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HOMOLOGY OF THE CLASSICAL GROUPS OVER THE DYER-LASHOF ALGEBRA(1,2)

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ABSTRACT. The action of the Dyer-Lashof algebra is computed on the homology of the infinite classical groups (including Spin), their classifying spaces, their homogeneous spaces, Im J, B Im J and $BBSO$. Some applications are given while applications by other authors appear elsewhere.

1. **Introduction.** In §2, we will show that the Dyer-Lashof algebra $\mathcal{R}$ acts on the homology of the infinite classical groups (including Spin), their classifying spaces, their homogeneous spaces, Im J, B Im J and $BBSO$. In Theorem 1 of that section we will also list the basic properties of the Dyer-Lashof operations which will be used extensively in the sequel.

In §3, we state Theorems 5, 6 and 7 which describe the action of the Dyer-Lashof algebra on $H_*(BU)$ and on $H^*(BU)$. From these three theorems we compute that $\mathcal{R}$-algebra indecomposables of $H_*(BU)$, the algebra of $\mathfrak{A}\mathcal{R}$-Hopf algebra endomorphisms of $H_*(BU)$ and the action of $\mathcal{R}$ on $H_*(BU \times Z) = H_*(\Omega^2 BU)$ in §3. We postpone the proofs of Theorems 5, 6 and 7 to §8. The proof of Theorem 7 contains an algorithm for computing the action of the Dyer-Lashof algebra on $H_*(BU)$ (see Theorem 97) which we apply in the mod 2 case to compute this action in dimensions less than or equal to twenty.

In §§4, 5 and 6 we show that the results of §3 imply similar results for the action of $\mathcal{R}$ on the homology of the remaining classifying spaces of the classical groups, the classical groups and the homogeneous spaces of the classical groups, respectively. §6 also contains a discussion of the action of $\mathcal{R}$ on $H_*(\text{Spin})$ and $H_*(B\ \text{Spin})$. In §7, we use the results of the preceding sections to investigate the action of $\mathcal{R}$ on $H_*(\text{Im } J)$, $H_*(B \text{ Im } J)$ and $H_*(BBSO)$.

The results of this paper have several applications. By T. tom Dieck [6, Theorem 17.2] the knowledge of the $\mathcal{R}$-action on $H_*(BO; Z_2)$ can be used to determine the normal characteristic numbers of the quadratic construction on a

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closed differentiable manifold M in terms of the normal characteristic numbers of M. In J. P. May [19], I. Madsen [14] and A. Tsuchiya [26] the knowledge of the $\mathscr{R}$-action on $H_*(O)$ and $H_*(BO)$ is used together with the maps $J_*: H_*(O) \to H_*(F)$ and $(BJ)_*: H_*(BO) \to H_*(BF)$ as part of their computation of $H_*(F)$ and $H_*(BF)$. M. Herrero [10] has computed homology operations on the homology of $BU \times Z$ and $BO \times Z$ with H-space structure induced by the tensor product of bundles. Her results use the $\mathscr{R}$-action on the homology of these spaces with H-space structure induced by the Whitney sum of bundles, i.e. the theorems of §§3, 4 and 8.

Throughout this paper $H_*(X)$ denotes the homology of X with Z_p-coefficients for p any prime. If a statement differs for p odd and $p = 2$ then the result for p odd will be stated, followed by the analogous statement for $p = 2$ in square brackets.

We will assume a familiarity with the structure of the homology and cohomology of the infinite classical groups, their classifying spaces and their homogeneous spaces as Hopf algebras over the Steenrod algebra. Three excellent references for these results are Séminaire Henri Cartan [4], A. Liulevicius [13] and E. Dyer and R. Lashof [8]. We will also use many elementary properties of the Steenrod algebra $\mathfrak{A}$ with no references. All such properties can be found in N. Steenrod and D. Epstein [24].

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2. The underlying geometry of the action of the Dyer-Lashof algebra on the homology of the classical groups. The results of the following sections are predicated on Theorems 1 through 4 which assert that the Dyer-Lashof algebra acts on the homology of the infinite classical groups, their classifying spaces and their homogeneous spaces with the product in homology induced by Whitney sum. These homology operations satisfy the usual properties and commute with the homomorphisms induced in homology by the structure maps of the two Bott spectra (see E. Dyer and R. Lashof [8]):

$$B_{2n} = BU \times Z, \qquad B_{2n+1} = U;$$

$$C_{8n} = BO \times Z, \quad C_{8n+1} = U/O, \quad C_{8n+2} = \mathrm{Sp}/U, \quad C_{8n+3} = \mathrm{Sp},$$
$$C_{8n+4} = B\mathrm{Sp} \times Z, \quad C_{8n+5} = U/\mathrm{Sp}, \quad C_{8n+6} = O/U, \quad C_{8n+7} = O.$$

There are two ways of obtaining this information. First, the Bott spectra show that all the spaces that we are interested in are infinite loop spaces, and hence the E_∞-operad $\mathcal{C}_\infty$ acts on these spaces (see Theorem 2). On the other hand, there is an E_∞-operad $\mathscr{L}$ which acts on these spaces such that J: $O \to F$ and BJ: BO

$\to BF$ are maps of $\mathcal{L}$-spaces (see Theorem 3). By Theorem 1, the Dyer-Lashof algebra acts on the homology of $\mathcal{C}$-spaces for E_∞-operads $\mathcal{C}$. Theorem 3 will also state that we obtain the same action of the Dyer-Lashof algebra from both of the above points of view. We will now define an E_∞-operad and explain the statements of Theorems 1 through 4.

The endomorphism operad $\mathcal{E}_X$ [17, Definition 1.2] consists of the spaces $\mathcal{E}_X(j)$ of based maps $X^j \to X$, $j \geq 1$, and $\mathcal{E}_X(0) = *$ where X^j is the j-fold Cartesian product of X with itself. The structure of $\mathcal{E}_X$ which makes it an operad consists of:

(1) The map γ: $\mathcal{E}_X(k) \times \mathcal{E}_X(j_1) \times \cdots \times \mathcal{E}_X(j_k) \to \mathcal{E}_X(j_1 + \cdots + j_k)$ defined by $\gamma(f; g_1, \ldots, g_k) = f(g_1 \times \cdots \times g_k)$. Note that γ satisfies an associativity formula.

(2) $1_X \in \mathcal{E}_X(1)$ satisfies $\gamma(1_X; g) = g$ and $\gamma(f; 1_X^k) = f$ for all $g \in \mathcal{E}_X(j)$ and all $f \in \mathcal{E}_X(k)$.

(3) The symmetric group Σ_j acts on $\mathcal{E}_X(j)$ by

$$(g\sigma)(x_1, \ldots, x_j) = g(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(j)}).$$

With the appropriate conventions, the map γ becomes equivariant in each of its two variables.

In general, an operad $\mathcal{C}$ [17, Definition 1.1] consists of nonempty spaces $\mathcal{C}(j)$ which have the same formal structure (1), (2) and (3) as the endomorphism operad. An operad is called Σ-free if Σ_j acts freely on $\mathcal{C}(j)$ for $j \geq 1$. A $\mathcal{C}$-space (X, θ) consists of Σ_j-equivariant maps θ_j: $\mathcal{C}(j) \to \mathcal{E}_X(j)$ which commute with γ. A map f: $(X, \theta) \to (X', \theta')$ of $\mathcal{C}$-spaces is a based map f: $X \to X'$ such that $f \circ \theta_j(c) = \theta'_j(c) \circ f^j$ for all $c \in \mathcal{C}(j)$ and all $j \geq 1$. Note that any $c \in \mathcal{C}(2)$ defines an H-space structure $\theta_2(c)$: $X \times X \to X$ on a $\mathcal{C}$-space X which by J. P. May [17, Lemma 1.9] is independent of c up to homotopy if all the $\mathcal{C}(j)$ are connected, $j \geq 2$.

An E_∞-operad $\mathcal{C}$ is a Σ-free operad $\mathcal{C}$ such that each $\mathcal{C}(j)$ is contractible. By the preceding remarks, if X a $\mathcal{C}$-space for an E_∞-operad $\mathcal{C}$ then $H_*(X)$ has a unique product induced by $\theta_2(c)$ for any $c \in \mathcal{C}(2)$. In addition, $H_*(X; Z_p)$ has Dyer-Lashof operations for all primes p because there are Σ_p-equivariant maps $\mathcal{C}(p) \times X^p \to X$ and $\mathcal{C}(p)/\Sigma_p = K(\Sigma_p, 1)$. More precisely, we state J. P. May [18, Theorem 1.1]:

Theorem 1. *Let $\mathcal{C}$ be an E_∞-operad, and let p be a prime number. The homology of a $\mathcal{C}$-space has Dyer-Lashof operations Q^n, $n \geq 0$, of degree $2n(p-1)$ [of degree n] which satisfy the following properties*:

(1) *The Q^n are natural with respect to the maps induced in homology by $\mathcal{C}$-maps.*

(2) *The Q^n are linear.*

(3) $Q^0(\phi) = \phi$ *and* $Q^n(\phi) = 0$ *if* $n > 0$ *where* $\phi \in H_0(X)$ *is the identity element for the multiplication in* $H_*(X)$.

(4) $Q^n(x) = x^p$ *if* $\deg x = 2n$ [*if* $\deg x = n$].

(5) $Q^n \circ \sigma_* = \sigma_* \circ Q^n$ *where* σ_*: $IH_*(X) \to H_*(\Omega X)$ *is the suspension map.*

(6) (*multiplicative Cartan formula*) $Q^r(xy) = \sum_{i=0}^r Q^i(x)Q^{r-i}(y)$.

(7) (*comultiplicative Cartan formula*)

$$\psi \circ Q^r(x) = \sum_{i=0}^{r} \sum Q^i(x') \otimes Q^{r-i}(x'') \quad \textit{where} \quad \psi(x) = \sum x' \otimes x''.$$

(8) $\chi \circ Q^r = Q^r \circ \chi$ *where* χ *is the conjugation on* $H_*(X)$.

(9) (*Adem relations*) *If* $a > pb$ *then*

$$Q^a \circ Q^b = \sum (-1)^{a+i}(pi - a, a - (p-1)b - i - 1)Q^{a+b-i} \circ Q^i$$

where $(i,j) = (i+j)!/i!j!$ *if* $i \geq 0$ *and* $j \geq 0$ *while* $(i,j) = 0$ *if* $i < 0$ *or* $j < 0$.

(10) (*Nishida relations*)

$$\mathcal{P}_*^s \circ Q^r = \sum (-1)^{i+s}(s - pi, r(p-1) - ps + pi)Q^{r-s+i} \circ \mathcal{P}_*^i$$

where $\mathcal{P}_*^s\colon H_*(X) \to H_*(X)$ *of degree* $-2s(p-1)$ [*of degree* $-s$] *is dual to* $\mathcal{P}^s$ *in the action of* $\mathfrak{A}^{\mathrm{op}}$ *on* $H_*(X)$.

The above properties of the Dyer-Lashof operations are proved by showing that the general algebraic considerations of J. P. May [16] are applicable to the homology of $\mathcal{C}$-spaces for $\mathcal{C}$ an E_∞-operad. Many of these basic properties of Dyer-Lashof operations were first proved in S. Araki and T. Kudo [3], E. Dyer and R. Lashof [9] and G. Nishida [21].

Recall that an infinite loop space B_0 is the first space in a sequence $\{B_n \mid n \geq 0\}$ where $B_n = \Omega B_{n+1}$ for $n \geq 0$. An infinite loop map $f_0\colon B_0 \to C_0$ between two infinite loop spaces is the first map in a sequence of maps $\{f_n\colon B_n \to C_n \mid n \geq 0\}$ such that $f_n = \Omega f_{n+1}$, $n \geq 0$. By J. P. May [15], these definitions are equivalent to the concepts of an Ω-spectrum and a map of Ω-spectra where homotopies replace the equalities. The first method of defining homology operations on the homology of the infinite classical groups, their classifying spaces, their homogeneous spaces Spin, $B\,$Spin, $BBSO$, $\operatorname{Im} J$ and $B \operatorname{Im} J$ is to use the Bott spectra to show that all the spaces in question are infinite loop spaces and that all the canonical maps and the Bott maps are infinite loop maps. The desired conclusions then follow from Theorem 1 and the following theorem.

Theorem 2. *There is an* E_∞*-operad* $\mathcal{C}_\infty$ *such that all infinite loop spaces are* $\mathcal{C}_\infty$*-spaces and all infinite loop maps are maps of* $\mathcal{C}_\infty$*-spaces.*

For a proof and discussion of this theorem see J. P. May [17, §§4 and 5].

Our second approach to defining homology operations on the homology of the spaces under consideration is based upon Theorem 1 and the following theorem.

Theorem 3. *There is an* E_∞*-operad* $\mathcal{L}$ *such that all of the classical groups* (*including Spin*), *their classifying spaces, their homogeneous spaces, F and BF are* $\mathcal{L}$*-spaces. All of the canonical maps, including* $J\colon O \to F$, $BJ\colon BO \to BF$, *and the Bott* *maps are maps of $\mathcal{L}$-spaces. Furthermore, the induced homology operations are the same as those induced by the E_∞-operad $\mathcal{C}_\infty$ via Theorem 2.*

For the definition of $\mathcal{L}$ and the proof of Theorem 3, see J. P. May [18, §6].

To define $BBSO$ as an $\mathcal{L}$-space we need to deloop BSO. That is we need the following result from J. P. May [17, §14].

Theorem 4. *Let $\mathcal{C}$ be an E_∞-operad, and let X be a connected $\mathcal{C}$-space. Then (up to weak homotopy equivalence) X is an infinite loop space such that $H_*(X)$ has the same homology operations induced by $\mathcal{C}$ and by $\mathcal{C}_\infty$ (the operad of Theorem 2 which acts on infinite loop spaces).*

The Dyer-Lashof algebra $\mathcal{R}$ is defined to be the quotient algebra F/J where F is the free associative algebra generated by $\{Q^r, \beta Q^{r+1} \mid r \geq 0\}$ [generated by $\{Q^r \mid r \geq 0\}$] and J is the ideal in F consisting of all elements of F which annihilate every element of every infinite loop space.

If $\mathcal{R}$ acts on $H_*(X)$ then $\mathcal{R}^{\mathrm{op}}$ acts on $H^*(X)$. We let $Q_*^r\colon H^*(X) \to H^*(X)$ denote the operation of degree $-2r(p-1)$ [of degree $-r$] which is dual to Q^r.

3. BU, BSU **and** $BU \times Z$. Recall that $H^*(BU) = P\{c_n \mid n \geq 1\}$ as algebras with $c_0 = 1$ and $\psi(c_n) = \sum_{i=0}^n c_i \otimes c_{n-i}$. c_n is called the nth Chern class and has degree $2n$. If we let $a_n = (c_1^n)^*$ and $p_n = c_n^*$ in the basis dual to the basis of $H^*(BU)$ which consists of monomials in the Chern classes, then $H_*(BU) = P\{a_n \mid n \geq 1\}$ as algebras with $\psi(a_n) = \sum_{i=0}^n a_i \otimes a_{n-i}$, and the primitive elements of $H_*(BU)$, $PH_*(BU)$, have a Z_p-basis $\{\mathfrak{p}_n \mid n \geq 1\}$. The three basic theorems whose proofs we postpone to §8 are:

Theorem 5. *In $H_*(BU)$ for $r \geq 0$ and $n \geq 1$,*

$$Q^r(\mathfrak{p}_n) = (-1)^{r+n}(n-1, r-n)\mathfrak{p}_{n+r(p-1)}$$
$$[Q^{2r}(\mathfrak{p}_n) = (n-1, r-n)\mathfrak{p}_{n+r}].$$

Theorem 6. *In $H_*(BU)$ for $r \geq 0$ and $n \geq 1$,*

$$Q^r(a_n) = (-1)^{r+n+1}(n, r-n-1)a_{n+r(p-1)} \quad \textit{modulo decomposables}$$
$$[Q^{2r}(a_n) = (n, r-n-1)a_{n+r} \quad \textit{modulo decomposables}].$$

Theorem 7. *In $H^*(BU)$ for $r \geq 0$ and $n \geq 1$,*

$$Q_*^r(c_n) = (-1)^{r+n}(n - r(p-1) - 1, pr - n)c_{n-r(p-1)}$$
$$[Q_*^{2r}(c_n) = (n-r-1, 2r-n)c_{n-r}].$$

In §11, as part of the proof of Theorem 6, we will produce an algorithm for computing $Q^r(a_n)$ inductively. We cannot produce a formula for $Q^r(a_n)$. However, we can say the following about which monomials im the a_k's can appear in $Q^r(a_n)$.

Theorem 8. *In $H_*(BU)$ for $r \geq n \geq 1$, $Q^r(a_n)$ $[Q^{2r}(a_n)]$ has no monomial summand of product filtration degree greater than $(r-n)(p-1)+p$ and the only summand of $Q^r(a_n)$ $[Q^{2r}(a_n)]$ of product filtration degree $(r-n)(p-1)+p$ is $a_n^p a_1^{(r-n)(p-1)}$. Furthermore if $\alpha a_{n_1} \cdots a_{n_t}$ with $0 \neq \alpha \in Z_p$ is a summand of $Q^r(a_n)$ $[Q^{2r}(a_n)]$ then some $n_i \geq n$.*

Proof. We prove this theorem by induction on $n \geq 1$. The assertion for $n = 1$ is clearly valid. Assume that $n \geq 2$ and that the theorem is valid for $Q^s(a_m)$ $[Q^{2s}(a_m)]$ if $m < n$. Among the monomials in the a_k's which appear with a nonzero coefficient in $Q^r(a_n)$ $[Q^{2r}(a_n)]$ choose M of highest product filtration degree containing the largest power of a_1. Write $M = \alpha a_{n_1}^{e_1} \cdots a_{n_t}^{e_t} a_1^s$ with $n_1 > n_{i-1} > 1$, $e_i \geq 1$, $t \geq 0$, $s \geq 0$ and $0 \neq \alpha \in Z_p$. Note that $t \geq 1$ by Theorem 7 since otherwise $Q_*^r(c_s)$ $[Q_*^{2r}(c_s)]$ would contain monomials in the Chern classes other than c_n. Then $\alpha a_{n_1-1}^{e_1} \cdots a_{n_t-1}^{e_t} \otimes a_1^{e_1+\cdots+e_t+s}$ appears on the left-hand side of the equation

$$\psi \circ Q^r(a_n) = \sum_{i=0}^{r} \sum_{j=0}^{n} Q^i(a_j) \otimes Q^{r-i}(a_{n-j}) \tag{1}$$
$$[\psi \circ Q^{2r}(a_n) = \sum_{i=0}^{r} \sum_{j=0}^{n} Q^{2i}(a_j) \otimes Q^{2r-2i}(a_{n-j})].$$

This term is matched by a term on the right-hand side of (1) which by the induction hypothesis must originate from $Q^{n-1}(a_{n-1}) \otimes Q^{r-n-1}(a_1)$ $[Q^{2n-2}(a_{n-1}) \otimes Q^{2r-2n-2}(a_1)]$. Hence $e_1 + \cdots + e_t + r = (p-1)(r-n) + p$, $M = a_n^p a_1^{(r-n)(p-1)}$ and $\alpha = 1$. If there are other monomial summands of $Q^r(a_n)$ $[Q^{2r}(a_n)]$ of product filtration degree $(r-n)(p-1)+p$, then choose the one $N \neq M$ which has the largest power of a_1. Write $N = \gamma a_{m_1}^{f_1} \cdots a_{m_u}^{f_u} a_1^v$ with $m_i > m_{i-1} > 1$, $f_i \geq 0$, $u \geq 1$, $v \geq 0$ and $0 \neq \gamma \in Z_p$. Then $\gamma a_{m_1-1}^{f_1} \cdots a_{m_u-1}^{f_u} \otimes a_1^{f_1+\cdots+f_u+v}$ is a summand of $\psi \circ Q^r(a_n)$ $[\psi \circ Q^{2r}(a_n)]$. Hence by our induction hypothesis applied to (1) we see that $f_1 + \cdots + f_u + v = (r-n)(p-1)+p$ and $N = M$, a contradiction. Thus, $a_n^p a_1^{(r-n)(p-1)}$ is the only monomial summand of $Q^r(a_n)$ $[Q^{2r}(a_n)]$ of product filtration degree $(s-n)(p-1)k+p$. Let $\lambda a_{k_1} \cdots a_{k_t}$ with $k_1 \geq \cdots \geq k_t$ and $0 \neq \lambda \in Z_p$ be the monomial summand of $Q^r(a_n)$ $[Q^{2r}(a_n)]$, for which $(k_1, \ldots, k_t)$ is least in the lexicographical order. Hence $\lambda a_{k_1-1} \cdots a_{k_t-1} \otimes a_1^t$ is a summand of $\psi \circ Q^r(a_n)$ $[\psi \circ Q^{2r}(a_n)]$. By our induction hypothesis applied to (1) we deduce that $k_1 - 1 \geq n-1$ and hence $k_1 \geq n$. This proves the last assertion of the theorem.

We now define and compute the indecomposable elements of $H_*(BU)$ over the Dyer-Lashof algebra.

Definition 9. Let ε: $S \to K$ and ε': $M \to K$ be augmented algebras over a field K. IS = Kernel ε and IM = Kernel ε' are called the augmentation ideals of S

and M respectively. Assume that M is a left S-module. We define the indecomposable elements of M over S by

$$Q_S M = IM/[(IS)(IM) + (IM)^2].$$

Theorem 10. (a) $Q_{\mathscr{R}} H_*(BU; Z_2) = \{a_{2^n} \mid n \geq 0\}$.

(b) *For p an odd prime, a Z_p-basis for $Q_{\mathscr{R}} H_*(BU; Z_p)$ is $\{a_{np^e} \mid n \neq 0 \bmod p$, $n = s(p-1) + r, 1 \leq r \leq p-1$ and if $s \neq 0$ then $s = \sum_{i=0}^k s_i p^i, 0 \leq s_i \leq p-1$, with $r \geq s_0 \geq \cdots \geq s_k \geq 1\}$.*

Proof. (a) By Theorem 6, there is an $\mathscr{R}$-indecomposable in degree $2n$ if and only if $(k, n-2k-1) = 0$ for all $k \geq 1$. It is easy to see that all such n are the powers of two.

(b) $a_{p^e n}$ is $\mathscr{R}$-decomposable if $s_0 = 0$ or $r < s_0$ since, by Theorem 6, $Q^{p^e s}(a_{rp^e}) = \alpha a_{np^e}$ modulo decomposables with $0 \neq \alpha \in Z_p$. If $k \geq 1$, $s_0 \neq 0$, $r \geq s_0$ and there is an s_j which is either zero or less than s_{j+1} (assume j is least with this property) then

$$Q^{\sum_{i=j+1}^n s_i p^{i+e}}\left(a_{rp^e + p^e(p-1)\sum_{i=0}^j s_i p^i}\right) = \gamma a_{np^e}$$

modulo decomposables for some $0 \neq \gamma \in Z_p$. Thus the claim for $Q_{\mathscr{R}} H_*(BU)$ contains $Q_{\mathscr{R}} H_*(BU)$. It remains to show that a_{np^e} is $\mathscr{R}$-indecomposable if $s = 0$ or if $r \geq s_0 \geq \cdots \geq s_k \geq 1$. That is, we must show that $Q^t(a_{np^e - t(p-1)})$ is decomposable for all $t \geq 1$ which by Theorem 6 is equivalent to showing that $(np^e - t(p-1), pt - np^e - 1) \equiv 0 \bmod p$ for all $t \geq 1$. This is clear if $p^e \nmid t$, so assume $t = p^e u$. Let $u = \sum_{i=0}^k u_i p^i$, $0 \leq u_i \leq p-1$ and $u_{-1} = r$. It can be shown by induction on i that if $(np^e - t(p-1), pt - np^e - 1) = (r + u(p-1), pu + s - r - 1) \not\equiv 0 \bmod p$ then

(i) $s_i \leq t_i = p^e u_i$.

(ii) If $i < n$ and

$$\begin{aligned} \delta_i &= 0 \qquad \text{if } u_{i-1} \geq u_i \\ &= 1 \qquad \text{if } u_{i-1} < u_i \end{aligned}$$

then

$$\left(u_i - \delta_i + (p-1)\left(\sum_{j=i+1}^k u_j p^{j-i-1}\right), \sum_{j=i+1}^k (s_j - pu_j)p^{j-i-1} - u_i + \delta_i - 1\right)$$
$$\not\equiv 0 \bmod p.$$

This implies that $s \leq t$, a contradiction. Hence $(np^e - t(p-1), pt - np^e - 1) \equiv 0 \bmod p$ for all $t \geq 1$.

We now determine the algebra of $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(BU; Z_p)$, first for $p = 2$ and then for odd primes p. The following terminology will prove useful.

Definition 11. Let A be a connected Hopf algebra over Z_p with $f\colon A \to A$ a map of Hopf algebras. f is said to be locally nilpotent if for all $a \in IA$ there is a natural number $n(a)$ with $f^{n(a)}(a) = 0$ where $f^0 = 1_A$, $f^1 = f$ and $f^k = f \circ f^{k-1}$ for $k \geq 2$. When f is locally nilpotent, every element of the power series ring $Z_p[[f]]$ is a well-defined Hopf algebra endomorphism of A. Addition in $Z_p[[f]]$ is given by Whitney sum, i.e. $g + h = \phi \circ g \otimes h \circ \psi$.

Let $F = \gamma \circ \alpha$ be the composite of the canonical maps $BU \xrightarrow{\alpha} B\,\mathrm{Sp} \xrightarrow{\gamma} BU$. In Z_2 homology, $F_*(a_{2k+1}) = 0$ and $F_*(a_{2k}) = a_k^2$. Hence F_* is locally nilpotent and $Z_2[[F_*]]$ is a well-defined algebra of $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(BU; Z_2)$.

If p is a prime number, let $Q_p = Z[1/q \mid q \text{ is prime}, q \neq p]$ as a subring of the rational numbers Q. By J. F. Adams [1, Lecture 4], complex K-theory with Q_p coefficients is a representable cohomology theory, and we denote the infinite loop space which represents this theory by BUQ_p. Observe that $H_*(BUQ_p; Z_p) = H_*(BU; Z_p)$.

Theorem 12. (a) $Z_2[[F_*]]$ *is the algebra of all* $\mathfrak{A}$*-Hopf algebra endomorphisms of* $H_*(BU; Z_2)$.

(b) $Z_2[[F_*]]$ *is the algebra of all* $\mathfrak{A}\mathscr{R}$*-Hopf algebra endomorphisms of* $H_*(BU; Z_2)$.

