26 1 0 0

Z_2 2/ 7 5 1 0
2/ 11 1 0 1
2/ 4 1 1 1
2/ 8 7 0 0
2/ 12 1 2 0
1/ 14 0 0 1
3/ 14 5 0 0
3/ 20 3 0 0
2/ 22 0 1 0
3/ 26 1 0 0

Z_4 4 6 1 0
6 3 2 0
14 5 0 0
26 1 0 0

Z_2 2/ 4 6 1 0
2/ 6 3 2 0
2/ 8 7 0 0
1/ 20 3 0 0
2/ 22 0 1 0
3/ 26 1 0 0

Z_2 2/ 5 3 0 1
2/ 11 1 0 1
2/ 6 3 2 0
2/ 8 7 0 0
2/ 12 1 2 0
1/ 22 0 1 0

Z_4 8 7 0 0
12 1 2 0

Z_4 6 3 2 0
8 7 0 0
26 1 0 0

Z_4	14 5 0 0	Z_4	8 7 0 0 26 1 0 0	Z_4	4 1 1 1 26 1 0 0
Z_4	26 1 0 0	Z_8	7 5 1 0	Z_8	11 1 0 1

Next we list the elements of the "cokernels" of the d^{16}-differentials from the 0 row to the 15 row. Note that $E^4_{*,15} = [Z_{32}\gamma_1 \oplus Z_{16}(2\gamma_1 M_1)] \otimes B\langle 2\rangle$. Thus, all the monomials below have coefficient γ_1.

(4.4.4) "COKERNEL $[d^{16}: E^{16}_{*,0} \longrightarrow E^{16}_{*,15}]$":

DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR
(22,15)	Z_2	8 1 0 0	(24,15)	Z_4	12 0 0 0	(26,15)	Z_8	10 1 0 0
(28,15)	Z_2	10/ 11 1 0 0 14/ 14 0 0 0		Z_{16}	14 0 0 0	(30,15)	Z_2	24/ 15 0 0 0 15/ 8 0 1 0 28/ 12 1 0 0
	Z_4	26/ 15 0 0 0 16/ 8 0 1 0 30/ 12 1 0 0		Z_{32}	12 1 0 0	(32,15)	Z_4	18/ 13 1 0 0 3/ 10 2 0 0
	Z_8	18/ 13 1 0 0 2/ 10 2 0 0	(34,15)	Z_2	9/ 8 3 0 0 1/ 10 0 1 0 9/ 14 1 0 0		Z_8	10 0 0 1
							Z_{32}	14 1 0 0
(36,15)	Z_2	22/ 11 0 1 0 20/ 15 1 0 0 18/ 12 2 0 0		Z_4	10/ 15 1 0 0 1/ 8 1 1 0 19/ 12 2 0 0		Z_{16}	8 1 1 0
	Z_{16}	14/ 15 1 0 0 10/ 8 1 1 0 4/ 12 2 0 0	(38,15)	Z_2	2/ 13 2 0 0 25/ 6 2 1 0 10/ 10 3 0 0 3/ 12 0 1 0		Z_4	26/ 13 2 0 0 26/ 6 2 1 0 8/ 10 3 0 0 4/ 12 0 1 0
	Z_{32}	6 2 1 0	(40,15)	Z_2	30/ 11 3 0 0 22/ 13 0 1 0 12/ 8 4 0 0 26/ 10 1 1 0 30/ 14 2 0 0		Z_4	12/ 13 0 1 0 29/ 8 4 0 0 20/ 10 1 1 0 30/ 14 2 0 0
	Z_8	22/ 13 0 1 0 12/ 8 4 0 0 26/ 10 1 1 0 30/ 14 2 0 0		Z_{16}	10 1 1 0 14 2 0 0		Z_{16}	10 1 1 0