(c) *Every element of* $Z_2[[F_*]]$ *is induced by an infinite loop endomorphism of* BUQ_2.

(d) *The elements of* $Z_2[[F_*]]$ *are induced by the stable* KUQ_2*-theory operations* $Z_2[[\psi + \psi^{-1}]]$.

Proof. (a) Let g be an $\mathfrak{A}$-Hopf algebra endomorphism of $H_*(BU; Z_2)$. Then g^* is an $\mathfrak{A}$-Hopf algebra endomorphism of $H^*(BU; Z_2)$. Let k_i be the coefficient of $c_1^{2^i}$ in $g^*(c_{2^i})$. We will prove that $g^* = \sum_{i=0}^{\infty} k_i F^{*i}$. It is well known that $Q_{\mathfrak{A}} H^*(BU; Z_2) = \{c_{2^n} \mid n \geq 0\}$. Thus, it suffices to show that $g^*(c_{2^n}) = (\sum_{i=0}^{\infty} k_i F^{*i})(c_{2^n})$, which we prove by induction on $n \geq 0$. The case $n = 0$ is immediate from the definition of k_0. Assume now that $g(c_{2^t}) = (\sum_{i=0}^{\infty} k_i F^{*i})(c_{2^t})$ if $0 \leq t < n$. Then $g(c_s) = (\sum_{i=0}^{\infty} k_i F^{*i})(c_s)$ if $1 \leq s < 2^n$. Hence $g(c_{2^n}) - (\sum_{i=0}^{\infty} k_i F^{*i})(c_{2^n})$ is primitive and therefore is zero by the definition of k_n.

(b) Every $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphism of $H_*(BU; Z_2)$ is an $\mathfrak{A}$-Hopf algebra endomorphism and every element of $Z_2[[F_*]]$ is a map of $\mathscr{R}$-modules. Hence (a) implies (b).

(c) and (d). $\psi + \psi^{-1}$ induces $F\colon BU \to BU$. Hence each element

$$\sum_{i=0}^{\infty} k_i(\psi + \psi^{-1})^i$$

induces an infinite loop endomorphism of BUQ_2 which induces $\sum_{i=0}^{\infty} k_i F^i$ in homology.

Corollary 13. (a) *$H^*(BU; Z_2)$ is an indecomposable $\mathfrak{A}$-Hopf algebra.*

(b) *If f is an $\mathfrak{A}$-Hopf algebra endomorphism of $H_*(BU; Z_2)$ then f is a homomorphism of $\mathscr{R}$-modules.*

We now want to prove the analogues of Theorem 12 and Corollary 13 for odd primes. Thus for the remainder of the discussion of BU, p will denote an odd prime and all homology will be understood to have Z_p-coefficients for p an odd prime. By J. F. Adams [1, Lecture 4], $BUQ_p = \prod_{i=0}^{p-2} BU_{p,i}$ as infinite loop spaces. Let E_i: $BUQ_p \to BU_{p,i}$ and J_i: $BU_{p,i} \to BUQ_p$ denote the canonical projections and injections respectively, $0 \le i \le p-2$. $H_*(BU_{p,i})$ is a polynomial algebra with indecomposable elements in all positive degrees $2n$ for which $n \equiv i \bmod p-1$. We will construct specific algebra generators in Lemma 17. First, however, we state the theorem and corollary which we are striving to prove.

Theorem 14. *Let p be an odd prime, and let $0 \le i \le p-2$. Then there are locally nilpotent $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms F_i of $H_*(BU_{p,i})$ such that:*

(a) *$Z_p[[F_i]]$ is the algebra of all $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BU_{p,i})$.*

(b) *$Z_p[[F_i]]$ is the algebra of all $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(BU_{p,i})$.*

(c) *The map $f \to E_{0*} \circ f \circ J_{0*} + \cdots + E_{p-2*} \circ f \circ J_{p-2*}$ is an isomorphism between $\prod_{i=0}^{p-2} Z_p[[F_i]]$ and*

(i) *the algebra of all $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BU)$,*

(ii) *the algebra of all $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(BU)$.*

Corollary 15. (a) *$H^*(BU_{p,i})$ is an indecomposable $\mathfrak{A}$-Hopf algebra.*

(b) *If f is an $\mathfrak{A}$-Hopf algebra endomorphism of $H_*(BU)$ or of $H_*(BU_{p,i})$ then f is a homomorphism of $\mathscr{R}$-modules.*

We begin our proof of Theorem 14 by defining the F_i.

Lemma 16. *Define an algebra endomorphism F of $H_*(BU)$ by $F(a_{pk}) = a_k^p$ and $F(a_n) = 0$ if n is not divisible by p. Let $F_i = E_{i*} \circ f \circ J_{i*}$, $0 \le i \le p-2$, be the corresponding algebra endomorphism of the $H_*(BU_{p,i})$. Then F and all the F_i are locally nilpotent morphisms of $\mathfrak{A}\mathscr{R}$-Hopf algebras.*

Proof. F and all the F_i are clearly locally nilpotent, and it thus suffices to show that F is a map of $\mathfrak{A}\mathscr{R}$-Hopf algebras. F is a map of coalgebras because both $F \otimes F \circ \psi(a_n)$ and $\psi \circ F(a_n)$ equal $\sum_{i=0}^{k} a_i^p \otimes a_{k-i}^p$ if $n = pk$, and they both equal zero if n is not divisible by p. We now recall that the action of $\mathfrak{A}^{\mathrm{op}}$ on $H_*(BU)$ is given by $\mathscr{P}_*^n(a_k) = (n, k-pn)a_{k-n(p-1)}$. Hence F is a homomorphism of $\mathfrak{A}$-modules because both $\mathscr{P}_*^n \circ F(a_k)$ and $F \circ \mathscr{P}_*^n(a_k)$ are equal to $(n, k-np)a_{t-n}^p$ if $n + k = pt$, while they both equal zero if $k + n$ is not divisible by p.

We digress to show that the Hopf algebra endomorphism F^* of $H^*(BU)$ is given by $F^*(c_{pk}) = c_k^p$ and $F^*(c_n) = 0$ if n is not divisible by p. The second assertion and the first assertion for $k = 1$ are clearly true. Now assume that $k > 1$ and that $F^*(c_{ph}) = c_h^p$ if $1 < h < k$. Then $F^*(c_{pk}) - c_k^p = \alpha \mathfrak{p}_{pk}$ since F^*

is a map of coalgebras where $\alpha \in Z_p$ and $\mathfrak{p}_{pk}$ is the basis element of $PH^{2pk}(BU)$. Hence $F(a_{pk})$ contains αa_1^{pk} as a summand, so $\alpha = 0$ since $k > 1$.

We now verify that F^* is a homomorphism of $\mathscr{A}$-modules by using Theorem 7. One finds that $Q_*^n \circ F(c_k)$ and $F \circ Q_*^n(c_k)$ are both equal to $(-1)^{k+n} \cdot (k - n(p-1) - 1, pn - k)c_t^p$ if $k + n = pt$, while they both equal zero if $k + n$ is not divisible by p.

The following lemma describes the algebraic structure of $H_*(BU_{p,i})$.

Lemma 17. *There are elements* $a_{e,r} \in H_{2p^er}(BU_{p,i})$ *if* $0 \leq i \leq p-2$, $e \geq 0$, $r \geq 1$, $r \not\equiv 0 \bmod p$ *and* $r \equiv i \bmod p-1$ *such that*:

(a) $H_*(BU_{p,i}) = P\{a_{e,r} \mid r \equiv i \bmod p - 1\}$ *as algebras.*

(b) $J_*(a_{0,r}) = \mathfrak{p}_r$ *if* $r \equiv i \bmod p-1$.

(c) $F_i(a_{0,r}) = 0$ *and* $F_i(a_{e,r}) = a_{e-1,r}^p$ *if* $e \geq 1$ *and* $r \equiv i \bmod p-1$.

Proof. We show that there are indecomposable elements $a_{e,r}$ in $H_{2p^er}(BU_{p,i})$ satisfying (b) and (c) by induction on p^er. Let $a_{0,r} = E_{i*}(\mathfrak{p}_r)$ if $r \geq 1$, $r \not\equiv 0 \bmod p$ and $r \equiv i \bmod p-1$. Clearly $a_{0,r}$ is indecomposable, $J_{i*}(a_{0,r}) = \mathfrak{p}_r$ and $F_i(a_{0,r}) = 0$. If $e \geq 1$ then we define $a_{e,r}$ as follows: Let $a'_{e,r}$ be any indecomposable element of $H_{2p^er}(BU_{p,i})$. Then

$$F_i(a'_{e,r}) = \alpha a_{e-1,r}^p + \sum_{j=1}^{s} \alpha_j a_{e_{j1},r_{j1}}^{p^{f_{j1}+1}} \cdots a_{e_{jt_j},r_{jt_j}}^{p^{f_{jt_j}+1}}$$

where $0 \neq \alpha \in Z_p$, $\alpha_j \in Z_p$, $f_{ji} \geq 0$ and $t_j \geq 2$ for all j. Now define

$$a_{e,r} = \alpha^{-1} a'_{e,r} - \alpha^{-1} \sum_{j=1}^{s} \alpha_j a_{e_{j1}+1,r_{j1}}^{p^{f_{j1}}} \cdots a_{e_{jt_j}+1,r_{jt_j}}^{p^{f_{jt_j}}}.$$

Clearly $a_{e,r}$ is indecomposable and $F_i(a_{e,r}) = a_{e-1,r}^p$.

Proof of Theorem 14. (a) Let g be an $\mathfrak{A}$-Hopf algebra endomorphism of $H_*(BU_{p,i})$, and let k_t be the coefficient of $a_{0,i'}^{p^t}$ in $g(a_{t,i'})$ where $i' = i$ if $i \neq 0$ and $i' = p - 1$ if $i = 0$. Reasoning as in the proof of Theorem 12(a), we see that $g = \sum_{t=0}^{\infty} k_t F_i^t$.

(b) The assertion follows from (a) and the fact that F_i is a homomorphism of $\mathscr{A}$-modules.

(c) The assertion follows from (a), (b) and the observation that if f is a Hopf algebra endomorphism of $H_*(BU)$ then $E_{j*} \circ f \circ J_{i*} = 0$ if $0 \leq i, j \leq p-2$ and $i \neq j$. If this last statement were false then choose $x \in H_*(BU_{p,i})$ of least degree with $E_{j*} \circ f \circ J_{i*}(x) \neq 0$. Note that x must be indecomposable and $E_{j*} \circ f \circ J_{i*}(x)$ must be primitive. Hence $i \equiv \deg x = \deg E_{j*} \circ f \circ J_{i*}(x) \equiv j \bmod p - 1$, a contradiction.

We now turn our attention to BSU. Recall that the map j: $BSU \to BU$ induces a monomorphism in homology and an epimorphism in cohomology. More specifically, $H^*(BSU)$ is the quotient Hopf algebra of $H^*(BU)$ modulo the ideal generated by c_1. Hence Theorem 7 with the added condition $c_1 = 0$ describes the action of $\mathscr{R}^{\mathrm{op}}$ on $H^*(BSU)$. Let $H_*(BSU; Z_p) = P\{a'_n \mid n \geq 2\}$ as algebras where $\deg a'_n = 2n$ and $j_*(a'_n) = a_n$ modulo decomposables if n is not a power of p.

Theorem 18. *In $H_*(BSU)$ for $r \geq 0$, $n \geq 2$ and $k \geq 1$:*

$Q^r(a'_n) = (-1)^{r+n+1}(n, r-n-1)a'_{n+r(p-1)}$ *modulo decomposables if n is not a power of p;*

$[Q^{2r}(a'_n) = (n, r-n-1)a'_{n+r}$ *modulo decomposables if n is not a power of* 2] *and $Q^r(a'_{p^k})$ is decomposable.*

Proof. $Q^r(a'_n)$, for n not a power of p, is evaluated by using Theorem 6 and the naturality of $\mathscr{R}$ with respect to j_*. Let $\mathfrak{p}'_i$, $\mathfrak{p}_i$ be basis elements for $PH^{2i}(BSU)$, $i \geq 2$, and $PH^{2i}(BU)$, $i \geq 1$, respectively with $j^*(\mathfrak{p}_i) = \mathfrak{p}'_i$ if i is not a power of p and $j^*(\mathfrak{p}_{p^k}) = 0$. Then $Q^r(a'_{p^k})$ $[Q^{2r}(a'_{2^k})]$ is decomposable if and only if $Q^r_*(\mathfrak{p}'_{p^k+r(p-1)}) = 0$ $[Q^{2r}_*(\mathfrak{p}'_{2^k+r}) = 0]$. If $p^k + r(p-1)$ is not a power of p then

$$Q^r_*(\mathfrak{p}'_{p^k+r(p-1)}) = j^*Q^r_*(\mathfrak{p}_{p^k+r(p-1)}) = 0$$

$$[Q^{2r}_*(\mathfrak{p}'_{2^k+r}) = j^*Q^{2r}_*(\mathfrak{p}_{2^k+r}) = 0]$$

because $j^*(\mathfrak{p}_{p^k}) = 0$. If $p^k + r(p-1) = p^t$ then $Q^r_*(\mathfrak{p}'_{p^k+r(p-1)}) = Q^r_*(\mathfrak{p}'^{p^{t-1}}_p) = 0$ $[Q^{2r}_*(\mathfrak{p}'_{2^t}) = Q^{2r}_*(\mathfrak{p}'^{2^{t-1}}_2) = 0]$.

Theorem 19. (a) $Q_{\mathscr{R}}H_*(BSU; Z_2) = \{a'_{2^m+2^n} \mid m \geq 0 \text{ and } n \geq 0\}$.

(b) *For p an odd prime, a Z_p-basis for $Q_{\mathscr{R}}H_*(BSU; Z_p)$ is given by $\{a'_n \mid a_n \in Q_{\mathscr{R}}H_*(BU; Z_p)$ and $n \geq 2\} \cup \{a'_{p^e[s(p-1)+1]} \mid e \geq 0,\ s \geq 0,\ (e,s) \neq (0,0)\}$.*

Proof. (a) By Theorem 18, if $k = 2^b + 2^c + 2^e h$ with $0 \leq b < c < e$ and $h \geq 1$ then $Q^{2^{e+1}h}(a'_{2^b+2^c}) = a'_k$ modulo decomposables while $Q^{2i}(a'_{2^b+2^c_{-i}})$ is always decomposable.

(b) By the proof of Theorem 10(b), it suffices to show that $Q^t(a'_{p^e[s(p-1)+1]-t(p-1)})$ is decomposable for all $t \geq 0$. This, however, follows easily from Theorem 18.

Observe that for p an odd prime $BSUQ_p = \prod_{i=0}^{p-2} BSU_{p,i}$ with $BSU_{p,i} = BU_{p,i}$ for $1 \leq i \leq p-2$. Let F' be the composite of the canonical maps $BSU \to BSO \to BSU$.

Theorem 20. (a) *The vector space of $\mathfrak{A}$-Hopf algebra maps from $H_*(BSU)$ to $H_*(BU)$ consists entirely of $\mathscr{R}$-module homomorphisms and is equal to the free left module with basis $\{j_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BU)$.*

(b) *Conjugation by j^* induces an isomorphism between the $\mathfrak{A}$-Hopf algebra endomorphisms of $H^*(BU)$ and $H^*(BSU)$.*

(c) *Conjugation by j^* induces an isomorphism between the $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H^*(BU)$ and $H^*(BSU)$.*

(d) $Z_2[[F'_*]]$ *is the algebra of $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(BSU; Z_2)$.*

(e) *The vector space of $\mathfrak{A}$-Hopf algebra maps from $H_*(BSU)$ to $H_*(BU)$ is the free right module with basis $\{j_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BSU)$.*

Proof. (a) Use the same idea as in the proofs of Theorems 12(a) and 14(a).

(b) Observe from Theorems 12(a) and 14(a) that if f is an $\mathfrak{A}$-Hopf algebra endomorphism of $H^*(BU)$ then $j^* \circ f \circ j^{*-1}$ is a well-defined $\mathfrak{A}$-Hopf algebra endomorphism of $H^*(BSU)$. Conjugation by j^* is clearly a monomorphism. This map is onto by (a) since if g is an $\mathfrak{A}$-Hopf algebra endomorphism of $H^*(BSU)$ then $g \circ j^*: H^*(BU) \to H^*(BSU)$ is a map of $\mathfrak{A}$-Hopf algebras, and hence $g \circ j^* = j^* \circ f$ for some $\mathfrak{A}$-Hopf algebra endomorphism f of $H^*(BU)$.

Corollary 21. (a) *$H^*(BSU; Z_2)$ and $H^*(BSU_{p,i}; Z_p)$ for p an odd prime are indecomposable $\mathfrak{A}$-Hopf algebras.*

(b) *Every $\mathfrak{A}$-Hopf algebra endomorphism of $H^*(BSU)$ is a homomorphism of $\mathscr{R}$-modules.*

We next consider $BU \times Z = \Omega U = \Omega^2 BU$. $H_*(BU \times Z) = H_*(BU) \otimes Z_p(Z)$ as Hopf algebras where $Z_p(Z)$ is the group algebra over Z_p of the additive groups of integers. Elements of $H_*(BU \times Z)$ are written as sums of $x \otimes [i]$ where $x \in H_*(BU)$ and $[i] \in Z \subset Z_p(Z)$. The canonical map $BU \to BU \times Z$ induces the map $x \to x \otimes [0]$ in homology. The action of $\mathfrak{A}^{\text{op}}$ on $H_*(BU)$ is given in terms of the action of $\mathfrak{A}^{\text{op}}$ on $H_*(BU)$ by the equation $\mathscr{P}_*^n(x \otimes [i]) = \mathscr{P}_*^n(x) \otimes [i]$.

The action of the Dyer-Lashof algebra on $H_*(BU \times Z)$ will be determined from our knowledge of the Dyer-Lashof operations on $H_*(BU)$ as soon as we compute $Q^n(1 \otimes [1])$ for all $n \geq 0$. This observation follows from the multiplicative Cartan formula, the fact that Dyer-Lashof operations commute with the conjugation χ and the equations

$$x \otimes [k] = x \otimes [0](1 \otimes [1])^k, \qquad x \otimes [-k] = x \otimes [0]\chi\{1 \otimes [1]^k\}$$

for $x \in H_*(BU)$ and $k > 0$.

Theorem 22. *In $H_*(BU \times Z)$ for $n \geq 0$,*

$$Q^n(1 \otimes [1]) = \gamma_n(\mathfrak{p}_{p-1}) \otimes [p]$$

$$[Q^{2n}(1 \otimes [1]) = a_n \otimes [2]]$$

where $\gamma_n(\mathfrak{p}_{p-1})$ is $(c_{p-1}^n)^*$ in the dual basis of the basis of monomials in the Chern classes.

Proof. We prove this theorem by induction on n. $Q^0(1 \otimes [1]) = (1 \otimes [1])^p = 1 \otimes [p]$. By the comultiplicative Cartan formula, $Q^1(1 \otimes [1])$ $[Q^2(1 \otimes [1])]$ is primitive so for some $\alpha \in Z_p$,

$$Q^1(1 \otimes [1]) = \alpha \mathfrak{p}_{p-1} \otimes [p] \qquad [Q^2(1 \otimes [1]) = \alpha a_1 \otimes [2]]. \tag{1}$$

Recall that $H_*(U) = E\{f_n \mid n \geq 1\}$ as Hopf algebras with $\deg f_n = 2n - 1$. Since $BU \times Z = \Omega U$, there is a suspension map $\sigma_*: IH_*(BU \times Z) \to H_*(U)$

which is given by

$$\sigma_*(a_n \otimes [k]) = f_{n+1} \quad \text{if } n > 0$$

and

$$\sigma_*(1 \otimes [k] - 1 \otimes [0]) = kf_1 \quad \text{if } k \neq 0.$$

Applying σ_* to equation (1) we see that

$$(2) \qquad Q^1(f_1) = \alpha f_p \qquad [Q^2(f_1) = \alpha f_3].$$

The universal U-bundle induces a suspension map $\sigma_*^U\colon IH_*(U) \to H_*(BU)$ which is given by $\sigma_*^U(f_n) = (-1)^{n+1}\mathfrak{p}_n$. Now we apply σ_*^U to (2) and obtain

$$Q^1(a_1) = \alpha a_1^p \qquad [Q^2(a_1) = \alpha a_1^2].$$

Hence $\alpha = 1$, and the theorem is true for $n = 0$ and $n = 1$.

Now assume that $n \geq 2$ and that the theorem is true in dimensions less than $2n$. Write $n = mp^e$ with m not divisible by p. By the comultiplicative Cartan formula,

$$(3) \qquad \begin{aligned} &Q^n(1 \otimes [1]) - \gamma_n(\mathfrak{p}_{p-1}) \otimes [p] = \lambda \mathfrak{p}_{n(p-1)} \otimes [p] \\ &[Q^{2n}(1 \otimes [1]) - a_n \otimes [2] = \lambda \mathfrak{p}_n \otimes [2]] \end{aligned}$$

for some $\lambda \in Z_p$.

Case 1. Assume that $e = 0$. As in the case $n = 1$, apply $\sigma_*^U \circ \sigma_*$ to (3) to obtain

$$\begin{aligned} &Q^n(a_1) - (-1)^{n+1}\mathfrak{p}_{n(p-1)+1} = n\lambda\mathfrak{p}_{n(p-1)+1} \\ &[Q^{2n}(a_1) + \mathfrak{p}_{n+1} = \lambda\mathfrak{p}_{n+1}]. \end{aligned}$$

Hence $\lambda = 0$ by Theorem 5.

To prove the two remaining cases of Theorem 22 we will use the following lemma.

Lemma 23. *The monomial summands of* $\mathcal{P}^r(c_n)$ $[\mathcal{P}^{2r}(c_n)]$ *have product filtration degree less than or equal to p.*

Proof. For $p = 2$, this lemma is an immediate consequence of the Wu formula. For odd primes p we prove this lemma by induction on $r + n$. If $r = 0$ or $n = 1$ then the assertion of the lemma is clearly true. Now assume that the lemma is true for $\mathcal{P}^s(c_m)$ if $s + m < r + n$, and assume that $r + n > 2$. If $\mathcal{P}^r(c_n) \neq 0$ then choose a summand $\alpha c_{i_1}^{e_1} \cdots c_{i_k}^{e_k} c_1^h$ of $\mathcal{P}^r(c_n)$ with $k \geq 0$, $h \geq 0$, $i_j > i_{j+1} > 1$ and $0 \neq \alpha \in Z_p$ as follows: Choose such a monomial with $e_1 + \cdots + e_k + h$ maximal and among all such monomials select the one with

$$(\overbrace{i_1,\ldots,i_1}^{e_1},\ldots,\overbrace{i_k,\ldots,i_k}^{e_k})$$

largest in the lexicographical order. By the Cartan formula,

$$\psi \circ \mathcal{P}^r(c_n) = \sum_{u=0}^{n} \sum_{v=0}^{r} \mathcal{P}^v(c_u) \otimes \mathcal{P}^{r-v}(c_{n-u}).$$

$\alpha c_{i_1-1}^{e_1} \cdots c_{i_k-1}^{e_k} \otimes c_1^{e_1+\cdots+e_k+h}$ is a summand of $\psi \circ \mathcal{P}^r(c_n)$. If $k \geq 1$ then by the induction hypothesis $e_1 + \cdots + e_k + h \leq p$. If $k = 0$ then $\alpha c_1^{h+r(p-1)}$ is a summand of $\mathcal{P}^r(c_n)$. Hence $\alpha \mathfrak{p}_n$ is a summand of $\mathcal{P}_*^r(a_{n+r(p-1)})$ when $\mathcal{P}_*^r(a_{n+r(p-1)})$ is written in terms of the dual basis of the basis of monomials in the Chern classes. However, by considering the map $CP^\infty \to BU$ one can show that $\mathcal{P}_*^i(a_k) = (i, k - pi)a_{k-i(p-1)}$. Thus $n = r = 1$, which is a contradiction to $n + r > 2$.

Proof of Theorem 22 (continued).

Case 2. Assume that $n = p^e$ and that $e \geq 1$.

We first show that if $p^e < k < p^{e+1}$ then

$$\text{(4)} \qquad \begin{aligned} &Q^k(1 \otimes [1]) = \gamma_k(\mathfrak{p}_{p-1}) \otimes [p] + \lambda \mathfrak{p}_{p-1}^{p^e} \gamma_{k-p^e}(\mathfrak{p}_{p-1}) \otimes [\mathfrak{p}] \\ &[Q^{2k}(1 \otimes [1]) = a_k \otimes [2] + \lambda a_1^{2^e} a_{k-2^e} \otimes [2]]. \end{aligned}$$

If (4) is not true then let k be the smallest integer for which it fails. By the comultiplicative Cartan formula,

$$\text{(5)} \qquad \begin{aligned} &Q^k(1 \otimes [1]) + \gamma_k(\mathfrak{p}_{p-1}) \otimes [p] - \lambda \mathfrak{p}_{p-1}^{p^e} \gamma_{k-p^e}(\mathfrak{p}_{p-1}) \otimes [p] = \mu \mathfrak{p}_{k(p-1)} \\ &[Q^{2k}(1 \otimes [1]) + a_k \otimes [2] + \lambda a_1^{2^e} a_{k-2^e} \otimes [2] = \mu \mathfrak{p}_k] \end{aligned}$$

for some $0 \neq \mu \in Z_p$. Write $k = p^f c$ with c not divisible by p. The application of σ_* to (5) shows that $e > f > 0$. Now apply $\mathcal{P}_*^{p^f}$ to (5), use the Nishida relations and the fact $\mathcal{P}_*^{p^f}(\mathfrak{p}_k) = \mathcal{P}_*^1(\mathfrak{p}_c)^{p^f} = c\mathfrak{p}_{k-p^f(p-1)}$ to obtain

$$\text{(6)} \qquad \begin{aligned} -(p^f, k(p-1) - p^{f+1})Q^{k-p^f}(1 \otimes [1]) + \mathcal{P}_*^{p^f}(\gamma_k(\mathfrak{p}_{p-1})) \otimes [p] \\ - \lambda \mathfrak{p}_{p-1}^{p^e} \mathcal{P}_*^{p^f}(\gamma_{k-p^e}(\mathfrak{p}_{p-1})) \otimes [p] = \mu c \mathfrak{p}_{k-p^f(p-1)} \otimes [p] \\ [0 = \mathcal{P}_*^{2^{f+1}} \circ Q^{2k}(1 \otimes [1]) + \mathcal{P}_*^{2^{f+1}}(a_k) \otimes [2] = \mu c \mathfrak{p}_{k-2^f} \otimes [2]]. \end{aligned}$$

Thus, if $p = 2$ then $\mu c = 0$, a contradiction. If p is odd then by Lemma 23 and the induction hypothesis the left-hand side of (6) contains no power of a_1 as a summand. However, the right-hand side of (6) contains $\mu c a_1^{k-p^f(p-1)}$ as a summand. Therefore $\mu c = 0$, a contradiction. Hence equation (4) is valid for all k between p^e and p^{e+1}.

Consider the following Adem relation:

(7)

$$Q^{p^{e+1}+(p-1)p^e} \circ Q^{p^e}(1 \otimes [1])$$
$$= \sum_{i=2p^e-p^{e-1}}^{p^{e+1}-1} (-1)^{i+1}(pi - 2p^{e+1} + p^e, p^{e+1} - i - 1)Q^{2p^{e+1}-i} \circ Q^i(1 \otimes [1])$$
$$[Q^{3\cdot 2^{e+1}} \circ Q^{2^{e+1}}(1 \otimes [1])$$
$$= \sum_{i=3\cdot 2^{e-1}}^{2^{e+1}} (2i - 3 \cdot 2^e, 2^{e+1} - i - 1)Q^{2^{e+3}-2i} \circ Q^{2i}(1 \otimes [1])].$$

By Theorems 5 and 8 the left-hand side of (7) contains $\lambda a_1^{p^{e+1}(p-1)} \otimes [p^2]$ as a summand while by our induction hypothesis and (4) no such summand appears on the right-hand side of (7). Hence $\lambda = 0$.

Case 3. Assume that $e \geq 1$ and that $m \geq 2$.

Apply $\mathcal{P}_*^{p^e}$ to (3) and obtain by the Nishida relations

$$(m-1)Q^{n-p^e}(1 \otimes [1]) - \mathcal{P}_*^{p^e}(\gamma_n(\mathfrak{p}_{p-1})) \otimes [p] = \lambda \mathcal{P}_*^{p^e}(\mathfrak{p}_{n(p-1)} \otimes [p]),$$

i.e.

$$(8)\quad \begin{aligned} &(m-1)\gamma_{n-p^e}(\mathfrak{p}_{p-1}) \otimes [p] - \mathcal{P}_*^{p^e}(\gamma_n(\mathfrak{p}_{p-1})) \otimes [p] = -\lambda m \mathfrak{p}_{(n-p^e)(p-1)} \otimes [p] \\ &[0 = 0 - \mathcal{P}_*^{2^{e+1}}(a_n) \otimes [2] = \lambda \mathcal{P}_*^{2^{e+1}}(\mathfrak{p}_n) \otimes [2] = \lambda \mathfrak{p}_{n-2^e} \otimes [2]]. \end{aligned}$$

Hence $\lambda = 0$ if $p = 2$. If p is odd then by Lemma 23 the left-hand side of (8) has no nonzero multiple of $a_1^{(n-p^e)(p-1)}$ as a summand while the right-hand side of (8) has $-\lambda m a_1^{(n-p^e)(p-1)}$ as a summand. Thus, $\lambda = 0$.

4. BSp, $B\mathrm{Sp} \times Z$, BO, BSO, **and** $BO \times Z$. We will use the results of the previous section on BU, BSU and $BU \times Z$ to prove analogous theorems about the classifying spaces of the other infinite classical groups.

We begin by introducing some notation. Let α: $BU \to B\mathrm{Sp}$, γ: $B\mathrm{Sp} \to BU$, μ: $BU \to BO$ and ν: $BO \to BU$ be the canonical maps. Define $b_n \in H_{4n}(B\mathrm{Sp})$ by $b_n = \alpha_*(a_{2n})$ for $n \geq 0$, and note that $\alpha_*(a_{2n+1}) = 0$ for $n \geq 0$. Then $H_*(B\mathrm{Sp}) = P\{b_n \mid n \geq 1\}$ as algebras and $\psi(b_n) = \sum_{i=0}^n b_i \otimes b_{n-1}$. Define $k_n \in H^{4n}(B\mathrm{Sp})$ by $k_n = (-1)^n \gamma^*(c_{2n})$ for $n \geq 0$, and note that $\gamma^*(c_{2n+1}) = 0$ for $n \geq 0$. Then $H^*(B\mathrm{Sp}) = P\{k_n \mid n \geq 1\}$ as algebras with $\psi(k_n) = \sum_{i=0}^n k_i \otimes k_{n-i}$. Use the dual basis of the basis of monomials in the k_n to define $\mathfrak{p}_n = k_n^*$ for $n \geq 1$, and note that $(k_1^n)^* = b_n$ for $n \geq 1$. Then $\{\mathfrak{p}_n \mid n \geq 1\}$ is a basis for $PH_*(B\mathrm{Sp})$ and $\gamma_*(\mathfrak{p}_n) = (-1)^n \mathfrak{p}_{2n}$ for $n \geq 1$.

For Z_p coefficients with p an odd prime, we now make the same definitions in $H_*(BO)$ and $H^*(BO)$. Thus, define $d_n \in H_{4n}(BO)$ by $d_n = \mu_*(a_{2n})$ for $n \geq 0$, and note that $\mu_*(a_{2n+1}) = 0$ for $n \geq 0$. Then $H_*(BO) = P\{d_n \mid n \geq 1\}$ as algebras with $\psi(d_n) = \sum_{i=0}^n d_i \otimes d_{n-i}$. Define the nth Pontryagin class P_n

$\in H^{4n}(BO)$ by $P_n = (-1)^n \nu^*(c_{2n})$ for $n \geq 0$, and note that $\nu^*(c_{2n+1}) = 0$ for $n \geq 0$. Then $H^*(BO) = P\{P_n \mid n \geq 1\}$ as algebras with $\psi(P_n) = \sum_{i=0}^n P_i \otimes P_{n-i}$. Using the dual basis of the basis of monomials in the Pontryagin classes, we observe that $(P_1^n)^* = d_n$ for $n \geq 1$, and we define $\mathfrak{p}_n = P_n^*$ for $n \geq 1$. Then $\{\mathfrak{p}_n \mid n \geq 1\}$ is a basis of $PH_*(BO)$ and $\nu_*(\mathfrak{p}_n) = (-1)^n \mathfrak{p}_{2n}$ for all $n \geq 1$.

Theorems 24 through 32 will be stated for the homology and cohomology of $B\mathrm{Sp}$ with Z_p-coefficients for all primes p. These theorems and their proofs for p an odd prime are also valid for the homology and cohomology of BO if b_n is replaced by d_n, k_n is replaced by P_n, α is replaced by μ and γ is replaced by ν.

Theorem 24. *In $H^*(B\mathrm{Sp})$ for $r \geq 0$ and $n \geq 1$,*

$$Q_*^r(k_n) = (-1)^{(1/2)r(p+1)}(2n - r(p-1) - 1, rp - 2n)k_{n-(1/2)r(p-1)}$$

$$[Q_*^{4r}(k_n) = (n - r - 1, 2r - n)k_{n-r}].$$

Proof. This theorem results from combining Theorem 7 with the definition $k_n = (-1)^n \gamma^*(c_{2n})$.

Theorem 25. *In $H_*(B\mathrm{Sp})$ for $r \geq 0$ and $n \geq 1$,*

(a)
$$Q^r(\mathfrak{p}_n) = (-1)^{(1/2)r(p+1)}(2n - 1, r - 2n)\mathfrak{p}_{n+(1/2)r(p-1)}$$
$$[Q^{4r}(\mathfrak{p}_n) = (n - 1, r - n)\mathfrak{p}_{n+r}]$$

(b)
$$Q^r(b_n) = (-1)^{r+1}(2n, r - 2n - 1)b_{n+(1/2)r(p-1)} \quad \textit{modulo decomposables}$$
$$[Q^{4r}(b_n) = (n, r - n - 1)b_{n+r} \quad \textit{modulo indecomposables}].$$

Proof. This theorem is proved by combining Theorems 5 and 6 with the facts $\gamma_*(\mathfrak{p}_n) = (-1)^n \mathfrak{p}_{2n}$ and $\alpha_*(a_{2n}) = b_n$ for $n \geq 1$.

Theorem 26. *In $H_*(B\mathrm{Sp})$ for $r \geq 2n \geq 2$ $[r \geq n \geq 1]$, $Q^r(b_n)$ $[Q^{4r}(b_n)]$ has no monomial summand of product filtration degree greater than $p + \frac{1}{2}(r - 2n)(p - 1)$ $[r - n + 2]$, and the only monomial summand of $Q^r(b_n)$ of product filtration degree $p + \frac{1}{2}(r - 2n)(p - 1)$ $[r - n + 2]$ is $b_n^p b_1^{(1/2)(r-2n)(p-1)}$ $[b_n^2 b_1^{r-n}]$. Furthermore, if $\lambda b_{n_1} \cdots b_{n_t}$ is a summand of $Q^r(b_n)$ $[Q^{4r}(b_n)]$ with $n_1 \geq \cdots \geq n_t$ and $0 \neq \lambda \in Z_p$ then $n_1 \geq n$.*

Proof. This theorem follows from Theorem 24 in the same way that Theorem 8 follows from Theorem 7.

Theorem 27. *$Q_{\mathscr{R}} H_*(B\mathrm{Sp})$ has Z_p-basis $\{b_{np^e} \mid n \not\equiv 0 \bmod p, n = s(p-1) + r, 1 \leq r \leq p - 1$ and if $s \neq 0$ then $s = \sum_{i=0}^k s_i p^i$, $0 \leq s_i \leq p - 1$, with $r \geq s_0 \geq \cdots \geq s_k \geq 1\}$ $[Q_{\mathscr{R}} H_*(B\mathrm{Sp}; Z_2) = \{b_{2^n} \mid n \geq 0\}]$.*

Proof. This theorem follows from Theorem 10 and the fact that $\alpha_*(a_{2n}) = b_n$ for $n \geq 0$.

Let $G = \alpha \circ \gamma\colon B\mathrm{Sp} \to B\mathrm{Sp}$. In homology with Z_2-coefficients, $G_*(b_{2n}) = b_n^2$ and $G_*(b_{2n-1}) = 0$ for $n \geq 1$. Hence G_* is locally nilpotent and $Z_2[[G_*]]$ is an algebra of $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(B\mathrm{Sp}; Z_2)$.

Theorem 28. (a) *$Z_2[[G_*]]$ is the algebra of all $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(B\mathrm{Sp}; Z_2)$.*

(b) *$Z_2[[G_*]]$ is the algebra of all $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(B\mathrm{Sp}; Z_2)$.*

(c) *Every element of $Z_2[[G_*]]$ is induced by an infinite loop endomorphism of $B\mathrm{Sp}Q_2$.*

Proof. (a) and (b) are proved in the same manner as Theorem 12.

(c) $\sum_{i=0}^{\infty} h_i G_*^i = [\gamma \circ g \circ \alpha \oplus h_0 1_{B\mathrm{Sp}Q_2}]_*$ where g is an infinite loop endomorphism of BUQ_2 which induces $\sum_{i=1}^{\infty} h_i F_*^{i-1}$ in homology and $\oplus$ denotes Whitney sum.

Corollary 29. (a) *$H_*(B\mathrm{Sp}; Z_2)$ is an indecomposable $\mathfrak{A}$-Hopf algebra.*

(b) *If g is an $\mathfrak{A}$-Hopf algebra endomorphism of $H_*(B\mathrm{Sp}; Z_2)$ then g is a homomorphism of $\mathscr{R}$-modules.*

Theorem 30. *Let p be an odd prime. There are sub-$\mathfrak{A}\mathscr{R}$-Hopf algebras $H_*(B\mathrm{Sp})_{p,i}$ of $H_*(B\mathrm{Sp})$ for $0 \leq i \leq \frac{1}{2}(p-3)$ such that:*

(a) *$H_*(B\mathrm{Sp}) = \bigotimes_{i=0}^{(1/2)(p-3)} H_*(B\mathrm{Sp})_{p,i}$.*

(b) *$\alpha_*\colon H_*(BU_{p,2i}) \cong H_*(B\mathrm{Sp})_{p,i}$ as $\mathfrak{A}\mathscr{R}$-Hopf algebras for $0 \leq i \leq \frac{1}{2}(p-3)$.*

(c) *$Z_p[[F_{2i}]]$ is isomorphic under conjugation by α_* to*

(i) *The algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(B\mathrm{Sp})_{p,i}$.*

(ii) *The algebra of $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(B\mathrm{Sp})_{p,i}$.*

(d) *$\prod_{i=0}^{(p-3)/2} Z_p[[F_{2i}]]$ is isomorphic under conjugation by α_* to*

(i) *The algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(B\mathrm{Sp})$.*

(ii) *The algebra of $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(B\mathrm{Sp})$.*

Proof. (a) and (b). Define $H_*(B\mathrm{Sp})_{p,i} = \alpha_*(H_*(BU_{p,2i}))$ for $0 \leq i \leq \frac{1}{2}(p-3)$.

(c) and (d). Combine (a) and (b) of this theorem with Theorem 14.

Corollary 31. (a) *$H_*(B\mathrm{Sp})_{p,i}$ is an indecomposable $\mathfrak{A}$-Hopf algebra if $0 \leq i \leq \frac{1}{2}(p-3)$.*

(b) *$H_*(B\mathrm{Sp})$ is an indecomposable $\mathfrak{A}$-Hopf algebra if $p = 3$.*

(c) *If g is an $\mathfrak{A}$-Hopf algebra endomorphism of $H_*(B\mathrm{Sp})$ or of $H_*(B\mathrm{Sp})_{p,i}$ for some $0 \leq i \leq \frac{1}{2}(p-3)$ then f is a homomorphism of $\mathscr{R}$-modules.*

Theorem 32. (a) *The vector space of $\mathscr{R}$-Hopf algebra maps from $H_*(BU)$ to $H_*(B\mathrm{Sp})$ are all $\mathscr{R}$-module homomorphisms and are equal to:*

(i) *The free left module with basis $\{\alpha_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(B\mathrm{Sp})$.*

(ii) *A cyclic right module over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BU)$ with generator α_*. If $p = 2$ then this module is free while if p is odd then this module is $\alpha_* \circ \prod_{i=0}^{(p-3)/2} Z_p[[F_{2i}]]$.*

(b) *The vector space of $\mathfrak{A}$-Hopf algebra maps from $H_*(B\mathrm{Sp})$ to $H_*(BU)$ are all $\mathcal{R}$-module homomorphisms and are equal to*:

(i) *A cyclic left module over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BU)$ with generator γ_*. If $p = 2$ then this module is free, while if p is odd then this module is $\prod_{i=0}^{(p-3)/2} Z_p[[F_{2i}]] \circ \gamma_*$.*

(ii) *The free right module with basis $\{\gamma_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BU)$.*

Proof. (a) Let f: $H_*(BU) \to H_*(B\mathrm{Sp})$ be a map of $\mathfrak{A}$-Hopf algebras. If $p = 2$, let $f(a_{2^n})$ have $h_n b_1^{2^{n-1}}$ as a summand for $n \geq 2$ and $h_n \in Z_2$. Then $f = (\sum_{n=1}^{\infty} h_n G_*^{n-1}) \circ \alpha_* = \alpha_* \circ (\sum_{n=1}^{\infty} h_n F_*^{n-1})$. If p is odd then f restricted to $H_*(BU_{p,i})$ is zero when i is odd. Our assertions now follow from Theorem 30(b).

(b) (i) The left module in question is isomorphic to the right module of $\mathfrak{A}$-Hopf algebra maps from $H^*(BU)$ to $H^*(B\mathrm{Sp})$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H^*(BU)$. Since $\gamma^*(c_{2n}) = (-1)^n k_n$ and $\gamma^*(c_{2n-1}) = 0$ for $n \geq 1$, the reasoning of the proof of (a)(i) applies to this case too.

(ii) Dualize the problem as above and then apply the reasoning of the proof of (a)(ii).

We now consider $B\mathrm{Sp} \times Z = \Omega(U/\mathrm{Sp})$. With the notation of Theorem 22, $H_*(B\mathrm{Sp} \times Z) = H_*(B\mathrm{Sp}) \otimes Z_p(Z)$ and the natural map $B\mathrm{Sp} \to B\mathrm{Sp} \times Z$ induces $x \to x \otimes [0]$ in homology. As in the case of $BU \times Z$, we will know the $\mathcal{R}$-action on $H_*(B\mathrm{Sp} \times Z)$ as soon as we evaluate $Q^n(1 \otimes [1])$ for $n \geq 0$. Similarly, there is $BO \times Z = \Omega(U/O)$. $H_*(BO \times Z) = H_*(BO) \otimes Z_p(Z)$ and the natural map $BO \to BO \times Z$ induces $x \to x \otimes [0]$ in homology. As above, we need to calculate $Q^n(1 \otimes [1])$ for $n \geq 0$ to know the action of the Dyer-Lashof algebra on $H_*(BO \times Z)$.

Theorem 33. (a) *In $H_*(B\mathrm{Sp} \times Z)$ for $n \geq 0$,*

$$Q^n(1 \otimes [1]) = 2^n \gamma_n(\mathfrak{p}_{(1/2)(p-1)}) \otimes [p]$$

$$[Q^{4n}(1 \otimes [1]) = b_n \otimes [2]].$$

(b) *In $H_*(BO \times Z)$ for $n \geq 0$,*

$$Q^n(1 \otimes [1]) = \gamma_n(\mathfrak{p}_{(1/2)(p-1)}) \otimes [p]$$

$$[Q^n(1 \otimes [1]) = e_n \otimes [2]].$$

Proof. (a) Let $\hat{\alpha}$: $BU \times Z \to B\,\mathrm{Sp} \times Z$ be the loops of the canonical projection $U \to U/\mathrm{Sp}$. Then $\hat{\alpha} = \alpha \times 1$. For p an odd prime $\alpha_*(\gamma_n(\mathfrak{p}_{2K})) = 2^n \gamma_n(\mathfrak{p}_K)$. Hence (a) follows from Theorem 22.

(b) Let $\hat{\mu}$: $BU \times Z \to BO \times Z$ be the loops of the canonical projection $U \to U/O$. Then $\hat{\mu}(x, n) = (\mu(x), 2n)$ for all $x \in BU$ and $n \in Z$. Hence, as in (a), it follows from Theorem 22 that

$$Q^n(1 \otimes [2]) = 2^n \gamma_n(\mathfrak{p}_{(p-1)/2}) \otimes [2p]$$
$$[Q^{2n}(1 \otimes [2]) = e_n^2 \otimes [4]].$$

Thus, $2^n \gamma_n(\mathfrak{p}_{(p-1)/2}) \otimes [2p] = Q^n(1 \otimes [2]) = \sum_{i+j=n} Q^i(1 \otimes [1]) \cdot Q^j(1 \otimes [1])$, and by induction on n we see that $Q^n(1 \otimes [1]) = \gamma_n(\mathfrak{p}_{(p-1)/2}) \otimes [p]$. [Thus, $Q^n(1 \otimes [1])^2 = Q^{2n}(1 \otimes [2]) = e_n^2 \otimes [4] = (e_n \otimes [2])^2$, and hence $Q^n(1 \otimes [1]) = e_n \otimes [2]$.]

Recall that

$$H^*(BO; Z_2) = P\{w_n \mid n \geq 1\}$$

as algebras with $\psi(w_n) = \sum_{i=0}^n w_i \otimes w_{n-i}$ where $w_n \in H_n(BO; Z_2)$ is the nth Stiefel-Whitney class. Define $e_n = (w_1^n)^*$ and $\mathfrak{p}_n = w_n^*$ in the dual basis of the basis of monomials in the Stiefel-Whitney classes. Then $H_*(BO; Z_2) = P\{e_n \mid n \geq 1\}$ as algebras with $\psi(e_n) = \sum_{i=0}^n e_i \otimes e_{n-i}$ and $PH_*(BO; Z_2) = \{\mathfrak{p}_n \mid n \geq 1\}$. $\nu^*(c_n) = w_n^2$ and $\mu_*(a_n) = e_n^2$ are the values of the canonical maps. Let $H = \mu \circ \nu$, so $H_*(e_{2n}) = e_n^2$ and $H_*(e_{2n-1}) = 0$, $n \geq 1$. Observe that $H^*(BSO; Z_2) = P\{w_n \mid n \geq 2\}$ as a quotient Hopf algebra of $H^*(BO; Z_2)$ and that $H_*(BSO; Z_2)$ is a sub-$\mathfrak{A}\mathscr{R}$-Hopf algebra of $H_*(BO; Z_2)$. We write $H_*(BSO; Z_2) = P\{e_n' \mid n \geq 2\}$ as algebras with $\deg e_n' = n$. j: $BSO \to BO$, μ': $BSU \to BSO$, ν': $BSO \to BSU$ and $H' = \mu' \circ \nu'$ will denote the canonical maps.

Theorem 34. *In $H^*(BO; Z_2)$ for $r \geq 0$ and $n \geq 1$,*

$$Q_*^r(w_n) = (n - r - 1, 2r - n) w_{n-r}.$$

This equation is also valid in $H^*(BSO; Z_2)$ if $r \neq n - 1$ since $Q_*^{n-1}(w_n) = 0$ in $H^*(BSO; Z_2)$.

Proof. By Theorem 7, $Q_*^r(w_n)^2 = \nu^* \circ Q_*^{2r}(c_n) = (n - r - 1, 2r - n) w_{n-r}^2$. $H^*(BO; Z_2)$ is a polynomial algebra over Z_2, and hence has unique square roots of squares. Thus, $Q_*^r(w_n) = (n - r - 1, 2r - n) w_{n-r}$.

Corollary 35. *In $H_*(BO; Z_2)$ for $r \geq 0$ and $n \geq 1$,*

$$Q^r(\mathfrak{p}_n) = (n - 1, r - n) \mathfrak{p}_{r+n}.$$

Proof. This corollary is proved by dualizing Theorem 34.

Theorem 36. *In $H_*(BO; Z_2)$ for $r \geq 0$ and $n \geq 1$,*

$$Q^r(e_n) = (n, r - n - 1) e_{r+n} \quad \textit{modulo decomposables,}$$

while in $H_*(BSO; Z_2)$ for $r \geq 0$ and $n \geq 2$,

$Q^r(e'_n) = (n, r-n-1)e'_{r+n}$ *modulo*

decomposables if n is not a power of two, and $Q^r(e'_{2n})$ is always decomposable .

Proof. Combine the map $\mu_*(a_n) = e_n^2$ with Theorem 6 and argue as in the proof of Theorem 34.

Theorem 37. *In $H_*(BO; Z_2)$ for $r \geq 0$ and $n \geq 1$, $Q^r(e_n)$ has no monomial summand of product filtration degree greater than $r-n+2$ and the only monomial summand of $Q^r(e_n)$ of product filtration degree $r-n+2$ is $e_n^2 e_1^{r-n}$. Furthermore, if $e_{n_1} \cdots e_{n_t}$ with $n_1 \geq \cdots \geq n_t$ is a summand of $Q^r(e_n)$ then $n_1 \geq n$.*

Proof. This theorem results from combining Theorem 8 with the fact $\mu_*(a_n) = e_n^2$ for $n \geq 1$.

Theorem 38. (a) $Q_{\mathscr{R}}H_*(BO; Z_2) = \{e_{2^n} \mid n \geq 0\}$.

(b) $Q_{\mathscr{R}}H_*(BSO) = \{e'_{2^m+2^n} \mid m \geq 0$ *and* $n \geq 0\}$.

Proof. This theorem follows from Theorems 10 and 19 and the facts $\mu_*(a_n) = e_n^2$, $n \geq 1$, and $\mu'_*(a'_n) = e_n'^2$, $n \geq 1$.

Theorem 39. (a) *$Z_2[[H_*]]$ is the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BO; Z_2)$, and every element of $Z_2[[H_*]]$ is a homomorphism of $\mathscr{R}$-modules.*

(b) *The vector space of $\mathfrak{A}$-Hopf algebra maps from $H_*(BU; Z_2)$ to $H_*(BO; Z_2)$ are all $\mathscr{R}$-module homomorphisms and are equal to*:

(i) *The free left module with basis $\{\mu_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BO; Z_2)$.*

(ii) *The free right module with basis $\{\mu_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BU; Z_2)$.*

(c) *The vector space of $\mathfrak{A}$-Hopf algebra maps from $H_*(BO; Z_2)$ to $H_*(BU; Z_2)$ are all $\mathscr{R}$-module homomorphisms and are equal to*:

(i) *The free left module with basis $\{\nu_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BU; Z_2)$.*

(ii) *The free right module with basis $\{\nu_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BO; Z_2)$.*

(d) *The vector space of $\mathfrak{A}$-Hopf algebra maps from $H_*(BSO; Z_2)$ to $H_*(BO; Z_2)$ are all $\mathscr{R}$-module homomorphisms and are equal to*:

(i) *The free left module with basis $\{j_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BO; Z_2)$.*

(ii) *The free right module with basis $\{j_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BSO; Z_2)$.*

(e) *$Z_2[[H'_*]]$ is the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(BSO; Z_2)$, and every element of $Z_2[[H'_*]]$ is a homomorphism of $\mathscr{R}$-modules.*

Proof. This theorem is proved in the same way as Theorems 20, 28 and 32.

Corollary 40. *$H_*(BSO; Z_2)$ and $H_*(BO; Z_2)$ are both indecomposable $\mathfrak{A}$-Hopf algebras.*

5. U, SU, Sp, O, SO, Spin **and** B Spin. We will use the results of §§3 and 4 together with various suspension maps to study the action of the Dyer-Lashof algebra on the homology of the infinite classical groups, the infinite spinor group and its classifying space.

We begin by considering U and SU. The Bott map from $BU \to \Omega(SU)$ induces an isomorphism in homology. Thus, there is a suspension map $\sigma_*: H_n(BU) \to H_{n+1}(SU)$ for $n \geq 2$. Define $f_{n+1} = \sigma_*(a_n)$ for $n \geq 1$. Then $H_*(SU) = E\{f_n \mid n \geq 2\}$ as Hopf algebras and j: $SU \to U$ induces a monomorphism in homology. For $n \geq 1$, let $\sigma_*^U: H_n(U) \to H_{n+1}(BU)$ denote the suspension map induced by the universal U-bundle. Then σ_*^U restricted to $H_1(U)$ is monic, and define $f_1 \in H_1(U)$ by $\sigma_*^U(f_1) = a_1$. For $n \geq 1$, $\sigma_*^U(f_n) = (-1)^{n+1}\mathfrak{p}_n$. Then $H_*(U) = E\{f_n \mid n \geq 1\}$ as Hopf algebras, and $H^*(U) = E\{f_n^* \mid n \geq 1\}$ as Hopf algebras using the dual basis of the basis of monomials in the f_n. $H^*(SU) = E\{f_n^* \mid n \geq 2\}$ as a quotient Hopf algebra of $H^*(U)$.