(42,15)	Z_2	14/ 11 1 1 0 9/ 6 5 0 0 13/ 8 2 1 0 6/ 12 3 0 0 13/ 14 0 1 0	Z_4	6/ 11 1 1 0 2/ 15 2 0 0 30/ 6 5 0 0 22/ 8 2 1 0 30/ 12 3 0 0 8/ 14 0 1 0	Z_8	18/ 15 2 0 0 24/ 6 5 0 0 6/ 8 2 1 0 14/ 12 3 0 0 20/ 14 0 1 0
	Z_8	1/ 6 5 0 0 2/ 12 3 0 0	Z_{32}	14 0 1 0	Z_{32}	12 3 0 0
(44,15)	Z_2	22/ 9 2 1 0 8/ 13 3 0 0 18/ 15 0 1 0 14/ 6 3 1 0 8/ 10 4 0 0 26/ 12 1 1 0	Z_4	20/ 13 3 0 0 30/ 15 0 1 0 7/ 6 3 1 0 13/ 10 4 0 0 27/ 12 1 1 0	Z_8	6/ 13 3 0 0 10/ 15 0 1 0 24/ 6 3 1 0 20/ 10 4 0 0 2/ 12 1 1 0
	Z_{16}	14/ 15 0 1 0 8/ 6 3 1 0 6/ 10 4 0 0 6/ 12 1 1 0	Z_{16}	10 4 0 0	Z_{32}	6 3 1 0
(46,15)	Z_2	12/ 11 4 0 0 24/ 13 1 1 0 24/ 4 4 1 0 3/ 8 0 0 1 4/ 8 5 0 0 4/ 10 2 1 0 8/ 14 3 0 0	Z_2	8/ 11 4 0 0 7/ 4 4 1 0 7/ 8 0 0 1 18/ 10 2 1 0	Z_4	10/ 11 4 0 0 28/ 13 1 1 0 12/ 4 4 1 0 12/ 8 0 0 1 14/ 8 5 0 0 12/ 10 2 1 0 4/ 14 3 0 0
	Z_8	30/ 13 1 1 0 14/ 4 4 1 0 18/ 8 0 0 1 16/ 8 5 0 0 14/ 10 2 1 0 16/ 14 3 0 0	Z_8	3/ 8 5 0 0 1/ 10 2 1 0 2/ 14 3 0 0	Z_{32}	14 3 0 0
					Z_{32}	10 2 1 0
(48,15)	Z_2	22/ 9 5 0 0 14/ 11 2 1 0 12/ 15 3 0 0 22/ 6 6 0 0 2/ 8 3 1 0 10/ 10 0 2 0 10/ 12 4 0 0 14/ 14 1 1 0	Z_4	16/ 11 2 1 0 28/ 15 3 0 0 25/ 6 6 0 0 26/ 8 3 1 0 31/ 10 0 2 0 20/ 12 4 0 0 18/ 14 1 1 0	Z_4	6/ 11 2 1 0 1/ 6 6 0 0 2/ 8 3 1 0 7/ 12 4 0 0 6/ 14 1 1 0
					Z_8	2/ 11 2 1 0 3/ 6 6 0 0
	Z_8	2/ 11 2 1 0	Z_{16}	30/ 15 3 0 0 28/ 6 6 0 0 26/ 8 3 1 0 14/ 10 0 2 0 8/ 12 4 0 0 22/ 14 1 1 0	Z_{16}	8 3 1 0 14 1 1 0
					Z_{32}	8 3 1 0

(50,15) Z_2
10/ 9 3 1 0
24/ 13 4 0 0
18/ 15 1 1 0
21/ 4 7 0 0
29/ 6 4 1 0
11/ 8 1 2 0
1/ 10 0 0 1
8/ 10 5 0 0
14/ 12 2 1 0