Theorem 41. *In $H_*(U)$ for $r \geq 0$ and $n \geq 1$, or in $H_*(SU)$ for $r \geq 0$ and $n \geq 2$,*

$$Q^r(f_n) = (-1)^{r+n}(n-1, r-n)f_{n+r(p-1)}$$

$$[Q^{2r}(f_n) = (n-1, r-n)f_{n+r}].$$

Proof. The Dyer-Lashof operations send primitive elements to primitive elements, commute with suspension, and $\sigma_*^U(f_n) = (-1)^{n+1}\mathfrak{p}_n$. Hence this theorem follows from Theorem 5.

Corollary 42. *In $H^*(U)$ for $r \geq 0$ and $n \geq 1$,*

$$Q_*^r(f_n^*) = (-1)^{r+n}(n - r(p-1) - 1, pr - n)f_{n-r(p-1)}^*$$

$$[Q_*^{2r}(f_n^*) = (n - r - 1, 2r - n)f_{n-r}^*].$$

The same result is valid in $H^*(SU)$ if $n \neq 1 + r(p-1)$ since $Q_*^r(f_{1+r(p-1)}^*) = 0$ $[Q_*^{2r}(f_{r+1}^*) = 0]$ in $H^*(SU)$.

Theorem 43. (a) $Q_{\mathscr{R}}H_*(U; Z_2) = \{f_1\}$.

(b) $Q_{\mathscr{R}}H_*(SU; Z_2) = \{f_{2^n+1} \mid n \geq 0\}$.

(c) *For odd primes p, a Z_p-basis for $Q_{\mathscr{R}}H_*(SU; Z_p)$ is given by $\{f_{p^eh+1} \mid h \geq 1$, $h \not\equiv 0 \bmod p$, $h = s(p-1) + r$, $1 \leq r \leq p-1$, $s = \sum_{i=0}^n s_ip^i$, $0 \leq s_i \leq p-1$, and if $s \neq 0$ then $r \geq s_0 \geq \cdots \geq s_n \geq 1\}$.*

(d) *For odd primes p, a Z_p-basis for $Q_{\mathscr{R}}H_*(U; Z_p)$ is given by $\{f_1\} \cup \{f_{p^eh+1} \mid h \geq 1$, $h \not\equiv 0 \bmod p$, $h = s(p-1) + r$, $1 \leq r \leq p-2$, and if $s \neq 0$ then $s = \sum_{i=0}^n s_ip^i$ with $0 \leq s_i \leq p-1$ and $r \geq s_0 \geq \cdots \geq s_n \geq 1\}$.*

Proof. (a) $Q^{2n}(f_1) = f_{n+1}$ for $n \geq 1$.

(b), (c). The map $\sigma_*: IH_*(BU) \to H_*(SU)$ induces an isomorphism on indecomposables. Hence $f_n \in Q_{\mathscr{R}}H_*(SU)$ if and only if $a_{n-1} \in Q_{\mathscr{R}}H_*(BU)$. Thus, (b) and (c) follow from Theorem 10.

(d) $Q^n(f_1) = (-1)^{n+1} f_{n(p-1)+1}$ for $n \geq 1$. Furthermore, if $k \not\equiv 0 \bmod p-1$ then $f_{k+1} \in Q_{\mathscr{R}} H_*(U; Z_p)$ if and only if $f_{k+1} \in Q_{\mathscr{R}} H_*(SU; Z_p)$. Thus, (d) follows from (c).

If p is an odd prime and $0 \leq i \leq p-2$, then define $H_*(U)_{p,i} = E\{f_{n+1} \mid n \geq 0$ and $n \equiv i \bmod p-1\}$ and $H_*(SU)_{p,i} = E\{f_{n+1} \mid n \geq 1$ and $n \equiv i \bmod p - 1\}$.

Theorem 44. (a) *The algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(U; Z_2)$ is isomorphic to $Z_2 \times Z_2$.*

(b) *The identity and zero maps are the only $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(U; Z_2)$.*

(c) *The identity and zero maps are the only $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(SU; Z_2)$.*

(d) *For p an odd prime, the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(U)_{p,i}$ is isomorphic to Z_p if $1 \leq i \leq p-2$ and is isomorphic to $Z_p \times Z_p$ if $i = 0$.*

(e) *For p an odd prime, the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(U; Z_p)$ is isomorphic to $\prod_{i=1}^{p} Z_p$.*

(f) *For p an odd prime, the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(SU)_{p,i}$ is isomorphic to Z_p for $0 \leq i \leq p-2$.*

(g) *For p an odd prime, the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(SU)$ is isomorphic to $\prod_{i=1}^{p-1} Z_p$.*

(h) *For p an odd prime, the algebra of $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(U)_{p,i}$ or of $H_*(SU)_{p,i}$ is isomorphic to Z_p.*

(i) *For p an odd prime, the algebra of $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(U; Z_p)$ or of $H_*(SU; Z_p)$ is isomorphic to $\prod_{i=1}^{p-1} Z_p$.*

Proof. (a) We identify the $\mathfrak{A}$-Hopf algebra endomorphisms g of $H^*(U; Z_2)$ with $Z_2 \times Z_2$ by mapping g to (α, γ) where $g(f_1^*) = \alpha f_1^*$ and $g(f_2^*) = \gamma f_2^*$. We prove by induction on n that $g(f_n^*) = \gamma f_n^*$ for $n \geq 2$. $Q_{\mathfrak{A}} H^*(BU) = \{c_{2^k} \mid k \geq 0\}$, and hence $Q_{\mathfrak{A}} H^*(U) = \{f_{2^k}^* \mid k \geq 0\}$. Thus, it suffices to show inductively that $g(f_{2^n}^*) = \gamma f_{2^n}^*$ for $n \geq 1$. However $\mathscr{P}^2(f_{2^n}^*) = f_{2^n+1}^*$ for $n \geq 1$ and $\mathscr{P}^4(f_{2^n-1}^*) = f_{2^n+1}^*$ for $n \geq 2$ from which our assertion follows.

(b) Since $Q^{2n}(f_1) = f_n$ for all $n \geq 1$, an $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphism of $H_*(U)$ is determined by its value on f_1.

(d) This assertion is proved in the same way as (a) using the following observations. First, a Z_p-basis of $Q_{\mathfrak{A}} H^*(U)$ is given by $\{f_{ip^n}^* \mid 1 \leq i \leq p-1$ and $n \geq 0\}$. Secondly,

$$\mathscr{P}^p(f_{ip^n-p+1}^*) = (ip^{n-1} - 1) f_{ip^n+p^2-2p+1}^*$$

if $1 \leq i \leq p-1, n \geq 2$ or $2 \leq i \leq p-1, n = 1$ while

$$\mathscr{P}^{p-1}(f_{ip^n}^*) = f_{ip^n+p^2-2p+1}^* \quad \text{if } 1 \leq i \leq p-1 \text{ and } n \geq 1.$$

(h) Since $Q^n(f_1) = (-1)^{n+1} f_{n(p-1)+1}$ for $n \geq 1$, an $\mathfrak{A}\mathcal{R}$-Hopf algebra endomorphism of $H_*(U)_{p,1}$ is determined by its value on f_1.

Corollary 45. (a) $H_*(U; Z_2) \cong H_*(SU; Z_2) \otimes Z_2 f_1$ *as* $\mathfrak{A}$*-Hopf algebras.*

(b) $H_*(U; Z_2)$ *is an indecomposable* $\mathfrak{A}\mathcal{R}$*-Hopf algebra and* $H_*(SU; Z_2)$ *is an indecomposable* $\mathfrak{A}$*-Hopf algebra.*

(c) *If p is an odd prime, then*

$$H_*(U; Z_p) \cong \bigotimes_{i=1}^{p-2} H_*(U)_{p,i} \otimes H_*(SU)_{p,0} \otimes Z_p f_1,$$

$$H_*(SU; Z_p) \cong \bigotimes_{i=0}^{p-2} H_*(SU)_{p,i}$$

are decompositions of $H_*(U; Z_p)$ *and* $H_*(SU; Z_p)$ *into indecomposable* $\mathfrak{A}$*-Hopf algebras.*

(d) *If p is an odd prime then* $H_*(U; Z_p) = \bigotimes_{i=0}^{p-2} H_*(U)_{p,i}$ *and* $H_*(SU; Z_p) = \bigotimes_{i=0}^{p-2} H_*(SU)_{p,i}$ *are decompositions of* $H_*(U; Z_p)$ *and* $H_*(SU; Z_p)$ *into indecomposable* $\mathfrak{A}\mathcal{R}$*-Hopf algebras.*

We will now consider the action of the Dyer-Lashof algebra on the homology of Sp with Z_p coefficients for all primes p and on the homology of SO with Z_p coefficients for odd primes p. Let γ: Sp $\to U$ and α: $U \to$ Sp be the canonical maps. γ induces a monomorphism in homology and we define $g_n \in H_{4n-1}(\mathrm{Sp})$ for $n \geq 1$ by $g_n = \gamma_*^{-1}(f_{2n})$. Then $H_*(\mathrm{Sp}) = E\{g_n \mid n \geq 1\}$ as Hopf algebras, and $H^*(\mathrm{Sp}) = E\{g_n^* \mid n \geq 1\}$ as Hopf algebras where we use the dual basis of the basis of $H_*(\mathrm{Sp})$ given by monomials in the g_n. Note that $\alpha_*(f_n) = 2g_{n/2}$ if n is even and p is odd, while $\alpha_*(f_n) = 0$ if n is odd or $p = 2$. We now let p be an odd prime, and we observe that the same situation occurs for $H_*(SO; Z_p)$. Let μ: $U \to SO$ and ν: $SO \to U$ be the canonical maps. Define $h_n \in H_{4n-1}(SO; Z_p)$ by $h_n = \nu_*^{-1}(f_{2n})$ for $n \geq 1$. Then $H_*(SO; Z_p) = E\{h_n \mid n \geq 1\}$ and $H^*(SO; Z_p) = E\{h_n^* \mid n \geq 1\}$ as Hopf algebras. Furthermore, $\mu_*(f_{2n}) = 2h_n$ and $\mu_*(f_{2n-1}) = 0$ for $n \geq 1$.

Theorems 46 through 51 will be stated for $H_*(\mathrm{Sp}; Z_p)$ and $H^*(\mathrm{Sp}; Z_p)$ for all primes p. However, these theorems and their proofs for p an odd prime are also valid for $H_*(SO; Z_p)$ and $H^*(SO; Z_p)$ if g_n, α and γ are replaced by h_n, μ and ν respectively.

Theorem 46. *In* $H_*(\mathrm{Sp})$ *for* $r \geq 0$ *and* $n \geq 1$,

$$Q^r(g_n) = (-1)^r(2n-1, r-2n) g_{n+(1/2)r(p-1)}$$

$$[Q^{4r}(g_n) = (n-1, r-n) g_{n+r}].$$

Proof. This theorem follows from Theorem 41 and the facts that γ_* is a monomorphism and $\gamma_*(g_n) = f_{2n}$ for $n \geq 1$.

Corollary 47. *In $H^*(\mathrm{Sp})$ for $r \geq 0$ and $n \geq 1$,*

$$Q_*^r(g_n^*) = (-1)^r(2n - r(p-1) - 1, pr - 2n)g^*_{n-(1/2)r(p-1)}$$

$$[Q_*^{4r}(g_n^*) = (n - r - 1, 2r - n)g^*_{n-r}].$$

Theorem 48. (a) $Q_{\mathcal{R}}H_*(\mathrm{Sp}; Z_2) = \{g_1\}$.

(b) *For p an odd prime, a Z_p-basis for $Q_{\mathcal{R}}H_*(\mathrm{Sp}, Z_p)$ is given by $\{g_n \mid 2n - 1 = p^e h, h \not\equiv 0 \bmod p, h = s(p-1) + r,\ 1 \leq r \leq p-2,$ and if $s \neq 0$ then $s = \sum_{i=0}^n s_i p^i$ with $0 \leq s_i \leq p-1$ and $r \geq s_0 \geq \cdots \geq s_n \geq 1\}$.*

Proof. $\gamma_*(g_n) = f_{2n}$ for $n \geq 1$. Hence $g_n \in Q_{\mathcal{R}}H_*(\mathrm{Sp}; Z_p)$ if and only if $f_{2n} \in Q_{\mathcal{R}}H_*(U)$. Thus, this theorem follows from Theorem 43.

If p is an odd prime and $0 \leq i \leq \frac{1}{2}(p-3)$ then define $H_*(\mathrm{Sp})_{p,i} = E\{g_n \mid n \geq 1$ and $n \equiv i \bmod \frac{1}{2}(p-3)\}$. Clearly γ_* induces an isomorphism of $\mathfrak{A}\mathcal{R}$-Hopf algebras from $H_*(\mathrm{Sp})_{p,i}$ to $H_*(U)_{p,2i+1}$.

Theorem 49. (a) *The algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(\mathrm{Sp}; Z_2)$ is isomorphic to $Z_2 \times Z_2$.*

(b) *The identity and zero maps are the only $\mathfrak{A}\mathcal{R}$-Hopf algebra endomorphisms of $H_*(\mathrm{Sp}; Z_2)$.*

(c) *If p is an odd prime and $0 \leq i \leq \frac{1}{2}(p-3)$ then the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(\mathrm{Sp})_{p,i}$ is isomorphic to Z_p.*

(d) *If p is an odd prime then the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(\mathrm{Sp}; Z_p)$ is isomorphic to $\prod_{i=0}^{(1/2)(p-3)} Z_p$.*

Proof. (a) Observe that $Q_{\mathfrak{A}}H^*(\mathrm{Sp}; Z_2) = \{g^*_{2^n} \mid n \geq 0\}$. If $n \geq 2$ then $\mathcal{P}^4(g^*_{2^n}) = g^*_{2^n+1}$ and $\mathcal{P}^8(g^*_{2^n-1}) = g^*_{2^n+1}$. Furthermore, $\mathcal{P}^r(g^*_1) = 0$ for all $r > 0$. Hence an $\mathfrak{A}$-Hopf algebra endomorphism of $H^*(\mathrm{Sp}; Z_2)$ is determined by its action on g_1 and g_2.

(b) $Q^{4n}(g_1) = g_{n+1}$ for all $n \geq 1$. Hence an $\mathfrak{A}\mathcal{R}$-Hopf algebra endomorphism of $H_*(\mathrm{Sp}; Z_2)$ is determined by its action on g_1.

(c), (d). These statements follow from Theorem 44 and the fact that $H_*(\mathrm{Sp})_{p,i} \cong H_*(U)_{p,2i+1}$ as $\mathfrak{A}\mathcal{R}$-Hopf algebras for $0 \leq i \leq \frac{1}{2}(p-3)$.

Corollary 50. (a) $H_*(\mathrm{Sp}; Z_2) \cong Z_2 g_1 \otimes E\{g_n \mid n \geq 2\}$ *as $\mathfrak{A}$-Hopf algebras.*

(b) $H_*(\mathrm{Sp}; Z_2)$ *is an indecomposable $\mathfrak{A}\mathcal{R}$-Hopf algebra.*

(c) *If p is an odd prime then $H_*(\mathrm{Sp})_{p,i}$, $0 \leq i \leq \frac{1}{2}(p-3)$, and $H_*(\mathrm{Sp}; Z_3)$ are indecomposable $\mathfrak{A}$-Hopf algebras.*

(d) *If p is an odd prime and f is an $\mathfrak{A}$-Hopf algebra endomorphism of $H_*(\mathrm{Sp}; Z_p)$ or of $H_*(\mathrm{Sp})_{p,i}$, $0 \leq i \leq \frac{1}{2}(p-3)$, then f is a homomorphism of $\mathcal{R}$-modules.*

Theorem 51. (a) *The vector space of $\mathfrak{A}$-Hopf algebra maps from $H_*(U)$ to $H_*(\mathrm{Sp})$ are all $\mathcal{R}$-module homomorphisms and are equal to:*

(i) *The free right module with basis $\{\alpha_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $\bigotimes_{i=0}^{(1/2)(p-3)} H_*(U)_{p,2i+1}$ $[H_*(SU)]$.*

(ii) *The free left module with basis* $\{\alpha_*\}$ *over the ring of* $\mathfrak{A}$*-Hopf algebra endomorphisms of* $H_*(\mathrm{Sp})$.

(b) *The vector space of* $\mathfrak{A}$*-Hopf algebra maps from* $H_*(\mathrm{Sp}; Z_2)$ *to* $H_*(U; Z_2)$ *is the free right module with basis* $\{\gamma_*\}$ *over the ring of* $\mathfrak{A}$*-Hopf algebra endomorphisms of* $H_*(\mathrm{Sp}; Z_2)$.

(c) γ_* *and* 0 *are the only* $\mathfrak{A}\mathscr{R}$*-Hopf algebra maps from* $H_*(\mathrm{Sp}; Z_2)$ *to* $H_*(U; Z_2)$.

(d) *If p is an odd prime then the vector space of* $\mathfrak{A}$*-Hopf algebra maps from* $H_*(\mathrm{Sp}; Z_p)$ *to* $H_*(U; Z_p)$ *are all* $\mathscr{R}$*-module homomorphisms and are equal to*:

(i) *The free right module with basis* $\{\gamma_*\}$ *over the ring of* $\mathfrak{A}$*-Hopf algebra endomorphisms of* $H_*(\mathrm{Sp}; Z_p)$.

(ii) *The free left module with basis* $\{\gamma_*\}$ *over the ring of* $\mathfrak{A}$*-Hopf algebra endomorphisms of* $\bigotimes_{i=0}^{(1/2)(p-3)} H_*(U)_{p,2i+1}$.

Observe that $f\colon H_*(\mathrm{Sp}; Z_2) \to H_*(U; Z_2)$, defined by $f(g_1) = 0$ and $f(g_n) = f_{2n}$ if $n \geq 2$, is a map of $\mathfrak{A}$-Hopf algebras. However, f cannot be written as γ_* followed by an $\mathfrak{A}$-Hopf algebra endomorphism of $H_*(U; Z_2)$.

We now consider $H_*(SO; Z_2)$ where the action of the Dyer-Lashof algebra is more complicated than on the homology of the other classical groups. Define $i_n \in H^{2n-1}(SO; Z_2)$ for $n \geq 1$ by $i_n = \sigma^*_{SO}(w_{n+1})$ where $\sigma^*_{SO}\colon IH^*(BSO; Z_2) \to H^*(SO; Z_2)$ is the suspension map induced from the universal SO-bundle. Then $H^*(SO; Z_2) = P\{i_n \mid n \geq 1\}$ as Hopf algebras. Recall that there are indecomposable elements $u_n \in H_n(SO; Z_2)$, $n \geq 1$, with $u_0 = 1$ and $\psi(u_n) = \sum_{t=0}^{n} u_t \otimes u_{n-t}$. Then $H_*(SO; Z_2) = E\{u_n \mid n \geq 1\}$ as algebras, and $PH_*(SO; Z_2) = \{\mathfrak{p}_n \mid n \geq 1\}$ where $\mathfrak{p}_n = \sum_{t=0}^{n-1} u_t u_{2n-t-1}$ has degree $2n-1$. Note that $\nu_*\colon H_*(SO; Z_2) \to H_*(U; Z_2)$ is the zero map and $\mu_*\colon H_*(U; Z_2) \to H_*(SO; Z_2)$ is given by $\mu_*(f_n) = \mathfrak{p}_n$, $n \geq 1$.

Theorem 52. *In* $H^*(SO; Z_2)$ *for* $r \geq 1$ *and* $n \geq 1$,

$$Q^{2r}_*(i_n) = (n-r-1, 2r-n)i_{n-r}$$

and

$$Q^{2r-1}_*(i_n) = (n-r, 2r-n-1)i_s^{2^t}$$

where $n - r = 2^{t-1}(2s-1)$.

Proof. This theorem follows from Theorem 34 and the observations that $\sigma^*_{SO}(w_{2n}) = i_n$ and $\sigma^*_{SO}(w_{2^k(2n-1)+1}) = i_n^{2^k}$.

Theorem 53. *In* $H_*(SO; Z_2)$ *for* $r \geq 0$ *and* $n \geq 1$,

(a) $Q^{2r}(\mathfrak{p}_n) = (n-1, r-n)\mathfrak{p}_{n+r}$.

(b) $Q^r(u_n) = (n, r-n-1)u_{r+n}$ *modulo decomposables.*

Proof. (a) This assertion is the dual statement of Theorem 52.

(b) This result follows from Corollary 35 and the fact $\sigma_*^{SO}(u_n) = \mathfrak{p}_{n+1}$ for $n \geq 1$.

In Theorem 56 we will use Theorem 53(b), the comultiplicative Cartan formula and various identities on binomial coefficients (see Lemma 55) to calculate the monomial summands of $Q^r(u_n)$ of product filtration degree two and three. Then we will show that $Q^r(u_n)$ has no monomial summands of product filtration degree greater than three.

Notation 54. Recall J. Adem's identities for binomial coefficients [2, Appendix]. For any integers a and b, define $\binom{a}{b}$ as the coefficient of x^b in the Taylor expansion of $(1+x)^a$. Note that $\binom{a}{b} = 0$ if $b < 0$ and if $a \geq 0$, $b \geq 0$ then $\binom{a}{b} = (b, a-b) = a!/b!(a-b)!$. J. Adem has proved the following relations:

$$\sum_{i=0}^{c} \binom{a-i-1}{i}\binom{b+i-1}{c-i} \equiv \binom{a+b-1}{c} \mod 2 \tag{1}$$

for all integers a, b and c.

$$\binom{a}{b} \equiv (-1)^b \binom{b-a-1}{b} \mod p \tag{2}$$

for p any prime, $a < 0$ and b any integer.

$$\sum_{i=0}^{c} \binom{a}{i}\binom{b}{c-i} = \binom{a+b}{c} \quad \text{for all integers } a, b \text{ and } c. \tag{3}$$

For example, (3) follows from the fact that the coefficient of x^c in $(1+x)^a \cdot (1+x)^b$ equals the coefficient of x^c in $(1+x)^{a+b}$.

Lemma 55. *For integers a, b and c the following identities are valid modulo* 2:

$$\text{(A)} \quad \sum_{i=0}^{b} \binom{a-i-1}{i}\binom{i-b-1}{c-i} = \binom{a-c-1}{b} \quad \textit{if } c \geq b \geq 0.$$

$$\text{(B)} \quad \sum_{i=a}^{c} \binom{a-i-1}{i}\binom{b+i-1}{c-i} = \binom{b-1}{c-a} \quad \textit{if } c \geq a,\ c \geq 0.$$

$$\text{(C)} \quad \sum_{i=0}^{c} \binom{a-i-1}{i}\binom{b+i-1}{c-i} = \binom{a+b-1}{c}.$$

Proof. (A) $\sum_{i=0}^{b} \binom{a-i-1}{i}\binom{i-b-1}{c-i} = \sum_{i=0}^{b} \binom{a-i-1}{i}\binom{c+b-i}{c-i}$ by (2) $= \sum_{i=0}^{b} \binom{a-i-1}{i}\binom{c+b-i}{b}$ $= \sum_{i=0}^{b} \binom{a-i-1}{i}\binom{i-c-1}{b}$ by (2) $= \binom{a-c-1}{b}$ by (1).

(B) $\sum_{i=a}^{c} \binom{a-i-1}{i}\binom{b+i-1}{c-i} = \sum_{j=0}^{c-a} \binom{-j-1}{j+a}\binom{a+b+j-1}{c-a-j}$ where $j = i - a$ $= \sum_{j=0}^{c-a} \binom{2j+a}{j+a} \cdot \binom{a+b+j-1}{c-a-j} = \sum_{j=0}^{c-a} \binom{2j+a}{j}\binom{a+b+j-1}{c-a-j}$ by (2) $= \sum_{j=0}^{c-a} \binom{-j-a-1}{j}\binom{a+b+j-1}{c-a-j} = \binom{b-1}{c-a}$ by (2) and (1).

(C) This identity follows from (1).

Theorem 56. *In $H_*(SO; Z_2)$ for $r \geq 0$ and $n \geq 1$,*

$$Q^r(u_n) = \sum_{\substack{a,b,c \geq 0 \\ a+b+c=r+n}} (n-a, r-n-b-1) u_a u_b u_c.$$

Proof. We prove this theorem by induction on $r+n$ and for fixed $r+n$ by induction on r. The cases $r = 0$ are trivial. Now assume the induction hypothesis. Let $\lambda_{i_1,\ldots,i_t}$ be the coefficient of $u_{i_1}\cdots u_{i_t}$ in $Q^r(u_n)$ if $1 \le i_1 < \cdots < i_t$ and $\sum_{s=1}^{t} i_s = r+n$.

We begin by calculating $\lambda_{t,r+n-t}$ for $1 \le t \le [\frac{1}{2}(r+n)]$ which the theorem asserts is $(n,r-t-n-1)+(n,t-2n-1)+(n-t,r-n-1)$. Consider the coefficient of $u_t \otimes u_{r+n-t}$ in the equation

$$(*)\qquad \psi \circ Q^r(u_n) = \sum_{i=0}^{r}\sum_{j=0}^{n} Q^i(u_j)\otimes Q^{r-i}(u_{n-j}).$$

$\lambda_{t,r+n-t} + (n,r-n-1) = \sum_{i=1}^{\min(n-1,t-1)}(i,t-2i-1)(n-i,r+2i-t-n-1)$. Hence

$$\begin{aligned}\lambda_{t,r+n-t} &= (n,r-n-1) + \sum_{i=0}^{n}\binom{t-i-1}{i}\binom{r+i-t-1}{n-i} + \binom{r-t-1}{n}\\ &\quad + (n,t-2n-1)\\ &\qquad\qquad \left[+\sum_{i=t}^{n}\binom{t-i-1}{i}\binom{r+i-t-1}{n-i} \quad \text{if } n \ge t\right]\\ &= (n,r-t-n-1)+(n,t-2n-1)+(n-t,r-n-1)\end{aligned}$$

by (C), (B).

We next calculate $\lambda_{a,b,r+n-a-b}$ for $1 \le a < b < r+n-a-b < r+n-2$ which the theorem asserts is $(n-b,r-a-n-1)+(n-a,r-b-n-1)+(n-a,a+b-2n-1)$. Consider the coefficient of $u_{r+n-a-b}\otimes u_a u_b$ in $(*)$ above.

$$\begin{aligned}&\lambda_{a,b,r+n-a-b} + \lambda_{a,r+n-a} + \lambda_{b,r+n-b}\\ &\qquad = \sum_{i=1}^{\min(n-1,r+n-a-b-1)} (i,r+n-a-b-2i-1)\\ &\qquad\quad \cdot [(n-i,b+2i-2n-1)+(n-i,a+2i-2n-1)\\ &\qquad\qquad\qquad + (n-a-i,a+b+2i-2n-1)].\end{aligned}$$

Thus, $\lambda_{a,b,r+n-a-b} = \lambda_{a,r+n-a} + \lambda_{b,r+n-b} + \sum_{i=0}^{n}\binom{r+n-a-b-i-1}{i}\binom{b-n+i-1}{n-i} + (n,b-2n-1) + (n,r-n-a-b-1)$ $[+\sum_{i=0}^{n-b}\binom{r+n-a-b-i-1}{i}\binom{b-n+i-1}{n-i}$ if $n \ge b]$ $[+\sum_{i=r+n-a-b}^{n}\binom{r+n-a-b-i-1}{i}\binom{b-n+i-1}{n-i}$ if $n \ge r+n-a-b]$ $+\sum_{i=0}^{n}\binom{r+n-a-b-i-1}{i}\cdot\binom{a-n+i-1}{n-i} + (n,a-2n-1) + (n,r-n-a-b-1)$ $[+\sum_{i=0}^{n-a}\binom{r+n-a-b-i-1}{i}\binom{a-n+i-1}{n-i}$ if $n \ge a]$ $[+\sum_{i=r+n-a-b}^{n}\binom{r+n-a-b-i-1}{i}\binom{a-n+i-1}{n-i}$ if $n \ge r+n-a-b]$ $+\sum_{i=0}^{n-a}\binom{r+n-a-b-i-1}{i}\binom{b-n+i-1}{n-a-i} + (n-a,a+b-2n-1)$ $[+\sum_{i=0}^{n-b}\binom{r+n-a-b-i-1}{i}\cdot\binom{b-n+i-1}{n-a-i}$ if $n \ge b]$ $= \lambda_{a,r+n-a} + \lambda_{b,r+n-b} + \binom{r-a-1}{n} + (n,b-2n-1) + \binom{r-a-b-1}{n-b} + \binom{b-n-1}{a+b-r} + \binom{r-b-1}{n} + (n,a-2n-1) + \binom{r-a-b-1}{n-a} + \binom{a-n-1}{a+b-r} + \binom{r-a-1}{n-a} + (n-a,a+b-2n-1) + \binom{r-b-1}{n-b}$ by (C), (A), (B), (C), (A), (B), (C), (A)

$$= (n-b, r-a-n-1) + (n-a, r-b-n-1)$$
$$+(n-a, a+b-2n-1) \qquad \text{if } a+b < r$$
$$= (n-a, a+b-2n-1) \qquad \text{if } a+b \geq r$$
$$= (n-b, r-a-n-1) + (n-a, r-b-n-1) + (n-a, a+b-2n-1).$$

We finally calculate that $\lambda_{a,b,c,r+n-a-b-c} = 0$ for $0 < a < b < c < r+n-a-b-c < r+n-3$. It is clear that our inductive method of calculation will then imply that $\lambda_{i_1,\ldots,i_t} = 0$ if $1 \leq i_1 < \cdots < i_t$, $\sum_{s=1}^{t} i_s = r+n$ and $t \geq 4$. Consider the coefficient of $u_{r+n-a-b-c} \otimes u_a u_b u_c$ in $(*)$ above.

$$\lambda_{a,b,c,r+n-a-b-c} + \lambda_{a,b,r+n-a-b} + \lambda_{a,c,r+n-a-c} + \lambda_{b,c,r+n-b-c}$$
$$= \sum_{i=1}^{\min(n-1,r+n-a-b-c-1)} (i, r+n-a-b-c-2i-1)$$
$$\cdot [(n-b-i, b+c+2i-2n-1) + (n-a-i, a+c-2n+2i-1)$$
$$+ (n-a-i, a+b-2n+2i-1)].$$

Thus,

$$\lambda_{a,b,c,r+n-a-b-c} = \lambda_{a,b,r+n-a-b} + \lambda_{a,c,r+n-a-c} + \lambda_{b,c,r+n-b-c}$$
$$+ \sum_{i=0}^{n-b} \binom{r+n-a-b-c-i-1}{i} \binom{c-n+i-1}{n-b-i}$$
$$+ (n-b, b+c-2n-1)$$
$$\left[+ \sum_{i=0}^{n-c} \binom{r+n-a-b-c-i-1}{i} \binom{c-n+i-1}{n-b-i} \text{ if } n \geq c\right]$$
$$+ \sum_{i=0}^{n-a} \binom{r+n-a-b-c-i-1}{i} \binom{c-n+i-1}{n-a-i}$$
$$+ (n-a, a+c-2n-1)$$
$$\left[+ \sum_{i=0}^{n-c} \binom{r+n-a-b-c-i-1}{i} \binom{c-n+i-1}{n-a-i} \text{ if } n \geq c\right]$$
$$\left[+ \sum_{i=r+n-a-b-c}^{n-a} \binom{r+n-a-b-c-i-1}{i} \right.$$
$$\left. \binom{c-n+i-1}{n-a-i} \text{ if } r \leq b+c\right]$$

$$+\sum_{i=0}^{n-a}\binom{r+n-a-b-c-i-1}{i}\binom{b-n+i-1}{n-a-i}$$

$$+(n-a,a+b-2n-1)$$

$$\left[+\sum_{i=0}^{n-b}\binom{r+n-a-b-c-i-1}{i}\binom{b-n+i-1}{n-a-i}\text{ if } n\geq b\right]$$

$$\left[+\sum_{i=r+n-a-b-c}^{n-a}\binom{r+n-a-b-c-i-1}{i}\binom{b-n+i-1}{n-a-i}\right.$$

$$\left.\text{if } r\leq b+c\right]$$

$$=\lambda_{a,b,r+n-a-b}+\lambda_{a,c,r+n-a-c}+\lambda_{b,c,r+n-b-a}+\binom{r-a-b-1}{n-b}$$

$$+(n-b,b+c-2n-1)+\binom{r-a-c-1}{n-c}$$

$$+\binom{r-a-b-1}{n-a}+(n-a,a+c-2n-1)$$

$$+\binom{r-b-c-1}{n-c}+\binom{c-n-1}{b+c-r}$$

$$+\binom{r-a-c-1}{n-a}+(n-a,a+b-2n-1)$$

$$+\binom{r-b-c-1}{n-b}+\binom{b-n-1}{b+c-r}$$

by (C), (A), (C), (A), (B), (C), (A), (B)

$$=0 \qquad \text{if } r>b+c$$

$$=\binom{c+n-r}{n-b}+\binom{c+n-r}{b+c-r}=0$$

if $r\leq b+c$ by (2) since $c>n>b$.

Theorem 57. $Q_{\mathscr{A}}H_*(SO;Z_2)=\{u_{2^n}\mid n\geq 0\}$.

Proof. $\sigma_*^{SO}(u_k)=\mathfrak{p}_{k+1}$ for $k\geq 1$. Hence $u_k\in Q_{\mathscr{A}}H_*(SO;Z_2)$ if and only if $Q_*^i(w_{k+1})=0$ for all $0\leq i<k$, i.e. if and only if $(k-i,2i-k-1)=0$ for all $0\leq i<k$. This occurs if and only if k is a power of two.

Theorem 58. (a) *The algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(SO;Z_2)$ is isomorphic to $Z_2\times Z_2$.*

(b) *The identity and zero maps are the only $\mathfrak{A}\mathcal{R}$-Hopf algebra endomorphisms of $H_*(SO; Z_2)$.*

Proof. This theorem follows from Theorem 44(a) and the fact that $\mu^*(i_n) = f_n^*$ for $n \geq 1$.

Corollary 59. (a) $H^*(SO; Z_2) = P\{i_1\} \otimes H^*(\text{Spin}; Z_2)$ *is a decomposition of $H^*(SO; Z_2)$ into indecomposable $\mathfrak{A}$-Hopf algebras.*
(b) *$H_*(SO; Z_2)$ is an indecomposable $\mathfrak{A}\mathcal{R}$-Hopf algebra.*

Theorem 60. (a) *The vector space of $\mathfrak{A}$-Hopf algebra maps from $H_*(U; Z_2)$ to $H_*(SO; Z_2)$ is equal to:*
(i) *The free right module with basis $\{\mu_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(U; Z_2)$.*
(ii) *The free left module with basis $\{\mu_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(SO; Z_2)$.*
(b) *μ_* and the zero map are the only maps of $\mathfrak{A}\mathcal{R}$-Hopf algebras from $H_*(U; Z_2)$ to $H_*(SO; Z_2)$.*
(c) *The zero map is the only map of Hopf algebras from $H_*(SO; Z_2)$ to $H_*(U; Z_2)$.*

Recall that $H_*(O; Z_p) = H_*(O; Z_p) \otimes Z_p(Z_2)$ as Hopf algebras where $Z_p(Z_2)$ is the group algebra of Z_2 over Z_p. For $x \in H_*(SO)$, we write x and $x \otimes [-1]$ to designate elements in the homology of the two components of O where we identify $H_*(SO)$ with its image in $H_*(O)$ under the canonical map. Since $x \otimes [-1] = x \cdot (1 \otimes [-1])$ for $x \in H_*(SO; Z_p)$, we will know the action of the Dyer-Lashof algebra on $H_*(O; Z_p)$ from Theorems 46, 56 and the multiplicative Cartan formula as soon as we compute $Q^n(1 \otimes [-1])$ for $n \geq 0$.

Theorem 61. *In $H_*(O; Z_p)$ for $n \geq 1$,*

$$Q^0(1 \otimes [-1]) = 1 \otimes [-1] \quad \text{and} \quad Q^n(1 \otimes [-1]) = 0$$
$$[Q^0(1 \otimes [-1] = 1 \quad \text{and} \quad Q^n(1 \otimes [-1]) = u_n].$$

Proof. We compute $Q^n(1 \otimes [-1])$ by induction on $n \geq 0$. $Q^0(1 \otimes [-1]) = (1 \otimes [-1])^p = (1 \otimes [-1]$ $[Q^0(1 \otimes [-1]) = (1 \otimes [-1])^2 = 1$. If $n \geq 1$ then the induction hypothesis and the comultiplicative Cartan formula imply that

$$Q^n(1 \otimes [-1] - 1) = \lambda h_{(1/2)n(p-1)} \otimes [-1] \; [Q^n(1 \otimes [-1] - 1) - u_n = \lambda \mathfrak{p}_{(1/2)(n+1)}]$$

for some $\lambda \in Z_p$ [with the convention that $\lambda = 0$ if n is even]. Now applying σ_*^0 to this equation, we see that $\lambda = 0$ [by Corollary 35] since

$$\sigma_*^0(1 \otimes [-1] - 1) = 0 \; [\sigma_*^0(1 \otimes [-1] - 1) = e_1]$$

and $\sigma_*^0(h_{(1/2)n(p-1)}) = (-1)^{(1/2)n(p-1)+1} \mathfrak{p}_{(1/2)n(p-1)}$ $[\sigma_*^0(\mathfrak{p}_n) = \mathfrak{p}_{2n}$ and $\sigma_*^0(u_n) = \mathfrak{p}_{n+1}]$.

Corollary 62. (a) $Q_{\mathscr{R}}H_*(O;Z_2) = \{1 \otimes [-1]\}$.

(b) *If p is an odd prime then*

$$Q_{\mathscr{R}}H_*(O;Z_p) = Q_{\mathscr{R}}H_*(SO;Z_p) \cup \{1 \otimes [-1]\}.$$

Let η: Spin $\to SO$ be the universal covering projection which induces $B\eta$: B Spin $\to BSO$. In homology with Z_p-coefficients for p an odd prime, η_* and $B\eta_*$ are isomorphisms of $\mathfrak{A}\mathscr{R}$-Hopf algebras, and hence Theorems 24 through 32 and 46 through 49 describe the action of the Dyer-Lashof algebra in this case. In cohomology with Z_2-coefficients, η^* and $B\eta^*$ are epimorphisms. Thus, write $H^*(\text{Spin};Z_2) = P\{i_n \mid n \geq 2\}$ and $H^*(B\,\text{Spin};Z_2) = P\{w_n \mid n \geq 4, n \neq 2^k + 1\}$ as quotient Hopf algebras of $H^*(SO;Z_2)$ and $H^*(BSO;Z_2)$ respectively. We can write $H_*(\text{Spin};Z_2) = E\{u'_n \mid n \geq 3$ and $n \neq 2^k,\ k \geq 2\}$ and $H_*(B\,\text{Spin};Z_2) = P\{e''_n \mid n \geq 4, n \neq 2^k + 1\}$ as algebras where $\eta_*(u'_n) = u_n$ and $(B\eta)_*(e''_n) = e'_n$ modulo decomposables if $n \neq 2^a + 2^b$, $0 \leq a \leq b$.

Theorem 63. (a) *In $H^*(\text{Spin};Z_2)$ for $n > r + 1 \geq 1$,*

$$Q_*^{2r}(i_n) = (n - r - 1, 2r - n)i_{n-r} \quad and \quad Q_*^{2r-1}(i_n) = (n - r, 2r - n - 1)i_s^{2^t}$$

where $n - r = 2^{t-1}(2s - 1)$ and $i_1 = 0$.

(b) *In $H^*(B\,\text{Spin};Z_2)$ for $n > r + 3 \geq 3$ and $n \neq 2^k + 1$,*

$$Q_*^r(w_n) = (n - r - 1, 2r - n)w_{n-r}$$

with the convention that $w_{2^k+1} = 0$ for $k \geq 2$.

Proof. η^* and $(B\eta)^*$ are epimorphisms, so this theorem follows Theorems 34 and 52.

Theorem 64. (a) *In $H_*(\text{Spin};Z_2)$ for $n \geq 3$ and $r \geq 0$,*

$$Q^r(u'_n) = (n, r - n - 1)u'_{n+r} \quad \text{modulo decomposables}.$$

(b) *In $H_*(B\,\text{Spin};Z_2)$ for $n \geq 4$, $n \neq 2^k + 1$ and $r \geq 0$,*

$$Q^r(e''_n) = (n, r - n - 1)e''_{n+r} \quad \text{modulo decomposables}$$

if n is not a power of two or a sum of two powers of two, for in that case, $Q^r(e''_n)$ is always decomposable.

Proof. If $n = 2^{k+1}$ or $n = 2^a + 2^b$, $0 < a < b$, and $k \geq 1$, and $r \geq 0$ then $Q_*^r(\mathfrak{p}_{n+r}) = 0$ and hence $Q^r(e''_n)$ is decomposable where $\{\mathfrak{p}_{n+r}\} = PH^{n+r}(B\,\text{Spin}; Z_2)$. $\eta_*(u'_n) = u_n$ modulo decomposables for $n \geq 4$ and $(B\eta_*)(e''_n) = e'_n$ modulo decomposables if $n \neq 2^{k+1}$, $n \neq 2^a + 2^b$ for all $0 < a < b$ and $k \geq 1$. Hence this theorem follows from Theorems 36 and 53.

Theorem 65. (a) $Q_{\mathscr{R}} H_*(\mathrm{Spin}; Z_2) = \{u'_{2^m+2^n} \mid 0 \leq m < n\}$.
(b) $Q_{\mathscr{R}} H_*(B\,\mathrm{Spin}; Z_2) = \{e_{2^a+2^b+2^c} \mid 0 \leq a \leq b < c\}$.

Proof. (a) From Theorem 64 we see that if $k = 2^m + 2^n h$ with $0 \leq m < n$ and h odd then $Q^{k-2^m-2^n}(u'_{2^m+2^n}) = u'_k$ modulo decomposables. Hence $\{u'_{2^m+2^n} \mid 0 \leq m < n\}$ generates $H_*(\mathrm{Spin}; Z_2)$ as an $\mathscr{R}$-algebra. By Theorem 64, it is easy to see that $u'_{2^m+2^n}$ is $\mathscr{R}$-indecomposable for $0 \leq m + n$.

(b) If $n = \sum_{i=1}^{t} 2^{n_i}$ with $t \geq 4$ then $Q^{2^{n_t}}(e''_{n-2^{n_t}}) = e''_n$ modulo decomposables. Hence $\{e''_{2^a+2^b+2^c} \mid 0 \leq a \leq b < c\}$ generates $H_*(B\,\mathrm{Spin}; Z_2)$ as an R-algebra. By Theorem 64 we see that $e''_{2^a+2^b+2^c}$ is R-indecomposable for all $0 \leq a \leq b < c$.

Let F''^* be the locally nilpotent endomorphism of $H^*(B\,\mathrm{Spin}; Z_2)$ defined by $F''^*(w_{2k}) = w_k^2$ and $F''^*(w_{2k+1}) = 0$ for $k \geq 2$ with the usual conventions. Thus, $Z_2[[F''^*]]$ is a ring of $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(B\,\mathrm{Spin}; Z_2)$.

Theorem 66. (a) *The identity and zero maps are the only $\mathfrak{A}$-Hopf algebra endomorphisms of* $H_*(\mathrm{Spin}; Z_2)$.

(b) *The natural inclusion η_* and the zero map are the only $\mathfrak{A}$-Hopf algebra maps from* $H_*(\mathrm{Spin}; Z_2)$ *to* $H_*(SO; Z_2)$.

(c) *The algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of* $H_*(B\,\mathrm{Spin}; Z_2)$ *are all homomorphisms of $\mathscr{R}$-modules, and this algebra is isomorphic to* $Z_2[[F''_*]]$.

(d) *The vector space of $\mathfrak{A}$-Hopf algebra maps from* $H_*(B\,\mathrm{Spin}; Z_2)$ *to* $H_*(BSO; Z_2)$ *are all homomorphisms of $\mathscr{R}$-modules, and this vector space is isomorphic to*:

(i) *The free right module with basis $\{B\eta_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of* $H_*(BSO; Z_2)$.

(ii) *The free left module with basis $\{B\eta_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of* $H_*(B\,\mathrm{Spin}; Z_2)$.

Proof. (a), (b). These assertions follow from Theorems 58 and 59.

(c), (d). As in the proof of Theorem 20, one first proves (d)(i). It then follows that conjugation by $B\eta^*$ is a well-defined isomorphism from the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H^*(BSO; Z_2)$ to the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H^*(B\,\mathrm{Spin}; Z_2)$. Thus, (c) follows from Theorem 39(e). Now (d)(ii) is clear.

Corollary 67. $H_*(\mathrm{Spin}; Z_2)$ *and* $H_*(B\,\mathrm{Spin}; Z_2)$ *are indecomposable $\mathfrak{A}$-Hopf algebras.*

6. Homogeneous spaces of the classical groups. We will use the results of the preceding sections to compute the action of the Dyer-Lashof algebra on the homology of U/Sp, SU/Sp, U/O, U/SO, SU/SO, Sp/O, Sp/SO, Sp/U, Sp/SU, SO/U, SO/SU, O/SU, O/U, SO/Sp and O/Sp. As applications we compute the $\mathscr{R}$-algebra indecomposables and the algebra of $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of the homology of each of these spaces. If H is a closed subgroup of a topological space G then we will let $\pi(G, H)$ denote the canonical projection from G to homogeneous space of left cosets G/H.

Define $j_n \in H_{4n-3}(U/\mathrm{Sp})$ for $n \geq 1$ by $j_n = \pi(U, \mathrm{Sp})_*(f_{2n-1})$. Then $H_*(U/\mathrm{Sp}) = E\{j_n \mid n \geq 1\}$ as Hopf algebras and $H_*(SU/\mathrm{Sp}) \cong E\{j_n \mid n \geq 2\}$ as Hopf algebras. If p is an odd prime, define $j_n \in H_{4n-3}(U/O; Z_p)$ by

$$j_n = \pi(U, O)_*(f_{2n-1}).$$

Then $H_*(U/O; Z_p) \cong H_*(U/SO; Z_p) \cong E\{j_n \mid n \geq 1\}$ as Hopf algebras and $H_*(SU/SO; Z_p) \cong E\{j_n \mid n \geq 2\}$ as Hopf algebras. We will state and prove Theorems 69 through 70 for $H_*(U/\mathrm{Sp}; Z_p)$ and $H_*(SU/\mathrm{Sp}; Z_p)$ with p any prime. However, if p is an odd prime, these theorems are also valid for $H_*(U/O; Z_p) \cong H_*(U/SO; Z_p) \cong H_*(U/\mathrm{Sp}; Z_p)$ as $\mathfrak{A}\mathscr{R}$-Hopf algebras and for $H_*(SU/SO; Z_p) \cong H_*(SU/\mathrm{Sp}; Z_p)$ as $\mathfrak{A}\mathscr{R}$-Hopf algebras.

Theorem 68. *In $H_*(U/\mathrm{Sp})$ for $n \geq 1$, $r \geq 0$ and in $H_*(SU/\mathrm{Sp})$ for $n \geq 2$, $r \geq 0$,*

$$Q^r(j_n) = (-1)^{r+1}(2n-2, r-2n+1) j_{n+(1/2)r(p-1)}$$

$$[Q^{4r}(j_n) = (n-1, r-n) j_{n+r}].$$

Proof. This theorem follows from Theorem 41 and the definition

$$j_n = \pi(U, \mathrm{Sp})_*(f_{2n-1}).$$

Theorem 69. (a) $Q_{\mathscr{R}} H_*(U/\mathrm{Sp}; Z_2) = \{j_1\}$.

(b) $Q_{\mathscr{R}} H_*(SU/\mathrm{Sp}; Z_2) = \{j_{2^n+1} \mid n \geq 0\}$.

(c) *For p an odd prime, a Z_p-basis for $Q_{\mathscr{R}} H_*(U/\mathrm{Sp}; Z_p)$ is given by $\{j_n \mid n \geq 2$, $2n-2 = p^e h$, $h \not\equiv 0 \bmod p$, $h = s(p-1)+r$, $2 \leq r \leq p-1$ and if $s \neq 0$ then $s = \sum_{i=0}^k s_i p^i$, $0 \leq s_i \leq p-1$ with $r \geq s_0 \geq \cdots \geq s_k \geq 1\} \cup \{j_1\}$.*

(d) *For p an odd prime, a Z_p-basis for $Q_{\mathscr{R}} H_*(SU/\mathrm{Sp}; Z_p)$ is given by $\{j_n \mid n \geq 2$, $2n-2 = p^e h$, $h \not\equiv 0 \bmod p$, $h = s(p-1)+r$, $1 \leq r \leq p-1$, and if $s \neq 0$ then $s = \sum_{i=0}^k s_i p^i$, $0 \leq s_i \leq p-1$ with $r \geq s_0 \geq \cdots \geq s_k \geq 1\}$.*

Proof. Clearly $Q_{\mathscr{R}} H_*(U/\mathrm{Sp}) = \pi(U, \mathrm{Sp})_*(Q_{\mathscr{R}} H_*(U))$ and $Q_{\mathscr{R}} H_*(SU/\mathrm{Sp}) = \pi(SU, \mathrm{Sp})_*(Q_{\mathscr{R}} H_*(SU))$. Hence this theorem follows from Theorem 43.

Theorem 70. (a) *The algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(U/\mathrm{Sp}; Z_2)$ is isomorphic to $Z_2 \times Z_2$.*

(b) *The identity and zero maps are the only $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(U/\mathrm{Sp}; Z_2)$.*

(c) *The identity and zero maps are the only $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(SU/\mathrm{Sp}; Z_2)$.*

(d) *If p is an odd prime then the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(U/\mathrm{Sp}; Z_p)$ is isomorphic to $\prod_{i=0}^{(1/2)(p-1)} Z_p$.*

(e) *If p is an odd prime then the algebra of $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphisms of $H_*(U/Sp; Z_p)$ is isomorphic to $\prod_{i=1}^{(1/2)(p-1)} Z_p$.*

(f) *If p is an odd prime then the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(SU/Sp; Z_p)$ is all $\mathscr{R}$-module homomorphisms and is isomorphic to $\prod_{i=0}^{(1/2)(p-3)} Z_p$.*

Proof. (a) From Theorem 44(a), it is clear that the vector space of $\mathfrak{A}$-Hopf algebra maps from $H_*(U; Z_2)$ to $H_*(U/Sp; Z_2)$ is isomorphic to $Z_2 \times Z_2$. This vector space is clearly the free module with basis $\{\pi(U, \mathrm{Sp})_*\}$ over the ring of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(U/Sp; Z_2)$.

(b) $Q^{4n}(j_1) = j_{n+1}$ for $n \geq 1$. Hence an $\mathfrak{A}\mathscr{R}$-Hopf algebra endomorphism of $H_*(U/Sp; Z_2)$ is determined by its value on j_1.

(c) This assertion follows from (a).

(d), (e), (f). $\pi(U, \mathrm{Sp})_*$ induces an isomorphism of $\mathfrak{A}\mathscr{R}$-Hopf algebras from $\bigotimes_{i=0}^{(1/2)(p-1)} H_*(U)_{p,i}$ to $H_*(U/Sp; Z_p)$ while $\pi(SU, \mathrm{Sp})_*$ induces an isomorphism of $\mathfrak{A}\mathscr{R}$-Hopf algebras from $\bigotimes_{i=0}^{(1/2)(p-1)} H_*(SU)_{p,i}$ to $H_*(SU/Sp; Z_p)$. Hence our assertions follow from Theorem 44.

Corollary 71. (a) *$H_*(U/Sp; Z_2) \cong H_*(SU/Sp; Z_2) \otimes Z_2 j_1$ as $\mathfrak{A}$-Hopf algebras, and $H_*(SU/Sp; Z_2)$ is an indecomposable $\mathfrak{A}$-Hopf algebra.*

(b) *$H_*(U/Sp; Z_2)$ is an indecomposable $\mathfrak{A}\mathscr{R}$-Hopf algebra.*

(c) *If p is an odd prime, then*

$$H_*(U/Sp; Z_p) \cong Z_p j_1 \otimes H_*(SU)_{p,0} \otimes \bigotimes_{i=1}^{(1/2)(p-3)} H_*(U)_{p,2i}$$

and

$$H_*(SU/Sp; Z_p) \cong \bigotimes_{i=0}^{(1/2)(p-3)} H_*(SU)_{p,2i}$$

as decompositions of $H_*(U/Sp; Z_p)$ and $H_*(SU/Sp; Z_p)$ into indecomposable $\mathfrak{A}$-Hopf algebras.