Z_2
24/ 9 3 1 0
6/ 13 4 0 0
8/ 15 1 1 0
31/ 4 7 0 0
19/ 6 4 1 0
4/ 8 1 2 0
1/ 10 5 0 0
2/ 12 2 1 0

Z_4
2/ 9 3 1 0
10/ 13 4 0 0
14/ 15 1 1 0
12/ 4 7 0 0
4/ 6 4 1 0
12/ 8 1 2 0
22/ 10 0 0 1
2/ 10 5 0 0
2/ 12 2 1 0

Z_4
6/ 13 4 0 0
4/ 4 7 0 0
14/ 6 4 1 0
4/ 8 1 2 0
24/ 10 0 0 1
12/ 10 5 0 0
24/ 12 2 1 0

Z_8
1/ 4 7 0 0
1/ 8 1 2 0
2/ 12 2 1 0

Z_8
3/ 6 4 1 0
2/ 12 2 1 0

Z_{16}
26/ 15 1 1 0
4/ 4 7 0 0
22/ 6 4 1 0
26/ 8 1 2 0
26/ 10 0 0 1
26/ 10 5 0 0
28/ 12 2 1 0

Z_{32}
12 2 1 0

Z_{32}
8 1 2 0

(52,15) Z_2
6/ 7 4 1 0
12/ 11 0 0 1
24/ 13 2 1 0
2/ 4 5 1 0
22/ 8 1 0 1
12/ 8 6 0 0
4/ 10 3 1 0
14/ 12 0 2 0
26/ 14 4 0 0
27/ 26 0 0 0

Z_2
30/ 7 4 1 0
30/ 11 0 0 1
8/ 11 5 0 0
24/ 13 2 1 0
30/ 4 5 1 0
20/ 8 1 0 1
8/ 8 6 0 0
20/ 10 3 1 0
26/ 12 0 2 0
14/ 14 4 0 0
16/ 26 0 0 0

Z_2
18/ 11 0 0 1
28/ 11 5 0 0
28/ 13 2 1 0
22/ 4 5 1 0
16/ 8 1 0 1
22/ 8 6 0 0
28/ 10 3 1 0
18/ 12 0 2 0
24/ 14 4 0 0

Z_4
26/ 11 5 0 0
18/ 13 2 1 0
11/ 4 5 1 0
7/ 8 1 0 1
31/ 8 6 0 0
30/ 10 3 1 0
11/ 12 0 2 0
6/ 14 4 0 0
24/ 26 0 0 0

Z_4
14/ 11 5 0 0
16/ 13 2 1 0
19/ 4 5 1 0
5/ 8 6 0 0
18/ 10 3 1 0
28/ 12 0 2 0
19/ 14 4 0 0
18/ 26 0 0 0

Z_8
4/ 11 5 0 0
22/ 13 2 1 0
8/ 4 5 1 0
7/ 8 1 0 1
1/ 8 6 0 0
26/ 10 3 1 0
24/ 12 0 2 0
6/ 14 4 0 0
2/ 26 0 0 0

Z_8
18/ 11 5 0 0
6/ 13 2 1 0
22/ 4 5 1 0
30/ 8 1 0 1
28/ 8 6 0 0
10/ 10 3 1 0
20/ 12 0 2 0
20/ 14 4 0 0
20/ 26 0 0 0

Z_{16}
22/ 13 2 1 0
4/ 4 5 1 0
24/ 8 1 0 1
26/ 8 6 0 0
26/ 10 3 1 0
16/ 12 0 2 0
2/ 14 4 0 0
6/ 26 0 0 0

Z_{16}
4 5 1 0

Z_{16}
8 6 0 0

Z_{32}
10 3 1 0

(54,15)	Z_2	4/ 9 6 0 0 4/ 11 3 1 0 22/ 13 0 2 0 14/ 15 4 0 0 9/ 6 2 0 1 16/ 6 7 0 0 28/ 8 4 1 0 2/ 10 1 2 0 7/ 12 0 0 1 22/ 12 5 0 0 20/ 14 2 1 0 9/ 24 1 0 0	Z_4	8/ 9 6 0 0 8/ 11 3 1 0 22/ 15 4 0 0 14/ 6 2 0 1 18/ 6 7 0 0 8/ 8 4 1 0 6/ 10 1 2 0 2/ 12 0 0 1 4/ 12 5 0 0 12/ 14 2 1 0 19/ 24 1 0 0	Z_4	20/ 9 6 0 0 4/ 11 3 1 0 22/ 13 0 2 0 24/ 15 4 0 0 26/ 6 2 0 1 4/ 6 7 0 0 28/ 8 4 1 0 6/ 10 1 2 0 22/ 12 0 0 1 8/ 12 5 0 0 8/ 14 2 1 0 22/ 24 1 0 0
	Z_4	16/ 9 6 0 0 16/ 11 3 1 0 22/ 15 4 0 0 18/ 6 2 0 1 20/ 6 7 0 0 20/ 8 4 1 0 12/ 10 1 2 0 2/ 12 0 0 1 2/ 12 5 0 0 12/ 14 2 1 0 18/ 24 1 0 0	Z_4	14/ 9 6 0 0 20/ 11 3 1 0 20/ 8 4 1 0 26/ 10 1 2 0 6/ 12 5 0 0 4/ 14 2 1 0	Z_8	3/ 6 7 0 0 3/ 10 1 2 0 3/ 12 5 0 0
			Z_8	2/ 11 3 1 0 3/ 8 4 1 0 3/ 10 1 2 0 2/ 14 2 1 0	Z_8	2/ 11 3 1 0 1/ 6 2 0 1 1/ 6 7 0 0
					Z_8	2/ 11 3 1 0 2/ 10 1 2 0
	Z_{32}	14 2 1 0	Z_{32}	6 7 0 0	Z_{32}	10 1 2 0