(d) *If p is an odd prime then $H_*(U/Sp; Z_p) \cong \bigotimes_{i=0}^{(1/2)(p-3)} H_*(U)_{p,2i}$ and $H_*(SU/Sp; Z_p) \cong \bigotimes_{i=0}^{(1/2)(p-3)} H_*(SO)_{p,2i}$ are decompositions of $H_*(U/Sp; Z_p)$ and $H_*(SU/Sp; Z_p)$ into indecomposable $\mathfrak{A}\mathscr{R}$-Hopf algebras.*

Let λ: $U/O \to BO$ be the canonical map. Then λ induces a monomorphism in homology with Z_2-coefficients. Define $k_n \in H_{2n-1}(U/O; Z_2)$ by $k_n = \lambda_*^{-1}(\mathfrak{p}_{2n-1})$ for $n \geq 1$. Then $H_*(U/O; Z_2) = P\{k_n \mid n \geq 1\}$ as Hopf algebras, $H_*(U/SO; Z_2) \cong P\{k_{3/2}, k_n \mid n \geq 2\} \otimes E\{x\}$ as Hopf algebras and $H_*(SU/SO; Z_2) \cong P\{k_{3/2}, k_n \mid n \geq 2\}$ as Hopf algebras where $k_{3/2} = \lambda_*'^{-1}(\mathfrak{p}_2)$, λ': $U/SO \to BSO$ and $x = \pi(U, SO)_*(f_1) \in H_1(U/SO; Z_2)$.

Theorem 72. *In $H_*(U/O; Z_2)$ if $r \geq 0$, $n \geq 1$ and in $H_*(U/O; Z_2)$ or $H_*(SU/SO; Z_2)$ if $r \geq 0$, $n \geq 1$,*

$$Q^r(k_{3/2}) = (r+1)k_t^{2^s} \quad \text{and} \quad Q^r(k_n) = (n-1, r-n)k_t^{2^s}$$

where $r + 2n - 1 = 2^s(2t - 1)$ *in* $H_*(U/O; Z_2)$ *or if* $t > 1$, *and* $r + 2n - 1 = 2^{s+1}$, $t = 3/2$ *otherwise. In* $H_*(U/SO; Z_2)$ *all Dyer-Lashof operations on* x *are zero.*

Proof. This theorem for $H_*(U/O; Z_2)$ and $H_*(SU/SO; Z_2)$ follows from Corollary 35 and the fact that λ_*, λ'_* are monomorphisms. Since $H_*(SU/SO; Z_2) \to H_*(U/SO; Z_2)$ is a monomorphism, $Q^r(k_n)$ is as claimed in $H_*(U/SO; Z_2)$. For $r \geq 0$, $Q^r(x)$ is primitive and in the kernel of λ'_*. Hence $Q^r(x) = 0$.

Theorem 73. (a) $Q_{\mathscr{R}} H_*(U/O; Z_2) = \{k_1\}$.

(b) $Q_{\mathscr{R}} H_*(U/SO; Z_2) = \{k_{2^n+1} \mid n \geq 0\} \cup \{k_{3/2}, x\}$.

(c) $Q_{\mathscr{R}} H_*(SU/SO; Z_2) = \{k_{2^n+1} \mid n \geq 0\} \cup \{k_{3/2}\}$.

Proof. (a) $Q^{2n}(k_1) = k_{n+1}$ for $n \geq 1$. Hence k_1 generates $H_*(U/O; Z_2)$ as an $\mathscr{R}$-algebra.

(b),(c). $\lambda'^{-1}_* \circ \sigma^{SO}_*$ induces an isomorphism from $Q_{\mathscr{R}} H_*(SO; Z_2)$ to $Q_{\mathscr{R}} H_*(SU/SO; Z_2)$. Thus, (c) follows from Theorem 57. Clearly $Q_{\mathscr{R}} H_*(U/SO; Z_2) = Q_{\mathscr{R}} H_*(SU/SO; Z_2) \cup \{x\}$.

Theorem 74. (a) *The identity and zero maps are the only* $\mathfrak{A}$*-Hopf algebra endomorphisms of* $H_*(U/O; Z_2)$ *and of* $H_*(SU/SO; Z_2)$.

(b) *The algebra of* $\mathfrak{A}$*-Hopf algebra endomorphisms of* $H_*(U/SO; Z_2)$ *are all* $\mathscr{R}$*-module homomorphisms and this algebra is isomorphic to* $Z_2 \times Z_2$.

Proof. (a) λ_* is an isomorphism of $\mathfrak{A}$-Hopf algebras from $H_*(U/O; Z_2)$ to $PH_*(BO; Z_2)$. If f is an $\mathfrak{A}$-Hopf algebra endomorphism of $H_*(U/O; Z_2)$ then $f_*(k_n)$ is primitive and hence equals either k_n or zero. Hence the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(U/O; Z_2)$ is a product of Z_2's, one factor for each k_n which is annihilated by $I\mathfrak{A}^{\text{op}}$. However, such k_n's correspond under λ_* and dualization to elements of odd degree in $Q_{\mathfrak{A}} H^*(BO; Z_2) = \{w_{2^k} \mid k \geq 0\}$. Now make this same argument with BO replaced by BSO to show that the algebra of $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(SU/SO; Z_2)$ is a product of Z_2's one factor for each element of degree two or of odd degree in $Q_{\mathfrak{A}} H^*(BSO; Z_2) = \{w_{2^k} \mid k \geq 1\}$.

(b) $H_*(U/SO; Z_2)$ is clearly isomorphic to $H_*(SU/SO; Z_2) \otimes E\{x\}$ as $\mathfrak{A}\mathscr{R}$-Hopf algebras.

Corollary 75. (a) $H_*(U/O; Z_2)$ *and* $H_*(SU/SO; Z_2)$ *are indecomposable* $\mathfrak{A}$*-Hopf algebras.*

(b) $H_*(U/SO; Z_2) \cong H_*(SU/SO; Z_2) \otimes E\{x\}$ *as* $\mathfrak{A}\mathscr{R}$*-Hopf algebras.*

The preceding considerations also apply to $H_*(\mathrm{Sp}/O; Z_2)$ and $H_*(\mathrm{Sp}/SO; Z_2)$. Let $\bar{\lambda}$: $\mathrm{Sp}/O \to BO$ and $\bar{\lambda}'$: $\mathrm{Sp}/SO \to BSO$ be the canonical maps. Then $\bar{\lambda}_*$ is a monomorphism, and let $\alpha_i = \bar{\lambda}^{-1}_*(\mathfrak{p}_{2i-1})$, $i \geq 1$, and $\gamma_i = \bar{\lambda}^{-1}_*(\gamma_2(\mathfrak{p}_{2i-1}))$, $i \geq 1$.

Then $H_*(\mathrm{Sp}/O; Z_2) = P[\alpha_i, \gamma_i | i \geq 1]$ as algebras with the α_i primitive and $\psi(\gamma_i) = \gamma_i \otimes 1 + \alpha_i \otimes \alpha_i + 1 \otimes \gamma_i$. Similarly, if $y = \pi(\mathrm{Sp}, SO)_*(g_1)$ then $H_*(\mathrm{Sp}/SO; Z_2) = P[\alpha'_i, \gamma'_i, \alpha'_{3/2}, \gamma'_{3/2} | i \geq 2] \otimes E[y]$ with y, α'_i primitive and $\psi(\gamma'_i) = \gamma'_i \otimes 1 + \alpha'_i \otimes \alpha'_i + 1 \otimes \gamma'_i$. Then by Corollary 35,

$$Q^r(\alpha_n) = (2n-2, r-2n+1)\alpha_s^{2^t} \text{ and } Q^r(\gamma_n) = (2n-2, r-2n+1)\gamma_s^{2^t}$$

for $r \geq 1, n \geq 1$ and $r + 2n - 1 = 2^t(2s-1)$. The above formulas also hold for $Q^r(\alpha'_n)$ and $Q^r(\gamma'_n)$ if $n > 1$ with the conventions that $(\alpha'_1)^2 = \alpha'_{3/2}$ and $(\gamma'_1)^2 = \gamma'_{3/2}$. Also, note that $y = \pi(\mathrm{Sp}, SO)_*(g_1)$, and hence $Q^r(y) = 0$ for all $r \geq 0$.

Let ξ: $\mathrm{Sp}/U \to BU$ be the canonical map. Then ξ_* is a monomorphism in homology. Define $n_i \in H_{4i-2}(\mathrm{Sp}/U; Z_2)$ for $i \geq 1$ by $n_i = \xi_*^{-1}(\mathfrak{p}_{2i-1})$. Then $H_*(\mathrm{Sp}/U; Z_2) = P\{n_i \mid i \geq 1\}$ as Hopf algebras, and $H_*(\mathrm{Sp}/SU; Z_2) = P\{n_{3/2}, n_i \mid i \geq 2\}$ as Hopf algebras.

Theorems 76 through 79 are proved in exactly the same way as Theorems 72 through 75.

Theorem 76. *In $H_*(\mathrm{Sp}/U; Z_2)$ if $r \geq 0$ and $k \geq 1$ and in $H_*(\mathrm{Sp}/SU; Z_2)$ if $r \geq 0$, $k \geq 2$,*

$$Q^{2r}(n_{3/2}) = (r+1)n_t^{2^s},$$
$$Q^{2r}(n_k) = (k-1, r-k)n_t^{2^s}$$

where $r + 2k - 1 = 2^s(2t-1)$ in $H_*(\mathrm{Sp}/U; Z_2)$ or if $t > 1$, and $r + 2k - 1 = 2^{s+1}$, $t = 3/2$ otherwise.

Theorem 77. (a) $Q_{\mathscr{R}} H_*(\mathrm{Sp}/U; Z_2) = \{n_1\}$.
(b) $Q_{\mathscr{R}} H_*(\mathrm{Sp}/SU; Z_2) = \{n_{2^i+1} \mid i \geq 0\} \cup \{n_{3/2}\}$.

Theorem 78. *The identity and zero maps are the only $\mathfrak{A}$-Hopf algebra endomorphisms of $H_*(\mathrm{Sp}/U; Z_2)$ and of $H_*(\mathrm{Sp}/SU; Z_2)$.*

Corollary 79. *$H_*(\mathrm{Sp}/U; Z_2)$ and $H_*(\mathrm{Sp}/SU; Z_2)$ are indecomposable $\mathfrak{A}$-Hopf algebras.*

If p is an odd prime then ξ: $\mathrm{Sp}/U \to BU$ induces a monomorphism in homology with Z_p coefficients. Define $m_k \in H_{4k-2}(\mathrm{Sp}/U; Z_p)$ by induction on k. Let $m_1 = \xi_*^{-1}(a_1)$ and $m_{k+1} = \xi_*^{-1}(a_{2k+1} - \sum_{i=1}^{k} m_i a_{2k-2i+2})$. Then $H_*(\mathrm{Sp}/U; Z_p) = P\{m_k \mid k \geq 1\}$ as algebras. ξ^* is an epimorphism, and $H^*(\mathrm{Sp}/U; Z_p) = H^*(BU)/I$ where I is the ideal generated by $\{\sum_{i=0}^{n} (-1)^i c_i c_{2n-i} \mid n \geq 1\}$. Hence $H^*(\mathrm{Sp}/U; Z_p)$ is generated as an algebra by the images of the odd Chern classes, so we know the action of $\mathscr{R}^{\mathrm{op}}$ on $H^*(\mathrm{Sp}/U; Z_p)$ by Theorem 7. Recall that $H_*(\mathrm{Sp}/SU; Z_p) \cong P\{m_k \mid k \geq 2\}$ as algebras and $H^*(\mathrm{Sp}/SU; Z_p)$ is isomorphic to the quotient algebra of $H^*(\mathrm{Sp}/U; Z_p)$ by the ideal generated by the first Chern class. Thus, we also know the action of $\mathscr{R}^{\mathrm{op}}$ on $H^*(\mathrm{Sp}/SU; Z_p)$. For p still an odd prime, $H_*(SO/U; Z_p) \cong H_*(\mathrm{Sp}/U; Z_p)$ as $\mathfrak{A}\mathscr{R}$-Hopf algebras and $H_*(SO/SU; Z_p) \cong H_*(\mathrm{Sp}/SU; Z_p)$ as $\mathfrak{A}\mathscr{R}$-Hopf algebras. Hence the preceding remarks apply to $H_*(SO/U; Z_p)$ and $H_*(SO/SU; Z_p)$ as well.

Theorem 80. *Let p be an odd prime. In $H_*(\mathrm{Sp}/U; Z_p) \cong H_*(SO/U; Z_p)$ for $r \geq 0$, $k \geq 1$ and in $H_*(\mathrm{Sp}/SU; Z_p) \cong H_*(SO/SU; Z_p)$ for $r \geq 0$, $k \geq 2$,*

$$Q^{4r}(m_k) = (2k-1, 4r-2k)m_{k+r} \quad \textit{modulo decomposables.}$$

Proof. ξ_* induces a monomorphism on indecomposables. Hence this theorem follows from Theorem 6.

Theorem 81. *Let p be an odd prime.*

(a) *A Z_p-basis for $Q_{\mathscr{R}}H_*(\mathrm{Sp}/U; Z_p) \cong Q_{\mathscr{R}}H_*(SO/U; Z_p)$ is given by $\{m_k \mid 2k-1 = hp^e,\ h \not\equiv 0 \bmod p,\ h = s(p-1)+r,\ 1 \leq r \leq p-1$, and if $s \neq 0$ then $s = \sum_{i=0}^n s_i p^i,\ 0 \leq s_i \leq p-1$ with $r \geq s_0 \geq \cdots \geq s_n \geq 1\}$.*

(b) *A Z_p-basis for $Q_{\mathscr{R}}H_*(\mathrm{Sp}/SU; Z_p) \cong Q_{\mathscr{R}}H_*(SO/SU; Z_p)$ is given by $\{m_k \mid m_k \in Q_{\mathscr{R}}H_*(\mathrm{Sp}/U; Z_p)$ and $k \geq 2\} \cup \{m_{p^e[s(p-1)+1]} \mid e \geq 0,\ s \geq 0,\ (e,s) \neq (0, 0)\}$.*

Proof. This theorem follows from Theorems 10 and 19.

Recall from Theorem 14 that for p an odd prime $H^*(BU; Z_p) \cong \bigotimes_{i=0}^{p-2} H^*(BU_{p,i}; Z_p)$ as $\mathfrak{A}\mathscr{R}$-Hopf algebras, and I is the ideal generated by the image of $\alpha^*: H^*(B\,\mathrm{Sp}; Z_p) \to H^*(BU; Z_p)$. That is, I is the ideal generated by $\bigotimes_{i=0}^{(1/2)(p-3)} H^*(BU_{p,2i}; Z_p)$. Hence $H^*(\mathrm{Sp}/U; Z_p) \cong \bigotimes_{i=0}^{(1/2)(p-3)} H^*(BU_{p,2i-1}; Z_p)$ as $\mathfrak{A}\mathscr{R}$-Hopf algebras. We thus have deduced the following theorem from Theorem 14.

Theorem 82. *Let p be an odd prime. Every $\mathfrak{A}$-Hopf algebra endomorphism of $H_*(\mathrm{Sp}/U; Z_p) \cong H_*(SO/U; Z_p)$ or of $H_*(\mathrm{Sp}/SU; Z_p) \cong H_*(SO/SU; Z_p)$ is a homomorphism of $\mathscr{R}$-modules and this algebra is isomorphic in both cases to $Z_p[[F_1, F_3, \ldots, F_{p-4}, F_{p-2}]]$.*

$\pi(SO, U)^*: H^*(SO/U; Z_2) \to H^*(SO; Z_2)$ is a monomorphism. Define $s_t \in H^{4t-2}(SO/U; Z_2)$ by $s_t = \pi(SO, U)^{*-1}(i_t^2)$. Then $H^*(SO/U; Z_2) = P\{s_t \mid t \geq 1\}$ as Hopf algebras. Let $V_t = \pi(SO, u)_*^{-1}(u_{2t}) \in H_{2t}(SO/U; Z_2)$. Then $H_*(SO/U; Z_2) = E\{V_t \mid t \geq 1\}$ as algebras and $\psi(V_t) = \sum_{j=0}^t V_j \otimes V_{t-j}$. Also, recall that $H^*(SO/SU; Z_2) \cong P\{i_1, s_t \mid t \geq 2\}$ and $H_*(SO/SU; Z_2) \cong E\{u_1, u_2, V_t \mid t \geq 2\}$ as algebras with u_1 primitive, $\psi(u_2) = u_2 \otimes 1 + u_1 \otimes u_1 + 1 \otimes u_2$ and $\psi(V_t) = \sum_{j=0}^t V_j \otimes V_{t-j}$. Note that for $n \geq 1$,

$$\pi(SO, SU)_*(u_{2n}) = V_n, \quad \pi(SO, SU)_*(u_{2n-1}) = V_{n-1}u_1$$

with the convention that $V_1 = u_2$.

Theorem 83. *In $H^*(SO/U; Z_2)$ for $r \geq 0$, $t \geq 1$ and in $H^*(SO/SU; Z_2)$ for $r \geq 0$, $t \geq 2$,*

$$Q_*^{4r}(s_t) = (t-r-1, 2r-t)s_{t-r} \quad \textit{and} \quad Q_*^{4r+2}(s_t) = (t-r-1, 2r-t+1)s_n^{2^k}$$

where $t + r = 2^{k-1}(2n - 1)$.

Proof. This theorem follows from Theorem 52 since $\pi(SO, U)^*$ and $\pi(SO, SU)^*$ are monomorphisms.

Theorem 84. (a) *In* $H_*(SO/U; Z_2)$ *for* $r \geq 0$, $t \geq 1$,

$$Q^{2r}(V_t) = \sum_{a+b+c=r+t} (t - a, r - t - b - 1) V_a V_b V_c.$$

(b) *In* $H_*(SO/SU; Z_2)$ *for* $r \geq 0$, $t \geq 1$,

$$Q^{2r}(V_t) = \sum_{a+b+c=r+t} (t - a, r - t - b - 1) V_a V_b V_c,$$

$$Q^{2r+1}(V_t) = \sum_{j=1}^{t+r-3} \sum_{a+b=t+r-j} [(t - j - 1, r - t - b) + (t - b, r - t - j - 1) + (t - b, r - t - a)] V_a V_b V_j u_1,$$

$$Q^{2r+1}(u_1) = (r + 1) \sum_{a+b=r+1} V_a V_b \quad \text{and} \quad Q^{2r}(u_1) = \sum_{a+b=r} V_a V_b u_1$$

with the convention $V_1 = u_2$.

Proof. This theorem follows from Theorem 56 since $\pi(SO, U)_*$ and $\pi(SO, SU)_*$ are onto.

Theorem 85. (a) $Q_{\mathscr{R}} H_*(SO/U; Z_2) = \{V_{2^n} \mid n \geq 0\}$.

(b) $Q_{\mathscr{R}} H_*(SO/SU; Z_2) = \{V_{2^n} \mid n \geq 1\} \cup \{u_1, u_2\}$.

Proof. Clearly $Q_{\mathscr{R}} H_*(SO/U; Z_2) = \pi(SO, U)_*(Q_{\mathscr{R}} H_*(SO; Z_2))$ and $Q_{\mathscr{R}} H_*(SO/SU; Z_2) = \pi(SO, SU)_*(Q_{\mathscr{R}} H_*(SO; Z_2))$. Hence this theorem is a consequence of Theorem 57.

Theorem 86. (a) *The identity and zero maps are the only* $\mathfrak{A}$*-Hopf algebra endomorphisms of* $H_*(SO/U; Z_2)$.

(b) *The algebra of* $\mathfrak{A}$*-Hopf algebra endomorphisms of* $H_*(SO/SU; Z_2)$ *is isomorphic to* $Z_2 \times Z_2$.

(c) *The identity and zero maps are the only* $\mathfrak{A}\mathscr{R}$*-Hopf algebra endomorphisms of* $H_*(SO/U; Z_2)$.

Proof. This theorem can be derived from Theorem 58.

Corollary 87. (a) $H_*(SO/U; Z_2)$ *is an indecomposable* $\mathfrak{A}$*-Hopf algebra.*

(b) $H^*(SO/SU; Z_2) \cong P\{i_1\} \otimes P\{V_t \mid t \geq 2\}$ *is a decomposition of* $H^*(SO/SU; Z_2)$ *into indecomposable* $\mathfrak{A}$*-Hopf algebras.*

(c) $H_*(SO/SU; Z_2)$ *is an indecomposable* $\mathfrak{A}\mathscr{R}$*-Hopf algebra.*

Note that $H_*(O/U) \cong H_*(SO/U) \otimes Z_p(Z_2)$ and $H_*(O/SU) \cong H_*(SO/SU) \otimes Z_p(Z_2)$ as Hopf algebras, where $Z_p(Z_2)$ is the group algebra of Z_2 over Z_p. For x an element of $H_*(O/U)$ or $H_*(O/SU)$ let x and $x \otimes [-1]$ designate elements in the homology of the two components. We will have complete information about the action of the Dyer-Lashof algebra on the homology of these two spaces as soon as we know $Q^r(1 \otimes [-1])$ for $r \geq 0$.

Theorem 88. *In $H_*(O/U)$ or $H_*(O/SU)$ for $n \geq 1$, $Q^0(1 \otimes [-1]) = 1 \otimes [-1]$ and $Q^n(1 \otimes [-1]) = 0$ [$Q^{2n}(1 \otimes [-1]) = V_n$ and*

$$Q^{2n-1}(1 \otimes [-1]) = 0 \quad \textit{in } H_*(O/U; Z_2)$$

while $Q^{2n}(1 \otimes [-1]) = V_n$ and $Q^{2n-1}(1 \otimes [-1]) = V_{n-1} u_1$ in $H_*(O/SU; Z_2)$ with the convention $V_1 = u_2$].

Proof. $\pi(O, U)_*(1 \otimes [-1]) = 1 \otimes [-1]$ and $\pi(O, SU)_*(1 \otimes [-1]) = 1 \otimes [-1]$. Hence this theorem follows from Theorem 61.

The preceding considerations also apply to $H_*(SO/\mathrm{Sp}; Z_2)$ and $H_*(O/\mathrm{Sp}; Z_2)$. Note that $\pi(SO, \mathrm{Sp})_*$ is an epimorphism. Let $u'_n = \pi(SO, \mathrm{Sp})_*(u_n)$. Then $H_*(SO/\mathrm{Sp}; Z_2) = E[u'_n | n \geq 1 \text{ and } n \not\equiv 3 \bmod 4]$ as algebras with $\psi(u'_n) = \sum_{i=0}^{n} u'_i \otimes u'_{n-i}$ using the convention $u'_{4k+3} = 0$, $k \geq 0$.

By Theorem 56, for $r \geq 0$ and $n \geq 1$,

$$Q^r(u'_n) = \sum_{\substack{a,b,c \geq 0 \\ a+b+c=r+n}} (n - a, r - n - b - 1) u'_a u'_b u'_c.$$

Observe that $H_*(O/\mathrm{Sp}; Z_2) = H_*(SO/\mathrm{Sp}; Z_2) \otimes Z_2(Z_2)$ and $Q^r(1 \otimes [-1]) = u'_r$ for $r > 0$ by Theorem 61.

7. Im J, B Im J **and** $BBSO$. We will calculate the action of the Dyer-Lashof algebra on $H_*(\mathrm{Im}\, J; Z_p)$ and $H_*(B\, \mathrm{Im}\, J; Z_p)$ for p an odd prime and on $H_*(BBSO; Z_p)$ for all primes p. In particular, J. Stasheff [22] showed that $H^*(B\, \mathrm{Im}\, J; Z_p) \cong H^*(BO)_{p,0} \otimes H^*(SU)_{p,0}$ and $H^*(\mathrm{Im}\, J; Z_p) \cong H^*(BO)_{p,0} \otimes H^*(SO)_{p,p-2}$ as Hopf algebras over the subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ generated by $\{\mathcal{P}^n \mid n \geq 0\}$. We will show that these decompositions are also valid over the subalgebra $\mathcal{R}'$ of $\mathcal{R}$ generated by $\{Q^n \mid n \geq 0\}$.

Recall from J. Stasheff [22] that there is a fibration $BUQ_p \xrightarrow{BJ} B\, \mathrm{Im}\, JQ_p \xrightarrow{BT} BBUQ_p$ and $BBUQ_p \simeq SUQ_p$ where $B\, \mathrm{Im}\, J = \prod_{p \text{ prime}} B\, \mathrm{Im}\, JQ_p$, $H^*(B\, \mathrm{Im}\, J; Z_p) = H^*(B\, \mathrm{Im}\, JQ_p; Z_p)$ and

$$\tilde{H}^*(B\, \mathrm{Im}\, JQ_p; Z_p) = 0$$

if q is prime, $q \neq p$. In both $H^*(BU; Z_p)$ and $H^*(B\, \mathrm{Im}\, J; Z_p)$ there are defined Wu classes q_n of degree $2n(p-1)$ for $n \geq 1$. If $t \geq n$ and

$$\Phi_t\colon H^i(BU(t); Z_p) \to H^{i+2t}(MU(t); Z_p)$$

is the Thom isomorphism given by $\Phi_t(x) = xU_t$ then $q_n = \Phi_t^{-1} \circ \mathcal{P}^n \circ \Phi_t(1)$. We have that $(BJ)^*(q_n) = q_n$ for $n \geq 1$ and

$$H^*(B\, \mathrm{Im}\, J; Z_p) = P\{q_n \mid n \geq 1\} \otimes E\{\beta q_n \mid n \geq 1\}$$

as algebras with $\psi(q_n) = \sum_{i=0}^{n} q_i \otimes q_{n-i}$ and $q_0 = 1$. Let $x_n = BT^*(f^*_{n(p-1)+1})$ for $n \geq 1$, so the x_n are primitive. Then $H^*(B\,\mathrm{Im}\,J; Z_p) = P\{q_n \mid n \geq 1\} \otimes E\{x_n \mid n \geq 1\}$ which is Stasheff's splitting of $H^*(B\,\mathrm{Im}\,J; Z_p)$ as Hopf algebras over $\mathfrak{A}'$ since $BJ^*\colon P\{q_n \mid n \geq 1\} \cong H^*(BO)_{p,0}$ and $BT^*\colon H^*(SU)_{p,0} \cong E\{x_n \mid n \geq 1\}$. We begin our investigation with the following useful lemma.

Lemma 89. *In the basis of $H^*(BU; Z_p)$ which is dual to the basis of monomials in the a_k, $q_n = (a^n_{p-1})^*$ for $n \geq 1$.*

Proof. Observe that if $y \in H^{2n}(BU; Z_p)$ and $n \equiv 0 \bmod p-1$ then $\bar{\psi}(y) = \psi(y) - y \otimes 1 - 1 \otimes y \in H^*(BU_{p,0}; Z_p) \otimes H^*(BU_{p,0}; Z_p)$ implies that $y \in H^{2n}(BU)_{p,0}$. Hence $(a^n_{p-1})^*$ and q_n are in $H^*(BU)_{p,0}$ for $n \geq 1$ because $a^*_{p-1} = \mathfrak{p}_{p-1} = q_1 \in H^{2(p-1)}(BU_{p,0}; Z_p)$. The last fact follows from $\Phi_1(q_1) = \mathcal{P}^1(U_1) = U_1^p = \Phi_1(c_1^{p-1}) = \Phi_1(\mathfrak{p}_{p-1})$. Thus, $H^*(BU_{p,0}; Z_p) = P\{q_n \mid n \geq 1\} = P\{(a^n_{p-1})^* \mid n \geq 1\}$ as algebras. Define a Hopf algebra automorphism S of $H^*(BU_{p,0}; Z_p)$ by $S(q_n) = (a^n_{p-1})^*$ for $n \geq 1$. We will show below that S is a homomorphism of $\mathfrak{A}$-modules. Hence S is the identity map by Theorem 30: $S = 1 + F_0^* G$ because S is an invertible element in $Z_p[[F_0^*]]$. However, if $G \neq 0$ then $S(q_{p^t})$ contains $\lambda c_1^{p^t(p-1)}$ for some $t \geq 1$ and $0 \neq \lambda \in Z_p$ when $S(q_{p^t})$ is written out in the basis of $H^*(BU; Z_p)$ consisting of monomials in the Chern classes. This is clearly absurd since $S(q_{p^t}) = (a^{p^t}_{p-1})^*$. Thus, S is the identity map and $q_n = (a^n_{p-1})^*$.

It remains to show that S is a homomorphism of $\mathfrak{A}$-modules. Let t be large and assume that $r \leq (p-1)n$. Then as in J. Milnor [20, p. 56] we see that

$$\begin{aligned}
\mathcal{P}^r(q_n)U_t &= \mathcal{P}^r(q_n U_t) - \sum_{i=0}^{r-1} \mathcal{P}^i(q_n)\mathcal{P}^{r-i}(U_t) \\
&= \mathcal{P}^r \circ \mathcal{P}^n(U_t) - \sum_{i=0}^{r-1} \mathcal{P}^i(q_n)q_{r-i}U_t \\
&= \sum_{j=0}^{[r/p]} (-1)^{r+j}(r-pj, (p-1)n - r + j - 1)\mathcal{P}^{r+n-j}(q_j U_t) \\
&\quad - \sum_{i=0}^{r-1} \mathcal{P}^i(q_n)q_{r-i}U_t \\
&= \sum_{j=0}^{[r/p]} \sum_{k=0}^{r+n-j} (-1)^{r+j}(r-pj, (p-1)n - r + j - 1)\mathcal{P}^k(q_j)q_{r+n-j-k}U_t \\
&\quad - \sum_{i=0}^{r-1} \mathcal{P}^i(q_n)q_{r-i}U_t.
\end{aligned}$$

From this equation it is easy to prove by induction on $n + r$ that for $n \geq 2$, $\mathcal{P}^r(q_n)$ when written as a polynomial in the q_k does not contain a nonzero multiple of q_1^{n+r} as a summand. Hence if $n \geq 2$ then neither $S \circ \mathcal{P}^r(q_n)$ nor $\mathcal{P}^r((a^n_{p-1})^*)$ when written as a polynomial in the Chern classes contains a nonzero multiple of $c_1^{(r+n)(p-1)}$ as a summand. Now we can prove that $S \circ \mathcal{P}^r(q_n)$

$= \mathcal{P}^r \circ S(q_n)$ for $n \geq 1$ and $r \geq 0$ by induction on $r + n$. The assertion is clearly true if $n = 1$ or $r = 0$. If $n > 1$ and $r > 0$ then by the induction hypothesis $\bar{\psi} \circ S \circ \mathcal{P}^r(q_n) = \bar{\psi} \circ \mathcal{P}^r \circ S(q_n)$, and hence $S \circ \mathcal{P}^r(q_n) - \mathcal{P}^r \circ S(q_n) = \alpha \mathfrak{p}_{(r+n)(p-1)}$ for some $\alpha \in Z_p$. However, $\mathfrak{p}_{(r+n)(p-1)}$ contains $c_1^{(r+n)(p-1)}$ as a summand so by the above remarks $\alpha = 0$. Thus, S is a homomorphism of $\mathfrak{A}$-modules.

Theorem 90. *For p an odd prime,*

$$H^*(B\operatorname{Im} J; Z_p) \cong H^*(BU_{p,0}; Z_p) \otimes H^*(SU)_{p,0}$$

as $\mathfrak{A}'\mathcal{R}'$-Hopf algebras. In particular for $r \geq 0$ and $n \geq 1$,

(1) $Q_*^r(q_n) = ((n - r)(p - 1) - 1, pr - n(p - 1))q_{n-r}$.

(2) $Q_*^r(x_n) = (-1)^{r+1}((n - r)(p - 1), pr - n(p - 1) - 1)x_{n-r}$ *if* $r < n$ *and* $Q_*^n(x_n) = 0$.

Proof. The value of $Q_*^r(x_n)$ follows from Corollary 42 and the definition $BT^*(f_{n(p-1)+1}^*) = x_n$. We next show that in $H^*(B\operatorname{Im} J; Z_p)$, $Q_*^r(q_n)$ is a polynomial in the q_k. This will prove that Stasheff's splitting is a splitting as $\mathcal{R}'$-Hopf algebras. We prove our assertion by induction on n and for fixed n by induction on r. If $n = 1$ or $r = 0$ then $Q_*^r(q_n) = 0$. By our induction hypothesis and the comultiplicative Cartan formula, $\psi \circ Q_*^r(q_n)$ is a sum of tensor products of polynomials in the q_k. Hence $Q_*^r(q_n)$ is also a polynomial in the q_k because all primitive elements in $H^m(B\operatorname{Im} J; Z_p)$ for $m \equiv 0 \bmod p - 1$ are in $P\{q_n \mid n \geq 1\}$. We now see that if (1) is valid in $H^*(BU; Z_p)$ then this equation is also valid in $H^*(B\operatorname{Im} J; Z_p)$. We prove (1) in $H^*(BU; Z_p)$ by induction on n and for fixed n by induction on r. If $n = 1$ or $r = 0$ then $Q_*^r(q_n) = 0$, while $Q_*^{n-1}(q_n) = (p - 2, n - p)q_1$ by Theorem 7 since $q_n = (-1)^{n+1}c_{n(p-1)}$ modulo decomposables. Now assume that $n - 1 > r \geq 1$. By the induction hypothesis and the comultiplicative Cartan formula,

$$Q_*^r(q_n) - ((n - r)(p - 1) - 1, pr - n(p - 1))q_{n-r} = \alpha \mathfrak{p}_{(n-r)(p-1)}$$

for some $\alpha \in Z_p$. $\mathfrak{p}_{(n-1)(p-1)} = a_{(n-1)(p-1)}^*$ and, by Lemma 89, $q_n = (a_{p-1}^n)^*$ and $q_{n-r} = (a_{p-1}^{n-r})^*$. Hence $Q^r(a_{(n-r)(p-1)})$ contains αa_{p-1}^n as a summand in $H_*(BU; Z_p)$. Thus, $\alpha = 0$ by Theorem 8 since $(n - r)(p - 1) > p - 1$.

The following is a summary of J. Stasheff's computation of $H^*(\operatorname{Im} J; Z_p)$ for odd primes p (see [22]). The following diagram commutes and the columns are fibrations.

$$\begin{array}{ccccc} U & \xrightarrow{J} & \operatorname{Im} J & \xrightarrow{T} & BU \\ \downarrow & & \downarrow & & \downarrow \\ PBU & \to & P\operatorname{Im} J & \to & PBBU \\ \downarrow & & \downarrow & & \downarrow \\ BU & \xrightarrow{BJ} & B\operatorname{Im} J & \xrightarrow{BT} & BBU \simeq SU \end{array}$$

The Eilenberg-Moore spectral sequences which converge to the cohomology of the fibers of the above fibrations collapse. Thus, $H^*(\operatorname{Im} J; Z_p) = P\{r_n \mid n \geq 1\} \otimes E\{s_n \mid n \geq 1\}$ as algebras where $r_n \in H^{2n(p-1)}(\operatorname{Im} J; Z_p)$ is defined by $r_n = T^*(q_n)$ and $s_n \in H^{2n(p-1)-1}(\operatorname{Im} J; Z_p)$ is defined by $s_n = \sigma^*_{\operatorname{Im} J}(q_n)$. Hence the s_n are primitive, and $\psi(r_n) = \sum_{i=0}^{n} r_i \otimes r_{n-i}$ for $n \geq 1$. Clearly, $J^*(r_n) = 0$ and $J^*(s_n) = f^*_{n(p-1)}$ for $n \geq 1$. As a consequence of all this, we have that

$$H^*(\operatorname{Im} J; Z_p) \cong H^*(BU_{p,0}; Z_p) \otimes H^*(U)_{p,p-2}$$

as $\mathfrak{A}'$-Hopf algebras.

Theorem 91. *For p an odd prime,*

$$H^*(\operatorname{Im} J; Z_p) \cong H^*(BU_{p,0}; Z_p) \otimes H^*(U)_{p,p-2}$$

as $\mathfrak{A}'\mathscr{R}'$-Hopf algebras. In particular, for $k \geq 0$ and $n \geq 1$,

$$Q^k_*(r_n) = ((n-k)(p-1) - 1, pk - n(p-1))r_{n-k}$$

and

$$Q^k_*(s_n) = (-1)^k((n-k)(p-1) - 1, pk - n(p-1))s_{n-k}.$$

Proof. We compute $Q^k_*(r_n)$ by Theorem 90 and the definition $r_n = T^*(q_n)$ for $n \geq 1$. s_n is primitive, so $Q^k_*(s_n)$ is also primitive, and hence $Q^k_*(s_n)$ is a Z_p-multiple of s_{n-k}. Thus, $Q^k_*(s_n)$ is determined by applying J^* and then using Corollary 42.

We will now compute the action of the Dyer-Lashof algebra on $H^*(BBSO)$. If p is an odd prime then $H^*(BBSO; Z_p) \cong H^*(SU/SO; Z_p)$ as $\mathfrak{A}\mathscr{R}$-Hopf algebras, and hence Theorems 68 through 71 are applicable to $H^*(BBSO; Z_p)$. We recall some of the work of R. Clough [5] and J. Stasheff [23], on $H_*(BBSO; Z_2)$. There is a fibration

$$SU \xrightarrow{\alpha} BBSO \xrightarrow{\gamma} B\,\mathrm{Spin}\,.$$

Define $e_n \in H^n(BBSO; Z_2)$ for $n \geq 4$ and $n \neq 2^k + 1$ by $e_n = \gamma^*(w_n)$. Clearly $\bar{\psi}(e_n) = \sum_{i=4}^{n-4} e_i \otimes e_{n-i}$ with the convention $e_{2^k+1} = 0$ for $k \geq 2$. Define $y^*_n \in H^{2^n+1}(BBSO; Z_2)$ for $n \geq 1$ by requiring that y^*_n be primitive and that $\alpha^*(y^*_n) = f^*_{2^{n-1}+1}$. Then

$$H^*(BBSO; Z_2) = E\{e_n \mid n \geq 4, n \neq 2^k + 1\} \otimes E\{y^*_n \mid n \geq 1\}$$

as algebras and $\mathscr{P}^{I_n}(y^*_1) = y^*_n$ for $n \geq 2$ where $I_n = (2^{n-1}, 2^{n-2}, \ldots, 4, 2)$.

Theorem 92. *In $H^*(BBSO;Z_2)$ for $r \geq 0$, $n \geq 4$, $n \neq 2^k+1$ and $m \geq 1$, $Q^r_*(y^*_m) = 0$ and $Q^r_*(e_n) = (n-r-1, 2r-n)e_{n-r}$ with the convention $e_1 = 0$, $e_{2^k+1} = 0$ for $k \geq 0$.*

Proof. By induction on m, the Nishida relations and the fact that $\mathcal{P}^{I_m}(y^*_1) = y^*_m$ for $m \geq 2$, it is easy to show that $Q^r_*(y^*_m) = 0$ for $r \geq 0$ and $m \geq 1$. $Q^r_*(e_n)$ is computed by using Theorem 63 and the definition $e_n = \gamma^*(w_n)$.

Define $y_n \in H_{2^n+1}(BBSO;Z_2)$ for $n \geq 1$ by $y_n = \alpha_*(f_{2^{n-1}+1})$. Let Z_k be the unique primitive element of $H_k(BBSO;Z_2)$ for k odd or $k = 2^n+2$, $k \geq 4$ and $n \geq 1$. Then

$$H_*(BBSO;Z_2) = E\{y_n \mid n \geq 1\}$$
$$\otimes\, P\{Z_k \mid k \text{ odd}, k \geq 4, k \neq 2^n+1, \text{ or } k = 2^n+2, n \geq 1\}$$

as Hopf algebras.

Theorem 93. *In $H_*(BBSO;Z_2)$ for $r \geq 0$, $n \geq 1$, k odd, $k \neq 2^n+1$, or $k = 2^n+2$ and $k \geq 4$, $Q^r(y_n) = 0$ and $Q^r(Z_k) = (k-1, r-k)Z_h^{2^t}$ where $r+k = 2^{t'}h'$, h' odd; $h = h'$ and $t = t'$ if $h' \neq 2^n+1$ for all $n \geq 1$ and $h' \neq 1$; $h = 2h'$ and $t = t'-1$ if $h' = 2^n+1$ for some $n \geq 1$; and $h = 4$ and $t = t'-2$ if $h' = 1$.*

Proof. This theorem is obtained by dualizing Theorem 92.

Theorem 94. $Q_{\mathcal{R}}H_*(BBSO;Z_2) = \{y_n \mid n \geq 1\} \cup \{Z_{2^a+2^b+1} \mid 0 \leq a < b\}$.

Proof. This theorem follows from Theorem 65(b) and 93.

Corollary 95. *$H_*(BBSO;Z_2)$ is an indecomposable $\mathfrak{A}$-Hopf algebra.*

Proof. $\mathcal{P}^{I_n}(y^*_1) = y^*_n$ for $n \geq 2$ and $\mathcal{P}^1(y^*_1) = e_4$. Furthermore, $E\{e_n \mid n \geq 4, n \neq 2^k+1\}$ is an indecomposable $\mathfrak{A}$-Hopf algebra because $H^*(B\,\mathrm{Spin};Z_2)$ is an indecomposable $\mathfrak{A}$-Hopf algebra by Corollary 67. Thus, the $\mathfrak{A}$-Hopf algebra $H^*(BBSO;Z_2)$ cannot be decomposed.

8. **Proofs of Theorems 5, 6 and 7.** The preceding sections have been based on Theorems 5, 6 and 7 which describe the action of the Dyer-Lashof algebra on $H_*(BU)$ and $H^*(BU)$. We will prove these theorems in this section. We will give a second proof of Theorems 5 and 6 in the case $p = 2$ which is based on cobordism theory. As part of our proof of Theorem 7 we will produce an algorithm for calculating the action of the Dyer-Lashof algebra on $H_*(BU)$ (see Theorem 97). In conclusion, we illustrate this algorithm by computing $Q^{2r}(a_n)$ in $H_*(BU;Z_2)$ for $r+n \leq 10$. We begin by recalling the action of $\mathfrak{A}^{op}$ on $PH_*(BU)$ which we then use in proving Theorem 5.

Lemma 96. *In $H_*(BU)$ for $r \geq 0$ and $n \geq 1$,*

$$\mathcal{P}_*^r(\mathfrak{p}_n) = (r, n - pr - 1)\mathfrak{p}_{n-r(p-1)}$$
$$[\mathcal{P}_*^{2r}(\mathfrak{p}_n) = (r, n - 2r - 1)\mathfrak{p}_{n-r}].$$

Proof. Note that $\mathcal{P}_*^r(a_k) = \alpha a_1$ with $0 \neq \alpha \in Z_p$ if and only if $k = p$ and $r = 1$ $[k = 2$ and $r = 2]$ because $\mathcal{P}^1(c_1) = c_1^p$ and $\mathcal{P}^r(c_1) = 0$ if $r \geq 2$ $[\mathcal{P}^2(c_1) = c_1^2$ and $\mathcal{P}^r(c_1) = 0$ if $r > 2]$. Now for all $k \geq 1$, a_1^k is a summand of $\mathfrak{p}_k$ and the only monomial summand of $\mathfrak{p}_n$ that can hit $a_1^{n-r(p-1)}$ under $\mathcal{P}_*^r$ is $(r, n - pr)n/[n - r(p - 1)]a_p^r a_1^{n-pr}$. Clearly

$$(r, n - pr)n/[n - r(p - 1)] \equiv (r, n - pr - 1) \bmod p.$$

Proof of Theorem 5. Define $\lambda_{r,n} \in Z_p$ by $Q^r(\mathfrak{p}_n) = \lambda_{r,n}\mathfrak{p}_{n+r(p-1)}$ $[Q^{2r}(\mathfrak{p}_n) = \lambda_{r,n}\mathfrak{p}_{n+r}]$ if r and n are positive integers, and let $\lambda_{r,n} = 0$ if r and n are rational numbers which are not positive integers. There is the Nishida relation

$$\mathcal{P}_*^1 \circ Q^{r+1} = rQ^r \qquad [\mathcal{P}_*^2 \circ Q^{2r+2} = rQ^{2r}].$$

Let $r \equiv \bar{r} \bmod p$ with $1 \leq \bar{r} \leq p$. Iterate the above Nishida relation $p - \bar{r}$ times

$$(r + p - \bar{r} - 1)!/(r - 1)!\, Q^r = \mathcal{P}_*^1 \circ \cdots \circ \mathcal{P}_*^1 \circ Q^{r+p-\bar{r}}$$
$$[rQ^{2r} = \mathcal{P}_*^2 \circ Q^{2r+2}].$$

Evaluate this identity on $\mathfrak{p}_n$ to obtain

$$\lambda_{r,n}\mathfrak{p}_{n+r(p-1)} = (r - 1)!/(r + p - \bar{r} - 1)!\, \lambda_{n+p-\bar{r},n}\mathcal{P}_*^1 \circ \cdots \circ \mathcal{P}_*^1(\mathfrak{p}_{n+(r+p-\bar{r})(p-1)})$$
$$[\bar{r}\lambda_{r,n}\mathfrak{p}_{n+r} = \lambda_{r+1,n}\mathcal{P}_*^2(\mathfrak{p}_{n+r+1})].$$

By Lemma 96,

$$(*) \qquad \lambda_{r,n} = (r - 1, n + p - \bar{r} - 1)/(n - 1, r + p - \bar{r} - 1)\lambda_{r+p-\bar{r},n}$$
$$[\lambda_{r,n} = n\lambda_{r+2-\bar{r},n}].$$

Let $n \equiv \bar{n} \bmod p$ with $1 \leq \bar{n} \leq p$ and write $n + p - \bar{n} = n'p^e$ with $n' \not\equiv 0 \bmod p$. We now show that if $s \geq 1$ and p divides s then

$$(**) \qquad \lambda_{s,n} = \lambda_{s/p^e,n}.$$

Let B, the Bott map, be given as the composite map

$$H_{2k}(BU) \xrightarrow{\cong} H_{2k}(\Omega SU) \xrightarrow{\sigma_*} H_{2k+1}(SU) \xrightarrow{j_*} H_{2k+1}(U) \xrightarrow{\mathcal{U}} H_{2k+2}(BU)$$

for $k \geq 1$. Recall that $B(a_n) = (-1)^n\mathfrak{p}_{n+1}$, and hence $B(\mathfrak{p}_n) = -n\mathfrak{p}_{n+1}$. Since $Q^s \circ B = B \circ Q^s$,

$$\begin{aligned}(-1)^{\bar{n}+1}&(n+p-\bar{n}-1)!/(n-1)!\,\lambda_{s,n}\mathfrak{p}_{n+s(p-1)+p-\bar{n}}\\ &= B\circ\cdots\circ B(\lambda_{s,n}\mathfrak{p}_{n+s(p-1)})\\ &= B\circ\cdots\circ BQ^s(\mathfrak{p}_n) \qquad [B\circ\cdots\circ B\circ Q^{2s}(\mathfrak{p}_n)]\\ &= (-1)^{\bar{n}+1}(n+p-\bar{n}-1)!/(n-1)!\,Q^s(\mathfrak{p}_{n+p-\bar{n}}) \qquad [Q^{2s}(\mathfrak{p}_{n+2-\bar{n}})]\\ &= (-1)^{\bar{n}+1}(n+p-\bar{n}-1)!/(n-1)!\,\lambda_{s,n+p-\bar{n}}\mathfrak{p}_{n+s(p-1)+p-\bar{n}}.\end{aligned}$$

Thus, $\lambda_{s,n} = \lambda_{s,n+p-\bar{n}} = \lambda_{(s/p)^e,n'}$ since $n+p-\bar{n} = p^e n'$ and $\mathfrak{p}_{n'p^e} = \mathfrak{p}_{n'}^{p^e}$. Thus, (**) has been proved. We combine (*) and (**) to prove Theorem 5 by induction on $r+n$. The case $r = n = 1$ is clear. Assume that Theorem 5 is valid for $Q^{r'}(\mathfrak{p}_{n'})$ if $r'+n' < r+n$. (*) reduces the computation of $\lambda_{r,n}$ to the computation of $\lambda_{r+p-\bar{r},n}$. However, (**) is applicable to $\lambda_{r+p-\bar{r},n}$, and hence we know $\lambda_{r+p-\bar{r},n}$ by the induction hypothesis. Thus,

$$\lambda_{r,n} = \frac{(-1)^{r-\bar{r}+n-\bar{n}}(r-1)!\,(n+p-\bar{r}-1)!}{(n-1)!\,(n+p-\bar{n}-1)!\,(r-\bar{r}-n+\bar{n})!}.$$

If $\bar{r} = \bar{n}$ then this equation becomes $\lambda_{r,n} = (-1)^{r+n}(n-1, r-n)$. If $\bar{r} > \bar{n}$ then

$$\begin{aligned}\lambda_{r,n} &= (-1)^{r-\bar{r}+n-\bar{n}}(n-1,r-n)\prod_{i=0}^{\bar{r}-\bar{n}-1}(r-n-i)\Big/\prod_{i=1}^{\bar{r}-\bar{n}}(n+p-\bar{n}+i)\\ &= (-1)^{r+n}(n-1,r-n)\end{aligned}$$

since for $0 \le i \le \bar{r}-\bar{n}-1$, $r-n-i \equiv -(n+p-\bar{n}-(\bar{r}-\bar{n}-i)) \bmod p$. If $\bar{r} < \bar{n}$ then $Q^r(\mathfrak{p}_n) = Q^r\circ B\circ\cdots\circ B(a_{n-\bar{n}}) = B\circ\cdots\circ B\circ Q^r(a_{n-\bar{n}}) = 0$ since $\deg Q^r(a_{n-\bar{n}}) \equiv -2\bar{r} \bmod p$ $[Q^{2r+1}(\mathfrak{p}_{2n}) = Q^{2r+1}(\mathfrak{p}_n^2) = 0]$. Furthermore, if $\bar{r} < \bar{n}$ then $(n-1, r-n) \equiv 0 \bmod p$. This completes the induction proof of Theorem 5.

I am grateful to I. Madsen for noticing that the inductive procedure for calculating the $\lambda_{r,n}$ which is given by (*) and (**) leads to binomial coefficients.

Proof of Theorem 6. Let $Q^r(a_n) = \alpha_{r,n}a_{n+r(p-1)} +$ decomposables $[Q^{2r}(a_n) = \alpha_{r,n}a_{n+r} +$ decomposables$]$ for some $\alpha_{r,n} \in Z_p$. Then

$$\begin{aligned}\alpha_{r,n}(-1)^n\mathfrak{p}_{n+r(p-1)+1} = B\circ Q^r(a_n)\ [B\circ Q^{2r}(a_n)] &= (-1)^nQ^r(\mathfrak{p}_{n+1})\\ [Q^{2r}(\mathfrak{p}_{n+1})] &= (-1)^{r+1}(n, r-n-1)\mathfrak{p}_{n+r(p-1)+1}.\end{aligned}$$

Hence $\alpha_{r,n} = (-1)^{r+n+1}(n, r-n-1)$.

Alternate proof of Theorems 5 and 6 when $p = 2$. By T. tom Dieck [6, Theorem 17.2], $\theta(P(r,n))$ is the coefficient of w^r in

$$\left[\sum_{k=0}^{\infty} w^kQ^{k+n}\circ\theta(RP^n)\right]\left[1+\sum_{i=1}^{\infty}a_iw^i\right]^{-r-1}$$

where $\theta\colon \mathfrak{N}^* \to H_*(BO; Z_2)$ is the normal characteristic number map, w is an indeterminate of degree 1 and $P(r,n)$ is the Dold manifold defined to be a suitable quotient of $S^r \times CP^n$. Thus,

$$\theta(P(r,n)) = Q^{r+n} \circ \theta(RP^n) + \text{decomposables}.$$

By R. Thom [25, Chapitre IV, §7], $\theta(RP^{2n}) = e_{2n} +$ decomposables, and by A. Dold [7], $\theta(P(2r,2n)) = (n, r-n-1)e_{2n+2r}$ +decomposables. Applying $\nu_*\colon H_*(BO; Z_2) \to H_*(BU; Z_2)$ we see that modulo decomposables,

$$(n, r-n-1)a_{n+r} = \nu_*((n, r-n-1)e_{2n+2r}) = \nu_* \circ \theta(P(2r,2n))$$
$$= \nu_* \circ Q^{2r+2n} \circ \theta(RP^{2n}) = Q^{2r+2n} \circ \nu_*(e_{2n}) = Q^{2r+2n}(a_n).$$

This proves Theorem 6, and the above proof that Theorem 5 implies Theorem 6 can be reinterpreted to show that Theorem 6 implies Theorem 5.