Next we list the "cokernels" of the d^{18}-differentials from the 0 row to the 17 row. Note that $E^4_{*,17} = [Z_2\alpha_2 \oplus Z_2\eta^2\gamma_1 M_1] \otimes B\langle 2\rangle$. Thus, the monomials below with an odd power of M_1 have coefficient $\eta^2\gamma_1$ and the monomials with an even power of M_1 have coefficient α_2.

(4.4.5) "COKERNEL $[d^{18}: E^{18}_{*,0} \longrightarrow E^{18}_{*,17}]$":

DEGREE	BASIS	DEGREE	BASIS	DEGREE	BASIS
(34,17)	14 1 0 0	(36,17)	15 1 0 0	(38,17)	19 0 0 0
(40,17)	17 1 0 0	(42,17)	15 2 0 0		14 0 1 0
(44,17)	19 1 0 0		15 0 1 0	(46,17)	23 0 0 0
	13 1 1 0		14 3 0 0	(48,17)	21 1 0 0
	15 3 0 0		14 1 1 0	(50,17)	15 1 1 0
	19 2 0 0		12 2 1 0	(52,17)	23 1 0 0
	13 2 1 0		17 3 0 0		19 0 1 0

Next we list the elements of the "cokernels" of the d^{20}-differentials from the 0 row to the 19 row. Recall that $E^4_{*,19} = [Z_4\beta_2 \oplus Z_8\beta_2 M_1] \otimes B\langle 2\rangle$. Thus, all the monomials below have coefficient β_2.

(4.4.6) "COKERNEL $[d^{20}: E^{20}_{*,0} \longrightarrow E^{20}_{*,19}]$":

DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR
(36,19)	Z_2	18 0 0 0	(38,19)	Z_4	16 1 0 0	(40,19)	Z_2	17 1 0 0
(42,19)	Z_2	14 0 1 0		Z_4	18 1 0 0	(44,19)	Z_2	2/ 19 1 0 0 3/ 22 0 0 0
	Z_8	19 1 0 0	(46,19)	Z_2	3/ 14 3 0 0 2/ 20 1 0 0		Z_4	20 1 0 0
(48,19)	Z_2	3/ 21 1 0 0 2/ 14 1 1 0 2/ 18 2 0 0		Z_2	2/ 14 1 1 0 1/ 18 2 0 0		Z_4	14 1 1 0
(50,19)	Z_2	1/ 15 1 1 0 2/ 22 1 0 0		Z_2	18 0 1 0 22 1 0 0		Z_4	16 3 0 0
							Z_4	22 1 0 0

Next we list the elements of the "cokernels" of the d^{24}-differentials from the 0 row to the 23 row. Note that $E^4_{*,23}$ contains $[Z_{16}\gamma_2 \oplus Z_8(2\gamma_2 M_1)] \otimes B\langle 2\rangle$ as a direct summand. Thus, all the monomials below have coefficient γ_2.

(4.4.7) "COKERNEL $[d^{24}: E^{24}_{*,0} \longrightarrow E^{24}_{*,23}]$":

DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR	DEGREE	GROUP	GENERATOR
(40,23)	Z_2	20 0 0 0	(42,23)	Z_4	18 1 0 0	(44,23)	Z_8	22 0 0 0
(46,23)	Z_2	2/ 23 0 0 0 6/ 20 1 0 0		Z_{16}	20 1 0 0			

The d^r-differentials, $r > 24$, which originate on the 0 row have zero "cokernels" in degrees less than 70.