We next produce an algorithm for computing $Q^r(a_n)$ $[Q^{2r}(a_n)]$ by induction on $n + r(p-1)$ and for fixed $n + r(p-1)$ by induction on n. In this procedure, the coefficients of the monomial summands of $Q^r(a_n)$ are determined by induction on their product filtration degree. Theorem 5 gives us the leading coefficient, the comultiplicative Cartan formula and the induction hypothesis gives us the coefficient of any decomposable monomial except for those of the form $a_{p^e}^d$, $e \geq 1$, $d \geq 1$. The coefficients of such monomials can be determined by an appropriate Nishida relation and the induction hypothesis. This algorithm will employ eight properties of the $\mathscr{A}$-action on $H_*(BU)$. This observation will be exploited to prove Theorem 7.

Theorem 97. *There is an algorithm for computing $Q^r(a_n)$ $[Q^{2r}(a_n)]$ by induction on $n + r(p-1)$ and for fixed $n + r(p-1)$ by induction on n which uses the following properties that the Q^r satisfy on $H_*(BU)$.*

(a) $Q^r\colon H_{2n}(BU) \to H_{2n+2r(p-1)}(BU)$ $[Q^{2r}\colon H_{2n}(BU) \to H_{2n+2r}(BU)]$ *are linear maps for $r \geq 0$, $n \geq 0$.*

(b) $Q^r(a_n) = 0$ $[Q^{2r}(a_n) = 0]$ *if $n > r \geq 0$.*

(c) *The Q^r satisfy the multiplicative Cartan formula on $H_*(BU)$.*

(d) *The Q^r satisfy the comultiplicative Cartan formula on $H_*(BU)$.*

(e) $Q^n(a_n) = a_n^p$ $[Q^{2n}(a_n) = a_n^2]$ *for $n \geq 1$.*

(f) *The Q^r satisfy the Nishida relations on $H_*(BU)$.*

(g) $Q^r(\mathfrak{p}_n) = (-1)^{r+n}(n-1, r-n)\mathfrak{p}_{n+r(p-1)}$ $[Q^{2r}(\mathfrak{p}_n) = (n-1, r-n)\mathfrak{p}_{n+r}]$ *for $r \geq 0$ and $n \geq 1$.*

(h) $Q^r(a_n) = (-1)^{r+n+1}(n, r-n-1)a_{n+r(p-1)}$ *modulo decomposables* $[Q^{2r}(a_n) = (n, r-n-1)a_{n+r}$ *modulo decomposables*$]$ *for $r \geq 0$ and $n \geq 1$.*

Notes. (1) By the proofs of Theorems 5 and 6, this theorem is valid if conditions (g) and (h) are replaced by (g ′) $Q^r \circ B = B \circ Q^r$ for all $r \geq 0$.

(2) The theorems of §§3 through 8 are valid for any homology operations Q^r, $r \geq 0$, which are defined on the spaces under consideration in those theorems and

(i) satisfy (a) through (h) (or (a) through (f) and (g′)) above,

(ii) are natural with respect to the suspension maps defined on the homology of the classical groups, and

(iii) are natural with respect to the canonical maps between the homology of the various spaces discussed in these theorems.

Proof. By (d), $Q^1(a_1) = a_1^p$ $[Q^2(a_1) = a_1^2]$. Now assume our induction hypothesis for computing $Q^r(a_n)$ $[Q^{2r}(a_n)]$. Write $Q^r(a_n) = \sum_M \xi_M^{(r,n)} M$ $[Q^{2r}(a_n) = \sum_M \xi_M^{(r,n)} M]$ where $\xi_M^{(r,n)} \in Z_p$ and the sum is taken over all monomials M in a_i of degree $2n + 2r(p-1)$. Let $n + r(p-1) = ep^d$ with $d \geq 0$, $e \geq 1$ and $e \not\equiv 0 \mod p$. There are six cases for determining the $\xi_M^{(r,n)}$.

Case 1. M is arbitrary and $n = 1$.

$a_1 = \mathfrak{p}_1$ so $Q^r(a_1) = (-1)^{r+1}\mathfrak{p}_{r(p-1)+1}$ $[Q^{2r}(a_1) = \mathfrak{p}_{r+1}]$ by (g).

Case 2. $M = a_{n+r(p-1)}$ and n is arbitrary.

By (h), $\xi_M^{(r,n)} = (-1)^{r+n+1}(n, r-n-1)$ in this case.

Case 3. $M = a_{ep^d - p^f}$, $1 \leq f \leq d$ and if $f = d$ then e cannot be written as $cp^{t+1} + \sum_{i=0}^{t} (p-1)p^i$ for any $1 \leq c \leq p-1$ and $t \geq -1$.

Let p^g be the smallest power of p whose coefficient is not $p-1$ in the p-adic expansion of ep^{d-f}. In particular, $p^g = 1$ if $f < d$. Now $\xi_M^{(r,n)}$ can be found by considering the coefficient of $a_{ep^{d-f} - p^g(p-1)}$ in the Nishida relation

$$\mathcal{P}_*^{p^{g+f}} \circ Q^r(a_n) = \sum (-1)^{i+r}(p^{g+f} - pi, r(p-1) - p^{g+f+1} + p^i)$$

$$\cdot (i, n-pi)Q^{r-p^{g+f}+i}(a_{n-i(p-1)})$$

$$[\mathcal{P}_*^{2^{g+f+1}} \circ Q^{2r}(a_n) = \sum (2^{g+f} - 2i, r - 2^{g+f+1} + 2i)$$

$$\cdot (i, n-2i)Q^{2r-2^{g+f+1}+2i}(a_{n-i})].$$

Case 4. M is arbitrary and if $d > 0$ then e cannot be written as $cp^{t+1} + \sum_{i=0}^{t} (p-1)p^i$ for any $1 \leq c \leq p-1$ and $t \geq -1$.

We calculate $\xi_M^{(r,n)}$ by induction on the product filtration degree of M, written PFD(M). In Case 2 we computed $\xi_M^{(r,n)}$ when PFD(M) = 1. Now assume that PFD(M) > 1 and that $\xi_N^{(r,n)}$ is known if PFD(N) < PFD(M). If $M = a_{ep^d - p^f}$ for some $1 \leq f \leq d$ then we found $\xi_M^{(r,s)}$ in Case 3. If $M \neq a_{ep^d - p^f}$ for all $1 \leq f \leq d$ then $\psi(M)$ contains a summand $M' \otimes M''$ with deg $M' > 0$, deg $M'' > 0$ and PFD(M') + PFD(M'') = PFD(M). We can now evaluate $\xi_M^{(r,n)}$ by considering the coefficient of $M' \otimes M''$ in the comultiplicative Cartan formula applied to $Q^r(a_n)$ $[Q^{2r}(a_n)]$. We obtain

$$\sum_{i=0}^{r} \sum_{j=0}^{n} Q^i(a_j) \otimes Q^{r-i}(a_{n-j}) = \sum_M \xi_M^{(r,n)} \psi(M)$$

$$\left[\sum_{i=0}^{r} \sum_{j=0}^{n} Q^{2i}(a_j) \otimes Q^{2r-2i}(a_{n-j}) = \sum_M \xi_M^{(r,n)} \psi(M)\right].$$

Case 5. M is arbitrary and $d > 0$, $e = cp^{t+1} + \sum_{i=0}^{t} (p-1)p^i$ with $1 \leq c \leq p-1$ and $t \geq 0$.

Define a Z_p-basis B of $H_{2n+2r(p-1)}(BU)$ to consist of $\mathfrak{p}_e^{p^d} = N_0$ and all the monomials in the a_i of degree $2n + 2r(p-1)$ except $a_e^{p^d}$. Let $Q^r(a_n) = \sum_{N\in B} \mu_N N$ $[Q^{2r}(a_n) = \sum_{N\in B} \mu_N N]$ where $\mu_N \in Z_p$. Use the method of Case 4 to find all of the μ_N except for μ_{N_0}. Let $(n + r - p^d)(p-1) = gp^h$ with $h \geq 1$, $g \geq 1$ and $g \not\equiv 0 \bmod p$. Then we can determine μ_{N_0} by considering the coefficient of $a_g^{p^h}$ in the Nishida relation

$$\mathcal{P}_*^{p^d} \circ Q^r(a_n) = \sum (-1)^{r+i}(p^d - pi, r(p-1) - p^{d+1} + pi) \cdot (i, n - pi)Q^{r-p^d+i}(a_{n-i(p-1)})$$

$$[\mathcal{P}_*^{2^{d+1}} \circ Q^{2r}(a_n) = \sum (2^d - 2i, r - 2^{d+1} + 2i)(i, n - 2i)Q^{2r-2^{d+1}+2i}(a_{n-i})].$$

Case 6. M is arbitrary and $d > 0$, $1 \leq e \leq p$.

The $\xi_M^{(r,n)}$ for M not equal to $M_0 = a_e^{p^d}$ can be found by the method of Case 4. Let p^h be the largest power of p that divides r. Use the methods of Cases 1 through 5 to determine $\xi_M^{(r+p^h,n)}$ for all $M = a_{k_1}^{e_1p^{d_1}} \cdots a_{k_t}^{e_tp^{d_t}}$ such that $1 \leq e_i \leq p-1$, $d > d_i \geq 0$, $k_i \geq 1$ and all the ordered triples $(d_i, e_i k_i)$ are distinct. Observing that $(p^h, r(p-1) - p^h) \not\equiv 0 \bmod p$, we find $\xi_{M_0}^{(r,n)}$ by considering the coefficient of $a_e^{p^d}$ in the Nishida relation

$$\mathcal{P}_*^{p^h} \circ Q^{r+p^h}(a_n) = -(p^h, r(p-1) - p^h)Q^r(a_n) + Q^{r+p^{h-1}}(a_{n-p^{h-1}(p-1)})$$

$$[\mathcal{P}_*^{2^{h+1}} \circ Q^{2r+2^{h+1}}(a_n) = Q^{2r}(a_n) + Q^{2r+2^h}(a_{n-2^{h-1}})]$$

where the second summand is omitted if $h = 0$.

Proof of Theorem 7. Define Z_p-linear maps $R_*^s\colon H^{2n}(BU) \to H^{2n-2s(p-1)}(BU)$ for $n \geq 0$ and $s \geq 0$ by $R_*^s(c_n) = (-1)^{s+n}(ps - n, n - s(p-1) - 1)c_{n-s(p-1)}$ if $(s,n) \neq (0,0)$ and $R_*^0(1) = 1$. Extend the domain of definition of these operations to all of $H^*(BU)$ by requiring that the R_*^s satisfy the multiplicative Cartan formula. Let $R^s = \operatorname{Hom}(R_*^s, 1)$ for $s \geq 0$. We will show that $R^s = Q^s$ $[R^s = Q^{2s}]$ as operations acting on $H_*(BU)$ by proving that the R^s satisfy properties (a) through (h) of Theorem 97. This will prove Theorem 7. (a), (b), (d) and (g) are immediate consequences of the definition of the R^s.

(c) To prove that the comultiplicative Cartan formula is valid for the R_*^s it suffices to show that for $ps \geq n \geq 1$,

$$\psi \circ R_*^s(c_n) = \sum_{i=0}^{s} \sum_{j=0}^{n} R_*^i(c_j) \otimes R_*^{s-i}(c_{n-j}).$$

That is, we must show that

$$\sum_{h=0}^{n-s(p-1)} (ps - n, n - s(p-1) - 1)c_h \otimes c_{n-s(p-1)-h}$$
$$= \sum_{i=0}^{s} \sum_{j=0}^{n} (pi - j, j - i(p-1) - 1) \cdot (ps - pi - n + j, n - j - (s-i)(p-1) - 1) \cdot c_{j-i(p-1)} \otimes c_{n-j-(s-i)(p-1)}.$$

This follows from the following computation:

$$\sum_{j-i(p-1)=h} ((pi-j,j-i(p-1))(ps-pi-n+j,n-j-(s-i)(p-1)-1)$$

$$= \sum_{i=h}^{s-1} \binom{i-1}{i-h}\binom{s-i-1}{ps-n-i+h} \quad \text{using Notation 54 (1)}$$

$$= \sum_{k=0}^{s-h-1} \binom{k+h-1}{k}\binom{s-k-h-1}{ps-n-k} \quad \text{where } k = i-h$$

$$= (-1)^{s+n} \sum_{k=0}^{ps-n} \binom{-h}{k}\binom{h-n+s(p-1)}{ps-n+k}$$

$$+ \sum_{k=ps-n+1}^{s-h-1} \binom{k+h-1}{k}\binom{s-k-h-1}{ps-n-k}$$

by Notation 54 (2)

$$= (-1)^{s+n}\binom{s(p-1)-n}{ps-n}$$

by Notation 54 (3) and the definition $\binom{a}{b} = 0$ if $b < 0$

$$= (-1)^{s+n}(ps-n,n-s(p-1)-1) \quad \text{by Notation 54 (2)}\,.$$

(e) We prove that $R^n(a_n) = a_n^p$ by induction on $n \geq 1$. Clearly $R^1(a_1) = a_1^p$ since $a_1 = \mathfrak{p}_1$ and $a_1^p = \mathfrak{p}_p$. $R^n(a_n) = a_n^p + \alpha\mathfrak{p}_{np}$ for some $\alpha \in Z_p$ by the comultiplicative Cartan formula and the induction hypothesis. By definition, $a_n = (c_1^n)^*$ and $\mathfrak{p}_{np} = c_{np}^*$. Hence $\alpha = 0$ since, for $n > 1$, $R_*^n(c_{np})$ does not contain a nonzero multiple of c_1^n as a summand.

(f) We show by induction on $n \geq 1$ that the R_*^s satisfy the Nishida relations when these relations are evaluated on c_n. This assertion implies (f) by the multiplicative Cartan formula. The Nishida relations are clearly true modulo decomposables because the R_*^s and the Q_*^s are equal modulo decomposables. Since the Nishida relations do not raise degree in cohomology, there are no nontrivial Nishida relations on c_n for $1 \leq n \leq p-1$. If $n-(s-k)(p-1) = 1$ then

$$R_*^s \circ \mathcal{P}^k(c_n) = (-1)^{s+1}(k,n-k-1)c_1 = (-1)^k(k,s(p-1)-pk)R_*^{s-k}(c_n)$$

$$[R_*^s \circ \mathcal{P}^{2k}(c_n) = (k,n-k-1)c_1 = (k,s-2k)R_*^{2s-2k}(c_n)].$$

If $n = p$ then the only nontrivial Nishida relation on c_p which is not of the above form is

$$R_*^p \circ \mathcal{P}^p(c_p) = c_1^p = \mathcal{P}^1 \circ R_*^1(c_p)$$

$$[R_*^2 \circ \mathcal{P}^4(c_2) = c_1^2 = \mathcal{P}^2 \circ R_*^2(c_2)].$$

Now assume that $n > p$ and that all Nishida relations are valid on c_m for $m < n$. By the induction hypothesis and the comultiplicative Cartan formula, if $s \geq k$ then

$$R_*^s \circ \mathcal{P}^k(c_n) - \sum (-1)^{k+i}(k - pi, s(p-1) - pk + pi)\mathcal{P}^i \circ R_*^{s-k+i}(c_n)$$
$$= \alpha \mathfrak{p}_{n-(a-k)(p-1)}$$
$$[R_*^s \circ \mathcal{P}^{2k}(c_n) - \sum (k - 2i, s - 2k + 2i)\mathcal{P}^{2i} \circ R_*^{s-k+i}(c_n) = \alpha \mathfrak{p}_{n+k-s}]$$

for some $\alpha \in Z_p$. If $n - (s-k)(p-1)$ is not divisible by p then $\mathfrak{p}_{n-(s-k)(p-1)}$ is indecomposable, and hence $\alpha = 0$ in this case. If $n - (s-k)(p-1) > p$ then $\alpha = 0$ by Lemma 23 since $\alpha \mathfrak{p}_{n-(s-k)(p-1)}$ has $\alpha c_1^{n-(s-k)(p-1)}$ as a summand. If $n - (s-k)(p-1) = p$ then by the reasoning of the preceding cases, all the Nishida relations are valid when they are evaluated on c_{n+i} for $1 \leq i \leq p-2$, or for $i = p-1$ if $n + p - 1 - (s-k)(p-1) \neq p$. Hence

$$\bar{\psi}\{R_*^{s+1} \circ \mathcal{P}^k(c_{n+p}) - \sum (-1)^{k+i}(k - pi, (s+1)(p-1) - p(k-i))\mathcal{P}^i$$
$$\circ R_*^{s-k+i+1}(c_{n+p})\}$$
$$= \alpha c_1^p \otimes c_1 + \alpha c_1 \otimes c_1^p$$
$$[\bar{\psi}\{R_*^{s+1} \circ \mathcal{P}^{2k}(c_{n+2}) - \sum (k - 2i, s + 1 - 2k + 2i)\mathcal{P}^{2i} \circ R_*^{s-k+i+1}(c_{n+2})\}$$
$$= \alpha c_1^2 \otimes c_1 + \alpha c_1 \otimes c_1^2].$$

Thus for some $\gamma \in Z_p$,

$$R_*^{s+1} \circ \mathcal{P}^k(c_{n+p}) - \sum (-1)^{k+i}(k - pi, (s+1)(p-1) - p(k-i))\mathcal{P}^i$$
$$\circ R_*^{s-k+i+1}(c_{n+p})$$
$$= \alpha c_1^{p+1} + \gamma \mathfrak{p}_{p+1}$$
$$[R_*^{s+1} \circ \mathcal{P}^{2k}(c_{n+2}) - \sum (k - 2i, s + 1 - 2k + 2i)\mathcal{P}^{2i} \circ R_*^{s-k+i+1}(c_{n+2})$$
$$= \alpha c_1^3 + \gamma \mathfrak{p}_3].$$

However, $\gamma = 0$ because $\mathfrak{p}_{p+1}$ is indecomposable. Hence $\alpha = 0$ by Lemma 23 since $\mathfrak{p}_{p+1}$ has c_1^{p+1} as a summand. Of course, for $p = 2$ use of the Wu formula simplifies this proof.

(h) We prove the following which is equivalent to (h):

$$(*) \qquad R_*^s(\mathfrak{p}_{s(p-1)+n}) = (-1)^{n+s+1}(n, s-n-1)\mathfrak{p}_n.$$

Our proof is by induction on $n \geq 1$ and for fixed n by induction on $r \geq 0$. If $0 \leq r \leq n$ then both sides of $(*)$ equal zero. Now assume the induction hypothesis, and assume that $r > n$. If $n \not\equiv 0 \bmod p$ then $\mathfrak{p}_n = (-1)^{n+1} n c_n$

modulo decomposables. Hence (*) follows from the definition of R_*^s in this case. If $n \equiv 0$ and $s \not\equiv 1 \bmod p$ then $R_*^s \circ \mathcal{P}^1(\mathfrak{p}_{n+(s-1)(p-1)}) = (s-1)R_*^{s-1}(\mathfrak{p}_{n+(s-1)(p-1)})$ $[R_*^s \circ \mathcal{P}^2(\mathfrak{p}_{n+s-1}) = R_*^{s-1}(\mathfrak{p}_{n+s-1})]$. Hence

$$-(s-1)R_*^s(\mathfrak{p}_{n+s(p-1)}) = (-1)^{n+s}(s-1)(n, s-n-2)\mathfrak{p}_n.$$

Thus since $s-1 \equiv s-n-1 \bmod p$, $R_*^s(\mathfrak{p}_{n+s(p-1)}) = (-1)^{n+s+1}(n, s-n-1)\mathfrak{p}_n$, and (*) is correct in this case. If $n \equiv 0$ and $s \equiv 1 \bmod p$ then

$$\begin{aligned} R_*^s(\mathfrak{p}_{n+s(p-1)}) &= (s-1)!/(s+p-2)!\, R_*^{s+p-1} \circ \mathcal{P}^1 \circ \cdots \circ \mathcal{P}^1(\mathfrak{p}_{n+s(p-1)}) \\ &[= R_*^{s+1} \circ \mathcal{P}^2(\mathfrak{p}_{n+s})] \\ &= R_*^{s+p-1}(\mathfrak{p}_{n+(s+p-1)(p-1)}) = (-1)^{s+n+1}(n, s+p-n-2)\mathfrak{p}_n \\ &= (-1)^{n+s+1} \\ &\quad (n, s-n-1)\mathfrak{p}_n \quad \text{by the induction hypothesis} \end{aligned}$$

since $\mathfrak{p}_{n+(s+p-1)(p-1)} = \mathfrak{p}^p_{n'+(s'+1)(p-1)}$ for $n = pn'$ and $s = ps' + 1$ and $(n', s'-n') = (n, s-n+1)$. This completes the proof of (*).

We have two methods for computing the Dyer-Lashof operations on the a_n in $H_*(BU)$. We know how these operations act on the basis of $H_*(BU)$ which consists of the dual basis of the monomials in the Chern classes. Then we can pass to the basis of monomials in the a_n by using the techniques of R. Van de Velde [27]. This procedure, however, is prohibitively difficult. An alternative method for calculating the $Q^r(a_n)$ is given by Theorem 97. It is an easy algorithm to use. In using either of these two methods, Theorem 8 reduces the work involved substantially. The following is a list of $Q^{2r}(a_n)$ in $H_*(BU; Z_2)$ for $1 \leq n \leq r$ and $n + r \leq 10$. Note that this list translates readily into a computation of $Q^{4r}(b_n)$ in $H_*(B\mathrm{Sp}; Z_2)$ and of $Q^r(e_n)$ in $H_*(BO; Z_2)$ for $1 \leq n \leq r$ and $n + r \leq 10$. More generally, the two methods of computation described above also apply to $H_*(B\mathrm{Sp})$ and to $H_*(BO)$.

$$Q^2(a_1) = a_1^2$$

$$Q^4(a_1) = a_3 + a_2a_1 + a_1^3$$

$$Q^4(a_2) = a_2^2$$

$$Q^6(a_1) = a_1^4$$

$$Q^6(a_2) = a_5 + a_4a_1 + a_3a_2 + a_2^2a_1$$

$$Q^8(a_1) = a_5 + a_4a_1 + a_3a_2 + a_3a_1^2 + a_2^2a_1 + a_2a_1^3 + a_1^5$$

$$Q^6(a_3) = a_3^2$$

$$Q^8(a_2) = a_6 + a_5a_1 + a_4a_2 + a_4a_1^2 + a_3a_2a_1 + a_2^3 + a_2^2a_1^2$$

$$
\begin{aligned}
Q^{10}(a_1) &= a_3^2 + a_2^2a_1^2 + a_1^6\\
Q^{8}(a_3) &= a_7 + a_6a_1 + a_5a_2 + a_4a_3 + a_3^2a_1\\
Q^{10}(a_2) &= a_5a_1^2 + a_4a_1^3 + a_3a_2a_1^2 + a_2^2a_1^3\\
Q^{12}(a_1) &= a_7 + a_6a_1 + a_5a_2 + a_4a_3 + a_5a_1^2 + a_3^2a_1 + a_3a_2^2 + a_4a_1^3\\
&\quad + a_3a_2a_1^2 + a_2^3a_1 + a_3a_1^4 + a_2a_1^5 + a_1^7\\
Q^{8}(a_4) &= a_4^2\\
Q^{10}(a_3) &= a_5a_3 + a_6a_1^2 + a_5a_2a_1 + a_4a_3a_1 + a_3^2a_2 + a_3^2a_1^2\\
Q^{12}(a_2) &= a_5a_3 + a_5a_2a_1 + a_4a_3a_1 + a_3^2a_2 + a_5a_1^3 + a_4a_2a_1^2\\
&\quad + a_2^4 + a_4a_1^4 + a_3a_2a_1^3 + a_2^3a_1^2 + a_2^2a_1^4\\
Q^{14}(a_1) &= a_1^8\\
Q^{10}(a_4) &= a_9 + a_8a_1 + a_7a_2 + a_6a_3 + a_5a_4 + a_4^2a_1\\
Q^{12}(a_3) &= a_6a_3 + a_6a_2a_1 + a_5a_3a_1 + a_5a_2^2 + a_4a_3a_2 + a_3^3\\
&\quad + a_6a_1^3 + a_5a_2a_1^2 + a_4a_3a_1^2 + a_3^2a_1^3\\
Q^{14}(a_2) &= a_9 + a_8a_1 + a_7a_2 + a_6a_3 + a_5a_4 + a_5a_2^2 + a_4^2a_1\\
&\quad + a_4a_2^2a_1 + a_3a_2^3 + a_5a_1^4 + a_2^4a_1 + a_4a_1^5 + a_3a_2a_1^4 + a_2^2a_1^5\\
Q^{16}(a_1) &= a_9 + a_8a_1 + a_7a_2 + a_6a_3 + a_5a_4 + a_7a_1^2 + a_5a_2^2 + a_4^2a_1\\
&\quad + a_3^3 + a_6a_1^3 + a_5a_2a_1^2 + a_4a_3a_1^2 + a_4a_2^2a_1 + a_3a_2^3 + a_3^2a_2a_1\\
&\quad + a_5a_1^4 + a_2^4a_1 + a_3a_2a_1^4 + a_4a_1^5 + a_3a_1^6 + a_2^2a_1^5 + a_2a_1^7 + a_1^9\\
Q^{10}(a_5) &= a_5^2\\
Q^{12}(a_4) &= a_{10} + a_9a_1 + a_8a_2 + a_6a_4 + a_8a_1^2 + a_7a_2a_1 + a_6a_3a_1\\
&\quad + a_5a_4a_1 + a_4^2a_2 + a_4^2a_1^2\\
Q^{14}(a_3) &= a_5a_3a_1^2 + a_3^2a_2^2 + a_6a_1^4 + a_5a_2a_1^3 + a_4a_3a_1^2 + a_3^2a_2a_1^2 + a_3^2a_1^4\\
Q^{16}(a_2) &= a_{10} + a_9a_1 + a_8a_2 + a_6a_4 + a_8a_1^2 + a_7a_2a_1 + a_6a_2^2 + a_6a_3a_1\\
&\quad + a_5a_4a_1 + a_4^2a_2 + a_5a_3a_1^2 + a_5a_2^2a_1 + a_4^2a_1^2 + a_4a_2^3\\
&\quad + a_5a_2a_1^3 + a_4a_3a_1^3 + a_4a_2^2a_1^2 + a_3^2a_2a_1^2 + a_3a_2^3a_1 + a_2^5\\
&\quad + a_5a_1^5 + a_4a_2a_1^4 + a_3^2a_1^4a_3a_2^2a_1^3 + a_2^4a_1^2 + a_4a_1^6 + a_3a_2a_1^5 + a_2^2a_1^6\\
Q^{18}(a_1) &= a_{10}^2 + a_4^2a_1^2 + a_3^2a_2^2 + a_2^3a_1^4 + a_2^4a_1^2 + a_2^2a_1^6 + a_1^{10}.
\end{aligned}
$$

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